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Borel sets and Baire functions

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BOREL SETS AND BAIRE FUNCTIONS

by

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CHAPTER I

INTRODUCTION

In this manuscript the general properties of the Borel sets and the Baire functions will be discussed, and several important relationships between the two will be shown.

The family of Borel sets is defined to be the collection of all sets of type F_α and G_α, for all ordinals $\alpha < \Omega$, where Ω is the smallest uncountable ordinal number. Sets of type F_0 are closed sets, and sets of type G_0 are open sets. The sets of type F_α and G_α, for any ordinal $\alpha < \Omega$, are then defined by transfinite induction and discussed in general in Chapter II. The Hausdorff sets of type P_α and Q_α are then defined by transfinite induction, and the relationships between the family of Borel sets and the family of Hausdorff sets are shown. It is then proven that these two families of sets are identical.

In Chapter IV, the Baire functions of type f_α, for all ordinals $\alpha < \Omega$, are defined by transfinite induction where a function of type f_0 is a continuous function. Relationships between the Borel sets and the Baire functions are then shown. One of the more important theorems proved in the final chapter is: If f is a Baire function defined on a complete metric space A, then

(1) f is continuous on a subset S of A relative to S,

(2) S is a countable intersection of open sets in A,

(3) S is dense and of the second category in A,

(4) the complement of S is of the first category in A.

It will be assumed that the reader is familiar with basic topological concepts and with the fundamental properties of cardinal numbers and ordinal numbers. We will now define some frequently used terms in order to facilitate the reading of the manuscript.

A set is any collection of objects which we shall call elements or points. If x is an element of the set E, we write $x \in E$. If x is not an element of the set E, we write $x \notin E$. A set E is said to be a subset of a set F if every element of E is an element of F, and we write $E \subset F$.

The union of a collection of sets is understood to be the set of all elements which belong to at least one of the sets over which the union is extended. The union of two sets E and F is denoted by $E \cup F$. The union of a finite collection of sets E_i, $i = 1, 2, \ldots, n$, is denoted by $\bigcup_{i=1}^{n} E_i$. The union of an infinite sequence of sets $E_1, E_2, E_3, \ldots$, written $\{E_n\}$, is denoted by $\bigcup_{n=1}^{\infty} E_n$. In general, let B be an arbitrary set, and suppose that with each element $b \in B$, there is ... et E_b. This yields an indexed collection of sets. The union of the sets of this collection is denoted by $\bigcup_{b \in B} E_b$.

The intersection of a collection of sets is understood to be the set of all elements which belong to each of the sets over which the intersection is extended. The intersection of two sets E and F is denoted by $E \cap F$, where $x \in [E \cap F]$ if and only if $x \in E$ and $x \in F$. The intersection of a finite collection of sets $E_1, E_2, \ldots, E_n$, is denoted by $\bigcap_{i=1}^{n} E_i$ and of an infinite sequence $\{E_n\}$ by $\bigcap_{n=1}^{\infty} E_n$. In a manner similar to that used for an arbitrary union we denote the intersection of an arbitrary indexed collection of sets by $\bigcap_{b \in B} E_b$.

For two sets E and F, the difference of the two sets is the set of all elements belonging to F but not to E, and is written F-E.

The largest (smallest) number in a set E of real numbers will be denoted by max $[E]$ (min $[E]$), if one exists. The least upper bound (greatest lower bound) of a set E of real numbers will be denoted by l.u.b.$[E]$ (g.l.b. $[E]$), if one exists.

A set A is a metric space if with any ordered pair of points x and y belonging to A there is associated a real number, called the distance between these points and denoted by $d(x,y)$, with the following properties:

(1) $d(x,y) \geq 0$ for all $x,y \in A$,

(2) $d(x,y) = 0$ if and only if $x = y$,

(3) $d(x,y) = d(y,x)$ for all $x,y \in A$.

(4) $d(x,z) \leq d(x,y) + d(y,z)$ for all $x,y,z \in A$.

In a metric space A, the complement of a set $E \subset A$, denoted by $C(E)$, is the difference $A-E$.

A point x is said to be the limit of an infinite sequence $\{x_n\}$ in a metric space A if for every $\varepsilon > 0$ there is some positive integer N such that if $n > N$ then $d(x,x_n) < \varepsilon$. If the sequence $\{x_n\}$ has a limit x, the sequence $\{x_n\}$ is said to converge to x, and we write $\lim_{n \to \infty} x_n = x$. Given a metric space A, then $\{x_n\}$ is a Cauchy sequence in A if for every $\varepsilon > 0$ there exists some positive integer N such that if $m,n > N$ then $d(x_m,x_n) < \varepsilon$. A metric space A is complete if every Cauchy sequence in A converges in A.

The set of all real numbers is a complete metric space, where $d(x,y) = |x-y|$ for any real numbers x and y.

The empty set is denoted by $\emptyset$. Sets E and F are said to be disjoint if $E \cap F = \emptyset$.

If $x \in A$, where A is any metric space, and if r is any real number, then the neighborhood of the point x with radius r is the set of all $y \in A$ such that $d(x,y) < r$, and this neighborhood is denoted by $N(x,r)$. A point x is an interior point of a set E

if and only if for some $r > 0$, $N(x,r) \subset E$. A set E is <u>open</u> if and only if every $x \in E$ is an interior point of E.

In a metric space A, a point x is called a <u>limit point</u> of a set E if and only if for every $r > 0$, $N(x,r) \cap E$ contains at least one point different from x. This is equivalent to saying $N(x,r) \cap E$ is an infinite set for each $r > 0$. A set E is said to be <u>closed</u> if and only if every limit point of E is contained in E.. The <u>derived set</u> E' of a set E is the set of all limit points of E. The <u>closure</u> $\bar{E}$ of a set E is the set $E \cup E'$.

Given sets E and F in a metric space A such that $E \subset F \subset A$, then E is <u>closed in F</u> (relative to F) if and only if $(E' \cap F) \subset E$. A set $E \subset F$ is <u>open in F</u> if and only if for every $x \in E$ there is some $r > 0$ such that $(N(x,r) \cap F) \subset E$. Set $E \subset F$ is <u>dense in F</u> if and only if for every $x \in F$ and every $r > 0$, $N(x,r) \cap E \neq \emptyset$. A set $E \subset F$ is <u>nowhere dense in F</u> if and only if for every $x \in F$ and every $\varepsilon > 0$ there is some $y \in F$ and some $\delta > 0$ such that $(N(y,\delta) \cap F) \subset (N(x,\varepsilon) \cap F)$ and $N(y,\delta) \cap E = \emptyset$.

A set E is said to be of the <u>first category in F</u> if it is the union of a sequence $\{E_n\}$ of nowhere-dense sets in F. If E is not of the first category in F, E is said to be of the <u>second category in F</u>.

It will be assumed throughout that we are working with real-valued functions. A function f defined on a set S is said to be continuous at a point $a \in S$ relative to S if and only if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that if $x \in S$ and $x \in N(a, \delta)$ then $|f(x) - f(a)| < \varepsilon$. A function f is said to be continuous on a set S relative to S if and only if f is continuous at every point $x \in S$ relative to S. If f is defined on a set S, and $F \subset S$, then f is continuous at a point $a \in S$ relative to F if and only if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that if $x \in [N(a, \delta) \cap F]$ then $|f(x) - f(a)| < \varepsilon$. A function f defined on a set S is said to be continuous on a set E relative to F, where $E \subset S$ and $F \subset S$, if and only if f is continuous at every point $x \in E$ relative to F.

A function f defined on a set S is said to be uniformly continuous on S relative to S if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that if $x, y \in S$ and $d(x,y) < \delta$ then $|f(x) - f(y)| < \varepsilon$.

If $\{f_n\}$ is a sequence of functions, where each function f_n is defined on a set S, then the sequence of functions $\{f_n\}$ is said to converge at $a \in S$ to a limit function f defined at a, and we write $f(a) = \lim_{n \to \infty} f_n(a)$ if and only if for every $\varepsilon > 0$ there exists an integer $N > 0$ such that if $n \geq N$ then

$|f_n(a) - f(a)| < \varepsilon$. The sequence of functions $\{f_n\}$ converges on a set $E \subset S$ to a limit function f defined on E, and we write $f = \lim_{n \to \infty} f_n$ on E, if and only if the sequence $\{f_n\}$ converges to f at every point $a \in E$.

The sequence of functions $\{f_n\}$, where each function f_n is defined on a set S, is said to <u>converge uniformly on the set S</u> to a limit function f defined on S, if and only if for every $\varepsilon > 0$ there exists an integer $N > 0$ such that if $n \geq N$ then $|f_n(x) - f(x)| < \varepsilon$ for all $x \in S$.

Any set which is finite or which can be put into one-to-one correspondence with the set of all positive integers is said to be <u>countable</u>. Any set which is not countable is said to be <u>uncountable</u>.

Two properties of ordinal numbers which will be relied upon heavily are:

(1) For every set of ordinal numbers, there is an ordinal number which is greater than every ordinal number in the set, and which is less than any other ordinal number with this same property, i.e. there is a definite next larger ordinal number for any set of ordinal numbers.

(2) If E is a countable set of ordinal numbers of countable sets, the next larger ordinal number is also an ordinal number of a countable set.

The ordinal number ω is the ordinal number of the set of all positive integers ordered according to increasing magnitude. ω is the smallest transfinite ordinal number. The set of all finite or countable ordinal numbers ordered according to increasing magnitude is a well-ordered set with ordinal number Ω. Ω is the smallest uncountable ordinal number.

The principle of transfinite induction will be relied upon heavily in the following work, and is as follows:

If M is any well-ordered set and if S is a subset of M such that

(1) if a is the first element of M, then $a \in S$, and

(2) for any element $y \in M$, if all elements $x \in M$ preceding y are in S, then $y \in S$,

then $S = M$.

The larger of two ordinal numbers α and β will be represented by max (α, β).

A sequence of sets $\{E_n\}$ is called nonincreasing (nondecreasing) if $E_i \supset E_{i+1}$ $(E_i \subset E_{i+1})$ for every positive integer i.

CHAPTER II

BOREL SETS F_α AND G_α

In this chapter we will define the Borel sets of type F_α and G_α, and will prove several important properties of these sets. It will be assumed throughout that we are working within a metric space A, unless otherwise stated.

Each ordinal $\alpha < \Omega$ will be designated as being either even or odd, but not both, by use of transfinite induction, in the following manner.

(1) $\alpha = 0$ is defined to be even, and not odd.

(2) Suppose that $\alpha < \Omega$, and that every ordinal $\beta < \alpha$ has been designated as being either even or odd, but not both.

(a) If α has no immediate predecessor, then α is designated as being even, and not odd.

(b) If α has an immediate predecessor, $\alpha -1$, then α is designated as being even (odd), and not odd (even), if $\alpha -1$ is odd (even).

Definition: A set is a Borel set of type F_0 (G_0) if and only if it is a closed (open) set. Suppose that $\alpha < \Omega$, and that Borel sets of type F_β and G_β have been defined for all $\beta < \alpha$.

(1) If α is odd, a Borel set of type F_α (G_α) is defined to be a countable union (intersection) of Borel sets, each of lower type F_β (G_β) for some

ordinal $\beta < \alpha$. We note that the sets of the countable union (intersection) need not all be of the same type; we require only that they all be of lower type than F_α (G_α) .

(2) If α is even, a Borel set of type F_α (G_α) is defined to be a countable intersection (union) of Borel sets, each of lower type F_β (G_β) , for some ordinal $\beta < \alpha$. The same remark applies as at the end of (1) .

We thus define Borel sets of type F_α and G_α by transfinite induction for all $\alpha < \Omega$.

Theorem 2.1: Every Borel set of type F_α (G_α) is a set of type F_β (G_β) if $\alpha < \beta < \Omega$.

Proof: Suppose set E is of type F_α , $\alpha < \beta < \Omega$. If β is odd, E is of type F_β since $E = \bigcup_{n=1}^{\infty} E_n$, where $E_n = E$ for each n. If β is even, E is of type F_β since $E = \bigcap_{n=1}^{\infty} E_n$, where $E_n = E$ for each n.

Suppose set E is of type G_α , $\alpha < \beta < \Omega$. If β is odd, E is of type G_β since $E = \bigcap_{n=1}^{\infty} E_n$, where $E_n = E$ for each n. If β is even, E is of type G_β since $E = \bigcup_{n=1}^{\infty} E_n$, where $E_n = E$ for each n.

We will designate Ω as being even since it has no immediate predecessor. Sets of type F_Ω and G_Ω will be defined to be countable intersections and countable unions of sets of type F_α and G_α , respectively, for $\alpha < \Omega$.

Theorem 2.2: Every set of type F_{Ω} (G_{Ω}) is of type F_{α} (G_{α}) for some $\alpha < \Omega$.

Proof: Suppose set E is of type F_{Ω}. Then $E = \bigcap_{n=1}^{\infty} E_n$, where for each n, E_n is of type F_{α_n}, $\alpha_n < \Omega$. Since for each n, α_n is an ordinal number of a countable set, the ordinal number α immediately succeeding the set of ordinals α_n, $n = 1, 2, \ldots$, is also an ordinal number of a countable set. Therefore E is a Borel set of type F_{α}, $\alpha < \Omega$, if α is even. If α is odd, then $\alpha + 1$ is even, and E is a Borel set of type $F_{\alpha+1}$.

It can be shown in a similar manner that no new sets are obtained by taking sets of type G_{Ω}.

Theorem 2.3: The complement of every set of type F_{α} (G_{α}) is a set of type G_{α} (F_{α}), for every $\alpha < \Omega$.

Proof: The theorem is true for $\alpha = 0$ since the complement of a closed set is an open set and the complement of an open set is a closed set. Now suppose the theorem is true for all ordinals $\beta < \alpha$ for some $\alpha < \Omega$.

Suppose α is even. If set E is of type F_{α}, then $E = \bigcap_{n=1}^{\infty} E_n$, where for each n, E_n is a Borel set of type F_{α_n}, $\alpha_n < \alpha$. By our induction assumption, each set $C(E_n)$ is of type G_{α_n}, $\alpha_n < \alpha$.

Therefore $C(E)$ is a Borel set of type G_α since $C(E) = C(\bigcap_{n=1}^{\infty} E_n) = \bigcup_{n=1}^{\infty} C(E_n)$.

Suppose set E is of type G_α, α even. Then $E = \bigcup_{n=1}^{\infty} E_n$, where for each n, E_n is a Borel set of type G_{α_n}, $\alpha_n < \alpha$. By our induction assumption, each set $C(E_n)$ is of type F_{α_n}, $\alpha_n < \alpha$. Therefore $C(E)$ is a Borel set of type F_α since $C(E) = C(\bigcup_{n=1}^{\infty} E_n) = \bigcap_{n=1}^{\infty} C(E_n)$.

If α is an odd ordinal, the induction process can be carried out in a similar manner.

Therefore, by transfinite induction, the theorem is true for every ordinal $\alpha < \Omega$.

<u>Theorem 2.4</u>: If $\alpha < \Omega$ is odd, the union (intersection) of a countable number of sets of type F_α (G_α) is a set of type F_α (G_α). If $\alpha < \Omega$ is even, the intersection (union) of a countable number of sets of type F_α (G_α) is a set of type F_α (G_α).

<u>Proof</u>: Suppose $\alpha < \Omega$ is odd, and $E = \bigcup_{n=1}^{\infty} E_n$, where for each n, E_n is of type F_α. Then for each n, $E_n = \bigcup_{m=1}^{\infty} E_{n,m}$, where for each m, $E_{n,m}$ is a set of type $F_{\alpha_{n,m}}$, $\alpha_{n,m} < \alpha$. Therefore E is a Borel set of type F_α since $E = \bigcup_{n=1}^{\infty} \bigcup_{m=1}^{\infty} E_{n,m}$, and the sets $\{E_{n,m}\}$ constitute a countable collection.

If $\alpha < \Omega$ is odd and $E = \bigcap_{n=1}^{\infty} E_n$, where for each n, E_n is a set of type G_α, then $C(E) = C(\bigcap_{n=1}^{\infty} E_n) = \bigcup_{n=1}^{\infty} C(E_n)$. Each set $C(E_n)$ is of type F_α by theorem 2.3. Therefore by the first part of the proof $C(E)$ is a Borel set of type F_α.

Therefore E is of type G_α by theorem 2.3.

The proof of the second part of the theorem where $\alpha < \Omega$ is even is entirely analogous.

<u>Theorem 2.5</u>: If $\alpha < \Omega$ is odd, the union (intersection) of a finite number of sets of type F_α (G_α) is a set of type F_α (G_α). If $\alpha < \Omega$ is even, the intersection (union) of a finite number of sets of type F_α (G_α) is a set of type F_α (G_α).

<u>Proof</u>: This theorem is an immediate consequence of theorem 2.4 .

<u>Theorem 2.6</u>: If $\alpha < \Omega$ is odd, the intersection (union) of a finite number of sets of type F_α (G_α) is a set of type F_α (G_α). If $\alpha < \Omega$ is even, the union (intersection) of a finite number of sets of type F_α (G_α) is a set of type F_α (G_α).

<u>Proof</u>: Suppose $\alpha < \Omega$ is odd, and the two sets A and B are of type F_α. Then $A = \bigcup_{n=1}^{\infty} A_n$, where for each n, A_n is of type F_{α_n}, $\alpha_n < \alpha$, and $B = \bigcup_{m=1}^{\infty} B_m$, where for each m, B_m is of type F_{β_m}, $\beta_m < \alpha$. Then $S = A \cap B = (\bigcup_{n=1}^{\infty} A_n) \cap (\bigcup_{m=1}^{\infty} B_m) = \bigcup_{n=1}^{\infty} \bigcup_{m=1}^{\infty} (A_n \cap B_m)$. Choose $\alpha_{n,m}$ to be an even ordinal such that $\alpha_{n,m} \geq \alpha_n$, $\alpha_{n,m} \geq \beta_m$, and $\alpha_{n,m} < \alpha$ for all indices n and m. Then each set $A_n \cap B_m$ is a Borel set of type $F_{\alpha_{n,m}}$, $\alpha_{n,m} < \alpha$, by theorem 2.5. Therefore set S is of type F_α.

