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# Boolean algebras and their topological duals

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# BOOLEAN ALGEBRAS AND THEIR TOPOLOGICAL DUALS

By

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E. T. W.

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NOTATION

| Symbol | Meaning |
|---|---|
| $\varepsilon$ | is an element of |
| $\wedge$ | meet, in a Boolean algebra |
| $\vee$ | join, in a Boolean algebra |
| A' or p' | set theoretic or Boolean complement of A or p |
| $P^-$ | the closure of P |
| P | the complement of the closure of P |
| $\cup$ | set theoretic union |
| $\cap$ | set theoretic intersection |
| $\emptyset$ | the empty set |
| $\subset$ | is a subset of |
| $p \leq q$ | q dominates p |

# CHAPTER I

## INTRODUCTION AND PRELIMINARIES

In 1847, an English elementary school teacher named George Boole published a thin volume, entitled The Mathematical Analysis of Logic, in which he gave a mathematical formulation of the laws of logic. His book aroused the admiration of August de Morgan and Boole was soon appointed Professor of Mathematics at Queen's University in Cork. Six years later he published a more comprehensive work, giving a more detailed development as well as many important applications. The algebraic system that grew out of this formulation has come to be called Boolean algebra.

This system also arises quite naturally in other ways. One of these is in lattice theory, a second is in ring theory. In this paper some of the theoretical aspects of Boolean algebras are developed after first giving a formulation of these algebras in ring-theoretic terms.

The year 1936 seems to have been a watershed in the development of the theory of Boolean algebras; for it was in that year that M. H. Stone, in his famous paper [5], opened a new era by showing that every Boolean algebra could be represented by a field of sets. Since

that time the work of Halmos, Sikorski, Tarski and Stone himself, among others, has elucidated many dark corners of this field.

This paper begins with the exposition of certain ideas and facts which are necessary to the development of the subsequent chapters. Chapter II provides a development of what has come to be called "the Stone representation theorem" from the algebraic standpoint. Ring Theory per se is not used. A dual representation theorem, relating topological spaces to Boolean algebras, is then proved and the setting is provided for Chapter III which examines this topological duality in more detail.

Applications of Boolean algebras to other branches of mathematics are not considered. For a glimpse at such applications the reader is referred to Sikorski's five volume [4]. Nor are applications to such things as electrical circuit theory and logic touched upon. For the reader who is interested in the role of Boolean algebras in applied mathematics Whitesitt's book would be very helpful [7].

Before discussing the general theory of Boolean algebras, it is well to recall certain facts. The following definitions, theorems, and other discussion are germane to the development of the material of the following chapters.

Definition 0.1. A ring is a set $\{R, +, \cdot\}$ in which R is a collection of elements, and + and · are binary operations such that

(i) $\{R, +\}$ is an abelian group

(ii) $\{R, \cdot\}$ is an abelian group

(iii) $a\cdot(b+c) = a\cdot b + a\cdot c$ for every $a, b, c \in R$

Henceforth a ring will be denoted by the capital letter naming its set of elements.

Definition 0.2. A ring R is said to be a ring with unit whenever $\{R, \cdot\}$ Has an identity element, 1.

Definition 0.3. A Boolean ring is defined to be a ring R with unit, such that every element is idempotent; that is $x^2 = x$ for every $x \in R$.

The existence of Boolean rings is easy to establish. The ring of integers, modulo 2, which shall hereafter be denoted merely by the symbol 2, is clearly idempotent and has a unit.

A slightly more sophisticated example of a Boolean ring is $2^X$, the set of all functions from any non-empty set X into 2, where 0 and 1 in $2^X$ are the functions defined by

$$0(x) = 0 \text{ for every } x \in X$$

$$1(x) = 1 \text{ for every } x \in X;$$

and the operations are defined pointwise.

Definition 0.4. A ring R is said to have characteristic

2 if $x + x = 0$ for every $x \in R$.

Theorem 0.1. Every Boolean ring R has characteristic 2 and is commutative.

Proof: $(x+y) = (x+y)^2 = x^2 + xy + yx + y^2$

$= x + xy + yx + y$. Thus

$0 = xy + yx$. Letting $x = y$,

$0 = x^2 + x^2 = x + x$. Thus R has characteristic 2.

Since $0 = xy + yx$, then $-(xy) = yx$. But R has characteristic 2, which implies that $x = -x$ for every $x \in R$. Therefore, $-(xy) = xy = yx$ and thus R is commutative.

Definition 0.5. Three operations called meet, join, and complement, and denoted by $\wedge$, $\vee$, and $'$ respectively, can be defined on a Boolean ring R as follows:

(i) $a \wedge b = ab$ for every a and b in R

(ii) $a \vee b = a + b + ab$ for every a and b in R

(iii) $a' = 1 + a$ for every $a \in R$.

It is easy to show that, given expressions between elements of a Boolean ring involving only the usual ring operations, these same expressions can be rewritten using meet join and complement. In fact,

$ab = a \wedge b$

$a + b = (a \wedge b') \vee (a' \wedge b)$.

If the defining conditions of a Boolean ring are

expressed in terms of meet, join, and complement, a large set of conditions results. From this set a subset, larger than necessary to derive the remaining ones, can be extracted. This subset of conditions follows.

(1) $0' = 1$ $\qquad$ $1' = 0$

(2) $a \wedge 0 = 0$ $\qquad$ $a \vee 1 = 1$

(3) $a \wedge 1 = a$ $\qquad$ $a \vee 0 = a$

(4) $a \wedge a' = 0$ $\qquad$ $a \vee a' = 1$

(5) $a'' = a$

(6) $a \wedge a = a$ $\qquad$ $a \vee a = a$

(7) $(a \wedge b)' = a' \vee b'$ $\qquad$ $(a \vee b)' = a' \wedge b'$

(8) $a \wedge b = b \wedge a$ $\qquad$ $a \vee b = b \vee a$

(9) $a \wedge (b \wedge c) = (a \wedge b) \wedge c$ $\qquad$ $(a \vee b) \vee c = (a \vee b) \vee c$

(10) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ $\qquad$ $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

There are many subsets of the above 10 conditions which will imply all of the others. The independent subset cited most often in the literature is (3), (4), (8), and (10). Implicit in (3) and (4) is the existence of the distinguished elements 0 and 1.

Now the object of study in this paper can be defined.

Definition 0.6. A Boolean algebra is a set A together with distinguished elements 0 and 1 (distinct), two binary operations $\wedge$ and $\vee$, and a unary operation $'$, satisfying the identities (1) through (10), given

above. Clearly then, every Boolean ring can be considered a Boolean algebra and vice-versa.

Principle of Duality: If in the identities (1)-(10), 0 and 1 are interchanged and if at the same time $\wedge$ and $\vee$ are interchanged those identities are merely permuted among themselves. Thus the same is true for all the statements derived from (1)-(10). As a consequence, it is sufficient to state and prove only half of the theorems. For example the absorptive law is proven as follows:

$$\begin{aligned} a &= 1 \wedge a \\ &= (1 \vee b) \wedge a \\ &= (1 \wedge a) \vee (b \wedge a) \\ &= a \vee (a \wedge b) \text{ and dually} \\ a &= a \wedge (a \vee b). \end{aligned}$$

Since $\wedge$ and $\vee$ are associative, it makes sense to write $p_1 \wedge p_2 \wedge \cdots \wedge p_n$ which will hereafter be denoted by $\bigwedge_{i=1}^{n} p_i$, or in cases where no confusion is possible, simply $\bigwedge_i p_i$. Analogously $p_1 \vee p_2 \vee \cdots \vee p_n$ may be written $\bigvee_{i=1}^{n} p_i$ or $\bigvee_i p_i$.

Definition 0.7. Let A be a class of subsets of a fixed space X such that A is closed with respect to the set-theoretic operations of finite union and intersection as well as complementation. Then A is said to be a field of sets. Clearly every field of sets is a Boolean algebra.

Definition 0.8. A field of sets $\mathcal{F}$ is said to be separating if, for every pair of distinct points x and y, there exists a set P belonging to $\mathcal{F}$ such that $x \in P$ and $y \in P'$.

Definition 0.9. Let X be a set. $P \subset X$ is said to be cofinite whenever P', relative to X, is finite.

It is easy to verify that the class of subsets of a nonempty set X, which are either finite or cofinite is a field of sets.

Definition 0.10. Let $(X, \mathcal{J})$ be a topological space. An open set $P \in \mathcal{J}$ is called regular open whenever P is the interior of its own closure.

