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Asymptotically good colorings of plane multigraphs

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Asymptotically good colorings of plane multigraphs

by

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Asymptotically good colorings of plane multigraphs

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A multigraph is a loopless graph with repeated edges allowed. Various chromatic numbers can be formulated as solutions of integer programming problems, or IPs. A specific chromatic number is *asymptotically good*, or a.g., if the solution of its IP approaches the solution of its linear relaxation when the relevant multigraph parameter grows without bound. Formally, a multigraph invariant β, which can be formulated as the solution to an IP problem, is asymptotically good in case $\beta/\beta^* \to 1$ as $\beta^* \to \infty$, where β^* is the solution of the linear relaxation of the IP defining β. The main goal of this work is to investigate several conjectures on the asymptotics of coloring plane multigraphs.

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CHAPTER I
INTRODUCTION

One of the most prominent graph-theoretic parameters, the chromatic number (definitions of terminology are deferred to §1.1) is often formulated as the solution of an *integer programming* problem, or IP. This approach is computationally difficult when the relevant coloring parameter is large (see [16]). In a 1996 paper [6], Kahn noticed the asymptotic behavior of the *chromatic index* of a *multigraph;* i.e., the solution of the IP defining the chromatic index can be approximated by the solution of the IP's linear relaxation, the computation of which may be completed in polynomial time [10]. In a following paper [7], he established the asymptotics of the *list-chromatic index* of multigraphs (see also [8] for related research). Based on Kahn's results, Kayll [11, 12] proved similar results about the asymptotics of the *total chromatic number* of a graph and of a multigraph. In this thesis we add two new parameters to the list, namely, the *entire chromatic number* and the *edge-face chromatic number* of a plane multigraph. The asymptotics of these two invariants were conjectured by Kayll [11] in 1997.

1.1 Terminology

In this thesis, *graph* means simple graph containing no loops or repeated edges; *multigraph* is used when repeated edges are allowed. Thus, every graph is a multigraph, but the converse is false. A graph is *planar* if it can be embedded in the plane. If a planar graph is embedded in the plane, then it is called a *plane* graph. Unless specified, G will always denote a multigraph. For graphs and multigraphs, we use $V(G)$, $E(G)$, and $\Delta(G)$ to denote the vertex set, edge set, and maximum degree of G. If it is clear from the context, '(G)' will be omitted. Two vertices are *adjacent* if they are joined by an edge, two edges are adjacent if they share a common vertex, and two faces are adjacent if their boundaries have at least one common edge. (Two faces touching only at a vertex are not adjacent; similarly, an edge and a face touching only at a vertex are not adjacent.) A vertex (or an edge) is *incident* to a face if it forms part of the boundary of the face. Also, the vertices u and v are each *incident* to the edge uv. See [3] for any omitted terminology.

In this work, we will be considering a variety of graph coloring parameters which we now define. An assignment of k colors to the vertices of G so that adjacent vertices receive different colors is called a (valid) *coloring* of G; when this is possible, G is said to be *k-colorable*. The least number n for which G is n-colorable is called the *vertex chromatic number*, or simply the *chromatic number* of G, and denoted by χ. The *chromatic index*, χ', of G is the least number of colors to ensure that the edge set E admits a valid coloring. (Some authors, e.g. [15], prefer to use χ_v to denote chromatic number and χ_e to denote chromatic index.) The *total chromatic number*, χ_t, of G is the least number of colors needed to color all the elements of $V \cup E$ such that no two adjacent or incident elements in $V \cup E$ receive the same color. The *entire chromatic number*, χ_{vef}, of a plane G is the least number of colors needed to color the vertices, edges, and faces of G, where incident or adjacent elements are colored differently (two faces touching only in a vertex may receive the same color; similarly, a face and an edge touching in only one vertex may get the same color). The *coupled (edge-face) chromatic number,* χ_{vf} (χ_{ef}), is the least number of colors needed to color the vertices (edges) and faces of G such that two incident or adjacent elements receive different colors. The *list-chromatic index,* χ'_l, of G is the least number t, such that, for any assignment of a list $\Lambda(e)$ of size t to every edge $e \in E(G)$, it is possible to color $E(G)$ so that every edge receives a color from its list. The case when all the lists $\Lambda(e)$ are identical of size χ'_l implies that $\chi'_l \geq \chi'$.

