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Algebraic methods in topoloogy

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ALGEBRAIC METHODS IN TOPOLOGY

by

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A. F. G. III

INTRODUCTION

This study of algebraic topology is concerned generally with the decomposition of geometric forms into the simplest of geometric forms, simplexes, with the peculiar geometric forms which admit of this decomposition, called polyhedra, and with the schemes of such decompositions, called complexes. Certain objects preserved under topological transformations known as their topological invariants are associated with polyhedra. Although the complete system of topological invariants of polyhedra is not yet known, certain invariants have been discovered and studied. Of these the homology groups are the most significant. The homology groups are abelian, admit of finite systems of generators and are capable of being determined by their numerical invariants. This study is particularly concerned, however, with the proof that the homology groups are truly topological invariants. To this end simplicial approximation mappings are devised to replace continuous mappings from one polyhedron to another which have the advantage of submitting completely to algebraic methods of treatment and which indeed provide algebraic relationships between the two polyhedra. The simplicial approximation mappings may be defined on the simplexes of a polyhedron or may be defined on smaller simplexes formed by a finite-order barycentric subdivision of the original simplexes. The extent to which the simplicial mappings approximate the continuous mappings depends on the order of the barycentric subdivision involved. After the homology groups of a complex are proved to be isomorphic under barycentric subdivision the proof that the homology groups of two complexes with homeomorphic polyhedra are isomorphic concludes the work.

DEFINITION 1: A set R^n of elements, referred to as points (or as vectors), is called a real linear space (or a real vector space) if it satisfies the following conditions:

a) the set R^n is an additive abelian group,

b) R^n is closed under operation on the left by real numbers in such a manner that if x and y are elements of R^n and if λ and μ are real numbers then:

1) $\lambda(x + y) = \lambda x + \lambda y$

2) $(\lambda + \mu)x = \lambda x + \mu x$

3) $\lambda(\mu x) = (\lambda\mu)x$

4) $1x = x$

DEFINITION 2: A system $x_1, x_2, \ldots, x_k$ of elements of R^n is a linearly independent system if the relation $\lambda^1 x_1 + \lambda^2 x_2 + \ldots + \lambda^k x_k = 0$ implies $\lambda^1 = \lambda^2 = \ldots = \lambda^k = 0$, where the $\lambda^1, \lambda^2, \ldots, \lambda^k$ are all real numbers. A system is linearly dependent in case it is not linearly independent.

DEFINITION 3: A linearly independent set $x_1, x_2, \ldots, x_k$ is maximal in case every set $x_1, x_2, \ldots, x_k, y$ is linearly dependent.

LEMMA 1: In a real linear space R^n every maximal linearly independent set has the same cardinality.

DEFINITION 4: The cardinality of a maximal linearly independent set of elements in R^n is called the dimension of R^n.

DEFINITION 5: A real linear space R^n whose dimension is n is called an n-dimensional linear space over the field of real numbers.

DEFINITION 6: A basis of R^n is a maximal system e_1, e_2, ..., e_n of linearly independent elements of the n-dimensional linear space R^n.

REMARK: By means of a basis of R^n it is possible to introduce co-ordinates. This is done in the following manner: if x is an element of R^n then there must exist a dependence relation $\lambda x + \lambda^1 e_1 + \lambda^2 e_2 + \ldots + \lambda^n e_n = 0$ because the system e_1, e_2, ..., e_n is maximally linearly independent. Since the basis e_1, e_2, ..., e_n is linearly independent we must have $\lambda \neq 0$. Solving for x we obtain $x = x^1 e_1 + x^2 e_2 + \ldots + x^n e_n$ where the x^i, $i = 1, 2, \ldots, n$, are real numbers and are referred to as the co-ordinates of the vector with respect to the basis e_1, e_2, ..., e_n. We often write x in terms of its co-ordinates: $x = (x^1, x^2, \ldots, x^n)$.

DEFINITION 7: A system of points x_0, x_1, ..., x_k of an n-dimensional linear space R^n is called independent (not to be confused with linearly independent) if the system of vectors

(1) $\qquad (x_1 - x_0), (x_2 - x_0), \ldots, (x_k - x_0)$

is linearly independent.

LEMMA 2: If x_0, x_1, ..., x_k are independent then $k \leq n$.

LEMMA 3: The points x_0, x_1, ..., x_k are independent if and only if the two relations (2) and (3):

(2) $\lambda^0 x_0 + \lambda^1 x_1 + \ldots + \lambda^k x_k = 0$

(3) $\lambda^0 + \lambda^1 + \ldots + \lambda^k = 0$

together imply that

(4) $\lambda^0 = \lambda^1 = \ldots = \lambda^k = 0$

where $\lambda^0, \lambda^1, \ldots, \lambda^k$ are real numbers.

<u>Proof</u>: a) Granted that $x_0, x_1, \ldots, x_k$ are independent we shall show that the two relations (2) and (3) together imply (4). Now if $x_0, x_1, \ldots, x_k$ are independent then the points

(1) $(x_1 - x_0), (x_2 - x_0), \ldots, (x_k - x_0)$

are linearly independent by definition of independence. Also, by (3) we may write (2) in the form $-(\lambda^1 + \lambda^2 + \ldots + \lambda^k)x_0 + \lambda^1 x_1 + \lambda^2 x_2 + \ldots + \lambda^k x_k = 0$ or $\lambda^1(x_1 - x_0) + \lambda^2(x_2 - x_0) + \ldots + \lambda^k(x_k - x_0) = 0$. Since system (1) is linearly independent the last relation implies that $\lambda^1 = \lambda^2 = \ldots = \lambda^k = 0$ and hence in view of (3) we see that $\lambda^0 = 0$.

b) Conversely, we shall show that if (2) and (3) together imply (4) then the system $x_0, x_1, \ldots, x_k$ is independent. Let us assume that

(5) $\lambda^1(x_1 - x_0) + \lambda^2(x_2 - x_0) + \ldots + \lambda^k(x_k - x_0) = 0.$

If we set $\lambda^0 = -(\lambda^1 + \lambda^2 + \ldots + \lambda^k)$ we may rewrite (5) in the form:

(2) $\lambda^0 x_0 + \lambda^1 x_1 + \ldots + \lambda^k x_k = 0$

where the numbers $\lambda^0, \lambda^1, \ldots, \lambda^k$ satisfy (3). But by assumption (2) and (3) together imply (4), so $\lambda^0 = \lambda^1 = \ldots = \lambda^k = 0$. That is, the system (1) is linearly independent. By definition of independence, then, $x_0, x_1, \ldots, x_k$ is an independent system.

<u>COROLLARY</u>: The independence of a system of points $x_0, x_1, \ldots, x_k$ is not affected by the order in which the points are enumerated.

<u>LEMMA 4</u>: The independence of a system of points implies the independence of every one of its subsystems.

<u>REMARK</u>: Geometrically the independence of the points $x_0, x_1, \ldots, x_k$ means that the hyperplane of least dimension which contains them is of dimension k. If the points $x_0, x_1, \ldots, x_k$ are dependent, then the hyperplane of least dimension which contains them has dimension less than k.

<u>DEFINITION 8</u>: The <u>rank</u> of an m x n matrix is the maximum number of linearly independent rows (or columns) if the rows (or columns) of the matrix are regarded as vectors in R^n (or R^m).

<u>LEMMA 5</u>: Let $x_0, x_1, \ldots, x_k$, $k \leq n$, be a system of points of an n-dimensional linear space R^n, and let $e_1, e_2, \ldots, e_n$ be a basis of this space. Suppose the co-ordinates of the points are determined by the $k + 1$ relations

(6) $\quad x_i = x_i^1 e_1 + x_i^2 e_2 + \ldots + x_i^n e_n, \quad i = 0, 1, \ldots, k.$

Introduce formally the numbers x_i^0 by letting

(7) $\quad x_i^0 = 1, \; i = 0, 1, \ldots, k.$

Denote the matrix

$$\begin{bmatrix} x_0^0 & x_0^1 & \cdots & x_0^n \\ x_1^0 & x_1^1 & \cdots & x_1^n \\ & & \cdots & \\ x_k^0 & x_k^1 & \cdots & x_k^n \end{bmatrix} = \left\| x_i^j \right\| , \quad \begin{matrix} i = 0, 1, \ldots, k \\ j = 0, 1, \ldots, n \end{matrix}$$

by the expression

$$N \begin{bmatrix} x_0 \\ x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_k \end{bmatrix} = N(X).$$

This matrix has $k + 1$ rows and $n + 1$ columns and $k + 1 \leq n + 1$. The points of the system $x_0, x_1, \ldots, x_k$ are independent if and only if the matrix $N(X)$ is of rank $k + 1$.

Proof: We divide the proof into two parts and prove in each case the contrapositive statement.

a) Assume the rank of $N(X)$ is not $k + 1$. We note that the rank of the matrix $N(X)$ cannot be greater than $k + 1$. Now if the rank of $N(X)$ is less than $k + 1$ then by definition of rank the $k + 1$ row vectors are not linearly independent. Then there must exist a linear dependence among the row vectors. That is, there must exist $k + 1$ numbers $\lambda^0, \lambda^1, \ldots, \lambda^k$, not all equal to zero, for which the following relation holds:

(8) $$\lambda^0 x_0^j + \lambda^1 x_1^j + \ldots + \lambda^k x_k^j = 0, \quad j = 0, 1, \ldots, n.$$

In other words, we have the relation:

$$\left[\lambda^0, \ \lambda^1 \ \ldots, \ \lambda^k\right] \begin{bmatrix} x_0^0 & x_0^1 & \ldots & x_0^n \\ x_1 & x_1^1 & \ldots & x_1^n \\ & & \ldots & \\ x_k^0 & x_k^1 & \ldots & x_k^n \end{bmatrix} = \left[0, \ 0, \ \ldots, \ 0\right] .$$

If we multiply relation (6) by λ^i we obtain $\lambda^i x_i = \lambda^i x_i^1 e_1 + \lambda^i x_i^2 e_2 + \ldots + \lambda^i x_i^n e_n$. Summing over i we obtain: $\lambda^0 x_0 + \lambda^1 x_1 + \ldots + \lambda^k x_k = \sum_{i=0}^{k} \lambda^i x_i =$

$$\sum_{j=1}^{n} \left(\sum_{i=0}^{k} \lambda^i x_i^j\right) e_j = \sum_{j=1}^{n} \left(\lambda^0 x_0^j + \lambda^1 x_1^j + \ldots + \lambda^k x_k^j\right) e_j = 0.$$

For j = 0 we see by (8) that $\lambda^0 x_0^0 + \lambda^1 x_1^0 + \ldots + \lambda^k x_k^0 = 0$. But by (7) we know that $x_i^0 = 1$, i = 0, 1, ..., k. Hence $\lambda^0 + \lambda^1 + \ldots + \lambda^k = 0$. Thus we have the situation that the two relations $\lambda^0 x_0 + \lambda^1 x_1 + \ldots + \lambda^k x_k = 0$ and $\lambda^0 + \lambda^1 + \ldots + \lambda^k = 0$ do not together imply that $\lambda^0 = \lambda^1 = \ldots = \lambda^k = 0$. But by the contrapositive of the sufficiency portion of Lemma 3 we know that if (2) and (3) together do not imply (4) then the system x_0, x_1, ..., x_k is not independent.

b) Assume that the points x_0, x_1, ..., x_k are not independent. There are always $k + 1$ numbers λ^0, λ^1, ..., λ^k which satisfy (2) and (3). But if x_0, x_1, ..., x_k is not an independent set then there exist λ^0, λ^1, ..., λ^k which satisfy (2) and (3) but which do not imply (4). That is, λ^0, λ^1, ..., λ^k are not all equal to zero. By substitution into (2) the expression for x_i given in (6) and by rewriting (3) by virtue of (7) in the form $\lambda^0 x_0^0 + \lambda^1 x_1^0 + \ldots + \lambda^k x_k^0 = 0$ we obtain

(8) $$\lambda^0 x_0^j + \lambda^1 x_1^j + \ldots + \lambda^k x_k^j = 0, \quad j = 0, 1, \ldots, n.$$

In other words, we have a nonzero vector $[\lambda^0, \lambda^1, \ldots, \lambda^k]$ such that $$|\lambda^0, \lambda^1, \ldots, \lambda^k| \begin{bmatrix} x_0^0 & x_0^1 & \ldots & x_0^n \\ x_1^0 & x_1^1 & \ldots & x_1^n \\ & & \ldots & \\ x_k^0 & x_k^1 & \ldots & x_k^n \end{bmatrix} =$$

$[0, 0, \ldots, 0]$. Thus the row vectors of $N(X)$ are not linearly independent. By definition of rank we know that $k + 1$ cannot be the rank of $N(X)$.

<u>LEMMA 6</u>: Given an n-dimensional linear space R^n and an integer $k \leq n$, the set of points

(9) $$u_0 = 0, u_1 = e_1, u_2 = e_2, \ldots, u_k = e_k,$$

where $e_1, e_2, \ldots, e_n$ constitute a basis of R^n, is an independent system of points.

<u>Proof</u>: The point $u_0 = 0$ is the zero of the additive abelian group R^n or, equivalently, the origin of co-ordinates of the linear space R^n. The vectors $(u_1 - u_0) = e_1$, $(u_2 - u_0) = e_2, \ldots, (u_k - u_0) = e_k$ are obviously linearly independent since $e_1, e_2, \ldots, e_k$ are a subset of the basis $e_1, e_2, \ldots, e_k, \ldots, e_n$. Hence by definition of an independent system of points we have the result that $u_0, u_1, \ldots, u_k$ are independent.

<u>DEFINITION 9</u>: A linear space R^n is a <u>linear Euclidean space</u> if there exists a function called <u>inner product</u> defined on

every pair of elements of R^n which associates with every two vectors x, y ε R^n a real number $x \cdot y$, called their inner product and which satisfies the following three relations for real numbers λ, μ:

a) linearity: $(\lambda x + \mu y) \cdot z = \lambda(x \cdot z) + \mu(y \cdot z)$

b) symmetry: $x \cdot y = y \cdot x$

c) nonnegativity: $x \cdot x \geq 0$

and where in the last relation equality occurs if and only if $x = 0$.

DEFINITION 10: Two vectors x and y are said to be orthogonal if $x \cdot y = 0$. In particular, the zero vector is orthogonal to every vector.

LEMMA 7: It is possible to introduce an inner product into every real linear space R^n.

Proof: If $e_1, e_2, \ldots, e_n$ is a basis of R^n, define the inner product for the basis vectors by putting $e_i \cdot e_j = \delta_{ij}$, where $\delta_{ii} = 1$ and where $\delta_{ij} = 0$ for $i \neq j$. If $x = x^1 e_1 + x^2 e_2 + \ldots + x^n e_n$ and $y = y^1 e_1 + y^2 e_2 + \ldots + y^n e_n$ are any two vectors of R^n, then we define $x \cdot y = x^1 y^1 + x^2 y^2 + \ldots + x^n y^n$. We now prove that the inner product as defined above satisfies the three relations of linearity, symmetry, and nonnegativity.

a) Linearity: $(\lambda x + \mu y) \cdot z = \left[(\lambda x^1 e_1 + \mu y^1 e_1) + (\lambda x^2 e_2 + \mu y^2 e_2) + \ldots + (\lambda x^n e_n + \mu y^n e_n)\right] \cdot \left[z^1 e_1 + z^2 e_2 + \ldots + z^n e_n\right] = \left[(\lambda x^1 + \mu y^1) e_1 + (\lambda x^2 +\right.$

$\mu y^2)e_2 + \dots + (\lambda x^n + \mu y^n)e_n] \cdot [(z^1)e_1 + (z^2)e_2 + \dots + (z^n)e_n] = (\lambda x^1 + \mu y^1)z^1 + (\lambda x^2 + \mu y^2)z^2 + \dots + (\lambda x^n + \mu y^n)z^n = (\lambda x^1 z^1 + \mu y^1 z^1) + (\lambda x^2 z^2 + \mu y^2 z^2) + \dots + (\lambda x^n z^n + \mu y^n z^n) = (\lambda x^1 z^1 + \lambda x^2 z^2 + \dots + \lambda x^n z^n) + (\mu y^1 z^1 + \mu y^2 z^2 + \dots + \mu y^n z^n) = \lambda(x \cdot z) + \mu(y \cdot z)$.

b) Symmetry: $x \cdot y = x^1 y^1 + x^2 y^2 + \dots + x^n y^n = y^1 x^1 + y^2 x^2 + \dots + y^n x^n = y \cdot x$.

c) Nonnegativity: $x \cdot x = x^1 x^1 + x^2 x^2 + \dots + x^n x^n = (x^1)^2 + (x^2)^2 + \dots + (x^n)^2 \geq 0$. In this relation the equality occurs only if $x = (0, 0, \dots, 0) = 0$.

<u>DEFINITION</u> <u>11</u>: An <u>orthonormal</u> <u>basis</u> in the linear Euclidean space R^n is a basis consisting of elements $e_1, e_2, \dots, e_n$, such that $e_i \cdot e_j = \delta_{ij}$.

<u>LEMMA</u> 8: It is always possible to introduce an orthonormal basis into the linear Euclidean space R^n.

<u>Proof</u>: Let $x_1, x_2, \dots, x_n$ be an arbitrary basis of R^n. Since $x_1, x_2, \dots, x_n$ is a basis, then $x_1 \neq 0$ and hence $x_1 \cdot x_1 \neq 0$. We may therefore put $e_1 = (x_1 \cdot x_1)^{-\frac{1}{2}} x_1$ and obtain the result that $e_1 \cdot e_1 = 1$. Now assume that the system $e_1, e_2, \dots, e_k$, where $e_i \cdot e_j = \delta_{ij}$, $k < n$, has been already constructed with all of its elements expressed linearly in terms of the vectors $x_1, x_2, \dots, x_k$. Because of this, then the vector $y = x_{k+1} - (\lambda^1 e_1 + \lambda^2 e_2 + \dots + \lambda^k e_k)$ differs from zero regardless of the choices for the λ^i, $i = 1, 2, \dots, k$. Let us choose the λ^i, $i = 1, 2, \dots, k$, such that $y \cdot e_1 = 0$ for $i = 1, 2, \dots, k$. For this it is sufficient to

set $\lambda^i = x_{k+1} \cdot e_i$. Now since $y \neq 0$ we may put $e_{k+1} = (y \cdot y)^{-\frac{1}{2}} y$ with the result that $e_i \cdot e_j = \delta_{ij}$, i, j = 1, 2, ..., k + 1. In this manner the system $e_1, e_2, \ldots, e_n$ may be constructed in a finite number of steps since n is finite. Thus the set $e_1, e_2, \ldots, e_n$ is an orthonormal set. To see that it is a linearly independent set and, indeed, a basis we merely consider the relation $\lambda^1 e_1 + \lambda^2 e_2 + \ldots + \lambda^n e_n = 0$. Taking the inner product with e_i yields the relation $(\lambda^1)(0) + (\lambda^2)(0) + \ldots + (\lambda^i)(1) + \ldots + (\lambda^n)(0) = 0$. Thus $\lambda^i = 0$, i 1, 2, ..., n. Therefore the elements $e_1, e_2, \ldots, e_n$ are linearly independent. But since there are n of them, the set is maximally linearly independent in R^n. That is, it is a basis.

DEFINITION 12: A metric on a space is a function of two variables associating with every pair of elements x and y of the space a nonnegative number $\rho(x,y)$, called the distance between x and y which satisfies the following three conditions:

a) $\rho(x,y) = \rho(y,x)$

b) $\rho(x,y) = 0$ if and only if $x = y$

c) $\rho(x,y) + \rho(y,z) \geq \rho(x,z)$.

A nonvacuous set in which a metric has been defined is called a metric space and its elements are called points. If a metric be introduced into a linear Euclidean space a linear Euclidean metric space is obtained. We define the distance between two sets A and B to be:

$$\rho(A,B) = \underset{\substack{x \in A \\ y \in B}}{\text{g.l.b.}}\ \rho(x,y).$$

<u>LEMMA</u> <u>9</u>: It is always possible to introduce a metric into a linear Euclidean space R^n.

<u>Proof</u>: Let $\rho(x,y) = +\left[(x - y)\cdot(x - y)\right]^{\frac{1}{2}}$.

a) To prove that $\rho(y,x) = \rho(x,y)$ we notice that by the axioms defining inner product we have $\rho(x,y) = +\left[(x - y)\cdot(x - y)\right]^{\frac{1}{2}} = +\left[(y - x)\cdot(y - x)\right]^{\frac{1}{2}} = \rho(y,x)$.

b) To prove that $\rho(x,y) = 0$ if and only if $x = y$, we note first that if $\rho(x,y) = 0$ then $+\left[(x - y)\cdot(x - y)\right]^{\frac{1}{2}} = 0$. Hence $(x - y)\cdot(x - y) = 0$. Thus by relation c) of the definition of inner product $x - y = 0$ and $x = y$. Conversely, if $x = y$ then $x - y = 0$ and $(x - y)\cdot(x - y) = 0$, so $\rho(x,y) = 0$.

c) To prove that $\rho(x,y) + \rho(y,z) \geq (x,z)$, we rewrite the above relation as follows: $\left[(x - y)\cdot(x - y)\right]^{\frac{1}{2}} + \left[(y - z)\cdot(y - z)\right]^{\frac{1}{2}} \geq \left[(x - z)\cdot(x - z)\right]^{\frac{1}{2}}$. By substitution of $x - y = u$ and $y - z = v$ we obtain $(u\cdot u)^{\frac{1}{2}} + (v\cdot v)^{\frac{1}{2}} \geq \left[(u + v)\cdot(u + v)\right]^{\frac{1}{2}}$. Since both sides of this inequality are nonnegative we may square them and obtain $u\cdot u + v\cdot v + 2\left[(u\cdot u)(v\cdot v)\right]^{\frac{1}{2}} \geq u\cdot u + 2(u\cdot v) + v\cdot v$, which is equivalent to $\left[(u\cdot u)(v\cdot v)\right]^{\frac{1}{2}} \geq u\cdot v$. Now since both sides of this relation are nonnegative we may square them and obtain $(u\cdot u)(v\cdot v) \geq (u\cdot v)^2$. This relation is the Schwartz inequality and may be proved in general in the following manner: Consider the identity $(u\cdot u)\lambda^2 + 2(u\cdot v)\lambda + (v\cdot v) =$

$(\lambda u + v) \cdot (\lambda u + v)$. Since the right-hand side is an inner product and is therefore nonnegative, both sides are nonnegative. Thus the discriminant of the polynomial in λ on the left must be nonnegative. That is, $[2(u \cdot v)]^2 - 4(u \cdot u)(v \cdot v) \leq 0$. Hence $(u \cdot v)(u \cdot v) \leq (u \cdot u)(v \cdot v)$.

<u>LEMMA</u> <u>10</u>: If the rank of a matrix is $k + 1$ then the matrix has a nonvanishing determinant of $k + 1$ columns.

<u>THEOREM</u> <u>1</u>: If $x_0, x_1, \ldots, x_k$, $k \leq n$, is a system of points of an n-dimensional linear space R^n, then an arbitrary neighborhood of each point x_i contains a point y_i such that the system $y_0, y_1, \ldots, y_k$ is independent.

<u>Proof</u>: Let $u_0, u_1, \ldots, u_k$ be a system of points known to be independent. This is possible by Lemma 6. Let t, where $0 \leq t \leq 1$, be a real parameter. Consider the points determined by

$$z_i(t) = tu_i + (1 - t)x_i, \quad i = 0, 1, \ldots, k. \tag{9}$$

Thus we have:

$$\begin{bmatrix} z_0(t) \\ z_1(t) \\ \cdots \\ z_k(t) \end{bmatrix} = \begin{bmatrix} tu_0 & (1-t)x_0 \\ tu_1 & (1-t)x_1 \\ & \cdots \\ tu_k & (1-t)x_k \end{bmatrix} = t\begin{bmatrix} u_0 \\ u_1 \\ \cdots \\ u_k \end{bmatrix} + (1-t)\begin{bmatrix} x_0 \\ x_1 \\ \cdots \\ x_k \end{bmatrix}.$$

Therefore:

$$N[Z(t)] = tN(U) + (1 - t)N(X). \tag{10}$$

Now since by hypothesis the points $u_0, u_1, \ldots, u_k$ are independent, it follows from Lemma 5 that the rank of the matrix $N(U)$ is equal to $k + 1$. Hence by Lemma 10 the matrix $N(U)$

contains a nonvanishing determinant of $k+1$ columns. Let $D(t)$ be the corresponding determinant of $N[Z(t)]$, the notation emphasizing its dependence on the parameter t. Now by (10) we see that $N[Z(1)] = N(U)$. Thus $D(1) \neq 0$. Therefore $D(t)$ does not vanish identically in t. Since $D(t)$ is a polynomial in t, there is an arbitrarily small positive number for which $D(s) \neq 0$. This means that the matrix $N\ Z(s)$ is of rank $k+1$, and hence we have the result that the system of points $y_0 = z_0(s)$, $y_1 = z_1(s)$, ..., $y_k = z_k(s)$ form an independent system. By (9) the point y_i is arbitrarily close to x_i for all $i = 0, 1, \ldots, k$ since s is arbitrarily small.

DEFINITION 13: A system of points $x_0, x_1, \ldots, x_m$ of an n-dimensional linear space R^n is said to be in general position in case each of its subsystems of $k+1$ points: $\xi_0, \xi_1, \ldots, \xi_k$, $k \leq n$, is independent.

LEMMA 11: If $m \leq n$, generality of position of the system $x_0, x_1, \ldots, x_m$ in the linear space R^n is equivalent to independence.

LEMMA 12: If $m \geq n$ and if every subsystem of exactly $n+1$ points $(k = n)$ of the system $x_0, x_1, \ldots, x_m$ in the linear space R^n is independent, then the system $x_0, x_1, \ldots, x_m$ is in general position, and conversely.

THEOREM 2: If $\{x_0, x_1, \ldots, x_m\} = S$ is a system of points in general position in the Linear Euclidean metric space R^n then there exists a positive number ϵ such that $\rho(x_i, y_i) < \epsilon$, $i = 0, 1, \ldots, m$ implies that the system of points

$\{y_0, y_1, \ldots, y_m\}$ is also in general position.

Proof: Let $\xi_i = x_{p_i}$, $i = 0, 1, \ldots, k$, where $k \leq n$, be an arbitrary subsystem S' of the system S. Now since the system S is in general position then the subsystem S' is an independent system by Lemma 12 and Lemma 4. Hence the matrix $N \begin{bmatrix} \xi_0 \\ \xi_1 \\ \cdots \\ \xi_k \end{bmatrix}$ is of rank $k+1$ by Lemma 5. Therefore one of the determinants, say D, composed of $k+1$ columns of this matrix is different from zero. Now since D is a polynomial in the co-ordinates of the points of the system S' it is a continuous function of these co-ordinates. Therefore there exists a positive number ϵ' such that for every system of points $\eta_i = y_{p_i}$, $i = 0, 1, \ldots, k$ with $\rho(\xi_i, \eta_i) \leq \epsilon$, the determinant formed for the points $\eta_0, \eta_1, \ldots, \eta_k$ is also different from zero. Hence the matrix $N \begin{bmatrix} \eta_0 \\ \eta_1 \\ \cdots \\ \eta_k \end{bmatrix}$ is of rank $k+1$, which means that the system $\eta_0, \eta_1, \ldots, \eta_k$ is independent. Thus a suitable ϵ' may be assigned to every subsystem S' of the system S and the required ϵ of the theorem may be obtained by choosing the smallest of the numbers ϵ'.

