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Calculus of variations

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CALCULUS OF VARIATIONS

by

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Introduction

Suppose there has been specified a set of arcs C, satisfying certain conditions, such that along each of them the integral $I(C) = \int f(x,y,z,y',z')\,dx$ has a well determined value. The arcs of this set will be called admissible arcs. A problem of the calculus of variations associated with such admissible arcs and their integrals is that of finding, in the class of admissible arcs joining two fixed points, one which gives the integral I(C) its smallest value. The problem so formulated is said to have fixed end-points. It may be modified by specifying the class of arcs, in which a minimum is sought, to be the class of admissible arcs joining a fixed point and a fixed curve, or a curve and a fixed surface. In these latter cases the problem is said to have variable end-points.

For our purpose we will suppose that there is a region R of the space of 5-tuples of real numbers (x,y,z,y',z') in which the integrand function $f(x,y,z,y',z')$ has continuous derivatives up to and including those of the fourth order. A point (x,y,z,y',z') interior to the region R is called an <u>admissible point</u>. An arc C is called <u>regular</u> if the functions $y(x)$, $z(x)$ defining it are single-valued and have continuous derivatives on the interval $x_1 \leq x \leq x_2$. The set of <u>admissible arcs</u> to be considered here is the set of continuous arcs each of which consists of a finite number of regular sub-arcs whose points (x,y,z,y',z') are admissible. We will now list some results of the problem with fixed end-points.

I The First Necessary Condition

An admissible arc E is said to satisfy condition I if there exist two constants c and d such that the equations

$$f_{y'} = \int_{x_1}^{x} f_y \, dx + c, \text{ and}$$

$$f_{z'} = \int_{x_1}^{x} f_z \, dx + d$$

are identities along E. Every admissible arc E which gives the integral I a minimum value must satisfy condition I.

Euler's Equations

On every sub-arc between corners of an admissible arc E which satisfies the condition I the functions $f_{y'}$, $f_{z'}$ have derivatives and the equations

$$\frac{d}{dx}(f_{y'}) = f_y \text{ , and}$$

$$\frac{d}{dx}(f_{z'}) + f_z$$

are satisfied.

The Weierstrass-Erdmann Corner Condition.

At each value x defining a corner of an admissible arc E that satisfies condition I, the right and left limits of the functions $f_{y'}$ and $f_{z'}$ are equal.

Hilbert's Differentiability Condition.

Let E be an admissible arc satisfying condition I. Then near every point (x, y, z, y', z') of E which is not a corner, and at which the determinant $f_{y'y'} \, f_{z'z'} - (f_{y'z'})^2$ is different from zero, the functions y(x) and z(x) defining E have continous nth derivatives when the integrand function f has all partial derivatives of orders $\leq$ n continuous near (x, y, z, y', z').

A sub-arc of E on which the determinant $f_{y'y'} \, f_{z'z'} - (f_{y'z'})^2$ is different from zero will be called non-singular.

An admissible arc defined by functions $y(x)$, and $z(x)$ having continuous first and second derivatives, and satisfying the equations

$$f_{y'x} + f_{y'y} Y' + f_{y'z} Z' + f_{y'y'} Y'' + f_{y'z'} Z'' - f_y = 0, \text{ and}$$

$$f_{z'x} + f_{z'y} Y' + f_{z'z} Z' + f_{z'y'} Y'' + f_{z'z'} Z'' - f_z = 0$$

will be called an extremal.

The above equations are the Euler differential equations in the expanded form. These are satisfied by every sub-arc of E (minimizing arc) along which the determinant $f_{y'y'} f_{z'z'} - (f_{y'z'})^2$ is different from zero. Euler's equations can be expressed in the above form when it is known that the functions $y(x)$, and $z(x)$ defining E have second derivatives.

Let $E(x,y,z,y',z',Y',Z') = f(x,y,z,Y',Z') - f(x,y,z,y',z') - (Y' - y')f_{y'}(x,y,z,y',z') - (Z' - z')f_{z'}(x,y,z,y',z')$.

II The Necessary Condition of Weierstrass.

An admissible arc E is said to satisfy condition II or the condition of Weierstrass if at every point (x,y,z,y',z') of E the condition

$$E(x,y,z,y',z',Y',Z') \geq 0$$

is satisfied for every admissible point (x,y,z,Y',Z') with $(Y',Z') \neq (y',z')$. Every arc E which minimizes the integral I must satisfy condition II.

III Legendre's Necessary Condition.

An admissible arc E is said to satisfy condition III or the condition of Legendre if at each point (x,y,z,y',z') of E the condition

$$\eta^2 f_{y'y'} + 2\eta\zeta f_{y'z'} + \zeta^2 f_{z'z'} \geq 0$$

is satisfied for every pair of real values η, ζ such that $\eta^2 + \zeta^2 = 1$, the arguments of the derivatives of f being the coordinates (x,y,z,y',z) of the point of E. Every arc E which minimizes the integral I must satisfy condition III.

In accordance with the symbolism used in Bliss, ([1] and [2])[1], we will denote by 1 and 2 the end-points $[x_1, y(x_1), z(x_1), y'(x_1), z'(x_1)]$ and $[x_2, y(x_2), z(x_2), y'(x_2), z'(x_2)]$, respectively, of an extremal arc connecting these end-points. The extremal arc will then be represented by the symbol E_{12}. Analogous symbols will be used for other arcs.

The terminology we will use will be consistent with that used in Bliss ([1], [2]), with a few exceptions.

A contact point of an extremal arc E_{12} with an envelope D, is said to be a <u>point conjugate to 1</u> on the arc E_{12}.

IV <u>Jacobi's Necessary Condition</u>

A non-singular extremal arc E_{12} is said to satisfy condition IV or the condition of Jacobi if it has on it between its end-points 1 and 2 no point conjugate to 1. Every non-singular minimizing arc E_{12} without corners is an extremal arc satisfying this condition.

If an arc E_{12} gives I a minimum value relative to the class of admissible arcs C_{12} in a sufficiently small neighborhood of the points (x, y, z, y', z') of E_{12}, then $I(E_{12})$ is said to be a <u>weak relative minimum</u>. A minimum provided by E_{12} relative to a class of admissible arcs C_{12}, restricted only to have their points (x, y, z) in a sufficiently small neighborhood F or E_{12} in xyz-space, is called a <u>strong relative minimum</u>.

The symbols <u>II</u>$'$ and <u>III</u>$'$ are used to denote the necessary conditions of Weierstrass and Legendre, respectively, with the equality signs excluded in their statements. Similarly, IV$'$ denotes Jacobi's condition (IV) strengthened to exclude points conjugate to 1 from the

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1) The symbol [n] will refer to the nth entry in the list of references at the end of this paper.

end-point 2 of an extremal arc E_{12} as well as from the interior of the arc.

Sufficient Conditions for a Weak Relative Minimum.

If an admissible arc E_{12} without corners satisfies conditions I, III$'$, and IV$'$, there exists a neighborhood R_1 of the values (x,y,z,y',z') belonging to E_{12} such that the inequality

$$I(C_{12}) > I(E_{12})$$

holds for every admissible arc C_{12} in R_1 and not identical with E_{12}.

An arc E_{12} is said to satisfy condition $\underline{II_N}$ if there is a neighborhood N of the elements (x,y,z,y',z') on E_{12} such that

$$E(x,y,z,y',z',Y',Z') \geqq 0$$

holds for all sets (x,y,z,y',z',Y',Z') with (x,y,z,y',z') admissible and in N and with (x,y,z,Y',Z') admissible and having

$$(Y',Z') \neq (y',z').$$

Condition $\underline{II'_N}$ is this condition with the equality excluded.

The notation $\underline{III_F}$ designates the property that the inequality

$$f_{y'y'}\eta^2 + 2f_{y'z'}\eta\zeta + f_{z'z'}\zeta^2 \geqq 0$$

holds for all admissible elements (x,y,z,y',z') with projections (x,y,z) in a neighborhood F of the arc E_{12} and for all pairs η, ζ such that $\eta^2 + \zeta^2 = 1$. The notation $\underline{III'_F}$ is used for this property with the equality sign excluded.

Sufficient Conditions for a Strong Relative Minimum

If an admissible arc E_{12} without corners is non-singular and satisfies conditions I, II_N, IV$'$, then there is a neighborhood F of E_{12} in xyz-space such that the relation

$$I(C_{12}) > I(E_{12})$$

holds for every admissible arc C_{12} in F not identical with E_{12}.

Let summarize with the table below.

TABLE OF NECESSARY AND OF SUFFICIENT CONDITIONS APPLICABLE TO ADMISSIBLE ARCS WITHOUT CORNERS

Type of Minimum	Necessary Conditions	Sufficient Conditions
Weak relative	I, III, IV	I, III$'$, IV$'$
Strong relative	I, II, III, IV	I, II_N, IV$'$ and E non-singular
Strong relative	I, II, III, IV	I, II_N, III$'$, IV$'$
Strong relative	I, II, III, IV	I, III'_F, IV$'$

For a detailed discussion of the preceeding results see Bliss, "Lectures in the Calculus of Variations", chapters I and II ([1]).

Section I

Problems in three-space with one end-point variable on a surface.

The problem to be studied in this section is that of finding in a certain class of admissible arcs joining a fixed surface S to a fixed point 2 one which minimizes the integral $I = \int_{x_1}^{x_2} f(x,y,z,y',z')dx$ in the space of 5-tuples (x,y,z,y',z') of real numbers.

Let E_{12} represent a particular admissible arc whose minimizing properties are to be studied.

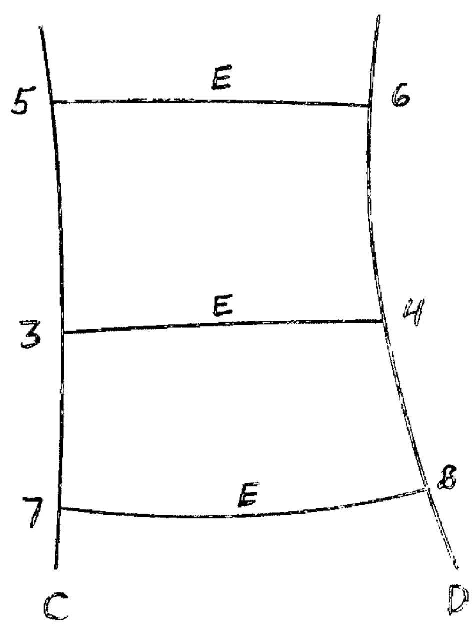

Fig. 1.1

Consider the variation of the value of the integral I taken along a variable arc E whose end-points describe two fixed curves C and D as shown in fig. 1.1. E may be taken in the form $y(x,a)$, $z(x,a)$, the displacement of E being caused by the variation of the value of the parameter a. If t is a parameter defining the position of the point 3 on C, then the coordinate x of the point 3 and the value of a defining arc E through 3 are functions of t, and the functions defining C may be written in the parametric form,

$$(1.1)\qquad \begin{aligned} & x_3(t), \\ & y\left[x_3(t), a(t)\right] = y_3(t), \\ & z\left[x_3(t), a(t)\right] = z_3(t). \end{aligned}$$

$$(1.2)\qquad \begin{aligned} & x_4(t), \\ & y\left[x_4(t), a(t)\right] = y_4(t), \\ & z\left[x_4(t), a(t)\right] = z_4(t). \end{aligned}$$

Since the point 4 on D is also determined when t is given, and by the same value of a as that corresponding to 3 we have D written in the parametric form (1.2).

Assume the functions $x_3(t)$, $x_4(t)$, and $a(t)$ defining the arcs C and D have continuous derivatives on an interval $(t' \leq t \leq t'')$. Assume for the values (x, a) specified by the conditions

$$\begin{aligned} & \left[x_3(t) \leq x \leq x_4(t)\right], \\ & a = a(t), \\ & (t' \leq t \leq t'') \end{aligned}$$

that the functions $y(x,a)$, $z(x,a)$ defining admissible arcs are without corners. Also assume that in a neighborhood of (x,a) the functions $y(x,a)$, $z(x,a)$ and their derivatives $y_x(x,a)$, $z_x(x,a)$ have continuous first partial derivatives with respect to a.

The value of the integral I taken along the arc E is a function I(t) defined by the equation

$$(1.3)\qquad I(t) = \int_{x_3}^{x_4} f\left[x, y(x,a), z(x,a), y'(x,a), z'(x,a)\right] dx$$

in which x_3, x_4, and a are the functions of t just described.

DEFINITION:

Consider two points 3 and 7 on the curve C described above. Let the point 3 be given by $t = t_3$, and similary let 7 be given by $t = t_7$.

Then we say that the point 3 precedes the point 7 on C if

$$t' \leq t_3 < t_7 \leq t''.$$

THEOREM (1.1)

The value of the integral I, taken along a variable arc E with the continuity properties described above and whose end-points 3 and 4 describe two fixed curves C and D, has the differential

$$(1.4) \qquad dI = \left[f dx + (dy - y' dx) f_{y'} + (dz - z' dx) f_{z'} \right]_3^4$$

at each position of the variable arc at which that arc satisfies Euler's differential equations. In the expression for dI the values (x, y, z, y', z') occurring in f and elsewhere are those belonging to E at the points 3 and 4 and dx, dy, dz are differentials belonging to C or D.

Proof

$$I(t) = \int_{x_3}^{x_4} f\left[x, y(x,a), z(x,a), y'(x,a), z'(x,a)\right] dx.$$

Let $u = x_4$, $v = x_3$ and write $G(u,v,a) = \int_v^u f dx$. Then

$$I'(t) = \frac{\partial G}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial G}{\partial v}\frac{\partial v}{\partial t} + \frac{\partial G}{\partial a}\frac{da}{dt}$$

$$= \left[f\frac{dx}{dt} \right]_3^4 + \frac{da}{dt}\int_{x_3}^{x_4} \left[f_y y_a + f_z z_a + f_{y'} y'_a + f_{z'} z'_a \right] dx.$$

At a point of arc E at which the Euler equations are satisfied,

$$\frac{d}{dx} f_{y'} - f_y = 0, \text{ and}$$

$$\frac{d}{dx} f_{z'} - f_z = 0. \quad \text{Then}$$

$$\frac{d}{dx}(f_{y'} y_a) = y_a \frac{d}{dx} f_{y'} + f_{y'} y'_a = f_y y_a + f_{y'} y'_a, \text{ and}$$

$$\frac{d}{dx}(f_{z'} z_a) = z_a \frac{d}{dx} f_{z'} + f_{z'} z'_a + f_z z_a + f_{z'} z'_a.$$

Therefore, for this particular arc E,

$$I'(t) = \left[f\frac{dx}{dt} + \frac{da}{dt}(f_{y'} y_a + f_{z'} z_a) \right]_3^4.$$

Now we have

$$y_3(t) = y[x_3(t), a(t)], \text{ and}$$

$$z_3(t) = z[x_3(t), a(t)]. \text{ Therefore}$$

$$dy_3 = y'dx_3 + y_a da, \text{ and}$$

$$dz_3 = z'dx_3 + z_a da. \text{ Similarly}$$

$$dy_4 = y'dx_4 + y_a da, \text{ and}$$

$$dz_4 = z'dx_4 + z_a da. \text{ Thus}$$

$$dI = \left[fdx + (dy - y'dx)f_{y'} + (dz - z'dx)f_{z'}\right]_3^4.$$

This completes the proof.