For simple plane graphs with maximum degree Δ, a few important bounds on the coloring parameters that we defined are the following: $\chi \leq 4$, $\chi_{vf} \leq 6$, and $\chi_e \leq \Delta+1$. The first bound is the Four Color Theorem [1]. (See [15] for further discussion and background on these bounds.) Recently, Sanders and Zhao [15] proved that, for a simple plane graph, if $\Delta \geq 8$, then $\chi_{ef} \leq \Delta + 2$, and in general $\chi_{ef} \leq \Delta + 3$, which was conjectured by Melnikov. Their proof partially relies on the Four Color Theorem. Two main results of our work are the establishment of the asymptotic behavior of χ_{vef} and χ_{ef} for plane multigraphs. Our arguments will use, but do not depend on, the Four Color Theorem.

A *stable set,* S, of G is a subset of V such that the induced subgraph of G on S is empty (contains no edges). A *matching*, M, of G is a subset of E such that no two edges of M share the same end. We use $\mathcal{M}$ to denote the family of matchings of G. Let $\delta(S)$ be the set of edges with one end in S. A *total stable set* of G is defined to be a subset of $E \cup V$, denoted by $M \cup S$, where $M(\subseteq E)$ is a matching, $S(\subseteq V)$ is a stable set and $M \cap \delta(S) = \emptyset$. We use $\mathcal{T}(= \mathcal{T}(G))$ to denote the family of total stable sets of G and T $(= M \cup S)$ to denote a member of $\mathcal{T}$. Let F be the set of all faces of G, N be a subset of F such that no two elements of N share the same edge and $\gamma(N)$ be the set of vertices and edges that are incident to N. An *entire stable set* is a subset of $E \cup V \cup F$ of the form $M \cup S \cup N$, where $M \cup S$ is a total stable set, N is described as above, and $(M \cup S) \cap \gamma(N) = \emptyset$. We use $\mathcal{R}(= \mathcal{R}(G))$ to denote the family of entire stable sets of G and $R\,(= M \cup S \cup N)$ to denote a member of $\mathcal{R}$. It is easy to see that $\mathcal{T} \subseteq \mathcal{R}$. Finally, *edge-face stable sets* are defined in the natural way; we use $\mathcal{U}$ for the family of those sets and U $(= M \cup N)$ to denote a member of $\mathcal{U}$.

1.2 Fractional coloring and asymptotically good invariants

All the chromatic numbers defined in the preceding section can be defined as solutions of IP problems (see [4] for omitted LP and IP terminology). For example, if f : $\mathcal{R} \rightarrow \{0,1\}$, then the entire chromatic number, χ_{vef}, of G can be formulated as the optimal solution of the IP problem:

$$\chi_{vef} = \min \sum_{R \in \mathcal{R}} f(R)$$

$$\text{subject to} \sum_{a \in R \in \mathcal{R}} f(R) = 1 \text{ , for each } a \in E \cup V \cup F.$$

The idea is that the members of $f^{-1}(\{1\})$ form the color classes of the entire coloring. The linear functional $\sum_{R \in \mathcal{R}} f(R)$ counts the number of colors used, while the equality constraints ensure that each vertex, edge and face appears in exactly one color class.

The linear relaxation of the problem above is formulated accordingly. If f : $\mathcal{R} \rightarrow [0,1]$, then the *fractional entire chromatic number*, χ^*_{vef}, is the optimal solution of the LP:

$$\chi_{vef}^* = \min \sum_{R \in \mathcal{R}} f(R)$$

$$\text{subject to} \sum_{a \in R \in \mathcal{R}} f(R) = 1 \text{ , for each } a \in E \cup V \cup F.$$

Clearly, $\chi_{vef}^* \leq \chi_{vef}$. Following the same routine, we can define χ, χ', χ_l', χ_t, χ_{vf}, χ_{ef}, and their fractional partners χ^*, χ'^*, $\chi_l'^*$, χ_t^*, χ_{vf}^*, χ_{ef}^*. A useful observation is that χ', χ_{vef}, χ_t and their fractional counterparts χ'^*, χ_{vef}^*, χ_t^* are at least Δ, since all edges incident to a vertex of maximum degree have to be colored differently.