Suppose $\alpha < \Omega$ is odd, and $S = A \cup B$ where sets A and B are both of type G_α. Then $C(S) = C(A) \cap C(B)$ where C(A) and C(B) are sets of type F_α by theorem 2.3. Therefore by the first part of the proof C(S) is a set of type F_α. Hence set S is of type G_α by theorem 2.3. If $\alpha < \Omega$ is even, it can be proved in a similar manner that the union of two sets of type F_α is a set of type F_α, and the intersection of two sets of type G_α is a set of type G_α.

We have shown the theorem is true for two sets. The proof is completed by the use of finite induction.

Theorem 2.7: For every ordinal $\alpha < \Omega$ every set of type F_α (G_α) is a set of type $G_{\alpha+1}$ ($F_{\alpha+1}$).

Proof: Suppose E is a closed set. If $E = \emptyset$, then $E = \bigcap_{n=1}^{\infty} E_n$, where $E_n = \emptyset$ for each n, is of type G_1 since $\emptyset$ is both open and closed. Suppose E is not empty. Then let $E_n = \bigcup_{x \in E} N(x, \frac{1}{n})$. Since each neighborhood $N(x, \frac{1}{n})$ is an open set, and the union of any collection of open sets is an open set, for each positive integer n, E_n is an open set. Also, $E = \bigcap_{n=1}^{\infty} E_n$, for if $x \in E$, then $x \in E_n$ for each n, and hence $x \in \bigcap_{n=1}^{\infty} E_n$. On the other hand, suppose $x \in \bigcap_{n=1}^{\infty} E_n$, then $x \in E_n$ for each n. Hence for each n there exists some $x_n \in E$ such that $d(x, x_n) < \frac{1}{n}$. Therefore $\lim_{n \to \infty} x_n = x$ and $x \in \bar{E}$, which means $x \in E$ since E is a closed set.

For any ordinal $\alpha < \Omega$, assume every set of type F_β is of type $G_{\beta+1}$, for all $\beta < \alpha$.

If $\alpha < \Omega$ is odd, and set E is of type F_α, then $E = \bigcup_{n=1}^{\infty} E_n$, where for each n, E_n is of type F_{α_n}, $\alpha_n < \alpha$. By our induction assumption, each set E_n is of type G_{α_n+1}, $\alpha_n + 1 < \alpha + 1$. Since $\alpha + 1$ is even, E is a set of type $G_{\alpha+1}$.

Suppose $\alpha < \Omega$ is even, and set E is of type F_α. Then $E = \bigcap_{n=1}^{\infty} E_n$, where for each n, E_n is of type F_{α_n}, $\alpha_n < \alpha$. By our induction assumption, each set E_n is of type $G_{\alpha_n + 1}$, where $\alpha_n + 1 < \alpha + 1$. Since $\alpha + 1$ is odd, E is a set of type $G_{\alpha + 1}$.

Therefore, by transfinite induction, every set of type F_α is a set of type $G_{\alpha+1}$ for all ordinals $\alpha < \Omega$.

Now consider E to be a set of type G_α, for any ordinal $\alpha < \Omega$. Then C(E) is a set of type F_α by theorem 2.3, and by the proof above also a set of type $G_{\alpha+1}$. Therefore by theorem 2.3 E is a set of type $F_{\alpha+1}$.

<u>Theorem 2.8:</u> The family of all Borel sets forms the smallest system of sets such that:

(1) All closed sets are in the system,

(2) the union of any countable collection of sets in the system is in the system,

(3) the intersection of any countable collection of sets in the system is in the system.

<u>Proof</u>: Condition (1) is satisfied by the system of Borel sets because of the definition of sets of type F_0. Suppose $E = \bigcup_{n=1}^{\infty} E_n$, where for each n, E_n is a Borel set. By our previous theorems, each E_n is of type F_{α_n}, for a certain ordinal $\alpha_n < \Omega$.

There exists an odd ordinal α such that $\alpha < \Omega$ and $\alpha_n < \alpha$ for every n. Therefore E is a Borel set of type F_α. Hence condition (2) is satisfied. Likewise, any countable intersection of Borel sets is a Borel set.

To prove that the system of Borel sets is the smallest system satisfying these conditions, we will show that any system satisfying these conditions contains all the Borel sets.

Suppose S is any system of sets satisfying the three conditions of the theorem. Every Borel set of type F_0 is in S because of condition (1). Assume that all sets of type F_β are in S, for all ordinals $\beta < \alpha < \Omega$. Suppose α is odd, and E is a Borel set of type F_α. Then $E = \bigcup_{n=1}^{\infty} E_n$, where for each n, E_n is a Borel set of type F_{α_n}, $\alpha_n < \alpha$. By our induction assumption each set E_n is in S. Then E is in S since S satisfies condition (2). On the other hand, suppose α is even and E is a Borel set of type F_α. Then $E = \bigcap_{n=1}^{\infty} E_n$, where for each n, E_n is a Borel set of type F_{α_n}, $\alpha_n < \alpha$. By our induction assumption each set E_n is in S. Then E is in S since S satisfies condition (3).

Therefore, by transfinite induction, all the Borel sets of type F_α, for any ordinal $\alpha < \Omega$, belong to S. Since every Borel set is a Borel set of type F_α, for some ordinal $\alpha < \Omega$, by theorem 2.7, S contains all the Borel sets.

We note that if in condition (1) of theorem 2.8, "closed" is replaced by "open", the resulting theorem is also true.

<u>Theorem 2.9</u>: A Borel set of type F_1 is either of the first category in a metric space A, or else contains a neighborhood.

<u>Proof</u>: Suppose E is any Borel set of type F_1. Then $E = \bigcup_{n=1}^{\infty} E_n$, where for each n, E_n is a closed set in the metric space A. If each set E_n is nowhere dense in A, then E is of the first category in A.

Suppose E is of type F_1 and not of the first category in A. Then there exists some positive integer N such that the set E_N is not nowhere dense in A, where $E = \bigcup_{n=1}^{\infty} E_n$, each set E_n closed in A. Then there is some $r > 0$ and some $x_0 \in A$ such that every nonempty open set $R \subset N(x_0, r)$ contains points of E_N. Hence, consider any $x_1 \in N(x_0, r)$. Then for every $\varepsilon > 0$, $N(x_1, \varepsilon) \cap E_N \neq \emptyset$. Therefore $x_1 \in \overline{E}_N$ which implies $N(x_0, r) \subset \overline{E}_N$. Hence $N(x_0, r) \subset E_N$, since E_N is a closed set, and $N(x_0, r) \subset E$.

<u>Definition</u>: Given any bounded set F, $\Delta(F) = \text{l.u.b.}[d(x,y)]$ where x and y range over F, is called the <u>diameter</u> of set F.

<u>Theorem 2.10 (Cantor's Theorem)</u>: Given a non-increasing sequence of nonempty sets $\{F_n\}$, where each F_n is a closed and bounded subset of the complete metric space A, with the additional requirement that $\lim_{n\to\infty} \Delta(F_n) = 0$, then $\bigcap_{n=1}^{\infty} F_n \neq \emptyset$.

<u>Proof</u>: Suppose $\{F_n\}$ is a nonincreasing sequence of nonempty sets satisfying the conditons of the theorem. For each set F_n, choose $x_n \in F_n$, thus obtaining a sequence of points $\{x_n\}$. Since $\lim_{n\to\infty} \Delta(F_n) = 0$, given any $\varepsilon > 0$ there is some $N > 0$ such that $\Delta(F_N) < \varepsilon$.

Since $\{F_n\}$ is a nonincreasing sequence of sets, $x_n \in F_n \in F_N$ for all $n \geq N$. Moreover, $\Delta(F_N) < \varepsilon$ and $x_n, x_m \in F_N$ imply that $d(x_n, x_m) < \varepsilon$ for all $n, m > N$. Therefore $\{x_n\}$ is a Cauchy sequence in A, and since A is a complete space, there is some $x \in A$ such that $\lim_{n\to\infty} x_n = x$.

Let M be any positive integer. If $n \geq M$, then $x_n \in F_n \subset F_M$. Therefore $x \in \overline{F}_M$, and since each set F_M is closed, $x \in F_M$ for any positive integer M. Therefore $x \in \bigcap_{n=1}^{\infty} F_n$, i.e. $\bigcap_{n=1}^{\infty} F_n \neq \emptyset$.

<u>Theorem 2.11 (Baire's Theorem)</u>: If H is a nonempty Borel set of type G_1 in a complete

metric space A, then H is of the second category relative to itself.

Proof: Suppose H is a nonempty Borel set of type G_δ, then $H = \bigcap_{n=1}^{\infty} G_n$, where for each n, G_n is open in A. Suppose H is of the 1st category relative to itself, then $H = \bigcup_{n=1}^{\infty} H_n$, where for each n, H_n is nowhere dense in H.

Suppose $x_0 \in H$, then consider $N(x_0, 1) \cap H$ which is open relative to H. Since H_1 is nowhere dense in H, there exists a nonempty set $F_1 \subset H$ open relative to H such that $F_1 \subset [N(x_0, 1) \cap H]$ and $F_1 \cap H_1 = \emptyset$.

Since F_1 is open in H, there is some set R_1 open in the space A such that $F_1 = R_1 \cap H$. Suppose $x_1 \in F_1$. Since R_1 is open in A, there is some $\delta_1 > 0$ such that $N(x_1, \delta_1) \subset R_1$.

Let $Q_1 = N(x_1, \delta_1) \cap H$. Set Q_1 is nonempty since $x_1 \in Q_1$. Also, since $F_1 \cap H_1 = \emptyset$ and $Q_1 \subset F_1 \subset [N(x_0, 1) \cap H]$, $Q_1 \cap H_1 = \emptyset$.

Now $x_1 \in F_1 \subset H$ implies $x_1 \in G_1$, and since G_1 is open in A there is some $\varepsilon_1 > 0$ such that $\overline{N(x_1, \varepsilon_1)} \subset G_1$ where $\varepsilon_1 < \min [\delta_1, \frac{1}{2}]$. Define the set $S_1 = N(x_1, \varepsilon_1) \cap H$. Then $\overline{S_1} \subset [\overline{N(x_1, \varepsilon_1)} \cap \overline{H}]$ where $\overline{N(x_1, \varepsilon_1)} \subset N(x_1, \delta_1)$. Therefore

$$[\overline{S_1} \cap H_1] \subset [\overline{N(x_1, \varepsilon_1)} \cap \overline{H} \cap H_1] = [\overline{N(x_1, \varepsilon_1)} \cap H_1]$$

$\subset [N(x_1, \delta_1) \cap H_1] = [N(x_1, \delta_1) \cap H \cap H_1] = [Q_1 \cap H_1]$, i.e. $\overline{S_1} \cap H_1 \subset Q_1 \cap H_1$ which implies $\overline{S_1} \cap H_1 = \emptyset$ since $Q_1 \cap H_1 = \emptyset$. Also, $x_1 \in F_1 \subset H$ and $S_1 = N(x_1, \varepsilon_1) \cap H$ imply $x_1 \in S_1$, therefore S_1 is nonempty. We also note

$\overline{S_1} \subset [\overline{N(x_1, \varepsilon_1)} \cap \overline{H}] \subset [G_1 \cap \overline{H}] \subset G_1$.

To summarize, we have the following situation: $S_1 = N(x_1, \varepsilon_1) \cap H$ is a nonempty bounded set, open in H, for which $\overline{S_1} \subset G_1$ and $\overline{S_1} \cap H_1 = \emptyset$.

Since the set H_2 is nowhere dense in H, and S_1 was constructed to be a nonempty set open in H, there exists a nonempty set $F_2 \subset H$ open in H, such that $F_2 \subset S_1$ and $F_2 \cap H_2 = \emptyset$.

Since F_2 is open in H, there is some set R_2 open in A such that $F_2 = R_2 \cap H$. Suppose $x_2 \in F_2$. Since R_2 is open in A, there is some $\delta_2 > 0$, where $\delta_2 < \varepsilon_1 - d(x_1, x_2)$, such that $N(x_2, \delta_2) \subset R_2$. Let $Q_2 = N(x_2, \delta_2) \cap H$. Set Q_2 is nonempty since $x_2 \in Q_2$. Also, since $F_2 \cap H_2 = \emptyset$ and $Q_2 \subset F_2 \subset S_1$, $Q_2 \cap H_2 = \emptyset$.

Now $x_2 \in F_2 \subset H$ implies $x_2 \in G_2$, and since set G_2 is open in A there is some $\varepsilon_2 > 0$ such that $\overline{N(x_2, \varepsilon_2)} \subset G_2$ where $\varepsilon_2 < \min [\delta_2, \frac{1}{4}]$.
$(\varepsilon_2 < \delta_2 < \varepsilon_1 - d(x_1, x_2))$

Define $S_2 = N(x_2, \varepsilon_2) \cap H$. Then $\overline{S_2} \subset [\overline{N(x_2, \varepsilon_2)} \cap \overline{H}]$ where $\overline{N(x_2, \varepsilon_2)} \subset N(x_2, \delta_2)$.

Therefore $\left[\overline{S_2} \cap H_2\right] \subset \left[\overline{N(x_2, \varepsilon_2)} \cap \overline{H} \cap H_2\right] = \left[\overline{N(x_2, \varepsilon_2)} \cap H_2\right]$

$$\subset \left[N(x_2, \delta_2) \cap H_2\right] = \left[N(x_2, \delta_2) \cap H \cap H_2\right]$$

$$= \left[Q_2 \cap H_2\right] = \phi .$$

Also, $x_2 \varepsilon S_2$, $\overline{S_2} \subset \left[\overline{N(x_2, \varepsilon_2)} \cap \overline{H}\right] \subset \left[G_2 \cap \overline{H}\right] \subset G_2$, and $\overline{S_2} \subset \overline{S_1}$, since $\varepsilon_2 < \varepsilon_1 - d(x_1, x_2)$ implies $S_2 \subset S_1$.

Thus, for n = 1, 2, 3, ... , we define nonempty sets $S_n = N(x_n, \varepsilon_n) \cap H$, where $0 < \varepsilon_n < \min\left[\delta_n, \frac{1}{2^n}\right]$ and $0 < \delta_n < \varepsilon_{n-1} - d(x_n, x_{n-1})$, with the following properties: S_n is nonempty, open in H, and bounded; $\overline{S_n} \subset G_n$, $\overline{S_n} \cap H_n = \phi$, and $\overline{S_n} \subset \overline{S_{n-1}}$.

Furthermore, the nonincreasing sequence $\{\overline{S_n}\}$ of nonempty-closed-bounded sets has the property that $\lim_{n \to \infty} \Delta(\overline{S_n}) = 0$. This is because the diameter $\Delta(\overline{S_n}) \leq 2 \cdot \varepsilon_n$, where $\varepsilon_n < \frac{1}{2^n}$, implies that $\lim_{n \to \infty} \varepsilon_n = 0$.

Therefore, by theorem 2.10, there is some point x such that $x \varepsilon \bigcap_{n=1}^{\infty} \overline{S_n}$, which implies $x \varepsilon H = \bigcap_{n=1}^{\infty} G_n$ since each $\overline{S_n} \subset G_n$. But $x \varepsilon \overline{S_n}$ implies also that $x \notin H_n$ for each n, since $\overline{S_n} \cap H_n = \phi$. Therefore $x \notin H = \bigcup_{n=1}^{\infty} H_n$.

Thus we have a contradiction, and we conclude that our assumption that H is of the first category relative to itself is false. Therefore H is of the second category relative to itself.

Theorem 2.12: If H is a Borel set of type G_1 in a complete metric space A, where $H = \bigcap_{n=1}^{\infty} H_n$ with each H_n open and dense in A, then H is also dense in A.

Proof: Assume $H = \bigcap_{n=1}^{\infty} H_n$, where for each n, H_n is open and dense in a complete metric space A. To prove H is also dense in A, we must show that every neighborhood in A contains a point of H.

Consider any $x_0 \varepsilon A$, and for any $\varepsilon_0 > 0$ some neighborhood $N(x_0, \varepsilon_0)$. Since H_1 is dense in A, there is some $x_1 \varepsilon N(x_0, \varepsilon_0)$ such that $x_1 \varepsilon H_1$. Since $N(x_0, \varepsilon_0)$ is an open set, there is some $\varepsilon_1 > 0$ such that $N(x_1, \varepsilon_1) \subset N(x_0, \varepsilon_0)$. Also since H_1 is open in A, there is some $\delta_1 > 0$ such that $N(x_1, \delta_1) \subset H_1$. Choose a γ_1, $0 < \gamma_1 <$ min $[\varepsilon_1, \delta_1, \frac{1}{2}]$. Then $\overline{N(x_1, \gamma_1)} \subset N(x_0, \varepsilon_0)$ and $\overline{N(x_1, \gamma_1)} \subset H_1$.