Since the interior of P may be written as $P'^{-'}$, P is regular if and only if $P = P^{-'-'}$. Hereafter $P^{-'}$ shall be denoted by $P^{\perp}$. Thus P is regular if and only if $P^{\perp} = P^{\perp}$.

Under suitable definitions of the Boolean operations and the distinguished elements, the class of all regular open sets of a non-empty topological space becomes a Boolean algebra. This algebra will be defined and put to use in Chapter III.

It turns out that regular open algebras are not the only areas of discussion in this paper that require some of the basic tools of the topologist's trade. A few of the notions most often used are given below.

Definition 0.11. A set P in a topological space is said to be nowhere dense if and only if the interior of its closure is empty.

Definition 0.12. A collection of subsets of a topological space is said to be a covering of a set A whenever A is contained in the union of that collection.

Definition 0.13. A topological space is compact if and only if each open cover has a finite subcover. A subset A of a topological space X is compact if and only if every covering by sets which are open in X has a finite subcovering.

Definition 0.14. A topological space is a $T_1$-space if and only if each set which consists of a single point is closed.

Definition 0.15. A topological space is regular if for each point x and each closed set A such that $x \notin A$, there are disjoint open sets U and V such that $x \in U$ and $A \subset V$.

Definition 0.16. A topological space X is said to be a Hausdorff space if and only if for every pair of distinct points, x and y, belonging to X, there exist disjoint open sets A and B such that $x \in A$ and $y \in B$. Obviously then, a regular $T_1$-space is Hausdorff.

Theorem 0.2. Each compact subset A of a Hausdorff

space X is closed.

Proof: Let $x \in A'$. For every $y \in A$ pick open sets $M_y(x)$, $N(y)$ such that $x \in M_y(x)$, $y \in N(y)$ and $M_y(x) \cap N(y) = \emptyset$. $\{N(y)\}_{y \in A}$ is an open cover for A. Thus there exists an open subcover $\{N(y_i)\}_{i=1}^{n}$. Then for each $N(y_i)$ pick the corresponding $M_{y_i}(x)$ such that $M_{y_i}(x) \cap N(y_i) = \emptyset$. Thus $[\bigcap_{i=1}^{n} M_{y_i}(x)] \cap [\bigcup_{i=1}^{n} N(y_i)] = \emptyset$ and $\bigcap_{i=1}^{n} M_{y_i}(x)$ is open. Therefore $[\bigcap_{i=1}^{n} M_{y_i}(x)] \cap A = \emptyset$. For each $x \in A'$ there exists such an open set containing it, and disjoint from A. Clearly, A' is the union of all such open sets. Hence $A = (A')'$ is closed.

Definition 0.17. A map from a topological space X into a topological space Y is continuous if and only if the inverse image of each open set is open.

The following well known and useful result is stated without proof. [2, p. 86]

Theorem 0.3. If X and Y are topological spaces, and if f is a function on X to Y, then the following statements are equivalent.

(i) The function f is continuous

(ii) The inverse image of each closed set is closed

(iii) The inverse image of each member of a subbase for the topology for Y is open.

(iv) For each $x \in X$ the inverse image of every

neighborhood of f(x) is a neighborhood of x.

(v) For each $x \in X$ and each neighborhood U of f(x) there is a neighborhood V of x such that $f(V) \subset U$.

Definition 0.18. A homeomorphism is a continuous one-to-one map f of a topological space X onto a topological space Y such that $f^{-1}$ is also continuous.

Using Theorem 0.2. the following useful result can be obtained [2, p. 14].

Theorem 0.4. Let f be a continuous function carrying the compact topological space X onto the topological space Y. Then Y is compact and if Y is Hausdorff and f is one to one, then f is a homeomorphism.

Definition 0.19.1. Suppose $X_0, X_1, \ldots, X_{n-1}$ are topological spaces. A base for the product topology for the cartesian product $X_0 \times X_1 \times \ldots \times X_{n-1}$ is the family of all products $U_0 \times U_1 \times \ldots \times U_{n-1}$ where each $U_i$ is open in $X_i$.

Such a definition, although sufficient for finite products, is inadequate for arbitrary products. Let $\{X_a\}_{a \in A}$ be an arbitrary family of topological spaces. Their cartesian product denoted by $X\{X_a\}_{a \in A}$, is the set of all functions x on A such that $x_a \in X_a$ for each $a \in A$. The projection map of the product into the a-th coordinate space is given by $P_a(x) = x_a$.

Definition 0.19.2. Let U be an open subset of $X_a$. Consider $P_a^{-1}[U]$. The family of all sets of this form is defined to be a subbase for the product topology for the cartesian product.

Definition 0.20. Let X be a topological space and let $Y \subset X$. U belongs to the relative topology for Y if and only if $U = V \cap Y$ where V is open in X.

Some of the most important results of the theory of Boolean algebras rely heavily on Zorn's lemma, or on another statement equivalent to the axiom of choice.

Zorn's lemma. If each chain in a partially ordered set has an upper bound, then there is a maximal element in the set.

Axiom of Choice. Let C be any collection of nonempty sets. Then there is a function F defined on C which assigns to each set A belonging to C an element F(A) in A.

In the context of the foregoing definitions the Axiom of Choice implies that the Cartesian product of a nonempty collection of nonempty sets is nonempty.

# CHAPTER II

## THE STONE REPRESENTATION THEOREM

Before discussing the famous Stone representation theorem it is well to develop some contiguous matter which elucidates much of the general theory.

Definition 1.1. A Boolean homomorphism is a mapping f from a Boolean algebra B, say, to a Boolean algebra A, such that

(1) $f(p \vee q) = f(p) \vee f(q)$,

(2) $f(p \wedge q) = f(p) \wedge f(q)$,

(3) $f(p') = (f(p))'$

whenever p and q are in B.

Special kinds of Boolean homomorphisms, namely monomorphisms, espmorphisms, and isomorphisms are defined in the usual manner.

Note that if f is a Boolean homomorphism, $f(0) = 0$ since $f(0) = f(p \wedge p') = f(p) \wedge f(p')$

$= f(p) \wedge (f(p))' = 0$.

Dually then, $f(1) = 1$ and thus there is no "trival" homomorphism, i.e. a mapping that sends everything onto zero since 0 and 1 are distinct.

The existence of Boolean homomorphisms which are not merely the identity homomorphism is shown in the following example.

Let X be a non-empty set. Let B be a field of subsets of X. Pick any element $x \in X$ and define f mapping B into 2 as follows: for $P \in B$, set

$$f(P) = \begin{cases} 1 \text{ if } x \in P \\ 0 \text{ if } x \notin P. \end{cases}$$

Note that $f(P \cap Q) = \begin{cases} 1 \text{ if } x \in P \cap Q \\ 0 \text{ if } x \notin P \cap Q \end{cases}$ and that

$f(P) \wedge f(Q) = \begin{cases} 1 \text{ if } x \in P \cap Q \\ 0 \text{ if } x \notin P \cap Q \end{cases}$ and thus $f(P \cap Q) = f(P) \quad f(Q)$.

Similarly, $f(P \cup Q) = f(P) \quad f(Q)$ and $f(P') = (f(P))'$.

In some cases condition (3) of definition 1.1 is difficult to verify. In such cases the following theorem is often useful.

Theorem 1.1. A mapping f between Boolean algebras which preserves 0, 1, $\wedge$ , and $\vee$ is a Boolean homomorphism.

Proof: Since f preserves $\wedge$ and 0, and since $(p \wedge p') = 0$ it is clear that

$$f(p) \wedge f(p') = f(p \wedge p') = f(0) = 0.$$

And likewise since $p \vee p' = 1$,

$$f(p) \vee f(p') = f(p \vee p') = f(1) = 1.$$

Thus $f(p') = (f(p))'$.

Note that a mapping that preserves either $\wedge$ and ' or $\vee$ and ' is also a Boolean homomorphism since, for example,

if, for all p, q, $f(p \vee q) = f(p)\ \ f(q)$ and $f(p') = (f(p))'$, then

$$f(p \wedge q) = f((p \wedge q)')') = (f(p \wedge q)')'$$
$$= (f(p' \vee q'))' = (f(p') \vee f(q'))'$$
$$= (f(p'))' \wedge (f(q'))' = f(p) \wedge f(q).$$

The existence of Boolean homomorphisms naturally motivates the following definitions.

Definition 1.2. If f is a Boolean homomorphism from B to A, the kernel of f is the set of all elements in B that f maps onto O in A.

Definition 1.3. A Boolean ideal in a Boolean algebra B is a subset M of B such that

(1) $O \in M$

(2) if $p \in M$ and $q \in M$, then $p \vee q \in M$,

(3) if $p \in M$ and $q \in B$, then $p \wedge q \in M$.