Kahn introduced the notion of *asymptotically good*, or a.g., behavior for multigraph coloring parameters (see e.g. [6, 8]). Let β be a multigraph invariant, such as χ_{vef}, that can be formulated as the optimal solution of an IP problem and let β^* be the optimal solution of the linear relaxation of the IP. We say β is asymptotically good if $\beta/\beta^* \longrightarrow 1$ as $\beta^* \longrightarrow \infty$; that is, for each $\varepsilon > 0$ there exists $B = B(\varepsilon)$ such that if $\beta^* > B$, then $(1+\varepsilon)^{-1} < \beta/\beta^* < 1+\varepsilon$. We often abbreviate $\beta/\beta^* \longrightarrow 1$ by $\beta \sim \beta^*$.

In this thesis, we are concerned with establishing which coloring parameters are a.g.

CHAPTER II
PRECURSORS TO OUR RESULTS

In this chapter, we will consider the asymptotics of χ', χ'_l, and χ_t. For multigraphs, Kahn [6, 7] proved that χ' and χ'_l are a.g. and asymptotic to each other.

Theorem 1 *[Kahn] For multigraphs, χ' is a.g.*

The proof of this theorem appeared in [6].

Theorem 2 *[Kahn] For multigraphs,*

$$\chi'_l \sim \chi'^* \text{ as } \chi'^* \to \infty.$$

Since $\chi'^* = \chi_l'^*$ (see [6]), Theorem 2 is really an assertion that χ'_l is a.g. The proof, to appear in [7], uses a method based on "hard-core" probability distributions (see [9]); this 30-page paper [7], together with Theorem 1, built a solid foundation for further research in the asymptotically good behavior of several other chromatic numbers.

In 1997, Kayll [11, 12] proved the following result, based on Theorem 2. We will use his strategy in establishing the a.g. behavior of χ_{vef} and χ_{ef}. The idea of the proof is to find a bridge, such that we can relate χ'^*, χ_t^*, χ_t and χ'_l in a "chain" ordered by " $\leq$ ". If we know that $\chi_t^* \to \infty$ implies $\chi'^* \to \infty$, then, by Theorem 2, we will be able to conclude that χ_t is a.g.

Theorem 3 *[Kayll] For multigraphs, χ_t is a.g. That is, for each $\varepsilon > 0$, there exists $C = C(\varepsilon)$ such that every multigraph G with $\chi_t^*(G) > C$ satisfies*

$$(1+\varepsilon)^{-1} < \frac{\chi_t}{\chi_t^*} < 1+\varepsilon. \tag{2.1}$$

To begin a sketch of Kayll's proof, it is easy to see that $(1+\varepsilon)^{-1} < \chi_t/\chi_t^*$ from the fact that $\chi_t^* \leq \chi_t$. Thus, we only need to establish the right-hand inequality of (2.1). By a mapping from $\mathcal{T}$ to $\mathcal{M}$, it can be shown that $\chi'^* \leq \chi_t^*$. Construction of such a mapping is the key to the proof. (We will see more details of such mappings in establishing the asymptotics of χ_{vef} and χ_{ef}.) It is well-known that $\chi_t \leq \chi_l' + 2$ (see e.g. [5] p. 87). Thus, we have $\chi'^* \leq \chi_t^* \leq \chi_t \leq \chi_l' + 2$. Kostochka proved that $\chi_t \leq \lfloor 3\Delta/2 \rfloor$ (see e.g. [5] p. 86). By the fact $\chi'^* \geq \Delta$, we then see that $\chi_t^* \to \infty$ forces $\chi'^* \to \infty$, which gives us $\chi_t^* \sim \chi_t$ by Theorem 2. From another point of view, the proof is to make a "sandwich" with χ_l' and χ'^* as "bread", χ_t^* and χ_t as ingredients, and use Theorem 2 to push them together.