<u>THEOREM 3</u>: If $\{x_0, x_1, \ldots, x_m\} = S$ is any system of points of the linear Euclidean metric space R^n and if ϵ is a positive number, then there exists a system of points $\{y_0, y_1, \ldots, y_m\}$ in general position such that $\rho(x_i, y_i) < \epsilon$, $i = 0, 1, \ldots, m$.

<u>Proof</u>: Let the collection of all subsystems $\xi_0, \xi_1, \ldots, \xi_k$, $k \leq n$, of the system S be denoted by $S_1, S_2, \ldots, S_r$. Now the system S_1 contains at most $n + 1$ points and it can therefore be transformed into an independent system of not more than $n + 1$ points by means of an arbitrarily small displacement by virtue of Theorem 1. Now assume that by means of arbitrarily small displacements of the whole system S, we have already obtained a position for which all of the subsystems $S_1, S_2, \ldots, S_s$, $s < r$, are independent. By Theorem 1 the system S_{s+1} can also be transformed into an independent system by an arbitrarily small displacement, and this displacement may be therefore chosen so small that the individual independences of the systems $S_1, S_2, \ldots, S_s$ achieved previously will not be disturbed. This is possible by Theorem 2. Since the induction on s is valid and we have the particular knowledge that $s = 1$ is a valid special case then validity is established for $s = 1, 2, \ldots$, and the theorem is proved. Thus any finite system of points of R^n can be brought into general position by an arbitrarily small displacement.

<u>DEFINITION 14</u>: Let a and b be two distinct points of the

Euclidean space R^n. The set of all points of R^n of the form $x = \lambda a + \mu b$ where λ and μ are real numbers satisfying the conditions

a) $\lambda + \mu = 1$

b) $\lambda \geq 0$

c) $\mu \geq 0$

will be called the <u>segment</u> $[a,b] = [b,a]$ with end points a and b.

<u>LEMMA</u> <u>13</u>: If $x = \lambda a + \mu b$ then $x = a + \mu(b - a)$, or substituting $u = b - a$, we obtain $x = a + \mu u$.

<u>LEMMA</u> <u>14</u>: If the segments $[a,b]$ and $[a,c]$ have a common point different from a then one of the segments is contained in the other. In particular, if the segments $[a,b]$ and $[a,c]$ coincide then $b = c$, and conversely.

<u>Proof</u>: Let $x = a + \mu u$, $0 \leq \mu \leq 1$. Let $y = a + tv$, $0 \leq t \leq 1$. Consider $u = b - a$ and $y = c - a$ to be any two points of the segments. Now if $x_0 = y_0 \neq a$ is a point common to the two segments then $x_0 = a + \mu_0 u = y_0 = a + t_0 v$ for some $\mu_0 \neq 0$ and some $t_0 \neq 0$. Hence $\mu_0 u = t_0 v$. Now if $\mu_0 = t_0$ then $u = v$ and $b = c$ and the two segments coincide. If $\mu_0 \neq t_0$ we may assume without loss of generality that $\mu_0 < t_0$. We obtain the relation $v = (\mu_0/t_0)u$ and any point y of the segment $[a,c]$ is of the form $a + t(\mu_0/t_0)u$. Since $\mu_0/t_0 < 1$ we have y belonging to the segment $[a,b]$

for $0 \leq t \leq 1$ and the second segment is a proper subset of the first.

DEFINITION 15: A set M of points of the linear Euclidean metric space R^n is called convex in case $a, b \in M$ implies $[a,b] \subset M$.

DEFINITION 16: The point a is called an interior point of the set Q if there exists a positive number ϵ such that $\rho(a,x) < \epsilon$ implies $x \in Q$.

DEFINITION 17: A set Q is an open set if every point of Q is an interior point.

DEFINITION 18: The set of points y such that $\rho(x,y) < \delta$ is called the spherical δ-neighborhood of the point x.

DEFINITION 19: If Q is any set then the complement of Q is the set of points x such that x does not belong to Q.

DEFINITION 20: If Q is the complement of an open set then Q is a closed set.

DEFINITION 21: Let Q be a subset of a Euclidean space. If $x, y \in Q$ implies that $\rho(x,y) < N$ for some real N then Q is a bounded set.

DEFINITION 22: A covering of a set Q is a collection of sets $P_1, P_2, \ldots$, which all have points in common with Q and such that $Q \subset P_1 \cup P_2 \cup \ldots$.

DEFINITION 23: A set S is a compact set if from every covering of S consisting of open sets a finite collection of sets can be selected which is also a covering of S.

LEMMA 15: If Q is a closed and bounded subset of a Euclidean space then Q is a compact set, and conversely.

LEMMA 16: A closed subset of a compact space is compact.

DEFINITION 24: A convex set W which is compact and which contains interior points is a convex body.

LEMMA 17: The set U of all interior points of a convex body W forms an open set in R^n and hence $V = W - U$ is compact.

DEFINITION 25: The set V is the frontier of the convex body W.

LEMMA 18: Let W be a convex body with interior U. If $a \in U$, $b \in W$, then every point c of the segment $[a,b]$ distinct from b is contained in U.

Proof: Let $c = \lambda a + \mu b$, with the conditions that $\lambda + \mu = 1$, $\lambda > 0$, $\mu > 0$. We wish to show that a neighborhood of c is contained in W. Now there exists a number $\delta > 0$ such that if $\rho(0,x) = \rho(a, a + x) < \delta$ then $a + x \in W$. This is because a is an interior point of W. Now, with the above conditions on λ and μ we have $\lambda(a + x) + \mu b \in W$, since $a + x$ and b are elements of the convex body W. But $\lambda(a + x) + \mu b = \lambda a + \mu b + \lambda x = c + \lambda x$. Now suppose that $\rho(0,y) < \lambda\delta$. Then $c + y = c + \lambda(y/\lambda)$ and $\rho(0,y/\lambda) < \delta$. Thus $c + y \in W$.

LEMMA 19: Let W be a convex body with interior U and frontier V. If $a \in U$ and b and c are two distinct points of V then the segments $[a,b]$ and $[a,c]$ have only one point in common: a.

Proof: Let $a \in U$ and b and c be two distinct points of

V. Now if we assume that $[a,b]$ and $[a,c]$ do have a common point different from a, then by Lemma 14 either they coincide and $b = c$ (which is not possible by the hypothesis) or one of them forms a proper subset of the other. If we assume that $[a,c] \subset [a,b]$, then $c \in [a,b]$ with $c \neq b$. Hence by Lemma 18 $c \in U$, $c \notin V$. But $c \in V$ by hypothesis. So we have a contradiction based on the assumption that $[a,b]$ and $[a,c]$ have a common point different from a.

LEMMA 20: Let W be a convex body with interior U and frontier V. If $a \in U$ and c is any point of W, then there is a point $b \in V$ such that the segment $[a,b]$ contains c.

Proof: a) Let $a \in U$ and let c be any point of W distinct from a. We shall determine the segment $[a,b]$, $b \in V$, which contains c. We set $c - a = v$, and consider the set of all points y of R^n of the form $y = a + tv$, $t \geq 0$. If t is sufficiently small, y is evidently in U, since a is an interior point of W. On the other hand, if t is sufficiently large, y cannot be in W since W is compact and hence bounded. Hence the compactness of W implies that there is a largest positive value $t = t_0$ for which $y \in W$. That is, $t_0 = \underset{(a+tv) \in W}{\text{l. u. b.}} \; t$. It is clear that $a + t_0 v = b$ is a frontier point of W, for otherwise t_0 would not be maximal. That is, if $a + t_0 v = b$ is not a frontier point of W then there exists an $\epsilon > 0$ such that the ϵ-neighborhood of $a + t_0 v$ contains only points of W. In particular, W will contain a point $a + t_0 v + \epsilon v = a + (t_0 + \epsilon) v \in W$. Thus $t = t_0$ would not be the largest positive value for which $y = a + tv \in W$. Thus b is a frontier point of W. Now since $c = a + v \in W$ it follows that $t_0 \geq 1$ for otherwise $c = a + 1(v)$ is a point of $[a,c] \subset W$ and t_0 is not maximal. If we set $t_0 v = u$ then the set of all points of $[a,b]$ can be written in the form $x = a + su$ where $0 \leq s \leq 1$. The point c is of the form $a + (1/t_0)u$, $0 < 1/t_0 \leq 1$, and therefore belongs to the segment $[a,b]$.

b) If $c = a$ the point c lies on any arbitrary segment $[a,b]$, $b \in V$.

<u>REMARK</u>: Since a convex body contains at least one interior point it contains an infinite number of interior points. All of these are on segments of the form $[a,b]$, where $b \in V$. Therefore V is nonvacuous.

<u>DEFINITION 26</u>: Let a_0, a_1, ..., a_r be a system of independent points of the n-dimensional linear Euclidean space R^n, $r \leq n$. The set of all the points x of the space R^n of the form

(11) $$x = \lambda^0 a_0 + \lambda^1 a_1 + \ldots + \lambda^r a_r$$

where λ^0, λ^1, ..., λ^r are real numbers which satisfy the conditions

(12) $$\lambda^0 + \lambda^1 + \ldots + \lambda^r = 1$$

(13) $$\lambda^i \geq 0, \quad i = 0, 1, \ldots, r,$$

is called an <u>r-dimensional simplex</u> or an <u>r-simplex</u>. We write $A^r = (a_0, a_1, \ldots, a_r)$. The ordered set λ^0, λ^1,

..., λ^r is a set of <u>barycentric</u> <u>co-ordinates</u> of x. The points a_0, a_1, ..., a_r of the simplex are called <u>vertices</u>.

<u>LEMMA</u> <u>21</u>: If $A^r = (a_0, a_1, \ldots, a_r)$ is a simplex in R^n consisting of the points of the form

(11) $$x = \lambda^0 a_0 + \lambda^1 a_1 + \ldots + \lambda^r a_r,$$

then for any $x \in A^r$ the numbers λ^0, λ^1, ..., λ^r are uniquely determined.

<u>Proof</u>: If

(14) $$x = \mu^0 a_0 + \mu^1 a_1 + \ldots + \mu^r a_r$$

where

(15) $$\mu^0 + \mu^1 + \ldots + \mu^r = 1$$

then subtraction of (11) from (14) yields $(\mu^0 - \lambda^0)a_0 + (\mu^1 - \lambda^1)a_1 + \ldots \quad (\mu^r - \lambda^r)a_r = 0$. But equations (12) and (15) imply that $(\mu^0 - \lambda^0) + (\mu^1 - \lambda^1) + \ldots + (\mu^r - \lambda^r) = 0$, and since the points a_0, a_1, ..., a_r are independent then $(\mu^i - \lambda^i) = 0$, $i = 0, 1, \ldots, r$. That is, $\mu^i = \lambda^i$, $i = 0, 1, \ldots, r$.

<u>DEFINITION</u> <u>27</u>: If u_0 and u_1 are two distinct points of R^n then $x = \frac{1}{2}(u_0 + u_1)$ is the point equidistant from u_0 and u_1 and is called the <u>midpoint</u> of the segment $[u_0, u_1]$.

<u>LEMMA</u> <u>21</u>: Every point x of the simplex A^r, which is not a vertex, is the midpoint of some segment whose end points belong to A^r.

<u>Proof</u>: Let $x = \lambda^0 a_0 + \lambda^1 a_1 + \ldots + \lambda^r a_r$ be any point of A^r but not a vertex of A^r. Then at least two of

its barycentric co-ordinates are greater than zero, because if only one of the barycentric co-ordinates is greater than zero then it must be equal to 1, in which case x would be a vertex. Also, if none of the barycentric co-ordinates is greater than zero we again have a contradiction because the sum of the barycentric co-ordinates would be zero and not 1. Let $\lambda^i > 0$ and $\lambda^j > 0$. Choose ϵ such that $0 < \epsilon < \lambda^i$ and $0 < \epsilon < \lambda^j$. Let $u_0 = x + \epsilon(a_i - a_j)$ and let $u_1 = x - \epsilon(a_i - a_j)$. Thus $u_0 = \lambda^0 a_0 + \lambda^1 a_1 + \ldots + (\lambda^i + \epsilon)a_i + \ldots + (\lambda^j - \epsilon)a_j + \ldots + \lambda^r a_r$. The sum of the barycentric co-ordinates of u_0 is 1 and yet, since $\lambda^j > \epsilon$, we know that each barycentric co-ordinate is positive. A similar situation obtains for u_1. Thus u_0 and u_1 both belong to A^r. Also we have $x = \frac{1}{2}(u_0 + u_1)$.

<u>LEMMA</u> <u>22</u>: A vertex of A^r is not the midpoint of any segment whose end points are in A^r.

<u>Proof</u>: The proof will be by proof of the contrapositive statement. Suppose some vertex of A^r, say a_k, is the midpoint of a segment whose end points are in A^r. Then $a_k = \frac{1}{2}(u_0 + u_1)$, where u_0 and u_1 are two distinct points of A^r and $u_0 = \lambda_0^0 a_0 + \lambda_0^1 a_1 + \ldots + \lambda_0^r a_r$ and $u_1 = \lambda_1^0 a_0 + \lambda_1^1 a_1 + \ldots + \lambda_1^r a_r$. Then there are two integers i, j, $i \neq j$, such that $\lambda_0^i > 0$ and $\lambda_1^j > 0$. Now by assumption, $a_k = \frac{1}{2}(u_0 + u_1) = \frac{1}{2}(\lambda_0^0 + \lambda_1^0)a_0 + \frac{1}{2}(\lambda_0^1 + \lambda_1^1)a_1 + \ldots + \frac{1}{2}(\lambda_0^r + \lambda_1^r)a_r$, where $\frac{1}{2}(\lambda_0^i + \lambda_1^i) > 0$ and $\frac{1}{2}(\lambda_0^j + \lambda_1^j) > 0$. But then although $a_k = 0(a_0) + 0(a_1) + \ldots + 1(a_k) + \ldots + 0(a_r)$, we

also have $a_k = \ldots + \frac{1}{2}(\lambda_0^i + \lambda_1^i)a_1 + \ldots + \frac{1}{2}(\lambda_0^j + \lambda_1^j)a_j + \ldots$. But by Lemma 21 the barycentric co-ordinates are uniquely determined. Hence we have a contradiction and the present lemma is proved.

<u>LEMMA</u> <u>23</u>: The vertices $a_0, a_1, \ldots, a_r$ are uniquely determined by the set A^r.

<u>Proof</u>: By the preceding two lemmas the vertices are precisely those points which are not midpoints of segments in A^r.

<u>REMARK</u>: By the definition of simplex, a 0-simplex (a_0) consists of the single point a_0. A 1-simplex (a_0, a_1) is the segment joining the points a_0 and a_1. A 2-simplex (a_0, a_1, a_2) is merely the triangle (including the interior) whose vertices are a_0, a_1, a_2. A 3-simplex (a_0, a_1, a_2, a_3) is the tetrahedron (including interior) whose vertices are a_0, a_1, a_2, a_3.

<u>LEMMA</u> <u>24</u>: A^r is a closed set.

<u>Proof</u>: Suppose $A^r = (a_0, a_1, \ldots, a_r) \subset R^n$. Let $\alpha_1, \alpha_2, \ldots, \alpha_r$ be an orthonormal basis in the space spanned by $(a_1 - a_0), (a_2 - a_0), \ldots, (a_r - a_0)$. Let $\alpha_1, \alpha_2, \ldots, \alpha_r, \epsilon_{r+1}, \epsilon_{r+2}, \ldots, \epsilon_n$ be an orthonormal basis for R^n.

a) Let x be some point not in the space spanned by $\alpha_1, \alpha_2, \ldots, \alpha_r$. Then $x = \lambda^1\alpha_1 + \lambda^2\alpha_2 + \ldots + \lambda^r\alpha_r + \mu^{r+1}\epsilon_{r+1} + \ldots + \mu^n\epsilon_n$, where some of the $\mu^i \neq 0$. Consider a positive number $\delta > 0$ such that $\delta < |\mu^i|$. Let $N_\delta(x)$ be a δ-neighborhood of x defined as:

all y such that $\rho(x,y) < \delta$. Now we assert that $N_\delta(x) \cap A^r = \emptyset$. This is true, for if we assume $y \in A^r$, so that $y = v^1\alpha_1 + v^2\alpha_2 + \dots + v^r\alpha_r + \eta^{r+1}\epsilon_{r+1} + \dots + \eta^n\epsilon_n$ where each $\eta^i = 0$, then $\rho(x,y) = \left[\sum_{i=1}^{r}(v^i - \lambda^i)^2 + \sum_{i=r+1}^{n}(\eta^i - \mu^i)^2\right]^{\frac{1}{2}} = \left[\sum_{i=1}^{r}(v^i - \lambda^i)^2 + \sum_{i=r+1}^{n}(\mu^i)^2\right]^{\frac{1}{2}} \geq \mu^i > \delta$. Thus y is not an element of $N_\delta(x)$. Therefore for any point x which is not in the space spanned by the vertices of A^r there is an open set containing x which contains no points of A^r.

b) Let x be a point in the space spanned by $(a_1 - a_0)$, $(a_2 - a_0)$, ..., $(a_r - a_0)$ but such that x is not contained in A^r. Obviously x is in the space spanned by $\alpha_1, \alpha_2, \dots, \alpha_r$. Now x may be expressed uniquely in the form $x = \lambda^0 a_0 + \lambda^1 a_1 + \dots + \lambda^r a_r$ provided that $\lambda^0 + \lambda^1 + \dots + \lambda^r = 1$. Then since x is not an element of A^r we must have at least one of the coefficients $\lambda^0, \lambda^1, \dots, \lambda^r$ less than zero. Let λ^i be less than zero. Since each element $a_0, a_1, \dots, a_r$ is a vector in R^n we may express any one of them in the form $a_j = \sum_{i=1}^{r}\mu_j^i\alpha_i$, $j = 0, 1, \dots, r$.

Then we have the relation $x = \lambda^0\sum_{i=1}^{r}\mu_0^i\alpha_i + \lambda^1\sum_{i=1}^{r}\mu_1^i\alpha_i + \dots + \lambda^r\sum_{i=1}^{r}\mu_r^i\alpha_i = \sum_{i=1}^{r}\xi^i\alpha_i$. This may be written:

$$\lambda^0\mu_0^1 + \lambda^1\mu_1^1 + \dots + \lambda^r\mu_r^1 = \xi^1$$

$$\lambda^0\mu_0^2 + \lambda^1\mu_1^2 + \dots + \lambda^r\mu_r^2 = \xi^2$$

$$\dots$$

$$\lambda^0\mu_0^r + \lambda^1\mu_1^r + \dots + \lambda^r\mu_r^r = \xi^r.$$

We also have the relation $\lambda^0 + \lambda^1 + \dots + \lambda^r = 1$. The above $r + 1$ relations in $r + 1$ unknowns (i.e., the unknowns $\lambda^0, \lambda^1, \dots, \lambda^r$) are soluble in terms of $\xi^1, \xi^2, \dots, \xi^r$. Thus each λ^i, $i = 0, 1, \dots, r$, is a linear and therefore a continuous function of the set $\xi^1, \xi^2, \dots, \xi^r$. Now if $y = \sum_{h=1}^{r} \eta^h \alpha_h$ and if $\rho(x,y) = \left[\sum_{h=1}^{r} (\eta^h - \xi^h)\right]^{\frac{1}{2}} < \delta$, then we also have $\eta^h - \xi^h < \delta$ for each h. If y is expressed in terms of $a_0, a_1, \dots, a_r$ in the form $y = \sum_{h=0}^{r} \mu^h a_h$ with the restriction that $\mu^0 + \mu^1 + \dots + \mu^r = 1$, then it is possible to choose δ such that if $\rho(x,y) < \delta$ then $|\mu^i - \lambda^i| < |\lambda^i|$, that is, so that $\mu^i < 0$. Thus if y is sufficiently close to x we have y not in A^r. Hence for every point x in the complement of A^r in the space spanned by $a_0, a_1, \dots, a_r$ we may find an open set which does not intersect A^r.

By the arguments a) and b) above, the complement of A^r is open. Hence A^r is closed.

<u>DEFINITION</u> <u>28</u>: The <u>interior</u> G^r of the simplex A^r is the set

of points whose barycentric co-ordinates are positive.

LEMMA 25: The interior G^r of the simplex A^r is an open set in A^r.

Proof: Suppose $A^r = (a_0, a_1, \ldots, a_r) \subset R^m$, $m \geq r$. Let $\epsilon > 0$. Let $e_1, e_2, \ldots, e_n$ be an orthonormal basis for R^m. Consider a point x belonging to G^r. Suppose that $x = \sum_{i=0}^{r} \lambda^i a_i = \sum_{j=1}^{m} \xi^j e_j$. Now by Lemma 24 each λ^i is a continuous function of the set $\xi^1, \xi^2, \ldots, \xi^m$. Because there exists a positive number $\eta > 0$ such that if $|\psi^i - \xi^i| < \eta$ for $i = 1, 2, \ldots, m$ and if $z = \sum_{i=1}^{m} \psi^i e_i = \sum_{i=0}^{r} \nu^i a_i$ then $|\nu^i - \lambda^i| < \epsilon$. Because there exists $\delta > 0$ such that if $\rho(x,y) < \delta$ and if $y = \sum_{i=0}^{m} \phi^i e_i$, then $|\phi^i - \xi^i| < \eta$ for $i = 1, 2, \ldots, m$. Let $N = \{y$ such that $\rho(x,y) < \delta\}$. Suppose $y \in N \cap A^r$. Then if $y = \sum_{i=0}^{r} \mu^i a_i$ we have $\mu^i > 0$, for $i = 0, 1, \ldots, r$. Thus if $\rho(x,y) < \delta$ we have the relation that y is an element of G^r. Thus G^r is an open set in A^r.

DEFINITION 29: If G^r is the interior of an r-simplex G^r is an r-dimensional open simplex (or an open r-simplex).

DEFINITION 30: Let $A^r - G^r = F^{r-1}$. Then F^{r-1} is called the frontier of A^r.

LEMMA 26: $\overline{G^r} = A^r$.

Proof: a) $G^r \subset A^r$ and therefore $\overline{G^r} \subset \overline{A^r} = A^r$.

b) Let $x \in A^r$. If $x \in G^r$ then certainly $x \in \overline{G^r}$ since $G^r \subset \overline{G^r}$. Now suppose that $x \in A^r - G^r$. We wish to show that within any neighborhood of x there is a point of G^r and that consequently $x \in \overline{G^r}$. Let $x = \sum_{i=0}^{r} \lambda^i a_i$, with some nonempty subset of the set of coefficients bearing the relation: $\lambda^{i_1} = \lambda^{i_2} = \ldots = \lambda^{i_k} = 0$. Now by a suitable renaming of the a_i's we may obtain the following relation: $x = \sum_{j=0}^{r} \lambda^j a_j$, where $\lambda^j = 0$, for $j = 0, 1, \ldots, k-1$, and where $\lambda^j > 0$, for $j = k, k+1, \ldots, r$, and $\sum_{j=k}^{r} \lambda^j = 1$. Suppose $\epsilon > 0$. Now let $y = x + (\epsilon/k)(a_0 + a_1 + \ldots + a_{k-1}) - [\epsilon/(r+1-k)][a_k + a_{k+1} + \ldots + a_r]$. In this expression for y the sum of the coefficients is 1. If $[\epsilon/(r+1-k)]$ is less than the minimum of λ^k, λ^{k+1}, $\ldots$, λ^r then all of the coefficients of y are positive. Therefore $y \in G^r$. But $y = \sum_{i=0}^{r} \mu^i a_i = \sum_{i=0}^{r} \eta^i e_i$ where e_0, $e_1, \ldots, e_r$ is an orthonormal basis for the space R^r. For a given ϵ each η^i depends continuously on the set $\mu^0, \mu^1, \ldots, \mu^r$. Now since $\rho(x,y)$ depends continuously on the η^i then $\rho(x,y)$ depends continuously on the set $\mu^0, \mu^1, \ldots, \mu^r$. If ϵ is a small enough number then $\rho(x,y)$ is arbitrarily small. Thus we may always find a point of G^r arbitrarily near to x. Therefore $x \in \overline{G^r}$ and finally, $A^r \subset \overline{G^r}$.

c) Since $A^r \subset \overline{G^r}$ and $\overline{G^r} \subset A^r$, then $A^r = \overline{G^r}$.

LEMMA 27: A^r is a compact set.

Proof: A^r is a closed, bounded subset of Euclidean n-space for $r \leq n$.

LEMMA 28: F^{r-1} is closed in A^r.

Proof: The complement of F^{r-1} is the open set G^r.

LEMMA 29: If G^r and H^r are two open simplexes which coincide then the corresponding closed simplexes A^r and B^r coincide. That is to say that an open simplex determines its vertices uniquely.

DEFINITION 31: A topological space X is a collection of points such that certain sets of these points are distinguished and called open sets with the requirements that:

a) the union of a collection of open sets is an open set,

b) the intersection of a finite collection of open sets is an open set,

c) the empty subset $\emptyset$ and X are open sets.

DEFINITION 32: A (single-valued) function f defined on a set A to a set B is a correspondence which assigns to every point x of A a (unique) point f(x) of B. If T is a subset of A then f(T) in B is the set of all f(a) for a in T. If S is a subset of B, then $f^{-1}(S)$ is the set of all a in A such that f(a) is in S. We call $f^{-1}(S)$ the inverse image of S.

We sometimes call a function a <u>mapping</u>.

<u>DEFINITION 33</u>: A mapping f of a topological space X into a topological space Y is a <u>continuous function</u> (or: <u>continuous mapping</u>) if the inverse image of each open set in Y is an open set in X.

<u>DEFINITION 34</u>: A topological space is a <u>Hausdorff space</u> (T_2-space, separated space) if whenever x and y are distinct points of the space there exist disjoint open sets U_x, U_y containing x and y respectively.

<u>DEFINITION 35</u>: A <u>homeomorphism</u> is a continuous one-to-one mapping f of a topological space X onto a topological space Y such that f^{-1} is also a continuous mapping.

<u>LEMMA 30</u>: If f is a continuous function which maps the compact topological space X onto the topological space Y then Y is compact. Furthermore, if Y is a Hausdorff space and if f is one-to-one then f is a homeomorphism.

<u>Proof</u>: Suppose $\mathfrak{A}$ is an open covering of Y. Since f is a continuous mapping then the family of all sets of the form $f^{-1}(A)$, where A is an element of $\mathfrak{A}$, is a collection $\mathfrak{B}$ of open sets in X. $\mathfrak{B}$ covers X, since otherwise there would be points x of X which map into points f(x) of Y but such that f(x) is not in some set of $\mathfrak{A}$. Thus $\mathfrak{B}$ is an open covering of X. Since X is a compact set we may select a finite open covering of X from among the sets of $\mathfrak{B}$. Call this finite open covering of X: $\mathfrak{B}'$. Consider the family of images of the type f(B), where B is a set in the collection

$\mathcal{B}'$. This collection of images forms a subfamily $\mathcal{O}\!\mathcal{L}'$ of the family $\mathcal{O}\!\mathcal{L}$. The finite collection $\mathcal{O}\!\mathcal{L}'$ covers Y because otherwise there would be a point y of Y whose inverse $f^{-1}(y)$ is in some set of the collection $\mathcal{B}'$ but such that $y = f\left[f^{-1}(y)\right]$ is not in some set of the collection $\mathcal{O}\!\mathcal{L}'$. Hence Y is compact. Now suppose that Y is a Hausdorff space and f is a one-to-one mapping. If A is a closed subset of X then A is compact by Lemma 16. By the first part of the present lemma f(A) is a compact subset of Y. By Lemma 15 f(A) is closed. Then $(f^{-1})^{-1}(A) = f(A)$ is closed for each closed set A. Since for any open set $B \subset X$, the set A = (complement of B) is closed, then $(f^{-1})^{-1}(A)$ is closed and the complement of $(f^{-1})^{-1}(A)$ is equal to $(f^{-1})^{-1}(B)$ and is open since the image is one-to-one. Therefore f^{-1} is continuous. By Definition 35 f is a homeomorphism.