COROLLARY (1.1)

If the ends of a variable extremal arc E describe two curves C and D, the difference between the values of I at two positions E_{56} and E_{78} of E, as show in fig. (1.1), is given by the formula

$$I(E_{56}) - I(E_{78}) = I^{\star}(D_{68}) - I^{\star}(C_{57}), \text{ where}$$

$$I^{\star} = \int \left[fdx + (dy - y'dx)f_{y'} + (dz - z'dx)f_{z'}\right].$$

Proof

The functions y(x,a), z(x,a) in the second member of 1.4 and their derivatives y′(x,a), z′(x,a) with respect to x are all functions of t calculable with the help of equations 1.1. The differentials dx, dy, dz are functions of t multiplied by dt, defined by the equations

$$dy = y'dx + y_a da, \text{ and}$$

$$dz = z'dx + z_a da.$$

Thus, the expressions of the two sides of 1.4 are functions of t multiplied by dt and can be integrated with respect to t from a value t' defining E_{56} of E to a value t'' defining E_{78}, as show in fig. (1.1).

$$I(E_{78}) - I(E_{56}) = \int_{t'}^{t''} \left[fdx + (dy - y'dx)f_{y'} + (dz - z'dx)f_{z'}\right]_3^4$$

$$= \int_{t'}^{t''} \left[fdx + (dy_4 - y'dx_4)f_{y'} + (dz_4 - z'dx_4)f_{z'}\right] -$$

$$\int_{4'}^{4''} [\, f dx_3 + (dy - y' dx_3) f_{y'} + (dz_3 - z' dx_3)\, f_{z'} \,]$$
$$= I^{\star}(D_{68}) - I^{\star}(C_{57}).$$

This completes the proof.

Let E_{12} represent a particular arc whose minimizing properties are to be studied. Let S be the fixed surface defined by the functions

(1.5) $\xi(\alpha, \beta)$,
$\eta(\alpha, \beta)$,
$\zeta(\alpha, \beta)$,

which have continuous derivatives of the third order near the values (α_1, β_1) which define the intersection point 1 of S and E_{12}.

Since every admissible arc joining the end-points 1 and 2 of E_{12} also joins the surface S with 2, it is evident that E_{12} must satisfy all the necessary conditions for a minimizing arc in the class of arcs joining its end-points.

Let Υ be an arbitrary curve on S through the point 1 defined by the functions

(1.6) $\alpha(a)$,
$\beta(a)$,

which have continuous second derivatives near the parameter value a_1, defining the point 1.

The curve Υ can be joined to the point 2 by a one-parameter family of admissible arcs containing E_{12} as a member. For example let E_{12} be defined by the functions

$y(x)$,
$z(x)$;
$(x_1 \leq x \leq x_2)$.

Then such a family is defined by the functions

(1.7) $$y(x) + \{\eta[\alpha(a), \beta(a)] - y(\xi)\} Q(x,a),$$ and

(1.8) $$z(x) + \{\zeta[\alpha(a), \beta(a)] - z(\xi)\} Q(x,a),$$ where

(1.9) $$Q(x,a) = \frac{x - x_2}{\xi[\alpha(a), \beta(a)] - x_2}.$$

For

$$x = \xi[\alpha(a), \beta(a)]$$

the arcs of this family intersect the curve and pass through the point 2, since

(1.10) $$Q(x,a) = \frac{x - x_2}{\xi[\alpha(a), \beta(a)] - x_2} = \frac{x - x_2}{x - x_2} = 1,$$ so that

(1.11) $$y = y(x) + \eta[\alpha(a), \beta(a)] - y(\xi)$$
$$= y(x) + \eta[\alpha(a), \beta(a)] - y(x) = \eta[\alpha(a), \beta(a)],$$ and

(1.12) $$z = z(x) + \zeta[\alpha(a), \beta(a)] - z(\xi)$$
$$= z(x) + \zeta[\alpha(a), \beta(a)] - z(x) = \zeta[\alpha(a), \beta(a)].$$

For $x = x_2$ we have

$$Q(x,a) = 0,$$

$$y = y(x_2),$$ and

$$z = z(x_2).$$

For the parameter value $a = a_1$ we have,

(1.13) $$y = y(x) + \{\eta[\alpha(a_1), \beta(a_1)] - y(\xi)\} Q(x,a_1)$$
$$= y(x) + \{\eta[\alpha(a_1), \beta(a_1)] - y[\alpha(a_1), \beta(a_1)]\} Q(x,a_1)$$
$$= y(x),$$ and

(1.14) $$z = z(x) + \{\zeta[\alpha(a_1), \beta(a_1)] - z(\xi)\} Q(x,a_1)$$
$$= z(x).$$

DEF.

An admissible arc E is said to satisfy the <u>Transversality Condition</u> at its intersection point 1 with a surface S if the equation

(1.15) $$(f - y'f_{y'} - z'f_{z'})dx + f_{y'}dy + f_{z'}dz = 0$$

is satisfied by every direction dx: dy: dz tangent to S at the point 1, the arguments (x,y,z,y',z') in f and its derivatives being those of the arc E_{12} at 1.

The family defined by (1.7) and (1.8) satisfies the conditions of theorem (1.1) and therefore the value of the integral I taken along a curve of the family is a function of the parameter a, whose differential at the value a_1 defining E_{12} in the family is given by equation (1.4). If $I(E_{12})$ is to be a minimum, this differential must vanish, and since $\mathcal{T}$ is an arbitrary curve on S through the point 1, the following theorem is established.

THEOREM (1.2)

Every minimizing arc for the problem of this section must satisfy the transversality condition.

DEF.

For differentiable functions $y(x,\alpha,\beta)$, $z(x,\alpha,\beta)$, we let

$$\Delta(x,\alpha,\beta) = \begin{vmatrix} y_\alpha(x) & y_\beta(x) \\ z_\alpha(x) & z_\beta(x) \end{vmatrix} .$$

DEF.

Let E_{12} be an extremal arc cut transverally by a non-singular surface S at the point 1 and not tangent to S at 1. If $\Delta(x,\alpha,\beta)$ is the determinant of the two parameter family of extremals

$$y = y(x,\alpha,\beta),$$
$$z = z(x,\alpha,\beta)$$

cut transverally by the surface S and containing E_{12} for parameter values α_1, β_1, then the points determined on E_{12} by the zeros of $\Delta(x,\alpha_1,\beta_1)$ are called focal points of S on E_{12}.

THEOREM (1.3)

Suppose E_{12} is a non-singular extremal arc cut transversally by the surface S at the point 1 and not tangent to S at the point 1. Also suppose S in non-singular at the point 1. Then there exists a two-parameter family of extremals

$$y(x, \alpha, \beta),$$
$$z(x, \alpha, \beta)$$

containing E_{12} for parameter values (α_1, β_1) and having the properties

1) the members of the family are cut transversally at $x = \xi(\alpha, \beta)$,

2) the functions $y(x, \alpha, \beta)$, $z(x, \alpha, \beta)$ and their first and second derivatives with respect to x have continuous second partial derivatives for values (x, α, β) in a neighborhood of those belonging to E_{12}, and

3) $$\Delta(x, \alpha, \beta) = \begin{vmatrix} y_\alpha & z_\alpha \\ y_\beta & z_\beta \end{vmatrix} \neq 0$$

identically along E_{12} .

<u>Proof</u>.

The surface S is defined by the functions

$$\xi(\alpha, \beta),$$
$$\eta(\alpha, \beta),$$
$$\zeta(\alpha, \beta).$$

If the two-parameter family of extremals exists it will be of the form $f(x,y,z,y',z')$ where

$$y = \eta(x, \alpha, \beta), \quad y' = \eta'(x, \alpha, \beta), \text{ and}$$
$$z = \zeta(x, \alpha, \beta), \quad z' = \zeta'(x, \alpha, \beta).$$

Consider the integrand of the Hilbert integral of this family:

(1.15) $$(f - y'f_{y'} - z' f_{z'})dx + f_{y'} dy + f_{z'} dz.$$

For $x = \xi(\alpha, \beta)$ the function (1.15) takes the form

$$P d\alpha + Q d\beta \text{ , where}$$

(1.16) $$P = f \xi_\alpha + (\eta_\alpha - \eta' \xi_\alpha) f_{y'} + (\zeta_\alpha - \zeta' \xi_\alpha) f_{z'} \text{ , and}$$

(1.17) $$Q = f\xi_\beta + (\eta_\beta - \eta'\xi_\beta)f_{y'} + (\zeta_\beta - \zeta'\xi_\beta)f_{z'} .$$

Transversality implies

$$P d\alpha + Q d\beta = 0$$

independent of $d\alpha$ and $d\beta$.

Therefore P = 0, and Q = 0, or

(1.18) $$f\xi_\alpha + (\eta_\alpha - \eta'\xi_\alpha)f_{y'} + (\zeta_\alpha - \zeta'\xi_\alpha)f_{z'} = 0, \text{ and}$$

(1.19) $$f\xi_\beta + (\eta_\beta - \eta'\xi_\beta)f_{y'} + (\zeta_\beta - \zeta'\xi_\beta)f_{z'} = 0.$$

Implicit function theorems assure us that solution's of 1.18, and 1.19

$$y' = \eta'(\alpha, \beta), \quad z' = \zeta'(\alpha, \beta)$$

exists provided there exists an initial solution $\alpha_1, \beta_1, \eta_1, \zeta_1$, at the point defined by (α_1, β_1) such that the element

$$[\xi(\alpha_1, \beta_1), \eta(\alpha_1, \beta_1), \zeta(\alpha_1, \beta_1), \eta', \zeta']$$

is admissible and makes the appropriate Jacobian different from zero at the point (α_1, β_1).

This Jacobian is

$$\frac{\partial(P, Q)}{\partial(y', z')} = \begin{vmatrix} \frac{\partial P}{\partial y'} & \frac{\partial P}{\partial z'} \\ \frac{\partial Q}{\partial y'} & \frac{\partial Q}{\partial z'} \end{vmatrix} .$$

We find that

$$\frac{\partial P}{\partial y'} = \xi_\alpha f_{y'} + (\eta_\alpha - \eta'\xi_\alpha)f_{y'y'} - \xi_\alpha f_{y'} + (\zeta_\alpha - \zeta'\xi_\alpha)f_{z'y'} ,$$

$$\frac{\partial P}{\partial z'} = \xi_\alpha f_{z'} + (\eta_\alpha - \eta'\xi_\alpha)f_{y'z'} - \xi_\alpha f_{z'} + (\zeta_\alpha - \zeta'\xi_\alpha)f_{z'z'} ,$$

$$\frac{\partial Q}{\partial y'} = \xi_\beta f_{y'} + (\eta_\beta - \eta'\xi_\beta)f_{y'y'} - \xi_\beta f_{y'} + (\zeta_\beta - \zeta'\xi_\beta)f_{z'y'} , \text{ and}$$

$$\frac{\partial Q}{\partial z'} = \xi_\beta f_{z'} + (\eta_\beta - \eta'\xi_\beta)f_{y'z'} - \xi_\beta f_{z'} + (\zeta_\beta - \zeta'\xi_\beta)f_{z'z'} .$$

Therefore,

$$\frac{\partial(P, Q)}{\partial(y', z')} = \begin{vmatrix} (\eta_\alpha - \eta' \xi_\alpha) & (\zeta_\alpha - \zeta' \xi_\alpha) \\ (\eta_\beta - \eta' \xi_\beta) & (\zeta_\beta - \zeta' \xi_\beta) \end{vmatrix} \cdot \begin{vmatrix} f_{y'y'} & f_{y'z'} \\ f_{z'y'} & f_{z'z'} \end{vmatrix} .$$

For $\alpha = \alpha_1$ and $\beta = \beta_1$ the family defines the arc E_{12} which by assumption is non-singular. Thus

$$\begin{vmatrix} f_{y'y'} & f_{y'z'} \\ f_{z'y'} & f_{z'z'} \end{vmatrix} \neq 0$$

at the point (α_1, β_1).

By assumption E_{12} is not tangent to S at the point 1. Therefore

$$\begin{vmatrix} 1 & \eta' & \zeta' \\ \xi_\alpha & \eta_\alpha & \zeta_\alpha \\ \xi_\beta & \eta_\beta & \zeta_\beta \end{vmatrix} = \begin{vmatrix} (\eta_\alpha - \eta' \xi_\alpha) & (\zeta_\alpha - \zeta' \xi_\alpha) \\ (\eta_\beta - \eta' \xi_\beta) & (\zeta_\beta - \zeta' \xi_\beta) \end{vmatrix} \neq 0.$$

The existence of the family is thus assured. The determinant $\Delta(x, \alpha, \beta)$ of the family is different from zero at the point 1 on E_{12}, since the identities

$$\eta(\alpha, \beta) = y(\xi, \alpha, \beta), \text{ and}$$
$$\zeta(\alpha, \beta) = z(\xi, \alpha, \beta)$$

show that

$$x = \xi(\alpha, \beta),$$
$$\eta_\alpha = y_\xi \xi_\alpha + y_\alpha = y' \xi_\alpha + y_\alpha, \text{ and}$$
$$\eta_\beta = y_\xi \xi_\beta + y_\beta = y' \xi_\beta + y_\beta .$$

Therefore

$$y_\alpha = \eta_\alpha - y' \xi_\alpha, \text{ and}$$
$$y_\beta = \eta_\beta - y' \xi_\beta .$$

Similarly

$$z_\beta = \zeta_\beta - z'\xi_\beta \quad , \text{ and}$$
$$z_\alpha = \zeta_\alpha - z'\xi_\alpha \quad .$$

Therefore

$$\begin{vmatrix} y_\alpha & y_\beta \\ z_\alpha & z_\beta \end{vmatrix} = \begin{vmatrix} (\eta_\alpha - y'\xi_\alpha) & (\eta_\beta - y'\xi_\beta) \\ (\zeta_\alpha - z'\xi_\alpha) & (\zeta_\beta - z'\xi_\beta) \end{vmatrix} = 0$$

by the previous argument.

The focal points of the surface S, are independent of the parametric representation chosen for S. For if, in the equations

$$\xi = \xi(\alpha, \beta),$$
$$\eta = \eta(\alpha, \beta), \text{ and}$$
$$\zeta = \zeta(\alpha, \beta)$$

defining S, the parameters α, β are replaced by functions

$$\alpha = \alpha(\gamma, \delta), \text{ and}$$
$$\beta = \beta(\gamma, \delta)$$

of two new parameters, the family

$$y(x, \alpha, \beta),$$
$$z(x, \alpha, \beta)$$

with the original parameters α, β will go into a second family

$$y(x, \gamma, \delta) = y,$$
$$z(x, \gamma, \delta) = z$$

with similar properties. The determinants $\Delta(x, \alpha, \beta)$ and $\Delta(x, \gamma, \delta)$ of the two families will differ only by a non-vanishing factor independent of x. The non-vanishing factor is $\alpha_\gamma\beta_\delta - \alpha_\delta\beta_\gamma$. This factor is the Jacobian of the transformation and therefore it is non-vanishing.

DEF.

A non-singular extremal arc E_{12} cut transversally by a non-singular surface S at the point 1, and not tangent to S at 1, is said to satisfy the focal-point condition if there exists no focal point of S on E_{12} between 1 and 2.