We proceed to define $\overline{N(x_n, \gamma_n)}$ by induction in the following manner. Assume $\overline{N(x_i, \gamma_i)}$, $i = 1, \ldots, n-1$, have been defined such that there is some $x_i \varepsilon N(x_{i-1}, \gamma_{i-1})$ such that $\overline{N(x_i, \gamma_i)} \subset \overline{N(x_{i-1}, \gamma_{i-1})}$ and $\overline{N(x_i, \gamma_i)} \subset H_i$. Since H_n is dense in A, for each n, there is some $x_n \varepsilon N(x_{n-1}, \gamma_{n-1})$ such that $x_n \varepsilon H_n$. Since $N(x_{n-1}, \gamma_{n-1})$ is open in A, there is some $\varepsilon_n > 0$ such that $N(x_n, \varepsilon_n) \subset N(x_{n-1}, \gamma_{n-1})$. Also since H_n is open in A, there is some $\delta_n > 0$

such that $N(x_n, \delta_n) \subset H_n$. Choose $\gamma_n < \min [\mathcal{E}_n, \delta_n, \frac{1}{2^n}]$, $\gamma_n > 0$. Then $\overline{N(x_n, \gamma_n)} \subset \overline{N(x_{n-1}, \gamma_{n-1})}$ and $\overline{N(x_n, \gamma_n)} \subset H_n$.

Thus we obtain a nonincreasing sequence $\{ \overline{N(x_n, \gamma_n)} \}$ of nonempty-closed-bounded sets with the additional property that the diameter $\Delta \overline{N(x_n, \gamma_n)} < \frac{1}{2^{n-1}}$ which implies $\lim_{n \to \infty} \Delta \overline{N(x_n, \gamma_n)} = 0$.

Therefore, by theorem 2.10, there is some $x \in A$ such that $x \in \bigcap_{n=1}^{\infty} \overline{N(x_n, \gamma_n)}$. Therefore $x \in N(x_o, \mathcal{E}_o)$, and since $\overline{N(x_n, \gamma_n)} \subset H_n$ for each n, $x \in H = \bigcap_{n=1}^{\infty} H_n$.

Theorem 2.13: If $H = \bigcap_{n=1}^{\infty} H_n$ in a complete metric space A, where each H_n is a Borel set of type G_1 and dense in A, then H is of type G_1 and dense in A.

Proof: Suppose $H = \bigcap_{n=1}^{\infty} H_n$, where for each n, H_n is of type G_1 and dense in the complete metric space A. Then for each n, $H_n = \bigcap_{m=1}^{\infty} H_{n,m}$, where for each m, $H_{n,m}$ is open in A. Furthermore, since each set H_n is dense in A and $H_n \subset H_{n,m}$ for all m, each set $H_{n,m}$ is dense in A. Also, by theorem 2.4, H is a Borel set of type G_1. Therefore, by theorem 2.12, H is dense in A.

Theorem 2.14: If E is a Borel set of type G_1 and dense in a complete metric space A, then E is of the second category in A and C(E) is of the first category in A.

<u>Proof</u>: Suppose E is any Borel set of type G_1 and dense in the complete metric space A. By theorem 2.3 the set C(E) is of type F_1 in A. By theorem 2.9 C(E) is either of the first category in A, or else contains a neighborhood. But C(E) cannot contain a neighborhood since E is dense in A. Therefore C(E) is of the first category in A, i.e. a countable union of nowhere dense sets in A.

Suppose E is also of the first category in A, i.e. a countable union of nowhere dense sets in A. Then the complete metric space A would be of the first category relative to itself since $A = E \cup C(E)$ implies A also is a countable union of nowhere dense sets in A. Therefore we have a contradiction since by Baire's Theorem 2.11 the complete metric space A must be of the second category relative to itself.

Therefore set E is not of the first category in A, and hence is of the second category in A.

<u>Theorem 2.15</u>: In the space A of all real numbers, the set of all rational (irrational) numbers is a Borel set of type F_1 (G_1), but is not a set of type G_1 (F_1).

<u>Proof</u>: Let R be the set of all rational numbers, then C(R) is the set of all irrational numbers. Since R is a countable set, and any countable

set is of type F_1 , R is a Borel set of type F_1 . (This is because each element of a countable set may be thought of as being a one-element set, which is closed.) Hence C(R) is a Borel set of type G_1 , by Theorem 2.3 .

Suppose R is also of type G_1 . Since the set of all real numbers A constitutes a complete metric space, and since R is dense in A, by theorem 2.14 R is of the second category in A. But since R is of type F_1 and does not contain a neighborhood, by theorem 2.9, R is of the first category in A. Therefore we have a contradiction, and the set R of all rational numbers is not a Borel set of type G_1 . Also, the set of all irrational numbers C(R) is not a Borel set of type F_1 .

CHAPTER III

HAUSDORFF SETS P_α AND Q_α

In this chapter we will define the Hausdorff sets of type P_α and Q_α, and will prove several important properties of these sets. We will also show that the family of Borel sets and the family of Hausdorff sets are identical. It will be assumed throughout that we are working within a metric space A, unless otherwise stated.

Hausdorff sets of type P_α and Q_α are defined by transfinite induction in the following manner.

<u>Definition</u>: A set is of type P_0 (Q_0) if and only if it is an open (closed) set. Suppose that $\alpha < \Omega$ and that Hausdorff sets of type P_β and Q_β have been defined for all ordinals $\beta < \alpha$. Then by transfinite induction a set is of type P_α (Q_α) if and only if $E = \bigcup_{n=1}^{\infty} E_n$ ($E = \bigcap_{n=1}^{\infty} E_n$), where for each n, E_n is a set of type Q_{α_n} (P_{α_n}), $\alpha_n < \alpha$. We note that the sets of the countable union (intersection) need not all be of the same type.

<u>Theorem 3.1</u>: For every $\alpha < \Omega$, every set of type P_β (Q_β) is also a set of type P_α (Q_α) if $\beta < \alpha$.

<u>Proof</u>: For $0 < \beta < \alpha < \Omega$, if set E is of type P_β, then $E = \bigcup_{n=1}^{\infty} E_n$, where for each n, E_n is of type Q_{β_n}, $\beta_n < \beta$. But $\beta_n < \beta$ implies $\beta_n < \alpha$, therefore E is of type P_α.

Likewise, for $0 < \beta < \alpha < \Omega$, if set E is of type Q_β, then $E = \bigcap_{n=1}^{\infty} E_n$, where for each n, E_n is of type P_{β_n}, $\beta_n < \beta$. Therefore $\beta_n < \alpha$ and set E is of type Q_α.

Suppose $\beta = 0$. A set E of type P_0 is an open set, and as such is a Borel set of type G_0. Then by theorem 2.7, E is of type F_1, i.e. a countable union of closed sets. Therefore, by definition, E is of type P_1. Likewise, if set E is of type Q_0, i.e. a closed set, then E is a Borel set of type F_0, and hence of type G_1 by theorem 2.7. Since E is a countable intersection of open sets, E is of type Q_1.

Now, if $0 < \alpha < \Omega$, then, since every set of type P_0 (Q_0) is of type P_1 (Q_1), since $0 < 1 \leq \alpha < \Omega$, and since $0 < \beta < \alpha < \Omega$ implies that every set of type P_β (Q_β) is of type P_α (Q_α), it follows that every set of type $P_0(Q_0)$ is of type P_α (Q_α).

This completes the proof of theorem 3.1.

Theorem 3.2: The complement of a set of type P_α (Q_α) is a set of type Q_α (P_α), for all ordinals $\alpha < \Omega$.

Proof: The theorem is true for $\alpha = 0$ since the complement of a closed set is an open set, and the complement of an open set is a closed set. Assume the theorem is true for all ordinals $\beta < \alpha$ where $\alpha < \Omega$.

Suppose set E is of type P_α . Then $E = \bigcup_{n=1}^{\infty} E_n$, where each E_n is of type Q_{α_n}, $\alpha_n < \alpha$. Therefore $C(E) = \bigcap_{n=1}^{\infty} C(E_n)$, where each set $C(E_n)$ is of type P_{α_n}, $\alpha_n < \alpha$, by our induction assumption. Therefore $C(E)$ is a set of type Q_α , $\alpha < \Omega$.

Suppose set E is of type Q_α . Then $E = \bigcap_{n=1}^{\infty} E_n$, where each E_n is of type P_{α_n}, $\alpha_n < \alpha$. Therefore $C(E) = \bigcup_{n=1}^{\infty} C(E_n)$, where each set $C(E_n)$ is of type Q_{α_n}, $\alpha_n < \alpha$, by our induction assumption. Therefore $C(E)$ is a set of type P_α , $\alpha < \Omega$.

Therefore the theorem follows by transfinite induction.

Theorem 3.3: For all ordinals $\alpha < \Omega$, the union (intersection) of a finite or countable collection of sets of type P_α (Q_α) is a set of type P_α (Q_α) .

Proof: The theorem is true for $\alpha = 0$ since the union of any collection of open sets is an open set and the intersection of any collection of closed sets is a closed set.

Suppose $0 < \alpha < \Omega$ and $E = \bigcup_{n=1}^{\infty} E_n$, where each set E_n is of type P_α. Then $E_n = \bigcup_{m=1}^{\infty} E_{n,m}$, where each set $E_{n,m}$ is of type $Q_{\alpha_{n,m}}$, $\alpha_{n,m} < \alpha$. Therefore $E = \bigcup_{n=1}^{\infty} \bigcup_{m=1}^{\infty} E_{n,m}$ is of type P_α. The same is true if E is a finite union of sets of type P_α.

Suppose $0 < \alpha < \Omega$ and $E = \bigcap_{n=1}^{\infty} E_n$, where each set E_n is of type Q_α. Then $E_n = \bigcap_{m=1}^{\infty} E_{n,m}$, where each set $E_{n,m}$ is of type $P_{\alpha_{n,m}}$, $\alpha_{n,m} < \alpha$. Therefore $E = \bigcap_{n=1}^{\infty} \bigcap_{m=1}^{\infty} E_{n,m}$ is of type Q_α. The same is true if E is a finite intersection of sets of type Q_α.

Theorem 3.4: For any ordinal $\alpha < \Omega$, the union (intersection) of a finite number of sets of type Q_α (P_α) is a set of type Q_α (P_α).

Proof: The theorem is true for $\alpha = 0$ since the union of a finite collection of closed sets is a closed set and the intersection of a finite collection of open sets is an open set. Suppose α is any ordinal number such that $0 < \alpha < \Omega$.

Suppose E and F are two sets of type P_α. Then $E = \bigcup_{n=1}^{\infty} E_n$, where each set E_n is of type Q_{α_n}, $\alpha_n < \alpha$, and $F = \bigcup_{m=1}^{\infty} F_m$, where each set F_m is of type Q_{β_m}, $\beta_m < \alpha$. Then

$E \cap F = \bigcup_{n=1}^{\infty} \bigcup_{m=1}^{\infty} (E_n \cap F_m)$. Let $\alpha_{n,m} = \max [\alpha_n, \beta_m]$, then by theorem 3.1 both E_n and F_m are sets of type $Q_{\alpha_{n,m}}$, $\alpha_{n,m} < \alpha$, and by theorem 3.3, $E_n \cap F_m$ is a set of type $Q_{\alpha_{n,m}}$, $\alpha_{n,m} < \alpha$. Therefore $E \cap F$ is a set of type P_α.

Suppose E and F are two sets of type Q_α. Then by theorem 3.2, $C(E)$ and $C(F)$ are two sets of type P_α. Therefore $[C(E) \cap C(F)]$ is a set of type P_α by the proof above, and by theorem 3.2 again, $C[C(E) \cap C(F)] = E \cup F$ is a set of type Q_α.

Now that the theorem has been shown true for two sets, the theorem can be proved for any finite number of sets by using finite induction.

<u>Theorem 3.5</u>: For all ordinals $\alpha < \Omega$, every set of type P_α (Q_α) is a set of type $Q_{\alpha+1}$($P_{\alpha+1}$).

<u>Proof</u>: Suppose $\alpha < \Omega$ and E is a set of type P_α. Then $E = \bigcap_{n=1}^{\infty} E_n$, where $E_n = E$ for each n, is of type $Q_{\alpha+1}$. Likewise, if E is a set of type Q_α, then $E = \bigcup_{n=1}^{\infty} E_n$, where $E_n = E$ for each n, is of type $P_{\alpha+1}$.

<u>Theorem 3.6</u>: For $\alpha < \Omega$, the difference of two sets of type P_α (Q_α) is both a set of type $P_{\alpha+1}$ and a set of type $Q_{\alpha+1}$.

<u>Proof</u>: Suppose $S = E - F$, where E and F

are sets of type P_α, $\alpha < \Omega$. Then $S = E \cap C(F)$ where E is of type $P_{\alpha+1}$ by theorem 3.1 and $C(F)$ is of type Q_α by theorem 3.2. But then $C(F)$ is of type $P_{\alpha+1}$ by theorem 3.5, and therefore S is of type $P_{\alpha+1}$ by theorem 3.4. Also, E is of type $Q_{\alpha+1}$ by theorem 3.5, and $C(F)$ is of type $Q_{\alpha+1}$, by theorem 3.1, since $C(F)$ was shown above to be of type Q_α. Therefore S is also of type $Q_{\alpha+1}$ by theorem 3.3.

The same can be shown true for the difference of two sets of type Q_α by taking set complements.

Theorem 3.7: If $\alpha < \Omega$ is any even (odd) ordinal, Borel sets of type F_α are identical to the Hausdorff sets of type Q_α (P_α), and Borel sets of type G_α are identical to the Hausdorff sets of type P_α (Q_α).

Proof: The theorem is true for $\alpha = 0$ by definition of the two families of sets. To prove the theorem for $\alpha < \Omega$, assume the theorem is true for all ordinals $\beta < \alpha$.

Suppose α is an even ordinal, and set S is of type F_α. Then $S = \bigcap_{n=1}^{\infty} S_n$, where for each n, S_n is of type F_{α_n}, $\alpha_n < \alpha$. If α_n is even, S_n will be of type Q_{α_n} by our induction assumption. Therefore S_n will be of type P_{α_n+1}, $\alpha_n + 1 < \alpha$, by theorem 3.5 If α_n is odd, S_n will be of type

P_{α_n}, $\alpha_n < \alpha$, by our induction assumption. In any case S is of type Q_α.

If α is even, and T is of type Q_α, then $T = \bigcap_{n=1}^{\infty} T_n$, where for each n, T_n is of type P_{α_n}, $\alpha_n < \alpha$. If α_n is even, T_n will be of type G_{α_n} by our induction assumption. Therefore T_n will be of type F_{α_n+1}, $\alpha_n+1 < \alpha$, by theorem 2.7. If α_n is odd, T_n will be of type F_{α_n}, $\alpha_n < \alpha$, by our induction assumption. In any case T is of type F_α.

Suppose α is an odd ordinal, and set S is of type F_α. Then $S = \bigcup_{n=1}^{\infty} S_n$, where for each n, S_n is of type F_{α_n}, $\alpha_n < \alpha$. If α_n is odd, S_n will be of type P_{α_n} by our induction assumption. Therefore S_n will be of type Q_{α_n+1}, $\alpha_n+1 < \alpha$, by theorem 3.5 If α_n is even, S_n will be of type Q_{α_n}, $\alpha_n < \alpha$, by our induction assumption. In any case S is of type P_α.

A set of type P_α is shown to be of type F_α, if α is an odd ordinal, by similar reasoning.

Suppose $\alpha < \Omega$ is any even ordinal and S is a Borel set of type G_α. Then by theorem 2.3, $C(S)$ is of type F_α. By the first part of the proof $C(S)$ is of type Q_α. Therefore by theorem 3.2 S is of type P_α. Likewise, a set of type P_α is shown to be a Borel set of type G_α, for any even ordinal $\alpha < \Omega$.

Similarly, it can be shown that the sets of type Q_{α} and the Borel sets of type G_{α} are identical, for any odd ordinal $\alpha < \Omega$, by taking complements.

Hence the theorem is true for all ordinals $\alpha < \Omega$ by transfinite induction.

CHAPTER IV

BAIRE FUNCTIONS

The Baire functions are a class of functions which are defined in an analogous manner to the Borel sets. It will be assumed throughout that we are working with real-valued functions defined on a metric space, unless otherwise stated.

Definition: A function is a Baire function of type f_0 if and only if it is a continous function. A function is a Baire function of type f_1 if and only if it is the limit of a convergent sequence of continuous functions. Suppose that $\alpha < \Omega$, and that Baire functions of type f_β have been defined for all ordinals $\beta < \alpha$. Then a function is a Baire function of type f_α if and only if it is the limit of a convergent sequence of functions, each of type f_β for some $\beta < \alpha$. We note that all of the functions of the convergent sequence need not be of the same type. By transfinite induction, this defines Baire functions of type f_α, for all ordinals $\alpha < \Omega$.

Theorem 4.1: For every $\alpha < \Omega$, every function of type f_β is also a function of type f_α if $\beta < \alpha$.

Proof: Suppose $f(x)$ is of type f_β, $\beta < \alpha < \Omega$.

Then $f(x)$ is of type f_α since $f(x) = \lim_{n\to\infty} f_n(x)$, where $f_n(x) = f(x)$ for each n.

Theorem 4.2: For every $\alpha < \Omega$, the sum and product of two functions of type f_α are Baire functions of type f_α.

Proof: The theorem is true for $\alpha = 0$ since the sum and product of two continuous functions are continuous.

Consider $\alpha < \Omega$ and assume the theorem is true for all ordinals $\beta < \alpha$. If $f(x)$ and $g(x)$ are both of type f_α, then $f(x) = \lim_{n\to\infty} f_n(x)$ and $g(x) = \lim_{n\to\infty} g_n(x)$ where, for each n, $f_n(x)$ is of type f_{α_n}, $\alpha_n < \alpha$, and $g_n(x)$ is of type f_{β_n}, $\beta_n < \alpha$. Let $\gamma_n = \max(\alpha_n, \beta_n)$ for each n. Since $\gamma_n < \alpha$ for each n, by our induction assumption each function $f_n(x) + g_n(x)$ or $f_n(x) \cdot g_n(x)$ is of type f_{γ_n}, $\gamma_n < \alpha$.