<u>LEMMA</u> <u>31</u>: Every two r-simplexes are homeomorphic.

<u>Proof</u>: Let R^{r+1} be the $(r+1)$-dimensional linear Euclidean vector space. Let $e_0, e_1, \ldots, e_r$ be an orthonormal system of vectors in R^{r+1}. Let $E^r = (e_0, e_1, \ldots, e_r) \subset R^{r+1}$ be the r-simplex with the points $e_0, e_1, \ldots, e_r$ as vertices. Now every point z of E^r is of the form $z = \lambda^0 e_0 + \lambda^1 e_1 + \ldots + \lambda^r e_r$ such that $\lambda^0 + \lambda^1 + \ldots + \lambda^r = 1$, and such that $\lambda^i \geq 0$, $i = 0, 1, \ldots, r$. Since $e_0, e_1, \ldots, e_r$ is an orthonormal system, the Euclidean co-ordinates of the point z relative to this system are the same as its barycentric co-ordinates in E^r. Thus we may

write $z = (\lambda^0, \lambda^1, \ldots, \lambda^r) = \lambda$ and $y = (\mu^0, \mu^1, \ldots, \mu^r) = \mu$. The distance between the two points λ and μ of E^r is $\rho(\lambda, \mu) = \left[(\lambda^0 - \mu^0)^2 + (\lambda^1 - \mu^1)^2 + \ldots + (\lambda^r - \mu^r)^2\right]^{\frac{1}{2}}$. This resulting metric defined for the points of the r-simplex E^r is a function of the intrinsic barycentric co-ordinates of the simplex. For any point x belonging to an r-simplex A^r there is exactly one point λ of E^r. That is to say, the mapping from A^r to E^r defined by the co-ordinates is continuous and one-to-one. By Lemma 30 we have then that the correspondence between A^r and E^r is a homeomorphism. Hence every two r-simplexes are homeomorphic and this homeomorphism may be achieved by the mapping which associates points having identical barycentric co-ordinates in the two simplexes.

<u>DEFINITION</u> <u>36</u>: The metric of Lemma 31 induced by the barycentric co-ordinates of the set E^r is called its <u>natural metric</u>.

<u>REMARK</u>: Let A^r $(a_0, a_1, \ldots, a_r)$ be an r-simplex in R^n, and let $\alpha_k = a_{i_k}$, $k = 0, 1, \ldots, s$; $0 \leq s \leq r$, be a subset of the vertices of A^r. Since the vertices $a_0, a_1, \ldots, a_r$ are independent, the vertices $\alpha_0, \alpha_1, \ldots, \alpha_s$ are also independent and hence $C^s = (\alpha_0, \alpha_1, \ldots, \alpha_s)$ is a simplex in R^n.

<u>DEFINITION</u> <u>37</u>: The simplex C^s defined in the preceding remark is an <u>s-dimensional</u> <u>face</u> (s-face) of the simplex A^r.

<u>LEMMA</u> <u>32</u>: $C^s \subset A^r$.

Proof: Let $A^r = (a_0, a_1, \ldots, a_r)$ be an r-simplex in R^n, and let $\alpha_k = a_{i_k}$, $k = 0, 1, \ldots, s$; $0 \leq s \leq r$, be a subset of the vertices. Let $C^s = (\alpha_0, \alpha_1, \ldots, \alpha_s)$ be a face of A^r. Let those of the numbers $0, 1, \ldots, r$ which are different from any of the set $i_0, i_1, \ldots, i_s$ be denoted by $j_1, j_2, \ldots, j_{r-s}$. Then an arbitrary point $x \in C^s$ is obtained by putting

(16) $\lambda^{j_k} = 0, \quad k = 1, 2, \ldots, r - s$

in the relation $x = \lambda^0 a_0 + \lambda^1 a_1 + \ldots + \lambda^r a_r$ which is part of the definition of an element of a simplex. Hence $C^s \subset A^r$.

LEMMA 33: C^s is determined in A^r by the system (16).

LEMMA 34: Every system such as (16) determines some face of A^r.

REMARK: Every vertex of a simplex A^r is a 0-face of A^r. A^r is its own r-face.

DEFINITION 38: A face of dimension less than r of an r-simplex A^r is a proper face.

DEFINITION 39: Two simplexes A and B of the linear Euclidean vector space R^n are said to be properly situated either if they are nonintersecting or if their intersection $A \cap B$ is a common face of A and B.

LEMMA 35: If C is a face of the simplex A, and if D is a face of the simplex B, with $A \cap B \subset C \cap D$ (that is, $A \cap B = C \cap D$), then the simplexes A and B are properly situated if and only if the faces C and D are properly situated simplexes.

LEMMA 36: Two faces of a simplex are always properly situated.

Proof: If C and D are two faces of the simplex A, then each of them is determined as the set of all x satisfying an equation of the form

(17) $$x = \lambda^0 a_0 + \lambda^1 a_1 + \dots + \lambda^r a_r$$

with

(18) $$\lambda^0 + \lambda^1 + \dots + \lambda^r = 1,$$

and

(19) $$\lambda^i \geq 0, \quad i = 0, 1, \dots, r$$

such that

(20) $$\lambda^{j_k} = 0, \quad k = 1, 2, \dots, r - s; \; s < r$$

for some $r - s$ indices j_k. The two systems which correspond to the two faces C and D respectively when combined yield either another system of the form (16) or they are inconsistent. If they yield another system of the form (16) then the relations which these two systems have in common define the dommon face of C and D. They are inconsistent if and only if the joint system contains all the relations $\lambda^i = 0$, $i = 0, 1, \dots, r$.

DEFINITION 40: A finite set K of simplexes of the linear Euclidean space R^n is a geometric complex (or merely: a complex) if K satisfies the two conditions:

a) if A is a simplex of K then every face of A is also in K

b) every two simplexes of K are properly situated.

<u>REMARK</u>: The set of 0-simplexes of a complex K is the set of all vertices of all simplexes contained in K.

<u>DEFINITION 41</u>: The maximum dimension of the simplexes of K is the <u>dimension of the complex</u> K. An n-dimensional complex K may be referred to as an n-complex.

<u>DEFINITION 42</u>: If K is a geometric complex situated in the linear Euclidean metric space R^n, then the set of all points contained in the simplexes of the complex K is called the <u>polyhedron</u> $|K|$.

<u>LEMMA 37</u>: $|K| \subset R^n$ is a metric space, provided K is non-vacuous.

<u>Proof</u>: $|K|$ is a subset of R^n which is a metric space. All properties of R^n considered as a metric space are hereditarily applicable in $|K|$ also.

<u>LEMMA 38</u>: $|K|$ is a compact space.

<u>Proof</u>: Since K is the set-theoretic union of a finite number of simplexes each of which is compact by Lemma 27 then the polyhedron $|K|$ consisting by definition of all the points contained in these simplexes is compact.

<u>REMARK</u>: If K and L are two complexes and f is a continuous mapping of the polyhedron $|K|$ into the polyhedron $|L|$ then we shall sometimes refer to f as a mapping of the complex K

into the complex L.

REMARK: The most elementary type of r-complex is the set T^r of all faces (proper or otherwise) of the simplex A^r. The set S^{r-1} of all proper faces of the simplex A^r also forms a complex.

LEMMA 39: $|T^r| = A^r$.

LEMMA 40: $|S^{r-1}| = F^{r-1}$.

LEMMA 41: In order to determine the complex K in R^n it is sufficient to list all the vertices of K and then to distinguish those sets of vertices whose spanning simplexes yield all the simplexes in K.

Proof: By the definition of geometric complex the vertices of each simplex in K are also vertices of K.

DEFINITION 43: A finite set $\mathcal{K}$ of elements α_0, α_1, ..., α_k is called an abstract complex with vertices α_0, α_1, ..., α_k in case $\mathcal{K}$ satisfies the conditions

a) certain nonempty subsets of the set $\mathcal{K}$ are distinguished and are called abstract simplexes of the abstract complex $\mathcal{K}$,

b) every subset of $\mathcal{K}$ consisting of a single element is a distinguished subset. Thus every vertex of $\mathcal{K}$ is also a simplex of $\mathcal{K}$.

c) If $\mathfrak{A}$ is a simplex of $\mathcal{K}$, then every nonempty subset of $\mathfrak{A}$, referred to as a face of the simplex $\mathfrak{A}$ is also a simplex of the complex $\mathcal{K}$.

The abstract simplex $\mathfrak{A}^r = (\alpha_0, \alpha_1, \ldots,$

σ_r) with $r + 1$ vertices is said to have dimension r. The maximum of the dimensions of the simplexes contained in the complex $\mathcal{K}$ is called the dimension of the complex $\mathcal{K}$. If the dimension of $\mathcal{K}$ is n then $\mathcal{K}$ is an n-complex.

DEFINITION 44: Let $\mathcal{K}$ be an abstract complex with vertices $\alpha_0, \alpha_1, \ldots, \alpha_k$, and let $E^k = (e_0, e_1, \ldots, e_k)$ be a k-simplex considered with its natural metric such that $E^k \subset R^{k+1}$. Let $\alpha_{i_j} = \vartheta_j$, $j = 0, 1, \ldots, r$, denote any subset of the vertices of $\mathcal{K}$, and let $e_{i_j} = v_j$, $j = 0, 1, \ldots, r$, denote the corresponding subset of the set of elements e_0, $e_1, \ldots, e_k$ of R^{r+1}. Assign the face $A^r = (v_0, v_1, \ldots, v_r)$ of the simplex E^k to the corresponding abstract simplex $\sigma^r = (\vartheta_0, \vartheta_1, \ldots, \vartheta_r)$ of the abstract complex $\mathcal{K}$. The set N of geometric simplexes thus obtained forms a geometric complex, since by Lemma 36 the faces of the simplex E^k are properly situated. The geometric complex N is called a geometric realization of the abstract complex $\mathcal{K}$. If the set $e_0, e_1, \ldots, e_k$ forms a basis of R^{r+1} then the geometric realization N is called the natural realization of the abstract complex $\mathcal{K}$, and the metric of the polyhedron $|N|$ induced by the metric of E^k will be called the natural metric corresponding to $\mathcal{K}$.

LEMMA 42: The point $\lambda^0 e_0 + \lambda^1 e_1 + \ldots + \lambda^k e_k = (\lambda^0, \lambda^1, \ldots, \lambda^k) = \lambda$ of the simplex E^k is contained in the polyhedron $|N|$ if and only if the relation that all co-ordinates equal zero except a certain set $\lambda^{i_j} \neq 0$, $j = 0, 1, \ldots, r$, implies that $\mathcal{K}$ contains the simplex $(\vartheta_0, \vartheta_1, \ldots, \vartheta_r)$.

<u>LEMMA</u> <u>43</u>: Every two geometric realizations of $\mathcal{K}$ are homeomorphic.

<u>Proof</u>: Let K be an arbitrary geometric realization of the abstract complex $\mathcal{K}$ in the Euclidean space R^m. Denote the vertices of K by $c_0, c_1, \ldots, c_k$, where c_i corresponds to $\mathcal{X}_i$. Assign to each point $\lambda = (\lambda^0, \lambda^1, \ldots, \lambda^k)$ of the polyhedron $|N|$ the point $\psi(\lambda) = \lambda^0 c_0 + \lambda^1 c_1 + \ldots + \lambda^k c_k \in R^m$. We shall show that the mapping ψ is a homeomorphism of $|N|$ onto $|K|$.

a) We first note that ψ is continuous.

b) We next show that the mapping from $|N|$ to $|K|$ is onto. Let $A^r = (v_0, v_1, \ldots, v_r)$ be a simplex of N and let $*A^r = (u_0, u_1, \ldots, u_r)$ be the corresponding simplex of K, where $u_i = c_{i_j}$, $j = 0, 1, \ldots, r$. Now if x is any point of A^r then x is of the form $x = \lambda^0 v_0 + \lambda^1 v_1 + \ldots + \lambda^r v_r$ and $\psi(x)$ is of the form $\psi(x) = \lambda^0 u_0 + \lambda^1 u_1 + \ldots + \lambda^r u_r$, which is obviously a uniquely determined element of $*A^r$. Conversely, any element $y = \mu^0 u_0 + \mu^1 u_1 + \ldots + \mu^r u_r \in *A^r$ has a unique inverse image in A^r; this unique inverse image is $\psi^{-1}(y) = \mu^0 v_0 + \mu^1 v_1 + \ldots + \mu^r v_r$. Since each simplex A^r in N has a corresponding simplex in K and conversely, it follows that $\psi(|N|) = |K|$ and incidentally that $\psi^{-1}(|K|) = |N|$ and hence that ψ^{-1} is a mapping from $|K|$ onto $|N|$.

c) It remains to be seen whether ψ is one-to-one from $|N|$ to $|K|$. To prove that each arbitrary point of $|K|$ has an inverse image of exactly one point, we let $\lambda = (\lambda^0, \lambda^1, \ldots, \lambda^k)$ and $\mu = (\mu^0, \mu^1, \ldots, \mu^k)$ be two points of $|N|$, and let λ^{i_q}, $q = 0, 1, \ldots, r$, and μ^{j_p}, $p = 0, 1, \ldots, s$, be the nonzero coefficients which appear in the expressions for λ and μ respectively. Let $e_{i_q} = t_q$, $q = 0, 1, \ldots, r$, and let $e_{j_p} = w_p$, $p = 0, 1, \ldots, s$. If $A^r = (t_0, t_1, \ldots, t_r)$ and $B^s = (w_0, w_1, \ldots, w_s)$ are the simplexes indicated in N, then obviously $\lambda \in A^r$ and $\mu \in B^s$, since λ and μ satisfy the conditions necessary for elements of the respective simplexes indicated. Indeed, λ is an element of the interior of A^r and μ is an element of the interior of B^s. Therefore by the definition of ψ and of $*A^r$ and $*B^s$, we know that $\psi(\lambda)$ and $\psi(\mu)$ are interior points of $*A^r$ and $*B^s$ respectively. Now $*A^r$ and $*B^s$ are properly situated since they are both simplexes in the complex K. Clearly if two properly situated simplexes have a common interior point then they coincide. Hence if $\psi(\lambda) = \psi(\mu)$ then $*A^r = *B^s$ and hence $A^r = B^s$. Now on any simplex $A^r = B^s$ we know that ψ is a one-to-one mapping onto the corresponding simplex $*A^r = *B^s$ in K. Therefore $\psi(\lambda) =$

$\psi(\mu)$ implies $\lambda = \mu$. Hence ψ is a one-to-one mapping. By Lemma 30 then ψ is a homeomorphism. Thus every geometric realization of the abstract complex $\mathcal{K}$ is homeomorphic to the natural realization N of $\mathcal{K}$, and therefore every two geometric realizations of $\mathcal{K}$ are homeomorphic.

<u>THEOREM</u> <u>4</u>: An abstract n-complex $\mathcal{K}$ can always be realized by a geometric complex K imbedded in the Euclidean space R^{2n+1} of dimension $2n+1$. To achieve this realization the vertices of the complex K may be chosen arbitrarily with the sole requirement that they be in general position.

<u>Proof</u>: If $\alpha_0, \alpha_1, \ldots, \alpha_k$ are the vertices of the abstract complex $\mathcal{K}$, let $c_0, c_1, \ldots, c_k$ be any system of points in general position in R^{2n+1} and consider the correspondence between α_i and c_i. If $\mathfrak{A}^r = (\alpha_0, \alpha_1, \ldots, \alpha_r)$ is any abstract simplex of $\mathcal{K}$, let $A^r = (a_0, a_1, \ldots, a_r)$ be the geometric simplex which spans the points $a_0, a_1, \ldots, a_r$ which in turn correspond to the vertices $\alpha_0, \alpha_1, \ldots, \alpha_r$. Because of condition c) of the definition of abstract complex we know that K fulfills condition a) of the definition of geometric complex. To show that K is a geometric complex it is only necessary then to show that K satisfies condition b) of the definition of a geometric complex. To do this let $\mathfrak{A}^r$ and $\mathfrak{B}^s$ be two simplexes of $\mathcal{K}$. Let A^r and B^s be the corresponding geometric simplexes of K, and let $d_0, d_1, \ldots, d_t$ be the set of all points which are vertices of one or more of the simplexes A^r and B^s. Now $r \leq n$ and $s \leq n$ be-

cause the dimension of $\mathcal{K}$ is n. Therefore $t \leq 2n + 1$. Thus $D^t = (d_0, d_1, \ldots, d_t)$ is a simplex in R^{2n+1}. D^t may or may not be in K. Since A^r and B^s are faces of D^t they are properly situated by Lemma 36. Therefore K also satisfies condition b) of the definition of geometric complex. Thus K is a geometric complex which realizes $\mathcal{K}$.

<u>LEMMA</u> <u>44</u>: If R is a linear Euclidean metric space, A is a set of points in R, and δ is a positive number, let $H(A,\delta)$ denote the set of all points of R whose distance from the set A is less than δ. Let $\overline{H}(A,\delta)$ denote the set of all points of R whose distance from the set A is less than or equal to δ. Then

a) $\overline{H(A,\delta)} \subset \overline{H}(A,\delta)$

b) $H(A,\delta)$ is open

c) $\overline{H}(A,\delta)$ is closed

d) if A consists of one point a then the diameter of $H(a,\delta)$ and the diameter of $\overline{H}(a,\delta)$ do not exceed 2δ.

<u>DEFINITION</u> <u>45</u>: Let R be a metric space and let $\Sigma = \{c_0, c_1, \ldots, c_k\}$ be a finite system of nonvacuous subsets of R. The system Σ is a <u>covering</u> of R if every point of R is contained in at least one of the sets of the system Σ. If the diameter of each set of the system Σ is less than some positive number ϵ then Σ is an <u>ϵ-covering</u> of the space R. We shall consider <u>open coverings</u>, which consist solely of open sets, and <u>closed coverings</u>, which consist solely of

closed sets. A covering is of order n if n is the maximum number of sets in which any point is contained and if this maximum is achieved by at least one point.

REMARK: Although the term "covering" has been used previously in a broader sense, for the present we shall limit its use to designate only finite coverings.

LEMMA 45: If R is a compact metric space and ϵ is a positive number then R has both an open and a closed ϵ-covering.

Proof: $R = \bigcup_{x \in R} H(x, \epsilon/3)$. Since R is compact, we may select a finite number of the sets in this union, say: $H(x_0, \epsilon/3), H(x_1, \epsilon/3), \ldots, H(x_k, \epsilon/3)$ which will form an open ϵ-covering of R. And since by Lemma 44 we know that $H(x_i, \epsilon/3) \quad \overline{H(x_i, \epsilon/3)} \quad \overline{H}(x_i, \epsilon/3)$, then we may let $\overline{H}(x_0, \epsilon/3), \overline{H}(x_1, \epsilon/3), \ldots, \overline{H}(x_k, \epsilon/3)$ be a closed covering of R.

DEFINITION 46: A compact metric space R has finite dimension r if the following conditions are satisfied:

a) For every $\epsilon > 0$ there exists a closed ϵ-covering of R of order less than or equal to $r + 1$.

b) There exists an $\epsilon > 0$ such that every closed ϵ-covering of R is of order greater than r. If no integer $r \geq 0$ exists which satisfies the conditions a) and b) then we say that R is of infinite dimension.

LEMMA 46: Let R and R' be two compact metric spaces and let f be a homeomorphism of R onto R'. Then R and R' are of equal dimension.

<u>Proof</u>: Since R is compact, f is uniformly continuous, because a continuous function from a compact metric space to a metric space is uniformly continuous. Hence for every positive number ϵ there exists a number δ such that when $\rho(x,y) < \delta$ then $\rho[f(x),f(y)] < \epsilon$. Now if $\Sigma = \{C_0, C_1, \ldots, C_k\}$ is a closed δ-covering of R, then the sets $f(C_i) = C_i'$, $i = 0, 1, \ldots, k$, form a closed ϵ-covering Σ' of R'. Since f is one-to-one, then Σ and Σ' are of the same order, and therefore the dimension of R' does not exceed the dimension of R. And since R and R' may be interchanged in this argument, they are of equal dimension.

<u>LEMMA</u> <u>47</u>: If R is a compact metric space and $\Sigma = \{C_0, C_1, \ldots, C_k\}$ is a closed ϵ-covering of R, then there exists a positive number δ such that the closed covering $\overline{\Sigma}_\delta$ consisting of the sets $F_i = \overline{H}(C_i, \delta)$, $i = 0, 1, \ldots, k$, is an ϵ-covering of the same order as the covering Σ. This implies that the open covering Σ_δ consisting of the sets $G_i = H(C_i, \delta)$, $i = 0, 1, \ldots, k$, is an open ϵ-covering of the same order as Σ.

<u>Proof</u>: Let $C_{p_i} = \Gamma_i$, $i = 0, 1, \ldots, n$, be any system Σ' of sets whose intersection is empty. Let $F_{p_i} = \Phi_i$, $i = 0, 1, \ldots, n$. We shall first show that there is a positive number η for which the intersection of the sets of the

sets of the system $\Sigma'_\eta = \{\Phi_0, \Phi_1, \ldots, \Phi_n\}$ is also empty. If the contrary were true, that is, if now such number η existed then for each positive integer m and for each $\eta = 1/m$ there would exist a point a_m which belonged to each set of the system $\overline{\Sigma}_\eta$. The sequence $\{a_i\}_{i = 1, 2, \ldots}$ would have a limit point since R is compact and since any infinite sequence in a compact space has at least one limit point. Let one of these limit points be denoted by a. Obviously a is in each set of the system $\overline{\Sigma}'$ since each set is a closed set. By the preceding argument we may assign a sufficiently small number $\eta > 0$ to each subsystem of the form Σ' of the system Σ, such that the sets of the system $\overline{\Sigma}'_\eta$ have a void intersection. Now Σ has only a finite number of subsets of the form Σ' and we can therefore select the smallest η for our δ. This value for δ will be sufficiently small to imply that the order of $\overline{\Sigma}_\delta$ is the same as the order of Σ. Furthermore, it is possible to choose the number δ small enough to insure that the diameter of each set F_i, $i = 0, 1, \ldots, k$, will be less than ϵ, because the diameter of each set C_i is less than ϵ. Thus the system $\overline{\Sigma}_\delta$ is an ϵ-covering.

<u>COROLLARY</u>: If the dimension of the compact metric space R is equal to r, then for every $\epsilon > 0$ there exists an open ϵ-covering of R whose order does not exceed $r + 1$.

<u>DEFINITION</u> <u>47</u>: Let $\Sigma = \{C_0, C_1, \ldots, C_k\}$ be a system of sets of the space R. With each set C_i we associate the

letter α_i and we let the set of letters $\alpha_0, \alpha_1, \ldots, \alpha_k$ represent the vertices of an abstract complex $\mathcal{K}$. We allow the subset of vertices α_{i_j}, $j = 0, 1, \ldots, s$, to define a simplex of $\mathcal{K}$ if and only if the sets C_{i_j}, $j = 0, 1, \ldots, s$, have a nonempty intersection. The abstract complex $\mathcal{K}$ is the <u>nerve</u> of the system Σ.

<u>LEMMA</u> <u>48</u>: If the system Σ is of order $r + 1$ then the nerve of Σ is of dimension r.

<u>DEFINITION</u> <u>48</u>: A continuous mapping f of a metric space R into a metric space S is called an <u>ϵ-mapping</u> if the complete inverse image $f^{-1}(z)$ of every point $z \in f(R)$ is of diameter less than ϵ in R.

<u>THEOREM</u> <u>5</u>: Let R be a compact metric space, $\Sigma = \{G_0, G_1, \ldots, G_k\}$ an open ϵ-covering of R, $\mathcal{K}$ the nerve of this covering, and K a geometric realization of $\mathcal{K}$ in some Euclidean space R^m, so that to each set G_i of Σ there corresponds a point c_i of R^m which is a vertex of the geometric nerve $\mathcal{K}$ of the covering Σ. Then there is a continuous ϵ-mapping f of R into $|K|$ for which $x \in G_p$ implies that $f(x)$ is contained in a simplex A^* of K with vertex c_p.

<u>Proof</u>: We define the real-valued function $\psi_i(x)$, $x \in R$, $i = 0, 1, \ldots, k$, to be the distance between the point x and the closed set $R - G_i$. The function $\psi_i(x)$ is a continuous function of R and is positive if and only if $x \in G_i$. The function $\psi_i(x) = 0$ if $x \in R - G_i$. Since every

point x is contained in at least one of the sets G_i of the system Σ, then the sum $\psi(x) = \psi_0(x) + \psi_1(x) + \ldots + \psi_k(x)$ is positive for every x. Let $\lambda^i(x) = \psi_i(x)/\psi(x)$. Then the function $\lambda^i(x)$ has the properties listed above for ψ_i, and also we have

(21) $$\lambda^0(x) + \lambda^1(x) + \ldots + \lambda^k(x) = 1.$$

Now let N be the natural realization of the abstract complex $\mathcal{K}$, and assign to each point $x \in R$ the point

(22) $$\lambda(x) = \lambda^0(x)e_0 + \lambda^1(x)e_1 + \ldots + \lambda^k(x)e_k$$

of the Euclidean space R^{k+1}. By relation (21) and the nonnegativeness of $\lambda^i(x)$ we see that $\lambda(x)$ is contained in the simplex $E^k \subset R^{k+1}$. It will be shown that $\lambda(x)$ is contained in the polyhedron $|N|$. Let $x \in R$ and denote by $\Sigma_x = \{G_{i_j},\ j = 0, 1, \ldots, r\}$ the set of all open sets of the system Σ which contain s. Because the open sets of the system Σ_x have a nonvacuous intersection, the simplex $\mathfrak{A} = (\alpha_{i_0}, \alpha_{i_1}, \ldots, \alpha_{i_r})$ is in $\mathcal{K}$ and hence the simplex A = $(e_{i_0}, e_{i_1}, \ldots, e_{i_r})$ is in N. However, λ^{i_j}, $j = 0, 1, \ldots,$ r, are those numbers of the sequence $\lambda^0(x), \lambda^1(x), \ldots, \lambda^k(x)$ which are nonzero, and therefore by (22) $\lambda(x) = \sum_{j=0}^{r} \lambda^{i_j} e_{i_j}$. This means that $\lambda(x) \in A \subset N$. Furthermore, if $x \in G_p$, then G_p is a set of the system Σ_x and e_p is a vertex of A. To show that the mapping λ of the space R into the polyhedron $|N|$ is an ϵ-mapping, let z be a point of $\lambda(R)$, let $\lambda^{-1}(z)$ be its complete inverse image in R, and let $x \in \lambda^{-1}(z)$. Some set of Σ, say G_p, contains x, and hence $\lambda^p(x) \neq 0$. If $y \in \lambda^{-1}(z)$ then $\lambda(x) = \lambda(y)$ and by relation (22) $\lambda^p(y) = \lambda^p(x) \neq 0$. Then $y \in G_p$. That is, $\lambda^{-1}(z) \subset G_p$; also since the diameter of G_p is less than ϵ and $\lambda^{-1}(z) \subset G_p$ then the diameter of $\lambda^{-1}(z)$ is also less than ϵ. Let ψ be the natural mapping from N to K. That is, $\psi \sum_{i=0}^{k} \lambda^i e_i = \sum_{i=0}^{k} \lambda^i c_i$, where $c_0, c_1, \ldots, c_k$ are the vertices of K. Define f to be the mapping from R to K such that $f(x) = \psi[\lambda(x)]$. Since ψ is a homeomorphism of N onto K, f satisfies the requirement of the theorem. Indeed, the mapping of R into $|K|$ is given by the equation $f(x) = \lambda^0(x)c_0 + \lambda^1(x)c_1 + \ldots + \lambda^k(x)c_k$.

