Oriented flow of rank 3 matroids

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8/98
ORIENTED FLOW OF RANK 3 MATROIDS

by

Matthew R. Edmonds

presented in partial fulfillment of the requirements

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A parameter of oriented matroids called the oriented flow number is defined and studied. It is an extension of the concept of the circular chromatic number of a graph to oriented matroids. Oriented matroids can be realized as signed pseudosphere arrangements. When the rank of the matroid is 3, the pseudosphere arrangements take the form of line arrangements in the plane, in which the lines are not necessarily straight, and each pair of lines intersects exactly once. Rank 3 oriented matroids are studied in this setting. It had been conjectured that the oriented flow number of all rank 3 matroids is at most 4. This is shown to in fact be the case. This is shown first for uniform rank 3 matroids, and then the proof is extended to all rank 3 orientable matroids. The proof relies upon simple geometric considerations of arrangements and orientations of small numbers of lines in the plane. Larger arrangements are then viewed as unions of these smaller arrangements. The bound on the oriented flow number is then found by orienting the smaller arrangements in an optimal way.
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Contents

1 Background 1

1.1 Matroids ................................................................. 1
1.2 Oriented Matroids ...................................................... 2
1.3 Topological Representation Theorem ............................. 4
1.4 Circular Chromatic Number in Graphs ............................ 7
1.5 Oriented Flow in Matroids .......................................... 8

2 Bounding the Oriented Flow Number of $U_{3,n}$ 11

3 Bounding the Oriented Flow Number of All Rank 3 Orientable Matroids 34

4 Potential Topics of Future Research 52

A Appendix: Simple Arrangements of Small Numbers of Lines 53
List of Figures

1. The relative discrepancy of adjacent faces in $U_{3,n}$ .................................. 13
2. The discrepancy at a vertex not lying on the boundary of an $n$-face . 17
3. Pseudoline arrangements of $U_{3,7}$ ................................................................. 23
4. Pseudoline arrangements of $U_{3,7}$ ................................................................. 24
5. Pseudoline arrangements of $U_{3,7}$ ................................................................. 25
6. An orientation of $U_{3,4}$ with minimal discrepancy ........................................... 26
7. Three non-isomorphic pseudoline arrangements of $U_{3,6}$ .......................... 30
8. An orientation of $\mathcal{H}$ with $|\delta(F^m)| = 4$ ............................................ 31
9. An orientation of $\mathcal{H}$ with $|\delta(F_3)| = |\delta(F_5)| = 4$ .......................... 32
10. The case when $m \geq 4$ .................................................................................. 35
11. The case when $m = 3$ ................................................................................... 36
12. The method of orientation of odd subarrangements of $\mathcal{H}$ ...................... 39
13. A vertex that gives rise to an odd subarrangement ......................................... 41
14. Orienting one odd subarrangement to balance the discrepancy of another odd subarrangement ................................................................. 46
15. A vertex that does not give rise to an odd subarrangement .......................... 51
16. The nonisomorphic simple arrangements of at most 6 lines ....................... 53
17. The nonisomorphic simple arrangements of 7 lines ....................................... 54
1 Background

1.1 Matroids

We assume a basic familiarity with the concepts of matroid theory. For a detailed treatment of the subject, see Oxley's *Matroid Theory* [6].

A matroid is a pair $\mathcal{M} = (E, \mathcal{S})$, where $E$ is a finite set, called the *ground set* of $\mathcal{M}$, and $\mathcal{S} \subseteq 2^{|E|}$, satisfying any of several equivalent axiom systems. It is the definition of a matroid in terms of its circuits that we will primarily be interested in. This is presented below.

Definition 1.1. A matroid $\mathcal{M}$ is an ordered pair $(E, \mathcal{C})$ consisting of a finite set $E$ and a collection $\mathcal{C}$ of subsets of $E$, called circuits, satisfying the following three conditions:

(C1) $\emptyset \notin \mathcal{C}$.

(C2) If $C_1$ and $C_2$ are members of $\mathcal{C}$ and $C_1 \subseteq C_2$, then $C_1 = C_2$.

(C3) If $C_1$ and $C_2$ are distinct members of $\mathcal{C}$ and $e \in C_1 \cap C_2$, then there is a member $C_3$ of $\mathcal{C}$ such that $C_3 \subseteq (C_1 \cup C_2) \setminus e$.

If a single element $e \in E$ forms a circuit in $\mathcal{M}$, then $e$ is called a *loop*.

An *independent set* of $\mathcal{M}$ is a subset of $E$ that does not contain a circuit. A *basis* is a maximal independent set. If $A \subseteq E$, then the rank of $A$ is $\rho(A) = \{|B \cap A| : \}

1
B is a basis of \( \mathcal{M} \). A hyperplane is a subset of \( E \) of rank \( \rho(\mathcal{M}) - 1 \).

If \( \mathcal{C} \) is the set of circuits of a matroid \( \mathcal{M} \) with ground set \( E \), then \( \mathcal{C}^* = \{ E \setminus C : C \in \mathcal{C} \} \) is the set of bases of a matroid on \( E \). We call this matroid the dual matroid of \( \mathcal{M} \) and denote it by \( \mathcal{M}^* \). The circuits, independent sets, bases, circuits, loops and hyperplanes of \( \mathcal{M}^* \) are called the cocircuits, coinddependent sets, cobases, coloops and cohyperplanes of \( \mathcal{M} \).

### 1.2 Oriented Matroids

The notation and definitions adopted in Sections 1.2 and 1.3 are all taken from Oriented Matroids by Björner et. al.[1], except for the information about pseudoline arrangements, which is from the work of Grünbaum [4, 5].

A signed set \( X \) is a set \( X \) together with a partition \( (X^+, X^-) \) of \( X \) into two subsets \( X^+ \) and \( X^- \) called the positive and negative elements of \( X \). The set \( X = X^+ \cup X^- \) is the support of \( X \). Two signed sets \( X \) and \( Y \) are equal if \( X^+ = Y^+ \) and \( X^- = Y^- \).

The opposite of a signed set \( X \), denoted by \( -X \), is the signed set with \( (-X)^+ = X^- \) and \( (-X)^- = X^+ \). Given a signed set \( X \) and a set \( A \), denote by \( \_A X \) the signed set with \( (\_A X)^+ = (X^+ \setminus A) \cup (X^- \cap A) \) and \( (\_A X)^- = (X^- \setminus A) \cup (X^+ \cap A) \). We say that the signed set \( \_A X \) is obtained from \( X \) by a reorientation on \( A \).
Definition 1.2. An oriented matroid $M$ is an ordered pair $(E, C)$ consisting of a finite set $E$ and a collection $C$ of signed subsets of $E$, called oriented circuits, satisfying the following three conditions:

(C0) $\emptyset \notin C$;

(C1) $X \in C \Rightarrow -X \in C$;

(C2) For all $X, Y \in C$, if $X \subseteq Y$, then $X = Y$ or $X = -Y$;

(C3) For all $X, Y \in C$ such that $X \neq -Y$ and $e \in X^+ \cup Y^-$, there is a $Z \in C$ such that $Z^+ \subseteq (X^+ \cup Y^+) \setminus \{e\}$ and $Z^- \subseteq (X^- \cup Y^-) \setminus \{e\}$.

An orientation of an (unoriented) matroid $M$ is a signing of the ground set that satisfies the conditions (C1)-(C3). A matroid is orientable if it has an orientation. A matroid is regular if it is representable over all fields. Not all matroids are orientable; however, all regular matroids are.

Let $M$ be an oriented matroid on a finite set $E$. Let $C$ be the set of signed circuits of $M$. Let $A$ be a subset of $E$, and let $-A C = \{-A X : X \in C\}$. Then it follows from the axioms that $-A C$ is also the set of circuits of an oriented matroid $-A M$. We say that $-A M$ is obtained from $M$ by a reorientation on $A$. Two oriented matroids $M$ and $M'$ are isomorphic up to reorientation if their underlying matroids $M$ and $M'$ are isomorphic. Sets of all matroids that are isomorphic up to reorientation are called reorientation classes of oriented matroids. An orientable matroid may have several
reorientation classes.

The collection of signed cocircuits of $M$ also satisfies the axioms (C0)-(C3) above and forms the set of circuits of the dual oriented matroid $M^* = (E,C^*)$.

1.3 Topological Representation Theorem

For $d \in \mathbb{Z}, d \geq -1$ let $S^d = \{ x \in \mathbb{R}^{d+1} : ||x|| = 1 \}$ denote the $d$-dimensional standard sphere. A subset $S$ of $S^d$ is called a pseudosphere if $S = h(S^{d-1})$ for some homeomorphism $h : S^d \to S^d$. If $S_e$ is a pseudosphere of $S^d$, choosing one of the two components of $S^d \setminus S_e$ to be the positive side $S_e^+$ yields a signed pseudosphere $\tilde{S}_e$. The negative side $S_e^-$ equals $S^d \setminus (S_e \cup S_e^+)$. We define a pseudosphere arrangement $\mathcal{H} = (S_e)_{e \in E}$ to be a finite set of pseudospheres $S_e$ in $S^d$ such that

(A1) Every non-empty intersection $S_A = \bigcap_{e \in A} S_e$ is homeomorphic to a sphere of some dimension, for $A \subseteq E$; and

(A2) For every non-empty intersection $S_A$ and every $e \in E$ such that $S_A \nsubseteq S_e$, the intersection $S_A \cap S_e$ is a pseudosphere in $S_A$ with sides $S_A \cap S_e^+$ and $S_A \cap S_e^-$. Every point $x \in S^d$ has an associated sign vector $X \in \{+, -, 0\}^E$, where $X_e$ indicates whether $x$ is on the positive side of $S_e$, the negative side of $S_e$, or lies on $S_e$. A signed
arrangement of pseudospheres is a pseudosphere arrangement of signed pseudospheres. If $\mathcal{H}$ is a signed arrangement of pseudospheres, then let $C(\mathcal{H})$ be the family of all sign vectors $X \in \{+,-,0\}^E$ that satisfy the following conditions:

(C1) $\bigcup_{e \in X} S^X_e = S^d$, where $S^X_e$ denotes $S^+_e$ or $S^-_e$, and

(C2) the support $X = \{e \in E : X_e \neq 0\}$ is minimal with property (C1).

The Topological Representation Theorem, proven by Folkman and Lawrence in 1978, describes the correspondence between oriented matroids and pseudosphere arrangements.

**Theorem 1.1 (Topological Representation Theorem).** [1]

1. If $\mathcal{H} = (S_e)_{e \in E}$ is a signed arrangement of pseudospheres in $S^d$, then $C(\mathcal{H})$ is the family of circuits of a rank $d + 1$ simple oriented matroid on $E$.
2. If $(E,C)$ is a rank $d + 1$ simple oriented matroid, then there exists a signed arrangement of pseudospheres $\mathcal{H}$ in $S^d$ such that $C = C(\mathcal{H})$.
3. $C(\mathcal{H}) = C(\mathcal{H}')$ for two signed arrangements $\mathcal{H}$ and $\mathcal{H}'$ in $S^d$ if and only if $\mathcal{H}' = h(\mathcal{H})$ for some self-homeomorphism $h$ of $S^d$.

One powerful consequence of this theorem is that there is a one-to-one correspondence between equivalence classes of arrangements of pseudospheres in $S^d$ and reorientation classes of simple rank $d + 1$ oriented matroids.