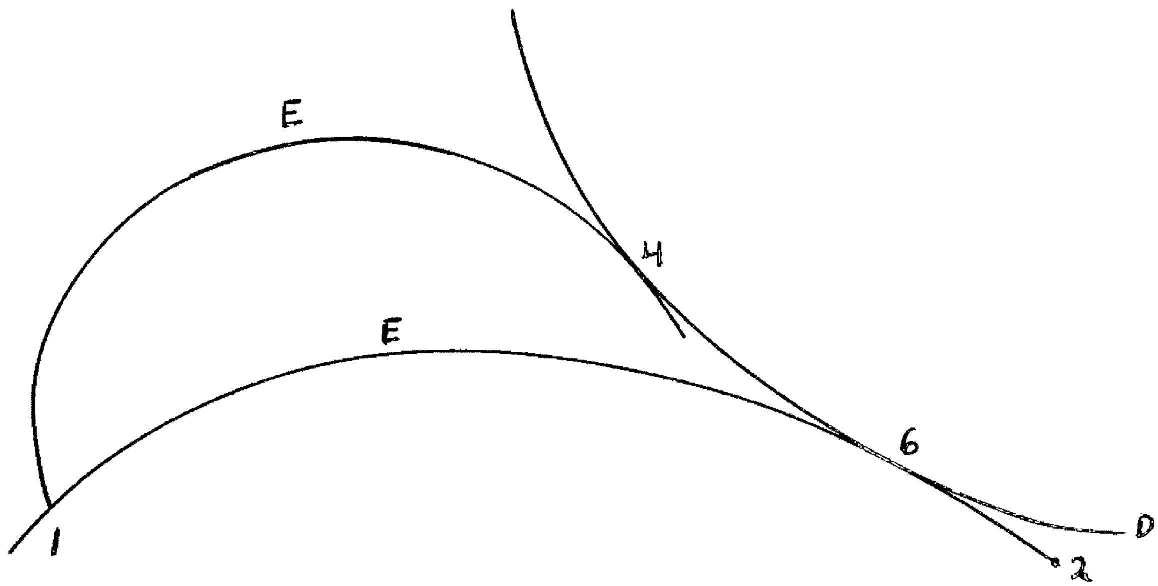

Fig. 1.2

THEOREM (1.4) (The Envelope Theorem)

Assume there exists a one-parameter family of extremal arcs with an envelope D touching the extremal arc E_{12} at the conjugate point 6, as shown in fig. 1.2. Then the equation

(1.20) $\quad I(E_{16}) = I(E_{14}) + I(D_{46})$

holds for every position of the point 4 preceding the point 6 on D.

Proof.

By Corollary 1.1 we have

$$I(E_{56}) - I(E_{34}) = I^*(D_{46}) - I^*(C_{35}).$$

In the special case when the curve C is a fixed point 1 and the variable extremal arc E is, in every position, tangent to the curve D, as shown in fig. 1.2, we have

$$I(E_{16}) - I(E_{14}) = I(D_{46})$$

since

$$I^{*}(C_{35}) = 0$$

because C degenerates to a point in this case.

The term $I^{*}(D_{46})$ is equal to $I(D_{46})$ because the direction $dx: dy: dz$ of the curve D coincides with the direction $1: y': z'$ of the variable extremal arc E at their intersection.

This completes the proof.

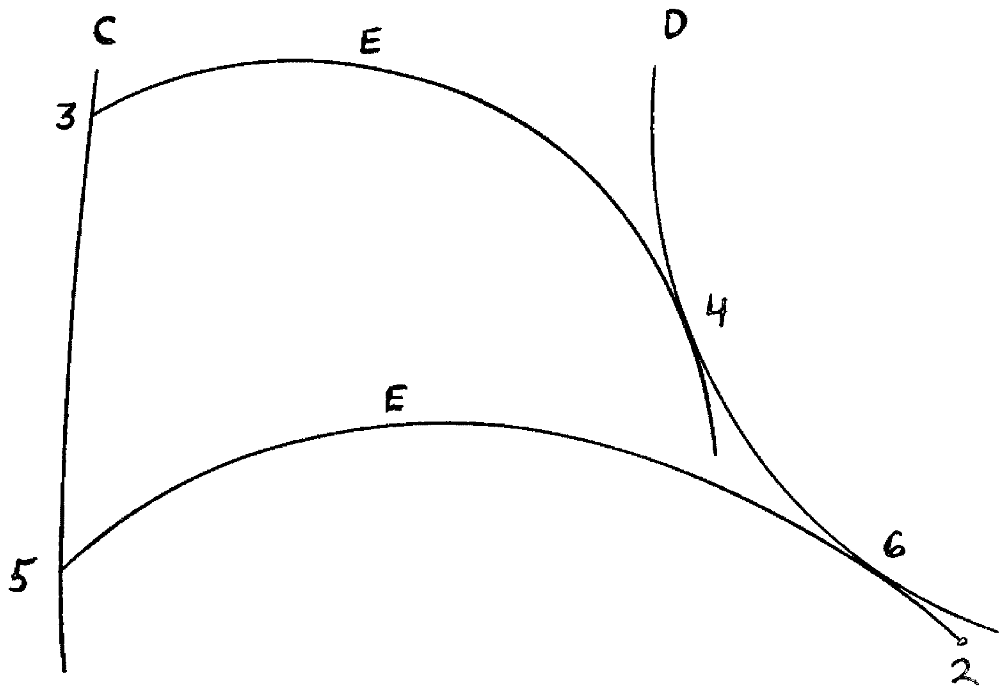

Fig. 1.3

THEOREM (1.5)

If each arc E of a one-parameter family of extremals is cut transversally by a curve C and if the family has further an envelope D, as shown in fig. 1.3, then the equation

(1.21) $$I(E_{56}) = I(E_{34}) + I(D_{46})$$

holds for every position of the point 4 preceding the point 6 on D.

Proof.

The proof is the same as for Theorem (1.4) except $I^{*}(C_{35})$

vanishes because of the transversality and not because C degenerates to a point.

THEOREM (1.6)

Every non-singular minimizing arc for the problem of this section must be an extremal and satisfy the focal point condition if we assume $\Delta_x(x_6, \alpha, \beta)$ is different from zero where 6 is a point conjugate to 1 on E_{12} defined by a zero x_6 of the determinant $\Delta(x, \alpha, \beta)$.

Proof

Let the extremal E_{12} be contained in a two-parameter family of extremals

(1.22) $$y = y(x, \alpha, \beta),$$
$$z = z(x, \alpha, \beta)$$

for parameter values α_1, β_1. All the extremals of this family pass through the point 2.

$$\Delta_x = \begin{vmatrix} y'_\alpha & y'_\beta \\ z_\alpha & z_\beta \end{vmatrix} + \begin{vmatrix} y_\alpha & y_\beta \\ z'_\alpha & z'_\beta \end{vmatrix} \neq 0 \text{ at}$$

$x = x_6$,

$\alpha = \alpha_1$, and

$\beta = \beta_1$.

Therefore y_α, y_β, z_α, z_β do not all vanish at (x_6, α_1, β_1). If $y_\alpha \neq 0$, for example, the first two of the differential equations

(1.23) $$\Delta_x dx + \Delta_\alpha d\alpha + \Delta_\beta d\beta = 0,$$

(1.24) $$y_\alpha \, d\alpha + y_\beta d\beta = 0,$$

(1.25) $$z_\alpha \, d\alpha + z_\beta d\beta = 0,$$

can be solved for $\frac{dx}{d\beta}$, $\frac{d\alpha}{d\beta}$ and determine a unique solution $x(\beta)$, $\alpha(\beta)$ through the initial point (x_6, α_1, β_1). For we have

$$\Delta_x \frac{dx}{d\beta} + \Delta_\alpha \frac{d\alpha}{d\beta} + \Delta_\beta = 0,$$

$$y_\alpha \frac{d\alpha}{d\beta} + y_\beta = 0,$$

and the determinant of this pair of equations is

$$\begin{vmatrix} \Delta_x & \Delta_\alpha \\ 0 & y_\alpha \end{vmatrix} = \Delta_x y_\alpha \neq 0.$$

Equation (1.25) is also satisfied identically by $x(\beta)$, $\alpha(\beta)$, as we see by considering the two cases which may occur.

Case I: $z_\alpha \neq 0$, Then

$$\frac{d\alpha}{d\beta} = -\frac{y_\beta}{y_\alpha}$$

from equation 1.24. Also

$\Delta = y_\alpha z_\beta - z_\alpha y_\beta = 0$, so that

$\frac{z_\beta}{z_\alpha} = \frac{y_\beta}{y_\alpha}$; but then

$z_\alpha \frac{d\alpha}{d\beta} + z_\beta = z_\alpha(-\frac{y_\beta}{y_\alpha}) + z_\beta$, and

$z_\alpha(-\frac{z_\beta}{z_\alpha}) + z_\beta = -z_\beta + z_\beta = 0.$

Case II: $z_\alpha = 0$. Then from

$y_\alpha z_\beta - z_\alpha y_\beta = 0$ we have

$y_\alpha z_\beta = 0$. Then

$y_\alpha \neq 0$, and necessarily

$z_\beta = 0$, so that

equation 1.25 is trivially satisfied.

In any case three functions, $x(t)$, $\alpha(t)$, and $\beta(t)$ are determined, t being α or β, which take the initial values x_1, α_1, β_1 for the value $t = t_1$.

We now have one-parameter family of extremals

(1.26) $y[x, \alpha(t), \beta(t)] = y(x,t),$

$z[x, \alpha(t), \beta(t)] = z(x,t).$

Let D be a curve defined as follows:

(1.27) $x(t),$

$y[x(t), t] = Y(t),$

$z[x(t), t] = Z(t).$

The fact that D is tangent at each of its points to one of the extremals 1.26 is expressed by the equations

(1.28) $x'(t) = \lambda ,$

$y_x x + y_t = \lambda y_x ,$

$z_x x + z_t = \lambda z_x ,$

where λ is a factor of proportionality and where the arguments of the derivatives of y and z are x(t), t.

The curve D satisfies equations 1.28 since the equations 1.23 and 1.24 show that the derivatives y_t , z_t vanish identically along it. It follows that the family 1.26 is a one-parameter family of extremals with an envelope D touching the extremal arc E_{12} at the conjugate point 6.

The envelope theorem then is applicable to the family and the theorem follows.

DEF.

By condition $\underline{I^{\dagger}}$ we shall mean the first necessary condition plus the transversality condition.

DEF.

The symbol $\underline{II^{*\prime}}$ will be used to denote the focal-point condition strengthened so as to exclude the point 2 on E_{12} from being a focal point of S.

DEF.

Let

$$y = y(x, \alpha, \beta),$$
$$z = z(x, \alpha, \beta)$$

be a two-parameter family of extremals, containing a particular arc E_{12} for values

$$x_1 \leq x \leq x_2, \quad \alpha_0, \quad \beta_0$$

and such that the functions y, z, y_x, z_x belonging to the family have continuous partial derivatives of at least the second order in a neighborhood of the values (x, α, β) belonging to E. Such a family is said to simply cover a region F of xyz-space for values (x, α, β) satisfying conditions of the form

$$x_1 - \epsilon \leq x \leq x_2 + \epsilon,$$
$$|\alpha - \alpha_0| \leq \epsilon,$$
$$|\beta - \beta_0| \leq \epsilon$$

if through each point (x,y,z) of F there passes one and only one of the extremals.

DEF.

Let

$$y = y(x, \alpha, \beta),$$
$$z = z(x, \alpha, \beta)$$

be a two parameter family of extremals that simply cover a field F. The functions

$$p(x,y,z) = y_x\,[x,\ \alpha(x,y,z),\ \beta(x,y,z)], \text{ and}$$
$$q(x,y,z) = z_x\,[x,\ \alpha(x,y,z),\ \beta(x,y,z)]$$

are called the slope functions of the family in F.

DEF.

A <u>field</u> is a region F of xyz-space with a pair of slope functions

$$p(x,y,z),$$
$$q(x,y,z)$$

having the following properties:

1) They are single valued and have continuous first partial derivatives in F;

2) The elements $[x, y, z, p(x,y,z), q(x,y,z)]$ defined by points (x,y,z) in F are all admissible, and

3) The Hilbert integral

$$\int [f dx + (dy - y'dx)f_{y'} + (dz - z'dx)f_{z'}]$$

is independent of the path in F, i.e. if the arguments y' and z' are replaced by the slope functions p and q, the integral has the same value on all arcs D_{34} in F having suitable continuity properties and the same end-points 3 and 4.

<u>THEOREM (1.7)</u>

If for a family of extremals

$$y = y(x, \alpha, \beta),$$
$$z = z(x, \alpha, \beta)$$

containing a particular arc E_{12} the determinant $\Delta(x, \alpha, \beta)$ is different from zero along E_{12}, then there is a region

$$x_1 - \epsilon \leqq x \leqq x_2 + \epsilon$$

of points (x, α, β) and a neighborhood F of E_{12} in xyz-space such that F is simply covered by the extremals for values (x, α, β) in

$$x_1 - \epsilon \leqq x \leqq x_2 + \epsilon$$

and further, such that in F the slope functions $p(x,y,z)$, $q(x,y,z)$ of the family, as well as the functions $\alpha(x,y,z)$ and $\beta(x,y,z)$, have continuous partial derivatives of the second order.

THEOREM (1.8)

If a two-parameter family

$$y(x, \alpha, \beta), \quad z(x, \alpha, \beta)$$

is cut by a surface S defined by the functions

$$(1.29) \qquad \begin{aligned} &\xi(\alpha, \beta) \\ &\eta(\alpha, \beta) = y[\xi(\alpha, \beta), \alpha, \beta] \\ &\zeta(\alpha, \beta) = z[\xi(\alpha, \beta), \alpha, \beta] \end{aligned}$$

and if on S the integral $I^{\star}$, formed with the slope functions

$$y_x(\xi, \alpha, \beta), \; z_x(\xi, \alpha, \beta)$$

of the intersecting extremals, is independent of the path, then every region F of xyz-space which is simply covered by the extremals is a field with the slope functions of the family, provided that the determinant $\Delta(x, \alpha, \beta)$ of the family is different from zero at each set of values x, α, β corresponding to a point on F.

THEOREM (1.9)

If the condition

$$\eta^2 f_{y'y'} + 2\eta\zeta f_{y'z'} + \zeta^2 f_{z'z'} > 0$$

is valid at every element (x,y,z,y',z') of an arc E_{12} for all values η, ζ such that

$$\eta^2 + \zeta^2 = 1,$$

then the inequality

$$E(x,y,z,y',z',Y',Z') > 0$$

will be satisfied at least for all elements (x,y,z,y',z') and (x,y,z,Y',Z') lying in a sufficiently small neighborhood N of those on E_{12} and having $(Y',Z') \neq (y',z')$.

The proofs of Theorems 1.7, 1.8, and 1.9 can be found in Bliss' "Lectures in the Calculus of Variations", ([1]) pages 38, 47, and 23 respectively.

THEOREM (1.10)

Let E_{12} be an admissible arc without corners cut at a single point 1 by a surface S which is non-singular and not tangent to the arc E_{12} at 1. If E_{12} satisfies the conditions I^{+}, III', $IV^{*\prime}$ there is a neighborhood R_1 of the values (x, y, z, y', z') belonging to E_{12} such that the inequality

$$I(C_{32}) > I(E_{12})$$

holds for every admissible arc C_{32} in R joining S with 2 and not identical with E_{12}.