DEFINITION 49: Let R be a metric space. A sequence $\{a_i\}_{i = 0, 1, \ldots}$ of points of R is called a Cauchy sequence if for every positive number $\epsilon > 0$ there exists a positive integer N such that $\rho(a_p, a_q) < \epsilon$ whenever $p, q > N$.

DEFINITION 50: A sequence $\{a_i\}_{i = 0, 1, \ldots}$ converges to the point p if for every $\epsilon > 0$ there exists a positive integer $N(\epsilon)$ such that if $n > N(\epsilon)$ then $\rho(a_n, p) < \epsilon$.

LEMMA 49: A Cauchy sequence cannot converge to more than one point.

DEFINITION 51: If every Cauchy sequence in R converges to a point in R then R is complete. If some Cauchy sequence in R does not converge to a point in R then R is an <u>incomplete</u> space.

<u>LEMMA 50</u>: A compact metric space is a complete space.

<u>LEMMA 51</u>: A Euclidean space is complete.

<u>LEMMA 52</u>: Suppose $R = G_0, G_1, \ldots, G_m, \ldots$ is a sequence of open sets of a complete metric space, each set of the sequence being everywhere dense in R. Then the intersection of all of these open sets is nonvacuous and is itself everywhere dense in R.

<u>Proof</u>: (To prove this lemma we shall let a_0 be any point of $G_0 = R$ and we shall let $\epsilon_0 > 0$ be any positive number. We shall show that there is a point a contained in all of the sets G_i, $i = 0, 1, \ldots$ such that $\rho(a_0, a) \leq \epsilon_0$. Since the point a_0 is any point of the space and a point auch as a can be exhibited to be an element of the set $F = \bigcap_{i=0,1,\ldots} G_i$ and to be arbitrarily close to, that is, within ϵ distance of, that point a_0 of R, then F is a set which is everywhere dense in R.) Suppose that the finite sequence of points $a_0, a_1, \ldots, a_n$ of the space R and that also the finite sequence of numbers $\epsilon_0, \epsilon_1, \ldots, \epsilon_n$ have been constructed to satisfy the following requirements:

a) $0 < \epsilon_0 < 1$, $\epsilon_i < 1/i$; $i = 1, 2, \ldots, n.$

b) $\overline{H}(a_i, \epsilon_i) \subset H(a_{i-1}, \epsilon_{i-1}) \cap G_i$, $i = 1, 2, \ldots, n.$

We shall extend the sequences $\{a_i\}$ and $\{\epsilon_i\}$.

Since G_{n+1} is dense in R then there exists a point a_{n+1} which belongs to $H(a_n, \epsilon_n) \cap G_{n+1}$; and since the intersection of two open sets is again an open set then $H(a_n, \epsilon_n) \cap G_{n+1}$ is an open set. Now if $H(a_n, \epsilon_n) \cap G_{n+1}$ is an open set then there exists a positive number $\epsilon_{n+1} < 1/(n+1)$ such that $\bar{H}(a_{n+1}, \epsilon_{n+1}) \subset H(a_n, \epsilon_n) \cap G_{n+1}$. Therefore the sequences $\{a_i\}$ and $\{\epsilon_i\}$ are infinite and they satisfy the condition b). Now if $p < q$, then $a_q \in H(a_p, \epsilon_p)$. Therefore $\rho(a_p, a_q) < \epsilon_p < 1/p$. Hence $\{a_i\}_{i = 0, 1, \ldots, m, \ldots}$ is a Cauchy sequence, and since R is a complete space by hypothesis we know by definition of complete that $\{a_i\}$ converges to some point $a \in R$. Obviously the sequence $\{a_i\}_{i = m, m+1, \ldots}$ also converges to the same point a. And furthermore this last sequence is contained in the closed set $\bar{H}(a_m, \epsilon_m)$. Therefore $a \in \bar{H}(a_m, \epsilon_m)$ and a is an element of each set G_m for $m = 0, 1, \ldots$. Moreover, because $a \in \bar{H}(a_0, \epsilon_0)$ we know that $\rho(a_0, a) \leq \epsilon_0$. Thus we have found a point a which is in every set G_m and which is arbitrarily close to an arbitrarily chosen point a_0. Thus the set $\bigcap_{i = 1, 2, \ldots} G_i$ is nonvacuous and is indeed dense in R.

<u>LEMMA</u> <u>53</u>: A continuous real-valued function defined on a compact space attains its maximum.

<u>DEFINITION</u> <u>52</u>: Let S be a metric space, R a compact metric space, and $\Phi(R,S)$ the set of all continuous mappings of R into S. If f and g are two mappings of Φ, then $\rho[f(x), g(x)]$ is a continuous real-valued function defined on the

compact space R and by Lemma 53 therefore attains its maximum. This maximum, denoted by $\rho(f,g)$, is the <u>distance between the two elements f and g in the space of continuous mappings Φ</u>.

<u>LEMMA 54</u>: The function $\rho(f,g)$ as defined above satisfies all of the axioms of a metric, and Φ is therefore a metric space. Furthermore, if S is complete then Φ is complete.

<u>Proof</u>: We first prove that Φ is a metric space. Clearly, $\rho(f,g) = 0$ if and only if $f = g$. Also $\rho(f,g) = \rho(g,f)$. Now let a be a point of R for which $\rho[f(x),h(x)]$ attains its maximum. Then $\rho[(f,h)] = \rho[f(a),h(a)] \leq \rho[f(a),g(a)] + \rho[g(a),h(a)] \leq \rho[(f,g)] + \rho[(g,h)]$. Thus the metric axioms are satisfied for Φ and Φ is a metric space. We next show that if S is complete then Φ is also complete. If f_0, f_1, ..., is a Cauchy sequence of Φ then for every positive ϵ there exists a positive integer N such that if m, n $>$ N then $\rho(f_m,f_n) < \epsilon$. Hence $\rho[f_m(x),f_n(x)] < \epsilon$ for any point $x \in R$. Therefore $f_0(x)$, $f_1(x)$, ... is a Cauchy sequence in the complete space S, and therefore converges to a point of S which we denote by f(x). Thus for m $>$ N we have

$$\rho[f_m(x),f(x)] \leq \epsilon, \quad \text{for all } x. \tag{23}$$

We now show that f is a continuous mapping of R into S. Since f_m is a continuous function at x, there exists a positive number δ such that $\rho[f_m(x),f_m(y)] < \epsilon$ for $\rho(x,y) < \delta$. But since relation (23) holds for the point y as well as for x, we have $\rho[f(x),f(y)] \leq \rho[f(x),f_m(x)] + \rho[f_m(x),f_m(y)] + \rho[f_m(y),f(y)] < 3\epsilon$. Therefore f is a continuous mapping and $f \in \Phi$. Furthermore, since relation (23) implies that $\rho(f_m,f) \leq \epsilon$ for $m > N$, the sequence f_0, f_1, ..., converges to f.

<u>REMARK</u>: It may be noted that the last point proved is a generalization of the theorem that a uniformly convergent sequence of continuous functions converges to a continuous function.

<u>LEMMA</u> <u>55</u>: If f is a continuous function from a compact metric space R to a metric space S then the complete inverse image of a point in S is a compact set in R.

<u>LEMMA</u> <u>56</u>: Let $\Phi(R,S)$ be the metric space of all continuous mappings of a compact metric space R into an arbitrary metric space S. Let Φ_ϵ be the set of all ϵ-mappings belonging to Φ. If $f \in \Phi_\epsilon$, then there exists a positive number δ_ϵ such that $x, y \in R$ and $\rho[f(x),f(y)] < \delta_\epsilon$ imply together that $\rho(x,y) < \epsilon$.

<u>Proof</u>: We shall prove the contrapositive statement. If there does not exist such a number δ_ϵ, then there is a sequence of positive numbers δ_1, δ_2, ..., δ_m, ..., which converges to zero, and a pair of points x_m, y_m for each positive integer m such that although $\rho[f(x_m),f(y_m)] < \delta_m$, it is nevertheless true also that $\rho(x_m,y_m) \geq \epsilon$. Now since R is a compact space, a convergent subsequence may be chosen from each of the two sequences $\{x_i\}$ and $\{y_i\}$, such that

these two convergent subsequences converge to points x and y respectively. Then we have two points x and y such that $\rho[f(x),f(y)] = 0$ and yet $\rho(x,y) \geq \epsilon$. This situation is in contradiction to the assumption that f was an ϵ-mapping.

LEMMA 57: If $\Phi(R,S)$ is the metric space of all continuous mappings of a compact metric space R into an arbitrary metric space S and Φ_ϵ is the set of all ϵ-mappings which all elements in $\Phi(R,S)$, then Φ_ϵ is an open set in $\Phi(R,S)$.

Proof: Assume that f is an element of Φ_ϵ. We shall show that if $\rho(f,g) < \delta/2$ then g is an ϵ-mapping. This will be sufficient to show that Φ_ϵ is an open set in Φ. Now suppose that g is an element of Φ such that $\rho(f,g) < \delta/2$. If $g(x) = g(y) = z$ then $\rho[f(x),f(y)] \leq \rho[f(x), g(x)] + \rho[g(y),f(y)] < \delta/2 + \delta/2 = \delta$. Since $f \in \Phi_\epsilon$ then $\rho(x,y) < \epsilon$. Now since the complete inverse image $g^{-1}(z)$ of the point $z \in g(R)$ is compact in R by Lemma 55, and since $\rho(x,y) < \epsilon$ for all $x, y \in g^{-1}(z)$, then the diameter of $g^{-1}(z)$ must be less than ϵ. So $g \in \Phi_\epsilon$ and Φ_ϵ is an open set in Φ.

DEFINITION 53: The diameter of a bounded subset of a metric space is $\underset{x,y \in A}{\text{l.u.b.}} \rho(x,y)$.

LEMMA 58: The diameter of a simplex A of R^n is equal to the maximum length of its 1-faces.

Proof: Let $A = (a_0, a_1, \ldots, a_r)$ be an r-simplex, and let x and y be any two points of A. Suppose the barycentric co-ordinates of x are $\lambda^0, \lambda^1, \ldots, \lambda^r$. Now the distance between x and y in R^n is given by the equation $\rho(x,y) = [(x - y)\cdot(x - y)]^{\frac{1}{2}}$. If the vector x is given an increment ξ then $\rho(x+\xi, y) = [(x - y)\cdot(x - y) + 2(x - y)\cdot\xi + \xi\cdot\xi]^{\frac{1}{2}}$. Now if x is not a vertex of A, at least two of its barycentric co-ordinates are different from zero. We may assume without loss of generality that these two positive co-ordinates are λ^0 and λ^1. Let ν be a positive number such that $\nu < \lambda^0/2$ and $\nu < \lambda^1/2$. Let $\xi = \epsilon\nu(a_0 - a_1)$ and $\epsilon = \pm 1$. Clearly $x + \xi$ is a point of A. If ϵ is chosen so that $(x - y)\cdot\xi \geq 0$ then for any value of ξ as restricted above we have $\rho(x+\xi, y) > \rho(x,y)$. Therefore if x is not a vertex of A the function $\rho(x,y)$ cannot attain its maximum value. It does, however, attain its maximum value when x and y are different vertices of the 1-face of A of greatest diameter.

THEOREM 6: A compact metric space R of dimension r can be mapped homeomorphically onto some subset of the Euclidean space R^{2r+1} of dimension $2r + 1$.

Proof: Let Φ be the space of all continuous mappings of the space R into the Euclidean space R^{2r+1}. Let Φ_ϵ be the set of all ϵ-mappings which belong to the set Φ. Certainly the following relations holds: $\Phi_{1/1} \supset \Phi_{1/2} \supset \ldots \supset \Phi_{1/m} \supset \ldots$. Let $\Psi = \bigcap_{i = 1, 2, \ldots} \Phi_{1/i}$. Since each element $h \in \Psi$ is an ϵ-mapping for every $\epsilon > 0$, then h is a one-to-one continuous mapping of the space R onto the subset h(R) of R^{2r+1}. And since R is compact, then h is a homeomorphism by Lemma 30 since R^{2r+1} is Hausdorff and since a subset h(R) of a Hausdorff space is Hausdorff. It is therefore sufficient to show that Ψ is nonvacuous. To show this we need only prove for any $\epsilon > 0$ that Φ_ϵ is everywhere dense in Φ. To show that Φ_ϵ is dense in Φ we shall show that if $g \in \Phi$, and if ϵ and η are any two positive numbers, then there exists an ϵ-mapping f of R into R^{2r+1} such that $\rho(g,f) < \eta$. Now since g is a continuous mapping from the compact metric space R into the Euclidean metric space R^{2r+1} clearly g is uniformly continuous because a continuous mapping from a compact metric space to a metric space is uniformly continuous. Since g is uniformly continuous there exists a positive number $\delta < \epsilon$ such that $\rho(x,y) < \delta$ implies that $\rho[g(x),g(y)] < \eta/6$. Now let $\Sigma = \{G_i\}_{i = 0, 1, \ldots, k}$ be an open δ-covering of R whose order does not exceed $r + 1$. This is possible since by the corollary to Lemma 47 if the dimension of the compact space R is equal to r than for any positive ϵ there exists an open ϵ-covering of R whose order does not exceed $r + 1$. Now since $\delta \leq \epsilon$, then Σ is also an ϵ-covering. And since the diameter of each set G_i is less than δ, the diameter of each set $g(G_i) = F_i$ is less than $\eta/6$. Now choose points c_i, $i = 0, 1, \ldots, k$, of R^{2r+1} so that the distance of c_i from F_i is less than $\eta/6$ and such that c_0, c_1, ..., c_k are in general position. This is possible by Theorem 3 and since the sets F_i are continuous functions of the nonvacuous sets G_i and are therefore nonvacuous themselves. Let the points c_0, c_1, ..., c_k be the vertices of the geometric nerve K of the covering Σ, and associate with each vertex c_i the element G_i of the covering Σ. It will be shown that the mapping f of the space R into the nerve K, constructed in Theorem 5 satisfies the requirements of the present theorem. Now since Σ is an open ϵ-covering then by Theorem 5 f is an ϵ-mapping. It suffices to show that $\rho(g,f) < \eta$. To this end we note first that since the diameter of a simplex is equal to the length of its longest 1-face by Lemma 58, it will suffice to estimate the diameters of the 1-simplexes of K in order to estimate the diameters of the simplexes of the complex K. If (c_m, c_n) is any 1-simplex of K, then the sets G_m and G_n have a common point and therefore the sets F_m and F_n have a nonempty intersection. Since the diameter of each of the sets F_m and F_n is less than $\eta/6$ and since also the distances from the points c_m and c_n to the sets F_m and F_n, respectively, are also less than $\eta/6$, it is apparent that $\rho(c_m,c_n) < 2\eta/3$. Therefore the diameter of every simplex of K is less than $2\eta/3$. Finally, let x be any point of R. There exists an open set G_m of the system Σ which contains x. Since $g(x) \in F_m$ and $\rho(c_m,F_m) < \eta/6$, clearly $\rho[g(x),c_p] < \eta/3$. But the point $f(x) \in A^*$ of K and c_m is a vertex of the simplex A^*. Furthermore, since the diameter of A^* is less than $2\eta/3$, clearly $\rho[c_p, f(x)] < 2\eta/3$. Therefore by the triangle inequality we have $\rho[g(x), f(x)] < \eta$. Since x is an arbitrary point of R then $\rho(g,f) < \eta$.

<u>DEFINITION</u> <u>54</u>: An arbitrary ordering of the set of all vertices of a simplex is a <u>vertex</u> <u>ordering</u> of the simplex. The simplex $A^r = (a_0, a_1, \ldots, a_r)$ is said to receive an <u>orientation</u> or to be <u>oriented</u> if each of the vertex orderings is assigned the sign + or - in such a way that orderings differing by an odd permutation receive opposite signs. A simplex clearly has two orientations. We may express this idea by writing $A^r = \epsilon(a_0, a_1, \ldots, a_r)$ where $\epsilon = +1$ if a positive orientation is meant and where $\epsilon = -1$ if a negative orientation is meant. If A^r is an oriented simplex then $-A^r$ will denote the oppositely oriented simplex. Although a 0-simplex $(a_0) = A^0$ may have only one vertex ordering it is assigned two opposite orientations $+(a_0)$ and $-(a_0)$ for the sake of consistency in notation.

<u>DEFINITION</u> <u>55</u>: If $A^r = (a_0, a_1, \ldots, a_r)$ is an r-simplex any one of its (r - 1)-faces may be obtained by the deletion of some one vertex from the sequence $\{a_i\}_{i = 0, 1, \ldots, r}$. The face obtained by deletion of the vertex a_i is the <u>face</u> <u>opposite</u> <u>the</u> <u>vertex</u> <u>a_i</u>. With the orientation $+A^r = \epsilon(a_0, a_1, \ldots, a_r)$ of the simplex A^r we associate the orientation $B_i^{r-1} = (-1)^i \epsilon(a_0, a_1, \ldots, \hat{a}_i, \ldots, a_r)$ of the (r - 1)-face

of A^r opposite the vertex a_i, where the notation $\hat{a}_i$ denotes the deletion of the element a_i from the sequence under consideration.

<u>LEMMA</u> <u>59</u>: The correspondence between the orientations of A^r and B_i^{r-1} is independent of the vertex ordering $a_0, a_1, \ldots, a_r$. That is, if B_i^{r-1} is a face of A^r then $-B_i^{r-1}$ is the corresponding face of $-A^r$ and conversely.

<u>DEFINITION</u> <u>56</u>: Two simplexes A^r and B^{r-1} are said to be <u>coherently</u> <u>oriented</u> if the vertices of B^{r-1} are a subset of the vertices of A^r and if the orientation of B^{r-1} is the same as the orientation of the $(r - 1)$-face of A^r with the same vertices as B^{r-1}.

<u>DEFINITION</u> <u>57</u>: Let $A_1^r, A_2^r, \ldots, A_{\alpha(r)}^r$ be the set of all arbitrarily oriented r-simplexes of a complex K and let G be an arbitrary additive abelian group. The linear form $x = g_1A_1^r + g_2A_2^r + \cdots + g_{\alpha(r)}A^r_{\alpha(r)}$ in the simplexes $A_1^r, A_2^r, \ldots, A^r_{\alpha(r)}$ with coefficients $g_1, g_2, \ldots, g_{\alpha(r)}$ in G, is an <u>r-dimensional</u> <u>chain</u> (or merely: r-chain) of the complex K over the coefficient group G. If $^*A_1^r, {}^*A_2^r, \ldots, {}^*A^r_{\alpha(r)}$ are the r-simplexes of K and if $x = g_1{}^*A_1^r + g_2{}^*A_2^r + \cdots + g_{\alpha(r)}{}^*A^r_{\alpha(r)}$, and if certain possibly different orientations of the simplexes be taken such that $^*A_i^r = \epsilon_i A_i^r$, then the chain x may also be written $x = \epsilon_1 g_1 A_1^r + \epsilon_2 g_2 A_2^r + \cdots + \epsilon_{\alpha(r)} g_{\alpha(r)} A^r_{\alpha(r)}$. With this convention the chains are independent of the choice of orientation. Addition of chains may be defined in the following manner if $x =$

$\sum_{i=1}^{\alpha(r)} g_i A_i^r$ and $y = \sum_{i=1}^{\alpha(r)} h_i A_i^r$ are two r-chains of K over G, let $x + y = \sum_{i=1}^{\alpha(r)} (g_i + h_i) A_i^r$. The set of all r-chains of K over G is denoted by $L^r(K,G)$ or by L^r when this notation does not lead to misunderstanding.

LEMMA 60: L^r is an additive abelian group.

REMARK: The group G used as the coefficient set in the formation of r-chains of a complex is usually taken to be the group G_0 of integers or the group G_m of residues modulo m. For brevity the group $L^r(K,G_m)$ is written L_m^r for $m = 0, 1, \ldots$. The groups G_0 and G_2 are particularly useful. If the coefficient group is G_2 then $g \mathcal{E} -g$ for all elements $g \mathcal{E} G_2$, and there is no need of distinguishing between the simplex A_i^r and $-A_i^r$, $i = 1, 2, \ldots, \alpha(r)$, in the chain $x = g_1 A_1^r + g_2 A_2^r + \cdots + g_{\alpha(r)} A_{\alpha(r)}^r$.

DEFINITION 58: If A^r is an oriented r-simplex of a complex K it may be regarded as a chain of K over G_0. The boundary of the oriented simplex A^r is defined as the $(r - 1)$-chain of K over G_0 given by the relation $\Delta(A^r) = \Delta A^r = B_0^{r-1} + B_1^{r-1} + \cdots + B_r^{r-1}$, where $\{B_i^{r-1}\}_{i = 0, 1, \ldots, r}$ is the set of all $(r - 1)$-faces of A^r oriented coherently with respect to A^r. If $r = 0$ we set $\Delta A^0 = 0$.

LEMMA 61: $\Delta(-A^r) = -\Delta A^r$.

DEFINITION 59: The boundary of an r-chain x of K over G is an extension of the idea of the boundary of an oriented simplex. We let $\Delta(x) = \Delta x = \sum_{i=1}^{\alpha(r)} g_1 \Delta A_i^r$.

<u>LEMMA</u> <u>62</u>: $\Delta(-x) = -\Delta x$ and $\Delta(x + y) = \Delta x + \Delta y$; that is, Δ is a homomorphism on the group of chains.

<u>LEMMA</u> <u>63</u>: $\Delta\Delta x = 0$.

<u>Proof</u>: It is sufficient to prove that $\Delta\Delta A^r = 0$ for any A^r. Let $A^r = +(a_0, a_1, \ldots, a_r)$ and let C_p^{r-1} and C_{pq}^{r-2} be the oriented simplexes obtained from A^r by omitting, respectively, the vertex a_p and the two vertices a_p and a_q, $p < q$. That is, let $C_p^{r-1} = +(a_0, a_1, \ldots, \hat{a}_p, \ldots, a_r)$ and $C_{pq}^{r-2} = +(a_0, a_1, \ldots, \hat{a}_p, \ldots, \hat{a}_q, \ldots, a_r)$. Then $\Delta A^r = \sum_{i=0}^{r} (-1)^i C_i^{r-1}$. Also $\Delta C_i^{r-1} = \sum_{j=0}^{i-1} (-1)^j C_{ji}^{r-2} + \sum_{j=i+1}^{r} (-1)^{j-1} C_{ij}^{r-2}$. Therefore $\Delta\Delta A^r = \sum_{j<i} (-1)^{i+j} C_{ji} + \sum_{i<j} (-1)^{j+i-1} C_{ij}^{r-2} = 0$.

<u>DEFINITION</u> <u>60</u>: An r-chain x is a <u>cycle</u> if its boundary is equal to zero. The set of all r-cycles of K over G is denoted by $Z^r(K,G)$, or merely by Z^r. We let Z_m^r denote $Z^r(K,G_m)$.

<u>LEMMA</u> <u>64</u>: Every boundary is a cycle.

<u>LEMMA</u> <u>65</u>: Z^r is a subgroup of L^r.

<u>Proof</u>: This follows from the fact that Δ is a group homomorphism of which Z^r is the kernel.

<u>COROLLARY</u>: If $r = 0$ then $Z^0 = L^0$.

Proof: Δ is the zero map on L^0.

DEFINITION 61: An r-cycle z of an n-complex K is homologous to zero if it is the boundary of an $(r+1)$-chain of K, $r = 0, 1, \ldots, (n-1)$. An n-cycle z of the n-complex K is homologous to zero only if it is equal to zero. We express the fact that z is homologous to zero symbolically by $z \sim 0$. We denote the set of all r-cycles of K over G which are homologous to zero by $H^r(K,G)$ or by H^r. In particular, we let H^r_m denote $H^r(K,G_m)$. Two r-cycles z_1 and z_2 are homologous and we write $z_1 \sim z_2$ in case their difference is homologous to zero. That is, if $z_1 - z_2 \sim 0$.

LEMMA 66: H^r is a subgroup of the group Z^r.

Proof: This lemma follows directly from the fact that Δ is a homomorphism and that H^r is the image of Z^r under Δ.

DEFINITION 62: Since H^r is a subgroup of Z^r the factor group $B^r = Z^r/H^r = B^r(K,G)$ exists. We call B^r the r-dimensional Betti group or the r-dimensional homology group of the complex K over G. We denote $B^r(K,G_m)$ by B^r_m. We note that the elements of the Betti group are classes or cosets of homologous cycles.

REMARK: If K is an n-complex and Δ is the boundary operator in K, then Δ is a homomorphism of the group L^r into the group L^{r-1}, $r = 1, 2, \ldots, n$. The subgroup Z^r is therefore the kernel of the homomorphism Δ in the group L^r and the subgroup $H^{r-1} \subset L^{r-1}$ is the image of L^r under Δ. The groups L^r/Z^r and H^{r-1} are therefore isomorphic.

<u>LEMMA</u> <u>67</u>: If x^*, y^*, z^* are elements of the Betti group B^r of a complex K and if x, y, z are cycles of K in the corresponding homology classes x^*, y^*, z^*, then the relations $x^* + y^* = z^*$ and $x + y \sim z$ are equivalent.

<u>DEFINITION</u> <u>63</u>: A <u>subcomplex</u> of a complex K is any complex all of whose simplexes are contained in K. The set of all simplexes of a complex K, whose dimension does not exceed r, is called the <u>r-dimensional</u> <u>skeleton</u> (or merely: <u>r-skeleton</u>) of the complex K.

<u>LEMMA</u> <u>68</u>: The r-skeleton of K is a subcomplex of K.

<u>DEFINITION</u> <u>64</u>: A complex K is <u>connected</u> in case it cannot be represented as the union of two nonempty subcomplexes L and M without common simplexes.

<u>LEMMA</u> <u>69</u>: A complex is connected if and only if given any two of its vertices a and e, there exists a sequence of vertices $\{a_i\}_{i = 1, 2, \ldots, q}$ such that $a_1 = a$ and $a_q = e$ and such that a_i, a_{i+1}, $i = 1, 2, \ldots, (q - 1)$ are the vertices of a 1-simplex of K.

<u>Proof</u>: Suppose that the complex K is not connected. Then K is the union of two disjoint nonvacuous subcomplexes L and M. Let a be a vertex of L and e be a vertex of M, and assume that a sequence as in the statement of the lemma exists for these vertices. Now if a_i is the last vertex of the sequence which is contained in L, then the simplex (a_i, a_{i+1}) which exists according to the condition above

cannot lie in L nor in M. Therefore if K is not connected the sequence is lacking for at least one pair of vertices of K. Therefore if the sequence exists then the complex is connected. Conversely, let us show that if the complex is connected then such a sequence exists. We assume that the complex K is connected. Let a be an arbitrary fixed vertex of K and denote by E the set of all vertices of K which can be joined to a by a sequence as in the statement of the lemma. Clearly, if a simplex A has at least one vertex in E then all its vertices are in E. Hence the set of all simplexes of K with vertices in E forms a subcomplex L of K. The set of all simplexes of K which are not in L also forms a subcomplex M of K. But M is vacuous because K is connected. Therefore E contains all the vertices of K, and the vertex a may be joined to an arbitrary vertex e by a sequence as in the statement of the lemma. Hence any pair of vertices of K may be joined by a sequence of vertices as required.

DEFINITION 65: Any connected subcomplex L of the complex K, such that K is the union of two disjoint complexes L and M, is a *component* of K.