When $d = 2$, the pseudosphere arrangements are pseudoline arrangements.
pseudoline arrangement in the real projective plane $\mathbb{P}^2$ is any family of simple closed curves in $\mathbb{P}^2$ such that every two curves have exactly one point in common, at which they cross each other. The arrangement determines a decomposition of $\mathbb{P}^2$ into open topological cells of dimensions 0, 1, and 2, respectively called vertices, line segments and faces of the arrangement. Two arrangements of pseudolines are isomorphic if and only if there exists an incidence-preserving one-to-one correspondence between the vertices, line segments and faces of one arrangement and those of the other. An arrangement of pseudolines is stretchable if it is isomorphic to an arrangement of straight lines. It is known that every pseudoline arrangement of at most 7 lines is stretchable [4]. An arrangement of straight lines in which no point belongs to more than two lines is called a simple arrangement. The different isomorphism types of simple arrangements of 7 lines or less are all known [4, 5], and shown in Figures 16 and 17 in the Appendix. In particular, up to isomorphism there is only one simple arrangement each of 1, 2, 3, 4 and 5 lines; 4 simple arrangements of 6 lines; and 11 simple arrangements of 7 lines. The numbers of isomorphism types for larger numbers of lines are not known. The uniform matroid $U_{3,n}$ is represented by pseudoline arrangements of $n$ lines in which no point belongs to more than two lines. When $n \leq 7$, these arrangements are all isomorphic to the simple arrangements shown in Figures 16 and 17. We will make use of this fact later on.
1.4 Circular Chromatic Number in Graphs

The circular chromatic number of a graph $G$ is a generalization of the chromatic number. Introduced by Vince in 1988 [7], it is denoted by $\chi^*(G)$, and defined as follows.

**Definition 1.3.** For two integers $1 \leq d \leq k$, a $(k,d)$-coloring of a graph $G$ is a coloring $c$ of the vertices of $G$ with colors $\{0, 1, 2, \ldots, k-1\}$ such that $(x, y) \in E(G) \Rightarrow d \leq |c(x) - c(y)| \leq k-d$. The circular chromatic number of $G$, $\chi^*(G)$, is the infimum of those $k/d$ for which there exists a $(k, d)$-coloring of $G$.

Note that a $(k, 1)$-coloring of $G$ is just an ordinary $k$-coloring. Regarding the relation of $\chi^*(G)$ to the chromatic number $\chi(G)$, it can be shown that $\chi(G) - 1 < \chi^*(G) \leq \chi(G)$.

Let $k$ be a positive integer. A $k$-flow in a graph $G$ is an orientation $\omega(G)$ together with a function $f : E(G) \rightarrow \{0, \pm 1, \pm 2, \ldots, \pm (k - 1)\}$ such that the net flow $\sum_{vu \in E(G)} f(vu) - \sum_{uv \in E(G)} f(uv) = 0$ for each $v \in V(G)$. The flow index $\xi(G)$ is the smallest $k$ for which $G$ has a nowhere-zero $k$-flow, that is, a $k$-flow with $f(e) \neq 0$ for all $e \in E(G)$.

Goddyn et. al. [3] give an equivalent definition of the circular chromatic number by relating it to nowhere-zero flows in graphs. They define a $(k, d)$-flow and the star
flow index of a graph $G$ as follows.

**Definition 1.4.** A $(k,d)$-flow in a graph $G$ is a $k$-flow $(\omega(G), f)$ such that $d \leq |f(e)| \leq (k-d)$ for all $e \in E(G)$. The star flow index $\xi^*(G)$ is the infimum of those $k/d$ for which there exists a $(k,d)$-flow in $G$.

Vertex colorings and nowhere-zero flows of graphs are dual concepts. If $G$ is a plane graph and $G^d$ is its planar dual, then $\chi(G) = \xi(G^d)$.

### 1.5 Oriented Flow in Matroids

Let $M = (E, C^*)$ be a regular matroid. An integer flow in $M$ is an orientation of $M$ together with a function $f : E \to \mathbb{Z}$ such that, for every cocircuit $B \in C^*$, $\sum_{e \in B^+} f(e) = \sum_{e \in B^-} f(e)$. A flow $f$ is nowhere-zero if $f(e) \neq 0$ for all $e \in E$. For integers $0 < d < k$, a $(k,d)$-flow is an integer flow with values in the set $\{ \pm d, \pm (d+1), \ldots, \pm (k-d) \}$, and a nowhere-zero $k$-flow is a $(k,1)$-flow. The **star flow index** $\xi^*(M)$ is the infimum of $k/d$ over all $(k,d)$-flows in $M$, and the **flow index** $\xi(M)$ is the minimum $k$ for which $M$ has a nowhere-zero $k$-flow. If $M$ has no coloops, then it is known that $M$ has a nowhere-zero $k$-flow for some $k$.

Goddyn et. al. proved the following result [3].

**Theorem 1.2.** A regular matroid $M$ has a $(k,d)$-flow if and only if there exists an orientation of $M$ such that, for any cocircuit $B$, $\frac{d}{k-d} \leq \frac{|B^+|}{|B^-|} \leq \frac{k-d}{d}$.
This implies the following, which gives an equivalent definition of the star flow index of a regular matroid.

**Corollary 1.1.** Let $M$ be a regular matroid, with $C^*$ the set of cocircuits. The star flow index $\xi^*(M)$ is the minimum over all orientations of $M$ of

$$\max_{B \in C^*} \text{imbal}(B),$$

where the imbalance of $B$ is defined as

$$\text{imbal}(B) = \frac{|B|}{\min\{|B^+|,|B^-|\}}.$$

Noting that a $(k,d)$-coloring of a graph $G$ induces a nowhere-zero $k$-flow in its cocographic matroid $M^*(G)$, Goddyn et. al. [3] showed that $\chi^*(G) = \xi^*(M^*(G))$, so that

$$\chi^*(G) = \min_{\omega(G)} \max_{C \in C} \left\{ \frac{|C|}{|C^+|, |C^-|} \right\},$$

where the minimum is over all orientations $\omega(G)$ of $M(G)$.

Goddyn et. al. [2] generalized this definition of $\chi^*$ to oriented matroids, defining the oriented flow number of an oriented matroid $M = (E,C^*)$ to be

$$\phi_o(M) = \min_{O} \max_{B \in C^*} \text{imbal}(B),$$

where the minimum ranges over the set of reorientations $O$ of $M$. Since reorientation classes of $M$ correspond to equivalence classes of pseudosphere arrangements of $M$, we have an equivalent definition which will be more suited to our purposes.
Definition 1.5. Let $\mathcal{H} = (H_e)_{e \in E}$ be an (unsigned) arrangement of pseudospheres with underlying matroid $\mathcal{M}$ such that $\mathcal{M}$ does not have a coloop. Let $\mathcal{C}^*$ be the set of unsigned cocircuits. Then we define the oriented flow number $\phi_o$ of $\mathcal{H}$ to be

$$\phi_o(\mathcal{H}) = \min_{\mathcal{H}} \max_{B \in \mathcal{C}^*} \text{imbal}(B),$$

where the minimum is taken over all signings of $\mathcal{H}$.

We also define the oriented flow number of an orientable matroid $\mathcal{M}$ as the minimum of $\phi_o(\mathcal{H})$ over all pseudosphere arrangements $\mathcal{H}$ of $\mathcal{M}$:

$$\phi_o(\mathcal{M}) = \min_{\mathcal{H}} \phi_o(\mathcal{H}).$$

Using probabilistic methods, Goddyn et. al. [2] proved the following bounds on $\phi_o(\mathcal{H})$.

Theorem 1.3. If $\mathcal{H}$ is a pseudoline arrangement whose underlying matroid $\mathcal{M} = (E, \mathcal{C}^*)$ is coloop-free and of rank 3, then $\phi_o(\mathcal{H}) \leq 17$, and furthermore, $|E| \geq 159 \Rightarrow \phi_o(\mathcal{H}) \leq 4$, and $|E| \geq 427 \Rightarrow \phi_o(\mathcal{H}) \leq 3$.

Theorem 1.4. If $\mathcal{H}$ is a pseudoline arrangement whose underlying matroid $\mathcal{M} = (E, \mathcal{C}^*)$ is coloop-free and of rank $r \geq 4$, then $\phi_o(\mathcal{H}) \leq 14r^2 \ln r$.

We explore the rank 3 case in depth, and show that $\phi_o(\mathcal{H}) \leq 4$ no matter the size of $E$. 

10
2 Bounding the Oriented Flow Number of $U_{3,n}$

Let $M$ be an oriented matroid on a finite set $E$, and let $\mathcal{H}$ be a projective pseudoline arrangement of $M$. Let $C^*$ be the set of signed cocircuits of $M$. Let $F$ be the set of faces of $\mathcal{H}$, and let $V$ be the set of vertices of $\mathcal{H}$. If $M$ has rank 3, each element of $M$ is represented by a pseudoline in $\mathcal{H}$. We will refer to pseudolines simply as lines, with the understanding that the lines may be curved. Each signed cocircuit (or bond) $B$ of $M$ is represented by a vertex $v_B$ in $\mathcal{H}$ in the following way: the elements of $B$ are precisely those represented by lines not passing through $v_B$. We define the degree of $v_B$, denoted $\deg(v_B)$, as the number of lines passing through $v_B$. Then $|B| = |E| - \deg(v_B)$. Two vertices are adjacent if they lie on the same line $l$, and no other vertex on $l$ lies between them. An (open) line segment $s$ of $l$ is the portion of $l$ lying between two adjacent vertices. Let $S$ be the set of line segments of $\mathcal{H}$. Two line segments are adjacent if their closures intersect at a vertex. Every line $l$ is made up of the union of line segments and the vertices that separate them. Let $p$ be any point in the plane. Define the discrepancy of $p$ in $\mathcal{H}$, denoted by $\delta_{\mathcal{H}}(p)$, to be the sum of the orientations of the elements of $M$ with respect to $p$ in $\mathcal{H}$. Each line $l$ contributes either 0, 1 or $-1$ to this sum, contributing 0 if and only if $p$ lies on the line represented by $l$. If so, we say that $l$ contains $p$. We call $|\delta_{\mathcal{H}}(p)|$ the absolute discrepancy of $p$ in $\mathcal{H}$. Note that $|\delta_{\mathcal{H}}(p)| \leq |E|$. If $F$ is any open face in $\mathcal{H}$, then
define the discrepancy of $F$ in $\mathcal{H}$, denoted by $\delta_{\mathcal{H}}(F')$, to be equal to $\delta_{\mathcal{H}}(p)$ for any $p \in F$. Two faces are adjacent if their closures intersect at a line segment. We call a face $F$ an $n$-face if $F$ is bounded by $n$ line segments. For any two adjacent faces $F_1$ and $F_2$, $\delta_{\mathcal{H}}(F_2) = \delta_{\mathcal{H}}(F_1) \pm 2$ since $F_1$ and $F_2$ lie on the same side of all lines except the one that divides them. If $s$ is a line segment separating adjacent faces $F_1$ and $F_2$, then define the discrepancy of $s$ in $\mathcal{H}$, denoted by $\delta_{\mathcal{H}}(s)$, to be equal to $\delta_{\mathcal{H}}(p)$ for any $p$ lying on $s$, and note that

$$\delta_{\mathcal{H}}(s) = \frac{\delta_{\mathcal{H}}(F_1) + \delta_{\mathcal{H}}(F_2)}{2}.$$  

Let $v_B$ be a vertex in $\mathcal{H}$ corresponding to a signed cocircuit $B$ of $M$. Define the discrepancy of $B$ in $\mathcal{H}$, denoted by $\delta_{\mathcal{H}}(B)$, to be equal to $\delta_{\mathcal{H}}(v_B)$. It follows from the definitions that $\delta_{\mathcal{H}}(B) = |B^+| - |B^-|$. When $\mathcal{H}$ is evident from the context it will be dropped from the notation and we will use the notation $\delta(p)$, $\delta(F')$, $\delta(s)$ and $\delta(B)$.

