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Balanced graphs and balanced matroids

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Balanced Graphs and Balanced Matroids
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Balanced Graphs and Balanced Matroids

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The idea of "balance" in graph theory originated with the study of random graphs. This idea was formulated first for graphs and then generalized to matroids. Matroids are useful in solving large problems often found in the fields of civil, electrical, and mechanical engineering, as well as computer science and mathematics.

After exploring which classed of graphs or matroids are balanced, a connection between graph balance and matroid balance is obtained. The main theorems concern constructions of matroids and the effect these constructions have on the property of balance.

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1 Introduction

The idea of balanced graphs originated with P. Erdős and A. Rényi in the late 1950's and early 1960's. At this time these two mathematicians wrote a series of papers [4, 5, 6] on the theory of random graphs; it was this work which prompted the study of balanced graphs. Since this time, the theory of random graphs and the use of balanced graphs have undergone enormous growth. Balanced graphs are important because of the 'nice' properties they possess. To obtain results on general graphs, it is often easier to find a proof for the balanced graph case and then to extend to the general case.

Since matroids are generalizations of graphs, it is natural to see which results for graphs may be extended to matroids. In order to motivate the research done in the field of matroids, a quote from K. Truemper [20] is provided:

> With matroids, one may formulate rather compactly and solve a large number of interesting problems in diverse fields such as civil, electrical, and mechanical engineering, computer science and mathematics.

In the early 1980's, D. Kelly and J. Oxley began to examine which results from the theory of random graphs and balanced graphs would generalize to matroids [7, 8, 9, 16]. Matroids are generalizations of graphs; therefore it is natural to see which ideas formulated for balanced graphs would also hold for balanced matroids. The next step would be to generalize these ideas to all matroids, if possible. The work in this area is generally new and quite sparse.

The purpose of this research thesis is to consider the work that has been done in the area of balanced graphs and balanced matroids and to see what generalizations and connections can be made. To begin with, we will find some families of graphs and matroids which are balanced, strictly balanced or can be shown to be neither. We will then consider the connection between graph balance and matroid balance. Next, we will look at various operations and determine if they preserve balance. Finally, we will end with some suggestions on further research in this area. Throughout all graphs will be nonempty and simple; likewise, all matroids will be nonempty and loopless. The reader is referred to [17, 24] for a more thorough discussion on graph and matroid theory.

2 Background

The probabilistic method, introduced by Erdős [3] in order to prove a lower bound on Ramsey numbers, was formulated in terms of a random graph. This is just one of the many motivations for the study of random graphs; the following quote from B. Bollobás [2] provides further motivation:

> Mathematicians who are not interested in graphs for their own sake should view the theory of random graphs as a modest beginning from which we can learn a variety of techniques and can find out what kind of results we should try to prove about more complicated random structures.

Random graphs have become a powerful tool in Ramsey theory, and the theory of random graphs itself has grown rapidly. A **random graph** $G_{n,p}$ is a subgraph of the complete graph K_n obtained by independent removal of each edge with probability $1-p$, where $p = p(n) \in (0,1)$. Let A be a fixed property which a graph may or may not possess and let $\Pr_{n,p(n)}(A)$ denote the probability that $G_{n,p}$ has property A. In [5], Erdős and Rényi studied the probable structure of a random graph. It has been shown for several properties A of graphs that there exists a function $t(n)$ such that

$$\lim_{n\to\infty} \Pr_{n,p(n)}(A) = \begin{cases} 0 & \text{if } \lim_{n\to\infty} \frac{p(n)}{t(n)} = 0, \\ 1 & \text{if } \lim_{n\to\infty} \frac{p(n)}{t(n)} = \infty. \end{cases}$$

If such a function exists, then it is called a **threshold function** for the property A.

Erdős and Rényi [4, 5, 6] set out to answer the following question about random graphs: What is the probability that a random graph on n vertices has a particular subgraph? They originally proved that $n^{-1/m(G)}$ is a threshold function for the existence of a given balanced graph as a subgraph of $G_{n,p}$ (here $m(G)$ denotes the maximum average degree of a subgraph). In 1991, Bollobás [1] generalized this result to all graphs. The concept of balanced graphs is interesting in its own right and essential for certain distributional results; therefore, it is of great importance that Erdős and Rényi introduced this notion.

Let G be a graph with $|E(G)|$ edges and $|V(G)|$ vertices. A graph H is a **subgraph** of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A graph H is a **proper subgraph** of G if $V(H) \subseteq V(G)$ and $E(H) \subset E(G)$. Define the **density** of G to be

$$d'(G) := \frac{2|E(G)|}{|V(G)|},$$

which is also called the average degree of G. We say G is **balanced** if

$$d'(H) \leq d'(G) \text{ for all non-empty subgraphs } H \subseteq G,$$

and **strictly balanced** if

$$d'(H) < d'(G) \text{ for all non-empty proper subgraphs } H \subset G.$$

Balanced graph theory results originated with the following theorem of Erdős and Rényi.

Theorem 1 ([5]) *If G is balanced, then*

$$\lim_{n\to\infty} \Pr(G_{n,p} \supset G) = \begin{cases} 0 & \text{if } p(n)n^{\frac{1}{d'(G)}} \to 0 \text{ as } n \to \infty, \\ 1 & \text{if } p(n)n^{\frac{1}{d'(G)}} \to \infty \text{ as } n \to \infty. \end{cases}$$

Bollobás [1] generalized this result to all graphs. In contrast to the existence problems answered in the theorems of Erdős, Rényi and Bollobás, balanced graphs are essential for distributional results as is seen in the work of A. Ruciński and A. Vince [19].

Balanced matroids will be defined in a similar fashion. A short discussion on matroid terminology is provided to assist the reader in developing an understanding of matroid theory. For a more complete discussion of matroids the reader is referred to [17, 24].

The theory of matroids is an abstract theory of dependence. It originated with an article by H. Whitney [25]. In this article, Whitney established four "cryptomorphic" definitions of the term matroid. There are now many more equivalent ways to define a matroid, which is one interesting and useful characteristic of matroids. Each matroid definition has a similar axiomatization which is generalized below [24]:

a1 a nontriviality or normalization condition to rule out degeneracy;

a2 description of the general mathematical structure;

a3 the characteristic axiom.

It is often necessary to convert from one axiom system to another; to do this we shall use the matroid cryptomorphisms found in [24]. The majority of matroid terminology will be given in terms of rank functions, circuits, and bases; "...although (the rank function) has little intuitive appeal, the rank function gives straightforward descriptions for all other axiomatizations [24]." These axioms are described below.

A **matroid** M, defined on the ground set E, is a pair $(E, \mathcal{C})$, where $\mathcal{C}$ is a collection of subsets of E called **circuits** which obey the following axioms:

c1 $\emptyset \notin \mathcal{C}$;

c2 for any two distinct $C_1, C_2 \in \mathcal{C}$, C_1 is not a proper subset of C_2;

c3 for any two distinct $C_1, C_2 \in \mathcal{C}$ and any $z \in (C_1 \cap C_2)$, there is a set $C_3 \in \mathcal{C}$ where $C_3 \subseteq (C_1 \cup C_2) - z$.

We shall denote the set of all circuits of M by $\mathcal{C}(M)$ or simply by $\mathcal{C}$ if the context is clear.