LEMMA 70: If $\{K_i\}_{i=1,2,\ldots,p}$ is the set of all components of K, then $K_i \cap K_j = \emptyset$, $i \neq j$, and $K = \bigcup_{i=1}^{p} K_i$.

Proof: a) First we prove that $K_i \cap K_j = \emptyset$, $i \neq j$. Assume $K_i \cap K_j \neq \emptyset$. Then, since K_i is a component, K is the union of two disjoint subcomplexes $K_i = L$ and M. If we denote $K_j \cap L$ by K'_j and $K_j \cap M$ by K''_j it is clear that K'_j and K''_j are disjoint subcomplexes of the complex K_j and that $K'_j \cup K''_j = K_j$. Now since K_j is connected, one of the subcomplexes K'_j or K''_j must be vacuous. Specifically, $K''_j = \emptyset$ because $K'_j = K_i \cap K_j$ which is nonempty by hypothesis. Therefore $K_j \subset K_i$. Similarly it may be proved that $K_i \subset K_j$. Thus $K_i = K_j$ and hence $i = j$.

b) Secondly we prove that $K = \bigcup_{i=1}^{p} K_i$ by showing that any arbitrary simplex A of K is contained in one of the components. Now if K is connected then there is exactly one component $K = K_1$ and the equality holds. If K is not connected, then $K = L \cup M$, where L and M are disjoint subcomplexes of K and where one of them, say L, contains A. If L is connected, then L is a component of K, and the simplex A is contained in one of the components of K. If L is not connected then L may be decomposed in the same manner as was K. This decomposition may be extended until a component containing A is found.

<u>THEOREM 7</u>: If $K_1, K_2, \ldots, K_p$ is the set of all components of a complex K, and if B^r, B^r_i are the Betti groups of the complexes K and K_i, respectively, then B^r is isomorphic to the direct sum $B^r_1 + B^r_2 + \ldots + B^r_p$.

<u>Proof</u>: Let L^r be the group of all r-chains of the complex K and let L^r_i be the subgroup of L^r consisting of all

chains of L^r in which the only simplexes appearing with nonvanishing coefficients are simplexes of the complex K_i. Clearly

(24) $\quad L^r = L_1^r + L_2^r + \ldots + L_p^r$

and L_i^r is the group of all r-chains of the complex K_i. Further, if we set $H_i^{r-1} = \Delta L_i^r$ then

(25) $\quad H_i^{r-1} \subset L_i^{r-1}$.

It will be shown that

(26) $\quad H_i^{r-1} = H_1^{r-1} + H_2^{r-1} + \ldots + H_p^{r-1}$.

Now if $x \in L^r$ and Δx is an element of H^{r-1} then by (24) we have $x = x_1 + x_2 + \ldots + x_p$, $x_i \in L_i^r$, and therefore

(27) $\quad \Delta x = \Delta x_1 + \Delta x_2 + \ldots + \Delta x_p$,

where $\Delta x_i \in H_i^{r-1}$. The uniqueness of the decomposition of Δx follows from relations (24) and (25). Now let Z_i^r be the kernel of the homomorphism Δ in the group L_i^r. That is, let Z_i^r consist of the elements of L_i^r which map to zero under the homomorphism Δ. That is, Z_i^r is the set of all r-cycles of L_i^r. We shall show that

(28) $\quad Z^r = Z_1^r + Z_2^r + \ldots + Z_p^r$.

Now if $z \in Z^r$, then the relation $z = x_1 + x_2 + \ldots + x_p$ where $x_i \in L_i^r$, holds because of (24). Therefore $\Delta x_1 + \Delta x_2 + \ldots + \Delta x_p = \Delta z = 0$. Now by (25) and (26) we see that $\Delta x_i = 0$, and thus that $x_i \in Z_i^r$. The uniqueness of this decomposition follows from (24). Finally, relations (26) and (28) imply that the group Z^r/H^r is isomorphic to the direct sum $Z_1^r/H_1^r + Z_2^r/H_2^r + \ldots + Z_p^r/H_p^r$.

DEFINITION 66: Let K be an arbitrary complex and let A_1^0, A_2^0, ..., A_α^0 be the set of all positively oriented 0-simplexes $A_i^0 = +(a_i)$ of K. If $x = g_1A_1^0 + g_2A_2^0 + \dots + g_\alpha A_\alpha^0$ is any 0-chain of K over G, we define the Kronecker index $I(x)$ of the chain x by setting $I(x) = g_1 + g_2 + \dots + g_\alpha$. We note that since a unique vertex ordering cannot exist for simplexes of dimension greater than zero it is possible to consider a positive orientation only for 0-simplexes. Hence the notion of index is not extended to chains of dimension greater than zero.

LEMMA 71: $I(x) + I(y) = I(x + y)$.

LEMMA 72: If $x \sim 0$, then $I(x) = 0$.

Proof: Let $A^1 = +(a, b)$ be any oriented 1-simplex of K, and let $A^0 = +(a)$ and $B^0 = +(b)$. Then according to the definition of a boundary, we have $\Delta(gA^1) = gB^0 - gA^0$. Thus $I\left[\Delta(gA^1)\right] = 0$. Finally, $I(\Delta y) = 0$ for any $y \in L^1$ because of Lemma 71.

LEMMA 73: If K is a connected complex then $I(x) = 0$ is equivalent to $x \sim 0$. Further, $B^0(K,G)$ is isomorphic to G.

Proof: Let a and e be any two vertices of K, and let $A^0 = +(a)$, $E^0 = +(e)$. Now since K is a connected complex, there exists a sequence of vertices $a_1 = a$, a_2, ..., $a_q = e$ such that a_i, a_{i+1}, $i = 1, 2, \dots, (q - 1)$, are vertices of a 1-simplex of K. If $A_i^1 = +(a_i, a_{i+1})$, $i = 1, 2, \dots, (q - 1)$, the boundary of any chain $y = gA_1^1 + gA_2^1 + \dots + gA_{q-1}^1$ is clearly $\Delta y = gE^0 - gA^0$. Now Δy is a 0-cycle because it has boundary zero. Also Δy is homologous to zero because it is the boundary of a 1-chain. Hence $gE^0 - gA^0$ is homologous to zero and we have $gE^0 \sim gA^0$. Therefore any 0-chain x over an arbitrary group G is homologous to a chain gA^0, $g \in G$. By Lemma 72, since $x \sim gA^0$ then $(x - gA^0) \sim 0$ and we have $I(x - gA^0) = 0$ and $I(x) - I(gA^0) = 0$. Hence $I(x) = g$. Therefore if $I(x) = 0$ then $x \sim 0$, and conversely. By Lemma 71 we know that the operator $I(\ )$ is a homomorphic mapping of the group $L^0 = Z^0$ into the group G. If $g \in G$ there exists a cycle gA^0 in Z^0 such that $I(gA^0) = g$. Therefore $I(Z^0) = G$. Also, the equivalence of the relations $I(x) = 0$ and $x \sim 0$ implies that H^0 is the kernel of the homomorphism $I(\ )$. Therefore Z^0/H^0 is isomorphic to G.

<u>THEOREM</u> <u>8</u>: The zero-dimensional Betti group of an arbitrary complex K over G is isomorphic to the direct sum $G + G + \ldots + G$, where the number of terms in the direct sum is the same as the number of components of the complex K.

<u>Proof</u>: This theorem is a direct consequence of Lemma 73 and Theorem 7.

<u>DEFINITION</u> <u>67</u>: An abelian group A is said to <u>admit</u> <u>of</u> <u>a</u> finite system of generators $x_1, x_2, \ldots, x_s$, $x_i \in A$, $i = 1, 2, \ldots, s$, if every $x \in A$ is of the form $x = \lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_s x_s$, where $\lambda_1, \lambda_2, \ldots, \lambda_s$ are integers.

<u>LEMMA</u> <u>74</u>: Every factor group and every subgroup of a group with a finite system of generators also admits of a finite system of generators.

The proof of this lemma for a factor group depends on the proposition that every homomorphic image of a finitely generated group is itself finitely generated. (See: Kurosh, <u>The Theory of Groups</u>, Vol. I (Chelsea, 1955) p. 50) The proof for the subgroup case may be found in Pontrjagin, <u>Topological Groups</u> (Princeton, 1939), p. 20.

<u>DEFINITION 68</u>: A group A generated by a single element x_1 is called a <u>cyclic group</u>. If the relation $\lambda x_1 = 0$, where λ is an integer, implies that $\lambda = 0$, then the generator x_1 and the group A itself are called <u>free</u> (or: <u>of order zero</u>). If there exists a positive integer λ such that $\lambda x_1 = 0$ and λ is the least positive integer which satisfies the condition, then the generator x_1 and the group A itself are said to be of <u>finite order</u> λ.

<u>LEMMA 75</u>: Every abelian group A with a finite system of generators is a direct sum of cyclic subgroups $A_1, A_2, \ldots, A_r; B_1, B_2, \ldots, B_q$, where each A_i is a free cyclic group and where each B_j is a cyclic group of finite order τ_j, with τ_{j+1} divisible by τ_j. Furthermore, if A has a finite system of linearly independent generators, then the decomposition of A into a direct sum exhibits no summands of finite order.

<u>DEFINITION 69</u>: The number r in Lemma 75 is called the <u>rank of the group A</u>. The numbers $\tau_1, \tau_2, \ldots, \tau_q$ are called the <u>torsion coefficients</u> of the group A. It may be noted that r equals zero in case all of the elements of the group

A are of prime order m. In this case all of the torsion coefficients $\tau_1, \tau_2, \ldots, \tau_q$ are equal to m and we call q the <u>rank</u> <u>modulo</u> <u>m</u> of the group A.

<u>LEMMA</u> <u>76</u>: If the coefficient group G is a cyclic group with generator g_1, then the group L^r admits of a finite system of generators $g_1A_1^r, g_1A_2^r, \ldots, g_1A_{\alpha(r)}^r$, where $\{A_i^r\}_{i=1, 2, \ldots, \alpha(r)}$ is the set of all arbitrarily oriented r-simplexes of the complex K.

<u>LEMMA</u> <u>77</u>: The subgroups Z^r and H^r of the group L^r, and the factor group $Z^r/H^r = B^r$ admit of a finite system of generators.

<u>Proof</u>: This lemma is a direct consequence of Lemma 74.

<u>DEFINITION</u> <u>77</u>: Let B_0^r be the r-dimensional Betti group of a complex K over the group of integers G_0. The rank of the group B_0^r is the <u>r-dimensional</u> <u>Betti</u> <u>number</u> of the complex K and is denoted by $p_0(r) = p(r)$. The torsion coefficients $\tau_1, \tau_2, \ldots, \tau_q$ of the group B_0^r are called the <u>r-dimensional</u> <u>torsion</u> <u>coefficients</u> of the complex K and are denoted by $\tau_1^r, \tau_2^r, \ldots, \tau_{q(r)}^r$.

<u>LEMMA</u> <u>78</u>: If the group G_m of residues modulo m, m a prime, is taken as the coefficient group, then every element of the group L_m^r, as well as every element of the subgroups Z_m^r and H_m^r of L_m^r and the factor group $B_m^r = Z_m^r/H_m^r$, is also of order m.

<u>DEFINITION</u> <u>71</u>: Let B_m^r be the r-dimensional Betti group of a complex K over the group G_m of residues modulo m, m a prime.

The rank modulo m of the group B_m^r is called the r-dimensional Betti number of the complex K modulo m and is denoted by $p_m(r)$.

THEOREM 9: The zero-dimensional Betti number $p_m(0)$ of an arbitrary complex K modulo m, $m = 0$ or a prime, is equal to the number p of components of the complex K. Furthermore, the zero-dimensional Betti group B_0^0 of the complex K over G_0 has no torsion coefficients.

Proof: By Theorem 8 the group B_m^0 is the direct sum of the groups C_1, C_2, ..., C_p, each of which is isomorphic to the group G_m of residues modulo m. If $m = 0$, then every group C_i is free. That is, if $m = 0$, then B_0^0 has no torsion coefficients, and its rank is p. If m is a prime, then every group C_i is of order m, and the rank modulo m of B_m^0 is p.

DEFINITION 72: If A_0 is an arbitrary abelian group, a system x_1, x_2, ..., x_s of elements of A_0 is linearly independent in case the relation $\lambda_1 x_1 + \lambda_2 x_2 + \dots + \lambda_s x_s = 0$, where each λ_i is an integer, implies that each λ_i is equal to zero. If the group A_0 admits of a finite maximal system of linearly independent elements x_1, x_2, ..., x_ρ, then A_0 is said to be of finite group dimension ρ, denoted by $\rho_0(A_0)$. If the group A_0 has n linearly independent elements for every positive integer n then $\rho_0(A_0) = \infty$. If A_m is a group and consists solely of elements of prime order m and if G_m is the group of residues modulo m then we define an operation of multiplication of $\lambda \varepsilon G_m$ by $x \varepsilon A_m$. Indeed, if β is any element of the residue class λ then the product βx clearly is independent of the choice of β and depends only on the class λ. We therefore define $\lambda x = \beta x$. Then a system $x_1, x_2, \ldots, x_s$ of elements of the group A_m is <u>linearly independent modulo m</u> in case the relation $\lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_s x_s = 0$, where each $\lambda_i \varepsilon G_m$, implies $\lambda_1 \equiv \lambda_2 \equiv \ldots \equiv \lambda_s \equiv 0$ modulo m. If A_m has a maximal system $x_1, x_2, \ldots, x_\rho$ of linearly independent elements modulo m then we define the <u>group dimension modulo m</u> of A_m to be ρ and we denote it by $\rho_m(A_m)$. However, if for every positive integer n there exists in the group A_m a system of n linearly independent elements modulo m we define $\rho_m(A_m)$ to be infinite.

<u>LEMMA 79</u>: Let A be an abelian group with elements of infinite order or all elements of prime order n and of finite group dimension. Then the group dimension of A is a group invariant. That is, any two maximal linearly independent sets have the same number of elements. In either case, A admits operators from an integral domain, in the first event from the integers, in the second from the prime field of n elements.

<u>Proof</u>: On the strength of the last statement of the lemma, which is obvious, we will employ in the proof operators λ, μ, ν chosen from the appropriate integral domain and will be assured that $\lambda\mu \neq 0$ if $\lambda \neq 0$ and $\mu \neq 0$.

This will enable us to prove both cases simultaneously. It might be remarked that in the case where all elements are of prime order, the group is a vector space and the usual vector space proof, somewhat simpler than the following, applies.

Let $a_1, a_2, \ldots, a_q$ and $b_1, b_2, \ldots, b_p$ be two maximal linearly independent sets. We shall show that $p \geq q$ by first assuming that $p < q$ and arriving at a contradiction. Then by virtue of the fact that the choice $p < q$ rather than $q < p$ was arbitrary, we will have $q = p$.

Consider the set $b_1, a_1, a_2, \ldots, a_q$. Since it is linearly dependent there is a nontrivial relation $\lambda b_1 + \lambda^1 a_1 + \lambda^2 a_2 + \ldots + \lambda^q a_q = 0$, and $\lambda \neq 0$ since the a_i are independent. Suppose λ^j is the nonzero coefficient whose index is least. Consider the set $b_1, a_1, a_2, \ldots, \hat{a}_j, \ldots, a_q$, where the symbol "$\wedge$" indicates omission as usual. We wish to show that this set is a maximal linearly independent set. Suppose that $\mu b_1 + \mu^1 a_1 + \mu^2 a_2 + \ldots + \mu^{j-1} a_{j-1} + \mu^{j+1} a_{j+1} + \ldots + \mu^q a_q = 0$. We note that $\mu \neq 0$. We have also $\lambda b_1 + \lambda^j a_j + \lambda^{j+1} a_{j+1} + \ldots + \lambda^q a_q = 0$ from an earlier relation. Multiplying the first of these last two relations by λ and the second by $-\mu$ we may add and obtain $\lambda\mu^1 a_1 + \lambda\mu^2 a_2 + \ldots + \lambda\mu^{j-1} a_{j-1} - \mu\lambda^j a_j + (\lambda\mu^{j+1} - \mu\lambda^{j+1}) a_{j+1} + \ldots + (\lambda\mu^q - \mu\lambda^q) a_q = 0$, where $\mu\lambda^j \neq 0$ since $\mu \neq 0$ and $\lambda^j \neq 0$. This is in contradiction to the independence of the a_i. Hence the representation $\mu b_1 +$

$\mu^1 a_1 + \mu^2 a_2 + \ldots + \mu^{j-1} a_{j-1} + \mu^{j+1} a_{j+1} + \ldots + \mu^q a_q =$ 0 is not possible and hence the set $b_1, a_1, a_2, \ldots, \hat{a}_j, \ldots, a_q$ is linearly independent. We show now that it is also a maximal linearly independent set. Let x be any element of A. Then there exists a nontrivial relation $\gamma x + \gamma^1 a_1 + \gamma^2 a_2 + \ldots + \gamma^j a_j + \ldots + \gamma^q a_q = 0$. Clearly $\gamma \neq 0$ since the a_i are independent. Now if $\gamma^j = 0$ then $x, b_1, a_1, a_2, \ldots, \hat{a}_j, \ldots, a_q$ is clearly a dependent set. We shall show that even if $\gamma^j \neq 0$ then $x, b_1, a_1, a_2, \ldots, \hat{a}_j, \ldots, a_q$ is a dependent set. If $\gamma^j \neq 0$ we can multiply by λ^j obtaining $\lambda^j \gamma x + \lambda^j \gamma^1 a_1 + \ldots + \lambda^j \gamma^j a_j + \ldots + \lambda^j \gamma^q a_q = 0$. Now since $\lambda b_1 + \lambda^j a_j + \lambda^{j+1} a_{j+1} + \ldots + \lambda^q a_q = 0$, we have $\lambda^j a_j = -\lambda b_1 - \lambda^{j+1} a_{j+1} - \ldots - \lambda^q a_q$. Substituting we obtain $\lambda^1 \gamma x + \lambda^j \gamma^1 a_1 + \ldots + \gamma^j(-\lambda b_1 - \lambda^{j+1} a_{j+1} - \ldots - \lambda^q a_q) + \ldots + \lambda^j \gamma^q a_q = 0$. Rearranging terms we obtain $\lambda^1 \gamma x - \gamma^j \lambda b_1 + \lambda^j \gamma^1 a_1 + \ldots + \hat{a}_j + (\lambda^j \gamma^{j+1} - \gamma^j \lambda^{j+1}) a_{j+1} + \ldots + (\lambda^j \gamma^q - \gamma^j \lambda^q) a_q = 0$, where $\lambda^j \gamma \neq 0$. Thus $x, b_1, a_1, a_2, \ldots, \hat{a}_j, \ldots, a_q$ is a linearly independent set and $b_1, a_1, a_2, \ldots, \hat{a}_j, \ldots, a_q$ is a maximal linearly independent set. The process may be applied p times, at each stage adding an element from the set of b_i's and deleting one element from the set of a_i's and obtaining a set of elements $b_p, b_{p-1}, \ldots, b_1, a_{i_1}, a_{i_2}, \ldots, a_{i_{q-p}}$, which must be a maximal linearly independent set and which contains at least one of the a_i, since $p < q$. But this is a contradiction to the maximality of $b_1, \ldots, b_p$; hence $p \geq q$.

__LEMMA__ __80__: The rank of a finitely generated abelian group A_0 is the same as the group dimension.

__Proof__: a) We consider first the case $A_0 = A_1 + A_2 + \ldots + A_r + B_1 + B_2 + \ldots + B_q$. Let x_i be a generator of the cyclic group A_i and let y_j be a generator of the cyclic group B_j. Suppose that

(29) $$\lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_r x_r = 0.$$

Now since A_0 is the direct sum as above, relation (29) implies that $\lambda_i x_i = 0$. But since x_i is a free generator, we have $\lambda_i = 0$. If $x = \lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_r x_r + \mu_1 y_1 + \mu_2 y_2 + \ldots + \mu_q y_q$ is any element of the group A_0 we may multiply x by τ_q and obtain the relation $\tau_q x = \tau_q \lambda_1 x_1 + \tau_q \lambda_2 x_2 + \ldots + \tau_q \lambda_r x_r + \tau_q \mu_1 y_1 + \tau_q \mu_2 y_2 + \ldots + \tau_q \mu_q y_q = \tau_q \lambda_1 x_1 + \tau_q \lambda_2 x_2 + \ldots + \tau_q \lambda_r x_r + 0 + 0 + \ldots + 0$. Thus the set $x, x_1, x_2, \ldots, x_r$ is linearly dependent and hence the set $x_1, x_2, \ldots, x_r$ forms a maximal linearly independent system.

b) The above proof may be easily adapted to the simple case where $A_0 = B_1 + B_2 + \ldots + B_q$; that is, where all of the elements are of prime order. In fact, the usual vector space proof applies.

REMARK: By Lemma 80 we may refer to the group dimension as the rank since the two terms have been shown to be equivalent.

LEMMA 81: Let m be zero or a prime number and let A_m be a group with the property that $mx = 0$ for all $x \in A_m$. That is, if $m \neq 0$ then every element of A_m is of order m, and if $m = 0$ then A_m is an arbitrary group. If B_m is any subgroup of A_m and if $C_m = A_m/B_m$, then

(30) $$\rho_m(A_m) = \rho_m(B_m) + \rho_m(C_m).$$

Proof: For convenience in notation we denote linear independence in the ordinary sense by linear independence modulo 0. We thus need not distinguish between the cases $m = 0$ and $m \neq 0$. Let

(31) $$y_1, y_2, \ldots, y_s$$

and

(32) $$z_1, z_2, \ldots, z_t$$

be two systems of elements of B_m and C_m respectively, let each of the systems be linearly independent modulo m, and let $x_i \in A_m$ be an element of the coset z_i. We will first show that the system

(33) $$x_1, x_2, \ldots, x_t, y_1, y_2, \ldots, y_s$$

is a linearly independent system modulo m in A_m. To this end we assume that

(34) $$\lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_t x_t + \mu_1 y_1 + \mu_2 y_2 + \ldots + \mu_s y_s = 0$$

where λ_j, $\mu_i \in G_m$. Relation (34) corresponds to the relation $\lambda_1 z_1 + \lambda_2 z_2 + \ldots + \lambda_t z_t = 0$ if we consider the factor group C_m. Then since $z_1, z_2, \ldots, z_t$ is a linearly independent system modulo m, we have $\lambda_1 = \lambda_2 = \ldots = \lambda_t = 0$. Therefore relation (34) may be written $\mu_1 y_1 + \mu_2 y_2 + \ldots + \mu_s y_s = 0$. Hence, since $y_1, y_2, \ldots, y_s$ is a linearly independent system modulo m, we have $\mu_1 = \mu_2 = \ldots = \mu_s = 0$. Therefore relation (34) implies that $\lambda_j = \mu_i = 0$. Hence system (33) is linearly independent. Now if either $\rho_m(B_m) = \infty$ or $\rho_m(C_m) = \infty$ then $\rho_m(A_m) = \infty$ and the lemma holds for this case. We next show that if both $\rho_m(B_m)$ and $\rho_m(C_m)$ are finite, and if systems (31) and (32) are both maximal linearly independent in their respective groups then system (33) is also maximal linearly independent modulo m in A_m. Suppose that $x \in A_m$ and that z is the coset of C_m which contains x. Since the set $z_1, z_2, \ldots, z_t$ is maximal linearly independent modulo m it is possible to find a set $\nu, \nu_1, \nu_2, \ldots, \nu_t$ such that $\nu, \nu_i \in G_m$, $i = 1, 2, \ldots, t$; $\nu \neq 0$ and such that

(35) $$\nu z + \nu_1 z_1 + \nu_2 z_2 + \ldots + \nu_t z_t = 0.$$

Thus

(36) $$\nu x + \nu_1 x_1 + \ldots + \nu_t x_t = y$$

where y is clearly some element of B_m. Now since $y_1, y_2, \ldots, y_s$ is a maximal linearly independent set modulo m we have

(37) $$\mu y + \mu_1 y_1 + \mu_2 y_2 + \ldots + \mu_s y_s = 0$$

for some set $\mu, \mu_1, \mu_2, \ldots, \mu_s$, where $\mu, \mu_i \varepsilon G_m$, $\mu \neq 0$. By relations (36) and (37) we may obtain

$$(38) \qquad \mu\nu x + \mu\nu_1 x_1 + \mu\nu_2 x_2 + \ldots + \mu\nu_t x_t + \mu_1 y_1 + \mu_2 y_2 + \ldots + \mu_s y_s = 0.$$

Now in this relation, if m = 0 then μ and ν are each integers different from zero. On the other hand, if $m \neq 0$ we have μ and ν clearly nonzero residues modulo m, for m a prime. In the first case the product $\mu\nu$ is obviously nonzero; in the second case $\mu\nu \not\equiv 0$ modulo m. In either event the system $x, x_1, x_2, \ldots, x_t, y_1, y_2, \ldots, y_s$ is a linearly dependent set modulo m and hence system (33) is maximal.

<u>THEOREM</u> <u>10</u>: Let K be an n-dimensional geometric complex, let $\alpha(r)$ be the number of r-simplexes of K, let p(r) be the r-dimensional Betti number of K, and let $p_m(r)$ be the r-dimensional Betti number of K modulo m, where m is a prime.

Then $\chi = \chi(K) = \sum_{r=0}^{n} (-1)^r \alpha(r) = \sum_{r=0}^{n} (-1)^r p(r) =$

$\sum_{r=0}^{n} (-1)^r p_m(r)$.

<u>Proof</u>: Let m be zero or a prime, let G_m be the group of residues modulo m, and let g be a generator of the group G_m. Let $A_1^r, A_2^r, \ldots, A^r_{\alpha(r)}$ be the set of all arbitrarily oriented r-simplexes of the complex K. Clearly the elements $gA_1^r, gA_2^r, \ldots, gA^r_{\alpha(r)}$ may be taken to be the generators of the group L_m^r of all r-chains of the complex K over the coefficient group G_m. The elements gA_i^r, i = 1, 2, ..., $\alpha(r)$,

are linearly independent modulo m. These elements form a maximal linearly independent set modulo m because they form a system of generators. Therefore

(39) $\rho_m(L^r_m) = \alpha(r).$

Now the following relation holds:

(40) $\rho_m(L^r_m) = \rho_m(Z^r_m) + \rho_m(L^r_m/Z^r_m)$, $r = 0, 1, \ldots, n$,

because of Lemma 81. Now if $r > 0$ then the groups L^r_m/Z^r_m and H^{r-1}_m are isomorphic by the remark following Definition 62 and equation (40) may be reduced to

(41) $\rho_m(L^r_m) = \rho_m(Z^r_m) + \rho_m(H^{r-1}_m)$, $r = 1, 2, \ldots, n$.

However if $r = 0$, then $Z^0_m = L^0_m$, and thus we have

(42) $\rho_m(L^0_m) = \rho_m(Z^0_m)$.

