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# Asymptotic approximations in computing

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# ASYMPTOTIC APPROXIMATIONS
# IN COMPUTING

by

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# INTRODUCTION

The motivation for this paper is twofold. First, many authors in applied mathematics, physics, astronomy, probability and statistics, engineering, etc., use the concepts of asymptotic series freely and presuppose some familiarity with the subject of the part of their readers. Second, asymptotic expansions are currently gaining wider use as an effective means of evaluating difficult functions in routines for automatic digital computers. Their use in such routines alleviates the necessity of storing large arrays of tabled function values and also obviates the need for having a routine in a program to interpolate for an intermediate value of an argument.

Although asymptotic approximations are not universally applicable to the problem of evaluating a complicated function, the frequent cases in which asymptotic methods do work, and the ease with which they can be applied, both recommend them.

# CHAPTER I

## HISTORICAL SKETCH AND EXAMPLES OF EARLY USES OF ASYMPTOTIC SERIES

### Euler's constant, $\gamma$

Prior to the theory of convergence of Abel and of Cauchy, mathematicians used many series in their work which were divergent. Some of those which were used for numerical computation belong to a class which are now known as asymptotic series. Before going to a more technical discussion, consider the following striking example of an asymptotic series used to compute the value of Euler's constant, $\gamma$ , which is defined by

$$(1.1)\quad \gamma = \lim_{n\to\infty}\left[1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots+\frac{1}{n}-\ln n\right] = \lim_{n\to\infty}\left[\sum_{k=1}^{n}\frac{1}{k}-\ln n\right].$$

This is an interesting limit. It is not at all obvious that it exists, nor does the definition give any hint as to how its value should be computed, since about $10^6$ terms would be needed to compute $\gamma$ to six digits using this definition. Even with a fast electronic computer this would not be a good way to calculate the value of $\gamma$.

About 1755 Leonhard Euler, by applying the device now often called the Euler-Maclaurin sum formula, derived the equation (see Bromwich, pages 324-325)

$$(1.2) \quad 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{n} - \ln n - \gamma = \frac{B_2}{2n^2} + \frac{B_4}{4n^4} + \ldots$$

where the $B_{2k}$ are Bernoulli numbers. The series on the right is not convergent; but Euler established that the error incurred in approximating $\gamma$ with the above equation by truncating the series at any particular term is less in absolute value than the next term of the series and that it is of the same sign. All of the terms in (1.2) except $\gamma$ and $\ln n$ are rational and can be calculated easily to any desired degree of precision. We can also determine $\ln n$ as accurately as we please. Therefore we can truncate the series, thereby removing any difficulties due to divergence, and then can determine $\gamma$ to within a tolerance represented by the first term omitted in the truncation. If none of the terms $B_{2k}/(2k\ n^{2k})$ is small in absolute value, then the fact that the error in $\gamma$ is less than the first term omitted is not computationally helpful. However, if the terms do become small in absolute value at some point in the series, then we can truncate the series, stopping just before the term least in absolute value. In this way, by Euler's result concerning the size of the error, we can be sure that the error is less than this least term. This is also the best approximation that can be obtained from this series, for this value of n.

## TABLE 1.0

## SYMBOLS NEEDING CLARIFICATION

| Symbol | Meaning, or how to be read |
| --- | --- |
| $\sim$ | "is asymptotic to" |
| 6D | The fractional part of a number is given to six decimal places. |
| 8S | This designates a number which is expressed to eight significant digits. |
| .9846E-14 | This is a common computer language symbolism for $0.9846 \times 10^{-14}$. |
| X(K) | In computer usage, this is equivalent to the ordinary subscripted variable, $x_K$. |
| ph Z | The phase, or argument, of the complex number, Z. |
| ln x | The natural logarithm of x. |
| log x | The logarithm to the base 10 of x. |
| x = 1(.05)2.5 | This designates the values of x from 1 to 2.5, in steps of .05, there being here thirty-one such values. |

It will be noted that in Euler's equation (1.2) n is a constant, and the various approximations that may be had for $\gamma$ by truncating the series at different places depend on the subscripts of the Bernoulli numbers. Therefore, if the precision guaranteed by the minimum term is not small enough, then one could choose a larger value of n, giving a larger number of terms to sum on the left- and right-hand sides of equation (1.2), but also holding the possibility that the least term of the series will be smaller than in the previous case.

By putting n = 10 in equation (1.2), Euler was able to compute the value of $\gamma$ to fifteen significant digits. Euler's computations have been repeated on an IBM 1620 computer, and are given in Table 1.1.

## Stirling's Approximation to ln(n!)

A second example that is probably more familiar to the reader is Stirling's approximation to the logarithm of the factorial function. In 1730 Stirling published in his <u>Methodus Differentialis</u> an infinite series for ln(n!) which is equivalent to the more modern notation

(1.3)
$$\ln(n!) = (n+\tfrac{1}{2})\ln(n+\tfrac{1}{2}) - n - \tfrac{1}{2} + \tfrac{1}{2}\ln(2\pi) + \sum_{k=1}^{\infty} \frac{B_{2k}(\tfrac{1}{2})}{(2k-1)(2k)(n+\tfrac{1}{2})^{2k-1}}$$

where $B_q(x)$ is the q-th Bernoulli polynomial. A lucid introduction to these numbers and polynomials is to be

TABLE 1.1 EULER'S CONSTANT, $\gamma$

```
 1/ 1= .1000000000                            E+01
 1/ 2= .5000000000                            E+00
 1/ 3= .3333333333333333333333333333E+00
 1/ 4= .250000000000   0                      E+00
 1/ 5= .2000000000     0                      E+00
 1/ 6= .1666666666666666666666666666E+00
 1/ 7= .1428571428571428571428571428E+00
 1/ 8= .1250000000000 0                       E+00
 1/ 9= .1111111111111111111111111111E+00
 1/10= .1000000000     0                      E+00
 1/2N= .5000000000     0                      E-01
 LOGN= .2302585092994045684017991454E+01

 1 + 1/2 + 1/3 +...+ 1/N - 1/2N - LOGN =
       .5763831609742082842359767980E+00
```

THE BERNOULLI TERMS, B(2K)/2KN**2K

| K | B(2K)/2KN**2K |
|---|---|
| 1 | .8333333333333333333333333330E-03 |
| 2 | -.8333333333333333333333333332E-06 |
| 3 | .3968253968253968253968253966E-08 |
| 4 | -.4166666666666666666666666666E-10 |
| 5 | .7575757575757575757575757575E-12 |
| 6 | -.2109279609279609279609279609E-13 |
| 7 | .8333333333333333333333333328E-15 |
| 8 | -.4432598039215686274509803921E-16 |
| 9 | .3053954330270119743803954330E-17 |
| 10 | -.2645621212121212121212121212E-18 |
| 11 | .2814601449275362318840579710E-19 |
| 12 | -.3607510546398046398046398046E-20 |
| 13 | .5482758333333333333333333330E-21 |
| 14 | -.9749368238505747126436781607E-22 |
| 15 | .2005269579668807894614346227E-22 |
| 16 | -.4723848677216299019607843134E-23 |
| 17 | .1263572479591666666666666666E-23 |
| 18 | -.3808793112524536881155302205E-24 |
| 19 | .1285085049930508333333333333E-24 |
| 20 | -.4824144835485017037158167035E-25 |
| 21 | .2004031065651625273810842166E-25 |
| 22 | -.9167743603195330775699275361E-26 |
| 23 | .4597988834365650349043794326E-26 |
| 24 | -.2518047192145109569708902331E-26 |
| 25 | .1500173349215392873371144015E-26 |
| 26 | -.9689957887463594065649794288E-27 |
| 27 | .6764588237929282099094524227E-27 |
| 28 | -.5089065946866228968976633291E-27 |

The last two digits of are in error due to a combination of asymptotic error and of round-off error.

Euler calculated those digits singly underlined and Gauss those which are doubly underlined.

THE APPROXIMATION TO EULER'S CONSTANT, GAMMA, IS
.5772156649015328606065120890E+00

TABLE 1.2

THE BERNOULLI NUMBERS, B(N)

| | |
|---|---|
| B( 0)= | .1000000000000000000000000000E+01 |
| B( 1)=- | .5000000000000000000000000000E+00 |
| B( 2)= | .1666666666666666666666666666E+00 |
| B( 4)=- | .3333333333333333333333333333E-01 |
| B( 6)= | .2380952380952380952380952380E-01 |
| B( 8)=- | .3333333333333333333333333333E-01 |
| B(10)= | .7575757575757575757575757575E-01 |
| B(12)=- | .2531135531135531135531135531E+00 |
| B(14)= | .1166666666666666666666666666E+01 |
| B(16)=- | .7092156862745098039215686274E+01 |
| B(18)= | .5497117794486215538847117794E+02 |
| B(20)=- | .5291242424242424242424242424E+03 |
| B(22)= | .6192123188405797101449275362E+04 |
| B(24)=- | .8658025311355311355311355311E+05 |
| B(26)= | .1425517166666666666666666666E+07 |
| B(28)=- | .2729823106781609195402298850E+08 |
| B(30)= | .6015808739006423683843038681E+09 |
| B(32)=- | .1511631576709215686274509803E+11 |
| B(34)= | .4296146430611666666666666666E+12 |
| B(36)=- | .1371165520508833277215908794E+14 |
| B(38)= | .4883323189735931666666666666E+15 |
| B(40)=- | .1929657934194006814863266814E+17 |
| B(42)= | .8416930475736826150005537098E+18 |
| B(44)=- | .4033807185405945541307681159E+20 |
| B(46)= | .2115074863808199160560145390E+22 |
| B(48)=- | .1208662652229652593460273119E+24 |
| B(50)= | .7500866746076964366855720075E+25 |
| B(52)=- | .5038778101481068914137893030E+27 |
| B(54)= | .3652877648481812335511043083E+29 |
| B(56)=- | .2849876930245088222626914643E+31 |

The last Bernoulli number in this table already contains round-off error, its numerator having exceeded the limitations of our 28S arithmetic.

Most other numbers involved in the calculation for $\gamma$ have some round-off error inherent in their final few digits.

found in Rainville (1960), (1967).[1] The series on the right is divergent; but Stirling was able to show that the error in approximating ln(n!) by truncating this series is of the same order as the first term omitted. For large n, theabsolute values of the terms in this series decrease very rapidly before they grow arbitrarily large. By taking the partial sum of the series up to, but omitting, the term of least absolute value, one obtains the best approximation afforded by this series for a particular value of n.

Following these and similar early uses of divergent and asymptotic series, the move to place analysis on a sounder basis by Abel, Cauchy and their contemporaries all but banished non-convergent series from mathematical work. Asymptotic series reappeared vigorously in the late nineteenth century. They occur in Stokes' work on the behavior of Bessel's functions and other functions for large values of the arguments. They are found in Stieltjes' work involving investigations of special functions and are there seen to be related to continued fractions. Asymptotic series are again encountered in the work of Henri Poincaré, to whom is due the credit for the modern definition of asymptotic equality, and for the introduction of the word "asymptotic." The preceding historical remarks are primarily from Bromwich (1926), Copson (1965), and Jeffreys (1962).[2]

Then in this century, with the development of atomic physics, modern diffraction theory, antenna design, and the space sciences and aeronautics, there has been increasing use of asymptotic approximations as a means of obtaining useful answers in a reasonable amount of time.

Let us return to a discussion of equation (1.2) from another standpoint. Equation (1.2) says that

$$\sum_{k=1}^{n} \frac{1}{k} - \ln n - \gamma - \frac{1}{2n} = \frac{B_2}{2n^2} + \frac{B_4}{4n^4} + \frac{B_6}{6n^6} + \ldots$$

The left-hand member of this equation is a well-defined (finite) real number for every finite integral value of n. The series on the right, as remarked earlier, is not convergent for any value of n. To see this, we can use the relation from Abramowitz and Stegun (1968):

$$(1.4) \qquad \frac{2(2k)!}{(2\pi)^{2k}} < \left|B_{2k}\right| < \frac{2(2k)!}{(2\pi)^{2k}} \frac{1}{1 - 2^{1-2k}} .$$

Taking the absolute value of the ratio of the k+1-st term to the k-th term of the series in (1.2) we have

$$\left|\frac{B_{2k+2}}{(2k+2)n^{2k+2}} \frac{2kn^{2k}}{B_{2k}}\right| < \frac{2(2k+2)!}{(2\pi)^{2k+2}n^{2k+2}(2k+2)} \frac{2kn^{2k}(2\pi)^{2k}}{2(2k)!}(1-2^{1-2k})$$

$$= \frac{2(k+1)(2k+1)}{(2\pi n)^2} \frac{2k}{2(k+1)} (1 - 2^{1-2k}) = \frac{2k(2k+1)}{(2\pi n)^2}(1 - \frac{1}{2^{2k-1}})$$

$$> \frac{2k(2k+1)}{(2\pi n)^2} (1 - \frac{1}{2}) = \frac{k(2k+1)}{(2\pi n)^2} \longrightarrow +\infty \quad \text{as } k \longrightarrow \infty .$$

Therefore, by the ratio test, the series diverges for all n. So the series on the right in (1.2) can never be a well-defined function of n. The reader can now see why Cauchy might have mistrusted such an "equation."

In what sense, then, can the expression (1.2) be interpreted as an equation? Rather than stretch logic any more or further overwork the symbol =, it seems better to introduce, as Poincaré did, a new symbol to denote the situation which we want to characterize in (1.2). Thus we use the symbol $\sim$, which is read "is asymptotic to" or as "is asymptotically equal to," and rewrite (1.2) as

$$(1.6) \quad \sum_{k=1}^{n}\frac{1}{k} - \ln n - \gamma - \frac{1}{2n} \sim \sum_{k=1}^{\infty}\frac{B_{2k}}{2kn^{2k}} \quad \text{for } n \to \infty .$$

The precise meaning of (1.6) is then taken to be as follows. Denote $\sum_{k=1}^{n}\frac{1}{k} - \ln n - \gamma - \frac{1}{2n}$ by E(n). By (1.6) is meant that, for each m = 1, 2, 3, ..., it is the case that (1.7) $\lim_{n\to\infty} \dfrac{E(n) - \sum_{k=1}^{m}\frac{B_{2k}}{2kn^{2k}}}{\frac{1}{n^{2m}}} = 0$. Alternatively, we say that the formal series $\sum_{k=1}^{\infty}\frac{B_{2k}}{2kn^{2k}}$ is an asymptotic expansion for E(n) as $n \to \infty$. Qualitatively this means that the difference between E(n) and the m-th partial sum of the series grows smaller much faster than the quantity $1/n^{2m}$, for each fixed m, as $n \to \infty$. To see that the situation in (1.7) actually holds for the series in question requires consideration of the remainder term in the Euler-Maclaurin sum formula. That, however, is extraneous to our present purpose, which is only to outline the description of asymptotic relationships.