Since $|B| = |B^+| + |B^-|,

$$|\delta(B)| = ||B^+| - |B^-|| = |B| - 2 \min\{|B^+|, |B^-|\}, \quad (2)$$

so that (1) becomes

$$\phi_o(\mathcal{H}) = \min_{\mathcal{H}} \max_{B \in \mathcal{C}^*} \frac{2|B|}{|B| - |\delta(B)|}. \quad (3)$$

We will show that $\phi_o(\mathcal{H}) \leq 4$ for all pseudoline arrangements $\mathcal{H}$ whose underlying matroid $M$ is coloop-free and has rank 3. The requirement that $\phi_o(\mathcal{H}) \leq 4$ is equiva-
lent to the requirement that $|\delta(B)| \leq \frac{|B|}{2}$ for all $B \in C^*$, since, given some signing of $\mathcal{H}$ that minimizes (3), we have:

$$\phi_0(\mathcal{H}) \leq 4$$

$$\iff \frac{2|B|}{|B| - |\delta(B)|} \leq 4 \text{ for all } B \in C^*$$

$$\iff 2|B| \leq 4|B| - 4|\delta(B)| \text{ for all } B \in C^*$$

$$\iff |\delta(B)| \leq \frac{|B|}{2} \text{ for all } B \in C^*.\quad (4)$$

We first examine the case where $M$ is the uniform rank 3 matroid $U_{3,n}$, with $n \geq 4$ (when $n \leq 3$, every element in $U_{3,n}$ is a coloop and the ratio is undefined). Let $\mathcal{H}$ be a
pseudoline arrangement of $U_{3,n}$. Let $C^*$ be the set of cocircuits and $\mathcal{F}$ the set of faces of $\mathcal{H}$. Let $B \in C^*$ and let $v$ be the vertex in $\mathcal{H}$ corresponding to $B$. Then $v$ is the intersection of exactly 2 lines $l_1$ and $l_2$, and $v$ is incident with four faces $F_1,\ldots,F_4$.

The faces $F_1,\ldots,F_4$ and the vertex $v$ lie on the same side of all lines except $l_1$ and $l_2$, so $\delta(F_1),\ldots,\delta(F_4)$ and $\delta(B)$ differ from each other only on account of the orientations of $l_1$ and $l_2$. Ordering $F_1,\ldots,F_4$ so that $\delta(F_1) \leq \delta(F_2) \leq \delta(F_3) \leq \delta(F_4)$, we must have $\delta(F_2) = \delta(F_3) = \delta(F_1) + 2$ and $\delta(F_4) = \delta(F_1) + 4$ since, as noted above, $\delta(F)$ differs by 2 for adjacent faces (see Figure 1). Now, $l_1$ and $l_2$ contribute 0 to $\delta(B)$, and they contribute 0 to $\delta(F_2)$ and $\delta(F_3)$ since $F_2$ and $F_3$ both lie on the positive side of one of $l_1, l_2$, and the negative side of the other. So $\delta(B) = \delta(F_2) = \delta(F_3)$, and furthermore, $\delta(B)$ is the average of $\delta(F_1),\ldots,\delta(F_4)$, since

$$\sum_{i=1}^{4} \frac{\delta(F_i)}{4} = \frac{\delta(F_1) + 2(\delta(F_1) + 2) + (\delta(F_1) + 4)}{4}$$

$$= \frac{4\delta(F_1) + 8}{4}$$

$$= \delta(F_1) + 2$$

$$= \delta(B).$$

Noting also that $|\delta(B)| = \max\{|\delta(F_1)|, |\delta(F_4)|\} - 2$, we have

$$\max_{B \in C^*} |\delta(B)| = \max_{F \in \mathcal{F}} |\delta(F)| - 2.$$  \hfill (6)

Since each cocircuit $B$ corresponds to a vertex $v_B$ in $\mathcal{H}$, and $\delta(B) = \delta(v_B)$, we also
have

$$\max_{v \in V} | \delta(v) | = \max_{F \in \mathcal{F}} | \delta(F) | - 2. \quad (7)$$

Let $F_1$ be a face with maximum absolute discrepancy and $F_2$ be any face adjacent to $F_1$. Let $s$ be the line segment separating $F_1$ and $F_2$. Then, since $| \delta(F_2) | = | \delta(F_1) | - 2,$

$$| \delta(s) | = \frac{| \delta(F_1) + \delta(F_2) |}{2} \leq \left| \frac{\delta(F_1)}{2} \right| + \left| \frac{\delta(F_2)}{2} \right|$$

$$= \left| \frac{\delta(F_1)}{2} \right| + \left( \left| \frac{\delta(F_1)}{2} \right| - 1 \right)$$

$$= | \delta(F_1) | - 1.$$

Since $| \delta(s) |$ depends directly on $| \delta(F_1) |$, and $F_1$ is a face with maximum absolute discrepancy, we have

$$\max_{s \in \mathcal{S}} | \delta(s) | \leq \max_{F \in \mathcal{F}} | \delta(F) | - 1. \quad (8)$$

Note that (6) holds only when $\mathcal{H}$ is a pseudoline arrangement of $U_{3,n}$, while (8) holds for all pseudoline arrangements of rank 3 matroids.

Before proceeding further, we will need a few lemmas.

**Lemma 2.1.** Let $\mathcal{H}$ be a pseudoline arrangement of $U_{3,n}$, with $n \geq 4$. Let $\mathcal{F}$ be the collection of faces and $\mathcal{C}^*$ the set of cocircuits of $\mathcal{H}$. If $\mathcal{F}$ contains an $n$-face, then
$U_{3,n}$ may be oriented so that

$$|\delta(F)| \leq \begin{cases} 2 & \text{if } n \text{ is even} \\ 3 & \text{if } n \text{ is odd} \end{cases}$$

for all faces $F \in \mathcal{F}$; furthermore,

$$|\delta(B)| = \begin{cases} 0 & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd} \end{cases}$$

for all cocircuits $B \in C^*$. 

Proof: Let $\mathcal{H}$ be a pseudoline arrangement of $U_{3,n}$ that contains an $n$-face $F_n$. Let $\mathcal{C}^*$ be the set of unsigned cocircuits of $U_{3,n}$. Without loss of generality, pick a line bounding $F_n$ and orient it outwards. Proceed clockwise around the boundary of $F_n$, alternately orienting lines inwards and outwards with respect to $F_n$. Since $F_n$ is an $n$-face, this completes the orientation of $\mathcal{H}$. Now, if $n$ is even, $|\delta(F_n)| = 0$, and if $n$ is odd, $|\delta(F_n)| = 1$. Let $B$ be a cocircuit in $\mathcal{H}$, and let $v_B$ be the vertex corresponding to $B$. Suppose $v_B$ lies on the boundary of $F_n$, and $n$ is even. Then $v_B$ is the intersection of two lines, one oriented inwards with respect to $F_n$ and one oriented outwards. So the four faces incident with $B$ have $\delta$-values $-2$, $0$, $0$ and $2$, and $|\delta(B)| = 0$. Now suppose $v_B$ lies on the boundary of $F_n$, and $n$ is odd. Then $\mathcal{H}$ has one more line oriented outwards with respect to $F_n$ than inwards, and $\delta(F_n) = -1$. If $v_B$ is the
Figures 2: Here $n = 9$ and $k = 5$. The sign vector of $v_B$ is given by $\bar{v}_B = (0, +, -, +, 0, -, +, -, +, -)$, and $\delta(B) = 1$.

intersection of an outward-oriented line and an inward-oriented line, then the four faces incident with $v_B$ have $\delta$-values $-3, -1, -1$ and $1$, and $|\delta(B)| = 1$. If $v_B$ is the intersection of two outward-oriented lines, then the four faces incident with $v_B$ have $\delta$-values $-1, 1, 1$ and $3$, and $|\delta(B)| = 1$.

Suppose $v_B$ does not lie on the boundary of $F_n$. Then $v_B$ is still the intersection of two lines that lie on the boundary of $F_n$, since the boundary of $F_n$ includes all lines in $\mathcal{H}$ (see Figure 2). Label the lines $l_1, \ldots, l_n$ of $\mathcal{H}$ so that $v_B$ is the intersection of $l_1$ and $l_k$ (here $k \neq 2$ and $k \neq n$). Without loss of generality, assume $l_1$ is
oriented outwards with respect to $F_n$. The lines $l_1$ and $l_k$ divide $\mathbb{P}^2$ into two projective half-planes. $F_n$ lies entirely within one of these half-planes. The remaining lines of $\mathcal{H}$ are partitioned into $\{l_2, \ldots, l_{k-1}\}$ and $\{l_{k+1}, \ldots, l_n\}$. One of the sign vectors of $\{l_2, \ldots, l_{k-1}\}$ or $\{l_{k+1}, \ldots, l_n\}$ agrees with its sign vector relative to $F_n$, and the other is its negative. Without loss of generality, assume $\{l_{k+1}, \ldots, l_n\}$ is oriented the same direction with respect to $v_B$ as it is with respect to $F_n$. Let $\bar{v}_B$ be the sign vector of $B$. If $n$ is even, then $\bar{v}_B = (0, -, +, \ldots, \pm, 0, \mp, \ldots, -, +)$. Now $l_k$ may be oriented outwards or inwards with respect to $F_n$, but in either case, $|\delta(B)| = 0$. If $n$ is odd, assume that both $l_1$ and $l_2$ are oriented outwards with respect to $F_n$. Then $\bar{v}_B = (0, -, +, \ldots, \pm, 0, \mp, \ldots, +, -, \mp, \ldots, +, -)$. Again, $l_k$ may be oriented outwards or inwards with respect to $F_n$, but in either case, $|\delta(B)| = 1$. So, considering all possible cases, we have

$$|\delta(B)| = \begin{cases} 
0 & \text{if } n \text{ is even} \\
1 & \text{if } n \text{ is odd} 
\end{cases} \quad (9)$$

for all $B \in C^*$, implying by (6) that

$$|\delta(F)| \leq \begin{cases} 
2 & \text{if } n \text{ is even} \\
3 & \text{if } n \text{ is odd} 
\end{cases}$$

for all $F \in \mathcal{F}$. 

\[ \square \]
The next corollary follows easily.