Condition (3) can be changed to an equivalent condition that is often more useful, if an order relation $\leq$, motivated by the concept of set inclusion, is introduced into every Boolean algebra A. Consider first the following simple result, where $p \in A$ and $q \in A$.

Theorem 1.2. $p\ \ q = p$ if and only if $p\ \ q = q$.

Proof: $p \wedge q = p \Rightarrow (p \wedge q) \vee q = p \vee q \Rightarrow p \vee q = q$ by the law of absorption. Interchanging $\vee$ and $\wedge$, and forming duals gives the converse.

Definition 1.4. Let p and q belong to Boolean

algebra A. Then, it is said that q dominates p (denoted by $p \leq q$) whenever $p \wedge q = p$, or, equivalently, whenever $p \vee q = q$.

It is easy to show that $\leq$ is a partial order, i.e., it is reflexive, antisymmetric, and transitive. Continuing to work in a fixed but arbitrary Boolean algebra A, the following theorem follows trivially.

Theorem 1.3. Let p, q, r, s be elements of A. Then

(1) $0 \leq p$ and $p \leq 1$.

(2) If $p \leq q$ and $r \leq s$, then $p \vee r \leq q \vee s$ and $p \wedge r \leq q \wedge s$.

(3) If $p \leq q$, then $q' \leq p'$.

(4) $p \leq q$ if and only if $p-q = 0$ where $p-q$ is defined to be $p \wedge q' \in A$.

Note that (4) implies that a Boolean homomorphism is order preserving since

$$p \leq q \Rightarrow p-q = 0 \Rightarrow f(p \wedge q') = 0$$
$$\Rightarrow f(p) \wedge f(q') = 0 \Rightarrow f(p) \wedge (f(q))' = 0$$
$$\Rightarrow f(p) - f(q) = 0 \Rightarrow f(p) \leq f(q).$$

Now condition (3) of definition 1.3. can be replaced by

(3') $p \in M$, $q \leq p \Rightarrow q \in M$,

for if $p \in M$ and $q \in B$, then $p \wedge q \in M$ and since $q \leq p \Rightarrow p \wedge q = q$ it follows that $q \in M$.

Definition 1.5. An ideal N of a Boolean algebra B is proper if $N \neq B$. A Boolean ideal is maximal if it

is a proper ideal that is not properly included in any other proper ideal.

The following theorem characterizes maximal ideals.

Theorem 1.4. An ideal M in a Boolean algebra B is maximal if and only if either $p \in M$ or $p' \in M$ but not both, for each $p \in B$.

Proof: Assume that for some $p_o \in B$, neither $p_o \in M$ nor $p_o' \in M$ (It will be shown that this implies M is not maximal.) Let N be the set of all elements of the form $p \vee q$ where $p \leq p_o$ and $q \in M$. N is an ideal since,

(1) $0 \leq p_o$, $0 \in M \Rightarrow 0 \vee 0 = 0 \in N$,

(2) $p_1 \vee q_1 \in N$, $p_2 \vee q_2 \in N \Rightarrow (p_1 \vee q_1) \vee (p_2 \vee q_2)$ $= (p_1 \vee p_2) \vee (q_1 \vee q_2) \in N$, since $p_1 \vee p_2 \leq p_o$ and $q_1 \vee q_2 \in M$, and

(3) $p \vee q \in N$, $r \in B \Rightarrow (p \vee q) \wedge r = (p \wedge r) \vee (q \wedge r) \in N$ since $p \wedge r \leq p \leq p_o$ and $q \in M$, $r \in B \Rightarrow q \wedge r \in M$.

Since $0 \leq p_o$, $M \subset N$. Also $p_o \in N$ and therefore $M \neq N$. It remains only to prove that N is properly contained in B. To prove this, assume $p_o' \in N$. Then, $p_o' = p \vee q$ for some $p \leq p_o$, $q \in M$, so that $p' \wedge p_o' = p' \wedge (p \vee q)$. But $p \leq p_o \Rightarrow p_o' \leq p'$ (by theorem 1.3). Therefore $p' \wedge p_o' = p_o'$ and thus $p_o' = p' \wedge (p \vee q) = (p' \wedge p) \vee (p' \wedge q) = 0 \vee (p' \wedge q)$. But $q \in M$, $p' \in B \Rightarrow p' \wedge q \in M$. Thus $p_o' \in M$ which contradicts the original assumption. Thus $p_o' \notin N$ and hence M is not maximal.

To prove the converse assume that always either p or $p' \in M$, and suppose N is an ideal properly containing M. It follows that N is not proper; for if $N \neq M$, there exists $p \in N - M$. Then $p' \in M$ so that $p' \in N$. But N is an ideal. Therefore $p \vee p' = 1$ is in N. Thus $N = B$.

It is helpful to notice, and not difficult to prove, that a subset M of a Boolean algebra B is a Boolean ideal if and only if it is an ideal in the Boolean ring. This fact makes the following theorem routine to prove by appealing to ring theory and using the usual canonical mapping.

Theorem 1.5. Every proper ideal is the kernel of some epimorphism.

Proof: Consider B as a Boolean ring. Let M be a proper ideal. Define the natural map $h:B \longrightarrow B/M$ by $h(b) = b + M$ for every $b \in B$. Thus $h^{-1}(0) = M$.

The representation theorem relates Boolean algebras to fields of sets. To get a clear description of these fields and the sets involved a special category of topological spaces is introduced.

Definition 1.6. A simultaneously open and closed subset of a topological space X is called a clopen set.

Definition 1.7. A compact Hausdorff space is said to be totally disconnected if every open set is the union of those clopen sets which it happens to include; i.e.,

the clopen sets form a base.

Definition 1.8. A Boolean space is a totally disconnected compact Hausdorff space.

It is obvious that the collection of all clopen sets of any topological space X constitute a field. Earlier (Chapter I) the algebraic Principle of Duality was formulated. Now the notion of a topological type of duality is introduced.

Definition 1.9. The field of all clopen sets in a Boolean space X is called the dual algebra of X.

Consider 2 with the discrete topology. Let I be an arbitrary set and let $2^I = \{x \mid x:I \longrightarrow 2\}$ (equivalently the cartesian product of as many copies of 2 as there are elements of I). $2^I$ with its product topology is Hausdorff and compact, by the well known theorem of Tychonoff [2, p. 143). The sets of the form $U_{i,\delta} = \{x \in 2^I : x(i) = \delta\}$ where $i \in I$ and $\delta \in 2$ constitute a subbase, i.e., finite intersections of them constitute a base. Note that the complements of sets of this form are also sets of this form and thus each such set is clopen. Clearly then $2^I$ is a Boolean space.

Definition 1.10. The Boolean space $2^I$, where I is an arbitrary set and 2 has the discrete topology is called a Cantor space.

As will be seen shortly, Cantor spaces give rise to

other Boolean spaces which can be used to shed light on various aspects of the theory of the topological duality mentioned above, as well as having a direct application in the proof of the representation theorem. First, however, note the following result, which is a useful tool in the study of Boolean spaces.

Theorem 1.6. If X is a compact Hausdorff space, and if A is a separating field of clopen subsets of X, then X is a Boolean space, and A is the field of all clopen subsets of X.

Proof: The proof of this theorem is dependent upon the fact that since A separates points, it also separates points and closed sets. To verify this assertion, let F be closed in X, $F \neq X$, and suppose $x_o \notin F$. A separates points in X, so that for each point $y \in F$, there exist disjoint clopen sets Cy, $y \in Cy$, and Dy, $x_o \in Dy$. The collection $\{Cy\}_{y \in F}$ covers F so that by compactness there exists a finite subcover made up of members of $\{Cy\}_{y \in F}$. The union C of this subcollection contains F and, being a finite union, is clopen. Each member Cy of this subcollection corresponds to some Dy and $Cy \cap Dy = \emptyset$. Therefore, if the intersection D of the finite collection of Dy's corresponding to the Cy's in the subcover of F is considered, D is clearly clopen and disjoint from C. Thus A separates points and closed

sets.

Now since every open set is the complement of a closed one, the preceding paragraph implies that A is a base for X, and thus X is Boolean. Let G be an arbitrary clopen set in X. Since G is both compact and open it is a finite union of members of A. But A is closed under finite unions. Hence G is in A.

An obvious corollary follows.

Corollary 1.6.1. If a field of clopen sets of a compact Hausdorff space is a base, then the space is Boolean and the field contains all the clopen sets.