An analogous restatement of (1.3), on Stirling's approximation, would be that

$$(1.8)\quad \ln n! + (n+\tfrac{1}{2}) - \left[(n+\tfrac{1}{2})\ln(n+\tfrac{1}{2}) + \ln\sqrt{2\pi}\right] \sim \sum_{k=1}^{m} \frac{B_{2k}(\tfrac{1}{2})}{2k(2k-1)n^{2k-1}} .$$

This means only that, analogous to (1.7), for each integer $m = 1, 2, 3, \ldots$

(1.9)

$$\lim_{n\to\infty} \frac{\ln n! + (n+\tfrac{1}{2}) - (n+\tfrac{1}{2})\ln(n+\tfrac{1}{2}) - \ln\sqrt{2\pi} - \sum_{k=1}^{m} \frac{B_{2k}(\tfrac{1}{2})}{2k(2k-1)n^{2k-1}}}{\frac{1}{n^{2m-1}}} = 0.$$

Again, to show that this condition is satisfied for the present case involves consideration of the remainder in the Euler-Maclaurin formula or a more difficult analysis involving the Gamma function.[3]

Occasionally one encounters expressions such as

$$(1.10)\quad n! = \left(\frac{n}{e}\right)^n \sqrt{2\pi n} \quad \text{or} \quad n! \doteq \left(\frac{n}{e}\right)^n \sqrt{2\pi n}\,\left(1 + \frac{1}{12\,n}\right) .$$

These are both simplified but obscurative forms of

$$(1.11)\quad \frac{n!}{\left(\frac{n}{e}\right)^n \sqrt{2\pi n}} \sim 1 + \frac{1}{12(n+1)} + \frac{1}{288(n+1)^2} - \cdots$$

where the rule for the formation of subsequent coefficients is itself rather obscure.[4] The expression

$$n! \sim \left(\frac{n}{e}\right)^n \sqrt{2\pi n} \quad \text{means that} \quad \frac{n!}{\left(\frac{n}{e}\right)^n \sqrt{2\pi n}} \sim 1,$$

i.e., that $$\lim_{n\to\infty} \frac{\dfrac{n!}{\left(\frac{n}{e}\right)^n \sqrt{2\pi n}} - 1}{\dfrac{1}{(n+1)^0}} = 0 .$$

Notice that $\lim_{n\to\infty} \frac{1}{(n+1)^0} = 1$ so that $\lim_{n\to\infty} \frac{n!}{\left(\frac{n}{e}\right)^n \sqrt{2\pi n}} = 1.$

This formulation does not give us any quantitative information about how closely $\left(\frac{n}{e}\right)^n \sqrt{2\pi n}$ approximates $n!$ for any finite n; and in any physical situation, it is only finite n in which we are really interested.

# CHAPTER II

## ASYMPTOTIC SEQUENCES AND EXPANSIONS

We now make a sketch of the definitions and results related to asymptotics which are found in the current literature and which are useful in applications. The material here basically follows the presentation of Copson (1965) and Erdélyi (1956).

### Asymptotic Sequences

In the examples considered in Chapter I, n appeared as the variable of major interest. We are now going to frame our definitions in terms of a variable x, which may be real or complex, or a positive integer. We shall most often regard x as a continuous real variable. At the end of Chapter I we mentioned an asymptotic expansion with the variable n "for $n \to \infty$." Similarly, we shall consider asymptotic expansions for $x \to \infty$, and more generally, for $x \to x_0$. This point $x_0$ may be any real number or $\pm\infty$.

If we are given a sequence of functions of x, $\{f_n(x)\}$, such that in some deleted neighborhood of $x_0$, for each $n = 1, 2, 3, \ldots$, we have

$$(2.1) \qquad \lim_{x \to x_0} \frac{f_{n+1}(x)}{f_n(x)} = 0,$$

then we say that $\{f_n(x)\}$ is <u>an asymptotic expansion for $x \longrightarrow x_0$.</u> As an example, $\left\{\frac{1}{n^{2k}}\right\}$ is an asymptotic sequence for $n \longrightarrow \infty$. In the definition, set x = n and n = k and then $f_k(n) = \frac{1}{n^{2k}}$. We also have

$$\frac{f_{k+1}(n)}{f_k(n)} = \frac{n^{2k}}{n^{2k+2}} = \frac{1}{n^2} \longrightarrow 0 \text{ as } n \longrightarrow \infty \text{ for}$$

each k = 1, 2, 3, . . .. Other examples frequently encountered are $\{x^n\}$ for $x \longrightarrow 0$, $\left\{\frac{1}{x^n}\right\}$ for $x \longrightarrow \infty$, $\{(x-b)^n\}$ as $x \longrightarrow b$, $\left\{\frac{1}{x^{a_n}}\right\}$ for $x \longrightarrow \infty$, where $\{a_n\}$ is a strictly increasing sequence of positive real numbers. This follows since

$$\frac{f_{n+1}(x)}{f_n(x)} = \frac{x^{a_n}}{x^{a_{n+1}}} = \frac{1}{x^{a_{n+1}-a_n}} = \frac{1}{x^{\epsilon}} \longrightarrow 0 \text{ as } x \longrightarrow \infty,$$

where $a_{n+1} - a_n = \epsilon > 0$ because $a_n$ is increasing. One other example is $\{x^n e^{-nx}\}$ for $x \longrightarrow \infty$. Note that, in these examples and in the definition, it is x which varies while n is fixed; that is, we fix attention on two adjacent elements of the sequence and then let $x \longrightarrow x_0$. This is to be contrasted with the situation of a convergent sequence such as $\{x^n\}$, which converges for $|x| < 1$. Here we fix a value of x such that $|x| < 1$, and then consider $x^n$ for $n = 1, 2, 3, \ldots$ . More will be said on this point later.

## Asymptotic Approximations

Given a function $F(x)$ defined on some set R, of which $x_o$ is a cluster point, and $\{f_n(x)\}$ is an asymptotic sequence for $x \longrightarrow x_o$, then we call the formal series (Formal means that we are not concerned with convergence or divergence.) $\sum_{n=1}^{N} a_n f_n(x)$ an asymptotic expansion in R, to N terms, of the function F(x) as $x \longrightarrow x_o$, relative to the asymptotic sequence $\{f_n(x)\}$ , if it is true that

$$(2.2) \qquad \lim_{x \to x_o} \frac{F(x) - \sum_{n=1}^{N} a_n f_n(x)}{f_N(x)} = 0.$$

Some modern books, for example Jeffreys (1962), use the expression asymptotic approximation to N terms for what we have just defined. One often finds omitted the words "in R," "as $x \longrightarrow x_o$," or "relative to the sequence $\{f_n(x)\}$," when it is supposed to be obvious what $x_o$, R, and $\{f_n(x)\}$ are.

In the terminology of the definition, returning to an earlier example, 1 is an asymptotic expansion to one term of $F(n) = \dfrac{n!}{\left(\frac{n}{e}\right)^n \sqrt{2\pi n}}$ , while $1 + \dfrac{1}{12(n+1)} + \dfrac{1}{288(n+1)^2}$ is an expansion to three terms, etc.

Asymptotic Expansions

If the series $\sum_{n=1}^{N} a_n f_n(x)$ is an asymptotic expansion to N terms of F(x) for $x \to x_o$ and for each value of N = 1, 2, 3, . . ., then we call the series a complete asymptotic expansion, or simply just an asymptotic expansion, of F(x) for $x \to x_o$ (in some set R).

Examples: If a function F(x) has a valid power series expansion in x, then this power series is an asymptotic expansion for $x \to 0$ of F(x) (with respect to the asymptotic sequence $\{x^n\}$). To see this let $F(x) = \sum_{n=o}^{\infty} a_n x^n$ be the power series in question. Then we have

$$\frac{F(x) - \sum_{n=o}^{N} a_n x^n}{x^N} = \frac{\sum_{n=N+1}^{\infty} a_n x^n}{x^N} = x \sum_{i=o}^{\infty} a_{N+1+i} x^i \to 0$$

as $x \to 0$. So we may write $F(x) \sim \sum_{n=o}^{\infty} a_n x^n$ for $x \to 0$.

Consider $F(x) = \frac{1}{1+x} = \frac{1}{1-(-x)} = 1-x+x^2-x^3+x^4- \ . \ . \ .,$ which is a valid power series expansion for $|x| < 1$. We also have $F(x) = \frac{1}{1+x} \sim 1 - x + x^2 - x^3 + \ldots$ for $x \to 0$.

Also, for $x \neq 0$, $F(x) = \frac{1}{1+x} = \frac{\frac{1}{x}}{1+\frac{1}{x}} = \frac{\frac{1}{x}}{1-\left(-\frac{1}{x}\right)} =$

$\frac{1}{x} - \frac{1}{x^2} + \frac{1}{x^3} - \frac{1}{x^4} + \ldots$ which is valid for $\left|\frac{1}{x}\right| < 1$, i.e.,

for $|x| > 1$. This time we have $\dfrac{F(x) - \sum_{n=1}^{N} \frac{(-1)^{n+1}}{x^n}}{\frac{1}{x^N}} =$

$$x^N \left[ F(x) - \sum_{n=1}^{N} \frac{(-1)^{n+1}}{x^n} \right] = x^N \sum_{n=N+1}^{\infty} \frac{(-1)^{n+1}}{x^n} = \frac{1}{x} \sum_{i=0}^{\infty} \frac{(-1)^{N+1+i}}{x^i}$$

which tends to zero as $x \to \infty$. So $F(x) \sim \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{x^n}$ as

as $x \to \infty$. Similarly for $|x| > 1$, $F(x) = \frac{1}{1+x} = \frac{x-1}{x^2-1} =$

$$(x-1)\frac{\frac{1}{x^2}}{1-\frac{1}{x^2}} = \sum_{n=1}^{\infty} (x-1)\frac{(-1)^{n+1}}{x^{2n}} \text{ ; and}$$

$$F(x) = \frac{1}{1+x} \sim \sum_{n=1}^{\infty} (x-1)\frac{(-1)^{n+1}}{x^{2n}} \quad \text{for } x \to \infty.$$

These asymptotic expansions and similar ones are of importance in obtaining expansions for more inaccessible functions, such as the examples in the next chapter.

When we have an asymptotic expansion for some function and when we are not particularly interested in the function itself, then we will sometimes call this expansion an <u>asymptotic</u> <u>series</u>. We are now in a position to be able to point out a distinction between convergent series and asymptotic series, which distinction may help illuminate why they are worthy of study in their own right.

Recall that $F(x) = e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$ is convergent for all x (real or complex). Theoretically then, this series could be used to compute $e^x$ for any (real) x to any desired degree of accuracy. It is probably not well understood that this series is not good for computing $e^x$ for "large" x, say $|x| > 3$. Assume, for example, that one wished to compute the quantity $e^{100\pi} = e^{314.15926\ldots}$. Looking at the absolute value of the ratio of the n+1-st term to the n-th term in the above power series, we have $\left|\frac{x^{n+1}}{(n+1)!}\frac{n!}{x^n}\right| = \frac{|x|}{n+1}$, regarding 1 and x as the 0-th and 1-st terms, respectively. This ratio is greater than one until $n+1 > |x|$, i.e., the terms of the series increase in absolute value until $n+1 > |x|$. In the present case of $e^{100\pi}$, the terms grow in size until the 314-th term, which term has a value of $\frac{(100\pi)^{314}}{314!}$ which is about equal to $10^{145}$. Clearly the above series is not the way to compute $e^{100\pi}$.[5]

On the other hand, if we desire to know $e^{\pi/100} = e^{0.031415926\cdots}$, then the ratio $\frac{|x|}{n+1}$ is about 0.03 when $n = 0$, i.e., the second term is only about 3% of the first term. The tenth term is only 0.3% of the ninth, etc. This discussion should convey that, for $x = 10^{-2}\pi$, the terms of the series become small very quickly. This is a desirable feature for a series to have if it is to be used for computation. Recall that $\{x^n\}$ is an asymptotic sequence for $x \longrightarrow 0$ and note that $e^x \sim \sum_{n=0}^{\infty} \frac{x^n}{n!}$ as $x \longrightarrow 0$. This fact is what makes the series rapidly convergent, for x near the origin, and thereby useful for computation. For when we get to the point in the series where the truncation error is less than our allowable error, then we can stop. If the terms grow small quickly, then we can afford to truncate after a fairly small number of terms.

Note that in using the series for an actual computation, we have not used the fact that the series is convergent. Rather we made use of the characteristic of the series that the terms grow small quickly and that an estimate of the upper bound of the error incurred by truncating the series is calculable.

In just the same way, if we have a function $F(x)$ and an asymptotic expansion of $F(x)$ for $x \longrightarrow x_o$, then for any x sufficiently close to $x_o$, we can be sure that the early terms of the series become small rapidly. If, in addition, we have some rule for finding an upper bound for the error incurred by stopping the sum at any term, which error usually depends in a simple way on the first term omitted or on the last term retained, then we have a means of making an approximation to F(x) of known minimum accuracy by summing relatively few terms of the formal series. Once more we have made no comment on possible convergence or divergence of the asymptotic series.

If an asymptotic expansion happens to be a convergent series for some value of x, then we know that eventually the terms must become arbitrarily small and remain so. With our backgrounds in rigorous analysis, this is a reassuring behavior for a series to exhibit, even if it is not computationally useful. However, if our asymptotic expansion happens to be divergent, then there is no guarantee that the terms will become arbitrarily small and keep getting smaller. In fact, the terms of many useful asymptotic series become arbitrarily large, after an initial rapid decrease.

As an example, in the next chapter we will derive the asymptotic series $\sum_{n=0}^{\infty} \frac{(-1)^n n!}{x^n}$, ralative to the sequence $\left\{\frac{1}{x^n}\right\}$, for $x \to \infty$. Here the coefficients are $a_n = (-1)^n n!$. The sequence $\left\{\frac{1}{x^n}\right\}$ we have seen to be asymptotic for $x \to \infty$.

Performing the ratio test on this series we have

$$\left| \frac{(n+1)!}{x^{n+1}} \cdot \frac{x^n}{n!} \right| = \frac{n+1}{|x|} \longrightarrow \infty \text{ as } n \longrightarrow \infty, \text{ no matter what}$$

x we choose such that $|x| < \infty$.