The notion of rank is very useful in linear algebra and other areas of mathematics. As it turns out it is also very important in the theory of matroids; thus we will refer to the rank function axioms quite frequently throughout this thesis.

Let E be a set. A function $\rho : 2^E \to Z$ is the **rank function** of a matroid on E if it satisfies the following three axioms:

r1 if $X \subseteq E$, then $0 \leq \rho(X) \leq |X|$;

r2 if $X \subseteq Y \subseteq E$, then $\rho(X) \leq \rho(Y)$;

r3 if X and Y are subsets of E, then $\rho(X \cup Y) + \rho(X \cap Y) \leq \rho(X) + \rho(Y)$.

We will denote the rank of a matroid M by $\rho(M)$, rather than $\rho(E)$, if the context is clear.

Bases of a matroid are defined as a collection of subsets of E which satisfy the following axioms:

b1 $\mathcal{B}$ is nonempty;

b2 if $B_1, B_2 \in \mathcal{B}$, and $x \in B_1 - B_2$, then there is an element y of $B_2 - B_1$ such that $(B_1 - x) \cup y \in \mathcal{B}$;

b3 if $B_1, B_2 \in \mathcal{B}$ and $B_1 \subseteq B_2$, then $B_1 = B_2$.

The reader is referred to [24] for a proof that the above sets of axioms are equivalent.

Two matroids $M_1 = (E_1, \mathcal{C}_1)$, and $M_2 = (E_2, \mathcal{C}_2)$ are **isomorphic** if there is a bijection ψ from E_1 to E_2 such that, for all $X \subseteq E_1, \psi(X)$ is circuit of M_2 if and only if X is a circuit of M_1. We will write $M_1 \cong M_2$ if M_1 is isomorphic to M_2.

Now that the basic definitions for a matroid have been given, we will look at a construction which will allow us to find "submatroids."

We will define a **submatroid** H of M as the matroid on the ground set $E' \subseteq E$ by defining its circuits:

$$\mathcal{C}(H) = \{C \subseteq E' : C \in \mathcal{C}(M)\}\,.$$

Such a matroid is often called a **restricted matroid**. The notation for submatroids to be used throughout will be similar to that for graphs; e.g., we will write $H \subseteq M$ if H is a submatroid of M. A proper submatroid is one for which $E' \subset E$. Only nonempty submatroids will be considered.

It is possible to extend the definition of balanced and strictly balanced to matroids. Recall the density of a graph $d'(G)$ is equal to twice the number of edges divided by the number of vertices. We would like to find analogous matroid notions for edges and vertices of graphs. It is easy to see how the number of edges in a graph can be analogous to the number of elements in a matroid, since these sets are the same size for graphic matroids. As matroids have no vertices, we will replace $|V(G)|$ by the rank of the matroid. It is natural to consider the rank of the matroid in this role, as the rank of a connected graph is $|V(G)| - 1$. The **density** of a matroid $M = (E, \mathcal{C})$ is

$$d(M) := \frac{|\, M \,|}{\rho(M)},$$

where $|M|$ is the size of the ground set E and $\rho(M)$ is the rank of the matroid. We say M is **balanced** if

$$d(H) \leq d(M) \text{ for all non-empty submatroids } H \subseteq M,$$

and **strictly balanced** if

$$d(H) < d(M) \text{ for all non-empty proper submatroids } H \subset M.$$

The properties above are defined by H. Narayanan and M. Vartak [15] as **molecular** and **atomic** rather than balanced and strictly balanced, respectively.

In [7, 8], Kelly and Oxley generalize some of the known results obtained for balanced graphs to balanced matroids. They begin with an analogue to Theorem 1.

Theorem 2 ([8]) *Let k and m be fixed positive integers with $k \leq m$ and suppose that $B_{k,m}$ denotes a non-empty set of balanced simple matroids each of which has m elements and rank k and is representable over $GF(q)$. Then a threshold function for the property that ω_r has a submatroid isomorphic to some member of $B_{k,m}$ is $q^{-rk/m}$.*

See page 11 for a more thorough discussion of ω_r. In [8] the following results are obtained from this theorem.

Corollary 3 *If k is a fixed positive integer, then a threshold function for the property that ω_r has a k-element independent set is q^{-r}.*

Corollary 4 *If $m \geq 2$ is a fixed integer, then a threshold function for the property that ω_r has an m-element circuit is $q^{-r(m-1)/m}$.*

Corollary 5 *Let k be a fixed positive integer. A threshold function for the property that ω_r contains a submatroid isomorphic to $PG(k-1, q)$ is $q^{-rk(q-1)/(q^k-1)}$.*

Corollary 6 *Let k be a fixed positive integer. A threshold function for the property that ω_r contains a submatroid isomorphic to the cycle matroid of the complete graph on $k+1$ vertices is $q^{-2r/(k+1)}$.*

To show that these results are valid, we are required to check that the appropriate submatroids are balanced. For example, in Corollary 3 the k-element independent set must be balanced; this will later be called a free matroid. Corollary 4 requires one to verify that an m-element circuit is balanced; this is precisely the uniform matroid of rank $m-1$ and size m. In Corollary 5, the projective geometry $PG(k-1, q)$ needs to be balanced, while in Corollary 6 it is required that the cycle matroid $M(K_n)$ is shown to be balanced. For a more thorough discussion of this material, the reader is referred to Propositions 11, 14, and 17.

3 Classes of Balanced Graphs

In this section, we consider which classes of graphs are balanced, strictly balanced or can be shown to be neither. There are many different classes of graphs which can be considered; only a few were chosen. It can be shown that trees, complete graphs, and cycles are strictly balanced. We will include a proof of these results for completeness. It can also be shown that complete bipartite graphs are strictly balanced and unicyclic graphs are balanced, but not strictly balanced [21].

A short review of graph theory terminology is included in part to introduce notation which is used throughout. A **cycle** C_n is a connected 2-regular graph, defined on $n \geq 3$ vertices. The graph C_n has exactly n edges, and consists of one cycle containing all edges. A **complete graph** K_n on n vertices is a graph in which every vertex is incident with every other vertex. In a complete graph each vertex has degree $n-1$ and there are $\frac{n(n-1)}{2}$ edges. A **tree** T_n is a connected graph on n vertices containing no cycles. A **unicyclic graph** is a connected graph which contains exactly one cycle and has minimum degree one.

Proposition 7 *All cycles are strictly balanced.*

Proof: Let G be a cycle defined on n vertices, hence having n edges. It is easily seen that the density or average degree of G is 2. We must show that the density of all proper subgraphs H of G have density less than 2. It is trivial to see that all proper subgraphs of G are forests; we will consider two cases.

Case one: Let H be a tree defined on $k \leq n$ vertices. Since H is a tree we know $|E(H)| = k - 1$, and thus

$$d'(H) = \frac{2(k-1)}{k} = 2 - \frac{2}{k} < 2 = d'(G).$$

Hence $d'(H) < d'(G)$.

Case two: Let H' be a forest with $p \geq 2$ connected components. Let n_i be the number of vertices in component $i = 1, 2 \ldots p$; hence there are $n_i - 1$ edges in component i. The density of H' is

$$\frac{2((\sum_1^p n_i) - p)}{\sum_1^p n_i} = 2 - (2p/\sum_1^p n_i) < 2.$$

Thus G is strictly balanced. □

Proposition 8 *All complete graphs are strictly balanced.*

Proof: Let G be a complete graph K_n. The density of G is $n - 1$, since this is the average degree. We must show for all proper subgraphs H we have $d'(H) < n - 1$. We claim the many possible cases reduce to the case where H is a complete graph on $k < n$ vertices. Assuming this claim is true, let H be a complete graph on $k < n$ vertices. Then $d'(H) = k - 1$, which is obviously less than $n - 1$. Thus K_n is strictly balanced.