**Corollary 2.1.** If $n \geq 4$, then

$$
\phi_o(U_{3,n}) = \begin{cases} 
2 & \text{if } n \text{ is even} \\
\frac{2n-4}{n-3} & \text{if } n \text{ is odd.}
\end{cases}
$$

**Proof:** Let $\mathcal{H}$ be a pseudoline arrangement of $U_{3,n}$. Let $\mathcal{C}^*$ be the set of unsigned cocircuits of $U_{3,n}$, and let $B \in \mathcal{C}^*$. First note that if $n$ is even, the smallest $|\delta(B)|$ can be is 0, and if $n$ is odd, the smallest $|\delta(B)|$ can be is 1. So,

$$
\phi_o(\mathcal{H}) = \min_{\mathcal{H}} \max_{B \in \mathcal{C}^*} \frac{2|B|}{|B| - |\delta(B)|} \geq \begin{cases} 
2 & \text{if } n \text{ is even} \\
\frac{2|B|}{|B|-1} & \text{if } n \text{ is odd.}
\end{cases}
$$

Since $|B| = n - 2$ for all $B \in \mathcal{C}^*$,

$$
\phi_o(\mathcal{H}) \geq \begin{cases} 
2 & \text{if } n \text{ is even} \\
\frac{2n-4}{n-3} & \text{if } n \text{ is odd.}
\end{cases}
$$

For each $n \geq 4$, there is a pseudoline arrangement $\mathcal{H}'$ of $U_{3,n}$ with an $n$-face. This may be constructed by arranging the $n$ lines of $U_{3,n}$ so they all border a single face. By Lemma 2.1, $\mathcal{H}'$ may be oriented so that the lower bounds of 0 and 1 for $|\delta(B)|$ are attained for all $B \in \mathcal{C}^*$. So

$$
\phi_o(\mathcal{H}') \leq \begin{cases} 
2 & \text{if } n \text{ is even} \\
\frac{2n-4}{n-3} & \text{if } n \text{ is odd}
\end{cases}
$$

19
and it follows that
\[ \phi_o(U_{3,n}) = \min_{\mathcal{H}} \phi_o(\mathcal{H}) = \phi_o(\mathcal{H}') \]

We now develop notation for a partition of pseudolines. Let \( M \) be a coloop-free orientable matroid of rank 3. Let \( \mathcal{H} \) be a pseudoline arrangement of \( M \) with orientation \( \mathcal{O} \). Let \( C^* \) be the set of signed cocircuits of \( \mathcal{H} \). Let \( \Pi = (\Pi_1, \ldots, \Pi_k) \) be a partition of the lines \( l_1, \ldots, l_n \) of \( \mathcal{H} \). Each \( \Pi_i \) gives rise to a subarrangement \( \mathcal{H}_i \) of \( \mathcal{H} \). For each \( i \), let \( \mathcal{F}_i \) be the collection of faces of \( \mathcal{H}_i \), let \( \mathcal{S}_i \) be the collection of line segments of \( \mathcal{H}_i \), let \( \mathcal{V}_i \) be the collection of vertices of \( \mathcal{H}_i \), and let \( \mathcal{O}_i \) be the orientation of \( \mathcal{H} \) restricted to \( \Pi_i \). Now, each face \( F \in \mathcal{F} \) is contained in some face \( F_i \in \mathcal{F}_i \) for each \( i \). Define \( \delta_{\mathcal{H}_i}(F) \) to be equal to \( \delta_{\mathcal{H}_i}(F_i) \). Let \( (F_1, \ldots, F_k) \) be the faces of the subarrangements \( \mathcal{H}_i \) that contain \( F \), with each \( F_i \in \mathcal{F}_i \). Then \( F = F_1 \cap \cdots \cap F_k \) and
\[ \delta_{\mathcal{H}}(F) = \sum_{i=1}^k \delta_{\mathcal{H}_i}(F_i), \]
implying
\[ |\delta_{\mathcal{H}}(F)| = \left| \sum_{i=1}^k \delta_{\mathcal{H}_i}(F_i) \right| \leq \sum_{i=1}^k |\delta_{\mathcal{H}_i}(F)|. \quad (10) \]
If \( s \) is a line segment in \( \mathcal{H} \), then for each \( \mathcal{H}_i \), \( s \) either lies in a face \( F_i \in \mathcal{F}_i \), or lies on a line segment \( s_i \in \mathcal{S}_i \). We define
\[ \delta_{\mathcal{H}_i}(s) = \begin{cases} \delta_{\mathcal{H}_i}(F_i) & \text{if } v_B \text{ lies in a face } F_i \in \mathcal{F}_i \\ \delta_{\mathcal{H}_i}(s_i) & \text{if } v_B \text{ lies on a line segment } s_i \in \mathcal{S}_i. \end{cases} \]
Then $\delta_\mathcal{H}(s) = \sum_{i=1}^{k} \delta_{\mathcal{H}_i}(s)$, and

$$|\delta_\mathcal{H}(s)| = \left| \sum_{i=1}^{k} \delta_{\mathcal{H}_i}(s) \right| \leq \sum_{i=1}^{k} |\delta_{\mathcal{H}_i}(s)|.$$ (11)

If $B \in C^*$, then for each $\mathcal{H}_i$, the vertex $v_B$ corresponding to $B$ either lies in a face $F_i \in \mathcal{F}_i$, is a vertex $v_i \in \mathcal{V}_i$, or lies on a line segment $s_i \in \mathcal{S}_i$. So, we define

$$\delta_{\mathcal{H}_i}(B) = \delta_{\mathcal{H}_i}(v_B) = \begin{cases} 
\delta_{\mathcal{H}_i}(F_i) & \text{if } v_B \text{ lies in a face } F_i \in \mathcal{F}_i \\
\delta_{\mathcal{H}_i}(s_i) & \text{if } v_B \text{ lies on a line segment } s_i \in \mathcal{H}_i \\
\delta_{\mathcal{H}_i}(v_i) & \text{if } v_B \text{ is a vertex } v_i \in \mathcal{H}_i.
\end{cases}$$

Then $\delta_\mathcal{H}(B) = \sum_{i=1}^{k} \delta_{\mathcal{H}_i}(B)$. So, we also have

$$|\delta_\mathcal{H}(B)| = \left| \sum_{i=1}^{k} \delta_{\mathcal{H}_i}(B) \right| \leq \sum_{i=1}^{k} |\delta_{\mathcal{H}_i}(B)|.$$ (12)

We now use Lemma 2.1 to prove a bound on $|\delta(F)|$ in the particular case when $\mathcal{H}$ is a pseudoline arrangement of $U_{3,7}$. This will be used in the proof of Theorem 2.1.

**Lemma 2.2.** Let $\mathcal{H}$ be a pseudoline arrangement of $U_{3,7}$, and let $\mathcal{F}$ be the collection of faces of $\mathcal{H}$. Then $\mathcal{H}$ may be oriented so that $|\delta(F)| \leq 3$ for all faces $F \in \mathcal{F}$.

**Proof:** There are only 11 nonisomorphic pseudoline arrangements of $U_{3,7}$ [4]. These are shown in Figure 17. Figure 17a contains a 7-face, Figures 17b-e contain a 6-face, and Figures 17f-k contain neither a 6-face nor a 7-face. By Lemma 2.1, the
arrangement containing a 7-face may be oriented so that it contributes at most 3 to $|\delta(F)|$ for all $F \in \mathcal{F}$. The arrangements that contain a 6-face can be partitioned into two pieces: the 6 lines bounding the 6-face and the one leftover line. By Lemma 2.1, the 6-line arrangement can be oriented so that it contributes at most 2 to $|\delta(F)|$ for all $F \in \mathcal{F}$, and the remaining line can be oriented arbitrarily, contributing 1 to $|\delta(F)|$ for all $F \in \mathcal{F}$. So, these arrangements can be oriented so that they contribute at most 3 to $|\delta(F)|$ for all $F \in \mathcal{F}$. The other 6 arrangements of $U_{3,7}$ are shown in Figures 3, 4 and 5. They have all been oriented so that $|\delta(F)| \leq 3$ for all $F \in \mathcal{F}$.

Consider $U_{3,4}$. Let $\mathcal{H}$ be the unique pseudoline arrangement of $U_{3,4}$ (up to isomorphism). It contains three 4-faces and four 3-faces and is shown in Figure 6. It is symmetric in the sense that each of the 4-faces is adjacent to the four 3-faces. Choose one of the 4-faces and orient the 4 lines bordering the face alternately with respect to this face. By Lemma 2.1, if the lines are oriented in this way, $|\delta(F)| \leq 2$ for all faces $F$ and $|\delta(B)| = 0$ for all cocircuits $B$. By Corollary 2.1, this orientation minimizes $U_{3,4}$.

Our strategy for finding an upper bound for $\phi_o(U_{3,n})$, with $n > 4$, relies on the fact that there is only one pseudoline arrangement for $U_{3,4}$, and we know what the optimal way to orient it is. Let $\mathcal{H}$ be a pseudoline arrangement with underlying matroid $U_{3,n}$.
Figure 3: An orientation of two of the pseudoline arrangements of $U_{3,7}$ satisfying $|\delta(F)| \leq 3$ for all $F \in \mathcal{F}$. These correspond to Figures 17f and 17g. The values of $\delta(F)$ for each face are shown.
Figure 4: An orientation of two of the pseudoline arrangements of $U_{3,7}$, corresponding to Figures 17h and 17i, satisfying $|\delta(F)| \leq 3$ for all $F \in \mathcal{F}$. 

24
Figure 5: An orientation of two of the pseudoline arrangements of $U_{3,7}$, corresponding to Figures 17j and 17k, satisfying $|\delta(F)| \leq 3$ for all $F \in \mathcal{F}$. 

25
Partition the lines of $\mathcal{H}$ into sets of four lines and orient them as described above, until there are either 0, 1, 2 or 3 lines remaining. Now, every face $F$ in $\mathcal{H}$ lies in a face of each arrangement of the 4-line sets we have chosen. We can think of $\mathcal{H}$ as some number of 4-line arrangements laid on top of each other, plus up to 3 leftover lines. So $\delta(F)$ is the sum of $\delta(F_i)$ for each face $F_i$ that contains $F$ and lies in our choices of 4-line arrangements (ignoring, for the moment, the leftover lines). The manner in which we partition the lines is irrelevant since every subset of 4 lines in $\mathcal{H}$ is isomorphic to the pseudoline arrangement of $U_{3,4}$ shown in Figure 6. From above, $|\delta(F_i)| \leq 2$ for all $F_i \in \mathcal{F}(U_{3,4})$. Using (10), this choice of orientation will be sufficient to show $\phi_o(\mathcal{H}) \leq 4$ for all pseudoline arrangements with underlying matroid $U_{3,n}$.
We are now ready to prove the main result of this section.

**Theorem 2.1.** Let $\mathcal{H}$ be a pseudoline arrangement with underlying matroid $U_{3,n}$. Then $\phi_o(\mathcal{H}) \leq 4$ for all $n \geq 4$, with equality possible only if $n \equiv 2 \pmod{4}$.