It has been shown (in the section following definition 1.1.) that if a Boolean algebra A is a field of subsets of a set X, the points of X define 2-valued homomorphisms on A. Therefore, if A is to be represented as the dual algebra of some Boolean space X it is reasonable to search for points from which to construct X among the 2-valued homomorphisms on A. That there is an overabundance of such points follows from the following theorem.

Theorem 1.7. For every non-zero element p of every Boolean algebra A, there exists a 2-valued homomorphism x on A such that $x(p) = 1$.

Proof: Let $\{A\sigma\}_{\sigma\varepsilon\Gamma}$ be a chain of proper ideals containing p'. That such a chain exists follows from the

fact that $\{q \in A \mid q \leq p'\}$ is an ideal. Let $A = \bigcup_{\sigma \in \Gamma} A_\sigma$ since $1 \notin A_\sigma$ for each $\sigma$, it follows that A is a proper ideal and an upper bound for the above chain. Therefore, by Zorn's lemma, there exists a maximal element M containing $p'$. Clearly $p \notin M$ for then $p' \vee p = 1 \in M$ and M would not be proper. Thus for every $p \in A$, there exists a maximal ideal that does not contain p. Hence there exists a mapping x from A to 2 defined by

$$x(q) = \begin{cases} 1 \text{ if } q \notin M \\ 0 \text{ if } q \in M \end{cases}$$

It is routine to show that x is a homomorphism.

Theorem 1.8. Let A be a Boolean algebra. Then M is a maximal ideal of A if and only if M is the kernel of a 2-valued homomorphism.

Proof: Let M be a maximal ideal and define $x: A \longrightarrow 2$ as in the previous theorem. Obviously M is the kernel of x.

Conversely let $M = \{p \in A : x(p) = 0\}$ where x is a 2-valued homomorphism on A. If this is not a maximal ideal then there exists a maximal ideal $M^*$ properly containing M. This implies the existence of an element $q \in M^*$ such that $x(q) = 1$. But then $x(q') = 0$ so that $q' \in M \subset M^*$ while, by theorem 1.4, $q' \notin M^*$, which is a contradiction. Therefore M is a maximal ideal.

Theorem 1.9. The set X of all 2-valued homomorphisms on a Boolean algebra A is a closed subset of $2^A$.

Proof: Let p be a fixed element of A. Let $x \in X$. Then $x(p) \in 2$. By the definition of the topology for $2^A$, the set $Q = \{ y \in 2^A \mid y(p) = x(p) \}$ is a basic open set. Considering p as a function of x, it turns out that p is continuous. To show this, note that $p: 2^A \longrightarrow 2$. Consider P an open subset of 2 and recall that 2 has the discrete topology. If P = 2 or P = $\emptyset$ the inverse image of P under p is clearly open and hence p is continuous. If $P = \{ \delta \}$, $\delta \in 2$, the inverse image of P under p is $\{ x \in 2^A \mid x(p) = \delta \}$ which is a set of the same form as Q, and therefore open. Hence p is continuous.

Therefore, since the set of points where two continuous functions are equal is always closed, the set $P = \{ x \mid x(p') = (x(p))' \}$ is closed in $2^A$. Thus $\bigcap_{p \in A} P$ is closed, and hence the set of 2-valued functions on A that preserve complementation is closed. A similar argument involving such sets as $\{ x \mid x(p \vee q) = x(p) \vee x(q) \}$ justifies the same conclusion for join-preserving functions. Since the intersection of these two sets of operation preserving 2-valued functions is the set of 2-valued homomorphisms X, it is clear that X is a closed subset of $2^A$.

As the following theorem indicates, X, the set of all 2-valued homomorphisms on A takes on the structure of a Boolean space in a natural way. First the following definition is made.

Definition 1.11. The set X of all 2-valued homomorphisms on a Boolean algebra A is called the dual space of A.

Theorem 1.10. Every closed subset Y of a Boolean space X is a Boolean space with respect to the relative topology. Also, every clopen set in Y is the intersection of Y with some clopen set of X.

Proof: (i) If the clopen sets form a base for X, their intersections with Y do the same for Y. Then by Corollary 1.6.1, Y is a Boolean space.

(ii) If Q is clopen in Y, then Q is open in Y and Q is closed in Y. Therefore there exists an open set U in X, such that $Q = Y \cap U$. Also there exists a closed set F in X such that $Q = Y \cap F$. Thus Q, as a subset of X, is closed and hence compact. The clopen subsets of U in X cover Q and thus, by the compactness of Q as a subset of X, there exists a finite collection of clopen subsets of U whose union, P say, covers Q. Then $Q \subset P \subset U$ and, as we have seen, $Y \cap U = Q$. Hence $Y \cap P = Q$ where P is clopen.

The path to the Stone representation theorem has

now been cleared. Its statement and proof follow.

Theorem 1.11. Let A be a Boolean algebra. If B is the dual algebra of the dual space X of A, then A is isomorphic to B.

Proof: Define $f:A \longrightarrow B$ by $f(p) = \{ x \in X : x(p) = 1 \}$. Note that $x(p)$ continuous in $X \Rightarrow f(p)$ clopen for every $p \in A$, and thus f maps A into B. The verification that f is a homomorphism is routine.

$$f(p \vee q) = \{ x : x(p \vee q) = 1 \} = \{ x : x(p) \vee x(q) = 1 \}$$
$$= \{ x : x(p) = 1 \} \cup \{ x : x(q) = 1 \} = f(p) \cup f(q)$$

$$\text{also } f(p') = \{ x : x(p') = 1 \} = \{ x : (x(p))' = 1 \}$$
$$= \{ x : x(p) = 0 \} = (f(p))'.$$

To show f is a monomorphism assume $f(p) = \{ x : x(p) = 1 \} = \emptyset$. If $p \neq 0$, theorem 1.7. tells us that $f(p) \neq \emptyset$. Hence $p = 0$. To show f is an epimorphism, note that since the range of a Boolean homomorphism is always a Boolean algebra, the clopen sets of the form $\{ x : x(p) = 1 \}$ constitute a field. Since two distinct homomorphisms must disagree on some element of A, the field is separating and thus by theorem 1.6. this field is B. Hence f maps A onto B.

Corollary 1.11.1. Every Boolean algebra is isomorphic to a field of sets.

The isomorphism of the previous theorem existing between two algebras suggests the following theorem concerning the topological equivalence between certain Boolean

spaces. This theorem has been called the dual representation theorem [2].

Theorem 1.12. Let X be a Boolean space. If Y is the dual space of the dual algebra A of X, and if $\varphi(x)$ is the 2-valued homomorphism that sends each element P of A onto 1 or 0, according as $x \in P$ or $x \notin P$, then is a homeomorphism from X onto Y.

Proof: According to theorem 0.4. it is only necessary to show that $\varphi$ is continuous, onto and one-to-one. Since the dual space is a Boolean space, $\varphi$ can be shown to be continuous by demonstrating that the inverse image under $\varphi$ of every clopen set in Y is clopen in X. Note that a subbasic clopen subset of Y is of the form $\{ y:y(P) = 1, P \in A\}$ (see preceding theorem). The inverse image under $\varphi$ this set is precisely P, which is clopen. To show $\varphi$ is onto, first note that if the clopen set $\{ y:y(P) = 1\}$ is non-empty so is its inverse image, since if not, there would exist a 2-valued homomorphism h such that $h(\emptyset) = 1$, which is impossible for homomorphisms. Now since the clopen sets form a base for Y, and since the inverse images of the non-empty ones of these are never empty, the range of $\varphi$ is dense; i.e., $(\varphi(X))^- = Y$. Therefore if $\varphi(X) = (\varphi(X))^-$ it follows that $\varphi$ is onto. To show that this is indeed the case, consider $\{\theta_\alpha\}_{\alpha \in \Gamma}$ an open

covering for $\varphi(X)$. Then $\varphi(X) \subset \bigcup_{\alpha\in\Gamma} \Theta_\alpha$, and $X \subset \varphi^{-1}(\bigcup_{\alpha\in\Gamma} \Theta_\alpha) = \bigcup_{\alpha\in\Gamma} \varphi^{-1}(\Theta_\alpha)$. But $\varphi$ is continuous so that $\varphi^{-1}(\Theta_\alpha)$ is open for each $\alpha \in \Gamma$. Therefore $\bigcup_{\alpha\in\Gamma} \varphi^{-1}(\Theta_\alpha)$ is an open covering for X. But X is compact so that there exists an open subcovering $\{\varphi^{-1}(\Theta_i)\}_{i=1}^{n}$ and $X \subset \bigcup_{i=1}^{n} \varphi^{-1}(\Theta_i) = \varphi^{-1}(\bigcup_{i=1}^{n}\Theta_i)$; and $\varphi(X) \subset \varphi(\varphi^{-1}(\bigcup_{i=1}^{n} \Theta_i))$ or $\varphi(X) \subset \bigcup_{i=1}^{n} \Theta_i$ and hence $\varphi(X)$ is compact. Then by theorem 0.2. $\varphi(X)$ is also closed, and therefore $\varphi(X) = (\varphi(X))^{-} = Y$.