Since $f(x) + g(x) = \lim_{n\to\infty} [f_n(x) + g_n(x)]$ and $f(x) \cdot g(x) = \lim_{n\to\infty} [f_n(x) \cdot g_n(x)]$, it follows that the functions $f(x) + g(x)$ and $f(x) \cdot g(x)$ are both Baire functions of type f_α, $\alpha < \Omega$.

Therefore, by transfinite induction, the theorem is true for all ordinals $\alpha < \Omega$.

Theorem 4.3: For every ordinal $\alpha < \Omega$, the difference of two functions of type f_α is a Baire function of type f_α.

__Proof__: The theorem is true for $\alpha = 0$ since the difference of two continuous functions is a continuous function. Consider any ordinal α, $0 < \alpha < \Omega$. Suppose $f(x)$ and $g(x)$ are both of type f_α. Since any constant function is continuous, -1 is of type f_0. By theorem 4.1, -1 is of type f_α. Hence $-g(x)$ is of type f_α by theorem 4.2. It then follows that $f(x) - g(x)$ is of type f_α by theorem 4.2.

__Theorem 4.4__: For every ordinal $\alpha < \Omega$, if $f(x)$ is of type f_α then $|f(x)|$ is of type f_α.

__Proof__: Consider the Baire function $f(x)$ of type f_α, $\alpha < \Omega$. If $\alpha = 0$, $f(x)$ is a continuous function. By definition this means given any x_0 and $\varepsilon > 0$ there exists some $\delta > 0$ such that if $d(x,x_0) < \delta$ then $|f(x) - f(x_0)| < \varepsilon$. Hence $\big||f(x)| - |f(x_0)|\big| \leq |f(x) - f(x_0)| < \varepsilon$ and the function $|f(x)|$ is also a continuous function. Therefore the theorem is true for $\alpha = 0$.

Assume the theorem is true for all ordinals $\beta < \alpha$. If $f(x)$ is of type f_α, $\alpha < \Omega$, then $f(x) = \lim_{n\to\infty} f_n(x)$ where for each n, $f_n(x)$ is of type f_{α_n}, $\alpha_n < \alpha$. Then by our induction assumption each function $|f_n(x)|$ is of type f_{α_n}, $\alpha_n < \alpha$. But $|f(x)| = |\lim_{n\to\infty} f_n(x)| = \lim_{n\to\infty} |f_n(x)|$, therefore

$|f(x)|$ is of type f_α. Therefore by transfinite induction, the theorem is true for all ordinals $\alpha < \Omega$.

Theorem 4.5: For every ordinal $\alpha < \Omega$, if $f(x)$ and $g(x)$ are of type f_α, then $\max[f(x),g(x)]$ and $\min[g(x),f(x)]$ are of type f_α.

Proof: By theorems 4.2, 4.3, and 4.4 the functions $f(x) + g(x)$ and $|f(x) - g(x)|$ are of type f_α, $\alpha < \Omega$. Hence the functions $\max[f(x),g(x)] = 1/2\,[f(x) + g(x)] + 1/2\,|f(x) - g(x)|$ and $\min[f(x),g(x)] = 1/2\,[f(x) + g(x)] - 1/2\,|f(x)-g(x)|$ are of type f_α, $\alpha < \Omega$, by theorems 4.1, 4.2, and 4.3.

Theorem 4.6: For every $\alpha < \Omega$, if $f(x)$ is of type f_α and never equal to 0, then $\frac{1}{f(x)}$ is of type f_α.

Proof: The theorem is true for $\alpha = 0$, as the reciprocal of a nonzero continuous function is a continuous function. Consider $\alpha < \Omega$, and suppose the theorem is true for all ordinals $\beta < \alpha$.

Suppose $f(x)$ is a Baire function of type f_α, $f(x) \neq 0$ for all x. Let $Q(x) = [f(x)]^2$. Since $f(x) \neq 0$ for all x, $Q(x) > 0$ for all x. Also, $Q(x)$ is of type f_α by theorem 4.2. We shall show that $\frac{1}{Q(x)}$ is of type f_α, and hence $\frac{1}{f(x)}$ is of type f_α, by theorem 4.2, since

$\frac{1}{f(x)} = f(x) \cdot \frac{1}{Q(x)}$.

Since $Q(x)$ is of type f_α, $Q(x) = \lim_{n \to \infty} f_n(x)$ where, for each n, $f_n(x)$ is of type f_{α_n}, $\alpha_n < \alpha$. Let $g_n(x) = \max [f_n(x), \frac{1}{n}]$, then $g_n(x) > 0$ for all x and $g_n(x)$ is of type f_{α_n}, $\alpha_n < \alpha$, by theorems 4.1 and 4.5 . Also, $\lim_{n \to \infty} g_n(x) = \max [Q(x), 0] = Q(x)$ and $\frac{1}{Q(x)} = \lim_{n \to \infty} \frac{1}{g_n(x)}$ where, for each n, $g_n(x) > 0$. But by our induction assumption each function $\frac{1}{g_n(x)}$ is of type f_{α_n}, $\alpha_n < \alpha$. Therefore the function $\frac{1}{Q(x)}$ is of type f_α .

Hence $\frac{1}{f(x)}$ is of type f_α, and the theorem is true for all ordinals $\alpha < \Omega$ by transfinite induction.

The following lemma will be needed to prove theorem 4.7.

Lemma 4.1: For every ordinal $\alpha < \Omega$, if $f(x)$ is of type f_α on a metric space A and $|f(x)| \leq k$ for every $x \in A$, $k > 0$, then $f(x) = \lim_{n \to \infty} f_n(x)$, where each function $f_n(x)$ is of type f_{α_n}, $\alpha_n < \alpha$, and $|f_n(x)| \leq k$ for every $x \in A$.

Proof: If the function $f(x)$ is of type f_α, $\alpha < \Omega$, then $f(x) = \lim_{n \to \infty} g_n(x)$ where for each n, $g_n(x)$ is of type f_{α_n}, $\alpha_n < \alpha$.

For each n define $h_n(x) = \min [g_n(x), k]$ and $f_n(x) = \max [h_n(x), -k]$ for $k > 0$. Then by definition $f_n(x) \leq k$ and $f_n(x) \geq -k$, therefore

$|f_n(x)| \leq k$ for every $x \in A$. By theorems 4.1 and 4.5 each function $f_n(x)$ is of type f_{α_n}, $\alpha_n < \alpha$. Now $f(x) = \lim_{n\to\infty} f_n(x)$ if for every $\varepsilon > 0$ and every $x_o \in A$ there exists some positive integer N depending upon x_o, such that if $n > N$ then $|f_n(x_o) - f(x_o)| < \varepsilon$.

Suppose any $\varepsilon > 0$ is given. Then since $f(x) = \lim_{n\to\infty} g_n(x)$, for any $x=x_o$ there is some $N > 0$, depending upon x_o, such that if $n > N$ then $|g_n(x_o) - f(x_o)| < \varepsilon$. Therefore $f(x_o) - \varepsilon < g_n(x_o) < f(x_o) + \varepsilon$ or $-k - \varepsilon < g_n(x_o) < k + \varepsilon$ since $|f(x_o)| \leq k$.

Suppose that $g_n(x_o) \leq k$, then $h_n(x_o) = g_n(x_o)$ by definition, and $|h_n(x_o) - f(x_o)| < \varepsilon$.

Suppose instead that $g_n(x_o) > k$, then by definition $h_n(x_o) = k$, and since $|f(x_o)| \leq k$, $0 \leq h_n(x_o) - f(x_o) < g_n(x_o) - f(x_o) < \varepsilon$.

Hence $|h_n(x_o) - f(x_o)| < \varepsilon$ for $n > N$ and for any value of $g_n(x_o)$.

Suppose $h_n(x_o) \geq -k$, then by definition $f_n(x_o) = h_n(x_o)$, and $|f_n(x_o) - f(x_o)| = |h_n(x_o) - f(x_o)| < \varepsilon$.

Suppose instead that $h_n(x_o) < -k$, then by definition $f_n(x_o) = -k$, and since $|f(x_o)| \leq k$, $0 \geq f_n(x_o) - f(x_o) > h_n(x_o) - f(x_o) > -\varepsilon$.

Therefore $|f_n(x_o) - f(x_o)| < \varepsilon$ for $n > N$ and for any value of $h_n(x_o)$. Therefore $\lim_{n\to\infty} f_n(x) = f(x)$ at x_o, and therefore at any point $x \in A$ since x_o was arbitrarily chosen.

Since the sequence $\{f_n(x)\}$ was constructed so that $|f_n(x)| \leq k$ for all $x \in A$, for each n, and each function $f_n(x)$ is of type $f_{\alpha_n}, \alpha_n < \alpha$, the lemma is proved.

Theorem 4.7: For every ordinal $\alpha < \Omega$, the limit of a uniformly convergent sequence of functions of type f_α, on a metric space A, is a Baire function of type f_α.

Proof: Let $\{f_n(x)\}$ be a uniformly convergent sequence of functions of type f_α, where $f(x) = \lim_{n\to\infty} f_n(x)$. Let any $\varepsilon > 0$ be given. Since $\{f_n(x)\}$ is uniformly convergent, there exists some $N > 0$ such that for every $n \geq N$, $|f(x) - f_n(x)| < \varepsilon/3$ for every $x \in A$.

Suppose $\alpha = 0$ and $n \geq N$. Now since each function $f_n(x)$ is continuous on A, if $a \in A$ there exists a $\delta > 0$ such that if $d(x,a) < \delta$ then $|f_N(x) - f_N(a)| < \varepsilon/3$. Suppose $d(x,a) < \delta$, then

$$|f(x) - f(a)| \leq |f(x) - f_N(x)| + |f_N(x) - f_N(a)| + |f_N(a) - f(a)|$$
$$< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon .$$

This shows that $f(x)$ is continuous at $a \in A$ and therefore continuous on the space A. Therefore the theorem is true for $\alpha = 0$.

Now suppose $0 < \alpha < \Omega$, and $f(x) = \lim_{n\to\infty} f_n(x)$ uniformly, with $f_n(x)$ of type f_α for each n. Choose a convergent series $\sum_{i=1}^{\infty} k_i$ of positive real numbers, say $k_i = 1/2^i$, such that $k_{i+1} < k_i$, for all i. Then $\lim_{i\to\infty} k_i = 0$. Now, from the definition of uniform convergence, for each $k_i > 0$ there exists some $m_i > 0$ such that if $n,m \geq m_i$ then $|f_n(x) - f_m(x)| < k_i$ for all $x \in A$, or $|f_n(x) - f_{m_i}(x)| < k_i$ for all $n \geq m_i$ and for all $x \in A$, where $m_i < m_{i+1}$ for all positive integers i.

We thus define a uniformly convergent subsequence $\{f_{m_i}(x)\}$ of $\{f_n(x)\}$ where $\lim_{i\to\infty} f_{m_i}(x) = \lim_{n\to\infty} f_n(x) = f(x)$. Let $\{f_{m_i}(x)\} = \{g_i(x)\}$, then $f(x) = \lim_{i\to\infty} g_i(x)$.

Now, $f(x) - g_1(x) = \sum_{n=1}^{\infty} [g_{n+1}(x) - g_n(x)]$ where, for every n and all $x \in A$, $|g_{n+1}(x) - g_n(x)| < k_n$. Hence, by the Weierstrass M-test, since $\sum_{i=1}^{\infty} k_i$ was constructed to be a convergent series of positive constants, the series $\sum_{n=1}^{\infty} |g_{n+1}(x) - g_n(x)|$ converges uniformly to $f(x) - g_1(x)$. Furthermore, since $\{f_{m_i}(x)\} = \{g_i(x)\}$, each function $g_i(x)$

is of type f_α, and the function $g_{n+1}(x) - g_n(x)$ is of type f_α for every n by theorem 4.3.

Therefore, by lemma 4.1, since $|g_{n+1}(x) - g_n(x)| < k_n$ for every $x \in A$, there exists a sequence $\{g_{n,m}(x)\}$ such that $|g_{n,m}(x)| < k_n$ for every m and $x \in A$, and $g_{n+1}(x) - g_n(x) = \lim_{m \to \infty} g_{n,m}(x)$ where each function $g_{n,m}(x)$ is of type $f_{\alpha_{n,m}}$, $\alpha_{n,m} < \alpha$.

Now define $h_m(x) = \sum_{n=1}^{m} g_{n,m}(x)$ and let $\alpha_m = \max[\alpha_{1,m}, \alpha_{2,m}, \ldots, \alpha_{m,m}]$. Then, by theorem 4.2, for every value of m, $h_m(x)$ is of type f_{α_m}, where $\alpha_m < \alpha$.

Let $\varepsilon > 0$ be given. Since $\sum_{i=1}^{\infty} k_i$ is a convergent series there exists some $M > 0$ such that $\sum_{n=M+1}^{\infty} k_n < \varepsilon/3$. Therefore for every $x \in A$,

$$\begin{aligned}
&\left| f(x) - g_1(x) - \sum_{n=1}^{M} [g_{n+1}(x) - g_n(x)] \right| \\
&= \left| \sum_{n=1}^{\infty} [g_{n+1}(x) - g_n(x)] - \sum_{n=1}^{M} [g_{n+1}(x) - g_n(x)] \right| \\
&\leq \sum_{n=M+1}^{\infty} |g_{n+1}(x) - g_n(x)| \\
&< \sum_{n=M+1}^{\infty} k_n < \varepsilon/3.
\end{aligned}$$

Now suppose $m > M$, then since $h_m(x) = \sum_{n=1}^{m} g_{n,m}(x)$,

$$\left| h_m(x) - \sum_{n=1}^{M} g_{n,m}(x) \right| \leq \sum_{n=M+1}^{m} |g_{n,m}(x)| < \sum_{n=M+1}^{m} k_n \leq \sum_{n=M+1}^{\infty} k_n < \varepsilon/3 \text{ for all } x \in A.$$

Also, for every $x \in A$,

$$\begin{aligned}
&\left| \sum_{n=1}^{M} [g_{n+1}(x) - g_n(x)] - \sum_{n=1}^{M} g_{n,m}(x) \right| \\
&= \left| \sum_{n=1}^{M} [g_{n+1}(x) - g_n(x) - g_{n,m}(x)] \right| \\
&\leq \sum_{n=1}^{M} |g_{n+1}(x) - g_n(x) - g_{n,m}(x)| .
\end{aligned}$$

Consider $x_o \in A$, then since for each n

$g_{n+1}(x) - g_n(x) = \lim_{m\to\infty} g_{n,m}(x)$, there exists some

$c_n > 0$ such that if $m \geq c_n$, then

$|g_{n+1}(x_o) - g_n(x_o) - g_{n,m}(x_o)| < \varepsilon/3M$ for

$n = 1, 2, .., M$. Let $M' = \max [c_1, c_2, \ldots, c_M]$,

then $m \geq M'$ implies $m \geq c_n$ for $n=1, 2, \ldots, M$.

Take $m \geq M'$, then $\sum_{n=1}^{M} |g_{n+1}(x_o) - g_n(x_o) - g_{n,m}(x_o)|$

$< M \cdot \frac{\varepsilon}{3M} = \frac{\varepsilon}{3}$ and

$$\left|\sum_{n=1}^{M} [g_{n+1}(x) - g_n(x)] - \sum_{n=1}^{M} g_{n,m}(x)\right| < \varepsilon/3$$

at $x_o \in A$.

Now, for $x_o \in A$, and $M > 0$ and $M' > 0$ defined as above, if $m > \max [M, M']$, then

$$\begin{aligned} &|f(x_o) - g_1(x_o) - h_m(x_o)| \\ \leq\ &\left|f(x_o) - g_1(x_o) - \sum_{n=1}^{M} [g_{n+1}(x_o) - g_n(x_o)]\right| \\ &+ \left|\sum_{n=1}^{M} [g_{n+1}(x_o) - g_n(x_o)] - \sum_{n=1}^{M} g_{n,m}(x_o)\right| \\ &+ \left|h_m(x_o) - \sum_{n=1}^{M} g_{n,m}(x_o)\right| \\ <\ &\varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon . \end{aligned}$$

This implies $f(x) - g_1(x) = \lim_{m\to\infty} h_m(x)$

where each function $h_m(x)$ is of type f_{α_m}, $\alpha_m < \alpha$.

Therefore $f(x) - g_1(x)$ is of type f_α, $\alpha < \Omega$,

and by theorem 4.2, since $g_1(x)$ is of type f_α,

$f(x)$ is of type f_α.

<u>Definition:</u> If a function $f(x)$ is defined on a set S, then $f(x)$ is <u>lower-semicontinuous at $c \in S$ relative to S</u> if for every $\varepsilon > 0$ there is a $\delta > 0$

such that if $x \in S$ and $d(x,c) < \delta$ then $f(x) > f(c) - \varepsilon$. The function $f(x)$ is <u>upper-semi-continuous at $c \in S$ relative to S</u> if for every $\varepsilon > 0$ there is a $\delta > 0$ such that if $x \in S$ and $d(x,c) < \delta$ then $f(x) < f(c) + \varepsilon$.

<u>Definition</u>: The function $f(x)$ is said to be <u>lower-semicontinuous</u> (upper-semicontinuous) o<u>n S relative to S</u> if it is lower-semicontinuous (upper-semicontinuous) at every $x \in S$ relative to S.

<u>Theorem 4.8</u>: If $\{f_n(x)\}$ is a nondecreasing sequence of continuous functions on a metric space A, and if for each $x_o \in A$, $\{f_n(x_o)\}$ is a sequence which is bounded above, then the sequence $\{f_n(x)\}$ converges to a function $f(x)$ which is lower-semicontinuous on A.