If E_{12} is non-singular and satisfies the conditions I^{+}, II_N, IV', then there exists a neighborhood F of the values (x, y, z) on E_{12} such that the inequality

$$I(C_{32}) > I(E_{12})$$

holds for every admissible arc C_{32} in F joining S with 2 and not identical with E_{12}.

Proof

Conditions I and III imply that E_{12} is a non-singular extremal arc and hence belongs to a two-parameter family

$$y(x, \alpha, \beta), \quad z(x, \alpha, \beta)$$

of extremals cut transversally by the surface S by theorem 1.6. The determinant $\Delta(x, \alpha_1, \beta_1)$ of this family is different from zero, not only at the point 1 but at every point of E_{12}, since the surface S has no focal point on E_{12}, by condition $IV^{*\prime}$. For ϵ small enough the extremal arcs defined in the family by values x, α, β satisfying the conditions

$$x_1 - \epsilon < x < x_2 + \epsilon ,$$

$$|\alpha - \alpha_1| < \epsilon,$$
$$|\beta - \beta_1| < \epsilon$$

will simply cover an open region F of xyz-space by theorem 1.7, therefore no two members of the family can intersect when ϵ is sufficiently small. By the usual implicit function theorems [1] applied to the equations

$$y = y(x, \alpha, \beta), \quad z = z(x, \alpha, \beta)$$

every point (x,y,z) which is covered by the extremals, has a neighborhood which is also covered, so that F is an open region. By theorem 1.8, the region F is a field with the slope functions of the family, since the extremals of the family are cut transversally by the surface S. The value of the Hilbert integral I^* with the slope functions

$$p(x,y,z), \quad q(x,y,z)$$

of the field is zero along every arc L in F on the transversal surface S by definition of transversality.

The equation

$$I(C_{32}) - I(E_{12}) = \int E(x,y,z,p,q,y',z')dx$$

holds for every admissible arc C_{32} in F joining the surface S with the point 2. For, if L_{13} is an arc on S in F joining the points 1 and 3, then,

$$I(C_{32}) - I(E_{12}) = I(C_{32}) - I^*(E_{12}) = I(C_{32}) - I^*(L_{13} + C_{32})$$

because the Hilbert integral with p,q is independent of path in a field. Choose R_1 so small that all its elements (x,y,z,y',z') and associated elements $[x,y,z,p(x,y,z), q(x,y,z)]$ are in the neighborhood N of theorem 1.9. Then

$$E > 0 \text{ unless}$$
$$y' = p, \text{ and}$$

$$z' = q$$

at every point of C_{12}. But the differential equations

$$y' = p(x,y,z) \text{ and}$$

$$z' = q(x,y,z)$$

have only one solution through the point 2 which is E_{12}. This completes the proof for the case I^+, III', $IV^{*\prime}$. For the case where II_N, IV', I^+ hold choose a neighborhood F so small that the condition

$$(x,y,z,p,q) \text{ a member of } F$$

implies

$$(x,y,z,p,q) \text{ a member of } N$$

where N is defined as in the definition of II_N. By the same reasoning as above

$$I(C_{32}) - I(E_{12}) > 0$$

unless

$$y' = p, \text{ and}$$

$$z' = q.$$

But then

$$C_{32} = E_{12}.$$

This completes the proof.

Section II

Problems in three space with one end-point on a curve.

The problem to be studied in this section is that of finding in a certain class of admissible arcs joining a fixed line L to a fixed point 2 one which minimizes the integral

$$I = \int_{x_1}^{x_2} f(x,y,z,y',z')dx$$

in the space of 5-tuples (x,y,z,y',z') of real numbers.

We will show that through a line L in three space it is possible to construct a surface S of the type discussed in section I. Consequently, this will reduce the problem of this section to that of section I.

It is obvious that for a minimizing arc E_{12} the conditions I, II, III must be satisfied. The transversality condition follows from theorem 1.2 if in the proof and statement of this theorem we replace the surface S by the curve L.

DEF.

If the determinant $\Delta(x, \alpha, \beta)$ is not identically zero along E_{12}, a point defined on E_{12} by

$$\Delta(x, \alpha_0, \beta_0) = 0$$

and such that $x \neq x_1$ is called a focal point of the curve L on E_{12}.

DEF.

A non-singular extremal arc E_{12} cut transversally by a non-singular curve L at the point 1, and not tangent to L at 1, is said to satisfy the focal point condition if there exists no focal point of L on E_{12} between 1 and 2.

Let the curve L be defined by the functions

$$(2.1) \qquad \xi(\alpha), \quad \eta(\alpha), \text{ and } \zeta(\alpha) \quad (\alpha' \leq \alpha \leq \alpha'')$$

which have continuous third order derivatives, and suppose that L does not intersect itself and has on it no singular points.

To formulate a necessary condition IV for this problem assume that the arc E_{12} whose minimizing properties are to be studied is a non-singular extremal arc cut transversally by L so that at the intersection point 1 of E_{12} and L the condition

$$(2.2) \qquad (f - y' f_{y'} - z' f_{z'})\xi_\alpha + f_{y'}\eta_\alpha + f_{z'}\zeta_\alpha = 0$$

is satisfied. Also assume that the function f is not zero at the element (x,y,z,y',z') belonging to the point 1 on E_{12}. This implies L is not tangent to E_{12} at the point 1 since (2.2) and the equality of the directions $1 : y : z$ and $\xi_\alpha : \eta_\alpha : \zeta_\alpha$ imply $f = 0$ at 1.

THEOREM (2.1)

Every non-singular minimizing arc for the problem of this section must be an extremal and satisfy the focal point condition.

If in the proof of Theorem 1.6 we replace the statement, "the surface S", by, "the curve L", the proof of this theorem is immediate.

DEF.

When the variables x, y, z, y', z' are replaced by the new set x, y, z, u, v for which u and v are defined by the equations

$$(2.3) \qquad u = f_{y'}(x,y,z,y',z'), \quad v = f_{z'}(x,y,z,y',z'),$$

then the new variables x, y, z, u, v are called canonical variables, and the equations of the extremals in terms of them are called canonical equations.

We will assume that the region R consists only of interior points (x,y,z,y',z') at which the determinant

$$f_{y'y'}\ f_{z'z'} - (f_{y'z'})^2$$

is different from zero, and that the equations 2.3 define a 1 - 1 correspondence between the points (x,y,z,y',z') of R and the points (x,y,z,u,v) of the region S into which R is transformed by means of equations 2.3. The equations 2.3 then have a single-valued solution [ ]

$$(2.4)\qquad y' = P(x,y,z,u,v),$$
$$z' = Q(x,y,z,u,v)$$

which also relate corresponding points of R and S.

This is a result of existence theorem's for implicit functions, the proofs of which can be found in Bliss "Lectures in the Calculus of Variations", p. 269.

DEF.

A function $H(x,y,z,u,v)$, the Hamiltonian Function, is defined by the equation

$$(2.5)\qquad H(x,y,z,u,v) = y'f_{y'} + z'f_{z'} - f$$
$$= Pu + Qv - f(x,y,z,P,Q), \text{ where } y' = P,\ z' = Q.$$

Taking partials of H we have

$$(2.6)\qquad H_u = P = y', \text{ and}$$
$$H_v = Q = z'.$$

Let

$$(2.7)\qquad \xi(\alpha,\beta),$$
$$\eta(\alpha,\beta),$$
$$\zeta(\alpha,\beta)$$

define a surface S, such that the points

(2.8) $[\xi, \eta, \zeta, \eta', \zeta']$

are admissible. Also assume the five functions $\xi, \eta, \zeta, \eta', \zeta'$ have continuous partial derivatives of the third order. The Hilbert Integral

$$\int [f dx + (dy - y' dx) f_{y'} + (dz - z' dx) f_{z'}]$$

then has the form

$$\int (P_0 d\alpha + Q_0 d\beta).$$

When we let

$$x = \xi(\alpha, \beta),$$
$$y = \eta(\alpha, \beta), \text{ and}$$
$$z = \zeta(\alpha, \beta),$$

then

$$dx = \xi_\alpha d\alpha + \xi_\beta d\beta,$$
$$dy = \eta_\alpha d\alpha + \eta_\beta d\beta, \text{ and}$$
$$dz = \zeta_\alpha d\alpha + \zeta_\beta d\beta.$$

The Hilbert Integral is then

$$\int [f \xi_\alpha d\alpha + f \xi_\beta d\beta + \eta_\alpha d\alpha f_{y'} + \eta_\beta d\beta f_{y'} - y' \xi_\alpha d\alpha f_{y'} - y' \xi_\beta d\beta f_{y'} + \zeta_\alpha d\alpha f_{z'} + \zeta_\beta d\beta f_{z'} - z' \xi_\alpha d\alpha f_{z'} - z' \xi_\beta d\beta f_{z'}].$$

$$= \int [f \xi_\alpha + (\eta_\alpha - \eta' \xi_\alpha) f_{y'} + (\zeta_\alpha - \zeta' \xi_\alpha) f_{z'}] d\alpha + [f \xi_\beta + (\eta_\beta - \eta' \xi) f_{y'} + (\zeta_\beta - \zeta' \xi_\beta) f_{z'}] d\beta.$$

Therefore we have

(2.9) $P_0 = f \xi_\alpha + (\eta_\alpha - \eta' \xi_\alpha) f_{y'} + (\zeta_\alpha - \zeta' \xi_\alpha) f_{z'}$, and

$Q_0 = f \xi_\beta + (\eta_\beta - \eta' \xi_\beta) f_{y'} + (\zeta_\beta - \zeta' \xi_\beta) f_{z'}$.

If there is a function

$$W(\alpha, \beta)$$

with continuous partial derivatives of the third order and such that

$P_0 = W_\alpha$, and

$$Q = W_\beta ,$$

then the Hilbert Integral is the integral of dW and is independent of the path in S. By theorem 1.8, the two parameter family of extremals with the initial elements 2.8 will form a field in every region F which it simply covers. If the surface S of 2.7 and the function $W(\alpha, \beta)$ are arbitrarily selected in advance in such a way that the surface is non-singular and the functions ξ, η, ζ, W have continuous partial derivatives of the third order, then the points 2.8 can be determined by solving for η' and ζ', as funtions of α and β, in the equations

(2.10) $$f\,\xi_\alpha + (\eta_\alpha - \eta'\xi_\alpha)f_{y'} + (\zeta_\alpha - \zeta'\xi_\alpha)f_{z'} = W_\alpha ,$$

(2.11) $$f\,\xi_\beta + (\eta_\beta - \eta'\xi_\beta)f_{y'} + (\zeta_\beta - \zeta'\xi_\beta)f_{z'} = W_\beta .$$

Implicit function theorems assure us that solutions

$$\eta'(\alpha, \beta),$$
$$\zeta'(\alpha, \beta)$$

exist, provided the equations 2.10 and 2.11 have an initial solution $(\alpha_0, \beta_0, \eta_0', \zeta_0')$ and that the point 2.8 is admissible and makes the functional determinant different from zero [1, p. 269].

Let us return to the problem of constructing a suitable surface. Consider a two-parameter family of extremals

(2.12) $$y(x, \alpha, \beta), \quad z(x, \alpha, \beta)$$

containing the arc E_{12} for values x, α, β satisfying conditions of the form

$$x_1 \leq x \leq x_2, \qquad \alpha = \alpha_0 , \qquad \beta = \beta_0$$

and such that each extremal of the family is cut transversally by the curve L at the point defined on the extremal by the value

$$x = \xi(\alpha).$$

To prove the existence of the family 2.12 consider a direction

$l : m : n$ transversal to E_{12} at the point 1 and not in the plane determined by the tangent to L and E_{12} at the point 1. Consider the equations

$$(2.13)\qquad \begin{aligned} -H\xi_\alpha + u\eta_\alpha + v\zeta_\alpha &= 0, \\ -Hl + um + vn &= \beta - \beta_0 \end{aligned}$$

where β_0 (constant) is selected arbitrarily. The arguments of H are

$$(x,y,z,y',z') = [\xi(\alpha), \eta(\alpha), \zeta(\alpha), u, v].$$

We have

$$-H\xi_\alpha + u\eta_\alpha + v\zeta_\alpha = 0, \text{ or}$$

$$-y'f_{y'}\xi_\alpha - z'f_{z'}\xi_\alpha + f\xi_\alpha + f_{y'}\eta_\alpha + f_{z'}\zeta_\alpha = 0,$$

and thus

$$(2.14)\qquad f\xi_\alpha + (\eta_\alpha - \eta'\xi_\alpha)u + (\zeta_\alpha - \zeta'\xi_\alpha)v = 0.$$

Similarly

$$-Hl + um + vn = \beta - \beta_0,$$

$$-y'f_{y'}l - z'f_{z'}l + fl + um + vn = \beta - \beta_0,$$

$$(2.15)\qquad fl + (m - y'l)u + (n - z'l)v = \beta - \beta_0, \text{ and thus}$$

$$(2.16)\qquad fl + (m - \eta'l)u + (n - \zeta'l)v = \beta - \beta_0.$$

Equations 2.13 are similar to equations 2.10 and 2.11 with the variables x,y,z,y',z' replaced by the canonical variables x,y,z,u,v. They have the special solution

$$(\alpha, \beta, u, v) = (\alpha_0, \beta_0, u_1, v_1)$$

where u_1 and v_1 are the values of $f_{y'}$ and $f_{z'}$ respectively at the point 1 on E_{12}. At the intersection point 1 of E_{12} and L the transversality condition is satisfied and therefore we have

$$(f - y'f_{y'} - z'f_{z'}) + f_{y'}\eta_\alpha + f_{z'}\zeta_\alpha = 0$$

at the point 1. Then

$$f\xi_\alpha - y'u\xi_\alpha - z'v\xi_\alpha + u\eta_\alpha + v\zeta_\alpha = 0, \text{ or}$$

$$f\xi_\alpha + (\eta_\alpha - \eta'\xi_\alpha)u + (\zeta_\alpha - \zeta'\xi_\alpha)v = 0.$$ Also

$$(f - y'f_{y'} - z'f_{z'})l + f_{y'}m + f_{z'}n = 0$$

at the point 1 and therefore

$$fl + (m - \eta'l)u + (n - \zeta'l)v = 0 = \beta - \beta_0$$

at the point 1 on E_{12}.