To show the claim is true, let H' be a graph on k vertices. It is obvious that $d'(H')$ is less than $d'(H)$ since both are defined on the same number of vertices and since H has more edges. □

Proposition 9 *All trees are strictly balanced.*

Proof: Let G be a tree on n vertices. The density of G is $\frac{2(n-1)}{n}$. All subgraphs of trees are forests; thus we shall consider two cases.

Case one: Let H be a proper subgraph of G on $k < n$ vertices such that H is a tree. The density of H is $\frac{2(k-1)}{k}$, which is strictly less than $\frac{2(n-1)}{n}$.

Case two: Let H be a forest with $p \geq 2$ components. Denote the number of vertices in each tree of H by n_i for $i = 1, 2, \ldots, p$; thus the number of edges in component i is $n_i - 1$. The density of H is $\frac{2((\sum_1^p n_i) - p)}{\sum_1^p n_i}$. We must show

$$\frac{2((\sum_1^p n_i) - p)}{\sum_1^p n_i} < \frac{2(n-1)}{n}.$$

Notice that $\sum_1^p n_i \leq n < np$ since $1 < p$. Therefore

$$2n \sum_1^p n_i - 2np < 2 \sum_1^p n_i(n-1),$$

and the result is immediate. □

4 Classes of Balanced Matroids

Next we consider classes of matroids that are balanced, strictly balanced, or neither. Of the many interesting classes of matroids, a few of our "favorites" are chosen. Before delving into this section, some additional matroid terminology is needed. It will be shown that it is not necessary to check the density of all submatroids of M in order to determine if M is (strictly) balanced. We will describe submatroids which are most dense; this is possible through the use of the closure operator.

A matroid **closure operator**, over the finite set E, is an operator cl: $2^E \rightarrow 2^E$ satisfying the following axioms:

cl1 for every $X \subseteq E, X \subseteq \mathrm{cl}(X)$;

cl2 for every $X, Y \subseteq E$, if $X \subseteq Y$, then $\mathrm{cl}(X) \subseteq \mathrm{cl}(Y)$;

cl3 for every $X \subseteq E, \mathrm{cl}[\mathrm{cl}(X)] = \mathrm{cl}(X)$;

cl4 for every $X \subseteq E$ and for every $y, z \in E$, if $y \in \mathrm{cl}(X \cup z) - \mathrm{cl}(X)$, then $z \in \mathrm{cl}(X \cup y) - \mathrm{cl}(X)$.

It is helpful to describe the closure operator in terms of the rank function of a matroid for $X \subseteq E$:

$$\mathrm{cl}(X) = \{x \in E : \rho(X) = \rho(X \cup x)\}.$$

A **flat** or **closed set** in a matroid M is a set $X \subseteq E$ such that $\mathrm{cl}(X) = X$. The closure of a set X is often denoted $\overline{X}$. Closed sets of rank k are called k-flats

To show a matroid is balanced we need only show that the density of closed sets satisfies the inequality required for all submatroids; this is suggested by the remarks below.

Lemma 10 *Let M be a matroid. For any submatroid $H \subseteq M$, we have $d(H) \leq d(\overline{H})$. Moreover if $H \subset \overline{H}$, then $d(H) < d(\overline{H})$.*

Proof: Notice that $d(H) = \frac{|H|}{\rho(H)} \leq \frac{|\overline{H}|}{\rho(H)} = \frac{|\overline{H}|}{\rho(\overline{H})} = d(\overline{H})$, where the inequality is strict if $|H| < |\overline{H}|$. □

According to Oxley [17], projective geometries "arise quite frequently in mathematics and are extremely natural to consider in matroid theory, their position among representable matroids being analogous to that of complete graphs in graph theory." Thus, it is natural to define a **random matroid** ω_r as a submatroid of a projective geometry $PG(r-1, q)$ obtained by independent removal of each element with probability $1 - p$. Here $PG(r-1, q)$ as usual denotes the projective geometry of rank r defined over the Galois field $GF(q)$ for q a prime power. For a more complete discussion of projective geometries, see [17].

Proposition 11 *The projective geometries $PG(r-1,q)$ are balanced.*

Proof: The number of elements of $PG(r-1,q)$ is $\frac{q^r-1}{q-1}$. Every k-flat of $PG(r-1,q)$ is isomorphic to $PG(k-1,q)$. Thus by Lemma 10 we need to show $\frac{q^k-1}{k} \leq \frac{q^r-1}{r}$, which one can check is true for $q \geq 2$, $1 \leq k \leq n$. □

Recall that Corollary 5 relied on the fact that $PG(r-1,q)$ is balanced. This condition has now been established.

Another interesting and closely related matroid to $PG(r-1,q)$ is the affine geometry. Affine geometries do not play the same role in matroid theory as projective geometries, but they are interesting in their own right because they form an important class of highly symmetric matroids. The affine geometry $AG(r-1,q)$ is obtained from $PG(r-1,q)$ by deleting from the latter all the points of a rank $r-1$ flat, also known as a hyperplane.

Proposition 12 *The affine geometries $AG(r-1,q)$ are balanced.*

Proof: The number of elements of $AG(r-1,q)$ is q^{r-1}. Every k-flat of $AG(r-1,q)$ is isomorphic to $AG(k-1,q)$. Thus by Lemma 10 we need to show $\frac{q^{k-1}}{k} \leq \frac{q^{r-1}}{r}$, which one can check is true for $q \geq 2$, $1 \leq k \leq n$. □

We now define another class of interesting matroids known as uniform matroids. These matroids are "uniform" because all of their circuits are the same size and every subset of this cardinality is a circuit. Let r, n be nonnegative integers with $r \leq n$. The **uniform matroid** $U_{r,n}$ is defined to be a rank r matroid defined on the ground set E, an n-element set, whose circuits are described by:

$$\mathcal{C}(U_{r,n}) = \begin{cases} \emptyset & \text{if } r = n, \\ \{X \subseteq E : |X| = r+1\} & \text{otherwise.} \end{cases}$$

Uniform matroids $U_{n,n}$ are precisely those matroids having no dependent sets, hence no circuits; they are called **free matroids**. The matroid $U_{1,n}$ is a multiple point. The uniform matroid $U_{2,n}$ is an n-point line. The uniform matroid $U_{m,m+1}$ is precisely the rank m matroid with exactly one circuit which contains all $m+1$ points. For example, $U_{1,2}$ is a double point; $U_{2,3}$ is three points on a line, no two on a point; and $U_{3,4}$ represents four points in a plane, no three on a line, no two on a point.

The following result allows us to classify the submatroids of uniform matroids.