If we introduce the notation $\rho_m(H^{-1}_m) = 0$ for the sake of uniformity then relations (41) and (42) may be written in virtue of (39) as

(43) $\alpha(r) = \rho_m(Z^r_m) + \rho_m(H^{r-1}_m)$, $r = 0, 1, \ldots, n$,

where $\rho_m(H^{-1}_m) = 0$. Now by virtue of Lemma 81 $\rho_m(Z^r_m) = \rho_m(H^r_m) + \rho_m(Z^r_m/H^r_m)$, $r = 0, 1, \ldots, n$, and furthermore, by virtue of Lemma 78 $\rho_m(H^r_m) + \rho_m(Z^r_m/H^r_m) = \rho_m(H^r_m) + \rho_m(B^r_m)$, $r = 0, 1, \ldots, n$. Moreover, $\rho_m(B^r_m) = \rho_m(r)$, $r = 0, 1, \ldots, n$, by virtue of Definition 71. Therefore, $\rho_m(Z^r_m) = \rho_m(H^r_m) + \rho_m(r)$, $r = 0, 1, \ldots, n$. Now by definition, $H^n_m = \{0\}$ and we thus have

(44) $\alpha(r) = \rho_m(r) + \rho_m(H^{r-1}_m) + \rho_m(H^r_m)$, $\rho_m(H^n_m) = 0$,
$r = 0, 1, \ldots, n$

by combining relation (43) and the fact that $\rho_m(Z^r_m) =$

$\rho_m(H_m^r) + \rho_m(r)$, $r = 0, 1, \ldots, n$. Finally, if we multiply relation (44) by $(-1)^r$ and sum over r we obtain the relation

$$\sum_{r=0}^{n} (-1)^r \alpha(r) = \sum_{r=0}^{n} (-1)^r \rho_m(r) + \sum_{r=0}^{n} (-1)^r \rho_m(H_m^{r-1}) + \sum_{r=0}^{n} (-1)^r \rho_m(H_m^r) = \sum_{r=0}^{n} (-1)^r \rho_m(r).$$

DEFINITION 73: The number $\chi = \chi(k)$ is the Euler characteristic of the complex K. It is sometimes referred to as the Euler-Poincaré characteristic.

DEFINITION 74: Let $A^r = (a_0, a_1, \ldots, a_r)$ be an r-dimensional simplex in R^m, let $B^s = (b_0, b_1, \ldots, b_s)$ be an s-dimensional simplex in R^n, and let f be a mapping, not necessarily one-to-one, which assigns to each vertex a_i some vertex b_j. If $x = \lambda^0 a_0 + \lambda^1 a_1 + \ldots + \lambda^r a_r$ is any point of A^r, let $f(x) = \lambda^0 f(a_0) + \lambda^1 f(a_1) + \ldots + \lambda^r f(a_r)$. This mapping when restricted to any individual vertex of A^r is the same as the mapping defined above on each vertex of A^r. The mapping f is a simplicial mapping of A^r into B^s.

LEMMA 82: If E^r and E^s are two simplexes whose vertices are orthonormal vectors in R^r and R^s respectively, then a simplicial mapping f from E^r to E^s is a continuous mapping of E^r into E^s.

Proof: a) We may assume without loss of generality that $r \geq s$. Otherwise E^r would map into a simplex which was a proper subset E^q of E^s with $r \geq q$. Suppose that $x = \lambda^1 e_1 + \lambda^2 e_2 + \ldots + \lambda^r e_r$ is any point of E^r. Then $f(x) =$

$\mu^1 \epsilon_1 + \mu^2 \epsilon_2 + \ldots + \mu^s \epsilon_s$ where each $\mu^i = \sum_{k=1}^{\beta(i)} \lambda^{ik}$.

b) Now there exists a positive number δ such that if $|\gamma^i - \mu^i| < \delta$ for i = 0, 1, ..., $\beta(i)$ then because of the continuity of the metrci function ρ, we have $\rho[f(x), \gamma^0 \epsilon_0 + \gamma^1 \epsilon_1 + \ldots + \gamma^s \epsilon_s] < \epsilon$.

c) Let $\beta = \max[\beta(i)]$. Let the point $s \in E^r$ be such that $\rho(x,s) < \delta/\beta$. Now if $s = \sigma^0 e_0 + \sigma^1 e_1 + \ldots + \sigma^r e_r$ such that $|\sigma^j - \lambda^j| < \delta/\beta$ for each j then $\rho[f(s), f(x)] = \rho[f(\sigma^0 e_0 + \sigma^1 e_1 + \ldots + \sigma^r e_r), f(\lambda^0 e_0 + \lambda^1 e_1 + \ldots + \lambda^r e_r)] = \rho[\gamma^0 \epsilon_0 + \gamma^1 \epsilon_1 + \ldots + \gamma^s \epsilon_s, \mu^0 \epsilon_0 + \mu^1 \epsilon_e + \ldots + \mu^s \epsilon_s]$, where $\gamma_i = \sum_{k=1}^{\beta(i)} \sigma^{ik}$ and $\mu^i = \sum_{k=1}^{\beta(i)} \lambda^{ik}$. But for each i we have $|\gamma^i - \mu^i| =$

$\left|\sum_{k=1}^{\beta(i)} (\sigma^{ik} - \lambda^{ik})\right| \leq \sum_{k=1}^{\beta(i)} |\sigma^{ik} - \lambda^{ik}| < \sum_{k=1}^{\beta(i)} \delta/\beta = [\beta(i)][\delta/\beta] \leq \delta$, whenever $\rho[f(s), f(x)] < \epsilon$ by a).

<u>LEMMA</u> <u>83</u>: A simplicial mapping f of A^r into B^s is a continuous mapping of A^r into B^s.

<u>Proof</u>: Suppose $A^r = (a_0, a_1, \ldots, a_r)$ and $B^s = (b_0, b_1, \ldots, b_s)$. Let $E^r = (e_0, e_1, \ldots, e_r)$ be the simplex whose vertices are orthonormal vectors in R^r. Let $E^s = (c_0, c_1, \ldots, c_s)$ be the simplex whose vertices are orthonormal vectors in R^s. The mapping g which maps A^r onto E^r is

bi-continuous and one-to-one by Lemma 31. The mapping i which maps E^s onto B^s is also continuous and one-to-one by the same lemma. Let $x = \lambda^0 a_0 + \lambda^1 a_1 + \dots + \lambda^r a_r$ be a point of A^r. Then the function g which maps x into $\lambda^0 e_0 + \lambda^1 e_1 + \dots + \lambda^r e_r$ is bi-continuous and one-to-one by Lemma 31. Let h be the simplicial mapping which maps $\lambda^0 e_0 + \lambda^1 e_1 + \dots + \lambda^r e_r \in E^r$ into $\lambda^0 h(e_0) + \lambda^1 h(e_1) + \dots + \lambda^r h(e_r) \in E^s$. This is clearly a continuous mapping by Lemma 82. Therefore $f = ihg$ is a continuous mapping from A^r to B^s.

<u>LEMMA</u> <u>84</u>: If $x \in A^r$, and f is a simplicial mapping of A^r into B^s, then $f(x) \in B^s$.

<u>Proof</u>: Let $x = \lambda^0 a_0 + \lambda^1 a_1 + \dots + \lambda^r a_r$. Now since f may map more than one vertex of A^r into a single vertex of B^s we may denote $f(x) = \lambda^0 f(a_0) + \lambda^1 f(a_1) + \dots + \lambda^r f(a_r)$ by $f(x) = \mu^0 b_0 + \mu^1 b_1 + \dots + \mu^s b_s$, where μ^j is the sum of all the λ^i for which a_i maps into b_j. Since the λ^i satisfy the restrictions on coefficients in the definition of a point of the simplex spanned by the vertices which appear with these coefficients we know that the μ^j also satisfy the restrictions. Therefore $f(x) \in B^s$.

<u>LEMMA</u> <u>85</u>: The set $f(A^r)$ is a face D^k of the simplex B^s, where D^k spans those vertices b_j which are of the form $f(a_i)$.

<u>Proof</u>: Since $f(x) = \mu^0 b_0 + \mu^1 b_1 + \dots + \mu^s b_s$ where μ^j is the sum of all the λ^i for which $f(a_i) = b_j$ then

$f(A^r) \subset D^k$. On the other hand let $D^k = (c_0, c_1, \ldots, c_k)$. Then for each b_i we may select d_i, a vertex of A^r such that $f(d_i) = c_i$. Let $x \in D^k$; $x = \lambda^0 c_0 + \lambda^1 c_1 + \ldots + \lambda^k c_k$. Then $f(\lambda^0 d_0 + \lambda^1 d_1 + \ldots + \lambda^k d_k) = \lambda^0 c_0 + \lambda^1 c_1 + \ldots + \lambda^k c_k = x$. Therefore $D^k \subset f(A^r)$. Thus $D^k = f(A^r)$.

<u>LEMMA</u> <u>86</u>: If f is a simplicial mapping of the simplex A^r into the simplex B^s and if g is a simplicial mapping of the simplex B^s into the simplex $C^t = (c_0, c_1, \ldots, c_t)$, then gf is a simplicial mapping of A^r into C^t.

<u>Proof</u>: Now if x is a point of A^r then $g[f(x)] = \lambda^0 g[f(a_0)] + \lambda^1 g[f(a_1)] + \ldots + \lambda^r g[f(a_r)]$. Since each point $f(a_i)$ is a vertex of B^s and since gf assigns to each point a_i a unique point in C^t, then gf is a simplicial mapping by definition.

<u>DEFINITION</u> <u>75</u>: Let K and L be two complexes and let f be a continuous mapping of the polyhedron $|K|$ into the polyhedron $|L|$. If f is simultaneously a simplicial mapping of the simplex A into some simplex B of L for every simplex A of K then f is a <u>simplicial mapping of the complex K into the complex L</u>.

<u>LEMMA</u> <u>87</u>: If K, L, M are complexes and if f is a simplicial mapping of the complex K into the complex L and if g is a simplicial mapping of the complex L into the complex M then gf is a simplicial mapping of the complex K into the complex M.

<u>Proof</u>: The lemma follows from Lemma 87 immediately.

LEMMA 88: If $a_0, a_1, \ldots, a_r$ are vertices of a simplex of the complex K and if f is a simplicial mapping of the complex K into the complex L then $f(a_0), f(a_1), \ldots, f(a_r)$ are vertices of a simplex of the complex L.

Proof: This lemma follows directly from the definition.

DEFINITION 76: A mapping f which assigns to every vertex of the complex K a vertex of the complex L in such a way as to satisfy Lemma 88 is called a simplicial vertex mapping of the complex K into the complex L. It may also be referred to as a simplicial mapping of the abstract complex $\mathcal{K}$ into the abstract complex $\mathcal{L}$, where $\mathcal{K}$ and $\mathcal{L}$ are the abstract complexes corresponding to the geometric complexes K and L. If two or more distinct vertices of the simplex A are mapped by f into a single vertex, then the simplex A is said to be degenerate under the mapping f (or merely: degenerate).

LEMMA 89: If K and L are two geometric complexes and f is a simplicial vertex mapping of the complex K into the complex L, then f can be extended to a unique mapping g of the whole polyhedron $|K|$ so that g is a simplicial mapping of the complex K into the complex L.

Proof: Let $a_0, a_1, \ldots, a_k$ be the vertices of the complex K, let $\mathcal{K}$ be the abstract complex corresponding to K, and let N be the natural realization of $\mathcal{K}$ in the simplex $E^k = (e_0, e_1, \ldots, e_k)$. We may allow a_i and e_i to correspond to the same vertex of the abstract complex $\mathcal{K}$ without loss of generality. Now the relation

(45) $\quad x = \lambda^0 a_0 + \lambda^1 a_1 + \ldots + \lambda^k a_k$

assigns to each point $\lambda = \lambda^0 e_0 + \lambda^1 e_1 + \ldots + \lambda^k e_k \in |N|$ a point $x \in |K|$. The resulting mapping from λ to x of the polyhedron $|N|$ into the the polyhedron $|K|$ is a one-to-one and bi-continuous mapping by Lemma 42. The relation

(46) $\quad g(x) = \lambda^0 f(a_0) + \lambda^1 f(a_1) + \ldots + \lambda^k f(a_k)$

defines a continuous mapping of λ into $g(x)$ of the polyhedron $|N|$ into the polyhedron $|L|$. Therefore relations (45) and (46) together define a continuous mapping g of $|K|$ into $|L|$. Clearly $g(a_i) = f(a_i)$ and hence $g = f$ when restricted to the vertices a_i of K. Further, if $A^r = (a_{i_0}, a_{i_1}, \ldots, a_{i_r})$ is a simplex of K, then g defines a simplicial mapping of A^r into a simplex B^s of L. Thus g is a simplicial mapping of K into L which coincides with f on the vertices of K. Furthermore, the mapping g is unique becauss if $f = g$ on all the vertices of some simplex A^r it can be extended to the entire simplex A^r so as to be a simplicial mapping on A^r in just one way, that is, by means of the relation $g(x) = \lambda^0 g(a_0) + \lambda^1 g(a_1) + \ldots + \lambda^r g(a_r)$. Since g and f coincide on every simplex of K, they are identical.

<u>REMARK</u>: We recall here that the set of all interior points of a simplex of a complex K is referred to as an open simplex of K.

<u>LEMMA 90</u>: Every point of the polyhedron $|K|$ is contained in exactly one open simplex of K. In other words, $|K|$ is the

set-theoretic union of all of the open simplexes of K.

<u>DEFINITION</u> <u>77</u>: If a is a vertex of the complex K, the set-theoretic union of all of the open simplexes of K with a as a vertex is called the <u>star</u> of the vertex a in K and is denoted by S(a).

<u>LEMMA</u> <u>91</u>: Every star S(a) of K is an open set in $|K|$.

<u>Proof</u>: Clearly $S(a) \subset |K|$. Let $F = |K| - S(a)$. Let K^* consist of all simplexes of K which do not have a as a vertex. Then $|K^*| = F$. This is because, by construction, F is the set-theoretic union of all open simplexes of K not having a as a vertex; but if A is an open simplex which does not have a as a vertex then none of its faces has a as a vertex. That is, $\bar{A} \subset F$. Therefore F is the set-theoretic union of all closed simplexes of K which do not have a as a vertex, and $F = |K^*|$. Now $|K^*|$ is a subcomplex of K and is hence compact and therefore closed. Also $|K|$ is closed. Therefore S(a) is open, since $F = |K| - S(a)$.

<u>DEFINITION</u> <u>78</u>: Let φ be a continuous mapping of a complex K into a complex L. Then φ is said to satisfy the <u>star condition</u> in case for every star S(a) of K there is at least one star S(b) of L such that $\varphi[S(a)] \subset S(b)$. We sometimes say that φ is a star-related continuous mapping in case it satisfies the star condition.

<u>DEFINITION</u> <u>79</u>: Let φ be a continuous mapping of the complex K into the complex L which satisfies the star condition. Assign to each vertex a of K any vertex f(a) of L for

which the following relation is satisfied: $\varphi[S(a)] \subset S[f(a)]$. It will be shown in the following theorem that f is a simplicial vertex mapping which can be extended by Lemma 89 to a simplicial mapping of the whole complex K into the complex L. With these conditions f is called a <u>simplicial approximation</u> to φ or we say that φ admits of a simplicial approximation f.

<u>THEOREM 11</u>: Let φ be a continuous mapping of the complex K into the complex L which satisfies the star condition. Assign to each vertex a of K any vertex f(a) of L for which the following relation is satisfied: $\varphi[S(a)] \subset S[f(a)]$. Then f is a simplicial vertex mapping of K into L and therefore can be extended by Lemma 89 to a simplicial mapping f of the whole complex K into the complex L. Then f is a simplicial approximation to φ and furthermore if $x \in |K|$, $D \in L$, and $\varphi(x) \in D$, then $f(x) \in D$.

<u>Proof</u>: Let $x \in |K|$. There is one and only one open simplex $A = (a_0, a_1, \ldots, a_r)$ of K which contains x and there is one and only one open simplex B of L which contains the point $\varphi(x)$. Now $x \in A \subset S(a_i)$ for $i = 0, 1, \ldots, r$, since $S(a_i)$ is the set-theoretic union of all open simplexes of K which have a_i as a vertex, and A is certainly one of these open simplexes. Furthermore, $\varphi(x) \in \varphi[S(a_i)]$. Also $\varphi[S(a_i)] \subset S[f(a_i)]$. Further, since $\varphi(x) \in B$, the open simplex B is contained in the star $S[f(a_i)]$, and therefore $f(a_i)$ is a vertex of B. Now x is any point of $|K|$ so A is

an arbitrary open simplex of K. Hence f maps the vertices of a simplex of K into the vertices of a simplex of L. That is, f is a simplicial mapping. Now since f is a simplicial mapping, we have the relation $f(\overline{A}) = C$ where C is a face of the closed simplex $\overline{B}$. Now if D is a simplex of L which contains $\varphi(x)$ and T is the complex consisting of D and all its proper faces, then $\overline{B} \in T$ since $\varphi(x)$ is contained in exactly one open simplex of L, namely B. Therefore $\overline{B}$ is a face of D. Hence C is a face of D also and $f(x) \in f(\overline{A}) = C \subset D$. It may be noted that the simplicial approximation f to the continuous mapping φ is not unique since there may be several stars S(b) which satisfy the condition $\varphi[S(a)] \subset S(b)$. That is, although the star condition is satisfied for φ it may not be uniquely satisfied.

<u>LEMMA</u> <u>92</u>: Let K, L, M be three complexes, and let φ, ψ be continuous mappings of K into L and L into M respectively. If f, g are simplicial approximations to φ and ψ respectively, then gf is a simplicial approximation to $\psi\varphi$.

<u>Proof</u>: If a is a vertex of K, then $\varphi[S(a)] \subset S[f(a)]$ and therefore $\psi\{\varphi[S(a)]\} \subset \psi\{S[f(a)]\} \subset S\{g[f(a)]\}$, and therefore gf is a simplicial approximation to the mapping $\psi\varphi$.

<u>REMARK</u>: The following ideas are applicable to abstract as well as geometric complexes and no distinction is made in the proofs.

<u>DEFINITION</u> <u>80</u>: Let f be a simplicial mapping of a complex K

into a complex L, and let $A^r = \epsilon(a_0, a_1, \ldots, a_r)$ be an oriented simplex of K. If all the vertices of A^r are mapped into distinct vertices of L (that is, if A^r is not degenerate under the mapping f) we define

(47) $\hat{f}(A^r) = \epsilon(b_0, b_1, \ldots, b_r) = B^r$

where $f(a_i) = b_i$. On the other hand, if A^r is in fact degenerate under the mapping f we define

(48) $\hat{f}(A^r) = 0.$

Now if $x = g_1 A_1^r + g_2 A_2^r + \ldots + g_{\alpha(r)} A^r_{\alpha(r)}$ is any r-chain of K over the coefficient group G, we associate with the chain x a chain $\hat{f}(x)$ of L of the same dimension and over the same group G by means of the relation:

(49) $\hat{f}(x) = g_1\hat{f}(A_1^r) + g_2\hat{f}(A_2^r) + \ldots + g_{\alpha(r)}\hat{f}(A^r_{\alpha(r)}).$

We say that the simplicial mapping f induces a <u>chain mapping</u> $\hat{f}$ given by relation (49).

<u>LEMMA 93</u>: $\hat{f}(x + y) = \hat{f}(x) + \hat{f}(y).$

<u>LEMMA 94</u>: $\hat{f}$ satisfies the following condition:

(50) $\Delta\hat{f}(x) = \hat{f}(\Delta x).$

<u>Proof</u>: If the lemma is true for $x = A^r$ it will be true in general for a chain x.

a) Now if A^r is not degenerate, relation (47) implies that $\hat{f}(A^r) = \sum_{i=0}^{r} \epsilon(-1)^i(b_0, b_1, \ldots, \hat{b}_i, \ldots, b_r).$

Moreover, $A^r = \sum_{i=0}^{r} \epsilon(-1)^i(a_0, a_1, \ldots, \hat{a}_i, \ldots, a_r).$ Since A^r is not degenerate, none of its faces is degenerate and

$$\hat{f}(\Delta A^r) = \sum_{i=0}^{r} \epsilon(-1)^i(b_0, b_1, \ldots, \hat{b}_i, \ldots, b_r) = \Delta f(A^r).$$

b) Let A^r be degenerate.

1) Let $f(A^r)$ have dimension $r - 1$. That is, suppose that exactly two vertices of A^r correspond to a single vertex of $f(A^r)$ and let all of the other vertices of A^r be mapped into distinct other vertices. We may assume without loss of generality that $f(a_0)$ and $f(a_1)$ each equal b and that all the other vertices $f(a_i) = b_i$, $i = 2, 3, \ldots, r$, are distinct and unequal to b. Now since A^r is degenerate, $\hat{f}(A^r) = 0$ and consequently $\Delta \hat{f}(A^r) = 0$. Therefore it is sufficient to show that $\hat{f}(\Delta A^r) = 0$. This may be achieved by first considering that $\Delta A^r = \sum_{i=0}^{r} \epsilon(-1)^i(a_0, a_1, \ldots, \hat{a}_i, \ldots, a_r)$. Since a_0 and a_1 are mapped to b, any simplex in the summation containing both of these vertices is degenerate. Thus there are just two values of i on the right-hand side of this equation which contribute non-degenerate simplexes. These values of i are 0 and 1. Therefore all simplexes other than $(a_1, a_2, \ldots, a_r)$ and $(a_0, a_2, \ldots, a_r)$ which appear in the summation are degenerate. Therefore $\hat{f}(A^r) = \epsilon(b, b_2, \ldots, b_r) - \epsilon(b_1, b_2, \ldots, b_r) = 0$.

2) If $f(A^r)$ has dimension less than $r - 1$ then all of its $(r - 1)$-faces are degenerate and relation (48) implies $\hat{f}(\Delta A^r) = 0$ and $\Delta \hat{f}(A^r) = 0$.

c) Thus in any case $\Delta \hat{f}(A^r) = \hat{f}\Delta(A^r)$.

<u>LEMMA</u> <u>95</u>: Let f be a simplicial mapping of a complex K into a complex L. If z is a cycle of K, then $\hat{f}(z)$ is a cycle of L.

<u>Proof</u>: Let z be a cycle of K. Then by definition of cycle, $\Delta z = 0$. Then since $\Delta \hat{f}(z) = \hat{f}(\Delta z)$ by the last lemma, we have $\Delta \hat{f}(z) = 0$ and hence $\hat{f}(z)$ is a cycle of L.

<u>LEMMA</u> <u>96</u>: Let f be a simplicial mapping of a complex K into a complex L. Let $z_1 \sim z_2$ be cycles in K. Then $\hat{f}(z_1) \sim \hat{f}(z_2)$.

<u>Proof</u>: Since $z_1 \sim z_2$ then $z_1 - z_2 = \Delta x$, where x is some chain in K. By Lemma 93 we have $\hat{f}(z_1 - z_2) = \hat{f}(z_1) - \hat{f}(z_2)$. Thus $\hat{f}(z_1) - \hat{f}(z_2) = \hat{f}(\Delta x)$. But by the last lemma $\hat{f}(\Delta x) = \Delta \hat{f}(x)$. Thus $\hat{f}(z_1) - \hat{f}(z_2) = \Delta \hat{f}(x)$, and $\hat{f}(z_1) \sim \hat{f}(z_2)$.

<u>COROLLARY</u>: $\hat{f}\left[Z^r(K)\right] \subset Z^r(L)$ and $\hat{f}\left[H^r(K)\right] \subset H^r(L)$.

<u>DEFINITION</u> <u>81</u>: Let f be a simplicial mapping of a complex K into a complex L, let $B^r(K)$ and $B^r(L)$ be the r-dimensional Betti groups of K and L over an arbitrary group G. If $z^* \in B^r(K)$ and if z is any cycle of the homology class z^*, set

(51) $$\tilde{f}(z^*) = \left[\hat{f}(z)\right]^*,$$

where $\left[\hat{f}(z)\right]^*$ is the homology class of $B^r(L)$ which contains the cycle $\hat{f}(z)$. In the next lemma it will be shown that the mapping $\tilde{f}$ of $B^r(K)$ into $B^r(L)$ defined by (51) is unique, and that it is indeed a homomorphism of $B^r(K)$ into $B^r(L)$. Therefore $\tilde{f}$ is referred to as the <u>induced</u> <u>homomorphism</u> <u>of</u>

the simplicial mapping f.

LEMMA 97: The mapping $\tilde{f}$ as defined above is unique and is, moreover, a homomorphism of $B^r(K)$ into $B^r(L)$.

Proof: a) Let z_1 and z_2 be two cycles of z^*. Then $z_1 \sim z_2$. Hence $\hat{f}(z_1) \sim \hat{f}(z_2)$. Therefore $[\hat{f}(z_1)]^* = [\hat{f}(z_2)]^*$. Thus $\tilde{f}$ is unique.

b) Let u^*, v^* be two homology classes of $B^r(K)$ such that $u^* + v^* = w^*$ and such that u, v are cycles of u^*, and v^* respectively. Then $w = (u + v) \varepsilon w^*$, and $\tilde{f}(w^*) = [\hat{f}(w)]^* = \hat{f}(u + v)^* = [\hat{f}(u) + \hat{f}(v)]^*$. The last equality holds because $\hat{f}$ is a homomorphsim. The last term of this equation is the homology class containing $[\hat{f}(u) + \hat{f}(v)]$. Now the class containing a sum is equal to the sum of the corresponding classes since $[\hat{f}(u) + \hat{f}(v)]^*$ is a sum in a factor group. Therefore $[\hat{f}(u) + \hat{f}(v)]^* = [\hat{f}(u)]^* + [\hat{f}(v)]^* = \tilde{f}(u^*) + \tilde{f}(v^*)$. Thus $\tilde{f}(w^*) = \tilde{f}(u^*) + \tilde{f}(v^*)$ and $\tilde{f}$ is a homomorphsim.

LEMMA 98: If K, L, M are three complexes and f and g are simplicial mappings of K into L and L into M respectively then the induced mappings of the simplicial mapping $e = gf$ satisfy the relations

(52) $\quad \hat{e} = \hat{g}\hat{f}$

and

(53) $\quad \tilde{e} = \tilde{g}\tilde{f}$.

Proof: a) It is sufficient to prove (52) for an oriented simplex A^r of K.

1) If A^r is not degenerate under e, then A^r is not degenerate under f and $f(A^r)$ is not degenerate under g. Then clearly, $\hat{e}(A^r) = \hat{g}\left[\hat{f}(A^r)\right]$. That is, relation (52) holds.

2) If A^r is degenerate under e then either A^r is degenerate under f or $f(A^r)$ is degenerate under g.

A) If A^r is degenerate under f then $\hat{f}(A^r) = 0$ and $\hat{g}\left[\hat{f}(A^r)\right] = 0$.

B) If $f(A^r)$ is degenerate under g then $\hat{g}\left[\hat{f}(A^r)\right] = 0$.

Thus in either case 1) or 2) $\hat{e}(A^r) = 0 = \hat{g}\hat{f}(A^r)$.

b) Relation (52) implies relation (53) because if $z^* \in B^r(K)$ and z is a cycle of z^* then $\tilde{e}(z^*) = \left[\hat{e}(z)\right]^* = \hat{g}\left[\hat{f}(z)\right]^* = \tilde{g}\left\{\left[\hat{f}(z)\right]^*\right\} = \tilde{g}\left[\tilde{f}(z^*)\right]$.

<u>DEFINITION</u> <u>82</u>: Let R^n be a Euclidean space, let F be any set of points in R^n, and let χ be a point of R^n. The point χ is said to be in <u>general</u> <u>position</u> <u>with</u> <u>respect</u> <u>to</u> <u>the</u> <u>set</u> <u>F</u> if, for any two distinct points x and y of F, the segments $[\chi, x]$ and $[\chi, y]$ have just the one point χ in common. If χ is in general position with respect to the set F, then the set of all points belonging to segments $[\chi, x]$, where x is any point of F, is called the <u>cone</u> <u>with</u> <u>vertex</u> <u>χ</u> <u>and</u> <u>base</u> <u>F</u>, and is denoted by $\chi(F)$.

<u>LEMMA</u> <u>99</u>: If χ is in general position with respect to the set F and if $G \subset F$, then $\chi(G) \cap F = G$.

LEMMA 100: If W is a convex body, if U is the set of interior points of W, if V is the frontier of W, and if χ is any point of U, then χ is in general position with respect to V and $W = \chi(V)$.