The functional determinant of equations 2.13 with respect to u and v is

$$(2.17)\quad \begin{vmatrix} (\eta_\alpha - \xi_\alpha H_u) & (\zeta_\alpha - \xi_\alpha H_v) \\ (m - l H_u) & (n - l H_v) \end{vmatrix} =$$

$$\begin{vmatrix} (\eta_\alpha - y'\xi_\alpha) & (\zeta_\alpha - z'\xi_\alpha) \\ (m - y'l) & (n - z'l) \end{vmatrix} =$$

$$\begin{vmatrix} 1 & \xi_\alpha & l \\ y' & \eta_\alpha & m \\ z' & \zeta_\alpha & n \end{vmatrix}$$

because on E_{12} we have

$$y' = H_u ,$$
$$z' = H_v .$$

The last determinant is different from zero at the point 1 because at that point the three ratios $(1 : y' : z')$, $(\xi_\alpha : \eta_\alpha : \zeta_\alpha)$, $(l : m : n)$ determine lines which are not coplanar, according to our hypothesis. Then, according to theorem 1.7 equations 2.12 have continuous second order partial derivatives near the values (α_0, β_0) and such that

$$u(\alpha_0, \beta_0) = u_1 ,$$
$$v(\alpha_0, \beta_0) = v_1 .$$

The family of extremals obtained by substituting the initial values

$$\xi(\alpha),\quad \eta(\alpha),\quad \zeta(\alpha),\quad u(\alpha, \beta),\ v(\alpha, \beta)$$

into the solutions of equation 2.6 is a two-parameter family 2.12 cut transversally by the arc L at the value

$$x = \xi(\alpha).$$

THEOREM (2.2)

If the Legendre condition III holds at the initial point 1 of the non-singular extremal arc E_{12}, then the family 2.12 has a determinant $\Delta(x, \alpha_0, \beta_0)$ which vanishes at x_1 but is different from zero near x_1, so that there is no focal point of the curve L determined by this family on E_{12} near the point 1.

Proof

Note that the equations

$$\eta(\alpha) = y\,[\xi(\alpha), \alpha, \beta] \quad \text{and}$$
$$\zeta(\alpha) = z\,[\xi(\alpha), \alpha, \beta]$$

hold along the curve defined by the functions 2.1. By differentiation with respect to α and β these equations evidently imply that at the point 1

$$\eta_\alpha = y_\xi \xi_\alpha + y_\alpha \ ;\ \eta_\beta = 0 = y_\beta \ ;\ \zeta_\alpha = z_\xi \xi + z_\alpha \ ;\ \zeta_\beta = 0 = z\ ,\ \text{or}$$

(2.18) $$y_\alpha = \eta_\alpha - y_\xi \xi_\alpha = c_1\ ,\quad y_\beta = 0,\ \text{and}$$

(2.19) $$z_\alpha = \zeta_\alpha - z_\xi \xi_\alpha = c_2\ ,\quad z_\beta = 0.$$

Therefore at the point 1 we have

$$\Delta(x, \alpha_0, \beta_0) = \begin{vmatrix} y_{\alpha_0} & y_{\beta_0} \\ z_{\alpha_0} & z_{\beta_0} \end{vmatrix} = 0.$$

By an application of Taylor's formula to the column $y_\beta\ ,\ z_\beta$ of $\Delta(x, \alpha, \beta)$

$$\begin{matrix} y_\alpha(x) & y_\beta(x) \\ z_\alpha(x) & z_\beta(x) \end{matrix} = \begin{vmatrix} y_\alpha(x) & y_\beta(x) + y_\beta[x + Q_1(x - x_1)\ (x - x_1)] \\ z_\alpha(x) & z_\beta(x) + z_\beta[x + Q_2(x - x_1)\ (x - x_1)] \end{vmatrix} =$$

$$(2.20)\qquad \begin{vmatrix} y_\alpha & y_\beta \\ z_\alpha & z_\beta \end{vmatrix} + (x - x_1)\begin{vmatrix} y_\alpha & y'_\beta[x + Q_1(x - x_1)] \\ z_\alpha & z'_\beta[x + Q_1(x - x_1)] \end{vmatrix} .$$

Then we have at the point (x_1, α_0, β_0),

$$(2.21)\qquad \begin{vmatrix} y_\alpha(x_1) & y_\beta(x_1) \\ z_\alpha(x_1) & z_\beta(x_1) \end{vmatrix} = (x_1 - x_1)\begin{vmatrix} y_\alpha & y'_\beta \\ z_\alpha & z'_\beta \end{vmatrix} = (x_1 - x_1)\begin{vmatrix} c_1 & y'_\beta \\ c_2 & z'_\beta \end{vmatrix} = 0.$$

If we differentiate equations 2.13 with respect to β we see

$$-H_\beta \xi_\alpha \quad -H\xi_{\alpha\beta} \quad + u_\beta\eta_\alpha \quad + u\eta_{\alpha\beta} \quad + v_\beta\zeta_\alpha \quad + v\zeta_{\alpha\beta} \quad = 0 \text{ and}$$

$$H = H[\xi(\alpha), \eta(\alpha), \zeta(\alpha), u, v], \text{ therefore}$$

$$-H_\beta \quad = -H_u u_\beta - H_v v_\beta = -y' u_\beta \quad - z' v_\beta \quad .$$

We now have

$$-y' u_\beta \xi_\alpha \quad - z' v_\beta \xi_\alpha \quad + u_\beta \eta_\alpha \quad + v_\beta \zeta_\alpha \quad = 0, \text{ or}$$

$$(\eta_\alpha \quad - y'\xi_\alpha)u_\beta \quad + (\zeta_\alpha \quad - z'\xi_\alpha)v_\beta \quad = 0, \text{ and thus}$$

$$(2.22)\qquad c_1 u_\beta + c_2 v_\beta \quad = 0.$$

Similarly, we have

$$-H_\beta l - Hl_\beta \quad + u_\beta m + um_\beta \quad + v_\beta n \quad + vn_\beta \quad = 1, \text{ or}$$

$$(m - y'l)u_\beta + (n - z'l)v_\beta \quad = 1$$

and we see that the derivatives u_β and v_β are not both zero and that the determinant 2.21 is

$$u_\beta y'_\beta \quad + v_\beta z'_\beta$$

except for a non-vanishing factor. If both c_1 and c_2 are zero, then

$$c_1 \quad = y_\alpha \quad = \quad \eta_\alpha \quad - y'\xi_\alpha \quad = 0, \text{ and}$$

$$c_2 \quad = z_\alpha \quad = \quad \zeta_\alpha \quad - z'\xi_\alpha \quad = 0, \text{ or}$$

$$\frac{\eta_\alpha}{\xi_\alpha} = y', \text{ and } \quad \frac{\zeta_\alpha}{\xi_\alpha} = z' \quad .$$

Then $(1, y', z')$, the direction of E_{12}, is the same as $\left(1, \frac{\eta_\alpha}{\xi_\alpha}, \frac{\zeta_\alpha}{\xi_\alpha}\right)$, the direction of L but this contradicts the hypothesis that L is not tangent to E_{12}. Therefore c_1 and c_2 are not both zero at the point 1. We will now show 2.21 is $u_\beta y'_\beta + v_\beta z'_\beta$ except for a non-vanishing factor. Let us consider the different cases which can occur.

<u>Case I.</u> $(\quad u_\beta \quad \neq 0, \quad c_1 \quad \neq 0 \quad)$.

This implies $c_2 \neq 0$ and $v_\beta \neq 0$. Then

$$c_1 z'_\beta - c_2 y'_\beta = c_2\left[\left(\frac{c_1}{c_2}\right)z'_\beta - y'_\beta\right] = -c_2\left[\left(\frac{z'_\beta}{u_\beta}\right)v_\beta + y'_\beta\right]$$

$$= \left(\frac{-c_2}{u_\beta}\right)\left(z'_\beta v_\beta + y'_\beta u_\beta\right).$$

Case II. $(v_\beta \neq 0,\ c_2 \neq 0)$.

This implies $c_1 \neq 0$ and $u_\beta \neq 0$. We then have the same argument as in case I.

Case III. $(u_\beta \neq 0,\ c_2 \neq 0)$

Then we have

$$c_1 z'_\beta - c_2 y'_\beta = c_2\left[\left(\frac{c_1}{c_2}\right)z'_\beta - y'_\beta\right] = c_2\left[-\left(\frac{v_\beta}{u_\beta}\right)z'_\beta - y'_\beta\right]$$

$$= \frac{-c_2}{u_\beta}\left(v_\beta z'_\beta + u_\beta y'_\beta\right).$$

Case IV. $(v_\beta \neq 0,\ c_1 \neq 0)$.

Then we have

$$c_1 z'_\beta - c_2 y'_\beta = c_1\left[z'_\beta - \left(\frac{c_2}{c_1}\right)y'_\beta\right] = c_1\left[z'_\beta + \left(\frac{u_\beta}{v_\beta}\right)y'_\beta\right]$$

$$= \frac{c_1}{v_\beta}\left(z'_\beta v_\beta + u_\beta y'_\beta\right).$$

We have

(2.22) $u = f_{y'}[\xi(\alpha), \eta(\alpha), \zeta(\alpha), y', z']$, where

$y' = y'[\xi(\alpha), \alpha, \beta]$, and

$z' = z'[\xi(\alpha), \alpha, \beta]$.

Also we have

(2.23) $v = f_{z'}[\xi(\alpha), \eta(\alpha), \zeta(\alpha), y', z']$, where

$y' = y'[\xi(\alpha), \alpha, \beta]$, and

$z' = z'[\xi(\alpha), \alpha, \beta]$.

Taking the partials with respect to β we see

(2.24) $u_\beta = f_{y'y'}\, y'_\beta + f_{y'z'}\, z'_\beta$, and

$v_\beta = f_{z'y'}\, y'_\beta + f_{z'z'}\, z'_\beta$, and then

(2.25) $u_\beta y'_\beta + v_\beta z'_\beta = f_{y'y'} y'^2_\beta + 2f_{y'z'} z'_\beta y'_\beta + f_{z'z'} z'^2_\beta$.

By the Legendre condition, i.e., condition III,

$$f_{y'y'} y'^2_\beta + 2f_{y'z'} z'_\beta y'_\beta + f_{z'z'} z'^2_\beta \geq 0, \text{ or}$$

$$f_{y'y'} \left(\frac{y'_\beta}{z'_\beta}\right)^2 + 2f_{y'z'} \left(\frac{y'_\beta}{z'_\beta}\right) + f_{z'z'} \geq 0$$

Because equation 2.25 is linear on the left in $\left(\frac{y'_\beta}{z'_\beta}\right)$, it is seen that the roots must be equal. But then

$$f^2_{y'z'} - f_{y'y'} f_{z'z'} = 0$$

and this implies that E_{12} is not non-singular which is a contradiction. Therefore we have

$$u_\beta y'_\beta + v_\beta z'_\beta = f_{y'y'} y'^2_\beta + 2f_{y'z'} z'_\beta y'_\beta + f_{z'z'} z'^2_\beta > 0$$

and evidently $\Delta(x, \alpha_0, \beta_0)$ vanishes at x_1 but is different from zero near x_1 .

DEF.

A surface S is said to be non-singular at a point

$$\xi(\alpha, \beta) = \xi(\alpha_0, \beta_0),$$
$$\eta(\alpha, \beta) = \eta(\alpha_0, \beta_0),$$
$$\zeta(\alpha, \beta) = \zeta(\alpha_0, \beta_0),$$

if the matrix $\begin{bmatrix} \xi_\alpha & \eta_\alpha & \zeta_\alpha \\ \xi_\beta & \eta_\beta & \zeta_\beta \end{bmatrix}$ is of rank two at the point (α_0, β_0).

<u>THEOREM (2.3)</u>

If a non-singular arc E_{12} is cut transversally by L at the point 1 and contains no focal point of L, then through the curve L there is a non-singular surface S transversal and not tangent to E_{12} at the point 1, and such that on E_{12} there is no focal point of S.

<u>Proof</u>.

Consider the functions $A(\alpha, \beta)$, $B(\alpha, \beta)$, $C(\alpha, \beta)$ defined by the equations

(2.26) $$A = l - H_u \frac{\beta - \beta_0}{f} ,$$

$$B = m - H_u \frac{\beta - \beta_0}{f}$$

$$C = n - H_v \frac{\beta - \beta_0}{f}$$

in which l, m, n are the values appearing in equations 2.13, and the arguments of H_u, H_v, and f are those associated with the solutions $u(\alpha, \beta)$, $v(\alpha, \beta)$ of equations 2.13. The surface defined by the functions

(2.27) $$\xi_0(\alpha, \beta, \epsilon) = \xi(\alpha) + \epsilon \int_{\beta_0}^{\beta} A(\alpha, \beta) d\beta ,$$
$$\eta_0(\alpha, \beta, \epsilon) = \eta(\alpha) + \epsilon \int_{\beta_0}^{\beta} B(\alpha, \beta) d\beta ,$$
$$\zeta_0(\alpha, \beta, \epsilon) = \zeta(\alpha) + \epsilon \int_{\beta_0}^{\beta} C(\alpha, \beta) d$$

has continuous derivatives of at least the second order in α, β, ϵ and contains the curve L for

$$\beta = \beta_0 .$$

Also this surface is non-singular along the curve L, provided

$$\epsilon \neq 0 .$$

We have

$$\xi_{0\alpha} = \xi'(\alpha) + \epsilon \int_{\beta_0}^{\beta} A_\alpha(\alpha, \beta) d\beta ,$$
$$\xi_{0\alpha\alpha} = \xi''(\alpha) + \epsilon \int_{\beta_0}^{\beta} A_{\alpha\alpha}(\alpha, \beta) d\beta ,$$
$$\xi_{0\beta} = \epsilon A(\alpha, \beta),$$
$$\xi_{0\beta\beta} = \epsilon A_\beta(\alpha, \beta),$$
$$\xi_{0\epsilon} = \int_{\beta_0}^{\beta} A(\alpha, \beta) d\beta ,$$
$$\xi_{0\epsilon\epsilon} = 0,$$
$$\eta_{0\alpha} = \eta'(\alpha) + \epsilon \int_{\beta_0}^{\beta} B_\alpha(\alpha, \beta) d\beta ,$$
$$\eta_{0\alpha\alpha} = \eta''(\alpha) + \epsilon \int_{\beta_0}^{\beta} B_{\alpha\alpha}(\alpha, \beta) d\beta ,$$
$$\zeta_{0\alpha} = \zeta'(\alpha) + \epsilon \int_{\beta_0}^{\beta} C_\alpha(\alpha, \beta) d\beta ,$$

$$\zeta_{o\alpha\alpha} = \zeta''(\alpha) + \epsilon\int_{\beta_0}^{\beta} C_{\alpha\alpha}(\alpha, \beta)d\beta,$$
$$\eta_{o\beta} = \epsilon B(\alpha, \beta),$$
$$\eta_{o\beta\beta} = \epsilon B_\beta(\alpha, \beta),$$
$$\eta_{o\epsilon} = \int_{\beta_0}^{\beta} B(\alpha, \beta)d\beta,$$
$$\eta_{o\epsilon\epsilon} = 0,$$
$$\zeta_{o\beta} = \epsilon C(\alpha, \beta),$$
$$\zeta_{o\beta\beta} = \epsilon C_\beta(\alpha, \beta),$$
$$\zeta_{o\epsilon} = \int_{\beta_0}^{\beta} B(\alpha, \beta)d\beta, \text{ and}$$
$$\zeta_{o\epsilon\epsilon} = 0.$$

For

$\beta = \beta_0$, we have

$$\xi_o(\alpha, \beta, \epsilon) = \xi(\alpha),$$
$$\eta_o(\alpha, \beta, \epsilon) = \eta(\alpha), \text{ and}$$
$$\zeta_o(\alpha, \beta, \epsilon) = \zeta(\alpha).$$