Therefore, to regard an asymptotic expansion of a function as a convergent or divergent infinite series is often misleading or confusing, and obscures the main purpose of employing an asymptotic expansion, which is to study the behavior of some F(x) as x approaches the cluster point $x_o$ about which the formal series is an asymptotic expansion. It would probably be better not to regard the

expression $F(x) \sim \sum_{n=1}^{\infty} a_n\, f_n(x)$ as having anything to do

with series at all, but rather to consider the expression as signifying a whole (infinite) class of finite-sum approximations to F(x), the elements of the class being

$\left\{ F(x) \sim \sum_{n=1}^{N} a_n\, f_n(x),\ E_N(x) \right\}$, where $E_N(x)$ is an upper

bound for the error in the corresponding approximation, the class being indexed by N = 1, 2, 3, . . ..

The terms and partial sums of an arbitrary convergent series, and the terms and partial sums of an asymptotic series which happens to be convergent, will behave about the same way, except that the asymptotic series will generally "converge faster" for a value of x close to the

$x_0$ about which it is an asymptotic expansion. By "converge faster" is meant that greater accuracy and smaller relative error are attained by the asymptotic expansion for partial sums to a fixed number of terms than are attained by the other series.

The contrast between convergent series and asymptotic series is probably best seen by a comparison of a convergent series with a divergent asymptotic series. Toward that end we shall consider the convergent power series expansion, a geometric series, for

$$0.857142\overline{857142} = \frac{6}{7} = \frac{1}{1 - \left(-\left(\frac{1}{6}\right)\right)} = 1 - \frac{1}{6} + \frac{1}{6^2} - \frac{1}{6^3} + \ldots,$$

and compare that with the asymptotic expansion for $10\ e^{10}\ E_1(10)$, where

$$z\ e^z\ E_1(z) = z\ e^z \int_z^{\infty} \frac{e^{-t}}{t} dt \sim 1 - \frac{1}{z} + \frac{2!}{z^2} - \frac{3!}{z^3} + \ldots.$$

Here $z$ is a complex number with $|\mathrm{ph}\, z| < \frac{\pi}{2}$, i.e., with $\mathrm{Re}(z) > 0$.[6] For the moment just accept this asymptotic series as a valid asymptotic expansion of the given function. In the next chapter we shall derive this expansion as one of our examples. This formal asymptotic series is divergent for all finite z, as the ratio test shows:

$$\left|\frac{(n+1)!}{z^{n+1}} \cdot \frac{z^n}{n!}\right| = \frac{n+1}{|z|} \longrightarrow \infty \text{ as } n \longrightarrow \infty.$$

Convergence for an infinite series is defined in terms of the convergence of the sequence of partial sums; so it is natural to look at the partial sums $S(N) = \sum_{n=1}^{N} \left(-\frac{1}{6}\right)^{n-1}$ of the geometric series for $\frac{1}{1+\frac{1}{6}}$. See Table 2.1, wherein both the partial sums, S(N), and the terms, A(N), are given to 15D. Since S(N+1) = S(N) + A(N+1) and since A(N+1) = S(N+1) - S(N), any entry, S(N), in the first column of Table 2.1 is the sum of the two entries, S(N-1) and A(N), in the row immediately above it. The A(N) represent the differences, S(N+1) - S(N), between successive partial sums. In Table 2.1 the partial sums increase and decrease by an amount, A(N+1), that decreases steadily in absolute value and finally disappears for N = 20, after which the partial sums no longer change. In actual fact, the terms A(N) are never really zero, but our computations were carried only to 15D. Therefore, for $N \geqslant 20$ we have that $|A(N+1)| < .5 \times 10^{-15}$. Since our numbers are printed out at fifteen decimal digits, these latter A(N+1) are rounded off to zero.

It is easier to see what is occurring if we do the problem once more, printing the S(N) out at 15D but now printing the A(N+1) at 15S, i.e., as $d\cdot 10^{-k}$ where $0<|d|<1$. Then the A(N+1) will not be rounded to zero when $|A(N+1)| < .5 \times 10^{-15}$. These computations are given in Table 2.2. Note that S(N) is already of maximum accuracy

TABLE 2.1

$$S(N) = \sum_{n=0}^{N-1} \left(-\frac{1}{6}\right)^n \quad \text{GEOMETRIC SERIES}$$

APPROXIMATIONS TO 6/7 = $0.857142\overline{857142}$

| N | S(N) | A(N+1) |
|---|---|---|
| 1 | 1.000000000000000 | -.166666666666666 |
| 2 | .833333333333333 | .027777777777777 |
| 3 | .861111111111111 | -.004629629629629 |
| 4 | .856481481481481 | .000771604938271 |
| 5 | .857253086419753 | -.000128600823045 |
| 6 | .857124485596707 | .000021433470507 |
| 7 | .857145919067215 | -.000003572245084 |
| 8 | .857142346822130 | .000000595374180 |
| 9 | .857142942196311 | -.000000099229030 |
| 10 | .857142842967281 | .000000016538171 |
| 11 | .857142859505453 | -.000000002756361 |
| 12 | .857142856749091 | .000000000459393 |
| 13 | .857142857208484 | -.000000000076565 |
| 14 | .857142857131919 | .000000000012760 |
| 15 | .857142857144680 | -.000000000002126 |
| 16 | .857142857142553 | .000000000000354 |
| 17 | .857142857142907 | -.000000000000059 |
| 18 | .857142857142848 | .000000000000009 |
| 19 | .857142857142858 | -.000000000000001 |
| 20 | .857142857142856 | 0.000000000000000 |
| 21 | .857142857142857 | 0.000000000000000 |
| 22 | .857142857142857 | 0.000000000000000 |
| 23 | .857142857142857 | 0.000000000000000 |
| 24 | .857142857142857 | 0.000000000000000 |

TABLE 2.2

$$S(N) = \sum_{n=0}^{N-1} \left(-\frac{1}{6}\right)^n \quad \text{GEOMETRIC SERIES}$$

APPROXIMATIONS TO 6/7 = $0.857142\overline{857142}$

| N | S(N) | A(N+1) |
|---|---|---|
| 1 | 1.000000000000000 | -.166666666666666E-00 |
| 2 | .833333333333333 | .277777777777777E-01 |
| 3 | .861111111111111 | -.462962962962962E-02 |
| 4 | .856481481481481 | .771604938271604E-03 |
| 5 | .857253086419753 | -.128600823045267E-03 |
| 6 | .857124485596707 | .214334705075445E-04 |
| 7 | .857145919067215 | -.357224508459076E-05 |
| 8 | .857142346822130 | .595374180765127E-06 |
| 9 | .857142942196311 | -.992290301275212E-07 |
| 10 | .857142842967281 | .165381716879202E-07 |
| 11 | .857142859505453 | -.275636194798670E-08 |
| 12 | .857142856749091 | .459393657997783E-09 |
| 13 | .857142857208484 | -.765656096662972E-10 |
| 14 | .857142857131919 | .127609349443828E-10 |
| 15 | .857142857144680 | -.212682249073047E-11 |
| 16 | .857142857142553 | .354470415121746E-12 |
| 17 | .857142857142907 | -.590784025202910E-13 |
| 18 | .857142857142848 | .984640042004851E-14 |
| 19 | .857142857142858 | -.164106673667475E-14 |
| 20 | .857142857142856 | .273511122779125E-15 |
| 21 | .857142857142857 | -.455851871298542E-16 |
| 22 | .857142857142857 | .759753118830903E-17 |
| 23 | .857142857142857 | -.126625519805150E-17 |
| 24 | .857142857142857 | .211042533008584E-18 |

when N = 21. The A(N+1) steadily decrease in size, and no longer influence the printed value of S(N) for $N \geq 21$. Also, this time none of the A(N+1) is printed as zero.

Now consider the computation for $10\ e^{10}\ E_1(10)$ as approximated asymptotically by (see Table 2.3)

$$T(N) = \sum_{n=1}^{N} C(n) = \sum_{n=0}^{N-1} \frac{(-1)^n\ n!}{10^n},$$ where $E_1(x)$ is the well-known Exponential Integral. The tabled value (Abramowitz and Stegun (1968), p. 243, Table 5.1) for $10\ e^{10}\ E_1(10)$ is 0.915633339. If we take the ratio of adjacent terms, we obtain $\left| \frac{(-1)^n\ n!}{10^n} \cdot \frac{10^{n-1}}{(-1)^{n-1}\ (n-1)!} \right| = \frac{n}{10}$. This ratio is less than one until n = 10. That is, the terms, C(n), decrease in absolute value until n = 10, so that C(9) and C(10) are the same size. After this the ratio is greater than one; and the terms begin to grow in absolute value, and continue to do so without bound. Therefore this asymptotic series is clearly divergent, for x = 10 and for any other value of x. But this asymptotic series is not an infinite series, in the sense of defining a certain real number as with a convergent series; it is merely a convenient way of denoting a countably infinite set of finite-sum approximations to $10\ e^{10}\ E_1(10)$ with known error bounds. These error bounds are (see next chapter)

TABLE 2.3

ASYMPTOTIC APPROXIMATIONS TO $x\, e^{x}\, E_1(x) =$

$$x\, e^{x} \int_{x}^{\infty} \frac{e^{-t}}{t} dt \sim \sum_{n=0}^{\infty} \frac{(-1)^n\, n!}{x^n} \quad \text{for} \quad x = 10.0$$

| N | | T(N) | C(N+1) | |
|---|---|---|---|---|
| 1 | | 1.000000000000000 | -.100000000000000 | |
| 2 | | .900000000000000 | .020000000000000 | |
| 3 | | .920000000000000 | -.006000000000000 | |
| 4 | | .914000000000000 | .002400000000000 | |
| 5 | | .916400000000000 | -.001200000000000 | |
| 6 | | .915200000000000 | .000720000000000 | |
| 7 | | .915920000000000 | -.000504000000000 | |
| 8 | | .915416000000000 | .000403200000000 | |
| 9 | → | .915819200000000 | -.000362880000000 | ← |
| 10 | | .915456320000000 | .000362880000000 | |
| 11 | | .915819200000000 | -.000399168000000 | |
| 12 | | .915420032000000 | .000479001600000 | |
| 13 | | .915899033600000 | -.000622702080000 | |
| 14 | | .915276331520000 | .000871782912000 | |
| 15 | | .916148114432000 | -.001307674368000 | |
| 16 | | .914840440064000 | .002092278988800 | |
| 17 | | .916932719052800 | -.003556874280960 | |
| 18 | | .913375844771840 | .006402373705728 | |
| 19 | | .919778218477568 | -.012164510040883 | |
| 20 | | .907613708436685 | .024329020081766 | |
| 21 | | .931942728518451 | -.051090942171709 | |
| 22 | | .880851786346742 | .112400072777760 | |
| 23 | | .993251859124502 | -.258520167388848 | |
| 24 | | .734731691735654 | .620448401733235 | |

The arrow on the right marks the term of least absolute value or, as in the present case of two adjacent terms of equal absolute value, the first such term. The arrow on the left marks the best partial sum approximation to the above function value. This partial sum includes all terms up to the term of least absolute value but omits that term.

$$\left| x\, e^{x}\, E_1(x) - \sum_{n=0}^{N} \frac{(-1)^n\, n!}{x^n} \right| < \frac{(n+1)!}{|x|^{n+1}} = \text{the absolute value}$$

of the first term omitted. The error is also the same sign as the first term omitted. Therefore $10\ e^{10}\ E_1(10)$ will always be in the interval (T(N), T(N) + C(N+1)). We have then the best accuracy attainable with this expansion, for this value of x = 10, when we take the partial sum up to but not including the term least in absolute value. Looking at Table 2.3 we have $10\ e^{10}\ E_1(10)$ in the interval (0.91581920, 0.91545632). The actual value is 0.91563334. If this approximation is good enough for our purposes, then we can be happy. If not, then we need some other method of approximation, perhaps a different asymptotic expansion, for x = 10, reserving the given asymptotic series for larger x, i.e., for x closer to $x_0 = +\infty$.

The main point to notice here is that, after N = 10, the values of C(N+1) continue to grow; and the variations in successive T(N) become increasingly larger. As $N \longrightarrow \infty$, $|T(N)|$ grows without bound; and the approximation to $10\ e^{10}\ E_1(10)$ becomes worse and worse. This is typical behavior for so-called "divergent" asymptotic series, these being the only type of series considered in works on asymptotics prior to about 1940. In such an asymptotic series the smallest variation in successive partial sums corresponds to the term of least absolute value. The best approximation is often taken to be the partial sum for which the term of least absolute value is the first term omitted in the partial sum. Sometimes this "best" value is "improved" by adding half of the smallest term. This new sum is usually closer to the actual function value. Other techniques are also used to improve the simple "best estimate" such as Euler's transformation, Shanks' various non-linear transformations, etc., some of which will be mentioned in an appendix.

Often we will have situations such as

$$\frac{F(x)}{G(x)} - H(x) \sim \sum_{n=o}^{\infty} a_n f_n(x) \quad \text{as } x \longrightarrow x_o. \quad \text{In such}$$

cases we permit ourselves license with the definition and write $F(x) \sim G(x)\left[H(x) + \sum_{n=o}^{\infty} a_n f_n(x)\right]$ as $x \longrightarrow x_o$.

Similar considerations hold for other combinations of functions. For example, we could rewrite (1.6) as

$$\sum_{k=1}^{n} \frac{1}{k} \sim \ln n + \gamma + \frac{1}{2n} + \sum_{k=1}^{\infty} \frac{B_{2k}}{2k\, n^{2k}} \text{ for } n \longrightarrow \infty,$$

and, as soon as $\gamma$ is a known quantity, use this expansion as a means of making an asymptotic estimate of the quantity $\sum_{k=1}^{n} \frac{1}{k}$ for large values of n.

# CHAPTER III

## METHODS OF OBTAINING ASYMPTOTIC EXPANSIONS AND APPROXIMATIONS

### Power series in x and $\frac{1}{x}$

There are several ways of generating asymptotic approximations to functions. If there are valid power series expansions for some $F(x)$ in x or in $\frac{1}{x}$, then these are asymptotic expansions for $F(x)$, as $x \longrightarrow 0$ or as $x \longrightarrow \infty$, respectively. If a function has a representation as a definite integral, then often an asymptotic expansion can be obtained from this integral. This sort of problem will be our primary interest in this chapter. Asymptotic series can also be obtained as formal solutions to differential equations. There are other techniques for obtaining expansions, mostly dealing with functions of a complex variable and contour integration, which will be mentioned in an appendix.