*Proof:* Let $\mathcal{F}$ be the collection of faces of $\mathcal{H}$ and $\mathcal{C}^*$ be the set of unsigned cocircuits of $U_{3,n}$. Note, if we can produce an orientation of $\mathcal{F}$ such that

$$\max_{F \in \mathcal{F}} |\delta(F)| \leq \frac{|B|+4}{2} \text{ for all } B \in \mathcal{C}^*, \quad (13)$$

then (6) implies

$$|\delta(B)| \leq \frac{|B|+4}{2} - 2 = \frac{|B|}{2} \quad \text{for all } B \in \mathcal{C}^*. $$

Thus, we have by (4),

$$\max_{F \in \mathcal{F}} |\delta(F)| \leq \frac{|B|+4}{2} \Rightarrow \phi_o(\mathcal{H}) \leq 4, \quad (14)$$

with $\phi_o(\mathcal{H}) = 4$ only if

$$\max_{F \in \mathcal{F}} |\delta(F)| = \frac{|B|+4}{2} \quad \text{for some } B \in \mathcal{C}^*. $$

To produce an orientation of $\mathcal{H}$ satisfying (13), we will partition and orient the lines of $\mathcal{H}$ as described in our discussion of strategy above. Four-line sets will be oriented as shown in Figure 6 until we are left with 0, 1, 2 and 3 leftover lines. This gives us
four cases to consider: \( n \equiv k \pmod{4} \), with \( k = 0, 1, 2 \) or 3. We show that in each case, the resulting orientation satisfies (13). Note that \( n = |B| + 2 \).

Case 1: \( n \equiv i \pmod{4} \), with \( i \in \{0, 1, 2\} \).

Partition the \( n \) lines of \( \mathcal{H} \) into sets \( \Pi_1, \ldots, \Pi_k, \Pi_{k+1} \), with \( n = 4k + i \) and \( |\Pi_j| = 4 \) for \( j \in \{1, \ldots, k\} \); and \( |\Pi_{k+1}| = i \). Each \( \Pi_i \) (except \( \Pi_{k+1} \)) gives rise to a subarrangement \( \mathcal{H}_i \) whose matroid is isomorphic to \( U_{3,4} \). These subarrangements may all be oriented so that \( |\delta_{\mathcal{H}_i}(F_i)| \leq 2 \) for all \( F_i \in \mathcal{F} \). The \( i \) leftover lines in \( \Pi_{k+1} \) may be oriented arbitrarily, contributing at most \( i \) to \( |\delta(F)| \) for all \( F \in \mathcal{F} \). So

\[
\max_{F \in \mathcal{F}} |\delta(F)| \leq \sum_{i=1}^{k+1} |\delta_{\mathcal{H}_i}(F_i)| \leq 2 \left( \frac{n-i}{4} \right) + i = \frac{n+i}{2} = \frac{|B|+2+i}{2}
\]

\[\Rightarrow \phi_\circ(\mathcal{H}) \leq 4 \text{ by (14)},\]

with equality only if \( i = 2 \).

Case 2: \( n \equiv 3 \pmod{4} \).

This case is not so straightforward. If the strategy followed in the preceding cases is used and the three leftover lines are oriented arbitrarily, one obtains an upper bound of \( \phi_\circ(U_{3,n}) \leq 5 \). So, instead of orienting 4-line sets until only 3 lines remain, we orient 4-line sets until 7 lines remain, and then consider the various pseudoline arrangements of \( U_{3,7} \). Formally, we partition the \( n \) lines of \( \mathcal{H} \) into sets \( \Pi_1, \ldots, \Pi_k \), with

28
\( n = 4k + 3, |\Pi_j| = 4 \) for \( j \in \{1, \ldots, k - 1\} \) and \( |\Pi_k| = 7 \). Each \( \Pi_i \) (except \( \Pi_k \)) gives rise to a subarrangement \( \mathcal{H}_i \) whose matroid is isomorphic to \( U_{3,4} \). As in Case 1, these subarrangements may all be oriented so that \( |\delta_{\mathcal{H}_i}(F_i)| \leq 2 \) for all \( F_i \in \mathcal{F}_i \). \( \Pi_k \) gives rise to a subarrangement \( \mathcal{H}_k \) whose matroid is isomorphic to \( U_{3,7} \). By Lemma 2.2, \( \mathcal{H}_k \) may be oriented so that \( |\delta_{\mathcal{H}_k}(F_k)| \leq 3 \) for all faces \( F_k \in \mathcal{F}_k \). So we have,

\[
\max_{F \in \mathcal{F}} |\delta(F)| \leq \sum_{i=1}^{k} |\delta_{\mathcal{H}_i}(F_i)| \leq 2 \left( \frac{n-7}{4} \right) + 3 = \frac{n-1}{2} = \frac{|B| + 1}{2}
\]

\[ \Rightarrow \phi_o(\mathcal{H}) < 4 \text{ by (14).} \]

In all cases, \( \phi_o(\mathcal{H}) \leq 4 \). Note also that equality is possible only if \( n \equiv 2 \) (mod 4). In Theorem 2.2 we will show that equality can be attained.

The next corollary actually follows from the proof of Theorem 2.1. We state it here because we will need it in the proof of Theorem 3.1.

**Corollary 2.2.** Let \( \mathcal{H} \) be a pseudoline arrangement with underlying matroid \( U_{3,n} \).

Then \( \mathcal{H} \) may be oriented so that

\[
|\delta(F)| \leq \frac{n+2}{2}
\]

for all \( F \in \mathcal{F} \).
Figure 7: Three non-isomorphic pseudoline arrangements of $U_{3,6}$.

Proof: In Case 1 of the proof of Theorem 2.1, we showed $H$ can be oriented so that $\max_{F \in \mathcal{F}} |\delta(F)| \leq \frac{n+i}{2}$, with $0 \leq i \leq 2$. In Case 2 we demonstrated an orientation with $\max_{F \in \mathcal{F}} |\delta(F)| \leq \frac{n-1}{2}$. Thus, $H$ can be oriented so that $|\delta(F)| \leq \frac{n+2}{2}$ for all $F \in \mathcal{F}$.

Next we will show that the upper bound $\phi_o(H) = 4$ can be attained for three of the four pseudoline arrangements of $U_{3,6}$. These are shown in Figure 7.

**Theorem 2.2.** Let $H$ be one of the pseudoline arrangements of $U_{3,6}$ shown in Figure 7. Then $\phi_o(H) = 4$.

Proof: Let $H$ be one of the pseudoline arrangements of $U_{3,6}$ shown in Figure 7. We will show that no matter the orientation, $|\delta(F)| \geq 4$ for some $F \in \mathcal{F}$. Then by (6)
Figure 8: An orientation of $\mathcal{H}$ with $|\delta(F'')| = 4$.

there exists $B \in C^*$ such that $|\delta(B)| \geq 2$, so that

$$\phi_0(\mathcal{H}) = \min_{\mathcal{H}} \max_{B \in C^*} \frac{2|B|}{|B| - |\delta(B)|} \geq \frac{2 \cdot 4}{4 - 2} = 4. (15)$$

Then by Theorem 2.1, $\phi_0(\mathcal{H}) = 4$.

In Figure 7 we see that each $\mathcal{H}$ contains at least one 5-face. Pick a 5-face $F' \in \mathcal{F}$. Pick a line bordering $F'$ and label it $l_1$. Proceed cyclically around the boundary of $F'$, labeling the four other lines bordering $F'$ as $l_2, l_3, l_4$ and $l_5$ cyclically. Label the one remaining line as $l_6$. Let $F_1, \ldots, F_5$ be the faces of $\mathcal{H}$ adjacent to $F'$ and incident respectively with $l_1, \ldots, l_5$. 31
We consider all possible orientations of the lines in $\mathcal{H}$ with respect to $F'$. Let $k$ be the number of lines oriented in the same direction with respect to $F'$. Then $k = 3, 4$ or $5$. Without loss of generality, say these $k$ lines are oriented outwards. If $k = 5$, then $\delta(F') = -4$ or $-6$, depending on the orientation of $l_6$. Suppose $k = 4$. Then $\delta(F') = -2$ or $-4$, depending on the orientation of $l_6$. If $\delta(F') = -2$ and $l_i$ is the inward-oriented line bordering $F'$, then $\delta(F_i) = -4$. Now, suppose $k = 3$. Then at least two outward oriented lines, say $l_1$ and $l_2$, are adjacent on the border of $F'$. Let $v$ be the vertex at the intersection of $l_1$ and $l_2$. Let $F''$ be the face adjacent to
both $F_1$ and $F_2$, and incident with $F'$ at $v$. Now, either $l_6$ is oriented inwards with respect to $F'$, or it is oriented outwards. First suppose it is oriented inwards. Then $|\delta(F')| = 0$, but $|\delta(F_1)| = |\delta(F_2)| = 2$ and $|\delta(F'')| = 4$. (see Figure 8). Now suppose $l_6$ is oriented outwards with respect to $F'$. Then $\delta(F') = -2$. If $l_i$ and $l_j$ are the two inward-oriented lines bordering $F'$, then $\delta(F_i) = \delta(F_j) = -4$ (see Figure 9).

In all cases, there exists a face $F \in \mathcal{F}$ such that $|\delta(F)| \geq 4$. 

\[ \square \]
3 Bounding the Oriented Flow Number of All Rank 3 Orientable Matroids

We now consider the general case when $M$ is a rank 3 matroid not equal to $U_{3,n}$. Then any pseudoline arrangement $\mathcal{H}$ of $M$ contains vertices of degree greater than two. We want to find a way to partition and orient the lines in $\mathcal{H}$ so that $\delta(B) \leq \frac{|B|}{2}$ for each cocircuit $B$, to prove that $\phi_o(M) \leq 4$. The following lemma will be crucial for our proof.

**Lemma 3.1.** Let $\mathcal{H}$ be a pseudoline arrangement of $U_{3,n}$, with $n \geq 5$ and $n \neq 7$. Let $F_1$ be any face in $\mathcal{F}$. Then $\mathcal{H}$ may be oriented so that

$$|\delta(F)| \leq \frac{n+2}{2} \text{ for all } F \in \mathcal{F} \text{ and } |\delta(F_1)| \leq \frac{n-2}{2}.$$

**Proof:** We will partition the lines of $\mathcal{H}$ into sets $\Pi_1, \ldots, \Pi_k$ in the manner described in the proof of Theorem 2.1. Since $n \geq 5$ and $n \neq 7$, $|\Pi_1| = 4$. In the proof of Theorem 2.1, we chose the set $\Pi_1$ arbitrarily. Here, we choose four lines that guarantee that $F_1$ lies in a 4-face of $\mathcal{H}_1$, and then orient them so that $|\delta_{\mathcal{H}_1}(F_1)| = 0$.

Let $F_1$ be an $m$-face in $\mathcal{H}$. If $m \geq 4$, choose $\Pi_1$ to be any four lines whose line segments bounding $F_1$ are consecutively adjacent. Then $F_1$ lies within a 4-face of $\mathcal{H}_1$ (see Figure 10). If $m = 3$, let $l_1, l_2$ and $l_3$ be the three lines bounding $F_1$. Consider
the subarrangement $\mathcal{H}'$ generated by $l_1, l_2, l_3$ and any two other lines $l_4$ and $l_5$ in $\mathcal{H}$. Now, $\mathcal{H}'$ is isomorphic to the pseudoline arrangement of $U_{3,5}$ shown in Figure 11 [4]. Note that $\mathcal{H}'$ contains a single 5-face $F_5$. Since $F_1$ is a 3-face, $F_1$ must be one of the five faces adjacent to $F_5$. Suppose $l_1$ is the line segment separating $F_1$ and $F_5$. Let $\Pi_1 = \{l_2, l_3, l_4, l_5\}$. Then $F_1$ lies in a 4-face of $\mathcal{H}_1$. By the symmetry of $\mathcal{H}'$, the argument holds for all 3-faces in $\mathcal{H}'$.