<u>Proof:</u> Suppose $\{f_n(x)\}$ is a sequence satisfying the conditions of the theorem. Then, for any $c \in A$, $\{f_n(c)\}$ is a nondecreasing sequence of real numbers, bounded above, and $f(c) = \lim_{n \to \infty} f_n(c)$ exists.

Now, to show $f(x)$ is lower-semicontinous at $c \in A$, let $\varepsilon > 0$ be given, then there is some $N > 0$ such that for all $n \geq N$, $|f_n(c) - f(c)| < \varepsilon/2$, so that $f_N(c) > f(c) - \varepsilon/2$ since $f(c) = \lim_{n \to \infty} f_n(c)$. Since $f_N(x)$ is continuous on A, there is a $\delta > 0$ such that if $d(x,c) < \delta$ then $|f_N(x) - f_N(c)| < \varepsilon/2$, i.e. $f_N(x) > f_N(c) - \varepsilon/2$.

Hence, for any $\varepsilon > 0$ there exists some $\delta > 0$ and some $N > 0$ such that if $d(x,c) < \delta$ then $f_N(x) > f_N(c) - \varepsilon/2 > f(c) - \varepsilon/2 - \varepsilon/2 = f(c) - \varepsilon$. But since $\{f_n(x)\}$ is a nondecreasing sequence, $f(x) \geq f_N(x)$ and $f(x) > f(c) - \varepsilon$ for all $x \in A$ such that $d(x,c) < \delta$. Therefore $f(x)$ is lower-semicontinuous at $c \in A$, and therefore lower-semicontinuous on A.

Theorem 4.9: Every lower-semicontinuous function $f(x)$ defined on a metric space A and bounded below, is the limit of a nondecreasing sequence of continuous functions on A.

Proof: Suppose $f(x)$ is bounded below and lower-semicontinuous on a metric space A. Define $g_n(x) = \text{g.l.b.} [f(y) + n \cdot d(x,y)]$ for every positive integer n, where y varies over the entire space A.

To show that each $g_n(x)$ is uniformly continuous, consider any two points $x_1, x_2 \in A$. Then

$$\begin{aligned} g_n(x_1) &= \text{g.l.b.} [f(y) + n \cdot d(x_1,y)] \\ &\leq \text{g.l.b.} [f(y) + n \cdot d(x_1,x_2) + n \cdot d(x_2,y)] \\ &= \text{g.l.b.} [f(y) + n \cdot d(x_2,y)] + n \cdot d(x_1,x_2) \\ &= g_n(x_2) + n \cdot d(x_1,x_2) \ . \end{aligned}$$

By interchanging x_1 and x_2, we obtain

$$g_n(x_2) \leq g_n(x_1) + n \cdot d(x_1,x_2) \ .$$

Therefore

$$| g_n(x_1) - g_n(x_2)| \leq n \cdot d(x_1,x_2) \quad \text{for each } n.$$

Let $\mathcal{E} > 0$ be given, then choose $\delta = \mathcal{E}/n$. Now, if $d(x_1,x_2) < \delta$, then $|g_n(x_1) - g_n(x_2)| < n\delta = \mathcal{E}$. Therefore for each n each function $g_n(x)$ is uniformly continuous on A, and therefore continuous on A.

Next, we note that $\{g_n(x)\}$ is a nondecreasing sequence, i.e. $g_{n+1}(x) \geq g_n(x)$ for all n, since

$$
\begin{aligned}
g_{n+1}(x) &= \text{g.l.b.} \left[f(y) + (n+1)\cdot d(x,y)\right] \\
&= \text{g.l.b.} \left[f(y) + n\cdot d(x,y) + d(x,y)\right] \\
&\geq \text{g.l.b.} \left[f(y) + n\cdot d(x,y)\right] \\
&= g_n(x) .
\end{aligned}
$$

Finally, we must show that $f(x) = \lim_{n\to\infty} g_n(x)$. This will be done by showing the following:

(1) $f(x) \geq \lim_{n\to\infty} g_n(x)$ for all $x \in A$, and

(2) $f(x) \leq \lim_{n\to\infty} g_n(x)$ for all $x \in A$.

<u>Proof of (1)</u>: For every n, $g_n(x) = \text{g.l.b.} \left[f(y) + n\cdot d(x,y)\right] \leq f(x)$. Therefore $f(x) \geq \lim_{n\to\infty} g_n(x)$.

<u>Proof of (2)</u>: Suppose $\mathcal{E} > 0$ is given. Since $f(x)$ is lower-semicontinuous on A, for any $x_o \in A$ there exists some $\delta > 0$ such that if $d(x_o,y) < \delta$ then $f(y) > f(x_o) - \mathcal{E}$.

Hence $\text{g.l.b.} \left[f(y) + n\cdot d(x_o,y)\right] \geq f(x_o) - \mathcal{E}$ if $d(x_o,y) < \delta$. Since $f(x)$ is bounded below, there exists some real number M such that $f(x) > M$ for every $x \in A$. Now, there exists some $N > 0$ such that $M + n\delta > f(x_o) - \mathcal{E}$ for all $n > N$.

Hence if $n > N$ and $d(x_o,y) \geq \delta$ then $f(y) + n \cdot d(x_o,y) > M + n\delta > f(x_o) - \varepsilon$. Hence

$$\text{g.l.b.} \left[f(y) + n.d(x_o,y)\right] \geq f(x_o) - \varepsilon \text{ if}$$

$d(x_o,y) \geq \delta$ and $n > N$.

Therefore $g_n(x_o) = \text{g.l.b.}\left[f(y) + n.d(x_o,y)\right] \geq f(x_o) - \varepsilon$ if $n > N$. Therefore $\lim_{n\to\infty} g_n(x) \geq f(x)$ for all $x \in A$.

It follows then that $f(x) = \lim_{n\to\infty} g_n(x)$.

<u>Definition</u>: For any real valued function $f(x)$ defined on a metric space A, the set $E[f(x) > k]$ ($E[f(x) \leq k]$) is the set of all $x \in A$ such that $f(x) > k$ ($f(x) \leq k$).

<u>Theorem 4.10</u>(a): The function $f(x)$ defined on a metric space A is lower-semicontinuous on A relative to A if and only if, for every real number k, the set $E[f(x) > k]$ ($E[f(x) \leq k]$) is open (closed) relative to A.

<u>Proof</u>: Suppose $f(x)$ is lower-semicontinuous on A relative to A. Let k be any real number and suppose $c \in A$, with $f(c) > k$. There is some $\varepsilon > 0$ such that $f(c) - \varepsilon > k$. Since $f(x)$ is lower-semicontinuous at $c \in A$ relative to A, there is some $\delta > 0$ such that if $d(x,c) < \delta$ then $f(x) > f(c) - \varepsilon$. Hence there is a neighborhood $N(c,\delta)$ such that for every $x \in N(c,\delta)$, $f(x) > f(c) - \varepsilon > k$. Thus $E[f(x) > k]$ is open relative to A.

Suppose the set $E[f(x) > k]$ is open relative to A for every real number k. Let $c \in A$ and $\varepsilon > 0$ be given. Then since $E[f(x) > f(c) - \varepsilon]$ is open relative to A, there is some $\delta > 0$ such that for every $x \in N(c, \delta)$, $f(x) > f(c) - \varepsilon$. Therefore f(x) is lower-semicontinuous at $c \in A$ relative to A, and therefore lower-semicontinuous on A relative to A.

Since $E[f(x) \leq k] = C(E[f(x) > k])$, and the complement of an open set is a closed set, it follows that f(x) is lower-semicontinuous on A if and only if for every real number k the set $E[f(x) \leq k]$ is closed relative to A.

Definition: For any real vlaued function f(x) defined on a metric space A, the set $E[f(x) < k]$ ($E[f(x) \geq k]$) is the set of all $x \in A$ such that $f(x) < k$ ($f(x) \geq k$).

Theorem 4.10 (b): The function f(x) defined on a metric space A is upper-semicontinuous on A relative to A if and only if, for every real number k, the set $E[f(x) < k]$ ($E[f(x) \geq k]$) is open (closed) relative to A.

Proof: The proof of this theorem is similar to that of theorem 4.10 (a).

Theorem 4.10 (c): A function f(x) is continuous on a metric space A relative to A if and only if, for every real number k, the sets $E[f(x) > k]$

($E\,[f(x) \leq k]$) and $E\,[f(x) < k]$ ($E[f(x) \geq k]$) are open (closed) relative to A.

Proof: The proof of this theorem is an immediate consequence of theorems 4.10 (a) and 4.10 (b) since a continuous function is one which is at the same time lower-semicontinuous and upper-semicontinuous.

The following theorem is a generalization of theorem 4.9.

Theorem 4.11: Every lower-semicontinuous function $f(x)$ defined on a metric space A (bounded or not) is the limit of a nondecreasing sequence of continuous functions on A.

Proof: Suppose $f(x)$ is unbounded and lower-semicontinuous on A.

Define $Q(x) = \dfrac{f(x)}{1 + |f(x)|}$. Then $|Q(x)| < 1$, $E\,[Q(x) > k]$ is empty if $k \geq 1$, and $E\,[Q(x) > k] =$ A if $k \leq -1$. Now, for $0 \leq k < 1$,

$$\begin{aligned} E\,[Q(x) > k] &= E\,[f(x) > k + k\cdot|f(x)|] \\ &= E\,[f(x) - k\cdot f(x) > k] \\ &= E\,[f(x) > \frac{k}{1 - k}] \end{aligned}$$

since $k \geq 0$ implies $Q(x) > 0$ and $f(x) > 0$.

For $-1 < k < 0$, and $-1 < Q(x) < 0$,

$$\begin{aligned} E\,[Q(x) > k] &= E\,[f(x) + k\cdot f(x) > k] \\ &= E\,[f(x) > \frac{k}{1 + k}] \end{aligned}$$ since $Q(x) < 0$

implies $f(x) < 0$. Therefore, for $-1 < k < 0$, and $|Q(x)| < 1$, $E\,[Q(x) > k] = E\,[f(x) > \frac{k}{1 + k}]$

since $f(x) \geq 0$ if $Q(x) \geq 0$, by definition.

But since $f(x)$ is lower-semicontinuous on A, $E\left[Q(x) > k\right] = E\left[f(x) > \frac{k}{1-k}\right]$ is open relative to A, for $0 \leq k < 1$, and $E\left[Q(x) > k\right] = E\left[f(x) > \frac{k}{1+k}\right]$ is open relative to A, for $-1 < k < 0$, by theorem 4.10 (a). Therefore $E\left[Q(x) > k\right]$ is open relative to A for all real numbers k. Therefore, by theorem 4.10 (a), $Q(x)$ is lowersemicontinuous on A.

Now, by theorem 4.9, since $Q(x)$ is a bounded lower-semicontinuous function on A, $Q(x)$ is the limit of a nondecreasing sequence $\{g_n(x)\}$ of continuous functions on A, i.e. $Q(x) = \lim_{n\to\infty} g_n(x)$, where for each n, $g_n(x)$ is continuous on A, and $g_{n+1}(x) \geq g_n(x)$ for all $x \in A$.

Since $\{g_n(x)\}$ is a nondecreasing sequence, $g_n(x) \leq Q(x) < 1$ for each n. Define $f_n(x) = \max\left[g_n(x), -1\right]$ for each n. Then for each n, $f_n(x)$ is continuous on A and $-1 \leq f_n(x) < 1$. Also, by definition, for all n, $f_n(x) \geq g_n(x)$, $f_n(x) \leq Q(x)$, and $\{f_n(x)\}$ is a nondecreasing sequence since $\{g_n(x)\}$ is a nondecreasing sequence, for all $x \in A$.

Since $Q(x) = \lim_{n\to\infty} g_n(x)$, for any given $\varepsilon > 0$ and $x_o \in A$ there exists some $N > 0$ such that if $n > N$ then $|g_n(x_o) - Q(x_o)| < \varepsilon$. But since

$g_n(x) \leq f_n(x) \leq Q(x)$ for all $x \in A$,

$Q(x_o) - \varepsilon < g_n(x_o) \leq f_n(x_o) < Q(x_o) + \varepsilon$, if $n > N$.

Therefore $|f_n(x_o) - Q(x_o)| < \varepsilon$ for $n > N$, and

$Q(x) = \lim_{n\to\infty} f_n(x)$ for all $x \in A$.

Now define $v_n(x) = \sum_{i=n}^{\infty} \frac{1}{2^{i-n+1}} \cdot f_i(x)$.

Since $\left| \frac{1}{2^{i-n+1}} \cdot f_i(x) \right| \leq \frac{1}{2^{i-n+1}}$ for $i = n$, $n + 1, \ldots$, and $\sum_{i=n}^{\infty} \frac{1}{2^{i-n+1}}$ is a convergent series of positive constants, then by the Weierstrass M-test, the series defining $v_n(x)$ converges uniformly on A. Therefore $v_n(x)$ is continuous on A for all n, since each function $v_n(x)$ is the sum of a uniformly convergent series of continuous functions on A.

Since $\{f_n(x)\}$ is a nondecreasing sequence and since $\sum_{i=1}^{\infty} \frac{1}{2^i} = 1$, we have for all n and all $x \in A$, $v_n(x) \geq \sum_{i=n}^{\infty} \frac{1}{2^{i-n+1}} \cdot f_n(x) = f_n(x) \cdot \sum_{i=n}^{\infty} \frac{1}{2^{i-n+1}} = f_n(x)$. But $v_n(x) \leq \sum_{i=n}^{\infty} \frac{1}{2^{i-n+1}} \cdot Q(x) = Q(x)$ for all n and $x \in A$. Therefore $Q(x) = \lim_{n\to\infty} v_n(x)$, since $f_n(x) \leq v_n(x) \leq Q(x)$ for all n and all $x \in A$, and $\lim_{n\to\infty} f_n(x) = Q(x)$.

Also, $v_{n+1}(x) = \sum_{i=n+1}^{\infty} \frac{1}{2^{i-n}} \cdot f_i(x) = \sum_{i=n}^{\infty} \frac{1}{2^{i-n+1}} \cdot f_{i+1}(x)$. Therefore $v_{n+1}(x) \geq v_n(x)$ for all n, i.e. since $\{f_n(x)\}$ is a nondecreasing sequence, so is $\{v_n(x)\}$.

Suppose for some $x_o \in A$, $v_n(x_o) = -1$. Then $f_i(x_o) = -1$ for $i = n, n+1, \ldots$, which implies

$Q(x_o) = -1$ since $\lim_{n\to\infty} f_n(x) = Q(x)$ for all $x \in A$. Thus we have a contradiction since $|Q(x)| < 1$ for all $x \in A$. Therefore $v_n(x) > -1$ for all n and all $x \in A$. Also, $v_n(x) \leq Q(x) < 1$ for all n and all $x \in A$; therefore $|v_n(x)| < 1$ for all n and all $x \in A$.

Define $F_n(x) = \dfrac{v_n(x)}{1 - |v_n(x)|}$. Then $F_n(x)$ is continuous on A since $v_n(x)$ is continuous on A, for all n. Furthermore, $\{F_n(x)\}$ is a nondecreasing sequence. Consider the following cases:

(1) Suppose $v_n(x) \geq 0$, then $v_{n+1}(x) \geq 0$ and $v_n(x) \cdot |v_{n+1}(x)| = |v_n(x)| \cdot v_{n+1}(x)$. Therefore $v_{n+1}(x) - |v_n(x)| \cdot v_{n+1}(x) \geq v_n(x) - v_n(x) \cdot |v_{n+1}(x)|$, and $F_{n+1}(x) = \dfrac{v_{n+1}(x)}{1 - |v_{n+1}(x)|} \geq \dfrac{v_n(x)}{1 - |v_n(x)|} = F_n(x)$.

(2) Suppose $v_n(x) < 0$ and $v_{n+1}(x) \geq 0$, then

$$F_{n+1}(x) = \frac{v_{n+1}(x)}{1 - |v_{n+1}(x)|} \geq 0 > \frac{v_n(x)}{1 - |v_n(x)|} = F_n(x) .$$

(3) Suppose $v_n(x) < 0$ and $v_{n+1}(x) < 0$, then $|v_n(x)| \geq |v_{n+1}(x)|$ and $1 - |v_{n+1}(x)| \geq 1 - |v_n(x)| > 0$, since $\{v_n(x)\}$ is a nondecreasing sequence. Therefore

$$\frac{1}{1 - |v_{n+1}(x)|} \leq \frac{1}{1 - |v_n(x)|} \text{ and } \frac{v_n(x)}{1 - |v_{n+1}(x)|} \geq \frac{v_n(x)}{1 - |v_n(x)|}$$

since $v_n(x) < 0$.

Therefore, since $\{v_n(x)\}$ is a nondecreasing sequence, $F_{n+1}(x) =$

$$\frac{v_{n+1}(x)}{1 - |v_{n+1}(x)|} \geq \frac{v_n(x)}{1 - |v_{n+1}(x)|} \geq \frac{v_n(x)}{1 - |v_n(x)|} = F_n(x),$$

where $v_n(x) < 0$ and $v_{n+1}(x) < 0$.

Therefore, $\{F_n(x)\}$ is a nondecreasing sequence.