Proof: This proposition follows directly from Lemma 19.

LEMMA 101: If $A^r = (a_0, a_1, \ldots, a_r)$ is an r-simplex of the n-dimensional Euclidean space R^n then A^r is a convex set. Further, if G^r is the set of interior points of A^r, if F^{r-1} is the frontier of A^r, and if χ is any point of G^r, then χ is in general position with respect to the set F^{r-1}, and $A^r = \chi(F^{r-1})$.

Proof: Let $a = \lambda^0 a_0 + \lambda^1 a_1 + \ldots + \lambda^r a_r$ and $b = \mu^0 a_0 + \mu^1 a_1 + \ldots + \mu^r a_r$ be two distinct points of A^r and let x be any point of the closed segment $[a,b]$. Then $x = \alpha a + \beta b$, $\alpha + \beta = 1$, $\alpha \geq 0$, $\beta \geq 0$, and $x = \gamma^0 a_0 + \gamma^1 a_1 + \ldots + \gamma^r a_r$ where

(54) $$\gamma^i = \alpha \lambda^i + \beta \mu^i, \quad i = 0, 1, \ldots, r.$$

Now $\sum_{i=0}^{r} \gamma^i = \alpha \sum_{i=0}^{r} \lambda^i + \beta \sum_{i=0}^{r} \mu^i = \alpha + \beta = 1$. Also $\gamma^i = \alpha \lambda^i + \beta \mu^i \geq 0$, because $\alpha, \lambda^i, \beta, \mu^i \geq 0$, $i = 0, 1, \ldots, r$. Therefore $x \in A^r$ by definition of simplex, and we have the barycentric co-ordinates of the point x of the closed segment $[a,b]$ expressed in terms of the barycentric co-ordinates of the points a and b by relation (54). Since a and b were any two distinct points of the simplex A^r and since x was any point of the closed segment $[a,b]$, and since we have shown that therefore x is a point of A^r it is clear by the definition of convex set that A^r is a convex set. It remains to be shown that if $\mathcal{K}$ is any point of the interior of A^r then $\mathcal{K}$ is in general position with respect to the frontier of A^r, and the cone $\mathcal{K}(F^{r-1}) = A^r$. Consider $A^r \subset R^r$. A^r is a convex body since A^r is a convex set which is compact and which contains by assumption at least one interior point, namely $\mathcal{K}$. Now G^r is the interior of A^r and F^{r-1} is the frontier of A^r. By Lemma 18 $\mathcal{K}(F^{r-1}) - F^{r-1} \subset G^r$, hence $\mathcal{K}(F^{r-1}) \subset G^r \cup F^{r-1} = A^r$. On the other hand, by Lemma 20 if c is any point of A^r and $\mathcal{K} \varepsilon G^r$, there exists a point b εF^{r-1} such that c $\varepsilon [a,b]$. That is, $A^r \subset \mathcal{K}(F^{r-1})$. Thus $A^r = \mathcal{K}(F^{r-1})$.

<u>LEMMA</u> <u>102</u>: If $A^r = (a_0, a_1, \ldots, a_r)$ is a simplex of the Euclidean space R^n, then a point $\mathcal{K} \varepsilon R^n$ is in general position with respect to the set A^r if and only if the system $\mathcal{K}, a_0, a_1, \ldots, a_r$, is independent. Clearly if $\mathcal{K}, a_0, a_1, \ldots, a_r$ is an independent set then $B^{r+1} = (\mathcal{K}, a_0, a_1, \ldots, a_r)$ is a simplex. Further, if the set $\mathcal{K}, a_0, a_1, \ldots, a_r$ is independent, then $\mathcal{K}(A^r) = B^{r+1}$.

<u>Proof</u>: The proof of this lemma begins with the proof that if $\mathcal{K}$ is in general position with respect to A^r then $\mathcal{K}, a_0, a_1, \ldots, a_r$ is an independent set and conversely. Then we show if $\mathcal{K}$ is in general position with respect to A^r then B^{r+1} is a simplex by means of the above short proof and

the definition of simplex. It is then shown that the cone $\mathcal{K}(A^r) = B^{r+1}$ by point set identity methods. Finally we show that if $\mathcal{K}$ is in general position with respect to A^r then $\mathcal{K}(A^r)$ is a simplex.

a) First we show that if $\mathcal{K}$ is in general position with respect to A^r then $\mathcal{K}$, a_0, a_1, ..., a_r are independent. We achieve this by showing that if $\mathcal{K}$, a_0, a_1, ..., a_r are linearly independent then the two segments $[\mathcal{K}, x_1]$ and $[\mathcal{K}, x_2]$, for x_1 and x_2 distinct points of A^r cannot intersect in a point y distinct from $\mathcal{K}$. That is, $\mathcal{K}$ is in general position with respect to A^r by definition. This is done by proof of the contrapositive statement. To this end we assume that A^r contains two distinct points x_1, x_2 such that the segments $[\mathcal{K}, x_1]$ and $[\mathcal{K}, x_2]$ intersect in a point y distinct from $\mathcal{K}$. Let the barycentric co-ordinates of the points x_j, j = 1, 2, be denoted by λ_j^0, λ_j^1, ..., λ_j^r. Then

$$(55) \qquad y = \alpha_j \mathcal{K} + \beta_j \lambda_j^0 a_0 + \beta_j \lambda_j^1 a_1 + \dots + \beta_j \lambda_j^r a_r, \quad \alpha_j \neq 1.$$

Subtracting relation (55) for j = 1 from relation (55) for j = 2, we obtain

$$(56) \qquad (\alpha_2 - \alpha_1)\mathcal{K} + (\beta_2 \lambda_2^0 - \beta_1 \lambda_1^0)a_0 + (\beta_2 \lambda_2^1 - \beta_1 \lambda_1^1)a_1 + \dots + (\beta_2 \lambda_2^r - \beta_1 \lambda_1^r)a_r = y - y = 0.$$

Now $(\alpha_2 - \alpha_1) = \sum_{i=0}^{r} (\beta_2 \lambda_2^i - \beta_1 \lambda_1^i) = 0$, and not all the coefficients of (56) are equal to zero. Hence the set

$\mathcal{K}$, a_0, a_1, ..., a_r is a linearly dependent set.

b) We next show that if $\mathcal{K}$ being in general position with respect to A^r implies that the points $\mathcal{K}$, a_0, a_1, ..., a_r are independent then $\mathcal{K}(A^r) = B^{r+1}$. We assume that if $\mathcal{K}$ is in general position with respect to A^r, then the points $\mathcal{K}$, a_0, a_1, ..., a_r are independent. Now if $x = \lambda^0 a_0 + \lambda^1 a_1 + \ldots + \lambda^r a_r \in A^r$ then any $y \in \mathcal{K}(x)$ is of the form $y = \alpha\mathcal{K} + \beta\lambda^0 a_0 + \beta\lambda^1 a_1 + \ldots + \beta\lambda^r a_r$. If this is so then $y \in B^{r+1}$ by definition of simplex (depending on the vertices being independent) by the present assumption provided only that $\mathcal{K}$ is in general position with respect to A^r.

1) Now let $z = \mu\mathcal{K} + \mu^0 a_0 + \mu^1 a_1 + \ldots + \mu^r a_r$ be an arbitrary point of B^{r+1}. If $\mu = 1$, then, and only then, $z = \mathcal{K}$. Assume $z \neq \mathcal{K}$. Then $\mu \neq 1$. Set

(57) $\alpha = \mu$, $\beta = 1 - \mu$, $\lambda^i = \mu^i/\beta$.

Then $z = \alpha\mathcal{K} + \beta\lambda^0 a_0 + \beta\lambda^1 a_1 + \ldots + \beta\lambda^r a_r$. Then $z \in \mathcal{K}(A^r)$. Hence $B^{r+1} \subset \mathcal{K}(A^r)$.

2) Now let z be any element of $\mathcal{K}(A^r)$. Then z will be of the form $z = \alpha\mathcal{K} + (1 - \alpha)x$, where $x \in A^r$. Thus $z = \alpha\mathcal{K} + (1 - \alpha)(\lambda^0 a_0 + \lambda^1 a_1 + \ldots + \lambda^r a_r)$. By definition of simplex z obviously belongs to $B^{r+1} = (\mathcal{K}, a_0, a_1, \ldots, a_r)$ since each coefficient in the expression for z is a nonnegative number, and the sum is $\alpha + (1 - \alpha)(\lambda^0 + \lambda^1 + \ldots + \lambda^r) = \alpha + (1 - \alpha)(1) = 1$. Therefore $\mathcal{K}(A^r) \subset B^{r+1}$.

3) Since $\mathcal{K}(A^r) \subset B^{r+1}$ and $B^{r+1} \subset \mathcal{K}(A^r)$, then $\mathcal{K}(A^r) = B^{r+1}$.

c) Finally we show that if $\mathcal{K}$ is indeed in general position with respect to A^r, then $\mathcal{K}$, a_0, a_1, ..., a_r are independent. The proof will be by proof of the contrapositive. We assume that the points $\mathcal{K}$, a_0, a_1, ..., a_r are dependent. Then by definition of independence and by the fact that a_0, a_1, ..., a_r are independent we have:

(58) $$\gamma\mathcal{K} + \gamma^0 a_0 + \gamma^1 a_1 + \ldots + \gamma^r a_r = 0,$$
$$\gamma + \gamma^0 + \gamma^1 + \ldots + \gamma^r = 0,$$
$$\gamma \neq 0.$$

System (58) remains true if multiplied by any arbitrary real number, and hence the coefficients may be assumed to be arbitrarily small. Now if $x_1 = \lambda_1^0 a_0 + \lambda_1^1 a_1 + \ldots + \lambda_1^r a_r$ is any interior point of A^r, let

(59) $$y = \alpha_1 \mathcal{K} + \beta_1 \lambda_1^0 a_0 + \beta_1 \lambda_1^1 a_1 + \ldots + \beta_1 \lambda_1^r a_r,$$
$$\alpha_1 \neq 1, \ \beta_1 \neq 1.$$

Now since all coefficients in relation (59) are positive, we may obtain the relation $y = (\alpha_1 + \mathcal{K}) + (\beta_1 \lambda_1^0 + \gamma^0) a_0 + (\beta_1 \lambda_1^1 + \gamma^1) a_1 + \ldots + (\beta_1 \lambda_1^r + \gamma^r) a_r$ with all coefficients positive by adding (58) and (59) provided only that the coefficients in (58) are small enough. Let $\alpha_2 = \alpha_1 + \gamma$, $\beta_2 = 1 - \alpha_2$, $\lambda_2^i = (\beta_1 \lambda_1^i + \gamma^i)/\beta_2$. Then $y = \alpha_2 \mathcal{K} + \beta_2 \lambda_2^0 a_0 + \beta_2 \lambda_2^1 a_1 + \ldots + \beta_2 \lambda_2^r a_r$. Let $x_2 = \lambda_2^0 a_0 + \lambda_2^1 a_1 + \ldots + \lambda_2^r a_r$. If we set $x_1 = x_2 = x$, then $y = \alpha_1 \mathcal{K} + \beta_1 x$, $y = \alpha_2 \mathcal{K} + \beta_2 x$. Then since $\alpha_1 \neq \alpha_2 =$

$\alpha_1 + \gamma$, we have two distinct expressions for $y \in (\mathcal{K}, x)$. But $(\mathcal{K}, x)$ is a 1-simplex and since α and β are barycentric co-ordinates with respect to it the existence of two representations for $y \in (\mathcal{K}, x)$ is not possible. Therefore $x_1 \neq x_2$. A contradiction thus results from the assumption that $\mathcal{K}$, a_0, a_1, ..., a_r are dependent and the lemma is proved.

<u>LEMMA</u> <u>103</u>: Let K be a complex in the Euclidean space R^n and let $\mathcal{K}$ be a point of R^n in general position with respect to the polyhedron $|K|$. $\mathcal{K}$ is in general position with respect to any simplex A of K. Furthermore, $\mathcal{K}(A)$ is a simplex in R^n.

<u>Proof</u>: $\mathcal{K}$ is in general position with respect to $|K|$. Furthermore $\mathcal{K}$ is in general position with respect to A and hence by the preceding lemma $\mathcal{K}(A)$ is a simplex in R^n.

<u>LEMMA</u> <u>104</u>: Let K be a complex in the Euclidean space R^n and let $\mathcal{K}$ be a point of R^n in general position with respect to the polyhedron $|K|$. Then the set of all simplexes of the form $\mathcal{K}(A)$, $A \in K$, and the set of faces of the simplexes of this form, form a complex which we may denote by $\mathcal{K}(K)$. Furthermore, $|\mathcal{K}(K)| = \mathcal{K}(|K|)$.

<u>Proof</u>: a) We need only show that the simplexes of $\mathcal{K}(K)$ are properly situated to prove that $\mathcal{K}(K)$ is a complex. First we note that if P and Q are two properly situated simplexes, then their respective boundaries consist of properly situated simplexes. Therefore it will be

sufficient to show that any two simplexes of the form $\mathcal{K}(A)$ and $\mathcal{K}(B)$, where A, B ε K, are properly situated. Now if $A \cap B = \emptyset$, then $\mathcal{K}(A) \cap \mathcal{K}(B) = \mathcal{K}$ and $\mathcal{K}$ is their common vertex. If $A \cap B = C$, then C is the common face of A and B and we have $\mathcal{K}(A) \cap \mathcal{K}(B) = \mathcal{K}(C)$.

b) To prove that $|\mathcal{K}(K)| = \mathcal{K}(|K|)$ we let A ε K. Then $A \subset |K|$ and therefore $\mathcal{K}(A) \subset \mathcal{K}(|K|)$. Hence $|\mathcal{K}(K)| \subset \mathcal{K}(|K|)$. Now if $y \;\varepsilon\; \mathcal{K}(|K|)$, then there exists a point $x \;\varepsilon\; |K|$ such that $y \;\varepsilon\; (\mathcal{K}, x)$ by definition of the cone $\mathcal{K}(|K|)$. Since x is an element of some simplex A of K we know that $y \;\varepsilon\; \mathcal{K}(A)$. Thus $\mathcal{K}(|K|) \subset |\mathcal{K}(K)|$.

c) Since $\mathcal{K}(|K|) \subset |\mathcal{K}(K)|$ and $|\mathcal{K}(K)| \subset \mathcal{K}(|K|)$, then $\mathcal{K}(|K|) = |\mathcal{K}(K)|$.

<u>LEMMA</u> <u>105</u>: Let K be a complex imbedded in the Euclidean space R^n, and let $\mathcal{K}$ be a point in general position with respect to the polyhedron $|K|$. If $A^r = \epsilon(a_0, a_1, \ldots, a_r)$ is any oriented simplex we may denote by $\mathcal{K}(A^r)$ the oriented simplex $\epsilon(\mathcal{K}, a_0, a_1, \ldots, a_r)$ of the complex $\mathcal{K}(K)$. If $x^r = g_1A_1^r + g_2A_2^r + \ldots + g_kA_k^r$ is any r-chain of K with coefficient group G and if $\mathcal{K}(x^r) = g_1\mathcal{K}(A_1^r) + g_2\mathcal{K}(A_2^r) + \ldots + g_k\mathcal{K}(A_k^r)$, then $\mathcal{K}(x^r)$ is an $(r+1)$-chain of $\mathcal{K}(K)$ over the coefficient group G. Furthermore, the following relations are satisfied:

(60) $\quad \Delta\mathcal{K}(x^r) = x^r - \mathcal{K}\Delta(x^r), \quad r > 0$

(61) $\quad \Delta\mathcal{K}(x^0) = x^0 - I(x^0)(\mathcal{K})$.

<u>Proof</u>: $\mathcal{K}(x^r)$ is an $(r+1)$-chain of $\mathcal{K}(K)$ over the

coefficient group G by definition of chain. The relations (60) and (61) are obvious for $x^r = A_i^r$, $i = 1, 2, \ldots, k$. The relations may be extended to an arbitrary chain x^r through multiplication of each term by the appropriate g_i and then summing over i.

<u>THEOREM</u> <u>12</u>: Let A^r be an r-simplex, let S^{r-1} be the complex consisting of all of the proper faces of A^r, and let T^r be the complex consisting of all of the faces of A^r, including A^r itself. Then every s-cycle z^s, $s > 0$, of T^r is homologous to zero. Furthermore, every s-cycle z^s, $0 < s < (r - 1)$ in S^{r-1} is homologous to zero, and every $(r - 1)$-cycle z^{r-1}, $(r - 1) > 0$, in S^{r-1} is of the form $z^{r-1} = g\Delta(A^r)$, where g is an element of the coefficient group chosen and A^r is the oriented simplex.

<u>Proof</u>: We may assume without loss of generality that the simplex $A^r = (a_0, a_1, \ldots, a_r)$ is imbedded in R^n and that a point $\mathcal{K}$ exists such that the system $\mathcal{K}, a_0, a_1, \ldots, a_r$ is independent. We define a simplicial mapping f of the complex $\mathcal{K}(T^r)$ into the complex T^r by letting $f(\mathcal{K}) = a_0$, $f(a_i) = a_i$, $i = 0, 1, \ldots, r$. Now if z^s, $s > 0$, is any cycle of T^r, set $v = \mathcal{K}(z^s)$ so that $\Delta v = z^s$. This is possible by relation (60) of Lemma 105. Then z^s is homologous to zero in $\mathcal{K}(T^r)$, and $\hat{f}(z^s) = z^s$ by construction. Also, since $z^s = \hat{f}(z^s) = \Delta\hat{f}(v) = \hat{f}(\Delta v) = \Delta\hat{f}(v) = \Delta u$ for some chain u of T^r, then z^s is homologous to zero in T^r also. Now if z^s, is any cycle of S^{r-1}, then $z^s = \Delta(u)$ where u is

some chain of T^r. If $s < (r - 1)$, then the chain u is contained in S^{r-1} by hypothesis and therefore z^s is homologous to zero in S^{r-1}. However, if $s = r - 1$, then since T^r contains precisely one simplex A^r of dimension r, we have $u = gA^r$, where here A^r may denote either of the two possible orientations of A^r. If $u = gA^r$, then $z^{r-1} = g\,\Delta(A^r)$.

REMARK: Zero-dimensional homologies in the complexes are considered in light of the ideas contained in the material commencing with Definition 63 and ending with Theorem 8. We may also note that T^r is always a connected set, and that S^{r-1} is a connected set except when $r - 1 = 0$, in which case $r = 1$ and S^{r-1} consists of two points.

DEFINITION 83: Let K be a geometric complex imbedded in the Euclidean space R^m. We define the complex K', imbedded also in R^m, to be the barycentric subdivision of K in the following manner : if K is a 0-complex, let $K' = K$. If K is not a 0-complex, assume that K is an $(n + 1)$-complex, $n = 0, 1, \ldots$. Assume further that the barycentric subdivision of an arbitrary n-complex has been defined already satisfying the following two conditions:

a) $|P'| = |P|$

b) if Q is a subcomplex of P, then Q' is a subcomplex of P'.

Then to define the barycentric subdivision of an $(n + 1)$-complex K, we let M be the n-skeleton of K, we let $A_1^{n+1}, A_2^{n+1}, \ldots, A_k^{n+1}$ be the set of all

$(n + 1)$-simplexes of K, we let S_i be the set of all proper faces of A_i^{n+1}, $S_i \subset M$, and we let $\varkappa_i$ be the point of A_i^{n+1} whose barycentric co-ordinates are all equal to $1/(n + 2)$. We define $\varkappa_i$ to be the <u>barycenter</u> (or merely: <u>center</u>) of A_i^{n+1}. Since S_i is an n-complex, by assumption its barycentric subdivision S_i' is already defined. Furthermore, $\varkappa(S_i')$ is a complex by Lemma 101. Finally we define K' to be the set of all simplexes contained in the complexes M' and $\varkappa_i(S_i')$, i $1, 2, \ldots, k$.

<u>LEMMA 106</u>: K' as defined above is a complex. K' satisfies conditions a) and b) in the definition of barycentric subdivision.

<u>Proof</u>: a) To prove that K' is a complex we must show first that K' satisfies the first requirement of a complex, that is, that if A is a simplex of K', then every face of A is also in K'. This is clear, because K' is the set of all simplexes contained in the complexes M' and $\varkappa_i(S_i')$, $i = 1, 2, \ldots, k$. We recall that S_i is the set of all proper faces of the $(n + 1)$-simplexes of K. Secondly, we must show that every two simplexes of K are properly situated. To this end we let P and Q be two simplexes of K' and we consider the following three possible cases:

1) P and Q are contained in M'. In this case P and Q are properly situated because M' is a complex by the inductive hypothesis in the definition of barycentric subdivision.

2) $P \in M'$, $Q \in \mathcal{K}_i(S_i')$. In this case we need only consider the case of the form $Q \in \mathcal{K}_i(B)$, where $B \in S_i'$, since all simplexes of $\mathcal{K}_i(S_i')$ are faces of simplexes of the form $\mathcal{K}_i(B)$, $B \in S_i'$. Now since $P \subset |M'| = |M|$ and since $\mathcal{K}_i(B) \subset A_i^{n+1}$ then clearly $P \cap Q \subset |M| \cap A_i^{n+1} = |S_i|$. Also $\mathcal{K}_i(B) \cap |S_i| = B$ implies that $P \cap Q \subset P \cap B$. Furthermore, P and B are properly situated since they are simplexes of M'. Therefore, by Lemma 35, P qnd Q are properly situated.

3) $P \in \mathcal{K}_i(S_i')$, $Q \in \mathcal{K}_j(S_j')$. If $i = j$ then P and Q are clearly properly situated, since they are contained in the same complex. If $i \neq j$, then we assume that $P = \mathcal{K}_i(A)$, $A \in (S_i')$, $Q = \mathcal{K}_j(B)$, $B \in S_j'$ and we have $P \subset A_i^{n+1}$, $Q \subset A_j^{n+1}$, $P \cap Q \subset A_i^{n+1} \cap A_j^{n+1}$. Now since $i \neq j$, we have $A_i^{n+1} \cap A_j^{n+1} \subset |S_i| \cap |S_j|$. Therefore $P \cap Q \subset |S_i| \cap |S_j|$. Hence $P \cap Q = P \cap |S_i| \cap Q \cap |S_j|$. Now $P \cap |S_i| = \mathcal{K}_i(A) \cap |S_i| = A$ and $Q \cap |S_j| = \mathcal{K}_j(B) \cap |S_j| = B$, and hence $P \cap Q = A \cap B$. Therefore P and Q are properly situated by Lemma 35 because A and B, being in M', are properly situated.

b) To prove that K' satisfies conditions a) and b) of the definition of the barycentric subdivision of an $(n+1)$-complex K we note first that

1) $|K| = |M| \cup A_1^{n+1} \cup A_2^{n+1} \cup \ldots \cup A_k^{n+1}$ and that $|K'| = |M'| \cup |\mathcal{K}_1(S_1')| \cup |\mathcal{K}_2(S_2')| \cup \ldots \cup |\mathcal{K}_k(S_k')|$. Now by the inductive hypothesis of the

definition, $|M| = |M'|$. By Lemma 101, Lemma 103, and Lemma 104 we know that $A_i^{n+1} = |\mathcal{K}_i(S_i')|$. Therefore $|K| = |K'|$ and K' satisfies condition a) of the definition.

2) Secondly, we let L be any subcomplex of K and we let N be the n-skeleton of L. Let A_1^{n+1}, A_2^{n+1}, ..., A_p^{n+1} be the set of all $(n+1)$-simplexes of L. Now L' consists of all the simplexes contained in the complexes N' and $\mathcal{K}_j(S_j')$, $j = 1, 2, \ldots, p$. Now N' is a subcomplex of M' because of the facts that N is a subcomplex of M and that the inductive hypothesis holds. Therefore, L' is a subcomplex of K' since $p \leq k$. Thus finally, K' satisfies condition b) of the definition.

<u>LEMMA</u> <u>107</u>: Let K be a complex in R^m and let

(62) $\quad A_0, A_1, \ldots, A_r$

be any sequence of simplexes of K in which A_{i+1} is a proper face of A_i, $i = 0, 1, \ldots, (r-1)$. Then

(63) $\quad (\sigma_0, \sigma_1, \ldots, \sigma_r)$

is a simplex of K', where σ_i is the barycenter of A_i. Conversely, every simplex P of K' can be shown to be a simplex of the form (63) where each σ_i is the barycenter of A_i and where (62) is a sequence of simplexes of K such that A_{i+1} is a proper face of A_i, $i = 0, 1, \ldots, (r-1)$.

<u>Proof</u>: The proof of this lemma is by induction on the dimension of K. We retain the notation in the definition of barycentric subdivision.

a) If the dimension of A_0 is less than $n+1$

then all of the simplexes of (62) are contained in M, and hence (63) is a simplex of $M' \subset K'$ by the inductive hypothesis in the definition of barycentric subdivision. If the dimension of A_0 is $n+1$ then $A_0 = A_i^{n+1}$, and $\sigma_0 = \mathcal{K}_i$. If $r = 0$, $(\sigma_0) \in \mathcal{K}_i(S_i') \subset K'$. If $r \neq 0$, then the sequence (62) is contained in S_i by definition of S_i and then $A = (\sigma_1, \sigma_2, \ldots, \sigma_r)$ is a simplex of S_i' by the inductive hypothesis and therefore $\mathcal{K}_i(A) = (\sigma_0, \sigma_1, \ldots, \sigma_r)$ is a simplex of $\mathcal{K}_i(S_i')$.

b) Conversely, if P is a simplex in K' then two cases may occur. Either $P \in M'$ in which case P is clearly determined by (62) by the inductive hypothesis, or the second of the two possible cases will occur and $P \in \mathcal{K}_i(S_i')$. Subordinate to this second possible case there are the following three subcases:

1) if $P \in S_i'$ then $P \subset M'$,

2) if $P = (\mathcal{K}_i)$ then the sequence consisting of the single simplex A_i^{n+1} determines P,

3) if $P = \mathcal{K}_i(A)$, where $A \subset S_i'$, then A is determined by some sequence $A_1, A_2, \ldots, A_r$ of faces of S_i because of the inductive hypothesis, and P is then determined by the sequence $A_i^{n+1}, A_1, A_2, \ldots, A_r$.

<u>DEFINITION</u> <u>84</u>: Let K be an arbitrary geometric complex, let $K^{(0)} = K$, and define $K^{(m)}$ to be the barycentric subdivision of $K^{(m-1)}$. Then the complex $K^{(m)}$ is the <u>barycentric subdivision of order m</u> of K (or less explicitly: a <u>subdivision</u>).

We shall have occasion to discuss subdivisions of complexes without the necessity of mentioning the order of the subdivision, and in such a case we shall use K with a lower case Greek superscript to signify the subdivision under consideration. Since in such a situation the superscript will not indicate a particular integer, if K_1 and K_2 are distinct complexes then their respective subdivisions K_1^α and K_2^α need not be of like order.

<u>THEOREM</u> <u>13</u>: Let K be an r-complex imbedded in R^n. If the diameter of every simplex of K does not exceed some positive number η, then the diameters of the simplexes of $K^{(m)}$ can be made arbitrarily small by taking a sufficiently large m.