Orient the lines of $\Pi_1$ alternately with respect to $F_1$, so that $\delta_{\mathcal{H}_1}(F_1) = 0$. This orientation is consistent with the method of orientation used in the proof of Theorem 2.1. By Corollary 2.2, $\mathcal{H}$ may now be oriented so that $|\delta(F)| \leq \frac{n+2}{2}$ for all $F \in \mathcal{F}$. For any given face $F$, the upper bound $\frac{n+2}{2}$ is attained precisely when $|\delta_{\mathcal{H}_1}(F)| = 2$.

Figure 10: The case when $m \geq 4$. 

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for all $i$. Now, since $|\delta_{H_1}(F_1)| = 0$, we have $|\delta(F_1)| \leq \frac{n+2}{2} - 2 = \frac{n-2}{2}$.

We now prove the main result of the thesis.

**Theorem 3.1.** Let $\mathcal{H}$ be a pseudoline arrangement with underlying matroid $M$ such that the rank of $M$ is 3 and $M$ does not have a coloop. Then $\phi_0(\mathcal{H}) \leq 4$.

*Proof:* Let $l_1, \ldots, l_n$ be the lines of $\mathcal{H}$. We may assume that there exists a vertex in $\mathcal{H}$ with degree greater than 2; if not, then $\mathcal{H}$ is isomorphic to a pseudoline arrangement of $U_{3,n}$ for some $n \geq 4$, and $\phi_0(\mathcal{H}) \leq 4$ by Theorem 2.1.
We define a partition \( \Pi = (\Pi_0, \Pi_1, \ldots, \Pi_k) \) of the lines of \( \mathcal{H} \) as follows. Find a vertex \( v_1 \) in \( \mathcal{H} \) of largest degree. If \( \deg(v_1) \) is odd, let \( \Pi_1 \) be the set of lines intersecting at \( v_1 \). If \( \deg(v_1) \) is even, let \( \Pi_1 \) be the set of lines intersecting at \( v_1 \) except one (that may be chosen arbitrarily). Let \( \mathcal{H}_1 \) be the subarrangement of \( \mathcal{H} \) generated by \( \Pi_1 \). We will say that \( v_1 \) gives rise to \( \mathcal{H}_1 \). Now consider the arrangement \( \mathcal{H} - \mathcal{H}_1 \). Find a vertex \( v_2 \) of largest degree in \( \mathcal{H} - \mathcal{H}_1 \). If \( \deg(v_2) \) is greater than 2, repeat the process; namely, if \( \deg(v_2) \) is odd, let \( \Pi_2 \) be the set of lines intersecting at \( v_2 \), and if \( \deg(v_2) \) is even, let \( \Pi_2 \) be the set of lines intersecting at \( v_2 \) except one. Let \( \mathcal{H}_2 \) be the subarrangement of \( \mathcal{H} \) generated by \( \Pi_2 \).

Continue the process, defining the sets \( \Pi_1, \ldots, \Pi_k \) until \( \mathcal{H} \setminus \{\mathcal{H}_1 \cup \ldots \cup \mathcal{H}_k\} \) contains no vertices of degree greater than two. Let \( \Pi_0 \) be the set of remaining lines and \( \mathcal{H}_0 \) the corresponding subarrangement of \( \mathcal{H} \). Let \( m = |\Pi_0| \). Note that if \( m \geq 3 \), then \( \mathcal{H}_0 \) is isomorphic to a pseudoline arrangement of \( U_{3,m} \). We will call the subarrangements \( \mathcal{H}_1, \ldots, \mathcal{H}_k \) the odd subarrangements of \( \mathcal{H} \) and \( \mathcal{H}_0 \) the uniform subarrangement of \( \mathcal{H} \).

If \( m \geq 4 \), we orient the uniform subarrangement \( \mathcal{H}_0 \) as in the proof of Theorem 2.1. Let \( F \in \mathcal{F}_0 \). By Corollary 2.2,

\[
|\delta_{\mathcal{H}_0}(F)| \leq \frac{m+2}{2}.
\]

If \( m \leq 3 \), then no matter the orientation of \( \mathcal{H}_0 \), \( |\delta_{\mathcal{H}_0}(F)| \leq m \) for all \( F \in \mathcal{F} \). Since
if $m$ equals 0, 1 or 2, we have

\[
|\delta_{\mathcal{H}_0}(F)| \leq \begin{cases} 
\frac{m+2}{2} & \text{if } m \neq 3 \\
3 & \text{if } m = 3.
\end{cases}
\] (16)

Let $s$ be a line segment in $\mathcal{H}_0$. By (8), $\max_{s \in \mathcal{S}} |\delta_{\mathcal{H}_0}(s)| \leq \max_{F \in \mathcal{F}_0} |\delta_{\mathcal{H}_0}(F)| - 1$, so that

\[
|\delta_{\mathcal{H}_0}(s)| \leq \begin{cases} 
\frac{m}{2} & \text{if } m \neq 3 \\
2 & \text{if } m = 3.
\end{cases}
\] (17)

Let $v$ be a vertex in $\mathcal{H}_0$. By (7), $\max_{v \in \mathcal{V}_0} |\delta_{\mathcal{H}_0}(v)| = \max_{F \in \mathcal{F}_0} |\delta_{\mathcal{H}_0}(F)| - 2$, so that

\[
|\delta_{\mathcal{H}_0}(v)| \leq \begin{cases} 
\frac{m-2}{2} & \text{if } m \neq 3 \\
1 & \text{if } m = 3.
\end{cases}
\] (18)

The vertex $v_B$ in $\mathcal{H}$ lies either in a face of $\mathcal{H}_0$ if it lies on zero lines of $\mathcal{H}_0$; a line segment of $\mathcal{H}_0$ if it lies on exactly one line of $\mathcal{H}_0$; or a vertex of $\mathcal{H}_0$ if it lies on two lines of $\mathcal{H}_0$. Let $m_0$ be the number of lines in $\mathcal{H}_0$ containing $v_B$. Note that $m_0 \in \{0, 1, 2\}$ since the degree of a vertex in $\mathcal{H}_0$ is at most 2. Combining (16), (17), and (18), we have

\[
|\delta_{\mathcal{H}_0}(B)| = |\delta_{\mathcal{H}_0}(v_B)| \leq \begin{cases} 
\frac{m+3}{2} - m_0 & \text{if } m \neq 3; \\
3 - m_0 & \text{if } m = 3.
\end{cases}
\] (19)

This implies, furthermore, that

\[
|\delta_{\mathcal{H}_0}(B)| \leq \frac{m+3}{2} - m_0
\] (20)
Figure 12: The method of orientation of odd subarrangements of \( \mathcal{H} \).

for all \( m \geq 0 \).

We orient the odd subarrangements \( \mathcal{H}_i \), for \( 1 \leq i \leq k \), of \( \mathcal{H} \) by alternating orientations as one rotates through the lines clockwise around \( v_i \) (see Figure 12). There are two possible ways to do this, yielding either \( \delta_{\mathcal{H}_i}(F) = 1 \) or \( \delta_{\mathcal{H}_i}(F) = -1 \) for any given face \( F \in \mathcal{F} \). When we need to specify which orientation is to be used, we will do so. Suppose \( s \) is a line segment in \( \mathcal{H}_i \) separating two faces \( F_1, F_2 \in \mathcal{F}_i \).

Then either \( \delta_{\mathcal{H}_i}(F_1) \) or \( \delta_{\mathcal{H}_i}(F_2) \) equals 1, and the other equals \(-1\). Since \( \delta_{\mathcal{H}_i}(s) \) is the average of \( \delta_{\mathcal{H}_i}(F_1) \) and \( \delta_{\mathcal{H}_i}(F_2) \), it follows that \( \delta_{\mathcal{H}_i}(s) = 0 \). Suppose \( v_B \) is a vertex in \( \mathcal{H} \). For each \( i \), \( v_B \) either equals \( v_i \), or \( v_B \) lies in a face or a line segment of \( \mathcal{H}_i \). For all \( i \) such that \( v_B \neq v_i \), let \( k_i \) be the number of lines of \( \mathcal{H}_i \) which contain \( v_B \). So, if \( v_B \) lies in a face of \( \mathcal{H}_i \), then \( k_i = 0 \), and if \( v_B \) lies in a line segment of \( \mathcal{H}_i \), then \( k_i = 1 \). If
\[ v_B = v_j \text{ for some } j, \text{ let } k_j = 0. \text{ Then } k_i \in \{0, 1\} \text{ for all } i, \text{ and} \]

\[
|\delta_{H_i}(B)| = |\delta_{H_i}(v_B)| = \begin{cases} 
1 & \text{if } k_i = 0 \text{ and } v_B \neq v_i \\
0 & \text{otherwise.}
\end{cases}
\]

Let \( k_B = \sum_{i=1}^{k} k_i. \) If \( v_B \neq v_i \) for all \( i, \) then \( k_B \) is the number of distinct odd subarrangements of \( H \) containing \( v_B. \) If \( v_B = v_j \) for some \( j, \) then \( k_B \) is the number of distinct odd subarrangements of \( H, \) other than \( H_j, \) which contain \( v_B. \)

In the proof of Theorem 2.1, we used the fact that since every vertex had degree two, \( |B| \) was the same for all \( B \in C^*. \) In the general case we do not have that advantage. We will consider two cases: cocircuits whose corresponding vertices in \( H \) give rise to an odd subarrangement, and cocircuits that do not. Further subcases are needed depending on the values of \( k \) and \( m. \)

Case 1: \( v_B = v_i \) for some \( 1 \leq i \leq k, \) so \( v_B \) gives rise to an odd subarrangement. Then \( v_B \) is the intersection of all of the lines in some odd subarrangement, say \( \Pi_1; \) \( m_0 \) lines of the uniform subarrangement \( H_0; \) and \( k_B \) lines of other distinct odd subarrangements (for example, see Figure 13). Order the odd subarrangements \( \Pi_1, \ldots, \Pi_k \) so that \( \Pi_2, \ldots, \Pi_{k_B+1} \) are the ones containing \( v_B. \) Note that \( m_0 \in \{0, 1\} \) since the partitioning of odd subarrangements leaves at most one line containing \( v_B \) in the uniform subarrangement. Furthermore, \( m_0 = 0 \) precisely when \( v_B \) lies in a face of \( H_0, \) and

40
Figure 13: An example where $v_B = v_1$; here $m_0 = 1$ and $k_B = 2$. The labels indicate which subarrangement each line is contained in.