Finally, $\lim_{n\to\infty} F_n(x) = \lim_{n\to\infty} \frac{v_n(x)}{1 - |v_n(x)|}$ where $|v_n(x)| < 1$. Hence $\lim_{n\to\infty} F_n(x) = \frac{Q(x)}{1 - |Q(x)|}$ since $\lim_{n\to\infty} v_n(x) = Q(x)$. Since by definition $Q(x) = \frac{f(x)}{1 + |f(x)|}$, $f(x) = Q(x)\cdot[1 + |f(x)|] = Q(x) + Q(x)\cdot|f(x)| = Q(x) + |Q(x)|\cdot f(x)$ where $Q(x)\cdot|f(x)| = |Q(x)|\cdot f(x)$ since $Q(x)$ and $f(x)$ have the same sign for all $x \in A$. Therefore

$Q(x) = f(x) - |Q(x)|\cdot f(x) = f(x)\cdot[1 - |Q(x)|]$

and $f(x) = \frac{Q(x)}{1 - |Q(x)|}$ for all $x \in A$.

Therefore $\lim_{n\to\infty} F_n(x) = f(x)$, where $\{F_n(x)\}$ is a nondecreasing sequence of continuous functions on A, and $f(x)$ is our original unbounded-lower-semicontinuous function defined on the metric space A.

CHAPTER V

RELATION BETWEEN BOREL SETS AND BAIRE FUNCTIONS

In this final chapter some of the important relationships between Borel sets and Baire functions will be discussed. It will be assumed throughout that we are working in a metric space, unless otherwise stated. It will be convenient to make the following definition.

<u>Definition</u>: For every ordinal $\alpha < \Omega$, a set S is of type A_α (B_α) if and only if there is some Baire function of type f_α and some real number k such that $S = E[f(x) > k]$ ($S = E[f(x) \geq k]$).

<u>Theorem 5.1</u>: For every $\alpha < \Omega$, the complement of a set of type A_α (B_α) is a set of type B_α (A_α).

<u>Proof</u>: Suppose $\alpha < \Omega$ and S is a set of type A_α. Then there is a Baire function $f(x)$ of type f_α and a real number k such that $S = E[f(x) > k]$. But $S = E[-f(x) < -k]$ and $C(S) = E[-f(x) \geq -k]$. By theorems 4.1 and 4.2, since $f(x)$ is of type f_α, $-f(x)$ is a Baire function of type f_α. Therefore $C(S)$ is a set of type B_α.

Likewise, the complement of a set of type B_α

is a set of type A_α .

The following lemma will be of use in the proof of theorem 5.2.

Lemma 5.1: If $f(x) = \lim_{n\to\infty} f_n(x)$ on A, then $E\left[f(x) > k\right] = \bigcup_{m=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} E\left[f_n(x) \geq k + \frac{1}{m}\right]$.

Proof: Suppose $x_o \in A$ and $f(x_o) > k$. There exists a positive integer m such that $f(x_o) > k + \frac{1}{m}$. Since $f(x_o) = \lim_{n\to\infty} f_n(x_o)$, for each $\varepsilon = f(x_o) - k - \frac{1}{m} > 0$ there is some $N > 0$ such that if $n \geq N$ then $|f(x_o) - f_n(x_o)| \leq \varepsilon$, implying $f_n(x_o) \geq f(x_o) - \varepsilon = k + \frac{1}{m}$. Therefore, if $x_o \in E\left[f(x) > k\right]$, then $x_o \in \bigcup_{m=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} E\left[f_n(x) \geq k + \frac{1}{m}\right]$.

Suppose that $x \in \bigcup_{m=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} E\left[f_n(x) \geq k + \frac{1}{m}\right]$ where $f(x) = \lim_{n\to\infty} f_n(x)$. Then, for some $m > 0$, $x \in \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} E\left[f_n(x) \geq k + \frac{1}{m}\right]$ and for some $N > 0$, $x \in \bigcap_{n=N}^{\infty} E\left[f_n(x) \geq k + \frac{1}{m}\right]$. This means $x \in E\left[f_n(x) \geq k + \frac{1}{m}\right]$ for all $n \geq N$, and since $f(x) = \lim_{n\to\infty} f_n(x)$, then $x \in E\left[f(x) \geq k + \frac{1}{m}\right] \subset E\left[f(x) > k\right]$.

Therefore $E\left[f(x) > k\right] = \bigcup_{m=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} E\left[f_n(x) \geq k + \frac{1}{m}\right]$.

Theorem 5.2: For every finite even (odd) ordinal α , every set of type A_α is a Borel set of type G_α (F_α) and every set of type B_α is a Borel set of type F_α (G_α) .

Proof: The theorem is true for $\alpha = 0$, since by theorem 4.10 (c), a function $f(x)$ is continuous if and only if for every real number k, the set of type A_o is open, and the set of type B_o is closed. Assume the theorem is true for all ordinals $\beta < \alpha$, where α is any finite ordinal.

Suppose first that α is some finite odd ordinal. Let S be any set of type A_α. There is then a function $f(x)$ of type f_α and a real number k such that $S = E\left[f(x) > k\right]$, where $f(x) = \lim_{n\to\infty} f_n(x)$, where for each n, $f_n(x)$ is of type f_{α_n}, $\alpha_n < \alpha$. By lemma 5.1,

$S = \bigcup_{m=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} E\left[f_n(x) \geq k + \frac{1}{m}\right]$. Since $\alpha_n \leq \alpha - 1$ for all n, each $f_n(x)$ is a Baire function of type $f_{\alpha-1}$. Then by our induction assumption, since $\alpha - 1$ is even, each set $E\left[f_n(x) \geq k + \frac{1}{m}\right]$ of type $B_{\alpha-1}$, is a Borel set of type $F_{\alpha-1}$. By theorem 2.4 the intersection of a countable number of Borel sets of type $F_{\alpha-1}$ is a set of type $F_{\alpha-1}$ where $\alpha-1$ is even. Therefore set S, as a countable union of Borel sets of type $F_{\alpha-1}$, where α is odd, is a Borel set of type F_α.

Suppose S is any set of type B_α where α is some finite odd ordinal. By theorem 5.1, $C(S)$ is of type A_α. Therefore by the first part of the proof, $C(S)$ is a Borel set of type F_α. Then by theorem 2.3, S is a Borel set of type G_α .

Suppose that α is some finite even ordinal and S is a set of type A_α . Then, as in the first part of the proof, for $f(x) = \lim_{n\to\infty} f_n(x)$ of type f_α , $S = \bigcup_{m=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} E\left[f_n(x) \geq k + \frac{1}{m}\right]$, where each $f_n(x)$ is of type $f_{\alpha-1}$. But now $\alpha - 1$ is odd and each set $E\left[f_n(x) \geq k + \frac{1}{m}\right]$ of type $B_{\alpha-1}$, is a Borel set of type $G_{\alpha-1}$ by our induction assumption. By theorem 2.4 the countable intersection of Borel sets of type $G_{\alpha-1}$ is a Borel set of type $G_{\alpha-1}$, since $\alpha-1$ is odd. Therefore set S, as a countable union of sets of type $G_{\alpha-1}$, where α is even, is a Borel set of type G_α .

Suppose S is a set of type B_α where α is some finite even ordinal. Then by theorem 5.1, C(S) is of type A_α . Therefore, by the above part of the proof, C(S) is a Borel set of type G_α . Therefore S is a Borel set of type F_α by theorem 2.3 .

Therefore the theorem is true for all finite ordinals α by finite induction.

The following lemma will be needed for the proof of the converse of theorem 5.2 .

<u>Lemma 5.2</u>: For every ordinal $\alpha < \Omega$, if S is a set of type A_α or B_α , there is a Baire function L(x) of type $f_{\alpha+1}$ in a metric space A,

such that $L(x) = 1$ for every $x \in S$ and $L(x) = 0$ for every $x \in C(S)$.

Proof: Suppose $\alpha < \Omega$ and S is a set of type A_α. Then there is a function $f(x)$ of type f_α and a real number k such that $S = E[f(x) > k]$. Define $g(x) = f(x) - k$. Since $f(x)$ is of type f_α, so is $g(x)$ by theorems 4.1 and 4.3. Furthermore, $S = E[f(x) > k] = E[g(x) > 0]$. Now define $h(x) = \max[g(x), 0]$. Then by theorems 4.1 and 4.5, $h(x)$ is also a Baire function of type f_α. Finally, define $f_n(x) = \min[n \cdot h(x), 1]$. Then $f_n(x)$ is a Baire function of type f_α for each n, and $L(x) = \lim_{n \to \infty} f_n(x)$ is a Baire function of type $f_{\alpha+1}$, for all $x \in A$.

Consider $x \in S$, then $g(x) > 0$, $h(x) = g(x)$, and $f_n(x) = \min[n \cdot g(x), 1]$. Therefore $L(x) = \lim_{n \to \infty} f_n(x) = 1$ for all $x \in S$. On the other hand, consider $x \in C(S)$. Then $g(x) \leq 0$, $h(x) = 0$, and $f_n(x) = \min[0, 1] = 0$. Therefore $L(x) = \lim_{n \to \infty} f_n(x) = 0$ for all $x \in C(S)$.

Suppose now that S is a set of type B_α, where $\alpha < \Omega$. Then by theorem 5.1, $C(S)$ is of type A_α. Therefore, by the preceding part of the proof, there is some function $g(x)$ of type $f_{\alpha+1}$ such that $g(x) = 1$ for all $x \in C(S)$ and $g(x) = 0$ for all $x \in S$. Hence $L(x) = 1 - g(x)$, which is also of type $f_{\alpha+1}$, is 1 for all $x \in S$ and 0 for all $x \in C(S)$.

Theorem 5.3: For every finite even (odd) ordinal α, every Borel set of type G_α (F_α) is of type A_α and every Borel set of type F_α (G_α) is of type B_α.

Proof: The theorem is true for $\alpha = 0$, for consider any open set E in a metric space A. For any $x \in A$, define $f(x) = \text{g.l.b.}[d(x,y)]$ for all $y \in C(E)$, where $f(y) = 0$ for all $y \in C(E)$ and $f(x) > 0$ for all $x \in E$. Therefore $E = E[f(x) > 0]$. Now for any given $\varepsilon > 0$ and $x_0 \in A$, choose $\delta = \varepsilon$. If $x \in N(x_0, \delta)$, then $d(x,x_0) < \delta = \varepsilon$. Therefore $f(x) = 0$ and $0 < f(x_0) < \varepsilon$ if $x \in C(E)$ and $x_0 \in E$, $f(x_0) = 0$ and $0 < f(x) < \varepsilon$ if $x \in E$ and $x_0 \in C(E)$, and $f(x_0) = 0$ and $f(x) = 0$ if $x \in C(E)$ and $x_0 \in C(E)$. For $x \in E$ and $x_0 \in E$, we have $f(x) < f(x_0) + \varepsilon$ since $d(x,y) \leq d(x,x_0) + d(x_0,y) < \varepsilon + d(x_0,y)$ for all $y \in C(E)$. Also, if $x \in E$ and $x_0 \in E$, $f(x) > f(x_0) - \varepsilon$ since $d(x,y) \geq d(x_0,y) - d(x,x_0) > d(x_0,y) - \varepsilon$ for all $y \in C(E)$. Therefore, $f(x)$ is continuous on A since for any $\varepsilon > 0$ and any $x_0 \in A$ if $x \in N(x_0, \delta)$ where $\delta = \varepsilon$ then $|f(x) - f(x_0)| < \varepsilon$.

Therefore, every Borel set of type G_0 is of type A_0, and by taking complements it can be shown that every Borel set of type F_0 is of type B_0, with the use of theorems 2.3 and 5.1. Assume the theorem is true for all ordinals $\beta < \alpha$, where α is a finite ordinal.

Suppose α is a finite odd ordinal. Let S be any Borel set of type F_α, then $S = \bigcup_{n=1}^{\infty} T_n$, where for each n, T_n is of type $F_{\alpha-1}$. By our induction assumption, each set T_n is also of type $B_{\alpha-1}$, since $\alpha-1$ is a finite even ordinal. Therefore, by lemma 5.2, for each n there is some function $f_n(x)$ of type f_α such that $f_n(x) = \frac{1}{2^n}$ for every $x \in T_n$, and $f_n(x) = 0$ for every $x \in C(T_n)$.

Let $f(x) = \sum_{n=1}^{\infty} f_n(x)$, i.e. $f(x) = \lim_{n\to\infty} s_n(x)$ where $s_n(x) = \sum_{i=1}^{n} f_i(x)$. Now for any $x \in A$, $|f_n(x)| \leq \frac{1}{2^n}$ for all n, where $\sum_{n=1}^{\infty} \frac{1}{2^n}$ is a convergent geometric series of positive constants. Therefore, by the Weierstrass M-test, f(x) is a uniformly convergent series for all $x \in A$. Therefore the sequence of partial sums $\{s_n(x)\}$ is uniformly convergent on A. Since each $f_n(x)$ is a Baire function of type f_α, and $s_n(x) = \sum_{i=1}^{n} f_i(x)$, each $s_n(x)$ is a Baire function of type f_α by theorem 4.2. Therefore, by theorem 4.7, f(x) is a Baire function of type f_α since it is the limit of a uniformly convergent sequence of Baire functions of type f_α. Finally, $S = E[f(x) > 0]$ since

$S = \bigcup_{n=1}^{\infty} T_n$ and $f(x) = \sum_{n=1}^{\infty} f_n(x)$ where each $f_n(x) > 0$ for all $x \in T_n$, and each $f_n(x) = 0$ for all $x \in C(T_n)$. Therefore the Borel set S of type F_α, where α is a finite odd ordinal, is a set of type A_α.

Suppose S is a Borel set of type G_α where α is a finite odd ordinal. Then $C(S)$ is a Borel set of type F_α, and by the first part of the proof also of type A_α. Therefore S is of type B_α by theorem 5.1.

Suppose next that α is a finite even ordinal. Let S be a Borel set of type G_α; then $S = \bigcup_{n=1}^{\infty} S_n$ where each S_n is of type $G_{\alpha-1}$. One now proceeds as before to show that every Borel set of type G_α is of type A_α and every Borel set of type F_α is of type B_α, if α is a finite even ordinal.

Therefore the theorem is true for all finite ordinals α by finite induction.

Theorem 5.4: For any finite even (odd) ordinal α, a set is of type A_α if and only if it is a Borel set of type G_α (F_α), and a set is of type B_α if and only if it is a Borel set of type F_α (G_α).

Proof: This theorem is a combination of theorems 5.3 and 5.2.

Theorem 5.5: If $f(x)$ is any Baire function of type f_α, where α is any finite even (odd) ordinal, then for every real number k the sets $E[f(x) > k]$ and $E[f(x) \geq k]$ are Borel sets of type G_α (F_α) and F_α (G_α) respectively.

Proof: Suppose α is any finite even (odd) ordinal, and $f(x)$ is any Baire function of type f_α. Then for every real number k the sets $E[f(x) > k]$ and $E[f(x) \geq k]$ are of type A_α and B_α respectively. But by theorem 5.2, this means that for every real number k the sets $E[f(x) > k]$ and $E[f(x) \geq k]$ are of type G_α (F_α) and F_α (G_α) respectively.

The following lemma will be needed for the proof of the converse of theorem 5.5.

Lemma 5.3: For $\alpha < \Omega$, if E and F are disjoint sets of type B_α in a metric space A, then there is a Baire function $g(x)$ of type f_α such that $g(x) = 1$ on E, $g(x) = 0$ on F, and $0 < g(x) < 1$ for all other $x \in A$.

Proof: Suppose $\alpha < \Omega$, and E and F are two disjoint sets of type B_α. Then there are Baire functions $f_1(x)$ and $f_2(x)$ of type f_α such that $E = E[f_1(x) \leq 0]$ and $F = E[f_2(x) \leq 0]$. The reason for this is that if a set S is of type B_α, then there is some Baire function $f(x)$ of type f_α and some real number k such that

$S = E\left[f(x) \geq k\right]$. Then $S = E\left[f(x) - k \geq 0\right] = E\left[k - f(x) \leq 0\right]$ where $k - f(x)$ is a Baire function of type f_α by theorems 4.1 and 4.3.

Let $g_1(x) = \max\left[f_1(x), 0\right]$ and $g_2(x) = \max\left[f_2(x), 0\right]$. If $x \varepsilon E$, then $x \varepsilon C(F)$, and $g_1(x) = 0$ and $g_2(x) > 0$. If $x \varepsilon F$, then $x \varepsilon C(E)$, and $g_2(x) = 0$ and $g_1(x) > 0$. If $x \notin E$ and $x \notin F$, then $x \varepsilon \left[C(E) \cap C(F)\right]$ and $g_1(x) > 0$ and $g_2(x) > 0$. The function $g_1(x) + g_2(x) > 0$ for all $x \varepsilon A$. Let $g(x) = \dfrac{g_2(x)}{g_1(x) + g_2(x)}$, which is a Baire function of type f_α by theorems 4.1, 4.2, 4.5, and 4.6. We note that $g(x) = 1$ for all $x \varepsilon E$, $g(x) = 0$ for all $x \varepsilon F$, and $0 < g(x) < 1$ for all other $x \varepsilon A$.

We are now ready to prove the converse of theorem 5.5.

Theorem 5.6: If α is a finite even (odd) ordinal and the function $f(x)$ is such that for every real number k the sets $E\left[f(x) > k\right]$ and $E\left[f(x) \geq k\right]$ are of type G_α (F_α) and F_α (G_α), respectively, then $f(x)$ is a Baire function of type f_α.

Proof: Suppose the sets $E\left[f(x) > k\right]$ and $E\left[f(x) \geq k\right]$ are of type G_α (F_α) and F_α (G_α), respectively, for every real number k and for some finite even (odd) ordinal α. Then by theorem

5.3 the sets $E[f(x) > k]$ and $E[f(x) \geq k]$ are of type A_α and B_α respectively, for every real number k. The sets $E[f(x) \leq k]$ for every real number k, as complements of sets of type A_α, are of type B_α by theorem 5.1.