Proposition 13 *Submatroids of uniform matroids are uniform; furthermore, these matroids are either full rank or free.*

Proof: Let H be a nonempty submatroid of $U_{r,n}$, defined on the ground set $E' \subseteq E$. If $\mathcal{C}(H) = \emptyset$, then H is a free matroid; so suppose $\mathcal{C}(H) \neq \emptyset$. Recall the definition of circuits of H : $\mathcal{C}(H) = \{C \subseteq E' : C \in \mathcal{C}(U_{r,n})\}$. If $X \in \mathcal{C}(H)$, then $X \in \mathcal{C}(U_{r,n})$ and $|X| = r+1$. Conversely, if $X \subseteq E'$ with $|X| = r+1$, then since $X \subseteq E$, we get $X \in \mathcal{C}(U_{r,n})$ and by definition, $X \in \mathcal{C}(H)$. Hence H is a uniform matroid of rank r.

□

Using Proposition 13, one can classify uniform matroids as balanced or strictly balanced under certain conditions.

Proposition 14 *The class of uniform matroids $U_{r,n}$ is strictly balanced when $r < n$, and balanced but not strictly balanced when $r = n$.*

Proof: Consider the matroid $U_{r,n}$ for $r < n$. The density of $U_{r,n}$ is n/r which is strictly greater than 1. Let H be a proper submatroid of $U_{r,n}$. By Proposition 13 we know

H is either a free matroid or a uniform rank r matroid. If H is free, then the density of H is 1, which is strictly less than $d(U_{r,n})$. On the other hand, if H is a proper submatroid of full rank, then the size of H is at most $n-1$. Hence the density is at most $\frac{n-1}{r}$, which is strictly less than $d(U_{r,n})$.

Consider the free matroid $U_{n,n}$. From Proposition 13 we know all non-empty submatroids K of $U_{n,n}$ are free. Hence $d(K) = 1 = d(U_{n,n})$; therefore $U_{n,n}$ is balanced, but not strictly balanced. □

Recall in Corollaries 3 and 4 it was required that we show free matroids and uniform matroids of the form $U_{r,r+1}$ are balanced; this has now been shown.

5 Connections Between Graph and Matroid Balance

In this section we explore the relationship between a balanced matroid and the graph associated with that matroid. A matroid that is isomorphic to the cycle matroid of a graph is called **graphic**. The cycle matroid of a graph G is denoted $M(G)$. We will answer the following question: if the matroid $M(G)$ is (strictly) balanced, does this imply that the graph G associated with $M(G)$ is (strictly) balanced? In order to answer this question it is important to fully understand the concepts behind the question.

If G is a graph, then adding isolated vertices to G will not alter the cycle matroid of G. For this reason, we shall assume all graphs have no isolated vertices. A graph H is **2-isomorphic** to the graph G if G can be transformed into H by a sequence of operations of the types described below:

Vertex Identification Let v and v' be vertices of distinct components of G. G is modified by identifying v and v' as a new vertex $\overline{v}$.

Vertex Cleaving This is the reverse operation of vertex identification.

Twisting The graph G is obtained from the disjoint graphs G_1 and G_2 by identifying the vertices u_1 of G_1 and v_2 of G_2 as the vertex u of G, and identifying vertices v_1 of G_1 and u_2 of G_2 as the vertex v of G; i.e. there is a **twisting** about $\{u, v\}$.

Since none of these operations alter the edge sets of the cycles, if G is 2-isomorphic to H, then $M(G) \cong M(H)$. Also, every graph without isolated vertices is 2-isomorphic to a connected graph. We now state Oxley's [17] version of the theorem which is needed to show the connection between graph and matroid balance.

Theorem 15 (Whitney's 2-Isomorphism Theorem) *Let G and H be graphs having no isolated vertices. Then $M(G)$ and $M(H)$ are isomorphic if and only if G and H are 2-isomorphic.*

If M is a graphic matroid, then Whitney's 2-Isomorphism Theorem assures us $M \cong M(G)$ for some connected graph G. Thus we will assume our graphs G are connected. We are now ready to show the relationship between graph density and matroid density.

Proposition 16 *If G is a connected graph and the cycle matroid $M(G)$ is (strictly) balanced, then G is (strictly) balanced.*

Proof: Let $M(G) = M$ be the cycle matroid of the connected graph G. Let N be a proper submatroid of M. It is well-known (see e.g. [17]) that N is graphic and, furthermore, $N = M(H)$ for some subgraph H of G. We know that $d(N) \leq d(M)$ because M is balanced; equivalently, we can write

$$\frac{|N|}{\rho(N)} \leq \frac{|M|}{\rho(M)}.$$

The graph density of G is $\frac{2|M|}{\rho(M)+1}$ and the graph density of H is $\frac{2|M|}{\rho(M)+k}$ where k is the number of connected components of H.

The comments preceding this proof imply it is sufficient to show

$$\text{if } \frac{|N|}{\rho(N)} \leq \frac{|M|}{\rho(M)}, \text{ then } \frac{2|N|}{\rho(N)+k} \leq \frac{2|M|}{\rho(M)+1}.$$

Now

$$|N|\rho(M) \leq |M|\rho(N)$$

implies that

$$|N|\rho(M) + |N| \leq |M|\rho(N) + |N| \leq |M|\rho(N) + |M| \leq |M|\rho(N) + k|M|,$$

since $|N| \leq |M|$ and $k \geq 1$. Therefore

$$\frac{2|N|}{\rho(N)+k} \leq \frac{2|M|}{\rho(M)+1},$$

and we have established the implication needed to show that G is balanced.

If $M(G)$ is strictly balanced then the fact that G is strictly balanced follows immediately from the argument above simply by replacing the inequalities with strict inequalities. □

The converse of the proposition above is not true. In general, if G is balanced, then $M(G)$ is not necessarily balanced. An example of this is found in Figure 1.

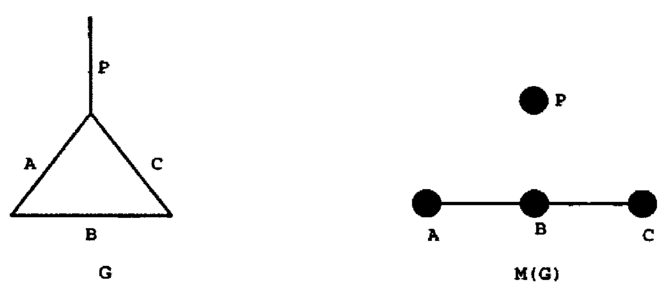

Figure 1: G is balanced; $M(G)$ is not balanced.

It is natural to ask what, if any, conditions can be imposed on G to yield a partial converse for Proposition 16. For example, one could consider two-connectedness of a graph G which forces $M(G)$ to be connected.

Conjecture 1 *If G is a two-connected (strictly) balanced graph, then $M(G)$ is (strictly) balanced.*

Now that we have introduced the class of graphic matroids, we are able to rely on the work of Narayanan and Vartak [15] to establish the following result.

Proposition 17 *The matroid $M(K_n)$ is balanced.*

Before giving the proof we need a theorem which characterizes balanced matroids in terms of their bases.

Theorem 18 ([15]) *Let M be a matroid defined on the ground set E. Then M is balanced if and only if there exist bases $B_1, B_2, \ldots, B_n$ of E such that each element of E belongs to precisely q of these bases.*

Now we are able to prove Proposition 17 and also satisfy the requirement needed to establish Corollary 6.

Proof of Proposition 17: If we consider the spanning trees of K_n, we can see because of the symmetry of the complete graph each edge is in precisely the same number of spanning trees. Spanning trees of a graph are equivalent to bases in the related cycle matroid; hence we can see that each element of $M(K_n)$ is in the same number of bases. Theorem 18 now implies the class of matroids $M(K_n)$ is balanced. $\square$

6 Graph and Matroid Operations

In this section, we explore how certain operations applied to graphs or matroids affect the property of balance. We will start by considering the results obtained by Veerapandiyan and Arumugam for graphs. These authors provide some necessary and sufficient conditions for graphs to be balanced. To begin with they characterize when a graph with more than one component is balanced.