Theorems 1, 2, and 3 begin a list of a.g. coloring parameters. In chapter III, we add χ_{vef} and χ_{ef} to this list.

CHAPTER III
ASYMPTOTICS OF THE ENTIRE CHROMATIC NUMBER AND THE EDGE-FACE CHROMATIC NUMBER

The most important theorem on planar graphs is probably the Four Color Theorem, which had been known as the Four Color Conjecture (4CC) for almost a hundred years.

Theorem 4 ***[The Four Color Theorem]*** *Every planar graph is 4-colorable.*

Using planar duality, Theorem 4 is easily seen to apply both to vertex colorings and to face colorings. Here is a brief history of this famous theorem. The 4CC was first raised by Guthrie in 1852 and it became well-known during the 1860's due to the interest of several famous mathematicians, such as DeMorgan and Cayley. In 1879, Kempe published a "proof" of the 4CC. It stood for about 10 years before Heawood discovered an error. Using Kempe's techniques, Heawood proved that every planar graph is 5-colorable. Heawood's result stood for about 86 years until Appel and Haken [1], with the aid of Koch, used about 1200 hours of computer time to check all of 1936 special cases. For a more comprehensive discussion on this theorem, please refer to [14].

The first of our two main results is the following theorem, which settles conjecture 5.2 from [11]. The proof follows Kayll's strategy, but uses χ_t^* and χ_t as "sandwich bread" and Theorem 3 to push them toward each other.

Theorem 5 *For plane multigraphs, χ_{vef} is a.g. That is, for each $\varepsilon > 0$, there exists $D = D(\varepsilon)$ such that every plane multigraph G with $\chi^*_{vef}(G) > D$ satisfies*

$$(1+\varepsilon)^{-1} < \frac{\chi_{vef}}{\chi^*_{vef}} < 1+\varepsilon.$$

Proof. Since χ^*_{vef} is the optimal solution of the linear relaxation of the IP defining χ_{vef}, we see that

$$\chi^*_{vef} \leq \chi_{vef}. \tag{3.1}$$

In light of Theorem 4, every valid total coloring can be expanded to a valid entire coloring using (at most) 4 additional colors for the faces; thus,

$$\chi_{vef} \leq \chi_t + 4. \tag{3.2}$$

An optimal fractional entire coloring, $f : \mathcal{R} \longrightarrow [0,1]$, can be "shifted" to a valid fractional total coloring, $h : \mathcal{T} \longrightarrow [0,1]$, as follows. Given $T = M \cup S \in \mathcal{T}$, let

$$h(T) = \sum_{R = T \cup N} f(R),$$

where the sum is taken over all $R \in \mathcal{R}$ of the specified form. It is easy to check that $\sum_{a \in T \in \mathcal{T}} h(T) = 1$, for each $a \in E \cup V$, so h gives us a valid fractional total coloring with

$$h(G) = \sum_{T \in \mathcal{T}} h(T) = \sum_{T \in \mathcal{T}} \left\{ \sum_{R = T \cup N} f(R) : R \in \mathcal{R} \right\} = \sum_{R \in \mathcal{R}} f(R) = f(G) = \chi^*_{vef},$$

and it follows that

$$\chi_t^* \leq \chi_{vef}^*. \tag{3.3}$$

For our final preliminary step, we will need a bound of the form $\chi_{vef}^* \leq \chi_t^* + c$ (c is a constant). For large Δ , this is easy, since, e.g., Borodin (see e.g. [5] p. 47) proved $\chi_{vef} \leq \Delta + 4$ when $\Delta \geq 7$. Since $\chi_t^* \geq \Delta + 1$, with (3.1) we obtain

$$\chi_{vef}^* \leq \chi_{vef} \leq (\Delta + 1) + 3 \leq \chi_t^* + 3,$$

provided $\Delta \geq 7$. As we prefer to avoid dependence on large Δ in our proof, we will instead obtain $\chi_{vcf}^* \leq \chi_t^* + c$ more directly.