The remainder of this chapter will deal with various examples of deriving asymptotic expansions for particular functions. Consider the function defined by

$\alpha(x) = \int_0^\infty \frac{e^{-xt}}{1 + t} dt$ for $x > 0$. One of the commonest ways to obtain an asymptotic expansion is to expand part of the integrand as a series (finite or not) and then to integrate term by term. Let us use the fact that

$1 - z + z^2 - z^3 + \ldots + (-z)^{N-1} = \dfrac{1 - (-z)^N}{1 - (-z)}$, which is valid for $z \neq -1$. Transforming this, we determine

$$(3.1)\quad \frac{1}{1+z} = 1 - z + z^2 - z^3 + \ldots + (-z)^{N-1} + \frac{(-z)^N}{1+z}.$$

Setting $z = t$ and substituting in the above integral we have

$$\alpha(x) = \int_0^\infty \frac{e^{-xt}}{1+t}\,dt = \int_0^\infty e^{-xt}\left[1-t+t^2-t^3+\cdots+(-t)^{N-1} + \frac{(-t)^N}{1+t}\right]dt$$

$$= \int_0^\infty e^{-xt}\,dt - \int_0^\infty e^{-xt}t\,dt + \int_0^\infty e^{-xt}\,t^2\,dt - \cdots$$

$$+ \int_0^\infty e^{-xt}\,(-t)^{N-1}\,dt + \int_0^\infty \frac{e^{-xt}\,(-t)^N}{1+t}\,dt.$$ Using the fact [7] that $\displaystyle\int_0^\infty e^{-xt}\,t^n\,dt = \frac{n!}{x^{n+1}}$, we then have

$$\alpha(x) = \frac{1}{x} - \frac{1!}{x^2} + \frac{2!}{x^3} - \frac{3!}{x^4} + \cdots + (-1)^{n-1}\frac{(N-1)!}{x^N} + (-1)^N \int_0^\infty \frac{e^{-xt}\,t^N}{1+t}\,dt.$$

The terms on the right up to the term with (N-1)! will be an asymptotic approximation to N terms of $\alpha(x)$ and the final integral will represent the error in the approximation. Now we need an approximation to the value of this integral. In the interval of integration, $1 + t \geqslant 1$ and also $\frac{1}{1+t} \leqslant 1$. Therefore, it is true that

$$\left| \int_0^\infty \frac{e^{-xt}\ t^N}{1+t}\, dt \right| \leqslant \int_0^\infty \frac{e^{-xt}\ t^N}{1}\, dt = \int_0^\infty e^{-xt}\ t^N\, dt = \frac{N!}{x^{N+1}} .$$

So the absolute value of the error is bounded above by that of the N+1-st term. Since the integral has a factor of $(-1)^N$, we see that the error has also the same sign as the N+1-st term. This particular type of error bound does not hold for all asymptotic expansions, but it does happen often enough to be looked out for.

To verify that this approximation is asymptotic, we could note that $\left\{\frac{1}{x^n}\right\}$ is an asymptotic sequence. We can also check that it satisfies the definition, (2.2):

$$\lim_{x \to \infty} \frac{\left| \alpha(x) - \sum_{n=0}^{N-1} \frac{(-1)^n\ n!}{x^{n+1}} \right|}{\frac{1}{x^N}} \leqslant \lim_{x \to \infty} \frac{x^N\ N!}{x^{N+1}} = 0 .$$

Since this approximation clearly holds for all integers $N = 1, 2, 3, \ldots$, we have a complete asymptotic expansion and we write $\alpha(x) = \int_0^\infty \frac{e^{-xt}}{1+t}\, dt \sim \sum_{n=0}^{\infty} \frac{(-1)^n\ n!}{x^{n+1}}$ .

This last expression is the form most commonly encountered in the literature on abstract analysis. There the writer usually desires an asymptotic expansion of a function, say $F(x) \sim \sum_{n=1}^{\infty} a_n f_n(x)$ for $x \longrightarrow x_o$; and he is really interested in the behavior of $F(x)$ only as $x$ tends to the limit, $x_o$. For example, see De Bruijn (1958).

The situation is quite different, however, when it is desired to use an asymptotic expansion for numerical calculation. Here it is not enough just to know that, for each $N = 1, 2, 3, \ldots$, $\left| F(x) - \sum_{n=1}^{N} a_n f_n(x) \right|$ can be made arbitrarily small by taking $x$ sufficiently close to $x_o$. That is, in this statement about the error, just as in the definition of asymptotic approximation, $N$ is fixed and $x$ can vary as $x \longrightarrow x_o$. In using this asymptotic approximation for numerical calculation, however, we want the value of $F(x)$ for some fixed $x$. Here $N$ is the only parameter which can vary. We need in this case an explicit estimate of the error in the approximation. It would therefore seem desirable in numerical work to emphasize the finiteness of the asymptotic sum and to write something similar to the following: For each $N = 1, 2, 3, \ldots$,

$$F(x) \sim \sum_{n=1}^{N} a_n f_n(x) = S_N(x); \text{ and } \left| F(x) - S_N(x) \right| < \left| a_{N+1} f_{N+1}(x) \right|$$

and is of the same sign.

This is not usually done in the available literature.

For example, in Abramowitz and Stegun (1968), and in Jahnke and Emde (1945), there are numerous complete asymptotic expansions given. However, to use them for calculation, one needs a concrete estimate of the error. This means doing some extra analysis in order to derive an error estimate.

Consider the $\int_x^\infty t^{-1}e^{x-t}\,dt$ which, setting $t = xv$ for $x > 0$, is equal to, with $v = 1 + w$,

$$x\int_1^\infty (xv)^{-1}e^{x-xv}\,dv = x\int_1^\infty \frac{e^{x(1-v)}}{xv}\,dv = \int_1^\infty \frac{e^{x(1-v)}}{v}\,dv =$$

$$\int_0^\infty \frac{e^{x(-w)}}{1+w}\,dw = \int_0^\infty \frac{e^{-xt}}{1+t}\,dt = \alpha(x) \text{ of a previous example.}$$

Therefore $\displaystyle\int_x^\infty \frac{e^{x-t}}{t}\,dt \sim \sum_{n=0}^{N} \frac{(-1)^{n-1}\,(n-1)!}{x^n}$ for each $N = 1, 2, 3, \ldots$; and the error is the same sign as and bounded above by the absolute value of $\dfrac{(-1)^N\,N!}{x^{N+1}}$, the N+1-st term of the expansion.

Very closely related to the above integral is the function, $E_1(x) = \displaystyle\int_x^\infty \frac{e^{-t}}{t}\,dt$, one of the several Exponential Integrals. This is the function of an example in Chapter II, where we were finding $10\,e^{10}\,E_1(10)$. See Abramowitz and Stegun (1968), Chapter 5, for a discussion of this class of functions.[8] We have that $E_1(x) =$

$$\int_x^\infty \frac{e^{-t}}{t}\,dt = e^{-x}\int_x^\infty \frac{e^x\, e^{-t}}{t}\,dt = e^{-x}\int_x^\infty \frac{e^{x-t}}{t}\,dt = e^{-x}\,\alpha(x),$$

where $\alpha(x)$ is as before.

We have already seen the use of term-by-term integration of a (finite) series expansion to derive an asymptotic expansion. Another common technique is the use of integration by parts. Let us apply this technique to the function in the last example, $E_1(x)$. Setting $e^{-t}\,dt = dv$ and $u = \frac{1}{t}$, we have $v = -e^{-t}$ and $du = \frac{-dt}{t^2}$, so that

$$\int_x^\infty \frac{e^{-t}}{t}\,dt = \Big[u\,v\Big]_{t=x}^{t=\infty} - \int_{t=x}^{t=\infty} v\,du = \left[-\frac{e^{-t}}{t}\right]_x^\infty - \int_x^\infty \frac{e^{-t^2}}{t^2}\,dt$$

$= \frac{e^{-x}}{x} - \int_x^\infty \frac{e^{-t}}{t^2}\,dt.$ Repeating the integration by parts

N - 1 times, we find that $E_1(x) = \int_x^\infty \frac{e^{-t}}{t}\,dt =$

$$\frac{e^{-x}}{x} - \frac{e^{-x}}{x^2} + \frac{2e^{-x}}{x^3} - \cdots + (-1)^{N-1}\,\frac{e^{-x}(N-1)!}{x^N} + (-1)^N N!\int_x^\infty \frac{e^{-t}}{t^{N+1}}\,dt$$

$$= e^{-x}\left[\frac{1}{x} - \frac{1}{x^2} + \frac{2!}{x^3} - \frac{3!}{x^4} + \cdots + (-1)^{N-1}\,\frac{(N-1)!}{x^N}\right] + (-1)^N N!\int_x^\infty \frac{e^{-t}}{t^{N+1}}\,dt.$$

We have $0 < x \leqslant t$ so that $\frac{1}{t} \leqslant \frac{1}{x}$ and

$$N!\int_x^{\infty}\frac{e^{-t}}{t^{N+1}}dt \leqslant N!\int_x^{\infty}\frac{e^{-t}}{x^{N+1}}dt = \frac{N!}{x^{N+1}}\int_x^{\infty}e^{-t}\,dt = \frac{N!}{x^{N+1}}\,e^{-x}.$$

Therefore, $E_1(x) = \displaystyle\int_x^{\infty}\frac{e^{-t}}{t}dt \sim e^{-x}\left[\frac{1}{x} - \frac{1}{x^2} + \frac{2!}{x^3} - \frac{3!}{x^4} + \cdots\right]$

as $x \longrightarrow \infty$, and the error of the sum up to and including the term in $\frac{1}{x^N}$ is the same sign as and bounded above by the absolute value of the next term, $(-1)^N \dfrac{e^{-x}\,N!}{x^{N+1}}$. This is the same expansion which we obtained before.

Another form of integral which one encounters is

$$\beta(z) = \int_0^{\infty}\frac{e^{-t}}{1+zt}dt = \frac{1}{z}\int_0^{\infty}\frac{e^{-t}}{\frac{1}{z}+t}dt = \frac{1}{z}\,e^{\frac{1}{z}}\int_{1/z}^{\infty}\frac{e^{-(t+\frac{1}{z})}}{t+\frac{1}{z}}d(t+\tfrac{1}{z})$$

$$= \frac{1}{z}\,e^{\frac{1}{z}}\int_{1/z}^{\infty}\frac{e^{-w}}{w}dw = \frac{e^{\frac{1}{z}}}{z}E_1(1/z).$$ $E_1(1/z)$ is the same Exponential Integral considered before.

Another related integral is $\lambda(x) = \displaystyle\int_0^{\infty}\frac{e^{-t}}{x+t}dt$, which is defined for $x > 0$. We can rewrite this as

$$\lambda(x) = \int_0^{\infty}\frac{e^{-t}}{x+t}dt = \frac{1}{x}\int_0^{\infty}\frac{e^{-t}}{1+\frac{1}{x}t}dt = \frac{1}{x}\beta\left(\frac{1}{x}\right) = \frac{1}{x}\,e^{x}\,E_1(x),$$

which is almost the same function which we approximated for $x = 10$ in Chapter II.

The examples so far can all be reduced to the Exponential Integral, $E_1(x)$; and the expansions found are asymptotic for $x \longrightarrow \infty$. $E_1(x)$ also has a valid power series expansion which is accordingly asymptotic for $x \longrightarrow 0$. We are considering only real x, but this power series for $E_1(x) = \int_x^\infty \frac{e^{-t}}{t} dt$ is valid for all complex x such that $|\mathrm{ph}\, x| < \pi$. Consult Abramowitz and Stegun (1968) again for a discussion; one can find a derivation in Franklin (1964), Art. 331, pp. 570-2. The expansion in question is $E_1(x) = -\gamma - \ln x + \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^N}{n\, n!}$. This time, verifying the definition (2.2):

$$\frac{\left| E_1(x) + \gamma + \ln x - \sum_{n=1}^{N} \frac{(-1)^{n+1} x^n}{n\, n!} \right|}{|x^N|} = \left| \frac{x^N \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{(N+n)(N+n)!}}{x^N} \right|$$

$$= \left| \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{(N+n)(N+n)!} \right| \longrightarrow 0 \quad \text{as } x \longrightarrow 0+.$$

Related to $E_1(x)$ are the so-called Incomplete Gamma functions, $\gamma(a, x) = \int_0^x e^{-t} t^{a-1} dt$ (for $x > 0$, $a > 0$)

and $\Gamma(a, x) = \Gamma(a) - \gamma(a, x) = \int_x^\infty e^{-t} t^{a-1} dt$, and

the often-used Error Function.[9]

## Asymptotic solutions of differential equations

Another source of asymptotic expansions is as formal solutions to differential equations. If for the equation $\frac{dy}{dx} = \frac{a}{x} + b\,y$, for $b > 0$, we assume a formal solution,

$$y = \sum_{n=0}^{\infty} \frac{a_n}{x^n},$$

then we find that we must have the series

$$y = -\frac{a}{b\,x}\left[ 1 - \frac{1}{bx} + \frac{2!}{(bx)^2} - \frac{3!}{(bx)^3} + \cdots \right].$$

This is just the expansion of $-\,a\,\lambda(b\,x) = -a\int_0^{\infty} \frac{e^{-t}}{bx + t}\,dt$. It can be verified that this function, $-\,a\,\lambda(b\,x)$, is a solution of the equation, $\frac{dy}{dx} = \frac{a}{x} + b\,y$.[10] Therefore, if we can find a differential equation which is satisfied by a function for which we desire an asymptotic expansion, then we can sometimes obtain the asymptotic expansion as a formal power series in x, in $\frac{1}{x}$, or in some auxiliary variable.

A second example of an asymptotic expansion derived from a differential equation is afforded by Bessel's differential equation, $y'' + \frac{1}{x} y' + y = 0$, with initial conditions $y(0) = 1$ and $y'(0) = 0$, the solution of which is $J_o(x)$, the Bessel function of the first kind. We desire an expansion which is asymptotic for $x \longrightarrow 0$, and toward that end assume a formal power series in x as a solution. Denote the series by $\sum_{n=o}^{\infty} a_n x^n$. Then we find from the initial conditions that $a_o = 1$, $a_1 = 0$. We also find that $a_n + (n + 2)^2 a_{n+2} = 0$ for $n = 0, 1, 2, \cdots$. This implies that $a_{2n+1} = 0$ and that $a_{2n} = \frac{(-1)^n x^{2n}}{2^2 4^2 \cdots (2n)^2}$ for $n = 1, 2, 3, \cdots$. Thus the series expansion is

$$y = \sum_{n=o}^{\infty} a_{2n} x^{2n} = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 4^2} - \frac{x^6}{2^2 4^2 6^2} + \cdots.$$

The ratio test gives $\left| \frac{a_{n+1} x^{2n+2}}{a_n x^{2n}} \right| = \left| \frac{x^{2n+2}\, 2^2 4^2 \cdots (2n)^2}{2^2 4^2 \cdots (2n+2)^2\, x^{2n}} \right|$

$$= \left| \frac{x^2}{2^2 (n+1)^2} \right| \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

So the series is convergent for all x, real or complex; and we can write $J_o(x) = \sum_{n=o}^{\infty} a_{2n} x^{2n}$. Since $\{x^{2n}\}$ is asymptotic for $x \longrightarrow 0$, we can also write $J_o(x) \sim \sum_{n=0}^{\infty} a_{2n} x^{2n}$. Or, verifying definition (2.2) directly:

$$\left| \frac{J_o(x) - \sum_{n=o}^{N} a_{2n} x^{2n}}{x^{2N}} \right| = \left| \frac{\sum_{n=N+1}^{\infty} a_{2n} x^{2n}}{x^{2N}} \right| = \left| x^2 \sum_{n=o}^{\infty} a_{N+1+n} x^{2n} \right| \longrightarrow 0$$

as $x \longrightarrow 0$, for each $N = 0, 1, 2, \cdots$.