Theorem 19 ([21]) *A graph G is balanced if and only if each component H of G is balanced and $d'(H) = d'(G)$.*

It is natural to ask if this result can be generalized to matroids; this leads to the following conjecture.

Conjecture 2 *A matroid M is balanced if and only if each component H of M is balanced and $d(H) = d(M)$.*

The following theorem is more specific than Theorem 19; when it is generalized it leads to interesting results. Recall that a unicyclic graph is a connected graph containing exactly one cycle and having minimum degree one.

Theorem 20 ([21]) *Let G be a connected graph with minimum degree 1. Then G is balanced if and only if G is either a tree or a unicyclic graph.*

It is not possible to find a direct matroidal analogue to the theorem above because there is no matroid notion analogous to degree. Consider the following assertion which is a matroidal statement of Theorem 20: The matroid M is balanced if and only if M is a free matroid or M contains exactly one cycle. It is clearly false, as seen in the example found in Figure 2; this matroid contains exactly one cycle $\{$A, B, C$\}$. This example will be generalized in Proposition 26.

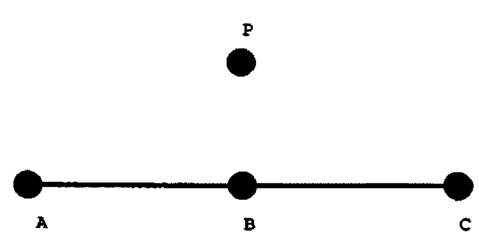

Figure 2: A matroid containing exactly one cycle which is not balanced.

Veerapandiyan and Arumugam [21] provided necessary and sufficient conditions for a graph with k components to be balanced. This is a generalization of Theorem 20.

Theorem 21 ([21]) *Let G be a graph with n vertices, k components and minimum degree 1. Then G is balanced if and only if each component of G is either a tree of order n/k or a unicyclic graph.*

Finally, Veerapandiyan and Arumugam established the following result on the effect of subdividing the edges of G to obtain the graph $S(G)$.

Theorem 22 ([21]) *A subdivision graph $S(G)$ is balanced if and only if G is balanced.*

We also consider which matroid constructions preserve balance. Narayanan and Vartak [15] considered the union of two matroids and the dual of a matroid and showed that these constructions preserve balance.

The union of two matroids on the same ground set is a generalization of direct sum, discussed on page 22. Let $M_1 = (E, \mathcal{C}_1)$ and $M_2 = (E, \mathcal{C}_2)$ be matroids defined on the same ground set E. The **matroid union** $M_1 \vee M_2 = (E, \mathcal{C})$ is a matroid on E with circuits in the set $\mathcal{C}$ which are minimal members of the set: $\{C : A \cap C$ contains a circuit of M_1, or $C - A$ contains a circuit of M_2, for all $A \subseteq C\}$.

Theorem 23 ([15]) *Let M_1, M_2 be matroids defined on the ground set E.*

- *If M_1, M_2 are balanced, then $M_1 \vee M_2$ is balanced.*
- *If M_1, M_2 are strictly balanced and $M_1 \vee M_2$ contains a circuit, then $M_1 \vee M_2$ is strictly balanced.*

Narayanan and Vartak also considered the operation of dualizing a matroid and its effect on balance. Let M be a matroid on the ground set E with rank function ρ. The **dual** $M^*(E)$ of M is a matroid on the set E with rank function ρ_{M^*} where

$$\rho_{M^*}(A) = \rho(E - A) + |A| - \rho(E),$$

and whose set of bases $\mathcal{B}^*$ is the set of all complements of the bases of M.

Theorem 24 ([15]) *If M is a (strictly) balanced matroid, without coloops, then M^* is (strictly) balanced.*

Likewise, it would be interesting to see if dualizing a planar balanced graph preserves balance.

Conjecture 3 *Let G be a planar graph containing no cut edge. If G is balanced, then G^* is balanced.*

We expand on the ideas previously presented and determine if there are other operations which preserve graph or matroid balance. There are many interesting operations for both graphs and matroids; and, of course, there is much room for further research in this area.

The operation of deletion is probably the most basic operation in matroid theory. This operation was briefly discussed earlier, when we described submatroids; we will expand on it now. One can think of deletion simply as "erasure" of an element or subset of the ground set E. Let $M = (E, \mathcal{C})$ and suppose that $X \subseteq E$. The **restriction** of M to X, denoted $M|X$, is the matroid $(X, \mathcal{C}(M|X))$, where $\mathcal{C}(M|X) = \{C \subseteq X : C \in \mathcal{C}(M)\}$. We also refer to this as the **deletion** of $E - X$ from M, which is denoted $M(E - X)$. The following example shows balance is not always preserved under deletion. The original matroid M is balanced but $M - P$ is not.

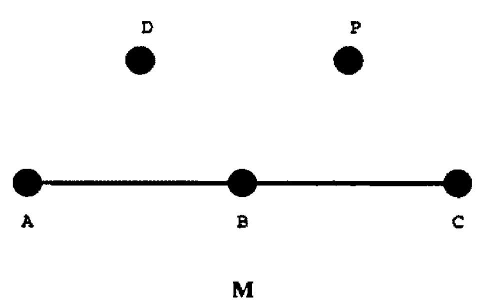

Figure 3: A balanced matroid with a submatroid which is not balanced.

We also consider how the operation of deletion affects balanced graphs. We ask the following question: If a matroid (graph) is balanced, are all submatroids (subgraphs) balanced? Obviously, from the example above we know this question is not always answered in the affirmative for balanced matroids and now show that the same holds for balanced graphs. We have shown that the complete graph K_5 is balanced, and the following example shows that not all subgraphs of K_5 are balanced.

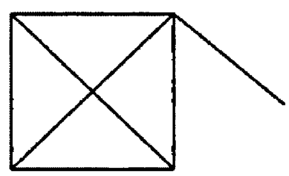

Figure 4: A subgraph of K_5 which is not balanced.

If deletion is viewed as erasure of elements, then the converse operation can be viewed as "adding" elements to a matroid. This operation is formally known as matroid extension. Let M be a matroid defined on the ground set E with rank function ρ_M. If N is a matroid defined on the ground set $E \cup E'$ with rank function ρ_N, then N is an **extension** of M by a subset E' if $N - E' = M$ and $\rho_M(M) = \rho_N(N)$. If the size of E' is one, then this is called a **single-element extension.** The set $\mathcal{K}$ of all flats of M can be partitioned into three types:

$$\begin{aligned} \mathcal{K}_1 = &\ \{K : K \text{ and } K \cup \{p\} \text{ are both flats of N } \} \\ \mathcal{K}_2 = &\ \{K : K \text{ is a flat of } N, \text{but } K \cup \{p\} \text{ is not}\} \\ \mathcal{K}_3 = &\ \{K : K \cup \{p\} \text{ is a flat of } N, \text{but } K \text{ is not}\}. \end{aligned}$$

If N is an extension of M, then N may be viewed as the result of a series of single-element extensions. We will consider the case when the element $\{p\}$ of extension is neither a loop nor an isthmus; otherwise the resulting matroid is isomorphic to $M \oplus U_{0,1}$ and $M \oplus U_{1,1}$, respectively. After the operation $\oplus$ is defined below, we show that if M is balanced, then $M \oplus U_{1,1}$ is not balanced.