Suppose first that $0 < f(x) < 1$ for all x in the metric space A. We will refer to this as case (1). For any positive integer N, the disjoint sets $E[f(x) \leq \frac{m}{N}]$ and $E[f(x) \geq \frac{m+1}{N}]$ are of type B_α for every $m = 0, 1, 2, \ldots, N-1$. By lemma 5.3, for each m there is a Baire function $g_{m,N}(x)$ of type f_α such that $g_{m,N}(x) = 1$ for all $x \in E[f(x) \geq \frac{m+1}{N}]$, $g_{m,N}(x) = 0$ for all $x \in E[f(x) \leq \frac{m}{N}]$, and $0 < g_{m,N}(x) < 1$ for all other $x \in A$. Let $h_N(x) = \frac{1}{N} \cdot \sum_{i=0}^{N-1} g_{i,N}(x)$ which, as a finite sum of Baire functions of type f_α, is of type f_α. Suppose $\frac{m}{N} \leq f(x) < \frac{m+1}{N}$. Then $g_{i,N}(x) = 1$ for $i = 0, 1, \ldots, m-1$, $g_{i,N}(x) = 0$ for $i = m+1, m+2, \ldots, N-1$, and $0 \leq g_{m,N}(x) < 1$. Therefore, for $\frac{m}{N} \leq f(x) < \frac{m+1}{N}$,

$$|f(x)-h_N(x)| < \frac{1}{N} \text{ for all } x \in A, \text{ since}$$

$$h_N(x) = \frac{1}{N}[m+g_{m,N}(x) + 0]$$

$$\text{implies } \frac{m}{N} \leq h_N(x) < \frac{m+1}{N}.$$

Since $|f(x) - h_n(x)| < \frac{1}{N}$ for all $n \geq N$ and for all $x \in A$, $\{h_n(x)\}$ is a uniformly convergent sequence of Baire functions of type f_α, where

$f(x) = \lim_{n\to\infty} h_n(x)$. Therefore $f(x)$ is also a Baire function of type f_α by theorem 4.7, where $0 < f(x) < 1$ for all $x \in A$. This completes the proof of case (1). We extend this to the general case by considering the additional three cases:

(2) $0 < f(x) < M$, where M is any positive real number,

(3) $-M_1 < f(x) < M_2$, where M_1, M_2 are positive real numbers, and

(4) $f(x)$ unbounded.

<u>Proof of case (2)</u>: Assume $0 < f(x) < M$ for all $x \in A$, where M is any positive real number. Define $Q(x) = \frac{f(x)}{M}$. Then $0 < Q(x) < 1$ for all $x \in A$. Suppose $A_k = E\left[Q(x) > k\right]$ and $B_k = E\left[Q(x) \geq k\right]$ where k is any real number. Then $A_k = E\left[\frac{f(x)}{M} > k\right]$ $= E\left[f(x) > kM\right]$ which is by assumption of type A_α for all real numbers k and M, and $B_k = E\left[\frac{f(x)}{M} \geq k\right]$ $= E\left[f(x) \geq kM\right]$ which is by assumption of type B_α for all real numbers k and M. Since $0 < Q(x) < 1$ for all $x \in A$, by case (1) we have that $Q(x)$ is of type f_α. But $f(x) = M \cdot Q(x)$, therefore $f(x)$ is a Baire function of type f_α, by theorems 4.1 and 4.2.

<u>Proof of case (3)</u>: Assume $-M_1 < f(x) < M_2$ on A, where M_1 and M_2 are any positive real numbers. Define $Q(x) = f(x) + M_1$. Then

$0 < Q(x) < M_1 + M_2 = M$ for all $x \in A$. Suppose $A_k = E[Q(x) > k]$ and $B_k = E[Q(x) \geq k]$ where k is any real number. Then $A_k = E[f(x) + M_1 > k] = E[f(x) > k - M_1]$ and $B_k = E[f(x) \geq k - M_1]$ which are by assumption of type A_α and B_α, respectively, for all real numbers k and M_1. Since $0 < Q(x) < M$ for all $x \in A$, by case (2) we have that $Q(x)$ is a Baire function of type f_α. But $f(x) = Q(x) - M_1$, therefore $f(x)$ is of type f_α by theorems 4.1 and 4.3.

Proof of case (4): Assume $f(x)$ is unbounded on A. Define $Q(x) = \frac{f(x)}{1+|f(x)|}$. Then $|Q(x)| < 1$ for $x \in A$. We will now show that $Q(x)$ is of type f_α. Suppose $A_k = E[Q(x) > k]$ and $B_k = E[Q(x) \geq k]$ where $0 \leq k < 1$. Then $A_k = E[f(x) > k + k \cdot |f(x)|] = E[f(x) - k \cdot |f(x)| > k]$. Since $k \geq 0$ and $Q(x) > k$, $f(x) > 0$ which implies $|f(x)| = f(x)$. Then $A_k = E[f(x) > \frac{k}{1-k}]$ and $B_k = E[f(x) \geq \frac{k}{1-k}]$ which are by assumption of type A_α and B_α, respectively, for all real numbers $k \neq 1$.

Now suppose $A_k = E[Q(x) > k]$ and $B_k = E[Q(x) \geq k]$ where $-1 < k < 0$ and $Q(x) < 0$. Then $A_k = E[\frac{f(x)}{1+|f(x)|} > k] = E[f(x) - k \cdot |f(x)| > k]$. Since $k < 0$ and $Q(x) < 0$, $f(x) < 0$ which implies $|f(x)| = -f(x)$. Then $A_k = E[f(x) + k \cdot f(x) > k] = E[f(x) > \frac{k}{1+k}]$ and $B_k = E[f(x) \geq \frac{k}{1+k}]$ which are by assumption of type A_α and B_α, respectively, for all real numbers $k \neq -1$.

Now suppose $A_k = E\left[Q(x) > k\right]$ and $B_k = E\left[Q(x) \geq k\right]$ where $-1 < k < 0$ and $Q(x) \geq 0$. Then $A_k = E\left[f(x) > k + k \cdot |f(x)|\right] = E\left[f(x) - k|f(x)| > k\right]$. Since $Q(x) \geq 0$, $f(x) \geq 0$, which implies $|f(x)| = f(x)$. Then $A_k = E\left[f(x) - k \cdot f(x) > k\right] = E\left[f(x) > \frac{k}{1-k}\right]$ and $B_k = E\left[f(x) \geq \frac{k}{1-k}\right]$. But since $-1 < k < 0$ implies $\frac{k}{1+k} < \frac{k}{1-k} < 0$, it is sufficient to say $A_k = E\left[f(x) > \frac{k}{1+k}\right]$ and $B_k = E\left[f(x) \geq \frac{k}{1+k}\right]$ which are by assumption of type A_α and B_α, respectively, for all real numbers $k \neq -1$.

Therefore $E\left[Q(x) > k\right]$ and $E\left[Q(x) \geq k\right]$ are of type A_α and B_α, respectively, for all real numbers k and all $x \in A$. We note that $E\left[Q(x) > k\right] = A$ for all $k \leq -1$ and $E\left[Q(x) > k\right] = \emptyset$ for all $k \geq 1$. Since $-1 < Q(x) < 1$ for all $x \in A$, by case (3) we have that $Q(x)$ is a Baire function of type f_α. But $f(x) = \frac{Q(x)}{1 - |Q(x)|}$ for all $x \in A$ since by definition $Q(x) = \frac{f(x)}{1 + |f(x)|}$ which implies

$$f(x) = Q(x) \cdot \left[1 + |f(x)|\right] = Q(x) + Q(x) \cdot |f(x)| = Q(x) + |Q(x)| \cdot f(x),$$

which implies

$$Q(x) = f(x) - |Q(x)| \cdot f(x) = f(x)\left[1 - |Q(x)|\right].$$

Therefore $f(x) = Q(x) \cdot \frac{1}{1 - |Q(x)|}$, where the second factor is a Baire function of type f_α by theorems 4.1, 4.3, 4.4 and 4.6. Therefore f(x) is of type f_α by theorem 4.2.

Theorem 5.7: For any real valued function $f(x)$ defined on a metric space A, the set D of all points where $f(x)$ is discontinuous on A is a Borel set of type F_1, and the set C of all points where $f(x)$ is continuous on A is a Borel set of type G_1.

Proof: Given the function $f(x)$ defined on the metric space A, we first define sets D_n such that a point $z \in D_n$ if and only if for every $\delta > 0$ there are at least two distinct points x and y such that $x, y \in N(z, \delta)$ and $|f(x) - f(y)| \geq \frac{1}{n}$. Let D be the set of all points where $f(x)$ is discontinuous on A, and $C = C(D)$ the set of all points where $f(x)$ is continuous on A. To prove that D is a Borel set of type F_1, we will show the following:

(1) $D = \bigcup_{n=1}^{\infty} D_n$, and

(2) each D_n is a Borel set of type F_o .

Proof of (1): Choose any $n = N$ and suppose $z \in D_N$. Then for any $\delta > 0$ there are two points x_δ and y_δ such that $x_\delta, y_\delta \in N(z, \delta)$ and $|f(x_\delta) - f(y_\delta)| \geq \frac{1}{N}$.

We would like to show that $z \in D$. Therefore assume $z \notin D$. Then $f(x)$ is continuous at $x = z$, and for $\varepsilon = \frac{1}{3N}$ there is some $\delta > 0$ such that if

$x \in N(z, \delta)$ then $|f(x) - f(z)| < \epsilon = \frac{1}{3N}$.
Hence, for the chosen points $x_\delta, y_\delta \in N(z, \delta)$ we have $|f(x_\delta) - f(z)| < \frac{1}{3N}$ and $|f(y_\delta) - f(z)| < \frac{1}{3N}$. Then

$$|f(x_\delta) - f(y_\delta)| \leq |f(x_\delta) - f(z)| + |f(z) - f(y_\delta)|$$
$$< \frac{1}{3N} + \frac{1}{3N} = \frac{2}{3N} < \frac{1}{N} .$$

But this is impossible since x_δ and y_δ were chosen to be points such that $|f(x_\delta) - f(y_\delta)| \geq \frac{1}{N}$. Therefore $z \in D$.

On the other hand, suppose $z \in D$. Then for some $\epsilon_0 > 0$ and every $\delta > 0$, $|f(x) - f(z)| > \epsilon_0$ for some $x \in N(z, \delta)$. Choose some positive integer $N \geq \frac{1}{\epsilon_0}$, then $\epsilon_0 \geq \frac{1}{N}$ and $|f(x) - f(z)| > \epsilon_0 \geq \frac{1}{N}$ for every $\delta > 0$ and some $x \in N(z, \delta)$. Therefore $z \in D_N$ and hence $z \in \bigcup_{n=1}^{\infty} D_n$.

<u>Proof of (2)</u>: For any positive integer N, suppose z is a limit point of D_N . Then for any $\delta > 0$ there is a point $a \neq z$ such that $a \in N(z, \delta)$ and $a \in D_N$. Choose $\epsilon > 0$ such that $0 < \epsilon < \delta - d(z,a)$. Then since $a \in D_N$ there are two points x and y such that $x,y \in N(a, \epsilon)$ and $|f(x) - f(y)| \geq \frac{1}{N}$. But $x,y \in N(z, \delta)$ since $N(a, \epsilon) \subset N(z, \delta)$. Therefore $z \in D_N$ since $\delta > 0$ was arbitrarily chosen, and each set D_n is closed.

Therefore D is a Borel set of type F_1 and $C = C(D)$ is a Borel set of type G_1 by theorem 2.3 .

The following definition and theorem will be needed for the proof of theorem 5.9.

Definition: Given $f(x) = \lim_{n\to\infty} f_n(x)$ on a metric space A, the sequence $\{f_n(x)\}$ is uniformly convergent at a single point $a \in A$ if for every $\varepsilon > 0$ there exists a $\delta > 0$ and some positive integer N such that if $x \in N(a, \delta)$ then $|f_N(x) - f(x)| \leq \varepsilon$.

Theorem 5.8: Given $f(x) = \lim_{n\to\infty} f_n(x)$ on a metric space A where for each n, $f_n(x)$ is continuous at $a \in A$, then $f(x)$ is continuous at a if and only if the sequence $\{f_n(x)\}$ is uniformly convergent at $a \in A$.

Proof: Suppose $f(x)$ is continuous at $a \in A$. Then for every $\varepsilon > 0$ there exists a $\delta_1 > 0$ such that $|f(x) - f(a)| \leq \varepsilon$ if $x \in N(a, \delta_1)$. Since $f(a) = \lim_{n\to\infty} f_n(a)$, there exists a positive integer N such that $|f_n(a) - f(a)| \leq \varepsilon$ for all $n \geq N$. Since each $f_n(x)$ is continuous at a, for $n = N$ there exists a $\delta_2 > 0$ such that $|f_N(x) - f_N(a)| \leq \varepsilon$ if $x \in N(a, \delta_2)$. Choose $\delta = \min[\delta_1, \delta_2]$; then for $x \in N(a, \delta)$,

$$|f_N(x) - f(x)| \leq |f_N(x) - f_N(a)| + |f_N(a) - f(a)| + |f(a) - f(x)|$$
$$\leq \varepsilon + \varepsilon + \varepsilon = 3\varepsilon .$$

Therefore the sequence $\{f_n(x)\}$ is uniformly convergent at $a \in A$.

On the other hand, suppose $\{f_n(x)\}$ is uniformly convergent at $x = a$. Then, by definition, for every $\varepsilon > 0$ there exists a $\delta_1 > 0$ and some positive integer N such that $|f_N(x) - f(x)| \leq \varepsilon$ if $x \in N(a, \delta_1)$. Since each $f_n(x)$ is continuous at $x = a$, for $n = N$ there exists a $\delta_2 > 0$ such that $|f_N(x) - f_N(a)| \leq \varepsilon$ if $x \in N(a, \delta_2)$. Choose $\delta = \min[\delta_1, \delta_2]$, then for $x \in N(a, \delta)$

$$|f(x) - f(a)| \leq |f(x) - f_N(x)| + |f_N(x) - f_N(a)| + |f_N(a) - f(a)|$$
$$\leq \varepsilon + \varepsilon + \varepsilon = 3\varepsilon .$$

Therefore $f(x)$ is continuous at $a \in A$.

Theorem 5.9: If $f(x)$ is a Baire function of type f_1 on a complete metric space A, the set of its points of continuity is dense in A.

Proof: Suppose $f(x)$ is a Baire function of type f_1 on a complete metric space A, then $f(x) = \lim_{n \to \infty} f_n(x)$ where for each n, $f_n(x)$ is continuous on A. Suppose H is any nonempty open set in A.

We now define for each m, a set $G(\frac{1}{m})$ by the condition that $a \in G(\frac{1}{m})$ if and only if $a \in H$ and there exists a $\delta > 0$ and some positive integer N such that if $x \in [N(a, \delta) \cap H]$ then $|f_N(x) - f(x)| \leq \frac{1}{m}$.

Let E be the set of all points in H at which

$f(x)$ is continuous relative to H. Then by theorem 5.8 E is the set of all points in H at which the sequence $\{f_n(x)\}$ is uniformly convergent relative to H. Moreover, $E = \bigcap_{n=1}^{\infty} G(\frac{1}{n})$ by definition of the sets $G(\frac{1}{m})$. Also, for each n, $G(\frac{1}{n})$ is open in H, for suppose $a \in G(\frac{1}{m})$. Then there is a $\delta > 0$ and some $N > 0$ such that if $x \in [N(a, \delta) \cap H]$ then $|f_N(x) - f(x)| \leq \frac{1}{m}$. Consider $b \in [N(a, \delta) \cap H]$ and choose $\delta' = \delta - d(a,b)$. Then $|f_N(x) - f(x)| \leq \frac{1}{m}$ for all $x \in [N(b, \delta') \cap H]$ since $[N(b, \delta') \cap H] \subset [N(a, \delta) \cap H]$. Hence if $b \in [N(a, \delta) \cap H]$, then $b \in G(\frac{1}{m})$, i.e. $[N(a, \delta) \cap H] \subset G(\frac{1}{m})$. Therefore $G(\frac{1}{n})$ is open in H for all values of n.

Since each set $G(\frac{1}{n})$ is open in H, $G(\frac{1}{n}) = H \cap R_n$ where each set R_n is open in A. Since H is open in A, each $G(\frac{1}{n})$ as the intersection of two open sets is also open in A.

We now define for each m, a set $K(\frac{1}{m})$ by the condition that $a \in K(\frac{1}{m})$ if and only if there is some $\delta > 0$ and some positive integer N such that if $x \in N(a, \delta)$ then $|f_N(x) - f(x)| \leq \frac{1}{m}$.

Let S be the set of all points in A at which $f(x)$ is continuous relative to A. Then by theorem 5.8, S is also the set of all points in A at which the sequence $\{f_n(x)\}$ is uniformly convergent. Therefore $S = \bigcap_{n=1}^{\infty} K(\frac{1}{n})$ where each set $K(\frac{1}{n})$ is open in A. Each set $K(\frac{1}{n})$ is seen to be open in A in the same way that $G(\frac{1}{n})$ was seen to be open in H.

We now have the following two relationships to show:

(1) $K(\frac{1}{n}) \cap H = G(\frac{1}{n})$, and

(2) $G(\frac{1}{n}) \neq \emptyset$ for all positive integers n.

<u>Proof of (1)</u>: Suppose first that $a \in [K(\frac{1}{m}) \cap H]$. Then $a \in K(\frac{1}{m})$ which implies there is some $\delta > 0$ and $N > 0$ such that if $x \in N(a, \delta)$ then $|f_N(x) - f(x)| \leq \frac{1}{m}$. If $x \in [N(a, \delta) \cap H]$, then $x \in N(a, \delta)$ and $|f_N(x) - f(x)| \leq \frac{1}{m}$. Hence since $a \in H$, $a \in G(\frac{1}{m})$.