Since X is a Boolean space, the clopen sets separate points. Thus distinct points of X will determine distinct 2-valued homomorphisms (points of Y) on A. Therefore is one-to-one.

# CHAPTER III

## DUALITY AND BOOLEAN SPACES

The notion of topological duality introduced in the last chapter suggests the possibility of gaining information about Boolean algebras through the study of Boolean spaces, and vice-versa. As Stone has shown [6], it is possible to dualize each Boolean algebraic concept. This dualization provides many helpful insights into the general theory of Boolean algebras. In this chapter some of the more far-reaching results of such dualization are examined. Before this examination proceeds, however, a few more fundamental facts and concepts about Boolean algebras must be introduced.

Definition 2.1. An element q of a Boolean algebra A is an upper bound of a subset E of that Boolean algebra whenever $p \leq q$ for every $p \in E$. If there exists an element $q_o$ in the set F of upper bounds for E, such that $q_o \leq q$ for every $q \in F$, $q_o$ is called the supremum of E. The lower bound and infimum of E are defined analogously.

Theorem 2.1. For each p and q in a Boolean algebra A, the set $\{ p, q \}$ has supremum $p \vee q$ and infimum $p \wedge q$.

Proof: $p \vee (p \vee q) = p \vee q$. Therefore $p \leq p \vee q$. Likewise $q \vee (p \vee q) = p \vee q$ so that $q \leq p \vee q$. Hence $p \vee q$ is an upper bound for $\{ p, q \}$. To show that $p \vee q$

is the least upper bound, i.e. the supremum, let r be any other upper bound of $\{p, q\}$. Then by theorem 1.3

$$p \leq r,\ q \leq r \Rightarrow p \vee q \leq r \vee r = r,$$

a dual argument shows that $p \wedge q$ is the infimum. This theorem generalizes to arbitrary finite subsets of A, and the infimum and supremum are denoted by $\bigwedge E$ and $\bigvee E$ respectively. In the infinite case the infima or suprema may not exist. For example, consider the finite-cofinite algebra of integers and observe that the collection of singletons of all even integers has no supremum since in this case it would contain the set-theoretic union which is the set of all even integers and there is no minimal cofinite set with this property. In general, if $\{p_i\}_{i \in \Gamma}$ is an arbitrary collection of elements of a Boolean algebra, the notation $\bigvee_i p_i$ is used for the supremum and $\bigwedge_i p_i$ is used for the infimum wherever they exist.

Definition 2.2. A Boolean algebra with the property that every subset has both an infimum and a supremum is called a complete Boolean algebra. The simplest example of a complete Boolean algebra is the field of all subsets of a set. A more intricate example of a complete Boolean algebra is given by the following two theorems.

Theorem 2.2. The class A of all regular open sets of a non-empty topological space X is a Boolean algebra

with respect to the distinguished elements and Boolean operations defined by

(1) $0 = \emptyset$

(2) $1 = X$

(3) $P \wedge Q = P \cap Q$

(4) $P \vee Q = (P \cup Q)^{\perp\perp}$

(5) $P' = P^{\perp}$

Proof: There are two parts to the proof. It must be shown that (a) the right sides of (1) - (5) are regular open sets, and (b) the Boolean axioms are satisfied by this definition. The proof employs several lemmas which are important in their own right. (a) The fact that $\emptyset$ and X are regular open sets is obvious.

Lemma 2.2.1. $P \subset Q \Rightarrow Q^{\perp} \subset P^{\perp}$.

Proof: Closure preserves set inclusion while complementation reverses it.

Lemma 2.2.2. If P is open, then $P \subset P^{\perp\perp}$.

Proof: Let P be an open set. Since $P \subset P^{-}$, taking complements gives $P^{\perp} \subset P'$. But P' is closed so that $P^{\perp -} \subset P'$. Taking complements again, the desired result is obtained, i.e., $P \subset P^{\perp\perp}$.

Lemma 2.2.3. If P is open, then $P^{\perp} = P^{\perp\perp\perp}$.

Proof: By lemma 2.2.2. $P \subset P^{\perp\perp}$. Therefore, by lemma 2.2.1 $P^{\perp\perp\perp} \subset P^{\perp}$. Also by lemma 2.2.2., if P is substituted for $P^{\perp}$, then $P^{\perp} \subset P^{\perp\perp\perp}$. Thus $P^{\perp} = P^{\perp\perp\perp}$.

Lemma 2.2.3 verifies that the right side of (5) is a regular open set, since $P^{\perp} = (P^{\perp})^{\perp\perp}$ implies that $P^{\perp}$ is regular. That the right side of (4) is regular open can be shown by noting that $(P \cup Q)^{\perp\perp} = [(P \cup Q)^{\perp}]^{\perp} = [(P \cup Q)^{\perp}]^{\perp\perp\perp} = [(P \cup Q)^{\perp\perp}]^{\perp\perp}$.

To demonstrate that $P \cap Q$ is regular the following lemma is employed.

Lemma 2.2.4. If P and Q are open, then $(P \cap Q)^{\perp\perp} = P^{\perp\perp} \cap Q^{\perp\perp}$.

Proof: Since $P \cap Q \subset P$ and $P \cap Q \subset Q$, lemma 2.2.1 implies that $(P \cap Q)^{\perp\perp} \subset P^{\perp\perp}$ and $(P \cap Q)^{\perp\perp} \subset Q^{\perp\perp}$. Thus $P \cap Q \subset P^{\perp\perp} \cap Q^{\perp\perp}$. To show the reverse inclusion note that since P is open $P \cap Q^{-} \subset (P \cap Q)^{-}$. Applying complementation, $(P \cap Q)^{\perp} \subset P' \cup Q^{\perp}$ so that since $P'$ is closed $(P \cap Q)^{\perp -} \subset P' \cup Q^{\perp -}$. Therefore,

$$(*) \qquad P \cap Q^{\perp\perp} \subset (P \cap Q)^{\perp\perp}.$$

Since $P^{\perp\perp}$ is also open it follows that P may be replaced by $P^{\perp\perp}$ in (*) so that $P^{\perp\perp} \cap Q^{\perp\perp} \subset (P^{\perp\perp} \cap Q)^{\perp\perp} = (Q \cap P^{\perp\perp})^{\perp\perp}$. But by (*), $(Q \cap P^{\perp\perp})^{\perp\perp} \subset (Q \cap P)^{\perp\perp\perp\perp}$. However since lemma 2.2.3 implies that $(Q \cap P)^{\perp\perp\perp\perp} = (P \cap Q)^{\perp\perp}$, it follows that $P^{\perp\perp} \cap Q^{\perp\perp} \subset (P \cap Q)^{\perp\perp}$. This lemma implies directly that the intersection of two regular open sets is regular and hence the right side of (3) is a regular open set.

(b) The verification of the Boolean axioms is trivial except for $p \vee p' = 1$, i.e., $(P \cup P^{\perp})^{\perp\perp} = X$. To verify this, the following lemma may be employed.

Lemma 2.2.5. The boundary of an open set is a nowhere dense closed set.

Proof: Let P be open. The boundary of $P = \bar{P} \cap \bar{P'}$. Since it is the intersection of two closed sets, it is obviously closed. To show $\bar{P} \cap \bar{P'}$ is nowhere dense assume there exists an open set Q such that $Q \subset \bar{P} \cap \bar{P'}$, $Q \neq \emptyset$. Then $Q \cap \bar{P} \neq \emptyset$. But since Q is contained in the boundary of P, $Q \cap P = \emptyset$. But this contradicts the definition of closure ($x \in \bar{P}$ if and only if every open set containing x intersects P). Therefore $Q = \emptyset$ and the boundary of P is nowhere dense.

Now it can easily be shown that $(P \cup P^{\perp})^{\perp\perp} = X$. Since the boundary of P is a nowhere dense closed set, $(\bar{P} \cap \bar{P'})' = P^{\perp} \cup P'^{\perp} = P^{\perp} \cup P$ is a dense open set. Therefore $(P \cup P^{\perp})^{-} = X$, which implies $(P \cup P^{\perp})^{\perp} = \emptyset$ finally, $(P \cup P^{\perp})^{\perp\perp} = \emptyset = X$.