Assume

$$(2.28) \qquad \begin{bmatrix} \xi_{o\alpha} & \eta_{o\alpha} & \zeta_{o\alpha} \\ \xi_{o\beta} & \eta_{o\beta} & \zeta_{o\beta} \end{bmatrix}$$

is not of rank two. Then

$$\xi_{o\alpha}\eta_{o\beta} - \xi_{o\beta}\eta_{o\alpha} = 0,$$
$$\xi_{o\alpha}\zeta_{o\beta} - \xi_{o\beta}\zeta_{o\alpha} = 0, \text{ and}$$
$$\eta_{o\alpha}\zeta_{o\beta} - \eta_{o\beta}\zeta_{o\alpha} = 0. \text{ Thus}$$
$$[\xi'(\alpha) + \epsilon\int_{\beta_0}^{\beta} A_\alpha d\beta]\epsilon B - \epsilon A[\eta'(\alpha) + \epsilon\int_{\beta_0}^{\beta} B_\alpha d\beta] = 0,$$
$$[\xi'(\alpha) + \epsilon\int_{\beta_0}^{\beta} A_\alpha d\beta]\epsilon C - \epsilon A[\zeta'(\alpha) + \epsilon\int_{\beta_0}^{\beta} C_\alpha d\beta] = 0, \text{ and}$$
$$[\eta'(\alpha) + \epsilon\int_{\beta_0}^{\beta} B_\alpha d\beta]\epsilon C - \epsilon B[\zeta'(\alpha) + \epsilon\int_{\beta_0}^{\beta} C_\alpha d\beta] = 0, \text{ or}$$
$$\epsilon[B\xi' - A\eta' + \epsilon(B\int_{\beta_0}^{\beta} A_\alpha d\beta - A\int_{\beta_0}^{\beta} B_\alpha d\beta)] = 0,$$
$$\epsilon[C\xi' - A\zeta' + \epsilon(C\int_{\beta_0}^{\beta} A_\alpha d\beta - A\int_{\beta_0}^{\beta} C_\alpha d\beta)] = 0, \text{ and}$$
$$\epsilon[C\eta' - B\zeta' + \epsilon(C\int_{\beta_0}^{\beta} B_\alpha d\beta - B\int_{\beta_0}^{\beta} C_\alpha d\beta)] = 0.$$

We have

$$\epsilon \neq 0;$$

therefore

$$B\xi' - A\eta' = 0$$

at

$$\beta = \beta_0 .$$

Similarly we have

$$C\xi' - A\zeta' = 0, \text{ and}$$

$$C\eta' - B\zeta' = 0$$

at

$$\beta = \beta_0 .$$

Then by substituting into equations 2.26 we see

$$m\xi' - l\eta' = 0,$$

$$n\xi' - l\zeta' = 0, \text{ and}$$

$$n\eta' - m\zeta' = 0$$

at

$$\beta = \beta_0 .$$

This implies

$$\begin{bmatrix} \xi' & \eta' & \zeta' \\ l & m & n \end{bmatrix}$$

is not of rank two. But then we see that the direction $l:m:n$ is in the plane determined by the tangents to L and $E_{/2}$ at $\beta = \beta_0$. This is a contradiction and therefore the matrix 2.28 is of rank two.

The equations

$$(2.29) \qquad -H\xi_{0\alpha} + u\eta_{0\alpha} + v\zeta_{0\alpha} = 0, \text{ and}$$

$$-HA + uB + vC = 0$$

with the arguments

$$(x, y, z, u, v) = (\xi_0, \eta_0, \zeta_0, u, v)$$

are equivalent to equations 2.13 when $\epsilon = 0$ and therefore have the initial solutions

$$(\alpha, \beta, \epsilon, u, v) = [\alpha, \beta, 0, u(\alpha, \beta), v(\alpha, \beta)]$$

on which the functional determinant of their first member with respect to u and v is 2.17 and is different from zero near the values (α_0, β_0). We have

$$-H[\xi_\alpha + \epsilon\int_{\beta_0}^{\beta} A_\alpha d\beta] + u[\eta_\alpha + \epsilon\int_{\beta_0}^{\beta} B_\alpha d\beta] + v[\zeta_\alpha + \epsilon\int_{\beta_0}^{\beta} C_\alpha d\beta] = 0,$$

and similarly we have

$$-H\left[1 - \frac{\beta-\beta_0}{f}\right] + u\left[m - H_u\frac{\beta-\beta_0}{f}\right] + v\left[n - H_v\frac{\beta-\beta_0}{f}\right] = 0.$$

For

$$\epsilon = 0,$$

$$\beta = \beta_0, \text{ and}$$

$$\alpha = \alpha_0$$

we have

$$-H\xi_\alpha + u\eta_\alpha + v\zeta_\alpha = 0, \text{ and}$$

$$-Hl + um + vn = 0.$$

Equations 2.29 have solutions

$$u(\alpha, \beta, \epsilon),$$

$$v(\alpha, \beta, \epsilon)$$

with continuous derivatives of the second order and reducing to the solutions

$$u(\alpha, \beta),$$

$$v(\alpha, \beta)$$

of equations 2.13 for

$$\epsilon = 0$$

When substituted with ξ_0, η_0, ζ_0 from 2.27 as initial values in the functions

(2.30) $y(x\ , a, b, c, d)$,

$z(x\ , a, b, c, d)$.

$u(x\ , a, b, c, d)$, and

$v(x\ , a, b, c, d)$,

which have the form of the solutions of the differential equations

$$\frac{dy}{dx} = H,$$

$$\frac{dz}{dx} = H,$$

$$\frac{du}{dx} = -H, \text{ and}$$

$$\frac{dv}{dx} = -H,$$

these solutions define a three-parameter family of extremals,

(2.31) $Y(x, \alpha , \beta , \epsilon)$,

$Z(x, \alpha , \beta , \epsilon)$.

When $\epsilon \neq 0$ this family is cut transversally at

$$x = \xi_o(\alpha , \beta , \epsilon)$$

by the surface 2.27. To see this, note that

$$\xi_{o\beta} = \epsilon A,$$

$$\eta_{o\beta} = \epsilon B, \text{ and}$$

$$\zeta_{o\beta} = \epsilon C,$$

and that

$$-Hdx + udy + vdz = 0$$

is the transversality condition. Using equations 2.29 we have

$$-H \xi_{o\alpha} + u\eta_{o\alpha} + v\zeta_{o\alpha} = 0, \text{ and}$$

$$-HA\epsilon + uB\epsilon + vC\epsilon = -H\xi_{o\beta} + u\eta_{o\beta} + v\zeta_{o\beta} = 0, \text{ or}$$

$$-H(\xi_{o\alpha} + \xi_{o\beta}) + u(\eta_{o\alpha} + \eta_{o\beta}) + v(\zeta_{o\alpha} + \zeta_{o\beta}) = 0$$

and then

$$-Hdx + udy + vdz = 0.$$

When $\epsilon = 0$, the three parameter family of extremals 2.31 contains the family 2.12 cut transversally at

$$x = \xi(\alpha)$$

by the curve L. It can now be seen that

$$(2.32) \qquad Y(\xi_0, \alpha, \beta, \epsilon) = \eta_0(\alpha, \beta, \epsilon),$$
$$Z(\xi_0, \alpha, \beta, \epsilon) = \zeta_0(\alpha, \beta, \epsilon),$$

$$(2.33) \qquad Y(x, \alpha, \beta, 0) = y(x, \alpha, \beta), \text{ and}$$
$$Z(x, \alpha, \beta, 0) = z(x, \alpha, \beta).$$

The determinant

$$(2.34) \qquad \Delta(x, \alpha, \beta, \epsilon) = Y_\alpha Z_\beta - Y_\beta Z_\alpha$$

for the family 2.31 has the expansion

$$(2.35) \qquad \Delta(x_1, \alpha_0, \beta_0, \epsilon) = \Delta(x, \alpha_0, \beta_0, 0) + \epsilon\Delta_\epsilon(x_1, \alpha_0, \beta_0, \theta_\epsilon), \text{ where } (0 < \theta < 1).$$

The first term on the right in 2.35 is the corresponding determinant for the family 2.12 and consequently is different from zero in the interval $x_1 \leq x \leq x_2$ because of the condition IV′. By differentiating equations 2.32 and using equations 2.27 we see

$$(2.36) \qquad Y_\alpha = Y_{\xi_0}\xi_{0\alpha} + Y_\alpha = \eta_{0\alpha},$$
$$Y_\beta = Y_{\xi_0}\xi_{0\beta} + Y_\beta = \eta_{0\beta}, \text{ and}$$
$$(2.37) \qquad Z_\alpha = Z_{\xi_0}\xi_{0\alpha} + Z_\alpha = \zeta_{0\alpha},$$
$$Z_\beta = Z_{\xi_0}\xi_{0\beta} + Z_\beta = \zeta_{0\beta}.$$

We now have

$$\Delta(x, \alpha, \beta, \epsilon) = \begin{vmatrix} Y_\alpha & Y_\beta \\ Z_\alpha & Z_\beta \end{vmatrix}$$

$$= \begin{vmatrix} (\eta_{0\alpha} - Y_{\xi_0}\xi_{0\alpha}) & (\eta_{0\beta} - Y_{\xi_0}\xi_{0\beta}) \\ (\zeta_{0\alpha} - Z_{\xi_0}\xi_{0\alpha}) & (\zeta_{0\beta} - Z_{\xi_0}\xi_{0\beta}) \end{vmatrix}$$

$$= \begin{vmatrix} [\eta_\alpha + \epsilon\int_{\beta_0}^{\beta} B_\alpha d\beta - Y_{\xi_0}(\xi_\alpha + \epsilon\int A_\alpha d\beta)] & [\epsilon B - Y_{\xi_0} A\epsilon] \\ [\zeta_\alpha + \epsilon\int_{\beta_0}^{\beta} C_\alpha d\beta - Z_{\xi_0}(\xi_\alpha + \epsilon\int A_\alpha d\beta)] & [\epsilon C - Z_{\xi_0} A\epsilon] \end{vmatrix}.$$

Therefore

$$\Delta_\epsilon(\xi_0, \alpha, \beta, \epsilon) = \begin{vmatrix} \frac{\partial}{\partial\epsilon}[\eta_\alpha + \epsilon\int B_\alpha d\beta - Y_{\xi_0}(\xi_\alpha + \epsilon\int A_\alpha d\beta)] & [\epsilon B - \epsilon Y_{\xi_0} A] \\ \frac{\partial}{\partial\epsilon}[\zeta_\alpha + \epsilon\int C_\alpha d\beta - Z_{\xi_0}(\xi_\alpha + \epsilon\int A_\alpha d\beta)] & [\epsilon C - Z_{\xi_0} A] \end{vmatrix}$$

$$+ \begin{vmatrix} [\eta_\alpha + \int B_\alpha d\beta - Y_{\xi_0}(\xi_\alpha + \epsilon\int A_\alpha d\beta)] & \frac{\partial}{\partial\epsilon}[\epsilon B - \epsilon Y_{\xi_0} A] \\ [\zeta_\alpha + \int C_\alpha d\beta - Z_{\xi_0}(\xi_\alpha + \epsilon\int A_\alpha d\beta)] & \frac{\partial}{\partial\epsilon}[\epsilon C - \epsilon Z_{\xi_0} A] \end{vmatrix}$$

$$= \begin{vmatrix} [\eta_{\alpha\epsilon} + \int B_\alpha d\beta - Y_{\xi_0\epsilon}\xi_\alpha - Y_{\xi_0}\xi_{\alpha\epsilon} - \int A_\alpha d\beta] & [\epsilon B - \epsilon Y_{\xi_0} A] \\ [\zeta_{\alpha\epsilon} + \int C_\alpha d\beta - Z_{\xi_0\epsilon}\xi_\alpha - Z_{\xi_0}\xi_{\alpha\epsilon} - \int A_\alpha d\beta] & [\epsilon C - \epsilon Z_{\xi_0} A] \end{vmatrix}$$

$$+ \begin{vmatrix} [\eta_\alpha + \epsilon\int B_\alpha d\beta - Y_{\xi_0}(\xi_\alpha + \epsilon\int A_\alpha d\beta)] & [B - AY_{\xi_0}] \\ [\zeta_\alpha + \epsilon\int C_\alpha d\beta - Z_{\xi_0}(\xi_\alpha + \epsilon\int A_\alpha d\beta)] & [C - AZ_{\xi_0}] \end{vmatrix},$$

and thus

$$\Delta_\epsilon(x, \alpha_0, \beta_0, 0) = \begin{vmatrix} (\eta_\alpha - y'\xi_\alpha) & (\zeta_\alpha - z'\xi_\alpha) \\ (m - y'l) & (n - lz') \end{vmatrix} \neq 0,$$

because of 2.17.

Therefore there exist positive constants h, k, such that $\Delta_\epsilon(x, \alpha_0, \beta_0, \theta_\epsilon)$ is different from zero for

$$x_1 \leq x \leq x_1 + h,$$

$$0 < |\epsilon| < k.$$

Hence, for

$$0 < |\epsilon| < k$$

and giving $\epsilon\Delta_\epsilon$ the same sign as $\Delta(x, \alpha_0, \beta_0, 0)$ on the interval

$$x_1 < x < x_1 + h,$$

the expression 2.35 will be different from zero on

$$x_1 \leq x < x_1 + h,$$

and it will also be different from zero on the whole interval

$$x_1 \leq x \leq x_2$$

if ϵ is still further restricted so that on the interval

$$x_1 + h \leq x \leq x_2$$

the second term in 2.35 has absolute value less then that of the first term.

This completes the proof.

THEOREM (2.4)

Let E_{12} be an admissible arc without corners in xyz-space, cut at a single point 1 by a non-singular curve L and such that the integrand function f is different from zero at the point 1 on E_{12}. Then for the problem of this section the conditions I, III$'$, IV$'$ are sufficient for $I(E_{12})$ to be a relative weak minimum, and the conditions I, II_N, IV$'$ with the non-singularity of E_{12} are sufficient for $I(E_{12})$ to be a strong relative minimum.

Proof.

This theorem is a consequence of Theorem 1.10 and Theorem 2.3.

PART II

Section I

Functions Spaces

In the ordinary theory of maxima or minima, the existence of a greatest or smallest value of a function in a closed domain is assured by the Bolzano-Weierstrass convergence theorem: a bounded set of points always contains a covergent sequence. This fact, together with the continuity of the function, serves to secure the existence of an extreme value.

In the calculus of variations the continuity of the function often has to be replaced by a weaker property, semi-continuity. Another difficulty in the calculus of variations arises from the fact that the Bolzano-Weierstrass convergence theorem does not hold if the elements of the set are no longer points on a line or in a n-dimensional space, but are functions, curves or surfaces.

There exists a remedy which very often proves sufficient in the direct methods of the calculus of variations. By a suitable restrictive condition imposed on the functions of a set, one can again obtain a theorem analogous to the Bolzano-Weierstrauss theorem, namely, Arzela's theorem, which will be considered in this section.

We will be concerned with the existence of a minimum of a functional on a function space. In particular our interest will be directed toward an arc length functional. Considering topological characteristics we will see what restrictions placed on a subset of the function space will assure that the functional assumes an extreme value on the subset.

DEF.

A set M in the metric space R is said to be relatively compact if every sequence of elements in M contains a subsequence which converges to some x in R.

DEF.

A set M in the metric space R is said to be compact if every sequence of elements in M contains a subsequence which converges to some x in M.

DEF.