Since this series is asymptotic for $x \longrightarrow 0$, we would expect it to be good for calculation only in the vicinity of $x_o = 0$. The values of $J_o(x)$ have been calculated for the arguments $x = 0(0.2)3$, using this asymptotic series expansion. The computations were performed on a Digital Equipment Corporation electronic computer, model PDP-11. The program was in the language called BASIC. Table 3.1 contains the results. The computed values are printed at 7S, the maximum accuracy attainable in BASIC. In the next-to-last column of Table 3.1, note the small numbers of terms which were required in the sums. This is one of the benefits of the series being asymptotic for $x \longrightarrow 0$. Note that, for larger x, more terms are needed in the sums. That is, the farther x is from $x_o = 0$, the longer it takes to reach the truncation point in the series for that particular value of x. Of course, the asymptotic series employed here is also convergent; so the terms keep diminishing in absolute value. In such a case, the point at which the series is truncated is dictated by the limits of the computer, rather than by the limitations of the

TABLE 3.1

ASYMPTOTIC APPROXIMATION TO $J_o(x)$

| Value of x | Tabled* Value of $J_o(x)$ | Computed Value of $J_o(x)$ | Number of Terms in Sum | Last Term Retained |
|---|---|---|---|---|
| 0.0 | 1.00000 00000 | 1.00000 0 | 2 | 0 |
| 0.2 | 0.99002 49722 | .99002 50 | 4 | .1736111E-10 |
| 0.4 | 0.96039 82267 | .96039 82 | 4 | .4444444E-8 |
| 0.6 | 0.91200 48635 | .91200 49 | 5 | -.4100625E-9 |
| 0.8 | 0.84628 73528 | .84628 74 | 5 | -.7281777E-8 |
| 1.0 | 0.76519 76866 | .76519 77 | 6 | .4709503E-9 |
| 1.2 | 0.67113 27443 | .67113 27 | 6 | .4199040E-8 |
| 1.4 | 0.56685 51204 | .56685 51 | 7 | -.2670001E-9 |
| 1.6 | 0.45540 21676 | .45540 22 | 7 | -.1731405E-8 |
| 1.8 | 0.33998 64110 | .33998 64 | 7 | -.9006043E-8 |
| 2.0 | 0.22389 07791 | .22389 08 | 8 | .6151187E-9 |
| 2.2 | 0.11036 22669 | .11036 23 | 8 | .2826454E-8 |
| 2.4 | 0.00250 76833 | .25076 83E-2 | 9 | -.2021791E-9 |
| 2.6 | -0.09680 49544 | -.96804 95E-1 | 9 | -.8539928E-9 |
| 2.8 | -0.18503 60334 | -.18503 60 | 9 | -.3241743E-8 |
| 3.0 | -0.26005 19549 | -.26005 20 | 10 | .2525219E-9 |

* Tabled values are from Abramowitz and Stegun (1968), Table 9.1, p. 390.

series.

Most students of mathematics have a good deal of experience with convergent series, but they probably have not worked seriously with many divergent series. We are, therefore, now going to consider some examples of divergent asymptotic series. This should enable the reader to see how a given expansion behaves for different values of the argument, and how expansions differ for various functions.

The first function to be discussed is a modification of the Error Function, erf(x), which is defined by

$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$, and the Complementary Error Function,

defined by $\mathrm{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt$. These two functions are related by the equation, $\mathrm{erf}(x) + \mathrm{erfc}(x) = 1$. We desire an asymptotic expansion of erf(x) for $x \longrightarrow \infty$ and use the equation, $\mathrm{erf}(x) = 1 - \mathrm{erfc}(x) = 1 - \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt$. We will actually derive an asymptotic expansion only for the integral, $\int_x^\infty e^{-t^2} dt$.

To obtain an expansion for $I = \int_x^\infty e^{-t^2} dt$, we use the technique of successive integration by parts, where we take $dv = -2te^{-t^2}$, $u = -\frac{1}{2t}$ for the first integration, and $u = \frac{(-1)^n \cdot 3\cdot 5\cdots(2n-1)}{t^{2n+1}}$ for the n-th integration by parts. This process gives us the following equation.

$$(3.2)\quad I = \int_x^{\infty} e^{-t^2}dt = \frac{e^{-x^2}}{2x}\left[1 - \frac{1}{2x^2} + \frac{3}{(2x^2)^2} - \frac{3\cdot 5}{(2x^2)^3} + \cdots + (-1)^n \frac{3\cdot 5\cdots(2n-1)}{(2x^2)^n}\right] + (-1)^n \frac{3\cdot 5\cdots(2n+1)}{2^{n+2}}\int_x^{\infty}\frac{-2te^{-t^2}}{t^{2n+3}}\,dt.$$

We now show that this last integral, representing the error in the approximation to I by the sum up to the term containing $1/(2x^2)^n$, has the same sign as, and is less in absolute value than, the n+1-st term:

$$\left|(-1)^n \frac{3\cdot 5\cdots(2n+1)}{2^{n+2}}\int_x^{\infty}\frac{-2te^{-t^2}}{t^{2n+3}}\,dt\right| \leqslant \left|(-1)^{n+1}\frac{3\cdot 5\cdots(2n+1)}{2^{n+1}\,x^{2n+2}}\int_x^{\infty}\frac{te^{-t^2}}{t}\,dt\right|$$

$$= \left|(-1)^{n+1}\frac{\left(\frac{1}{2}\right)_{n+1}}{x^{2n+2}}\,\frac{1}{2x}\int_x^{\infty} 2te^{-t^2}dt\right| = \left|(-1)^{n+1}\frac{\left(\frac{1}{2}\right)_{n+1}}{x^{2n+2}}\,\frac{1}{2x}\int_{t=x}^{t=\infty} -\,d(e^{-t^2})\right|$$

$$= \left|(-1)^{n+1}\frac{3\cdot 5\cdots(2n+1)}{(2x^2)^{n+1}}\,\frac{1}{2x}\,e^{-x^2}\right| = \left|\text{n+1-st term}\right|.$$

Therefore, the maximum number of significant digits of accuracy in I can be determined by the ratio of the least term to the first term inside the brackets in (3.2). That is, if the n+1-st term is the minimum term, and if this minimum is, say, $- 0.3 \times 10^{-17}$, then the sum up to and including the n-th term should have 17S, assuming that we can determine $e^{-x^2}$ that accurately. It is the quantity

$$S = S(x) = 2x\,e^{x^2} I \sim 1 - \frac{1}{2x^2} + \frac{3}{(2x^2)^2} - \frac{3\cdot 5}{(2x^2)^3} + \cdots$$

which has been computed and tabulated in Tables 3.2 to 3.5.

The next function which we shall discuss is

$F(x) = \int_0^\infty \frac{e^{-xt}}{1+t^2}\,dt.$ Using (3.1), with $z = t^2$ and $N = n+1$, we have $\frac{1}{1+t^2} = 1 - t^2 + t^4 - \cdots + (-t^2)^n + \frac{(-t^2)^{n+1}}{1+t^2}$.

Substituting this in the above integral and integrating term by term, and using the result of Note 7, we have

$$F(x) = \int_0^\infty \frac{e^{-xt}}{1+t^2}\,dt = \int_0^\infty e^{-xt}\left[1 - t^2 + \cdots + (-t^2)^n + \frac{(-t^2)^{n+1}}{1+t^2}\right] dt =$$

$$\int_0^\infty e^{-xt}\,dt - \int_0^\infty e^{-xt}t^2\,dt + \cdots + \int_0^\infty e^{-xt}(-t^2)^n\,dt + \int_0^\infty \frac{e^{-xt}(-t^2)^{n+1}}{1+t^2}\,dt =$$

$$\frac{1}{x} - \frac{2!}{x^3} + \frac{4!}{x^5} - \cdots + (-1)^n \frac{(2n)!}{x^{2n+1}} + (-1)^{n+1} \int_0^\infty \frac{e^{-xt}\,t^{2n+2}}{1+t^2}\,dt.$$

The error in the sum up to the term in $1/x^{2n+1}$ is the final integral above. Neglecting the sign of this integral we have, since $\frac{1}{1+t^2} \leqslant 1$ for $0 \leqslant t < \infty$,

TABLE 3.2

THE ASYMPTOTIC EVALUATION OF S = S(X) = $2xe^{x^2}I$

THE CURRENT VALUE OF X IS 2.00

| I | C(I), TERM OF SERIES |
|---|---|
| 1 | .1000000000000 E+01 |
| 2 | -.12500000000000 00 E+00 |
| 3 | .4687500000000000000 E-01 |
| 4 | -.2929687500000000000 0 E-01 |
| 5 | → .25634765625000000000000 E-01 |
| 6 | -.28839111328125000000000000E-01 |
| 7 | .39653778076171875000000000E-01 |
| 8 | -.64437389373779296875000000E-01 |
| 9 | .12082010507583618164062500E+00 |
| 10 | -.25674272328615188598632812 50E+00 |
| 11 | .60976396780461072921752929 67E+00 |
| 12 | -.16006304154871031641960144 03E+01 |

IMIN= 5 AND THE MINIMUM TERM IS

C( 5)= .25634765625000000000000 0E-01

N= 12 AND THE SUM OF 4 TERMS IS

.8925781250000000000 E+00

THE EULER TRANSFORMATION OF SERIES FROM IMIN TO N

| I | J | TERM B(J) OF EULER TRANSFORMATION |
|---|---|---|
| 5 | 1 | .12817382812500000000000000E-01 |
| 6 | 2 | -.80108642578125000000000000E-03 |
| 7 | 3 | .95129013061523437500000000E-03 |
| 8 | 4 | -.39741396903991699218750000E-03 |
| 9 | 5 | .35223551094532012939453125 00E-03 |
| 10 | 6 | -.29748596716672182083129882 81E-03 |
| 11 | 7 | .31459519959753379225730895 85E-03 |
| 12 | 8 | -.36716583736051688902080059 37E-03 |

JMIN= 6 AND THE MINIMUM B(J) IS

B( 6)= -.2974859671667218208312988281E-03

| | |
|---|---|
| HEAD= | .8925781250000000000 0 E+00 |
| STIELTJES TAIL= | .12817382812500000000000000E-01 |
| EULER TAIL= | .12922408059239387512207031 25E-01 |
| STIELTJES SUM= | .90539550781250000000000000E+00 |
| EULER SUM= | .90550053305923938751220703 12E+00 |

THE SUM TO 4 TERMS OF THE ASYMPTOTIC SERIES FOR

S( 2.00) IS .8925781250000000000 E+00

TABLE 3.3

THE ASYMPTOTIC EVALUATION OF S = S(X) = $2xe^{x^2}I$

THE CURRENT VALUE OF X IS 5.00

| I | C(I), TERM OF SERIES |
|---|---|
| 1 | .1000000000000 E+01 |
| 2 | -.2000000000000 E-01 |
| 3 | .12000000000000 0 E-02 |
| 4 | -.12000000000000 E-03 |
| 5 | .16800000000000 00 E-04 |
| 6 | -.302400000000000*00 E-05 |
| 7 | .66528000000000000000 E-06 |
| 8 | -.1729728000000000000 0 E-06 |
| 9 | .5189184000000000000 0 E-07 |
| 10 | -.176432256000000000000 E-07 |
| 11 | .6704425728000000000000 E-08 |
| 12 | -.2815858805760000000000000000E-08 |
| 13 | .1295295050649600000000000000E-08 |
| 14 | -.6476475253248000000000000000F-09 |
| 15 | .3497296636753920000000000000E-09 |
| 16 | -.2028432049317273600000000000E-09 |
| 17 | .1257627870576709632000000000E-09 |
| 18 | -.8300343945806283571200000000E-10 |
| 19 | .5810240762064398499840000000E-10 |
| 20 | -.4299578163927654899881600000E-10 |
| 21 | .3353670967863570814107648000E-10 |
| 22 | -.2750010193648128067568271360E-10 |
| 23 | .2365008766537390138108713369E-10 |
| 24 | -.2128507889883651124297842032E-10 |
| 25 | .2000797416490632056839971510E-10 |
| 26 | → -.1960781468160819415703172079E-10 |
| 27 | .1999997097524035840172355 20E-10 |
| 28 | -.2119996923375477952258269651E-10 |
| 29 | .2331996615713025747484096616E-10 |
| 30 | -.2658476141912849352131870142E-10 |
| 31 | .3137001847457162235515606767E-10 |
| 32 | -.3827142253897737927329040255E-10 |
| 33 | .4822199239911149788434590720E-10 |
| 34 | -.6268859011884494724964957936E-10 |
| 35 | .8400271075925222931453057032E-10 |
| 36 | -.1159237408477680764540521870E-09 |
| 37 | .1646117120038306685647541055E-09 |
| 38 | -.2403330995255927761045409940E-09 |
| 39 | .3604996492883891641568114910E-09 |
| 40 | -.5551694599041193128014896960E-09 |
| 41 | .8771677466485085142263537196E-09 |
| 42 | -.1421011749570583793046693025E-08 |
| 43 | .2358879504287169096457510421E-08 |

## TABLE 3.3 -- Continued

```
IMIN= 26 AND THE MINIMUM TERM IS
             C( 26)=   -.1960781468160819415703172079E-10
N= 43 AND THE SUM OF   25 TERMS IS
                        .9810943073251908705630383417E+00
```

```
THE EULER TRANSFORMATION OF SERIES FROM IMIN TO N
```

| I | J | TERM B(J) OF EULER TRANSFORMATION |
|---|---|---|
| 26 | 1 | -.9803907340804097078515860395E-11 |
| 27 | 2 | .9803907340804097078515860250E-13 |
| 28 | 3 | -.1009802456102821999087133625E-12 |
| 29 | 4 | .7009793748674929411138840000E-14 |
| 30 | 5 | -.3520092930715711056041118125E-14 |
| 31 | 6 | .5972001137114111714892542187E-15 |
| 32 | 7 | -.2416966590440407816658730468E-15 |
| 33 | 8 | .6725257249840997190591914062E-16 |
| 34 | 9 | -.2700665200784435050249953125E-16 |
| 35 | 10 | .9971336641206337337898554687E-17 |
| 36 | 11 | -.4325152642535195639425683593E-17 |
| 37 | 12 | .1905415719053024818070190429E-17 |
| 38 | 13 | -.9140124060610672284936523437E-18 |
| 39 | 14 | .4571529878016297852904663085E-18 |
| 40 | 15 | -.2422529194943787817313232421E-18 |
| 41 | 16 | .1342547649455980166447387695E-18 |
| 42 | 17 | -.7795691505729220241954803466E-19 |
| 43 | 18 | .4720654436020210946312332153E-19 |

```
JMIN= 18 AND THE MINIMUM B(J) IS
             B( 18)=    .4720654436020210946312332153E-19

             HEAD=      .9810943073251908705630383417E+00
  STIELTJES TAIL=      -.9803907340804097078515860395E-11
      EULER TAIL=      -.9802956154027991294616984798E-11
   STIELTJES SUM=       .9810943073153869632222342447E+00
       EULER SUM=       .9810943073153879144090103505E+00

THE SUM TO  25 TERMS OF THE ASYMPTOTIC SERIES FOR
      S(   5.00) IS     .9810943073251908705630383417E+00
```