Theorem 2.3. The regular open algebra of a topological space is a complete Boolean algebra. The supremum and infimum of a family $\{P_i\}$ of regular open sets are $(\cup_i P_i)^{\perp\perp}$ and $(\cap_i P_i)^{\perp\perp}$ respectively.

Proof: Let $(\cup_i P_i)^{\perp\perp} = P$. Since for each i, $P_i \subset \cup_i P_i$, lemma 2.2.2 implies that $P_i \subset P$. Thus P is

an upper bound. To show it is the supremum, let Q be any other upper bound. Then $P_i \subset Q$ for each i. Thus $U_i P_i \subset Q$ which by lemma 2.2.1. implies that $(U_i P_i)^{\perp\perp} \subset Q^{\perp\perp} = Q$. With the aid of the following lemma, which comprises infinite versions of the DeMorgan laws, the assertion about infima proceeds dually.

Lemma 2.3.1. If $\{p_i\}$ is a family of elements in a Boolean algebra then,

$(\bigvee_i p_i)' = \bigwedge_i p_i'$ and $(\bigwedge_i p_i)' = \bigvee_i p_i'$.

These equations are to be interpreted in the sense that if either term in either equation exists, then so does the other term of that equation, and the two terms are equal.

Proof: Let $p = \bigvee_i p_i$. Then since $p_i \leq p$ for every i, it follows that $p' \leq p_i'$ for every i. $\therefore p' \leq \bigwedge_i p_i'$. Let $q = \bigwedge_i p_i'$. Then $q \leq p_i'$ for each i and hence $p_i \leq q'$ for each i which by the definition of supremum implies that $p \leq q'$, and hence $q \leq p'$ or $\bigwedge_i p_i' \leq p'$. The second equality follows from a dual argument.

A useful fact, which follows directly from lemma 2.31. is the following:

Corollary 2.3.1. If every subset of a Boolean algebra has a supremum (or else if every subset has an infimum), then the algebra is complete.

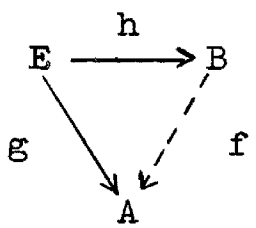
Regular open algebras are often useful in the verification

of conjectures concerning completeness. Two further notions useful in the consideration of topological duality are the concepts of free Boolean algebras and atomicity.

Definition 2.3. A subset E of a Boolean algebra A is said to generate A, if A is the smallest algebra that contains E.

Definition 2.4. The set E of generators of a Boolean algebra B is called free if every mapping from E to an arbitrary Boolean algebra A can be extended to a homomorphism on B.

This means that for every mapping g, such that $g:E \longrightarrow A$ there exists a homomorphism f such that $f(p) = g(p)$ for every $p \in E$. The following commutative diagram illustrates the situation. h is the identity mapping.




$$(f \circ h)p = g(p)$$

The phrases "E freely generates B" and "B is free on E," are used in this setting. It is easy to show that f is uniquely determined by h and g.

To show that such objects actually exists, let $E = \emptyset$. If $B = 2$, define f by $f(0) = 0$, $f(1) = 1$. Clearly 2 is freely generated by $\emptyset$. A less trivial

example is considered toward the end of this chapter.

Definition 2.5. An element q of a Boolean algebra A is said to be an atom of A if $q \neq 0$ and if there are only two elements p such that $p \leq q$, namely 0 and q. An example of an atom is a singleton in a field of sets.

Definition 2.6. A Boolean algebra A is atomic if every non-zero element dominates at least one atom. A is said to be non-atomic if it has no atoms.

Theorem 2.4. In an atomic algebra every element is the supremum of the elements it dominates.

Proof: Clearly each element p is an upper bound of the atoms it dominates. Let r be any upper bound of these atoms. To show $p \leq r$ assume it is not. Then $p - r \neq 0$ and, since the algebra is atomic, there exists an atom q such that $q \leq p - r$. But $p - r \leq p$. Thus $q \leq r$. But then $q = q \wedge r \leq (p-r)\ r = 0$. Thus $q \leq 0$ and hence q is not an atom, a contradiction. Therefore the assumption is false and $p \leq r$.

Theorem 2.5. Every finite Boolean algebra A is atomic.

Proof: Let $p \in A$. Assume p is not an atom. Consider the maximal descending chain from p to 0. Since this chain must be finite, there is a least non-zero element q such that $0 \leq q \leq p$. Clearly q is an atom, and A is atomic.

It has been shown that every Boolean algebra is isomorphic to a field of sets. One might wonder whether such an isomorphism preserves completeness. Surprisingly enough it doesn't in general. The following result is helpful in searching for a counter-example.

Theorem 2.6. A complete field of sets is atomic.

Proof: Let A be a nonempty member of a complete field of sets. Let $x_o \in A$. Let $\{B_i\}$ be the collection of all members of the field which contain $x_o$. Completeness implies that $A_o = \bigcap_i B_i$ exists. Assume $A_o$ is not an atom. Then there exists non-empty $A_1 \subset A_o$, $A_1 \neq A_o$. Since $x_o \notin A_1$, $x_o \in A_1'$. Therefore $x_o \in A_o \cap A_1'$, which is properly contained in $A_o$, contrary to the definition of $A_o$. Hence $A_o$ is an atom. Clearly $A_o \subset A$, and thus the field is atomic.

Taking up the search for a counter-example, note that since an isomorphism between two Boolean algebras is order-preserving and one-to-one, atomicity is obviously preserved. Thus a complete Boolean algebra which is not atomic will, as a result of theorem 2.6., be isomorphic to a non-complete field of sets. Therefore the search for a counter-example reduces to a search for a complete Boolean algebra which is not atomic. As has been shown, the regular open algebra of a topological space is complete.

If the real line, with the usual topology, is chosen for that space, the fact that any open interval is regular and dominates no atom indicates that such a regular open algebra is not atomic. In fact it is non-atomic.

Returning to a consideration of topological duality note the following result.

Theorem 2.7. A Boolean algebra A is finite if and only if it is the dual of a discrete space X.

Proof: Suppose A is finite. Let $x \in X$. Let $y \in X - \{x\}$. Then there exists $U_y \in A$ such that $x \in U_y$, $y \in U_y'$. Clearly, $\bigcap_{y \in X} U_y = \{x\}$. But since A is finite there are only finitely many distinct $U_y$'s and thus $\{x\}$ is open. Therefore X is discrete. Conversely is X is discrete compactness implies that X is finite and the dual algebra is easily seen to be finite also.

The dual concept for atomicity is not quite so obvious.

Theorem 2.8. Let X be a Boolean space and A its dual algebra. Then A is atomic if and only if the isolated points of X are dense.

Proof: Recall that an isolated point of X is a point $x \in X$ such that $\{x\}$ is open. An atom of A is a nonempty clopen set in X and, being the supremum of the

atoms it dominates, it is a singleton and thus an isolated point in X. Since the clopen sets form a base for X, then A is atomic if and only if the isolated points are dense.

Theorem 2.9. A is non-atomic if and only if its dual space X has no isolated points.

Proof: As was shown in the previous theorem, every atom in A is an isolated point in X. Therefore if A is not non-atomic X has at least one isolated point. Conversely, assume A is non-atomic and that X has an isolated point x, say, then $\{ x \}$ would be clopen. Thus $\{x\} \in A$ and has a copious supply of sub-elements. In X, $\{x\}$ is the union of these sub-elements as clopen sets, and thus can't be a singleton.

The dual space of a finite algebra has been shown to be a discrete space. Next the dual space of a countable algebra is examined.

Theorem 2.10. A dual algebra A of a Boolean space X is countable if and only if X is metrizable.

Proof: The proof makes use of the well known metrization theorem due to Urysohn which is stated as a lemma.

Lemma 2.10.1. A regular $T_1$ space X has a countable base if and only if it is metrizable and separable. [2, p. 125.]

Since a Boolean space X is compact and Hausdorff it follows that it is regular and $T_1$. Therefore since the dual algebra may be considered a basis for X half of the theorem follows directly from lemma 2.10.1. To prove the converse, assume that X is metrizable. If it can be shown that X is also separable it will follow from the lemma that X has a countable base.