Let M be any set in the metric space R and let ϵ be a positive number. Then the set A in R is said to be an ϵ-net with respect to M if for an arbitrary point x in M at least one point a, an element in A, can be found such that

$$\rho(a, x) < \epsilon.$$

DEF.

A subset M of R is said to be totally bounded if R contains a finite ϵ-net with respect to M for every $\epsilon > 0$. Note that the points of the ϵ-net are required only to be in R, not necessarily in M.

DEF.

A sequence $\{x_n\}$ of points of a metric space R is a fundamental sequence if it satisfies the Cauchy criterion, that is for arbitrary $\epsilon > 0$ there exists an interger N_ϵ such that

$$\rho(x_{n'}, x_{n''}) < \epsilon$$

for all

$$n', n'' \geq N_\epsilon.$$

DEF.

If every fundamental sequence in the space R converges to an element in R, then R is said to be complete.

THEOREM 3.1

A necessary and sufficient condition that a subset M of a complete metric space R be relatively compact is that M be totally bounded.

Proof. (Necessity).

Let us assume that M is not totally bounded. Then, by definition, for some $\epsilon > 0$ a finite net N_ϵ cannot be found in M. Let us take an arbitrary point x_1, in M. By our assumption, a point x_2 in M can be found such that

$$\rho(x_1, x_2) \geq \epsilon .$$

Similarly, a point x_3 in M can be found such that

$$\rho(x_1, x_3) \geq \epsilon , \text{ and}$$

$$\rho(x_2, x_3) \geq \epsilon ,$$

for otherwise the points x_1 and x_2 would form an ϵ-net in M. If we continue this process we obtain a sequence

$$x_1, x_2, \ldots\ldots\ldots, x_n, \ldots\ldots$$

of points in R such that

$$\rho(x_n, x_m) \geq \epsilon , \text{ for}$$

$$m \neq n.$$

But then it is impossible to select any convergent subsequence from such a sequence and M could not be relatively compact. This is a contradiction and our assumption is wrong. Therefore M is totally bounded.

(Sufficiency)

Let R be complete and M totally bounded. Let $\{x_n\}$ be a sequence of points in M. Let us set

$$\epsilon_1 = 1,$$

$$\epsilon_2 = 1/2$$

....

....

$\epsilon_k = 1/k$

...

and construct for every ϵ_k a corresponding ϵ_k-net in M, say

$$\{ a^k_1, a^k_2, \ldots\ldots, a^k_{n_k} \} .$$

Describe about each of the points which form a 1-net in M a sphere of radius 1. Since these spheres cover M and are finite in number at least one of them, say S_1, contains an infinite subsequence

$$x^1_1, \quad x^1_2, \ldots\ldots, x^1_n, \ldots\ldots$$

of the sequence $\{ x_n \}$. Further, about each of the points which form a $\frac{1}{2}$-net in R we describe a sphere of radius $\frac{1}{2}$. Since the number of spheres in finite, at least one of them, say S_2, contains an infinite subsequence

$$x^2_1, x^2_2, x^2_3, \ldots\ldots x^2_n, \ldots\ldots$$

of the sequence $\{ x^1_n \}$. Similarly, we find a sphere S_3 of radius $\frac{1}{3}$ containing an infinite subsequence

$$x^3_1, x^3_2, \ldots\ldots x^3_n, \ldots\ldots$$

of the sequence $\{ x^2_n \}$. We continue this process and obtain an infinte set of sequences,

$$\begin{array}{l} x^1_1, x^1_2, \ldots\ldots, x^1_n, \ldots \\ x^2_1, x^2_2, \ldots\ldots, x^2_n, \ldots \\ \cdot, \cdot, \ldots\ldots, \cdot, \ldots \end{array}$$

$$. \; , \; . \; , \; .... \; , \; . \; , \; ...$$

$$x_1^m, \; x_2^m, \; ...., \; x_n^m, \; ...$$

$$. \; , \; . \; , \; .... \; , \; . \; , \; ...$$

We now choose from the above sequences the sequence

$$x_1^1, \; x_2^2, \; x_3^3, \; ......, \; x_n^n, \; ...$$

This sequence is fundamental because all of its terms beginning with x_n^n lie in the interior of a sphere S_n of radius $\frac{1}{n}$. Since R is complete, this sequence has a limit point x in R. Then by definition M is relatively compact.

DEF.

Let C[a,b] denote the set of all continuous functions defined on the segment [a,b] with distance function

$$\rho(f,g) = \sup \left\{ |g(t) - f(t)| \; ; \; a \leq t \leq b \right\} .$$

The space C[a,b] forms a metric space. A sequence in C[a,b] is convergent if and only if it is uniformly convergent in the usual terminology.

DEF.

A family $\{Q(x)\}$ of functions defined on a closed interval is said to be uniformly bounded if there exists a number M such that

$$|Q(x)| < M$$

for all x and for all Q belonging to the given family.

DEF.

A family of functions on [a,b] is said to be equicontinuous if for every $\epsilon > 0$ there is a $\Delta > 0$ such that

$$|Q(x_1) - Q(x_2)| < \epsilon$$

for all x_1, x_2 in [a,b] satisfying

$$|x_1 - x_2| < \Delta \text{ , simultaneously}$$

for all Q in the given family.

DEF.

Let C_{XY} denote the set of all continuous mappings

$$y = f(x)$$

of a compact set X into a compact set Y(a metric space), and let

$$\rho(f,g) = \sup \{ \rho[f(x), g(x)] \text{ ; x a member of X} \} .$$

It is readily verified that

(1) $\rho(f,g) = 0$ if and only if $f = g$,

(2) $\rho(f,g) = \rho(g,f)$, and

(3) $\rho(f,g) + \rho(g,h) \geq \rho(f,h)$.

Thus C_{XY} is a metric space.

THEOREM 3.2

If f is a sequence of continuous functions on a metric space X to a metric space Y such that

$$\lim_n f_n = f_0$$

uniformly on X, then f_0 is continuous.

Proof. (ref. "Measure and Integration Theory", Munroe p. 43).

This theorem, together with the completeness of the reals, implies $C[a,b]$ is complete.

THEOREM 3.3 (Generalized Theorem of Arzela).

A necessary and sufficient condition that a set D contained in C_{XY} be relatively compact in C_{XY} is that the family of functions D be equicontinuous.

Proof. (Sufficiency).

Let us embed C_{XY} in M_{XY}, where M_{XY} is the space of all mappings of the compact space X into the compact space Y with its metric defined in the same way as that of C_{XY}. Then, by theorem 3.2, we see that C_{XY} is closed in M_{XY}, and therefore relative compactness of D in M_{XY} will imply relative compactness of D in C_{XY}.

Let $\epsilon > 0$ be chosen arbitrarily and choose Δ such that

$$\rho(x', x'') < \Delta \qquad ; \; x', x'' \text{ in } X$$

implies

$$\rho[f(x'), f(x'')] < \epsilon$$

for all f in D. Let the points

$$x_1, x_2, \ldots, x_n$$

form a $\left(\frac{\Delta}{2}\right)$-net in X. Then X can be represented as the union of non-intersecting sets ϵ_i such that if x and y are members of ϵ_i then

$\rho(x,y) < \Delta$. For example we can take

$$\epsilon_i = S(x_i, \tfrac{\Delta}{2}) - \bigcup_{j<i} S(x_j, \tfrac{\Delta}{2}), \text{ where}$$

$$S(x, \epsilon) = \{ y \in X \mid \rho(x,y) < \epsilon \}$$

We now consider in the compact set Y a finite ϵ-net

$$y_1, y_2, \ldots, y_m.$$

Denote by L the totality of functions g(x) in M_{XY} which assume the values y_j(constants) on the sets ϵ_i. The number of such functions is finite. Let f be a member of D. For every point x_i among

$$x_1, x_2, \ldots, x_n$$

we can find a point y_{j_i} among

$$y_1, y_2, \ldots y_m$$

such that

$$\rho[f(x_i), y_{j_i}] < \epsilon .$$

Let $g(x)$, a member of L, be chosen such that

$$g(x_i) = y_{j_i} \quad \text{for each } i. \quad \text{Then}$$

$$\begin{aligned}\rho[f(x), g(x)] &\leq \rho[f(x), f(x_i)] + \rho[f(x_i), g(x_i)] \\ &\quad + \rho[g(x_i), g(x)] \\ &= \rho[f(x), f(x_i)] + \rho[f(x_i), y_{j_i}] < 2\epsilon\end{aligned}$$

because

$$\rho[g(x), g(x_i)] = 0$$

if i is chosen such that x is a member of ϵ_i. Then

$$\rho(f,g) < 2\epsilon$$

for at least one g, a member of L, and L is then a 2ϵ-net for D in M_{xy} and consequently forms a 2ϵ-net in C_{XY}. By definition D is then totally bounded. Using theorem 3.1, D is thus relatively compact.

(Necessity)

Let D, a subset of C_{XY}, be relatively compact in C_{XY}. Then there exists a finite $(\frac{\epsilon}{3})$-net in D and if f is a member of D.

$$\rho(f, f_i) < \frac{\epsilon}{3}$$

for at least one f_i where

$$f_1, f_2, \ldots, f_n$$

is the $(\frac{\epsilon}{3})$-net in D. Each of the functions f is continuous and therefore uniformly continuous on the compact set X. Then for each i,

$$i = 1, 2, \ldots, n,$$

there is a $\Delta_i > 0$ such that

$$|f_i(x') - f_i(x'')| < \frac{\epsilon}{3} \quad \text{if}$$

$$|x' - x''| < \Delta_i.$$

Let $\Delta = \min \Delta_i$. Then if x_1, x_2 are members of D and

$$|x_1 - x_2| < \Delta,$$

$$|f(x_1) - f(x_2)| = |f(x_1) - f_i(x_1)| + |f_i(x_1) - f_i(x_2)| + |f_i(x_2) - f(x_2)|$$

$$\leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon .$$

Then by definition the set D is equicontinuous.

DEF.

A function f(x) is said to be lower (upper) semicontinuous at the point x_o if for arbitrary $\epsilon > 0$ there exists a Δ-neighborhood of x_o in which

$$f(x) > f(x_o) - \epsilon \quad ,$$

$$(f(x) < f(x_o) + \epsilon).$$

DEF.

Denote by M the space of all bounded real-valued functions of a real variable with metric

$$\rho(f,g) = \sup \{ |f(x) - g(x)| \} .$$

DEF.

A curve in a topological space X is a continuous function

$$f : [a,b] \longrightarrow X.$$

DEF.

We shall define the length of the curve

$y = f(x)$, $(a \leq x \leq b)$ in the plane

as the functional

$$L_a^b(f) = \sup \left\{ \sum_{i=1}^{N} [(x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1}))^2]^{1/2} \right\}$$

where the least upper bound is taken over all possible subdivisions $[a = x_o < x_1 < \ \ldots \ < x_n = b]$ of the closed interval $[a,b]$.

For continuous functions it coincides with the value of the limit

$$\lim_{N\to\infty} \sum_{i=1}^{N} [(x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1}))^2]$$

as $\max_{i=1,2,\ldots N} \{|x_i - x_{i-1}|\}$ goes to 0. For functions with continuous derivatives it can be written in the form

$$\int_a^b [1 + f'^2(x)]^{1/2} \, dx.$$

THEOREM 3.4

The functional $L_a^b(f)$ is lower semicontinuous in M.

Proof.

Let us choose a subdivision of $[a,b]$ such that

(3.1) $$\sum_{i=1}^{N} [(x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1}))^2]^{1/2} > L_a^b(f) - \frac{\epsilon}{2}.$$

Let $A = (x_i - x_{i-1})$,

$B = [f(x_i) - f(x_{i-1})]$, and

$C = [g(x_i) - g(x_{i-1})]$.

Then

$$|(A^2 + B^2)^{1/2} - (A^2 + C^2)^{1/2}|$$

$$= \left| \frac{[(A^2 = B^2)^{1/2} - (A^2 + C^2)^{1/2}][(A^2 + B^2)^{1/2} + (A^2 + C^2)^{1/2}]}{[(A^2 + B^2)^{1/2} + (A^2 + C^2)^{1/2}]} \right|$$

$$= \left| \frac{(A^2 + B^2) - (A^2 + C^2)}{(A^2 + B^2)^{1/2} + (A^2 + C^2)^{1/2}} \right|$$

$$+ \left| \frac{B^2 - C^2}{(A^2 + B^2)^{1/2} + (A^2 + C^2)^{1/2}} \right|$$

$$\left| \frac{(B - C)(B + C)}{(A^2 + B^2)^{1/2} + (A^2 + C^2)^{1/2}} \right| .$$

It is obvious that

$$B + C < (A^2 + B^2)^{1/2} + (A^2 + C^2)^{1/2} \quad . \text{ Therefore}$$

$$\frac{B + C}{(A^2 + B^2)^{1/2} + (A^2 + C^2)^{1/2}} < 1, \text{ and then}$$

$$\left| \frac{(B - C)(B + C)}{(A^2 + B^2)^{1/2} + (A^2 + C^2)^{1/2}} \right| < |B - C| \quad .$$

Let

$$|B - C| \leq \Delta' ,$$

and then let

$$\Delta' = \frac{\epsilon}{2n}. \text{ Then}$$

$$\left| (A^2 + B^2)^{1/2} - (A^2 + C^2)^{1/2} \right| < \frac{\epsilon}{2n}.$$

Substituting for A, B, and C we see that

$$[(x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1}))^2]^{1/2}$$
$$- [(x_i - x_{i-1})^2 + (g(x_i) - g(x_{i-1}))^2]^{1/2} < \frac{\epsilon}{2n}.$$

If we sum from n = 1 to N we see

$$\sum_{i=1}^{N} [(x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1}))^2]^{1/2}$$
$$- \sum_{i=1}^{N} [(x_i - x_{i-1})^2 + (g(x_i) - g(x_{i-1}))^2]^{1/2} < \frac{n\epsilon}{2n} = \frac{\epsilon}{2}, \text{ or}$$

$$L_a^b(g) \geq \sum_{i=1}^{N} [(x_i - x_{i-1})^2 + (g(x_i) - g(x_{i-1}))^2]^{1/2}$$
$$> \sum_{i=1}^{N} [(x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1}))^2]^{1/2} - \frac{\epsilon}{2}$$

$$> L_a^b(f) - \frac{\epsilon}{2} - \frac{\epsilon}{2} = L_a^b(f) - \epsilon ,$$

provided g is chosen such that

$\rho(f,g) < \frac{\epsilon}{4n}$. This follows from inequality 3.1.

Hence we choose $\Delta = \frac{\epsilon}{4n}$.

Theorems 3.5 and 3.6 are standard elementary results, and will be stated without proof.

THEOREM 3.5

A real valued function which is lower (upper) semicontinuous on a compact set K is bounded below (above) on K.

Proof. [3, p. 66]

THEOREM 3.6

A lower (upper) semicontinuous function defined on a compact set K attains its greatest lower (least upper) bound on K.

Proof. [3, p. 66]

Let K be a compact metric space and let C_K be the space of continuous real functions defined on K with distance function

$$\rho(f,g) = \sup_{x \in K} \{|f(x) - g(x)|\}.$$

THEOREM 3.7

A necessary and sufficient condition that a subset D of C_K be relatively compact is that the family of functions D be uniformly bounded and equicontinuous.