$m_0 = 1$ when $v_B$ lies on a line segment of $\mathcal{H}_0$. Also, $0 \leq k_B \leq k - 1$. Now, $v_B$ misses $m - m_0$ lines of $\mathcal{H}_0$; all of the lines in $k - k_B - 1$ subarrangements; and all but one line in $k_B$ subarrangements. So $v_B$ misses at least $(m - m_0) + 2k_B + 3(k - k_B - 1)$ lines of $\mathcal{H}$, implying

$$|B| \geq (m - m_0) + 2k_B + 3(k - k_B - 1).$$

(21)

Since $v_B$ lies on at least one line of $\mathcal{H}_i$ for $1 \leq i \leq k_B + 1$, $k_B + 1$

$$\sum_{i=1}^{k_B+1} |\delta_{\mathcal{H}_i}(B)| = \sum_{i=1}^{k_B+1} 0 = 0.$$

Since $v_B$ lies in a face of $\mathcal{H}_i$ for $k_B + 2 \leq i \leq k$,

$$\sum_{i=k_B+2}^{k} |\delta_{\mathcal{H}_i}(B)| = \sum_{i=k_B+2}^{k} 1 = k - k_B - 1.$$
So,

\[ |\delta(B)| \leq \sum_{i=0}^{k} |\delta_{H_i}(B)| \leq |\delta_{H_0}(B)| + k - k_B - 1. \]  \hfill (22)

We want to show that \(|\delta(B)| \leq \frac{|B|}{2}\). Suppose

\[ |\delta_{H_0}(B)| \leq \frac{1}{2}(m - m_0 + k + k_B - 1). \]

Then (22) implies that

\[
|\delta(B)| \leq \frac{1}{2}(m - m_0 + 3k - k_B - 3) \\
= \frac{1}{2}((m - m_0) + 2k_B + 3(k - k_B - 1)) \\
\leq \frac{|B|}{2}.
\]

So,

\[ |\delta_{H_0}(B)| \leq \frac{1}{2}(m - m_0 + k + k_B - 1) \Rightarrow |\delta(B)| \leq \frac{|B|}{2}. \]  \hfill (23)

We examine subcases for various values of \(m\) and \(k\).

Case 1.1: \(m = 3\). Then \(|\delta_{H_0}(B)| \leq 3 - m_0\) by (19). If \(k \geq 4\), then \(3 - m_0 \leq \frac{1}{2}(3 - m_0 + k + k_B - 1)\), and \(|\delta(B)| \leq \frac{|B|}{2}\) by (23). Suppose \(k < 4\). The unique uniform arrangement \(H_0\) with 3 lines has exactly four 3-faces. It may be oriented so that \(|\delta(F^*)| = 3\) for any chosen face \(F^*\), and \(|\delta(F)| = 1\) for the other three faces. If \(k < 4\), there must be at least one face \(F^*\) in \(H_0\) that does not contain \(v_i\) for all \(i\). Reorient \(H_0\) so that \(|\delta(F^*)| = 3\) and \(|\delta(F)| = 1\) for the other three faces. If \(m_0 = 0\),
then $v_B$ falls within a face $F$ of $\mathcal{H}_0$, and $|\delta_{\mathcal{H}_0}(B)| = |\delta_{\mathcal{H}_0}(F)| = 1$. Then, since $k \geq 1$ and $m = 3$,
\[
\frac{1}{2} (m - m_0 + k + k_B - 1) \geq \frac{3}{2} \geq 1 = |\delta_{\mathcal{H}_0}(B)|,
\]
and $|\delta(B)| \leq \frac{|B|}{2}$ by (23).

Suppose $m_0 = 1$. Then $v_B$ lies on a line segment $s$ of $\mathcal{H}_0$, and $|\delta_{\mathcal{H}_0}(B)| \leq 2$. If $k \geq 3$, then $2 \leq \frac{1}{2} (m - m_0 + k + k_B - 1)$, and $|\delta(B)| \leq \frac{|B|}{2}$ by (23). Suppose $k = 1$. Let $F_1$ and $F_2$ be the two faces in $\mathcal{H}_0$ bordering $s$. Reorient $\mathcal{H}_0$ so that $\delta(F_1) = 1$ and $\delta(F_1) = -1$. Then $\delta_{\mathcal{H}_0}(B) = \delta_{\mathcal{H}_0}(s) = 0$, and $0 \leq \frac{1}{2} (m - m_0 + k + k_B - 1) = 1 + \frac{1}{2} k_B$, implying $|\delta(B)| \leq \frac{|B|}{2}$ by (23). Suppose $k = 2$. Then $|B| \geq 5 - k_B \geq 4$ by (21). Now,
\[
\delta(B) = \delta_{\mathcal{H}_0}(B) + \delta_{\mathcal{H}_1}(B) + \delta_{\mathcal{H}_2}(B).
\]
Without loss of generality, suppose $v_B$ gives rise to $\mathcal{H}_1$. Then $\delta_{\mathcal{H}_1}(B) = 0$. Now, $\delta_{\mathcal{H}_0}(B) = -2, 0$ or $2$, depending on the orientation of $\mathcal{H}_0$, and
\[
|\delta_{\mathcal{H}_0}(B)| = \begin{cases} 
0 & \text{if } k_B = 1 \\
1 & \text{if } k_B = 0.
\end{cases}
\]
If $\delta_{\mathcal{H}_0}(B) = 0$, then
\[
|\delta(B)| = |\delta_{\mathcal{H}_0}(B)| \leq 1 < \frac{|B|}{2}.
\]
If $|\delta_{\mathcal{H}_0}(B)| = 2$ and $k_B = 1$, then
\[
|\delta(B)| = |\delta_{\mathcal{H}_0}(B)| = 2 \leq \frac{|B|}{2}.
\]

43
Suppose $|\delta_{H_0}(B)| = 2$ and $k_B = 0$. We reorient $H_2$ to "balance" the discrepancy of $v_B$. That is, if $\delta_{H_0}(B) = 2$, reorient $H_2$ so that $\delta_{H_2}(B) = -1$, and if $\delta_{H_0}(B) = -2$, reorient $H_2$ so that $\delta_{H_2}(B) = 1$. Then

$$|\delta(B)| = |\delta_{H_0} + \delta_{H_2}| = 1 < \frac{|B|}{2}.$$ 

Case 1.2: $m \neq 3$. Then $|\delta_{H_0}(B)| \leq \frac{m+2}{2} - m_0$ by (19). If $k \geq 3$, then $\frac{m+2}{2} - m_0 \leq \frac{1}{2}(m - m_0 + k + k_B - 1)$, and $|\delta(B)| \leq \frac{|B|}{2}$ by (23).

Suppose $k \in \{1, 2\}$. There are various subcases, depending on the value of $m$, and most fixing a special orientation.

Case 1.2a: $m = 0$. Then $H$ is partitioned into odd subarrangements only. We may assume $k = 2$ since if $k = 1$, then $M$ has rank 2. Now, $|B| \geq 3k - k_B - 3$ by (21), and $|\delta(B)| \leq k - k_B - 1$ by (22). Since $k = 2$,

$$1 \leq k + k_B$$

$$\Rightarrow \frac{1}{2} \leq \frac{1}{2}k + \frac{1}{2}k_B$$

$$\Rightarrow k - k_B - 1 \leq \frac{3}{2}k - \frac{1}{2}k_B - \frac{3}{2}$$

$$\Rightarrow |\delta(B)| \leq \frac{|B|}{2}.$$ 

Case 1.2b: $m = 1$. Then $H_0$ consists of a single line. We may assume $k = 2$ since if $k = 1$, then $M$ contains a coloop or has rank 2. Now, no matter the orientation of
\( \mathcal{H}_0, |\delta_{\mathcal{H}_0}(B)| \) equals 1 if \( m_0 = 0 \) (so \( v_B \) lies in a face of \( \mathcal{H}_0 \)), and equals 0 if \( m_0 = 1 \) (so \( v_B \) lies on the single line in \( \mathcal{H}_0 \)). So \( |\delta_{\mathcal{H}_0}(B)| = 1 - m_0 \). Then 

\[
1 - m_0 \leq 1 - \frac{m_0}{2} + \frac{k_B}{2} = \frac{1}{2}(1 - m_0 + 2 + k_B - 1),
\]

implying \( |\delta(B)| \leq \frac{|B|}{2} \) by (23).

Case 1.2c: \( m = 2 \). Then \( \mathcal{H}_0 \) has two faces \( F_1, F_2 \) and two possible orientations. Under one orientation, \( |\delta_{\mathcal{H}_0}(F_1)| = 0 \) and \( |\delta_{\mathcal{H}_0}(F_2)| = 2 \). Under the other orientation, \( |\delta_{\mathcal{H}_0}(F_1)| = 2 \) and \( |\delta_{\mathcal{H}_0}(F_2)| = 0 \). If \( k = 1 \), then \( v_B \) gives rise to the only odd subarrangement. Now, \( v_B \) cannot lie on a line segment of \( \mathcal{H}_0 \) since, if it did, \( M \) would contain a coloop. So \( \mathcal{H}_0 \) can be reoriented so that \( |\delta_{\mathcal{H}_0}(F)| = 0 \) for whichever face \( F \in \mathcal{F}_0 \) contains \( v_B \), so that \( |\delta_{\mathcal{H}_0}(B)| = 0 \), hence \( |\delta(B)| = |\delta_{\mathcal{H}_0}(B)| = 0 \) and \( |\delta(B)| \leq \frac{|B|}{2} \).

Suppose \( k = 2 \). Then \( |B| \geq 5 - m_0 - k_B \geq 3 \) by (21). Let \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) be the two odd subarrangements of \( \mathcal{H} \). Without loss of generality, suppose \( v_B \) gives rise to \( \mathcal{H}_1 \). Now, \( |\delta_{\mathcal{H}_0}(B)| \leq 2 \), \( |\delta_{\mathcal{H}_1}(B)| = 0 \), and \( |\delta_{\mathcal{H}_2}(B)| \leq 1 \). Suppose \( |\delta_{\mathcal{H}_2}(B)| = 0 \). If \( m_0 = 0 \), then \( |B| \geq 4 \), and

\[
|\delta(B)| = |\delta_{\mathcal{H}_0}(B)| \leq 2 \leq \frac{|B|}{2}.
\]

If \( m_0 = 1 \), then \( v_B \) lies on a line segment of \( \mathcal{H}_0 \), and \( |\delta_{\mathcal{H}_0}(B)| = 1 \). Then

\[
|\delta(B)| = |\delta_{\mathcal{H}_0}(B)| = 1 < \frac{3}{2} \leq \frac{|B|}{2}.
\]
Figure 14: $\mathcal{H}_0$ is shown in bold. $\mathcal{H}_2$ is oriented to balance the discrepancy of $v_1$.

Now, suppose $|\delta_{\mathcal{H}_2}(B)| = 1$. We reorient $\mathcal{H}_2$ to “balance” the discrepancy of $v_B$. So, if $\delta_{\mathcal{H}_0}(B)$ is positive, reorient $\mathcal{H}_2$ so that $\delta_{\mathcal{H}_2}(B) = -1$, and if $\delta_{\mathcal{H}_0}(B)$ is negative, reorient $\mathcal{H}_2$ so that $\delta_{\mathcal{H}_2}(B) = 1$ (see Figure 14). If $\delta_{\mathcal{H}_0}(B) = 0$, then

$$|\delta(B)| = |\delta_{\mathcal{H}_2}(B)| = 1 < \frac{3}{2} \leq \frac{|B|}{2}.$$  

Suppose $\delta_{\mathcal{H}_0}(B)$ is positive. Then $\delta_{\mathcal{H}_0}(B) = 2$ if $v_B$ lies in a face of $\mathcal{H}_0$ and 1 if $v_B$ lies on a line segment of $\mathcal{H}_0$. Orient $\mathcal{H}_2$ so that $\delta_{\mathcal{H}_2}(B) = -1$. Then

$$\delta(B) = \delta_{\mathcal{H}_0}(B) + \delta_{\mathcal{H}_1}(B) + \delta_{\mathcal{H}_2}(B) \leq 2 + 0 - 1 = 1,$$

and $|\delta(B)| < \frac{3}{2} \leq \frac{|B|}{2}$. If $\delta_{\mathcal{H}_0}(B)$ is negative, orient $\mathcal{H}_2$ so that $\delta_{\mathcal{H}_2}(B) = 1$, and the
result is the same.