To show X is separable, let n be a positive integer. Then for each $x \in X$, there exists an open neighborhood of radius less than $\frac{1}{m}$ containing x. Since this collection of open neighborhoods covers X, and since X is compact, there exists a finite subcollection of these open neighborhoods which covers X. From each neighborhood in this subcollection pick a point and call this set of points $S_n$. Thus for every positive integer n there exists a finite set $S_n$ such that $d(x, S_n) < \frac{1}{n}$ for every $x \in X$. Let $S = \bigcup_{n=1}^{\infty} S_n$. Then S is countable. Pick any $x \in X$. Let $\varepsilon > 0$. Then there exists a positive integer N such that $n \geq N$ implies that $\frac{1}{n} \leq \varepsilon$. Hence there exists a $y_n \in S_n$ such that $d(x, y_n) < \varepsilon$. Thus $x \in S^-$ so that $X = S^-$ and X is separable. Therefore, by the lemma, X has a countable base.

If X has a countable base, then every base contains a countable subclass which is itself a base. Therefore, since the dual algebra is a base it contains

a countable subclass E which itself is a base. Consider B, the gield of sets generated by E. Clearly $B \subset A$. But then by Corollary 1.6.1., $B = A$. If it can be shown that B is countable, the theorem will have been proven. The following lemma provides such a result.

Lemma 2.10.2. If E is a countable class of sets, then B, the field of sets generated by E, is countable.

Proof: For any class of sets C, let $C^*$ denote the collection of all finite unions of differences of members of C. If C is countable, it is obvious that $C^*$ is countable.

Without loss of generality it may be assumed that E contains the empty set. Let $E_0 = E$, $E_n = E^*_{n-1}$ for $n = 1, 2, \ldots$ . Clearly then $E \subset \bigcup_{n=0}^{\infty} E_n \subset B$, and $\bigcup_{n=0}^{\infty} E_n$, being a countable union of countable sets, is countable. Note that $E_0 \subset E_1 \subset E_2 \subset \ldots$ . Therefore if $P \in \bigcup_{n=0}^{\infty} E_n$ and $Q \in \bigcup_{n=0}^{\infty} E_n$, there exists a positive integer N such that $P \in E_N$ and $Q \in E_N$. Thus $P-Q \in E_{N+1}$ and since $P \cup Q = (P-\emptyset) \cup (Q-\emptyset)$ it follows that $P \cup Q \in E_{N+1}$. Also $\emptyset - Q = Q' \in E_{N+1}$. Thus $P \cap Q = (P' \cup Q')'$. $\in E_{N+1}$. Therefore $\bigcup_{n=0}^{\infty} E_{N+1}$ is closed under finite unions, intersections and complementation and thus is a field of sets. Since B is the smallest field containing E, $\bigcup_{n=0}^{\infty} E_n = B$ and the proof of the lemma is complete.

The duality theory for ideals is an outgrowth of the following definition.

Definition 2.7. If X is a Boolean space with dual algebra A, the dual of an ideal M in A is the union of the clopen sets belonging to M; and the dual of an open subset U of X is the class of all clopen subsets contained in U.

The following theorem follows directly from this definition.

Theorem 2.11. The dual of every ideal is an open set and the dual of every open set is an ideal. If M and N are ideals with duals U and V respectively, then a necessary and sufficient condition that $M \subset N$ is that $U \subset V$.

Definition 2.8. Let $p \in A$, A a Boolean algebra. The ideal $M = \{q : q \in A, q \leq p\}$ is called a principal ideal. p is said to generate M.

Note that if M is a principal ideal $\bigvee M$ is its generator p. Thus the dual of M, being the union of the clopen sets belonging to M, is p, which is clopen. Theorem 2.10 also implies that the dual of a maximal ideal is a maximal open set, i.e., the complement of a singleton. Thus the dualization of the assertion that every proper ideal is contained in a maximal ideal becomes a triviality.

Theorem 2.12. A Boolean algebra A is finite if and only if every ideal M is a principal ideal.

Proof: Assume A is finite. Suppose M is generated by $\{ p_1, p_2, \ldots, p_n \}$. Then $p = \bigvee_{i=1}^{n} p_i$ exists and p generates M. Conversely, assume every ideal is principal. Then the complement of every singleton in the dual space X is the dual of a principal ideal and is therefore clopen. Thus every singleton is open and X is discrete. Then by theorem 2.7. A is finite.

Theorem 1.12. suggests a dual correspondence between 2-valued homomorphisms on Boolean algebras and continuous 2-valued functions on Boolean spaces. The following terminology and notation facilitates the investigation of this correspondence.

Definition 2.9. A pairing of a Boolean algebra A and a Boolean space X is a function that associates with every pair (p, x), where $p \in A$, $x \in X$ an element of 2 in the following manner. Denote the value of the function by $\langle p, x \rangle$ ; then the requirements on the function can be expressed as follows: (1) $\langle p, x \rangle$ is continuous in x, and by suitable choice of p, every 2-valued continuous function on X has this form; (2) $\langle p, x \rangle$ determines a homomorphism in p, and, by a suitable choice of x, every 2-valued homomorphism on A has this form.

Suppose now that A is a Boolean algebra and X is a Boolean space and suppose that $\langle p, x \rangle$ represents all continuous 2-valued functions on X and all 2-valued homomorphisms on A. In addition, suppose B and Y are similarly paired.

Theorem 2.13. There is a one-to-one correspondence between all continuous mappings from X into Y and all homomorphisms f from B into A such that

$$(*) \qquad \langle q, \varphi(x) \rangle = \langle f(q), x \rangle$$

identically for all $q \in B$ and all $x \in X$. Each of and f is called the dual of the other.

Proof: Fix $\varphi$ and consider $\langle q, \varphi(x) \rangle$. As a function of q for fixed x, it corresponds to an element of Y, namely $\varphi(x)$. This yields nothing new. Next consider $\langle q, \varphi(x) \rangle$ as a function of x for fixed q. Note that this is a composite of two continuous functions namely $x \longrightarrow \varphi(x)$ and $y \longrightarrow \langle q, y \rangle$ where $\varphi(x) = y$. Hence it is continuous and 2-valued. Since this is true, it is given by a unique element $p \in A$ so that

$$\langle q, \varphi(x) \rangle = \langle p, x \rangle$$

identically in x. Denoting the passage from q to p by f it is easy to show that f is a Boolean homomorphism.

$$\langle f(q \vee r), x \rangle = \langle q \vee r, \varphi(x) \rangle = \langle q, \varphi(x) \rangle \vee \langle r, \varphi(x) \rangle$$
$$= \langle f(q), x \rangle \vee \langle f(r), x \rangle = \langle f(q) \vee f(r), x \rangle ;$$

so that f preserves $\vee$ . Also, f preserves complementation since,

$$\langle f(q'), x \rangle = \langle q', \varphi(x) \rangle = \langle q, \varphi(x) \rangle ' = \langle f(q), x \rangle '.$$

To make this more intuitive, if the pairings are given by evaluating the characteristic function of the first coordinate at the second coordinate, $(*)$ can be expressed as follows:

$\varphi(x) \in Q$ if and only if $x \in \varphi^{-1}(Q)$.

This means that f is the restriction of $\varphi^{-1}$ to the class of clopen subsets of X.

Now fix f and consider $\langle f(q), x \rangle$ . As a function of x for fixed q, it is an element of A, namely f(q). This yields nothing new. Next consider $\langle f(q), x \rangle$ as a function of q for fixed x. The result is a composite of two homomorphisms: $q \longrightarrow f(q) \longrightarrow \langle f(q), x \rangle$ , and as such is a 2-valued homomorphism on B. Therefore it is given by a unique member of Y, y say, so that

$$\langle f(q), x \rangle = \langle q, y \rangle$$

identically in q. Denote the passage from x to y by $\varphi$ . To show $\varphi$ is continuous, note that the definition of $\varphi$ implies

$$\varphi^{-1}(\{ y: \langle q, y \rangle = 1 \}) = \{ x: \langle f(q), x \rangle = 1 \}$$

Since every clopen subset of Y is given by some $q \in B$, and since the clopen sets form a basis for Y, $\{ y: \langle q, y \rangle = 1 \}$

is a basic set in Y. In theorem 1.11. it was pointed out that the right side of the above equality is clopen. Therefore $\varphi$ is continuous.

The definition of $\varphi$ can be made more intuitive by considering (*) and evaluating the second coordinate at the first coordinate. The result is

$$(\varphi(x))q = x(f(q)).$$

Since this is true for all $q \in B$ it follows that $\varphi(x) = x \circ f$.

To verify that this correspondence is one-to-one let and $\Theta$ both be duals of f. Then $\langle q, \varphi(x) \rangle = \langle f(q), x \rangle = \langle q, \Theta(x) \rangle$. Thus $\varphi = \Theta$. Let f and g be duals of . Then $\langle f(q), x \rangle = \langle q, \varphi(x) \rangle = \langle g(q), x \rangle$ and therefore f = g. And finally the theorem is proved [1, pp. 85-8].