Proof. (Sufficiency)

Assume the family of functions D is uniformly bounded and equicontinuous. Then by theorem 3.2 D is relatively compact.

(Necessity)

Let the set D be relatively compact in C_K. Then, by theorem 3.1 for each $\epsilon > 0$ there exists a finite $(\frac{\epsilon}{3})$-net

$$q_1, q_2, \ldots, q_k$$

in D. Each of the functions q_i, being a continuous function on a compact set, is bounded, i.e.

$$|q_i| \leq M_i.$$

Let

$$M = \max M_i + \frac{\epsilon}{3}.$$

By definition of an $(\frac{\epsilon}{3})$-net, for every q in C_K we have at least one q_i such that

$$\rho(q, q_i) = \sup |q(x) - q_i(x)| < \frac{\epsilon}{3}$$

Consequently

$$|q| < |q_i| + \frac{\epsilon}{3} < M_i + \frac{\epsilon}{3} < M.$$

Thus, D is uniformly bounded.

Each of the functions q_i is continuous and consequently uniformly continuous on the compact set K. Then for a given $\frac{\epsilon}{3}$ there exists a Δ_i such that

$$|q_i(x_1) - q_i(x_2)| < \frac{\epsilon}{3} \quad \text{if}$$

$$|x_1 - x_2| < \Delta_i.$$

Set

$$\Delta = \min_{i=1,\ldots,n} \Delta_i.$$

Then for

$$|x_1 - x_2| < \Delta$$

and for any q in D, taking q_i so that

$$\rho(q, q_i) < \frac{\epsilon}{3}, \text{ we have}$$

$$|q(x_1) - q(x_2)| = |q(x_1) - q_i(x_1) + q_i(x_1) - q_i(x_2)$$

$$+ q_i(x_2) - q(x_2)|$$

$$= |q(x_1) - q_i(x_1)| + |q_i(x_1) - q_i(x_2)|$$

$$+ |q_i(x_2) - q(x_2)|$$

$$< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon .$$

Thus D is equicontinuous.

Theorem 3.7 is Arzela's theorem for continuous functions defined on an arbitrary compact set.

DEF.

Two continuous functions

$P = f_1(t')$ and

$P = f_2(t'')$

defined, respectively, on the closed intervals

$a' \leq t' \leq b'$ and

$a'' \leq t \leq b''$

are said to be <u>equivalent</u> if there exist two non-decreasing functions

$t' = Q_1(t)$, and

$t'' = Q_2(t)$

defined on a closed interval

$a \leq t \leq b$

and possessing the properties

$Q_1(a) = a'$,

$Q_1(b) = b'$;

$Q_2(a) = a''$,

$Q_2(b) = b''$; and

$f_1[Q_1(t)] = f_2[Q_2(t)]$

for all t contained in $[a,b]$.

For an arbitrary function

$$P = f_1(t')$$

defined on a closed interval $[a', b']$ we can find a function which is equivalent to it and which is defined on the closed interval

$$[a, b] = [0,1].$$

It is sufficient to set

$$t' = Q_1(t) = (b' - a')t + a',$$

$$t'' = Q_2(t) = t.$$

We will assume a $<$ b.

Thus, we will consider the space C_{IR} of continuous mappings of the closed interval

$$I = [0,1]$$

into the space R (reals) with the metric

$$\rho(f,g) = \sup_t \rho[f(t), g(t)].$$

We say that the sequence of curves

$$L_1, L_2, \ldots\ldots, L_n, \ldots$$

converges to the curve L if the curve L_n can be represented parametrically in the form

$$P = f_n(t):$$

$$(0 = t = 1)$$

and the curve L in the form

$$P = f(t);$$

$$(0 \leq t \leq 1),$$

so that

$$\lim_n \rho(f, f_n) = 0.$$

THEOREM 3.8

If the sequence of curves

$$L_1, L_2, \ldots\ldots, L_n, \ldots$$

lying in the relatively compact subset K of a complete metric space can be represented parametrically by means of equicontinuous functions defined on the closed interval $[0,1]$, then this sequence contains a convergent subsequence.

Proof.

This is a direct result of Theorem 3.3.

The length of a curve given parametrically by means of the function

$$P = f(t),$$

$$a \leq t \leq b,$$

is the least upper bound of sums of the form

$$\sum_{i=1}^{N} \rho[f(t_{i-1}), f(t_i)]$$

where

$$a = t_o \leq t_1 \leq \ldots\ldots \leq t_n = b.$$

It is easy to see that the length of a curve does not depend on the choice of its parametric representation.

THEOREM 3.9

If the sequence of curves L_n, represented parametrically by functions defined on $[0,1]$, converges to the curve L, then the length of L is not greater than the lim inf. of the sequence of lengths of the curves L_n.

Proof.

The proof of this is similar to that of theorem 3.4.

Let us now consider curves of finite length or rectifiable curves. Consider a curve defined parametrically by means of the function

$$P = f(t),$$

$$a \leq t \leq b.$$

The function f, considered only on the closed interval $[a, T]$, where

$$a \leq T \leq b,$$

defines an "initial segment" of the curve from the point

$$P_a = f(a)$$

to the point

$$P_T = f(T).$$

Let

$$s = Q(t)$$

be its length. Then

$$P = g(s) = f\left[Q^{-1}(s)\right]$$

is a new parametric representation of the same curve. Let s run through the closed interval

$$0 \leq s \leq S,$$

where S is the length of the entire curve. This representation satisfies the requirement

$$\rho\left[g(s_1),\ g(s_2)\right] \leq |s_2 - s_1|$$

(the length of the curve is not less than the length of the chord). Going over to the closed interval $[0,1]$ we obtain the parametric representation

$$P = F(\tau) = g(s),$$

$$\tau = \frac{s}{S},$$

which satisfies the following Lipschitz Condition

$$[F(\tau_1), F(\tau_2)] = S \; |\tau_1 - \tau_2| \;.$$

Thus for all curves of length S such that

$$S \leq M,$$

where M is a constant, a parametric representation on the closed interval $[0,1]$ by means of equicontinuous functions is possible.

THEOREM 3.10

If two points A and B in the relatively compact set K can be connected by a continuous curve of finite length, then among all such curves there exists one of minimal length.

Proof.

Let Y be the greatest lower bound of the lengths of curves which connect A and B in K. Let the lengths of the curves

$$L_1, L_2, \ldots, L_n, \ldots$$

connecting A with B tend to Y. By theorem 3.8 it is possible to select a convergent subsequence from the sequence $\{L_n\}$. By theorem 3.9 the limit curve of this subsequence cannot have length greater than Y. This completes the proof.

PART II

Section II

<u>Direct Methods of the Calculus of Variations</u>

The direct methods in the calculus of Variations represent a relatively modern trend which has established the calculus of variations in a dominating position in mathematical analysis.

The general points of view in the calculus of variations are relevant for various domains of mathematics, namely the formation of invariants and covariants in function spaces, and the characterization of mathematical entities by extremum properties. We shall concentrate on the second topic.

In the mathematical treatment of physical phenomena it is often expedient to use formulations by means of which the quantities under consideration appear as extrema. An example of that is Fermat's Principle in optics.

The classical methods of the calculus of variations can be considered as indirect methods, in contrast to the modern direct methods.

Generally speaking the direct methods aim at solving boundary value problems of differential equations by reducing them to equivalent extremum problems of the calculus of variations, and then attacking these problems directly.

The most notable example of the direct approach goes back to Gauss and William Thompson. They considered the boundary value problem of the Laplace equation.

$$(4.1) \qquad \Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

for a domain G in the xy-plane, under the condition that the function u be defined in G and be equal to a prescribed continuous function on the boundary. The classical formalism of the calculus of variations for the integral

$$(4.2) \qquad D[\Phi] = \iint_G (\Phi_x^2 + \Phi_y^2)\,dxdy$$

shows that if u(x,y) furnishes the minimum of the integral when all functions Φ which are continuous in G and on its boundary, attain the prescribed boundary values, and possess continuous first and second derivatives in G are admitted to competition, then u(x,y) is the solution of the boundary value problem

$$\Delta u = 0 \text{ in } G, \; u(p) = f(p) \text{ for p on the boundary of G.}$$

Gauss and Thompson thought that, since the integral $D[\Phi]$ is positive, it must have a minimum. This reasoning was later resumed by Dirichlet, and a decisive use of it, under the name of Dirichlet's Principle was made by Bernhard Riemann. To make Dirichlet's Principle true the existence of a minimum, rather than a greatest lower bound, has to be established. Let us look at some examples where we have a greatest lower bound but a minimum does not exist.

a) Find the shortest curve from A to B with the condition that it be perpendicular to AB at A and B.

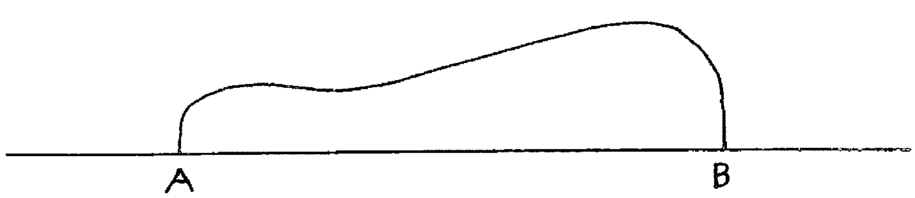

Fig. 4.1

The length of the admissible curves has a greatest lower bound, namely AB ; however, no shortest curve exists.

b) Find a function $\Phi(x)$ continuous, having a piecewise continuous derivative, for which the integral

$$I = \int_{-1}^{1} x^2 \left[\Phi'(x) \right]^2 dx$$

attains the smallest possible value, with the boundary conditions

$$\Phi(-1) = -1 ;$$
$$\Phi(1) = 1 .$$

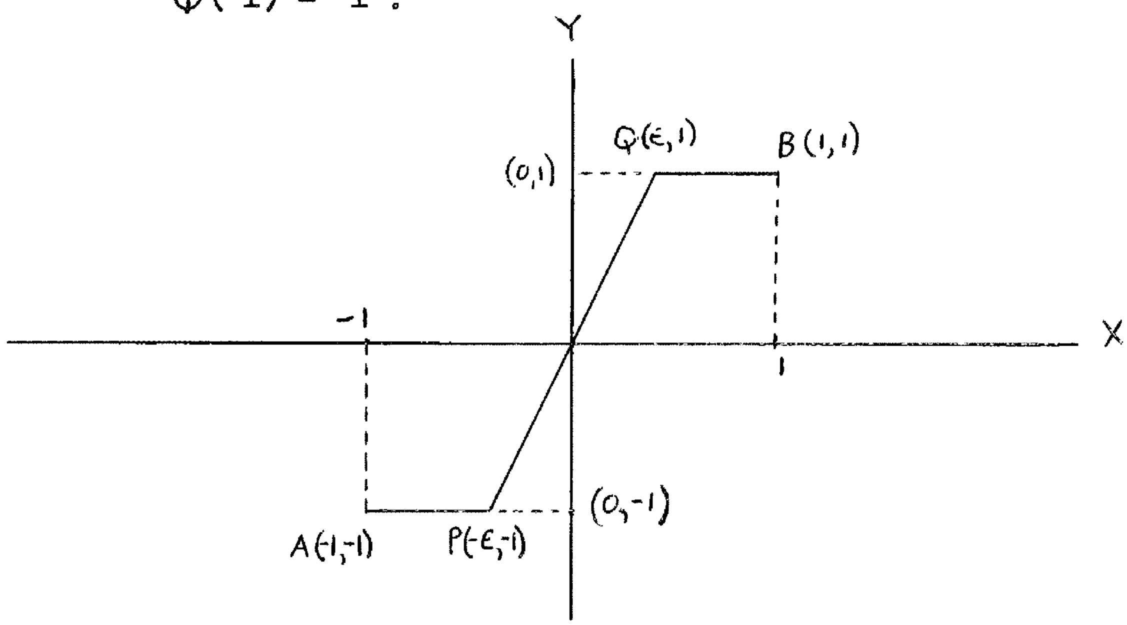

Fig. 4.2

The integral is always positive and has a greatest lower bound, namely, 0. Let $\Phi(x)$ be the function whose graph is APQB in fig. 4.2 We have

$$\Phi(x) = \frac{1}{\epsilon} x \qquad \text{for} \qquad -\epsilon < x < \epsilon$$

$$\Phi(x) = 1 \qquad \text{for} \qquad \epsilon < x < 1 \qquad \text{and}$$

$$\Phi(x) = -1 \qquad \text{for} \qquad -1 < x < -\epsilon \qquad .$$

Therefore

$$\Phi'(x) = \frac{1}{\epsilon} \quad \text{for} \quad -\epsilon < x < \epsilon \quad \text{and}$$

$$\Phi'(x) = 0 \quad \text{for} \quad \epsilon < x < 1 \quad \text{and} \quad -1 < x < -\epsilon \quad .$$

Hence

$$I = \int_{-1}^{1} x^2 \quad \Phi'(x)^2 dx \quad = \int_{-\epsilon}^{\epsilon} \frac{x^2}{\epsilon^2} dx \quad = \frac{2\epsilon}{3} \quad ,$$

and it can be made arbitrarily small. But the only function for which

$I = 0$ is

$\Phi(x) = c,$

and it is obvious that this does not satisfy the given boundary conditions.

Three related goals are envisaged by direct methods of calculus of variations.

1) Existence proofs for solutions of boundary value problems,

2) Analysis of the properties of these solutions, and

3) Numerical procedures for calculating the solutions.

Let us consider the following problem:

Among all continuous, closed curves C having a given length L, find one which makes the enclosed area A(C) a maximum.

By the classical methods of calculus of variations it can be shown that if a solution exists it is a circle. We will be concerned with proving the existence of the solution.

Any admissible curve C can be enclosed completely in a circle of radius $\frac{L}{2}$. Thus we have

$$A(C) \leq \pi \frac{L^2}{4} \quad ,$$

so that a least upper bound M exists for all the areas, and a maximizing sequence

$$C_1, C_2, \ldots\ldots, C_n, \ldots$$

of admissible curves exists such that

$$A_n(C_n) \rightarrow M \qquad \text{as} \qquad n \rightarrow \infty .$$

Each curve C_n can be assumed to be a convex curve, for if not, it could be replaced by a convex admissible curve of larger area, i.e. C_n can be replaced by its "convex hull" denoted by $\bar{C}_n$ (the least convex polygon and its interior points that contain C_n). Then $\bar{C}_n$, whose length may be less than L, is magnified into a similar admissible curve of length L, denoted by C_n. We now have

$$A(C_n) < A(C_n) < A(C_n).$$

We make the assumption that all the curves C_n be within a single circle of radius smaller than $\frac{L}{2}$. Thus we have a sequence of convex curves C_n lying in a closed domain, so that (by theorem 3.8) there is a subsequence which converges to a closed curve C. Since the area of a sequence of convex curves depends continuously on the curves, and the areas A_n of C_n converge to M, we have

$$A(C) = M.$$

If C_n converges to C, then

$$\underline{\lim}\, L(C_n) \geq L(C)$$

by the lower semi-continuity of length. In the present case, we have

$$L(C) \leq L.$$

The equality sign, however, must hold, since, if

$$L(C) < L,$$

C could be magnified into a curve of length L, whose area would then exceed M. Thus the existence of a curve of length L and enclosing maximum area is established.

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