Case 1.2d: \( m = 4 \). Then \( \mathcal{H}_0 \) is isomorphic to the unique pseudoline arrangement of \( U_{3,4} \). Note that \( \mathcal{H}_0 \) contains a 4-face. Orient \( \mathcal{H}_0 \), following the procedure in Lemma 2.1, so that \( |\delta_{\mathcal{H}_0}(F)| \leq 2 \) for all faces \( F \in \mathcal{F}_0 \). Then \( |\delta_{\mathcal{H}_0}(B)| \leq 2 - m_0 \). Since \( k \geq 1 \),

\[
k \geq 1 - (m_0 + k_B)
\]

\[
\Rightarrow \frac{1}{2} - \frac{m_0}{2} - \frac{k_B}{2} \leq \frac{k}{2}
\]

\[
\Rightarrow |\delta_{\mathcal{H}_0}(B)| \leq 2 - m_0 \leq \frac{3}{2} - \frac{m_0}{2} + \frac{k_B}{2} + \frac{k}{2}
\]

\[
\Rightarrow |\delta_{\mathcal{H}_0}(B)| \leq \frac{1}{2}(4 - m_0 + k_B + k - 1)
\]

\[
\Rightarrow |\delta(B)| \leq \frac{|B|}{2} \text{ by (23)}.
\]

Case 1.2e: \( m = 7 \). Then, by Lemma 2.2, \( \mathcal{H}_0 \) may be oriented so that \( |\delta_{\mathcal{H}_0}(F)| \leq 3 \) for all faces \( F \in \mathcal{F} \). Then \( |\delta_{\mathcal{H}_0}(B)| \leq 3 - m_0 \). Since \( k \geq 1 \),

\[
k \geq -m_0 - k_B
\]

\[
\Rightarrow -\frac{m_0}{2} - \frac{k_B}{2} \leq \frac{k}{2}
\]

\[
\Rightarrow |\delta_{\mathcal{H}_0}(B)| \leq 3 - m_0 \leq 3 - \frac{m_0}{2} + \frac{k}{2} + \frac{k_B}{2}
\]

\[
\Rightarrow |\delta_{\mathcal{H}_0}(B)| \leq \frac{1}{2}(7 - m_0 + k + k_B - 1)
\]
\[
\Rightarrow |\delta(B)| \leq \frac{|B|}{2} \text{ by (23)}.
\]

Case 1.2f: \(m \geq 5\) and \(m \neq 7\). First suppose \(k = 1\). Then \(v_B\) gives rise to the only odd subarrangement of \(\mathcal{H}\). Necessarily, \(k_B = 0\). Suppose that \(m_0 = 0\), so that \(v_B\) lies in a face of \(\mathcal{H}_0\). Then by Lemma 3.1, \(\mathcal{H}_0\) may be reoriented so that \(|\delta_{\mathcal{H}_0}(B)| \leq \frac{m-2}{2}\). Since \(|B| = m\) and \(|\delta(\mathcal{H})(B)| = |\delta_{\mathcal{H}_0}(B)|\), we have

\[
|\delta(B)| \leq \frac{m-2}{2} < \frac{m}{2} = \frac{|B|}{2}.
\]

Now suppose \(k = 1\) and \(m_0 = 1\), so that \(v_B\) lies on a line segment \(s\) of \(\mathcal{H}_0\). Then \(|B| = m - 1\). Let \(l\) be the line in \(\mathcal{H}\) containing \(s\). Then \(v_B\) lies in a face of \(\mathcal{H}_0 \setminus l\). If \(m \geq 6\) then \(|\mathcal{H}_0 \setminus l| \geq 5\), and by Lemma 3.1, \(\mathcal{H}_0 \setminus l\) can be reoriented so that

\[
|\delta(B)| = |\delta_{\mathcal{H}_0}(B)| = |\delta_{\mathcal{H}_0 \setminus l}(B)| \leq \frac{(m-1) - 2}{2} = \frac{m-3}{2} < \frac{m-1}{2} = \frac{|B|}{2}.
\]

Suppose \(m = 5\). Then \(\mathcal{H}_0\) is isomorphic to the unique pseudoline arrangement of \(U_{3,5}\). Note that \(\mathcal{H}_0\) contains a 5-face, so by Lemma 2.1, \(\mathcal{H}_0\) can be reoriented so that \(|\delta_{\mathcal{H}_0}(F)| \leq 3\) for all \(F \in \mathcal{F}_0\), implying \(|\delta_{\mathcal{H}_0}(s)| \leq 2\). So,

\[
|\delta(B)| = |\delta_{\mathcal{H}_0}(B)| = |\delta_{\mathcal{H}_0}(s)| \leq 2 = \frac{|B|}{2}.
\]

Now suppose \(k = 2\). Then \(|B| \geq m + 3 - k_B - m_0\) by (21). Let \(\mathcal{H}_1\) and \(\mathcal{H}_2\) be the two odd subarrangements of \(\mathcal{H}\). Without loss of generality, suppose \(v_B\) gives rise to
Then $\delta(B) = \delta_{\mathcal{H}_0}(B) + \delta_{\mathcal{H}_2}(B)$. By (19), we have

$$|\delta_{\mathcal{H}_0}(B)| \leq \frac{m + 2}{2} - m_0.$$ 

Note also that $|\delta_{\mathcal{H}_2}(B)| = 1 - k_B$. So, if $k_B = 1$ then

$$|\delta(B)| \leq \frac{m + 2}{2} - m_0 \leq \frac{m}{2} + 1 - \frac{m_0}{2} \leq \frac{|B|}{2}.$$ 

If $k_B = 0$ then, as in Case 1.2c, reorient each odd subarrangement to "balance" the discrepancy of the vertex corresponding to the other. If $\delta_{\mathcal{H}_0}(B) = 0$, then

$$|\delta(B)| = |\delta_{\mathcal{H}_2}(B)| = 1 < \frac{m}{2} + \frac{3}{2} - \frac{m_0}{2} \leq \frac{|B|}{2}$$

since $m \geq 5$. If $\delta_{\mathcal{H}_0}(B)$ is positive, reorient $\mathcal{H}_2$ so that $\delta_{\mathcal{H}_2}(B) = -1$, and if $\delta_{\mathcal{H}_0}(B)$ is negative, reorient $\mathcal{H}_2$ so that $\delta_{\mathcal{H}_2}(B) = 1$. Then by (22),

$$|\delta(B)| = |\delta_{\mathcal{H}_0}(B)| - 1,$$

so that

$$|\delta(B)| \leq \left( \frac{m + 2}{2} - m_0 \right) - 1 = \frac{m}{2} - m_0 < \frac{m}{2} + \frac{3}{2} - \frac{m_0}{2} \leq \frac{|B|}{2}.$$ 

This completes the proof of Case 1. The cases in which $\mathcal{H}$ was reoriented are all disjoint: these are cases 1.1, 1.2c, and 1.2f. In all cases, the reorientation was done in a manner consistent with the method of proof of Theorem 2.1, so the bounds on
\( \delta_{H_0}(B) \) which we used are all valid. \( H \) is not reoriented in Case 2, so there will be no ambiguity regarding the orientation of \( H \).

Case 2: \( v_B \neq v_i \) for \( 1 \leq i \leq k \). Then \( v_B \) does not give rise to an odd subarrangement, and \( v_B \) lies on \( m_0 \) lines contained in \( \Pi_0 \), and \( k_B \) lines contained in distinct odd subarrangements (for example, see Figure 15). Consider the orientation of \( H \) given in Case 1. Reorder the odd subarrangements \( H_1, \ldots, H_k \) so that the first \( k_B \) of these are the ones containing \( v_B \). We must have \( 0 \leq m_0 \leq 2 \), \( 0 \leq k_B \leq k \), and \( m_0 + k_B \geq 2 \). Now, \( v_B \) misses \( m - m_0 \) lines of the uniform subarrangement \( H_0 \). There are at least 3 lines in each odd subarrangement, and \( v_B \) misses all the lines in \( k - k_B \) subarrangements, and all but one line in \( k_B \) subarrangements. So \( v_B \) misses at least \( (m - m_0) + 2k_B + 3(k - k_B) \) lines of \( H \), implying

\[
|B| \geq (m - m_0) + 2k_B + 3(k - k_B). \tag{24}
\]

Since \( v_B \) lies on a line segment of \( H_i \) for \( 1 \leq i \leq k_B \),

\[
\sum_{i=1}^{k_B} |\delta_{H_i}(B)| = \sum_{i=1}^{k_B} 0 = 0.
\]

Since \( v_B \) lies on a face of \( H_i \) for \( k_B + 1 \leq i \leq k \),

\[
\sum_{i=k_B+1}^{k} |\delta_{H_i}(B)| = \sum_{i=k_B+1}^{k} 1 = k - k_B.
\]
Using the last two relations, and (20) to bound $|\delta(\mathcal{H}_0)|$, we find that

$$|\delta(B)| \leq \sum_{i=0}^{k} |\delta_{\mathcal{H}_i}(B)| \leq \left( \frac{m + 3}{2} - m_0 \right) + (k - k_B).$$

Comparing (24) and (25), we see that $|\delta(B)| \leq \frac{|B|}{2}$ provided that

$$\left( \frac{m + 3}{2} - m_0 \right) + (k - k_B) \leq \frac{1}{2} (m - m_0) + k_B + \frac{3}{2} (k - k_B).$$

But the latter inequality is equivalent to $m_0 + k_B \geq 3 - k$, which follows from $m_0 + k_B \geq 2$ and $k \geq 1$.

This completes the proof of the theorem. In all cases, $|\delta(B)| \leq \frac{|B|}{2}$, so by (4), $\phi_0(\mathcal{H}) \leq 4$. 

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Figure 15: An example where $v_B \neq v_i$ for all $i$; here $m_0 = 2$ and $k_B = 4$. The labels indicate which subarrangement each line is contained in.
4 Potential Topics of Future Research

There are many outstanding questions about the oriented flow number. First, can we find a natural bound for $\phi_o(\mathcal{H})$ in higher ranks? It does not seem that the geometric methods used here could be applied to higher rank matroids, except possibly to rank 4 matroids. In rank 3, how does the upper bound on $\phi_o(\mathcal{H})$ decrease as the number of elements in the matroid increases? Goddyn et. al. [2] showed that $\phi_o(\mathcal{H}) \leq 3$ when $|E| \geq 427$. Perhaps this can be improved. Also, are there any other arrangements $\mathcal{H}$, other than the three shown here, for which $\phi_o(\mathcal{H}) = 4$?

The methods of proof used here suggest algorithms for generating relatively "balanced" orientations of line arrangements. There may be applications for this which are completely unrelated to matroid theory.
Appendix: Simple Arrangements of Small Numbers of Lines

Figure 16: The nonisomorphic simple arrangements of at most 6 lines.
Figure 17: The nonisomorphic simple arrangements of 7 lines.
References