Problem 4 *Find hypotheses on a matroid M to ensure that if M is a balanced, then any extension N of M is balanced.*

Perhaps when N is a free extension of M, no additional hypothesis on M are required; in any case, more thorough consideration of Problem 4 is needed.

Matroids may also be constructed through an operation known as the **parallel extension**. This can be defined briefly as adding elements in parallel to some existing element or, less formally, as doubling one or more elements. Parallel elements refer to multiple points of a matroid; hence the resulting matroid is a multiple point matroid. Theorem 25 is a matroidal analogue to Theorem 22.

Theorem 25 ([15]) *Let M be a matroid. The matroid $M(k)$ obtained by replacing each element of M by k parallel elements is balanced if and only if M is balanced.*

Another matroid construction is that of direct sums. Let $M_1 = (E_1, \mathcal{C}_1)$ and $M_2 = (E_2, \mathcal{C}_2)$ be matroids. The **direct sum** $M_1 \oplus M_2 = (E_1 \uplus E_2, \mathcal{C})$ is a matroid defined on the disjoint union of the ground sets E_1 and E_2, whose circuit family $\mathcal{C}$ is described by

$$\mathcal{C}_{M_1 \oplus M_2} = \{C : C \in \mathcal{C}_1 \text{ or } C \in \mathcal{C}_2\}$$

and whose rank function, denoted $\rho_{(M_1 \oplus M_2)}$, is defined for $A \subseteq E_1 \uplus E_2$ as:

$$\rho_{(M_1 \oplus M_2)}(A) = \rho_{M_1}(A \cap E_1) + \rho_{M_2}(A \cap E_2).$$

If two matroids are balanced, their direct sum is not necessarily balanced. This is demonstrated by example found in Figure 5. $M_1 \cong U_{2,3}$ is balanced $M_2 \cong U_{1,1}$ is also balanced, but the direct sum of $M_1 \oplus M_2$ is not balanced. In this example, we could have substituted any balanced matroid M for $U_{2,3}$, as seen in the result found below.

Proposition 26 *If M is a balanced matroid with rank r and size n, where $r < n$, then $M \oplus U_{1,1}$ is not balanced; thus direct sum does not in general preserve balance.*

Proof: Let M be a balanced matroid and let S denote the direct sum $M \oplus U_{1,1}$. Then $d(S)$ is $\frac{n+1}{r+1}$. Suppose S is balanced; then for all submatroids $H \subseteq S$, we have

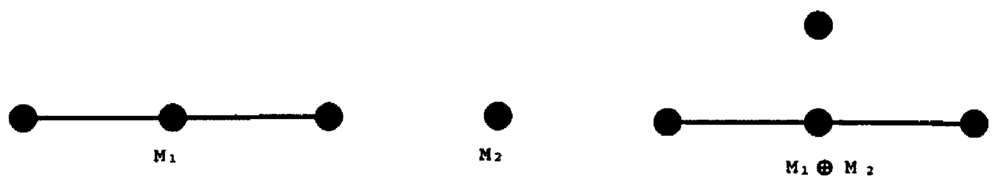

Figure 5: The direct sum of $U_{2,3}$ and $U_{1,1}$ is not balanced.

$d(H) \leq d(S)$. If $H \cong M$, then $d(H) = \frac{n}{r}$ which by our assumption must be at most $\frac{n+1}{r+1}$. This leads to a contradiction and hence proves the result. □

With some restrictions we can find cases when direct sum preserves balance.

Theorem 27 *The direct sum of the uniform matroids $U_{k,n}$ and $U_{r,m}$ is balanced if and only if $d(U_{k,n}) = d(U_{r,m})$.*

Before we prove Theorem 27, it will be necessary to characterize the closed sets of $M_1 \oplus M_2$, which are the members of $\{K_1 \uplus K_2 : K_1$ is closed in M_1 and K_2 is closed in $M_2\}$.

Lemma 28 *The closed sets of $U_{k,n} \oplus U_{r,m}$ are of the following forms: $U_{l,l}$ for $l < k+r$; $U_{s,s} \oplus U_{r,m}$ for $s < k$; $U_{t,t} \oplus U_{k,n}$ for $t < r$; and $U_{k,n} \oplus U_{r,m}$.*

Proof: Let $M_1 = U_{k,n}$, $M_2 = U_{r,m}$, and $S = M_1 \oplus M_2$. Let $K = K_1 \uplus K_2$ be closed in S, where $K_i \subseteq M_i$ for $i = 1, 2$. Thus K_1 is closed in M_1, and K_2 is closed in M_2. Proposition 13 implies that K_1 is isomorphic either to $U_{k,n}$ or to $U_{s,s}$ for $s < k$; likewise, K_2 is isomorphic either to $U_{r,m}$ or to $U_{t,t}$ for $t < r$.

Since the disjoint union of two matroids is simply their direct sum, the closed sets described above can be combined in the following ways:

- $U_{s,s} \uplus U_{t,t} \cong U_{s+t,s+t}$ for $s < k$ and $t < r$;
- $U_{s,s} \uplus U_{r,m} \cong U_{s,s} \oplus U_{r,m}$ for $s < k$;

- $U_{t,t} \uplus U_{k,n} \cong U_{t,t} \oplus U_{k,n}$ for $t < r$;
- $U_{k,n} \uplus U_{r,m} \cong U_{k,n} \oplus U_{r,m}$. □

Proof of Theorem 27: Let $M_1 = U_{k,n}$ and $M_2 = U_{r,m}$. Let $S = M_1 \oplus M_2$; note that $d(U_{k,n}) = d(U_{r,m})$ is equivalent to the condition $km = rn$. The density of S is $\frac{n+m}{k+r}$. For the "only if" direction, assume S is balanced. We know that for all submatroids H of S, $d(H) \leq d(S)$. In particular, $d(M_1) = \frac{n}{k} \leq d(S)$, implying $rn \leq km$. Also, $d(M_2) = \frac{m}{r} \leq d(S)$, implying $km \leq rn$.

For the "if" direction, we will appeal to both Lemmas 10 and 28 to show that the density of any closed set of S is at most the density of S. Let K be a closed set of S. We will consider each possible form of K separately.

Case **a**: $K \cong U_{l,l}$ for $l < r + k$. Since $d(K) = 1$, it is sufficient to show $1 \leq \frac{n+m}{k+r}$. To see this is the case, notice $k \leq n$ and $r \leq m$, implying

$$k + r \leq n + r \leq n + m,$$

and the condition is satisfied.

Case **b**: $K \cong U_{s,s} \oplus U_{r,m}$ for $s < k$. Thus $d(K) = \frac{m+s}{r+s}$. The assumption $km = rn$ and the facts that $k \leq n$ and $r \leq m$ imply

$$km + rm + ks + rs \leq rn + rm + ns + ms,$$

or

$$(k + r)(m + s) \leq (r + s)(n + m).$$

Hence the desired inequality is immediate.

Case **c**: $K \cong U_{t,t} \oplus U_{k,n}$ for $t < r$. The proof here is left to the reader, as it is similar to the argument in Case b.