TABLE 3.4

THE ASYMPTOTIC EVALUATION OF S = S(X) = $2 x e^{x^2} I$

THE CURRENT VALUE OF X IS 7.50

| I | C(I), TERM OF SERIES |
|---|---|
| 1 | .1000000000000 E+01 |
| 2 | -.8888888888888888888888888888E-02 |
| 3 | .2370370370370370370370370369E-03 |
| 4 | -.1053497942386831275720164608E-04 |
| 5 | .6555098308184727937814357560E-06 |
| 6 | -.5244078646547782350251486048E-07 |
| 7 | .5127543565513387186912564135E-08 |
| 8 | -.5925161453482136304876740776E-09 |
| 9 | .7900215271309515073168987701E-10 |
| 10 | -.1193810307664548944389980363E-10 |
| 11 | .2016212964055682661636411278E-11 |
| 12 | -.3763597532903940968387967717E-12 |
| 13 | .7694466067270279313148733998F-13 |
| 14 | -.1709881348282284291810829777E-13 |
| 15 | .4103715235877482300345991463E-14 |
| 16 | -.1057846594137306548533633354E-14 |
| 17 | .2914955059400578044848234129E-15 |
| 18 | -.8550534840908362264888153443E-16 |
| 19 | .2660166394949266260187425514E-16 |
| 20 | -.8748991698944260055727532800E-17 |
| 21 | .3032983788967343485985544704E-17 |
| 22 | -.1105354091979209626003620736E-17 |
| 23 | .4224908973787201237169394812E-18 |
| 24 | -.1689963589514880494867757924F-18 |
| 25 | .7060292329528834067447521992E-19 |
| 26 | -.3075149547972558838256031800E-19 |
| 27 | .1394067795080893340013934416E-19 |
| 28 | -.6567608279047764179621202136E-20 |
| 29 | .3210830714201129154481476599E-20 |
| 30 | -.1626620895195239771503948143E-20 |
| 31 | .8531771805912807779967372481E-21 |
| 32 | -.4626116268094944662915641966E-21 |
| 33 | .2590625110133169011232759500E-21 |
| 34 | -.1496805619188053206490038822E-21 |
| 35 | .8914309020942183540874008983E-22 |
| 36 | -.5467442866177872571736058842E-22 |
| 37 | .3450563942210035134162312689E-22 |
| 38 | -.2239032602500733909278656232E-22 |
| 39 | .1492688401667155939519104154E-22 |
| 40 | -.1021662283807742287493075731E-22 |
| 41 | .7174339592961034729951376243E-23 |
| 42 | -.5165524506931945005564990894E-23 |
| 43 | .3811009191780901648550171059E-23 |

TABLE 3.4 -- Continued

| | | |
|---|---|---|
| 44 | | -.287942916712334791223790702E-23 |
| 45 | | .222675855590872238546398142 9E-23 |
| 46 | | -.176161343534112259827817197 4E-23 |
| 47 | | .142494953436481916838501021 8E-23 |
| 48 | | -.117795828174158384586494178 0E-23 |
| 49 | | .994720326804004136508173058 6E-24 |
| 50 | | -.857669970666563566589269214 9E-24 |
| 51 | | .754749574186575938598556908 9E-24 |
| 52 | | -.677597395491948175986259960 2E-24 |
| 53 | | .620378059872628107791864692 9E-24 |
| 54 | | -.579019522547786233939073713 2E-24 |
| 55 | | .550711901445450018057607887 1E-24 |
| 56 | | -.533578642289369350829148974 9E-24 |
| 57 | ⟶ | .526464260392177759484760321 7E-24 |
| 58 | | -.528804101549476327304692589 6E-24 |
| 59 | | .540555303806131356300352424 8E-24 |
| 60 | | -.562177515958376611072366521 7E-24 |
| 61 | | .594658883547082815267658809 6E-24 |
| 62 | | -.639588665859529072421215252 9E-24 |
| 63 | | .699283608006418452513862009 7E-24 |
| 64 | | -.776981786673798280570957788 4E-24 |
| 65 | | .877126105845087836733436792 1E-24 |
| 66 | | -.100577126803570071945434085 4E-23 |
| 67 | | .117116476544601594887572134 9E-23 |
| 68 | | -.138457701159395663289307501 6E-23 |
| 69 | | .166149241391274795947169001 9E-23 |
| 70 | | -.202332853960930195953441362 3E-23 |
| 71 | | .249993481782838197666918660 8E-23 |
| 72 | | -.313325163834490541075871388 0E-23 |
| 73 | | .398271097140730198878663186 4E-23 |
| 74 | | -.513327191870274478554721440 1E-23 |
| 75 | | .670747530710491985311502681 6E-23 |
| 76 | | -.888367840674340496101456884 8E-23 |

IMIN= 57 AND THE MINIMUM TERM IS
C( 57)= .5264642603921777594847603217E-24
N= 76 AND THE SUM OF 56 TERMS IS
.9913382208415630736318990439E+00

THE EULER TRANSFORMATION OF SERIES FROM IMIN TO N

| I | J | TERM B(J) OF EULER TRANSFORMATION |
|---|---|---|
| 57 | 1 | .2632321301960888797423801608E-24 |
| 58 | 2 | -.5849602893246419549830669750E-27 |
| 59 | 3 | .1176420137419557709465945850E-26 |
| 60 | 4 | -.2872804976461019378916841875E-28 |
| 61 | 5 | .1651552326990506394678858125E-28 |
| 62 | 6 | -.1134640652678372527575014062E-29 |
| 63 | 7 | .4199615253451032501043492187E-30 |

TABLE 3.4 -- Continued

| | | |
|---|---|---|
| 64 | 8 | -.5358827992392900351887265625E-31 |
| 65 | 9 | .1651893660583419245225605468E-31 |
| 66 | 10 | -.3116749748389760904947265625E-32 |
| 67 | 11 | .9170235546565258169672851562E-33 |
| 68 | 12 | -.2220732460193617257705078125E-33 |
| 69 | 13 | .6703958727403299030261230468E-34 |
| 70 | 14 | -.1914377262420510652972412109E-34 |
| 71 | 15 | .6128109373794952019897460937E-35 |
| 72 | 16 | -.1967395261742344963989257812E-35 |
| 73 | 17 | .6752319826225148138732910156E-36 |
| 74 | 18 | -.2374371716582633183326721191E-36 |
| 75 | 19 | .8742456056352372494888305664E-37 |
| 76 | 20 | -.3317718022407703957366943359E-37 |

JMIN= 20 AND THE MINIMUM B(J) IS
B( 20)= -.3317718022407703957366943359E-37

| | |
|---|---|
| HEAD= | .9913382208415630736318990439E+00 |
| STIELTJES TAIL= | .2632321301960888797423801608E-24 |
| EULER TAIL= | .2638106234000007473837435749E-24 |
| STIELTJES SUM= | .9913382208415630736318993071E+00 |
| EULER SUM= | .9913382208415630736318993077E+00 |

THE SUM TO 56 TERMS OF THE ASYMPTOTIC SERIES FOR
S( 7.50) IS .9913382208415630736318990439E+00

TABLE 3.5

THE ASYMPTOTIC EVALUATION OF S = S(X) = $2xe^{x^2}I$

THE CURRENT VALUE OF X IS 10.00

| I | C(I), TERM OF SERIES |
|---|---|
| 1 | .1000000000000 E+01 |
| 2 | -.5000000000000 E-02 |
| 3 | .75000000000000 0 E-04 |
| 4 | -.1875000000000000 E-05 |
| 5 | .656250000000000000 F-07 |
| 6 | -.29531250000000000000 0 E-08 |
| 7 | .16242187500000000000 E-09 |
| 8 | -.1055742187500000000000000 E-10 |
| 9 | .7918066406250000000000000000E-12 |
| 10 | -.6730356445312500000000000000E-13 |
| 11 | .6393838623046875000000000000E-14 |
| 12 | -.6713530554199218750000000000E-15 |
| 13 | .7720560137329101562500000000E-16 |
| 14 | -.9650700171661376953125000000E-17 |
| 15 | .1302844523174285888671875000E-17 |
| 16 | -.1889124558602714538574218750F-18 |
| 17 | .2928143065834207534790039062E-19 |
| 18 | -.4831436058626442432403564452E-20 |
| 19 | .8455013102596274256706237791F-21 |
| 20 | -.1564177423980310737490653991E-21 |
| 21 | .3050145976761605938106775282E-22 |
| 22 | -.6252799252361292173116889328E-23 |
| 23 | .1344351839257677817220561205E-23 |
| 24 | -.3024791638329775088746262711E-24 |
| 25 | .7108260350074971458553717370E-25 |
| 26 | -.1741523785768366007345660755E-25 |
| 27 | .4440885653709338418731434925E-26 |
| 28 | -.1176834698232974680963830255E-26 |
| 29 | .3236295420140680372650533201E-27 |
| 30 | -.9223441947400939062054019622F-28 |
| 31 | .2720915374483277023305935788E-28 |
| 32 | -.8298791892173994921083104153E-29 |
| 33 | .2614119446034808400141177808E-29 |
| 34 | -.8495888199613127300458827876E-30 |
| 35 | .2846122546870397645653707338E-30 |
| 36 | -.9819122786702871877505290316E-31 |
| 37 | .3485788589279519516514378062E-31 |
| 38 | -.1272312835087024623527747992E-31 |
| 39 | .4771173131576342338229054970E-32 |
| 40 | -.1836901655656891800218186163E-32 |
| 41 | .7255761539844722610861835343E-33 |
| 42 | -.2938583423637112657399043313F-33 |
| 43 | .1219512120809401752820602974E-33 |

TABLE 3.5 -- Continued

| | |
|---|---|
| 44 | -.5182926513439957449487562639E-34 |
| 45 | .2254573033346381490527089747E-34 |
| 46 | -.1003284999839139763284554937E-34 |
| 47 | .4564946749268085922944724963E-35 |
| 48 | -.2122700238409659954169297107E-35 |
| 49 | .1008282513244588478230416125E-35 |
| 50 | -.4890170674236254119417518206E-36 |
| 51 | .2420634483746945789111671511E-36 |
| 52 | -.1222420414292207623501394113E-36 |
| 53 | .6295465133604869261032179681E-37 |
| 54 | -.3305119195142556362041894332E-37 |
| 55 | .1768238769401267653692413467E-37 |
| 56 | -.9636901293236908712623653395E-38 |
| 57 | .5348480217746484335506127634E-38 |
| 58 | -.3021891323026763649560962113E-38 |
| 59 | .1737587510740389098497553214E-38 |
| 60 | -.1016488693783127622621068630E-38 |
| 61 | .6048107728009609354595358348E-39 |
| 62 | -.3659105175445813659530191800E-39 |
| 63 | .2250349682899175400611067957E-39 |
| 64 | -.1406468551811984625381917473E-39 |
| 65 | .8931075304006102371175175953E-40 |
| 66 | -.5760543571083936029407988489E-40 |
| 67 | .3773156039059978099262232460E-40 |
| 68 | -.2509148765974885436009384585E-40 |
| 69 | .1693675417033047669306334594E-40 |
| 70 | -.1160167660667637653474839196E-40 |
| 71 | .8063165241640081691650132412E-41 |
| 72 | -.5684531495356257592613343350E-41 |
| 73 | .4064440019179724178718540495E-41 |
| 74 | -.2946719013905300029570941858E-41 |
| 75 | .2165838475220395521734642265E-41 |
| 76 | -.1613549664039194663692308487E-41 |
| 77 | .1218229996349591971087692907E-41 |
| 78 | -.9319459472074376578820850738E-42 |
| 79 | .7222581090857643398586159321E-42 |
| 80 | -.5669726156323250067890135066E-42 |
| 81 | .4507432294276983803972657377E-42 |
| 82 | -.3628482996892971962197989188E-42 |
| 83 | .2957213642467772149191361138E-42 |
| 84 | -.2439701255035912023082872980E-42 |
| 85 | .2037150547954986539274198938E-42 |
| 86 | -.1721392213021963625686698102E-42 |
| 87 | .1471790342133778899962126877E-42 |
| 88 | -.1273098645945718748467239748E-42 |
| 89 | .1113961315202503904908834779E-42 |
| 90 | -.9858557639542159558443187794E-43 |
| 91 | .8823409087390232804806653075E-43 |
| 92 | -.7985185224088160688350021032E-43 |

TABLE 3.5 -- Continued

| | | |
|---|---|---|
| 93 | | .73064444800406670298402692 44E-43 |
| 94 | | -.67584611440376170026022490 50E-43 |
| 95 | | .63191611696751718974331028 61E-43 |
| 96 | | -.59716073053430374430742822 03E-43 |
| 97 | | .57028849766026007581359395 03E-43 |
| 98 | | -.55032840024215097316011816 20E-43 |
| 99 | | .53657019023609719883111520 79E-43 |
| 100 | | -.52852163738255574084864847 97E-43 |
| 101 | ——→ | .52587902919564296214440523 73E-43 |
| 102 | | -.52850842434162117695512726 34E-43 |
| 103 | | .53643605070674549460945417 23E-43 |
| 104 | | -.54984695197441413197469052 66E-43 |
| 105 | | .56909159529351862659380469 50E-43 |
| 106 | | -.59470071708172696479052590 62E-43 |
| 107 | | .62740925652122194785400483 10E-43 |
| 108 | | -.66819085819510137446451514 50E-43 |
| 109 | | .71830517255973397754935378 08E-43 |
| 110 | | -.77936111222731136564104885 21E-43 |
| 111 | | .85340041788890594537694849 30E-43 |
| 112 | | -.94300746176724106964152808 47E-43 |
| 113 | | .10514533198704737926503038 14E-42 |
| 114 | | -.11828849848542830167315917 90E-42 |
| 115 | | .13425744578096112239903566 81E-42 |
| 116 | | -.15372477541920048514689583 99E-42 |