On the other hand, suppose $a \in G(\frac{1}{m})$, then $a \in H$ and there is some $\delta_0 > 0$ and $N > 0$ such that if $x \in [N(a, \delta_0) \cap H]$ then $|f_N(x) - f(x)| \leq \frac{1}{m}$. Also, since H is open in A, for $a \in H$ there is some $\delta_1 > 0$ such that $N(a, \delta_1) \subset H$. Choose $\delta = \min [\delta_0, \delta_1]$. Then $|f_N(x) - f(x)| \leq \frac{1}{m}$ if $x \in N(a, \delta)$. Therefore $a \in K(\frac{1}{m})$ and $a \in [K(\frac{1}{m}) \cap H]$.

Therefore $[K(\frac{1}{n}) \cap H] = G(\frac{1}{n})$ for all n.

<u>Proof of (2)</u>: To show $G(\frac{1}{n}) \neq \emptyset$ for all n, define sets $F_m(\frac{1}{k})$ by requiring that $x \in F_m(\frac{1}{k})$ if and only if $x \in H$ and $|f_m(x) - f_n(x)| \leq \frac{1}{k}$ for all $n \geq m$.

Suppose $a \in H$. Since $f(x) = \lim_{n\to\infty} f_n(x)$ on H, for $\varepsilon = \frac{1}{2k}$ there is some $m > 0$ such that if $n \geq m$ then $|f_n(a) - f(a)| \leq \varepsilon = \frac{1}{2k}$. Hence $|f_m(a) - f_n(a)| \leq |f_m(a) - f(a)| + |f(a) - f_n(a)| \leq 2\varepsilon = \frac{1}{k}$, for all $n \geq m$. Therefore $a \in H$ implies that $a \in F_m(\frac{1}{k})$. Therefore $H = \bigcup_{n=1}^{\infty} F_n(\frac{1}{k})$.

But each function $f_n(x)$ is continuous on H relative to H since each function $f_n(x)$ is continuous on A. Hence for $x \in H$ it follows from theorems 4.4 and 4.3 that $|f_m(x) - f_n(x)|$ is a continuous function on H relative to H, for all values of n and m. Therefore by theorem 4.10 (c) each set $F_{m,n}(\frac{1}{k}) = E\,[\,|f_m(x) - f_n(x)| \leq \frac{1}{k}\,] \cap H$ is closed in H, and $F_m(\frac{1}{k}) = \bigcap_{n=m}^{\infty} F_{m,n}(\frac{1}{k})$ is closed relative to H.

Since H is a nonempty set of type G_1 in A, H is of the second category relative to itself, by Baire's theorem 2.11. Therefore there is some positive integer t such that $F_t(\frac{1}{k})$ is not nowhere dense relative to H.

Since $F_t(\frac{1}{k})$ is both closed and not nowhere dense in H, $F_t(\frac{1}{k})$ contains a nonempty neighborhood relative to H, i.e. for some $b \in H$ and some $\delta > 0$,

$[N(b, \delta) \cap H] \subset F_t(\frac{1}{k})$. If $x \in [N(b, \delta) \cap H]$ then $x \in F_t(\frac{1}{k})$ and $|f_t(x) - f_n(x)| \leq \frac{1}{k}$ for all

$n \geq t$. Therefore $|f_t(x) - f(x)| \leq \frac{1}{k}$ for all $x \in [N(b, \delta) \cap H]$ since $f(x) = \lim_{n\to\infty} f_n(x)$ on H. Therefore $b \in G(\frac{1}{k})$ and $G(\frac{1}{n}) \neq \emptyset$ for all n.

Since $K(\frac{1}{n}) \cap H = G(\frac{1}{n}) \neq \emptyset$, for each n, and since H is an arbitrary open set in A, each set $K(\frac{1}{n})$ is dense in A. Therefore set S is dense in A by theorem 2.12, since $S = \bigcap_{n=1}^{\infty} K(\frac{1}{n})$ is a Borel set of type G_1, where each set $K(\frac{1}{n})$ is open and dense in A.

It is of interest to note that theorem 5.9 is also true if the metric space A is merely <u>locally complete</u>, i.e. if for each point $p \in A$ there is an open set G containing p, $G \subset A$, such that $\overline{G}$ is a complete metric space.

<u>Theorem 5.10</u>: For any ordinal $\alpha < \Omega$, if f(x) is a Baire function of type f_α on a complete metric space A, there is a Borel set S of type G_1, dense and of the second category relative to A, whose complement C(S) is of the first category relative to A, such that f(x) is continuous on S relative to S.

<u>Proof</u>: Suppose f(x) is a Baire function of type f_α on a complete metric space A, $\alpha < \Omega$. If $\alpha = 0$, then f(x) is continuous on A. Then S = A, where A is of type G_1, dense, and of the second category relative to itself, and $C(S) = \emptyset$

is of the first category relative to A. Assume the theorem is true for all ordinals $\beta < \alpha$.

If $f(x)$ is a Baire function of type f_α on A, then $f(x) = \lim_{n\to\infty} f_n(x)$ where for each n, $f_n(x)$ is of type f_{α_n}, $\alpha_n < \alpha$. By our induction assumption, each $f_n(x)$ is continuous on some set C_n relative to C_n, where each C_n is of type G_1, dense, and of the second category relative to A, and where each set $D_n = A - C_n$ is of the first category relative to A. Also, since each set C_n is of type G_1 in A and each set $D_n = C(C_n)$, then each D_n is of type F_1 in A by theorem 2.3.

Consider $D_o = \bigcup_{n=1}^{\infty} D_n$; then D_o is of the first category relative to A, and, by theorem 2.4, is a Borel set of type F_1. Let $C_o = A - D_o$. Then $C_o = C(\bigcup_{n=1}^{\infty} D_n) = \bigcap_{n=1}^{\infty} C_n$. Therefore C_o is a Borel set of type G_1 and dense in A by theorem 2.13.

We have $C_o \subset C_n$ for each n since $C_o = \bigcap_{n=1}^{\infty} C_n$. Therefore each function $f_n(x)$ is continuous on C_o relative to C_o, and $f(x) = \lim_{n\to\infty} f_n(x)$ on C_o.

Let S be the set of points in C_o at which $f(x)$ is continuous relative to C_o. By theorem 5.7, S is a Borel set of type G_1 in C_o, which implies $S = \bigcap_{n=1}^{\infty} K_n$, where for each n, K_n is open in C_o. This means that for each K_n there is some set M_n open in A such that $K_n = C_o \cap M_n$. Therefore

$S = \bigcap_{n=1}^{\infty}(C_o \cap M_n) = C_o \cap M$, where $M = \bigcap_{n=1}^{\infty} M_n$ is a Borel set of type G_1 in A. Since C_o is a Borel set of type G_1 in A, S is a Borel set of type G_1 in A by theorem 2.5.

Suppose $H \subset C_o$ is any nonempty set open in C_o; then $H = C_o \cap Q$ where Q is some open set in A. Since Q is of type G_1 in A, by theorem 2.1, and C_o is of type G_1 in A, H is a Borel set of type G_1 in A by theorem 2.5. Also, since $H \subset C_o$, $f(x) = \lim_{n\to\infty} f_n(x)$ on H since $f(x) = \lim_{n\to\infty} f_n(x)$ on C_o.

We now define for each m, a set $G(\frac{1}{m})$ by the condition that $a \in G(\frac{1}{m})$ if and only if $a \in H$ and there exists a $\delta > 0$ and some positive integer N such that if $x \in [N(a, \delta) \cap H]$ then $|f_N(x) - f(x)| \leq \frac{1}{m}$.

Let E be the set of all points in H at which $f(x)$ is continuous relative to H. Then, by theorem 5.8, E is the set of all points in H at which the sequence $\{f_n(x)\}$ is uniformly convergent relative to H. Moreover, $E = \bigcap_{n=1}^{\infty} G(\frac{1}{n})$ by definition of the sets $G(\frac{1}{m})$. Also, for each n, $G(\frac{1}{n})$ is open in H. (This is shown in exactly the same way as in the proof of theorem 5.9.)

Since each set $G(\frac{1}{n})$ is open in H, $G(\frac{1}{n}) = H \cap R_n$ where each set R_n is open in A. Since each set R_n is of type G_1 in A, by theorem 2.1, and H is of type G_1 in A, each set $G(\frac{1}{n})$ is of type G_1 in A by theorem 2.5 . We now define for each m, a set $K(\frac{1}{m})$ by the condition that $a \in K(\frac{1}{m})$ if and only if $a \in C_o$ and there is some $\delta > 0$ and some positive integer N such that if $x \in [N(a, \delta) \cap C_o]$ then $|f_N(x) - f(x)| \leq \frac{1}{m}$.

Since S is the set of all points in C_o at which f(x) is continuous relative to C_o , again by the use of theorem 5.8 , we have that S is also the set of all points in C_o at which the sequence $\{f_n(x)\}$ is uniformly convergent relative to C_o. Therefore $S = \bigcap_{n=1}^{\infty} K(\frac{1}{n})$ where each set $K(\frac{1}{n})$ is open in C_o . Each set $K(\frac{1}{n})$ is seen to be open in C_o in the same way that $G(\frac{1}{n})$ was seen to be open in H . Since $K(\frac{1}{n})$ is open in C_o , $K(\frac{1}{n}) = C_o \cap S_n$ where each set S_n is open in A. Since each set S_n is of type G_1 in A, by theorem 2.1, and C_o is of type G_1 in A, then each set $K(\frac{1}{n})$ is of type G_1 in A by theorem 2.5.

We now have the following two relationships to show:

(1) $K(\frac{1}{n}) \cap H = G(\frac{1}{n})$, and

(2) $G(\frac{1}{n}) \neq \emptyset$ for all positive integers n.

Proof of (1): Suppose first that $a \in [K(\frac{1}{m}) \cap H]$. Then $a \in K(\frac{1}{m})$ which implies there is some $\delta > 0$

and $N > 0$ such that if $x \in [N(a, \delta) \cap C_o]$ then $|f_N(x) - f(x)| \leq \frac{1}{m}$. Also, $a \in H$ and $[N(a, \delta) \cap H] \subset [N(a, \delta) \cap C_o]$ since $H \subset C_o$. Therefore, if $x \in [N(a, \delta) \cap H]$, then $x \in [N(a, \delta) \cap C_o]$ and $|f_N(x) - f(x)| \leq \frac{1}{m}$. Therefore $a \in G(\frac{1}{m})$.

On the other hand suppose $a \in G(\frac{1}{m})$, then $a \in H$ and there is some $\delta_o > 0$ and $N > 0$ such that if $x \in [N(a, \delta_o) \cap H]$ then $|f_N(x) - f(x)| \leq \frac{1}{m}$. Also, since H is open in C_o, for $a \in H$ there is some $\delta_1 > 0$ such that $[N(a, \delta_1) \cap C_o] \subset H$. Choose $\delta = \min [\delta_o, \delta_1]$. If $x \in [N(a, \delta) \cap C_o]$, then $x \in [N(a, \delta_1) \cap C_o] \subset H$ and $x \in [N(a, \delta_o) \cap H]$. Then $|f_N(x) - f(x)| \leq \frac{1}{m}$ where $a \in H \subset C_o$, if $x \in [N(a, \delta) \cap C_o]$. Therefore $a \in K(\frac{1}{m})$ and $a \in [K(\frac{1}{m}) \cap H]$.

Therefore $[K(\frac{1}{n}) \cap H] = G(\frac{1}{n})$ for all n.

Proof of (2): To show $G(\frac{1}{n}) \neq \emptyset$ for all n, we proceed exactly the same as we did in the same part of the proof of theorem 5.9, with the only difference being that each function $f_n(x)$ is continuous on H relative to H since each function $f_n(x)$ is continuous on C_o relative to C_o. This is because $H \subset C_o$.

Since $K(\frac{1}{n}) \cap H = G(\frac{1}{n}) \neq \emptyset$, for each n, and since H is an arbitrary open set in C_o, each set

$K(\frac{1}{n})$ is dense in C_o . But C_o is dense in A, therefore each set $K(\frac{1}{n})$ is dense in A.

Thus set S is of type G_1 and dense in A by theorem 2.13 , since $S = \bigcap_{n=1}^{\infty} K(\frac{1}{n})$, where each set $K(\frac{1}{n})$ is of type G_1 and dense in A.

By theorem 2.14 , S is of the second category relative to A and C(S) is of the first category relative to A . Since f(x) is continuous on S relative to C_o , where $S \subset C_o$, then f(x) is continuous on S relative to S .

Therefore the theorem is true for all $\alpha < \Omega$ by transfinite induction.

Theorem 5.11: If the set of points of discontinuity of f(x) defined on a metric space A is countable, then f(x) is a Baire function of type f_1 on A .

Proof: Let D be the set of all points where f(x) is discontinuous on a metric space A, and suppose D is a countable set. Let C be the set of all points where f(x) is continuous on A; then $C = C(D) = A - D$.

Since any countable set is a Borel set of type F_1, D is a Borel set of type F_1. (This is because each point of a countable set may be thought of as being a one-point set, which is closed. Then any countable set is the union of a countable number of closed sets, and therefore a Borel set of type F_1 .)

For any real number k, consider $B = B_1 \cup B_2$ where $B = E[f(x) > k]$, $B_1 = B \cap D$, and $B_2 = B \cap C$. Suppose $a \in B_2$; then $f(a) > k$ and $a \in C$. But $a \in C$ implies that $f(x)$ is continuous at $x = a$. Therefore, let $\varepsilon = f(a) - k > 0$; then there is some $\delta > 0$ such that if $x \in N(a, \delta)$ then $|f(x) - f(a)| < \varepsilon$, i.e. $f(x) > f(a) - \varepsilon = k$ for all $x \in N(a, \delta)$. Therefore $N(a, \delta) \subset B$. Let $H = \bigcup_{a \in B_2} N(a, \delta)$. Then H is open in A and $B_2 \subset H \subset B$.

Now consider $B = H \cup (B - H)$. Since $B_2 \subset H \subset B$, then $(B - H) \subset (B - B_2) = B_1$. Since $B_1 \subset D$, then $(B-H) \subset D$. But D is a countable set, therefore $(B-H)$ is at most countable and hence a Borel set of type F_1. Also, H, being an open set in A, is of type G_0, and hence by theorem 2.7 is of type F_1.

Therefore, by theorem 2.5, $B = H \cup (B-H)$ as the union of two sets of type F_1 is also of type F_1 for any real number k, i.e. $E[f(x) > k]$ is a Borel set of type F_1 for every real number k.

Likewise, a similar argument shows $E[f(x) < k]$ is a Borel set of type F_1 for every real number k. Therefore its complement, $E[f(x) \geq k]$, is a Borel set of type G_1 for every real number k, by theorem

2.3 .

Hence $f(x)$ is a Baire function of type f_1 by theorem 5.6.

Theorem 5.12: If B is a countable set in a metric space A, and $f(x)$ defined on A is continuous on $A-B$ relative to $A-B$, then $f(x)$ is a Baire function of type f_2 on A.

Proof: If B is a countable set, it is a Borel set of type F_1. Suppose $f(x)$ is continuous on $A-B$ relative to $A-B$, where A is some metric space.

For any real number k, consider $M = M_1 \cup M_2$ where $M = E[f(x) > k]$, $M_1 = M \cap B$, and $M_2 = M \cap (A-B)$. The set M_2 is open relative to $A-B$, by theorem 4.10 (c), since $f(x)$ is continuous on $A-B$ relative to $A-B$. Hence $M_2 = (A-B) \cap H$ for some open set H in A. Since B is of type F_1, $A-B$ is of type G_1 by theorem 2.3. Also since H is open in A, H is of type G_0, and hence of type G_1. Therefore by theorem 2.5, $M_2 = (A-B) \cap H$ as the intersection of two Borel sets of type G_1 is also of type G_1, and hence of type G_2.

Since $M_1 \subset B$, and B is a countable set, M_1 is at most countable and hence a Borel set of type F_1. Therefore by theorem 2.7, M_1 is a Borel set of type G_2.

Therefore $M = E\,[f(x) > k]$ is a Borel set of type G_2, by theorem 2.5, since M is the union of two sets of type G_2. A similar argument shows $E\,[f(x) < k]$ is a Borel set of type G_2 for every real number k. Therefore its complement, $E\,[f(x) \geq k]$, is a Borel set of type F_2 for every real number k, by theorem 2.3.

Hence $f(x)$ is a Baire function of type f_2, by theorem 5.6.

It is of interest to note the following example of a function satisfying the conditions of the last theorem, which is therefore a Baire function of type f_2, but is not of type f_1.

Choose the space A to be the set of all real numbers. Define $f(x) = 0$ if x is an irrational number, and $f(x) = 1$ if x is a rational number. Then $f(x)$ is discontinuous at every point of A. Moreover, $f(x)$ is continuous on the irrationals relative to the irrationals, where the rationals constitute a countable set. Therefore by the last theorem, $f(x)$ is a Baire function of type f_2 on A. If $f(x)$ were also of type f_1, its points of continuity would be dense in A by theorem 5.9. But $f(x)$ is discontinuous everywhere on A, hence $f(x)$ is not of type f_1 on A.

BIBLIOGRAPHY

1. Goffman, Casper. *Real Functions*. New York: Rinehart, 1953.

2. Sierpinski, Waclaw. *General Topology*. Translated by C. Cecilia Krieger. Toronto: University of Toronto Press, 1952.

3. Kamke, E. *Theory of Sets*. New York: Dover Publications, Inc., 1950.

4. Hausdorff, Felix. *Set Theory*. Translated by John R. Aumann. New York: Chelsea Publishing Company, 1957.