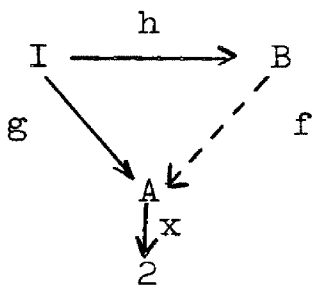
Two important theorems which are helpful in the determination of whether or not a given algebra has a free set of generators follow.

Theorem 2.14. For every set I, the dual algebra of the Cantor space $2^I$ is freely generated by a set of the same power as I.

Proof: Let $Y = 2^I$, and suppose B is the dual algebra of Y. Define a function $h: I \longrightarrow B$ by $h(i) = \{ y \in Y : y_i = 1 \}$. As was pointed out in definition

1.9. such sets are clopen. h is one-to-one for if $h(i) = h(j)$, $i = j$. Therefore $h(I)$ has the same power as I. Since the field generated by $h(I)$ is a base for Y, it follows from corollary 1.6.1 that $h(I)$ generates B. It remains to show that B is free on $h(I)$.

Let A be an arbitrary Boolean algebra. Let g be an arbitrary mapping from I into A. Let X be the dual space of A and for each $x \in X$ write $\varphi(x) = x \circ g$. Then $\varphi(x): I \longrightarrow 2$ or $\varphi(x) \in 2^I$ for every $x \in X$, so that $\varphi$ maps X into Y. Recall that




$$(**) \qquad \varphi^{-1}(h(i)) = \{x : x\ g \in h(i)\} = \{x : (x \circ g)(i) = 1\}.$$

Then since the elements of $h(I)$ and their complements form a subbase for Y, it follows that $\varphi$ is continuous. Then by the previous theorem there exists a unique dual homomorphism from B into A. Therefore $\varphi(x) \in h(i)$ if and only if $x(f(h(i))) = 1$. But by $(**)$, $\varphi(x) \in h(i)$ if and only if $x(g(i)) = 1$ and hence $x(f(h(i))) = x(g(i))$ for all $x \in X$. Thus $f \circ h = g$, or f is the homomorphic extension required for B to be freely generated by I.

Theorem 2.15. Let I be a countably infinite set. Then the Cantor space $2^I$ is homeomorphic to the Cantor middle-third set C.

Proof: Recall that the Cantor middle-third set is the set of all numbers of the form $\sum_{i=1}^{\infty} \frac{C_i}{3^i}$ where $C_i = 0$ or $2$.

Define $f: 2^I \longrightarrow$ by:

$f(a_1, a_2, \ldots, a_n, a_{n+1}, \ldots) = \sum_{i=1}^{\infty} \frac{2a_i}{3^i}$.

Let $x = (a_1, a_2, \ldots, a_n, a_{n+1}, \ldots)$, $f(x) = p$. Pick any neighborhood N of p in C. Then there exists a positive integer n such that $p \in (p - \frac{1}{3^{n-1}}, p + \frac{1}{3^{n-1}}) \cap C \subset N$. Let $A = \{y \in 2^I : y = (b_1, b_2, \ldots, b_n, b_{n+1}, \ldots), a_i = b_i$ for $i \leq i \leq n\}$. By the definition of the product topology on $2^I$, A is open. Now consider f(A). Since the first n terms in the expansions of two members of f(A) agree, the distance from p to any member of f(A) is less than or equal to $\frac{1}{3^n}$. This follows from the fact that $\sum_{i=n+1}^{\infty} \frac{2a_i}{3^i} \leq \frac{2}{3^n} \cdot \frac{1}{2} = \frac{1}{3^n}$. Therefore $f(A) \subset \left[(p - \frac{1}{3^{n-1}}, p + \frac{1}{3^{n-1}}) \cap C\right]$ and by theorem 0.3., f is continuous. Since f is obviously one-to-one and onto, theorem 0.4. implies that f is a homeomorphism.

Theorem 2.16. A finite Boolean algebra A is free if and only if the number of its atoms is a power of 2.

Proof: Let A be a free, finite Boolean algebra. Then by theorem 2.14., there exists a Cantor space $2^Q$ such that A is freely generated by a set with the same power, n, say, as Q. Without loss of generality let

this set be Q. There are $2^n$ elements in $2^Q$ and each of these is clopen and thus an isolated point, and hence an atom of A.

The proof of sufficiency requires the following lemma.

Lemma 2.15. If a finite (and therefore atomic) Boolean algebra A has k atoms, it has $2^k$ elements.

Proof: Let $Q = \{ a_1, a_2, \ldots, a_k \}$ be the set of all atoms. Consider $\mathcal{P}(Q)$, the power set of Q. Let M and N be distinct members of $\mathcal{P}(Q)$. By theorem 2.4., $\bigvee M \in A$ and $\bigvee N \in A$. Suppose $\bigvee M = \bigvee N$. Without loss of generality, suppose $a_m \in M$ and $a_m \notin N$. Then

$$a_m \wedge \bigvee M = a_m = a_m \wedge \bigvee N = 0,$$

which is a contradiction. Thus $\bigvee M \neq \bigvee N$ and thus there are as many elements (suprema of atoms) in A as there are subsets of the set of atoms, i.e., $2^k$.

Continuing the proof of the theorem, suppose there are $2^n$ elements in the set of atoms, Q. Then the algebra generated by Q has $2^{2^n}$ elements. Consider the Cantor space $2^Q$. Being finite $2^Q$ is discrete and thus every subset is clopen and hence its dual algebra has $2^{2^n}$ elements. This is obviously A and by theorem 2.14., A is free.

Theorem 2.17. A countable non-atomic Boolean algebra is free.

Proof: Let A be a countable non-atomic Boolean algebra. Let X be the dual space of A. By lemma 2.10.1, X is metrizable, for A is a countable basis for X. Also, by theorem 2.9., X has no isolated points. Now consider the Cantor middle-third set C. It is well known that C is metric and compact and has no isolated points. Since C is homeomorphic to the Boolean space $2^I$, where I is countably infinite, then clopen sets separate points in C. Thus C is totally disconnected. A topological lemma is now invoked to show that X is homeomorphic to C.

Lemma 2.17.1. All metrizable, totally disconnected compact spaces without isolated points are homeomorphic [3, p. 58]. Thus by theorem 2.15. all dual spaces of countable Boolean algebras are homeomorphic to the Cantor space $2^I$ so that their dual algebras are isomorphic to a freely generated one.

Finally, to round out the discussion in this chapter, the dualization of the concept of a complete Boolean algebra will be examined.

Theorem 2.18. If $\{P_i\}$ is a family of elements in the dual algebra A of Boolean space X, and if $U = \bigcup_i P_i$, then a necessary and sufficient condition that $\{P_i\}$ have a supremum in A is that $U^-$ be open. If the condition is satisfied, then $\bigvee_i P_i = U^-$.

Proof: (necessity) Let $\bigvee_i P_i = P$, i.e., suppose

the supremum exists and is equal to P. Since P is closed (clopen) and includes each of the $P_i$, $U \subset P$. $P - \bar{U}$ is open since it is the intersection of an open set with P (also open). If $P - U \neq \emptyset$, it includes a non-empty clopen set Q. Then P - Q is clopen and includes all the $P_i$'s and is properly contained in P. This contradicts the fact that P was a supremum. (Sufficiency) Let $\bar{U}$ be open, then it is clopen and includes all the $P_i$'s. If P is a clopen set that contains all of the $P_i$'s, then $U \subset P$. But P is closed so that $\bar{U} \subset P$. But P was an arbitrary clopen set containing all of the $P_i$'s. Therefore $\bar{U}$ is $\bigvee_i P_i$.

Corollary 2.18.1. If a family of elements in the dual algebra of a Boolean space has a supremum, then that supremum differs from the set-theoretic union by a nowhere dense closed set.

Proof: $\bigvee_i P_i - U$ = boundary of U. By lemma 2.2.5., the boundary of an open set is a nowhere dense closed set.

Definition 2.10. A Hausdorff space is extremally disconnected if and only if every open set has open closure.

Theorem 2.19. The dual algebra A of a Boolean space X is complete if and only if X is extremally disconnected.

Proof: Assume A is complete and let U be an open set in X. Next consider all of the clopen subsets $P_i$

contained in U. Then $\bigvee_i P_i = U^-$ is open.

Conversely suppose the closure of every open subset of X is clopen. Let $\{ P_i \}$ be a family of elements of A. Then if $U = \bigcup_i P_i$, it follows that $U^-$ is open which by theorem 2.17. implies that $\bigvee_i P_i$ exists.

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