IMIN=101 AND THE MINIMUM TERM IS

C(101)= .5258790291956429621444052373E-43

N=116 AND THE SUM OF 100 TERMS IS

.9950731878244697473807371968E+00

THE EULER TRANSFORMATION OF SERIES FROM IMIN TO N

| I | J | TERM B(J) OF EULER TRANSFORMATION |
|---|---|---|
| 101 | 1 | .2629395145978214810722026186E-43 |
| 102 | 2 | -.6573487864945537026805065250E-46 |
| 103 | 3 | .6622789023932628554506104125E-46 |
| 104 | 4 | -.1156523021238855420653522500E-47 |
| 105 | 5 | .5169483296666271114493937500E-48 |
| 106 | 6 | -.2319656762988766421414718750E-49 |
| 107 | 7 | .7099759730654749659007031250E-50 |
| 108 | 8 | -.5786907056945943271320312500E-51 |
| 109 | 9 | .1459466967500054063009375000E-51 |
| 110 | 10 | -.1777654872576711631006835937E-52 |
| 111 | 11 | .4128186741349059668066406250E-53 |
| 112 | 12 | -.6611435220650035381274414062E-54 |
| 113 | 13 | .1515408879058368436767578125E-54 |
| 114 | 14 | -.2930561115606490667724609375E-55 |
| 115 | 15 | .6903207609974079257202148437E-56 |

TABLE 3.5 -- Continued

```
116       16              -.15261789421853924392700195 31E-56

   JMIN= 16 AND THE MINIMUM B(J) IS
                 B( 16)=   -.152617894218539243927001953 1E-56


            HEAD=          .995073187824469747380737196 8E+00
   STIELTJES TAIL=         .262939514597821481072202618 6E-43
       EULER TAIL=         .262937883529481715945612779 8E-43
    STIELTJES SUM=         .995073187824469747380737196 8E+00
        EULER SUM=         .995073187824469747380737196 8E+00

   THE SUM TO 100 TERMS OF THE ASYMPTOTIC SERIES FOR
          S(  10.00) IS    .995073187824469747380737196 8E+00
```

$$\int_0^\infty \frac{e^{-xt}\ t^{2n+2}}{1 + t^2} dt \leqslant \int_0^\infty e^{-xt}\ t^{2n+2}\, dt = \frac{(2n+2)!}{x^{2n+3}}\ .$$ Again,

therefore, the error has the same sign as, and is less in absolute value than, the first term omitted in the sum.

So we write $F(x) = \displaystyle\int_0^\infty \frac{e^{-xt}}{1 + t^2} dt \sim \frac{1}{x} \sum_{n=0}^{N} \frac{(-1)^n\ (2n)!}{x^{2n}}$ and

the error, $E_N(x)$, is the same as, and less in absolute value than, the next term, $\dfrac{(-1)^{N+1}\ (2N+2)!}{x^{2N+3}}$. This

approximation holds for $N = 0, 1, 2, \ldots$ and so is a complete expansion. This series is asymptotic for $x \to \infty$, and we would accordingly expect its accuracy to improve as x increases. Since the minimum term (in absolute value) is a measure of the maximum accuracy attainable, we have tabulated the minimum terms in the expansion of F(x) for various values of x in Table 3.6.

One can see from the table that this expansion for F(x) is practically useful only for $x > 20$, whereas some of the previous expansions for $x \to \infty$ were useful for smaller x.

TABLE 3.6

ACCURACY OF THE EXPANSION OF THE FUNCTION

$$F(x) = \int_0^\infty \frac{e^{-xt}}{1 + t^2}\,dt \sim \sum_{n=0}^{\infty} \frac{(-1)^n\,(2n)!}{x^{2n+1}}$$

| X | Sum up to the Least Term | Least Term | Number, n, of Least Term | Relative Error in Sum | Number of Significant Digits |
|---|---|---|---|---|---|
| 5 | -.08 | .384 E-1 | 2 | .5 | $1^-$ |
| 10 | -.0179 | -.363 E-3 | 5 | .02 | $3^-$ |
| 20 | -.485995 E-2 | .232 E-7 | 10 | .5 E-5 | $6^-$ |
| 50 | -.79620508313550446 E-3 | -.349 E-20 | 25 | .4 E-17 | 17 |
| 100 | -.19976071600381751317096l7055 E-3 | .933 E-42 | 50 | .5 E-38$^+$ | 26$^*$ |

$^-$ Superscript minus means the final digit is probably in error.

$^+$ The actual error is not really so small. See next note.

* The asymptotic error is considerably less than the 28S of our calculations. Therefore, the actual error is due mostly to round-off and, after fifty terms, likely affects the last two digits of the twenty-eight.

# CHAPTER IV

# CONCLUSION

We have seen that a power series in x is asymptotic for $x \longrightarrow 0$ and is useful for computational purposes for values of x close to the origin. An asymptotic expansion for $x \longrightarrow \infty$ is similarly useful for large x. Where one has computational difficulties is for x in the midrange, say for $2 < x < 20$. If an integral such as that defined by $F(x) = \int_0^\infty f(x, t)\,dt$ is to exist, then the integrand must have a strong decay factor, such as $e^{-t}$, and have the horizontal axis as an asymptote for $t \longrightarrow \infty$.

Such a geometric situation as an asymptote can often be closely approximated by a rational function much more easily than by a power series in x. For rational functions can have asymptotes in any position in the plane. On the other hand, no polynomial ever had an asymptote; and for computational purposes, a power series in x is only a very high order polynomial.

Another means of approximation in the midrange is the use of continued fractions. These, however, are difficult to apply when one desires fixed minimum accuracy. These and other techniques are discussed in the current journal literature and in various textbooks, such as Acton (1970), Hamming (1962), and Hildebrand (1956).

One seldom obtains something for nothing. Therefore, when an asymptotic expansion, designed really only for investigating the behavior of a function for values of an argument tending to some limit, happens also to be useful for calculation in some sizable neighborhood about this limit, then this would seem to be an occasion for gladness. We should, therefore, make thankful use of such asymptotic methods when they are applicable and look for the next easier method when they are not.

# APPENDIX I

## THE IMPROVEMENT OF ASYMPTOTIC APPROXIMATIONS AND THE TRANSFORMATION OF SERIES

At the end of Chapter II mention was made of some techniques to improve an asymptotic approximation, such as Euler's transformation and Shanks' transformations. We now comment briefly on these transformations.

If for some function of x, one has power series expansions in x and in 1/x which are asymptotic for $x \rightarrow 0$ and for $x \rightarrow \infty$, respectively, then there is usually some midrange of x, $a < x < b$, wherein neither series gives satisfactory service. What is often done in practice is to sum a convenient number of terms of the ascending series, the power series in x, and then to apply the Euler transformation to the remaining "tail" of this series.

Then, for the same value of x, one sums the descending series, the series in 1/x, to a number of terms which is determined by the behavior of the particular series. If this series is divergent, then the sum is taken up to but omitting the term of least absolute value. The Euler transformation is then applied to the diverging tail.

The Euler terms obtained in this way usually become smaller in absolute value for a time and then begin to grow in size, reflecting the behavior of the original series. The partial sum of these is taken, first omitting the smallest term. The Euler transformation is then applied to this new diverging tail. The process is continued until good sense or fatigue dictates a halt. All these partial sums are then added together and the result called the corrected sum of the descending series. This prolonged process, hopefully, results in a greater number of significant digits than were had from the original series.

After this, the results obtained from the ascending and descending series are compared. The digits for which these two values agree are then very likely to be correct.

Alternatively, sometimes the ascending (convergent) series is summed to a convenient number of terms and the tail "Eulered." Then the process is repeated, summing a different number of initial terms and Eulering that tail. The digits which agree in these two quantities are then regarded as being correct.

For series of specific type, the Euler transformation often induces convergence in divergent series and can accelerate convergence in convergent series. Fortunately, the more slowly the original series converges, the more rapidly convergent is the Euler-transformed series.

Regarding x as a complex variable, if the series satisfies the requirements associated with the Euler transformation, then the transformation just extends the region of convergence of the series. The most helpful sources on this topic seem to me to be Knopp (1954), pp. 244-7, 262ff, and 468ff; Bromwich (1926), pp. 62-8 and 318f; and all of Rosser (1951). Ames (1901-02) discusses the Euler transformation without calling it by name. The content of this paper is troubled with errors and is covered much better in more recent literature.

Since Rosser (1951) is current and well worth reading, an error there should be noted. On page 56, column 2, line 9 from the bottom, an additional factor of x is needed. This line should read:

$$+ \; x \, \frac{(-x)^N}{1 + x} \sum_{m=0}^{\infty} \left[\frac{x}{1 + x}\right]^m \frac{m!}{(N + 1) \cdot \, \cdot \, \cdot (N + m + 1)} \, .$$

In Rosser's accompanying numerical example, his choice of x = 1 allows the error to remain hidden.

Apparently not much work has been done on the problem of optimum separation of a series into a head and a tail, in connection with the use of Euler's transformation. The author of this present paper has performed some numerical experiments along these lines, but so far there are no illuminating results.

Also mentioned before were Shanks' transformations.

These non-linear sequence-to-sequence transformations are discussed in Shanks (1955). An example of such a transformation is $\{S_n\} \rightarrow \{T_n\}$ which is defined by the equation $T_n = \dfrac{S_{n-1}\, S_{n+1} - S_n^{\,2}}{S_{n-1} + S_{n+1} - 2S_n}$. The reader will note that this is just Aitken's $\delta^2$ process, but it occurs naturally as a specific instance of Shanks' more general transformations. Applications are given in Shanks' paper on inducing and accelerating convergence, summing numerical series, generating continued fractions from series, and the generation of a sequence of rational function approximations to power series, useful in conjunction with the difficult midrange evaluation of asymptotic series mentioned earlier. Also included are some examples of the use of these transformations to detect errors in tables and in formulae.

The theory developed by Shanks is uniformly applicable to summing divergent series and accelerating the convergence of slowly convergent series. This is not possible in using linear transformations.

The literature on linear transformations of series and sequences is large and has been long in the making. In contrast, the literature on non-linear sequence transformations is small and recent. Shanks' paper on the subject constitutes a major portion of such work.

Other methods of transforming series include converting the series into a contour integral and evaluating the integral by means of Cauchy's theorem, sometimes leading to a new series which is more rapidly convergent. Another technique is the use of Mellin transforms, a class of integral transforms discussed in Macfarlane (1949).

In much of the recent literature on asymptotic expansions, use is made of a technique called converging factors, due to John R. Airey. Essentially this method takes the divergent tail of a series and expresses it as the first term of the tail times a numerical factor, which is derived from a transformation of the original series. This method is discussed at length in Airey (1937), in Dingle (1958) and (1959), and is mentioned in the monograph, Erdélyi (1956).

## APPENDIX II

## ADVANCED TECHNIQUES FOR OBTAINING ASYMPTOTIC EXPANSIONS

There are several additional techniques for obtaining asymptotic expansions of functions of a complex variable. The first is known as Watson's Lemma, which gives conditions under which the term-wise Laplace transform of a series gives an asymptotic expansion for the Laplace transform of the sum of the series. That is, for an appropriate f(t), $F(x) = \int_0^\infty e^{-xt} f(t)\,dt \sim \sum_n \frac{a_n \Gamma(n/r)}{x^{n/r}}$.

A simple example is discussed in Rainville (1960), on pp. 41 ff.

The secondtechnique is known as the method of steepest descent, and originated with G. F. B. Riemann and P. Debye. There are also the methods of Laplace and stationary phase and the saddle-point method. These are discussed in Erdélyi (1956), Jeffreys (1962), Copson (1965), Jeffreys and Jeffreys (1956), Friedman (1959), and in Evgrafov (1961). These books, as well as much of the late journal literature, are concerned with asymptotic approximations which hold uniformly in some region of the complex plane.

## APPENDIX III

## RECENT WORK IN THE LITERATURE

Recent papers on asymptotics, or on topics which are ancillary to asymptotics or to other subjects mentioned in this paper, are included in the bibliography. It is by no means intended to be a complete list, but rather only a representative view of the extent to which the ideas of asymptotics have become part of the fabric of current mathematical and scientific work.

# NOTES

1. In addition to Rainville (1960), p. 299, and (1967), pp. 144-7, one can find more discussion in Franklin (1964), pp. 546-52, and Exercises 8-11 on p. 576; in Bromwich (1926), pp. 297-303; in Courant (1957), Vol. I, pp. 421-2, and Examples 3 and 4 on p. 446; in Pierpont (1959), p. 289 and pp. 310-7; in Abramowitz and Stegun (1968), Ch. 23; in Whittaker and Watson (1952), Art. 7.2 on pp. 125-6; and in Knopp (1954), pp. 183, 203-4, 237, 479, 523, and 534-5.

When using these numbers, one should be aware that various systems of notation exist for them. One system has $B_0=1$, $B_1=-1/2$, $B_2=1/6$, $B_{2k+1}=0$ for $k \geq 1$, $B_4=-1/30$, $B_6=1/42$, $B_8=-1/30$, etc., with a continuing and regular alternation in sign. Rainville, Franklin, Knopp and this paper use this system. Another system disregards the first two numbers, 1 and -1/2, and then orders the remaining non-zero Bernoulli numbers, those with even subscripts in the previous system, in serial order, beginning with $B_1=1/6$. In this system the Bernoulli numbers are all positive, using the absolute values of the numbers in the first system.

2. See Bromwich (1926), Ch. XII; Copson (1965), Ch. 1; Jeffreys (1962), Ch. 1; and Knopp (1954), Ch. XIV.

3. See Bromwich (1926), pp. 329-31, p. 340; and Whittaker and Watson (1952), Art. 12.33, pp. 251-2.

4. See Salzer (1954), p. vii; Davis (1933), p. 180; and Wrench (1968), pp. 618-9.

5. To compute $e^{100\pi}$ we would use $\log_{10}(e^x) = x \log_{10} e$ and $e^x = \exp_{10}(x \log e) = 10^{x \log e}$. Such computations constitute one of the principal uses left for logarithms to the base ten after the wide-spread availability of calculators and computers. Another use is the computation of $\ln x$ for large x, using $\ln x = (\ln 10)(\log x)$.

6. There is an error in Abramowitz and Stegun, (1968), p. 231, entry (5.1.51), where it is stated that the expansion is valid for Z in the sector defined by $|\arg Z| < \frac{3}{2}\pi$. It should be $|\arg Z| < \frac{\pi}{2}$. Compare with (5.1.4), p. 228, of the same work.