We will define a fractional entire coloring $f : \mathcal{R} \rightarrow [0, 1]$ starting from an optimal fractional total coloring $h : \mathcal{T} \rightarrow [0, 1]$ and using Theorem 4, which guarantees that the faces of G may be properly colored using at most 4 colors. Denote the face color classes by $\{N_i\}_{i=1}^{s}$ ($s \leq 4$); no two faces within an N_i share a common edge. We are ready to expand h to a fractional entire coloring. Define $f : \mathcal{R} \rightarrow [0, 1]$ by

$$f(R) = \begin{cases} h(R), & \text{if } R \in \mathcal{T} \text{ , i.e. } R = M \cup S \\ 1, & \text{if } R = N_i,\ 1 \leq i \leq s \\ 0, & \text{otherwise.} \end{cases}$$

Clearly f is a fractional entire coloring; moreover,

$$f(G) = \sum_{R \in \mathcal{R}} f(R) = \sum_{R \in \mathcal{T}} h(R) + s = h(G) + s = \chi_t^* + s \leq \chi_t^* + 4.$$

Thus,

$$\chi_{vef}^* \leq \chi_t^* + 4, \tag{3.4}$$

as desired.

To complete the proof, we need to establish

$$(1+\varepsilon)^{-1} < \frac{\chi_{vef}}{\chi_{vef}^*} < 1+\varepsilon \tag{3.5}$$

for any given $\varepsilon > 0$, provided χ_{vef}^* is sufficiently large. By (3.1), the left-hand inequality in (3.5) is clear, so we will work to obtain the right-hand inequality.

Given $\varepsilon > 0$, let $\gamma = \varepsilon/2$ and choose C large enough so that (according to Theorem 3) if $\chi_t^* > C$, then $\chi_t/\chi_t^* < (1+\gamma)$. If $\chi_{vef}^* > D := \max\{C+4, 8/\varepsilon+4\}$, then since $\chi_{vef}^* - 4 \leq \chi_t^*$ (by (3.4)), we see that χ_t^* exceeds both C and $8/\varepsilon = 4/\gamma$. Thus, provided $\chi_{vef}^* > D$, we have

$$\chi_{vef} \leq \chi_t + 4 < (1+\gamma)\chi_t^* + \gamma\chi_t^* = (1+\varepsilon)\,\chi_t^* \leq (1+\varepsilon)\chi_{vef}^*,$$

justifying the inequality, respectively, by (3.2), the preceding two sentences, and (3.3). Comparing the extremes of the last chain of inequalities yields the right-hand bound in (3.5). ■

Remark. We did not really need the full power of Theorem 4 for our proof. If the constant 4 in the bounds (3.2), (3.4) were replaced by another constant, our asymptotic arguments would still be valid. Thus, for example, the simpler Five Color Theorem of Heawood (mentioned at the start of this chapter) would suffice for our purposes.

Our second main result is an analogue for χ_{ef} and partially settles Conjecture 5.3 from [11].

Theorem 6 *For plane multigraphs, χ_{ef} is a.g. That is, for each $\varepsilon > 0$, there exists $D = D(\varepsilon)$ such that every plane multigraph G with $\chi^*_{ef}(G) > D$ satisfies*

$$(1+\varepsilon)^{-1} < \frac{\chi_{ef}}{\chi^*_{ef}} < 1+\varepsilon.$$

The proof is similar to that of Theorem 5, but changes the "bread" to χ'^* and χ', and use Theorem 1 to push instead of Theorem 3.