Case **d**: Here $K \cong S$ and the desired inequality is obvious.

Therefore the direct sum of M_1 and M_2 is balanced if and only if their densities are equal. □

The next operations to be considered are the parallel and series connections of two graphs or matroids. To describe these operations, it is perhaps easiest first to consider graphs and then to generalize to matroids. Let p_i be an arbitrary edge of the graph G_i, for $i = 1, 2$. Arbitrarily assign a direction to p_i and label its tail u_i and its head v_i. To form the **parallel connection** of G_1 and G_2 with respect to the directed edges p_1 and p_2, begin by deleting the edge p_1 from G_1 and the edge p_2 from G_2; then identify the vertices u_1, u_2 as the vertex u and v_1, v_2 as the vertex v. The parallel connection is then completed by adding the new edge p joining the vertices u and v. To obtain the **series connection**, begin by deleting the edge p_1 from G_1 and the edge p_2 from G_2; then identify u_1 and u_2 as the vertex u. The series connection is completed by adding a new edge p joining v_1 and v_2. To illustrate these definitions, we offer the following examples.

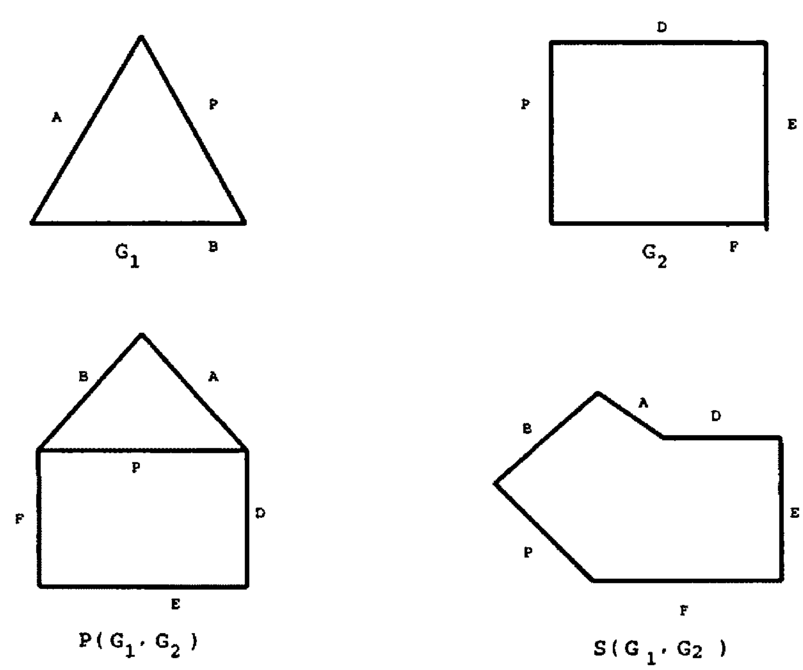

Figure 6: Parallel and series connections of graphs.

The graphs of the parallel and series connections of G_1 and G_2 with respect to the edges p_1 and p_2 will be denoted respectively by $P((G_1;p_1),(G_2;p_2))$ and $S((G_1;p_1),(G_2;p_2))$. The connecting edges are usually arbitrary and so an abuse of notation allows us to write $P(G_1,G_2)$ and $S(G_1,G_2)$.

It is easy to see that the series connection of two cycles remains balanced. This result is immediate from the description of series connection.

Proposition 29 *The series connection of C_n and C_m is balanced.*

Proof: From the definition of series connection of two graphs, it is obvious that $S(C_n,C_m)$ is isomorphic to C_{n+m-1}. Hence the series connection is balanced. □

A **base pointed matroid** M is a pair $(M(E),p)$ with $p \in E$. The **parallel connection** of two base pointed matroids may be described cryptomorphically in terms of their circuits or closed sets.

Proposition 30 ([24]) *Let (M_1,p_1) and (M_2,p_2) be two matroids defined on the ground sets E_i, $i = 1,2$, neither of whose basepoints p_i is an isthmus. Then the parallel connection of M_1 and M_2 can be specified in the following ways:*

- *Circuits of the parallel connection:*
  $\{C : C \text{ is a circuit of } M_1 \text{ or } M_2\} \cup \{C_1 \cup C_2 : C_1 \uplus p \text{ is a circuit of } M_1, \text{ and } C_2 \uplus p \text{ is a circuit of } M_2\}$.
- *Closed sets of the parallel connection:*
  $\{K : K \cap E_1 \text{ is closed in } M_1 \text{ and } K \cap E_2 \text{ is closed in } M_2\}$.
- *Rank function of the parallel connection for $A_i \subseteq M_i$, $i = 1$ or 2:*

$$\rho_P(A_1 \cup A_2) = \begin{cases} \rho_{M_1}(A_1 \cup p) + \rho_{M_2}(A_2 \cup p) - 1 & \text{if } \rho_{M_i}(A_i \cup p) = \rho_{M_i}(A_i) \\ & \text{for } i = 1 \text{ or } 2 \\ \rho_{M_1}(A_1) + \rho_{M_2}(A_2) & \text{otherwise.} \end{cases}$$

(In particular, for any closed set K of the parallel connection,

$$\rho_P(K) = \rho_{M_1}(K \cap E_1) + \rho_{M_2}(K \cap E_2) - \rho_P(K \cap \{p\}).$$

If M_1 and M_2 are two matroids whose ground sets meet in a single element p, then it is convenient to denote the parallel connection by $P((M_1;p_1),(M_2;p_2))$ and the series connection, which is defined below, by $S((M_1;p_1),(M_2;p_2))$. If the context is clear or if the choice of p is arbitrary we will use the notation $P(M_1,M_2)$ or $S(M_1,M_2)$.

Series and parallel connections are dual operations, related by the following theorem.

Theorem 31 ([17]) *Let M_1 and M_2 be matroids with basepoint p. Then*

$$S(M_1,M_2)=[P(M_1^*,M_2^*)]^*$$

and

$$P(M_1,M_2)=[S(M_1^*,M_2^*)]^*.$$

The operation of parallel connection does not always preserve balance. To see this the reader is asked to show that $P(U_{2,7},U_{2,3})$ is not balanced. We begin to explore when balance can be preserved by considering uniform matroids.

Theorem 32 *The parallel connection of two free matroids is balanced.*

Proof: We will show that the parallel connection of two free matroids is free; once this has been accomplished, Proposition 14 will give the theorem. Let M_1 and M_2 be free matroids; hence the parallel connection $P(M_1,M_2)$ contains no circuits. $\square$

The next result generalizes when the parallel connection of two balanced matroids preserves balance.

Theorem 33 *Let $r\le n\le m$; then the parallel connection $P(U_{r,n},U_{r,m})$ is balanced if and only if $m\le \frac{r}{r-1}(n-1)$.*

To see that Theorem 33 is not trivial, the reader should try showing that the parallel connection of two n-point lines is balanced.

The proof of the Theorem 33 relies heavily on the characteristics of closed sets of parallel connection.