7. For $x > 0$, $\int_0^\infty e^{-xt}\, t^n\, dt = \int_0^\infty \frac{e^{-(xt)}\,(xt)^n}{x^{n+1}}\, x\, dt =$

$$\frac{1}{x^{n+1}} \int_0^\infty e^{-w}\, w^n\, dw = \frac{1}{x^{n+1}} \Gamma(n+1) = \frac{n!}{x^{n+1}}$$, setting $w=xt$.

8. The reader might wonder why, when we are interested in $E_1(x)$, the table contains values of $x\, e^x\, E_1(x)$. The answer is that the "auxiliary" function, $x\, e^x\, E_1(x)$, is better behaved, i.e., is simpler or more nearly linear on the tabled interval, than the original function, $E_1(x)$. Interpolation in the table is then easier and more accurate; and $E_1(x)$ is easily recoverable from $x\, e^x\, E_1(x)$, since $e^x$ is well tabulated.

As another example of an auxiliary function for the purposes of tabulation, suppose that some $F(x)$ has a singularity at, say, the origin, and that it behaves like $-\ln x$ there. Then the auxiliary function $F(x) + \ln x$ would be much easier to table in the vicinity of the origin than $F(x)$ would be. See Abramowitz and Stegun (1968), p. x; Fox (1956), pp. 5-7; and Goodwin and Staton (1948), p. 320.

9. See Copson (1965), Ch. 3; Jeffreys (1962), p. 2; Copson (1935), p. 230; and Abramowitz and Stegun (1968), pp. 230 and 260.

10. To verify that $-a \int_0^\infty \frac{e^{-t}}{bx + t}\, dt$ is a solution of the equation $y' = \frac{a}{x} + b\, y$, set $F(x) = -a \int_0^\infty \frac{e^{-t}}{bx + t}\, dt$.

Then we have $F'(x) = ab \int_0^\infty \frac{e^{-t}\, dt}{(bx + t)^2}$. Now we have, integrating by parts on the right, with $u = e^{-t}$ and $dv = \frac{dt}{(bx + t)^2}$, $du = -e^{-t}dt$, $v = \frac{-1}{bx + t}$ and

$$F'(x) = ab \int_0^\infty \frac{e^{-t}dt}{(bx + t)^2} = ab\left[\left(\frac{-e^{-t}}{bx + t}\right)_0^\infty - \int_0^\infty \frac{e^{-t}dt}{bx + t}\right] =$$

$$ab\left(\frac{1}{bx}\right) - ab \int_0^\infty \frac{e^{-t}dt}{bx + t} = \frac{a}{x} + b(-a) \int_0^\infty \frac{e^{-t}dt}{bx + t} = \frac{a}{x} + b\,F(x).$$

Therefore, $y = F(x)$ is a solution of the differential equation, $y' = \frac{a}{x} + b\,y$.

COMPUTER PROGRAM LISTINGS

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PROGRAM FOR TABLE 1.1
ASYMPTOTIC APPROXIMATION OF EULER'S CONSTANT

*2805
C      1 + 1/2 + 1/3 +...+ 1/N - 1/2N - LOGN +
C        B(2)/2N**2 + B(4)/4N**4 +... = GAMMA
       DIMENSION X(300),B(60),C(60),R(60)
       TYPE 160
       ACCEPT 150,N
       Y=N
       DO 10 I=1,N
       FI=I
       X(I)=1./FI
       PUNCH 100,I,X(I)
    10 CONTINUE
       SUM=0.
       DO 15 I=1,N
       SUM=SUM+X(I)
    15 CONTINUE
       SUM2=SUM-.5*X(N)-LOG(Y)
       Z1=.5*X(N)
       Z2=LOG(Y)
       PUNCH 110,Z1,Z2,SUM2
       Z3=1.
       B(1)=-.5
       B(2)=1./6.
       B(4)=-1./30.
       B(6)=1./42.
       B(8)=B(4)
       B(10)=5./66.
       B(12)=-691./2730.
       B(14)=7./6.
       B(16)=-3617./510.
       B(18)=43867./798.
       B(20)=-174611./330.
       B(22)=854513./138.
       B(24)=-236364091./2730.
       B(26)=8553103./6.
       B(28)=-23749461029./870.
       B(30)=8615841276005./14322.
       B(32)=-7709321041217./510.
       B(34)=2577687858367./6.
       B(36)=-26315271553053477373./1919190.
       B(38)=2929993913841559./6.
       B(40)=-261082718496449122051./13530.
       B(42)=1520097643918070802691./1806.
       B(44)=-27833269579301024235023./690.
       B(46)=596451111593912163277961./282.
       B(48)=-5609403368997817686249127547./46410.
       B(50)=495057205241079648212477525./66.
       B(52)=-801165718135489957347924991 9.E2/1590.
       B(54)=29149963634884862421418123 81.E4/798.
       B(56)=-2479392929313226753685415 74.E7/870.
```

```
      PUNCH 119,Z3
      IDUM=1
      PUNCH 120,IDUM,B(1)
      DO 20 I=1,28
      J=2*I
      PUNCH 120,J,B(J)
   20 CONTINUE
      PUNCH 125
      DO 25 I=1,27
      JTOP=2*(I+1)
      JBOT=2*I
      FI=I
      R(I)=((B(JTOP)/B(JBOT))/Y**2)*(FI/(FI+1.))
      PUNCH 130,I,R(I)
   25 CONTINUE
      PUNCH 140
      DO 30 I=1,28
      IBY2=2*I
      FI=IBY2
      C(I)=B(IBY2)/(FI*Y**IBY2)
      PUNCH 145,I,C(I)
   30 CONTINUE
      TYPE 135
      ACCEPT 150,K
      BERNOU=0.
      DO 35 I=1,K
      BERNOU=BERNOU+C(I)
   35 CONTINUE
      SUM3=SUM2+BERNOU
      IMIN=K+1
      PUNCH 155,SUM3,C(IMIN)
  100 FORMAT(26X,2H1/I2,1H= E34.28)
  110 FORMAT(26X,5H1/2N= E34.28/26X,5HLOGN= E34.28 /
     1/26X39H1 + 1/2 + 1/3 +...+ 1/N - 1/2N - LOGN =
     2   /31X,E34.28    )
  119 FORMAT(//27X,27HTHE BERNOULLI NUMBERS, B(N)/
     1    25X6HB( 0)=E34.28 )
  120 FORMAT(25X,2HB(I2,2H)= E34.28 )
  125 FORMAT(///12X, 15HTHE VALUES OF I 12X,4HR(I) /)
  130 FORMAT(19X,I2,10X,F9.2 )
  135 FORMAT(42H WHAT IS K, FOR SUM OF B-TERMS UP TO B(2K))
  140 FORMAT(///24X34HTHE BERNOULLI TERMS, B(2K)/2KN**2K//
     1    20X1HK12X13HB(2K)/2KN**2K/ )
  145 FORMAT(19X,I2,10X,E34.28 )
  150 FORMAT(I2)
  155 FORMAT(39HTHE APPROXIMATION TO EULER'S CONSTANT,
     110HGAMMA, IS /31XE34.28//3X17HAND THE ERROR IS
     2  11HBOUNDED BY  E34.28 )
  160 FORMAT(8HN=   (I2))
      STOP
      END
```

PROGRAM FOR GEOMETRIC SERIES, TABLE 2.1

```
*1505
      DIMENSION A(100),S(100)
      PUNCH 1
      A(1)=1.
      S(1)=1.
      DO 100 I=2,25
      A(I)=-A(I-1)/6.
      S(I)=S(I-1)+A(I)
      J=I-1
      PUNCH 5,J,S(J),A(I)
  100 CONTINUE
    1 FORMAT(19X,1HN,11X,4HS(N),19X,6HA(N+1) //)
    5 FORMAT(17X,I3,5X,F18.15,5X,F18.15)
      STOP
      END
```

FOR TABLE 2.2, THIS NEXT CARD REPLACING THE OLD ONE.

```
    5 FORMAT(17X,I3,5X,F18.15,6X,E21.15 )
```

PROGRAM FOR TABLE 2.3, ASYMPTOTIC APPROXIMATION

TO $x\, e^{x}\, E_1(x)$

```
*1505
      DIMENSION A(100),S(100)
      PUNCH 1
      A(1)=1.
      S(1)=1.
      DO 100 I=2,25
      F=I-1
      A(I)=-F*0.1*A(I-1)
      S(I)=S(I-1)+A(I)
      J=I-1
      PUNCH 5,J,S(J),A(I)
  100 CONTINUE
    1 FORMAT(19X,1HN,11X,4HT(N),19X,6HC(N+1)   //)
    5 FORMAT(17X,I3,5X,F18.15,5X,F18.15)
      STOP
      END
```

PROGRAM FOR TABLES 3.2 TO 3.5
ASYMPTOTIC APPROXIMATION TO S = S(X) = $2xe^{x^2}I$

```
*2805
      DIMENSION C(100),A(100),B(100),U(100),V(100),H(10)
      PUNCH 48
      TYPE 10
      ACCEPT 11, X
      TYPE 12
      ACCEPT 13, LL,LU
      PUNCH 43,X
      C(1)=1.
      IDUM=1
      PUNCH 15,IDUM,C(1)
  100 DO 150 I=LL,LU
      FI=I
      C(I)=-C(I-1)*(FI-1.5)/X**2
      PUNCH 15,I,C(I)
  150 CONTINUE
      TYPE 12
      ACCEPT 13, LL, LU
      IF(LL) 300,300,200
  200 CONTINUE
      GO TO 100
  300 CONTINUE
      N=LU
      IMIN=1
      EM9=ABS(C(1))
      DO 325 I=2,N
      IF(ABS(C(I))-EM9) 330,325,325
  330 IMIN=I
      EM9=ABS(C(I))
  325 CONTINUE
      PUNCH 30, IMIN, IMIN, C(IMIN)
      K=IMIN-1
      HEAD=0.
      DO 210 I=1,K
      HEAD=HEAD+C(I)
  210 CONTINUE
      PUNCH 35,N,K,HEAD
      H(1)=HEAD
      NTAIL=N-K
      DO 215 I=1,NTAIL
      LDUM=I+K
      A(I)=C(LDUM)
  215 CONTINUE
      PUNCH 20
      B(1)=A(1)*.5
```

```
      JDUM=K+1
      IDUM=1
      PUNCH 21,JDUM,IDUM,B(1)
      B(2)=(A(1)+A(2))*.25
      JDUM=K+2
      IDUM=2
      PUNCH 21,JDUM,IDUM,B(2)
      U(1)=1.
      U(2)=1.
      DO 335 I=3,NTAIL
      IDUM=I-1
      DO 340 J=2,IDUM
      V(J)=U(J)+U(J-1)
  340 CONTINUE
      DO 345 J=2,IDUM
      U(J)=V(J)
  345 CONTINUE
      U(I)=1.
      T9=A(1)
      DO 350 J=2,I
      T9=T9+U(J)*A(J)
  350 CONTINUE
      B(I)=T9/2.**I
      KDUM=K+I
      PUNCH 21,KDUM,I,B(I)
  335 CONTINUE
      DO 360 I=1,NTAIL
      IF(ABS(B(I)))360,360,370
  370 TMIN=ABS(B(I))
      JMIN=I
      NDUM=I+1
      GO TO 375
  360 CONTINUE
      TYPE 50
      GO TO 351
  375 DO 380 J=NDUM,NTAIL
      IF(ABS(B(J))-TMIN)385,380,380
  385 JMIN=J
      TMIN=ABS(B(J))
  380 CONTINUE
      PUNCH 51,JMIN,JMIN,B(JMIN)
      KDUM=JMIN-1
  351 T9=0.
      DO 355 I=1,KDUM
      T9=T9+B(I)
  355 CONTINUE
      TAIL=T9
      H(2)=TAIL
      SUM=HEAD+TAIL
```

```
      STAIL=.5*C(IMIN)
      PUNCH 44,HEAD,STAIL
      STSUM=STAIL+H(1)
      PUNCH 45,H(2),STSUM
      EUSUM=HEAD+H(2)
      PUNCH 46,EUSUM
      PUNCH 49,K,X,HEAD
   10 FORMAT(   16HX= (F7.2)+---.--   )
   11 FORMAT(F7.2)
   12 FORMAT(18H LL, LU ARE (2I3) )
   13 FORMAT( 2I3 )
   15 FORMAT(12XI3,22XE34.28 )
   20 FORMAT(//15X34HTHE EULER TRANSFORMATION OF SERIES
     1 15H FROM IMIN TO N //14X1HI7X1HJ14X
     2 33HTERM B(J) OF EULER TRANSFORMATION /)
   21 FORMAT(12XI3,5XI3,14XE34.28 )
   22 FORMAT(/)
   30 FORMAT(15X5HIMIN= I3,24H AND THE MINIMUM TERM IS /
     1 27X2HC(I3,2H)=3XE34.28 )
   35 FORMAT(15X2HN=I3,16H AND THE SUM OF I3,9H TERMS IS
     1 /37XE34.28 )
   42 FORMAT(///)
   43 FORMAT(15X26HTHE CURRENT VALUE OF X IS F7.2//
     1      14X1HI24X20HC(I), TERM OF SERIES /)
   44 FORMAT(25X5HHEAD=7XE34.28/
     1 15X15HSTIELTJES TAIL=7XE34.28 )
   45 FORMAT(19X11HEULER TAIL=7XE34.28 /
     1 16X14HSTIELTJES SUM=7XE34.28 )
   46 FORMAT(20X10HEULER SUM=7XE34.28 //)
   48 FORMAT(15X39HTHE ASYMPTOTIC EVALUATION OF S = S(X) =)
   49 FORMAT(15X11HTHE SUM TO I3,
     1 35H TERMS OF THE ASYMPTOTIC SERIES FOR /
     2 22X2HS(F7.2,6H) IS  E34.28 )
   50 FORMAT(16HALL B'S ARE ZERO  )
   51 FORMAT(/15X5HJMIN=I3,24H AND THE MINIMUM B(J) IS /
     1 27X2HB(I3,5H)=    E34.28//)
      STOP
      END
```

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## Abbreviations used in this bibliography

Bull. AMS

Bulletin of the American Mathematical Society

J.Math.Phys.

Journal of Mathematics and Physics

Math.Comp.

Mathematics of Computation

MTAC

Mathematical Tables and Other Aids to Computation

NBS AMS

National Bureau of Standards - Applied Mathematics Series

Phil.Mag.

Philosophical Magazine

Phil.Trans.Roy.Soc.

Philosophical Transactions of the Royal Society

Proc.Camb.Phil.Soc.

Proceedings of the Cambridge Philosophical Society

Proc.London Math.Soc.

Proceedings of the London Mathematical Society

Proc.Roy.Soc.

Proceedings of the Royal Society

SIAM

Society for Industrial and Applied Mathematics

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