Proof. Since χ^*_{ef} is the optimal solution of the linear relaxation of the IP defining χ_{ef}, we have

$$\chi^*_{ef} \leq \chi_{ef}. \tag{3.6}$$

Again using Theorem 4, every valid edge coloring can be expanded to a valid edge-face coloring using (at most) 4 additional colors for the faces; thus,

$$\chi_{ef} \leq \chi' + 4. \tag{3.7}$$

A valid fractional edge coloring $g : \mathcal{M} \to [0,1]$ can be obtained from an optimal fractional edge-face coloring $h : \mathcal{U} \to [0,1]$ by defining, for $M \in \mathcal{M}$,

$$g(M) = \sum_{U = M \cup N} h(U),$$

where the sum is taken over all $U \in \mathcal{U}$ of the specified form. That g is a fractional edge coloring is easy to check; moreover,

$$g(G) = \sum_{M \in \mathcal{M}} g(M) = \sum_{M \in \mathcal{M}} \left\{ \sum_{U = M \cup N} h(U) : U \in \mathcal{U} \right\} = \sum_{U \in \mathcal{U}} h(U) = h(G) = \chi_{ef}^{*},$$

so that

$$\chi'^{*} \leq \chi_{ef}^{*}. \tag{3.8}$$

Using an argument analogous to that leading to (3.4), we may obtain

$$\chi_{ef}^{*} \leq \chi'^{*} + 4. \tag{3.9}$$

We are now equipped to complete the proof, for which we need to establish

$$(1+\varepsilon)^{-1} < \frac{\chi_{ef}}{\chi_{ef}^{*}} < 1 + \varepsilon \tag{3.10}$$

for arbitrary $\varepsilon > 0$. By (3.6), it is clear that $(1+\varepsilon)^{-1} < \chi_{ef}/\chi_{ef}^{*}$ when $\varepsilon > 0$, so we focus on the right-hand inequality in (3.10).

Given $\varepsilon > 0$, let $\gamma = \varepsilon/2$, and choose C large enough (according to Theorem 1) to ensure that if $\chi'^{*} > C$, then $\chi'/\chi'^{*} < (1+\gamma)$. If $\chi_{ef}^{*} > D := \max\{C+4, 8/\varepsilon + 4\}$, then since $\chi_{ef}^{*} - 4 \leq \chi'^{*}$ (by (3.9)), we see that χ'^{*} exceeds both C and $8/\varepsilon = 4/\gamma$. Thus, as long as $\chi_{ef}^{*} > D$, we have

$$\chi_{ef} \leq \chi' + 4 < (1+\gamma)\chi'^{*} + \gamma\chi'^{*} = (1+\varepsilon)\,\chi'^{*} \leq (1+\varepsilon)\chi_{ef}^{*},$$

where the inequalities are justified, respectively, by (3.7), the two preceding sentences, and (3.8). This chain of inequalities yields the right-hand inequality in (3.10), as desired. ■

As noted prior to the statement of Theorem 6, the result partially settles Conjecture 5.3 from [11]. That conjecture also concerned the asymptotics of the coupled chromatic number, χ_{vf}, of a plane multigraph. Observe that Theorem 4 implies that $\chi_{vf} \leq 8$; thus, χ_{vf} and χ^*_{vf} never grow without bound, so that part of the conjecture is not interesting to investigate.

CHAPTER IV
CONCLUSION

We have provided background on and investigated various chromatic numbers. Prior to this work, it was known that for multigraphs, χ', χ'_l and χ_t are each a.g. In this thesis, we have added χ_{vef} and χ_{ef} for plane multigraphs to this list of a.g. parameters. We also pointed out that since χ_{vf} is bounded by an absolute constant, we are not interested in considering the a.g. behavior of this parameter. Since both χ and χ_f (defined in the natural way) for plane multigraphs are bounded by 4 (by Theorem 4), we also need not consider the a.g. behavior of these parameters. Thus, the investigation of the a.g. behavior of chromatic numbers of plane multigraphs is complete. This of course leaves open the general question (for multigraphs): when does χ exhibit asymptotically good behavior? Our work provides several partial answers to this question. We offer the problem of completely answering it to future researchers. We also believe the results on plane multigraphs can be generalized to multigraphs embedded on surfaces of higher genus, as discussed, e.g., in [13].

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