Lemma 34 *The closed sets of $P(U_{k,n}, U_{r,m})$ are $U_{l,l}$ for $l \leq k+r-2$, $U_{r,m} \oplus U_{l,l}$ for $l \leq k-2$, $U_{k,n} \oplus U_{l,l}$ for $l \leq r-2$, and the entire parallel connection.*

Proof: Let $M_1 = U_{k,n}$ and $M_2 = U_{r,m}$ and $P = P(M_1, M_2)$. Let $K = K_1 \cup K_2$ be a closed set in P such that $K_1 = K \cap M_1$ and $K_2 = K \cap M_2$. Throughout this proof we assume $t < k$ and $s < r$. By Proposition 13 we know K_1 is isomorphic to either $U_{t,t}$ or $U_{k,n}$; likewise K_2 is isomorphic to either $U_{s,s}$ or $U_{r,m}$.

If $p \notin K$ then it is obvious that $K_1 \cong U_{t,t}$ and $K_2 \cong U_{s,s}$. Hence the union of K_1 and K_2 is disjoint and equivalent to the direct sum; therefore $K \cong U_{l,l}$ for some $l \leq r+k-2$.

On the other hand, let $p \in K$; then there are four case to consider, arising from the following: $K_1 \cong U_{t,t}$ or $U_{k,n}$, and $K_2 \cong U_{s,s}$ or $U_{r,m}$.

Case **a**: If $K_1 \cong U_{t,t}$ and $K_2 \cong U_{s,s}$, then the union of K_1 and K_2 is isomorphic to $U_{l,l}$ for some $l \leq r+k-2$.

Case **b**: If $K_1 \cong U_{k,n}$ and $K_2 \cong U_{r,m}$, then $K \cong P$.

Case **c**: If $K_1 \cong U_{t,t}$, $K_2 \cong U_{r,m}$ and $K_1 \cap K_2 = p$, then K can be written as the disjoint union $(K_1 - p) \uplus K_2$. Since K_1 is free and closed in M_1, we know $K_1 - p$ is also free and closed in M_1. Thus $K \cong (U_{t-1,t-1} \oplus U_{r,m})$ for $t < k$.

Case **d**: Let $K_1 \cong U_{k,n}$ and $K_2 \cong U_{s,s}$. The proof of this case is left to the reader to check since it is similar to the argument in case c. □

Now we have the necessary tools to prove Theorem 33.

Proof of Theorem 33: The result is almost immediate in the "only if" direction. Let $P = P(U_{r,n}, U_{r,m})$ be balanced. From Proposition 30 it is easy to see the density of P is $\frac{m+n-1}{2r-1}$. The assumption that P is balanced implies that all submatroids H of P have density at most $\frac{m+n-1}{2r-1}$. Consider the submatroid $U_{r,m}$; the density of this submatroid must satisfy $d(U_{r,m}) = \frac{m}{r} \leq \frac{m+n-1}{2r-1}$. Expressing this inequality as an upper bound for m gives the desired inequality:

$$m \leq \frac{r}{r-1}(n-1).$$

For the converse, we rely on Lemmas 34 and 10; therefore it is only required that we show the closed sets have density at most $\frac{n+m-1}{2r-1}$. Let K be a closed set of P. We will consider the four possible forms of closed sets described in Lemma 34.

Case **a**: Let $K \cong U_{l,l}$ for $l \leq 2r-2$; Since $d(K) = 1$, it is necessary to show $1 \leq \frac{m+n-1}{2r-1}$. Since $r \leq n \leq m$, we have $2r \leq n+m$. Thus $2r-1 \leq n+m-1$, as needed.

Case **b**: In this situation we have $K \cong P$ and the inequality is obvious.

Case **c**: Let $K \cong (U_{l,l} \oplus U_{r,m})$ for $l \leq r-2$. It will suffice to show $d(K) = \frac{m+l}{r+l} \leq \frac{n+m-1}{2r-1}$. The assumption gives

$$rm - m \leq rn - r$$

and

$$rm - m + rm + 2rl - l \leq rn - r + rm + ml + nl - l$$

since $r \leq n \leq m$. Thus, by observing

$$2r(m+l) - (m+l) \leq r(m+n-1) + l(n+m-1)$$

we see that

$$(2r-1)(m+l) \leq (r+l)(m+n-1),$$

and the desired inequality is obtained.

Case **d**: Let $K \cong (U_{r,n} \oplus U_{l,l})$ for $l \leq r-2$. The proof is left to the reader since it is similar to the one in case c.

This completes the proof of Theorem 33. □

Recall that the operations of taking series and parallel connections are dual to one another; also the dual of a balanced matroid is balanced. Using these facts we are able to establish the following result.

Corollary 35 *Let $r \leq n \leq m$; the series connection of $U_{n-r,n}$ and $U_{m-r,m}$ is balanced if and only if $m \leq \frac{r}{r-1}(n-1)$.*

Before Corollary 35 can be proved, we need to establish a fact about duals of uniform matroids.

Proposition 36 *The dual of the uniform matroid $U_{r,n}$ is the uniform matroid $U_{n-r,n}$.*

Proof: Consider $U_{r,n}$. The bases of $U_{r,n}$ are all the r-element subsets of an n-element set E. Hence, $\mathcal{B}^*(U_{r,n})$ consists of all the $(n-r)$-element subsets of E. Thus, $U^*_{r,n} = U_{n-r,n}$. □

Now using Theorem 31 and Theorem 24, the proof of Corollary 35 is immediate from Theorem 33.

7 Final Remarks

The theory of balanced matroids is relatively new and provides the researcher with many interesting facets to consider. A few conjectures and problems have been mentioned in the exposition (see pages 16, 18 20 21). We would like to conclude with some remarks which lead to interesting areas which could point to further research.

We were particularly interested in finding classes of matroids which are balanced or strictly balanced. Of course, there are many classes of matroids which were not considered. For example, the interested reader might consider $M(\mathcal{W}_n)$, the rank-n wheel, $\mathcal{W}^n$, the rank-n whirl, or the Pappus and non-Pappus matroids, to suggest a few.

We were also interested in determining how specific constructions or operations affect the balance of a matroid. There is a wealth of unanswered questions in this area.

The various articles cited in this thesis can be used to find direction to a variety of research opportunities focusing on the probabilistic method, threshold functions and their relationship to balanced matroids.

The main focus of this thesis is matroid theory; for those interested in graph theory, similar questions can be asked and explored for balanced graphs. There are also many closely related topics which were not discussed. Slight variations of the definition of balanced graphs lead to closely related ideas which have been considered by various authors.

Strongly balanced graphs have been researched by Ruciński and Vince [18] as well as Veerapandiyan and P. Ramachandran in [22]. For a nonempty graph G, define $d^*(G) = \frac{|E(G)|}{|V(G)|-1}$. Such a G is **strongly balanced** if $d^*(H) \leq d^*(G)$ for every nonempty subgraph H of G. It has been shown that the following classes of graphs are strongly balanced: maximal planar graphs, maximal outerplanar graphs, and maximal acyclic graphs.

Another closely related idea is that of k-balanced graphs. The concept of k-balanced graphs was introduced by Veerapandiyan, Ramachandran and Arumugam [23].

Let k be a nonnegative integer. For any graph G with $|V(G)| > k$, let $d_k(G) = \frac{|E(G)|}{|V(G)|-k}$. The graph G is k-**balanced** if $|V(G)| > \frac{k(k+1)}{2}$ and $d_k(H) \leq d_k(G)$ for every subgraph H of G with $|V(H)| > k$. Thus, a 0-balanced graph is simply a balanced graph and a 1-balanced graph is a strongly balanced graph.

There is an abundance of potential research problems in the area of balanced graphs and balanced matroids. The concepts introduced above will provide a natural launchpad for the interested reader.

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