<u>Proof</u>: By Lemma 58 the diameter of any simplex of $K^{(1)} = K'$ does not exceed the length of its 1-simplex of maximum length. Let (σ_0, σ_1) be any 1-simplex of K', where σ_0 is the center of a simplex $A_0 = (a_0, a_1, \ldots, a_s)$, and σ_1 is the center of a proper face $A_1 = (a_0, a_1, \ldots, a_t)$ of A_0. This is possible by Lemma 107. If $A = (a_{t+1}, a_{t+2}, \ldots, a_s)$ and σ is the center of A then by Lemma 101, relation (54), clearly $\sigma_0 = [(t+1)/(s+1)]\sigma_1 + [(s-t)/(s+1)]\sigma$. Therefore the point σ_0 divides the segment $[\sigma_1, \sigma]$ in the ratio $(s-t)/(t+1)$, and hence $\rho(\sigma_0, \sigma_1) = [(s-t)/(s+1)]\rho(\sigma_1, \sigma)$. But $\rho(\sigma_1, \sigma)$ does not exceed the diameter of A_0 because σ_1 and σ are contained in A_0, and therefore $\rho(\sigma_0, \sigma_1) \leq [(s-t)/(s+1)]\eta$. Thus the diameter of any 1-simplex of K' does not

exceed $[r/(r+1)]\eta$ since $0 \leq s \leq r$ and $0 \leq t \leq s-1$. Therefore the diameter of any r-simplex of K' does not exceed $[r/(r+1)]\eta$ by Lemma 58. Now since this argument holds for the barycentric subdivision of any complex, even a complex which is a barycentric subdivision itself, the diameter of every simplex of $K^{(m)}$ is less than or equal to $[r/(r+1)]^m\eta$.

<u>DEFINITION</u> <u>85</u>: Let K be a complex of arbitrary dimension. Let K' be the barycentric subdivision of K. If x is a 0-chain of K we define the barycentric subdivision x' of the chain x to be x' = x. For chains of higher dimension than zero, we assume that the barycentric subdivision of the n-chains of K has been defined and we define the barycentric subdivision of A where A is an oriented (n + 1)-simplex of K. We let x = A, we let S be the set of all proper faces of A and we let $\mathcal{K}$ be the center of A. Thus $\mathcal{K}(S') \subset K'$. Now by the hypothesis, the barycentric subdivision $(\Delta A)'$ of the boundary ΔA of A has already been defined because $(\Delta A)'$ is a chain of S'. Now set $A' = \mathcal{K}[(\Delta A)']$ as in Lemma 105 and let $x' = g_1A_1' + g_2A_2' + \ldots + g_kA_k'$ whenever $x = g_1A_1 + g_2A_2 + \ldots + g_kA_k$ is an (n + 1)-chain of K.

<u>LEMMA</u> <u>108</u>: Any chain x of K satisfies the relation

$$(64) \qquad \Delta(x') = (\Delta x)'.$$

<u>Proof</u>: The lemma is obvious for a 0-chain. Assume that it is true for an arbitrary n-chain. Clearly

$$(65) \qquad \Delta A' = \Delta \mathcal{K}[(\Delta A)'] = (\Delta A)' - \mathcal{K}[\Delta(\Delta A)'].$$

Now ΔA is n-dimensional and thus by the assumption $\Delta(\Delta A)' = (\Delta\Delta A)' = 0$. Therefore relation (65) is of the same form as (64). Hence the relation is true for an oriented $(n+1)$-simplex and by extension is true for any $(n+1)$-chain.

DEFINITION 86: If $K^{(m)}$ is the barycentric subdivision of order m of K, we define $x^{(m)}$ inductively by letting $x^{(0)} = x$ and $x^{(m+1)} = (x^{(m)})'$. If K^{α} denotes $K^{(m)}$ then we let x^{α} denote $x^{(m)}$.

LEMMA 109: $\Delta(x^{\alpha}) = (\Delta x)^{\alpha}$.

Proof: This follows immediately from Lemma 108.

LEMMA 110: If z is a cycle of K then z^{α} is a cycle of K^{α}. If z_1 and z_2 are homologous cycles in K then z_1^{α} and z_2^{α} are homologous cycles in K^{α}.

Proof: This lemma is an immediate consequence of Lemma 109.

THEOREM 14: Let K be any complex, and let K^{α} be a subdivision of K. If $\bar{z}$ is any r-dimensional homology class of K and z is any cycle belonging to $\bar{z}$ then we denote the homology class of K^{α} which contains the cycle z^{α} by $\overline{z^{\alpha}}$. Then $\bar{z}$ is mapped into $\overline{z^{\alpha}}$ by a single-valued mapping and an isomorphism of $B^r(K)$ onto $B^r(K^{\alpha})$.

Proof: The proof of the theorem is based on two lemmas which follow.

LEMMA 111: If K is a complex, K^{α} a subdivision of K and x a chain of K^{α} whose boundary Δx is of the form $\Delta x = z^{\alpha}$, where z is a cycle of K, then there is a chain y of K whose boundary is z and such that $x - y^{\alpha}$ is a cycle homologous to zero in K^{α}.

<u>Proof</u>: The proof of this lemma is based on the following lemma.

<u>LEMMA</u> <u>112</u>: Lemma 111 implies Theorem 14.

<u>Proof</u>: If $\varphi(z) = z^{\alpha}$ then φ assigns an element of $B^r(K^{\alpha})$ to each element of $B^r(K)$. It is clear that φ is a homomorphism of $B^r(K)$ into $B^r(K^{\alpha})$. In order to prove the lemma it is sufficient to show that φ is an isomorphism of $B^r(K)$ onto $B^r(K^{\alpha})$. We prove first that φ is an isomorphism of $B^r(K)$ onto a subset (which we shall subsequently prove not to be proper) of $B^r(K^{\alpha})$. Now if $z^{\alpha} = 0$ then $z^{\alpha} \sim 0$ in K^{α}. That is, a chain x of K^{α} exists such that $\Delta x = z^{\alpha}$. By Lemma 111 then, corresponding to the chain x there exists a chain y of K such that $\Delta y = z$. That is, such that $z \sim 0$ in K or $z = 0$. Thus $\varphi(z) = 0$ implies $z = 0$. Thus φ is an isomorphism. Secondly we prove that φ is a mapping of $B^r(K)$ onto $B^r(K^{\alpha})$. Suppose that r is an element of $B^r(K^{\alpha})$. That is, r is a class of cosets of homologous cycles of K^{α}. Let x be an element of r. If we represent the trivial cycle of K by 0, then $\Delta x = 0^{\alpha}$ because x is a cycle of K^{α}. Thus Lemma 111 is applicable with $z = 0$. Therefore there exists a chain y of K such that $\Delta y = 0$ and such that $x - y^{\alpha} \sim 0$ in K . Clearly $\varphi(u) = r$ where u is the homology class of K which contains y. Therefore φ maps

$B^r(K)$ onto all of $B^r(K^\alpha)$. Hence Lemma 111 implies Theorem 14.

REMARK: We now prove Lemma 112:

__Proof__: The proof will be limited to the case of $K^\alpha = K'$ since an obvious induction on the order of the subdivision will extend the conclusion to any required order. Now in the case of $K^\alpha = K'$ we shall prove the present lemma by induction on the dimension of the complex K. Clearly if K is a 0-complex the lemma holds. We assume now that the present lemma holds for every n-complex. By Lemma 113 then Theorem 14 holds for any n-complex also. Let K be an $(n+1)$-complex, let M be its n-skeleton, and let A_1^{n+1}, A_2^{n+1}, ..., A_k^{n+1} be the set of all arbitrarily oriented $(n+1)$-simplexes of K. Let S_i be the set of all proper faces of A_i^{n+1} and let $\mathcal{K}_i$ be the barycenter of A_i^{n+1}. Let $T_i = \mathcal{K}_i(S_i')$. By assumption the present lemma and Theorem 14 both hold for S_i since S_i is of dimension n. Now therefore the homology properties of the complex S_i' are the same as those of the complex S_i which have been discussed in Theorem 12. We now consider the homology properties of T_i. Let x_i be an r-chain of T_i of the form $\mathcal{K}_i(z_i)$ where z_i is a chain of S_i', with the property that its boundary Δx_i is contained in S_i'. Then the following relations hold:

a) if $r \leq n$ then there exists a chain y_i in S_i' such that $x_i - y_i$ is a cycle homologous to zero in T_i,

b) if $r = n + 1$, then $x_i = g_i(A_i^{n+1})'$, where g_i

is an element of the coefficient group over which the chains are defined.

Before proof of a) and b) we note that the two following relations are valid:

c) If $r = 1$ then $\Delta x_i = \Delta \mathcal{K}_i(z_i) = z_i - I(z_i)\mathcal{K}_i$, by Lemma 105. Since Δx_i is contained in S_i' then $I(z_i)\mathcal{K}_i = 0$. That is, if $r = 1$ then z_i is a 0-cycle of S_i' with Kronecker index zero.

d) If $r > 1$, then $\Delta x_i = \Delta \mathcal{K}_i(z_i) = z_i - \mathcal{K}_i(\Delta z_i)$. Thus $\mathcal{K}_i(\Delta z_i) = 0$ and $\Delta z_i = 0$. That is, if $r > 1$, then z_i is an $(r - 1)$-cycle of S_i'.

We now prove relation

a) If $r \leq n$ then the dimension of z_i is less than or equal to $n - 1$. Since we have assumed that the present lemma holds for every n-complex then Theorem 14 also holds. Thus the cycle z_i is homologous to zero in S_i' if $r > 1$ because every cycle of dimension greater than zero and less than n is homologous to zero in S_i by Theorem 12. However, if $r = 1$, then by c) we have $I(z_i) = 0$. Now since $n \geq r = 1$, then S_i is connected, and hence $z_i \sim 0$ in this case also by Lemma 73. Therefore there exists a chain y_i in S_i' whose boundary is z_i, and if $v_i = \mathcal{K}_i(y_i)$ we have the relation $\Delta v_i = y_i - \mathcal{K}_i(\Delta y_i) = y_i - x_i$. That is, $x_i - y_i \sim 0$ in T_i.

We now prove relation

b) If $r = n + 1$ then the dimension of z_i is n.

If $n = 0$, then $I(z_i) = 0$ and then obviously z_i is of the form $g_i \Delta(A_i')$. Thus b) holds in this case because $\Delta(A_i') = (\Delta A_i)'$. If $n > 0$ then there exists a cycle u_i in S_i such that $z_i \sim u_i'$ in S_i' by Theorem 14. However, the homology is an equality since z_i has the same dimension as S_i' and thus $z_i = u_i'$. By Theorem 12 the cycle u_i of S_i is of the form $g_i \Delta A_i^{n+1}$. Therefore $z_i = g_i(\Delta A_i^{n+1})'$. That is, $x_i = g_i \mathcal{K}_i[(\Delta A_i^{n\ 1})]' = g_i(A_i^{n+1})'$.

We consider one more relation

e) In a) and b) it was assumed that $x_i = \mathcal{K}_i(z_i)$ and therefore that the dimension of x_i was at least one. We shall prove here that if x_i is a 0-chain of T_i then there is a 0-chain y_i of S_i' such that $x_i - y_i \sim 0$ in T_i. To this end let a be a vertex of S_i'. Then $+(\mathcal{K}_i, a)$ is a simplex of T_i whose boundary is $+(a) - (\mathcal{K}_i)$. That is $(\mathcal{K}_i) \sim (a)$ in T_i. If we replace $+(\mathcal{K}_i)$ by $+(a)$ in x_i the required chain y_i is obtained.

We now apply relations a), b), and e) to the present lemma. If x is a chain of K' which satisfies the present lemma then z' is a chain of M' provided only that $\Delta x = z'$ since the dimension of Δx is less than or equal to n and hence z is a chain of **M**. Let x_i be the sum of all the members of the linear form x which contain simplexes with vertex $\mathcal{K}_i$. Then x_i is a chain of T_i. We shall prove that Δx_i is a chain of S_i'. Indeed, the chain $x - x_i$ does not contain such simplexes with vertices $\mathcal{K}_i$ and hence the boundary $\Delta x - \Delta x_i$ does not contain such simplexes. The chain Δx is contained in M' and also contains no simplexes with vertex K_i. Hence the difference of these chains $\Delta x_i = \Delta x - (\Delta x - \Delta x_i)$ contains no simplex with vertex K_i. Therefore the chain Δx_i is contained in S_i' and a), b), and c) are applicable to it. We now consider two cases. In the first case we consider the dimension of the chain x to be less than $n + 1$. Then by a) and c) there exists a chain y_i of S_i' such that $x_i - y_i \sim 0$ in T_i. If we set $x^* = x - (x_1 - y_1) - \dots - (x_k - y_k)$ then $\Delta x^* = \Delta x - z'$ and hence $x^* - x \sim 0$ in K'. The chain x^* has no simplex with vertex K_i, $i = 1, 2, \dots, k$, and hence x^* is contained in M'. Thus the present lemma applies to x^*. That is, there is a chain y of M such that $\Delta y = z$ and $x^* - y' \sim 0$ in M'. Hence $x - y' \sim 0$ in K', and the lemma holds for chains of dimension less than $n + 1$. In the second case we consider the dimension of the chain x to be $n + 1$. By b) x_i is of the form $x_i = g_i(A_1^{n+1})'$. If $y = g_1A_1^{n+1} + g_2A_2^{n+1} + \dots + g_kA_k^{n+1}$ then the chain $x - y'$ clearly contains no simplex with vertex K_i, $i = 1, 2, \dots, k$. Since $x - y'$ is an $(n + 1)$-chain it is equal to zero and therefore $x = y'$. Now the boundary $\Delta(x - y') = (z - \Delta y)' = 0$ and thus $\Delta y = z$, since the barycentric subdivision of a chain is zero only if the chain is zero. Hence the lemma holds for dimension of x exactly $n + 1$. This completes the proof of the present lemma, which by Lemma 112 implies the validity of Theorem 14.

REMARK: If K and L are two complexes, if K^{α} and L^{β} are subdivisions of K and L respectively, and if f is a simplicial mapping of K^{α} into L^{β}, then the mapping f induces a homomorphism $\tilde{f}$ of the group $B^r(K^{\alpha},G)$ into the group $B^r(L^{\beta},G)$. This consequence is stated in Lemma 97. However there exists an isomorphism between $B^r(K^{\alpha},G)$ and $B^r(K,G)$ and also an isomorphism between $B^r(L^{\beta},G)$ and $B^r(L,G)$. These two similar isomorphisms are in consequence of Theorem 14. Therefore, since isomorphic systems are abstractly identical, we may consider $\tilde{f}$ to be a homomorphism of $B^r(K,G)$ into $B^r(L,G)$. This idea may be expressed more explicitly in the following manner: let X be any element of the group $B^r(K,G)$, let x be a cycle belonging to the class X, and let x^{α} be the subdivision of x in K^{α}. Now $\hat{f}(x^{\alpha})$ is a cycle of L^{β} by Lemma 95. Furthermore, by Theorem 14 there exists a cycle y of L whose subdivision y^{β} is homologous to $\hat{f}(x^{\alpha})$ in L^{β}. If N is the homology class of elements of L which contains y then $\tilde{f}(X) = N$.

LEMMA 113: If K and L are two complexes and if φ is a continuous mapping of the polyhedron $|K|$ into the polyhedron $|L|$ then there exists an integer $m \geq 0$ for which the mapping φ of the complex $K^{(m)}$ into L satisfies the star condition.

Proof: We shall first show that there exists a positive number ϵ such that every subset F of the polyhedron $|L|$ of diameter less than ϵ is contained in one of the stars S(b) of the complex L. To prove that such an ϵ

exists we assume the contrary; that is, we assume that for every positive integer t there exists a set F_t of $|L|$ of diameter less than 1/t which is not contained in any one of the stars of the complex L. Now since $|L|$ is a compact set and since the diameters of the sets F_t approach zero as a limit, then there exists a point c in $|L|$ such that any neighborhood of c contains an infinite number of the sets in the sequence $\{F_t\}_{t\ 1,\ 2,\ \ldots}$. If we take as a neighborhoof of c any neighborhood containing the star S(b) of K which contains c, then at least one of the sets F_t is contained in S(b). This is a contradiction and hence the required number ϵ exists. Now since $|K|$ is compact, φ is uniformly continuous, and there is therefore a positive number δ such that

(66) $$\rho\left[\varphi(x), \varphi(y)\right] < \epsilon$$

provided only that x, y $\varepsilon\ |K|$ and that $\rho(x,y) < \delta$. Now if η is the maximum diameter of any simplex of K let m be a large enough positive integer that $\left[n/(n+1)\right]^m \eta < \delta/2$. Then by Theorem 13 every star S(a) of $K^{(m)}$ has diameter less than δ. But the diameter of $\varphi\left[S(a)\right]$ is less than ϵ by relation (66) and thus $\varphi\left[S(a)\right]$ is contained in at least one star of L. Therefore φ which maps $K^{(m)}$ into L satisfies the star condition.

<u>LEMMA</u> <u>114</u>: If K' is the barycentric subdivision of an arbitrary complex K then every star of K' is contained in some star of K.

Proof: Let σ be any vertex of K' and let A be the simplex of K whose center is σ. If $B = (\sigma_0, \sigma_1, \ldots, \sigma_r)$ is any simplex of the star $S(\sigma)$ of K', then let $\sigma = \sigma_i$. The open simplex B is contained in the open simplex A_0 by Lemma 107. Furthermore, since $A_i = A$ is a face of A_0, the star $S(\sigma)$ is contained in the union S(A) of all open simplexes which have A as a face. If a is a vertex of A then the star S(a) of K contains S(A), and thus $S(\sigma) \subset S(a)$.

LEMMA 115: Let f be a simplicial mapping of a complex K into a complex L and let K^α be a subdivision of K. If $K^\alpha \neq K$, then f is not a simplicial mapping of K^α into L, but it is a star-related mapping and hence by Theorem 11 there exists a simplicial mapping f^α of K^α into L which approximates f. Furthermore, if x is a chain of K, we have

(67) $\qquad \hat{f}^\alpha(x^\alpha) = \hat{f}(x).$

In particular, if $L = K$ and if f is the identity mapping of K onto itself we have $\hat{f}^\alpha(x^\alpha) = x$.

Proof: We show first that f satisfies the star condition. Let a be a vertex of K. If $f(a) = b$ and if A is any open simplex of K which has a as a vertex then f(A) is an open simplex of L with vertex b. This is a consequence of the definition of simplicial mapping. Thus $f[S(a)] \subset S(b)$ and hence by Lemma 114 we know that the mapping f of K^α into L is also a star-related mapping. We now prove relation (67) by means of an induction on the number of dimensions of the chain x. The relation is clearly valid for a 0-chain. We assume that relation (67) holds for any (r - 1)-chain. Let T be the set of all faces of an oriented r-simplex A of K. Then $f(|T|) = D$ where D is a simplex of L. Now f maps all vertices of T^{α} into points of D. Hence every simplex of T^{α} is mapped by f either into D or into a face of D by Theorem 11. We consider two cases.

a) If the dimension of the simplex D is less than r then all of the r-simplexes of T^{α} are degenerate under f^{α} and we have $\hat{f}^{\alpha}(A^{\alpha}) = 0$ which implies $\hat{f}(A) = 0$.

b) If the dimension of the simplex D is r then $\hat{f}(A)$ is the simplex D itself oriented in some particular way. However

(68) $\hat{f}^{\alpha}(A^{\alpha}) = k\hat{f}(A)$

for some integer k since every r-simplex of T^{α} either maps onto D or is degenerate under the mapping f^{α}. We need only show that $k = 1$ to complete the proof. Now applying the boundary operator Δ to relation (68) we obtain

(69) $\Delta\hat{f}^{\alpha}(A^{\alpha}) = \hat{f}^{\alpha}(\Delta A^{\alpha}) = \hat{f}^{\alpha}\left[(\Delta A)^{\alpha}\right] = k\Delta\hat{f}(A) = k\hat{f}(\Delta A).$

Replacing ΔA by x we obtain $\hat{f}^{\alpha}(x^{\alpha}) = k\hat{f}(x)$ from the equality of the third and fifth terms in (69). Hence $k = 1$, since the inductive hypothesis implies that (67) holds for any (r - 1)-chain such as x. Therefore

(70) $\hat{f}^{\alpha}(A_i^{\alpha}) = \hat{f}(A_i)$

for any arbitrary oriented simplex A_i of K. If we multiply

(70) by g_i and sum over i we obtain (67) for an arbitrary r-chain.

LEMMA 116: Let f be a simplicial mapping of a complex K into a complex L, let K^α be a subdivision of K, and let f^α be a simplicial mapping of K^α into L which approximates f. The mapping f^α induces a homomorphism $\hat{f}^\alpha$ of $B^r(K^\alpha)$ into $B^r(L)$. This induced homomorphism may also be considered to be a homomorphism of $B^r(K)$ into $B^r(L)$ by the second remark following Lemma 112. However f also induces a homomorphism $\hat{f}$ of $B^r(K)$ into $B^r(L)$. We shall prove here that $\hat{f}$ and $\hat{f}^\alpha$ are identical. Furthermore, if $L = K$ and f is the identity mapping of K onto itself then the homomorphism $\tilde{f}^\alpha$ is the identity mapping of $B^r(K)$ onto itself.

Proof: If $x^* \varepsilon B^r(K)$ and if x is a cycle of the homology class x^*, then $\hat{f}^\alpha(x^\alpha)$ is a cycle of L and the homology class $[\hat{f}^\alpha(x^\alpha)]^*$ which contains it is $\tilde{f}^\alpha(x^*)$ by the second remark following Lemma 112. However , $\hat{f}(x)$ is a cycle of L and the homology class $[\hat{f}(x)]^*$ which contains it is $\tilde{f}(x^*)$. Thus $\hat{f}^\alpha(x^\alpha) = \hat{f}(x)$ by Lemma 115 and hence $\tilde{f}^\alpha(x^*) = \tilde{f}(x^*)$.

THEOREM 15: If $|K_1|$ and $|K_2|$ are homeomorphic polyhedra, then their corresponding complexes K_1 and K_2 have equal dimensions.

Proof: We prove the contrapositive and assume without loss of generality that the dimension n of the complex K_1 is greater than the dimension of K_2. Let φ be a homeomorphism of K_2 onto K_1. We may choose subdivisions K_2^{α} and K_1^{α} of K_2 and K_1 respectively which are of sufficiently high order to insure that simplicial mappings f of K_2^{α} into K_1 and g of K_1^{α} into K_2^{α} exist and approximate φ and φ^{-1} respectively. This is true by Theorem 11 and Lemma 115. Then the mapping fg of K_1^{α} into K_1 approximates the identity mapping $\varphi^{-1}\varphi$ and hence by Lemma 115 we have

$$\hat{f}\left[\hat{g}(x^{\alpha})\right] = x \tag{71}$$

holds for every chain x of K_1. But we also have the condition that every n-simplex of K_1^{α} is degenerate under the mapping g since K_2^{α} does not contain any simplexes of dimension n. Therefore $\hat{g}(x^{\alpha}) = 0$ whenever x is an n-chain of K. Thus $\hat{f}\left[\hat{g}(x^{\alpha})\right] = 0$. But this is a contradiction to relation (71) because clearly x is a nontrivial n-chain of K_1. This proves the theorem by reduction to an absurdity. It may be noted that we are enabled by this theorem to refer to the dimension of a polyhedron.

<u>THEOREM</u> <u>16</u>: Let φ_1 be a homeomorphism of a complex K_1 onto a complex K_2, let $\varphi_1^{-1} = \varphi_2$, and let K_1^{α} and K_2^{α} be subdivisions of the given complexes for which the mapping φ_1 of K_1^{α} into K_2^{α} is a star-related mapping. If the mapping f_1^{α} of K_1^{α} into K_2^{α} is a simplicial approximation to φ_1 then we have the following three relations:

a) the homomorphism $\tilde{f}^{\alpha}$ of $B^r(K_1)$ into $B^r(K_2)$ is independent of the choice of K_1^{α}, K_2^{α}, and f_1^{α} and hence can be denoted by the relation $\tilde{\varphi}_1\left[B^r(K_1)\right] \subset B^r(K_2)$,

b) the homomorphism $\tilde{\varphi}_1$ is an isomorphism of $B^r(K_1)$ onto $B^r(K_2)$,

c) if the isomorphism $\tilde{\varphi}_2$ of $B^r(K_2)$ onto $B^r(K_1)$ is induced by φ_2 in the same manner, then the isomorphisms $\tilde{\varphi}_1$ and $\tilde{\varphi}_2$ are inverse to each other.

<u>Proof</u>: We first introduce two additional subdivisions K_1^β and K_2^β of the given complexes for which the mapping φ_1 of K_1^β onto K_2^β satisfies the star condition, and we let f_1^β of K_1^β into K_2^β be a simplicial approximation to φ_1. We may assume without loss of generality that K_1^β is a finer barycentric subdivision that K_1^α since they are clearly subdivisions of the same complex K_1. We may choose a subdivision K_2^γ of K_2 fine enough to insure that the mapping φ_2 of K_2^γ onto K_1^β is star-related. Let f_2^α be a simplicial approximation to φ_2 of K_2^γ into K_1^α and let f_2^β be a simplicial approximation of K_2^γ into K_1^β. Let K_1^γ be a subdivision of K_1 such that the mapping φ_1 of K_1^γ into K_2^γ is star-related. Let f_1^γ be a simplicial approximation to φ_1 of K_1^γ into K_2^γ. The following diagram with the given homeomorphisms written above the arrows and with the simplicial approximations written below outlines the mappings and subdivisions mentioned above

$$\begin{tikzcd}
K_2^{\alpha} & K_1^{\alpha} \arrow[l, "\varphi_1", "f_1^{\alpha}"'] & & \\
& & K_2^{\gamma} \arrow[ul, "\varphi_2", "f_2^{\alpha}"'] \arrow[dl, "\varphi_2"', "f_2^{\beta}"] & K_1^{\gamma}. \arrow[l, "\varphi_1", "f_1^{\gamma}"'] \\
K_2^{\beta} & K_1^{\beta} \arrow[l, "\varphi_1", "f_1^{\beta}"'] & &
\end{tikzcd}$$

These simplicial mappings induce homomorphisms of the corresponding Betti groups of the subdivided complexes which can be outlined as follows:

$$\begin{tikzcd}
B^r(K_2^{\alpha}) & B^r(K_1^{\alpha}) \arrow[l, "\tilde{f}_1^{\alpha}"'] & & \\
& & B^r(K_2^{\gamma}) \arrow[ul, "\tilde{f}_2^{\alpha}"'] \arrow[dl, "\tilde{f}_2^{\beta}"] & B^r(K_1^{\gamma}). \arrow[l, "\tilde{f}_1^{\gamma}"'] \\
B^r(K_2^{\beta}) & B^r(K_1^{\beta}) \arrow[l, "\tilde{f}_1^{\beta}"'] & &
\end{tikzcd}$$

The corresponding Betti groups of the original complexes for which there are also induced homomorphisms (by the second remark following Lemma 112) may be outlined in the following manner:

$$B^r(K_2) \longleftarrow B^r(K_1) \longleftarrow B^r(K_2) \longleftarrow B^r(K_1).$$

Now the mapping $f_1^{\alpha} f_2^{\alpha}$ of K_2^{γ} into K_2^{α} is a simplicial approximation to $\varphi_1 \varphi_2$ by Lemma 92. Thus the homomorphism $\tilde{f}_1^{\alpha} \tilde{f}_2^{\alpha}$ of $B^r(K_2)$ onto itself is the identity homomorphism. Now the kernel of the homomorphism $\tilde{f}_1^{\alpha} \tilde{f}_2^{\alpha}$ contains the kernel of $\tilde{f}_2^{\alpha}$. But since $\tilde{f}_2^{\alpha} \tilde{f}_2^{\alpha}$ is the identity homomorphism it has kernel zero. Hence

(72) the kernel of the homomorphism $\tilde{f}_2^{\alpha}$ is zero.

Since $\tilde{f}_1^{\alpha} \tilde{f}_2^{\alpha}$ maps $B^r(K_2)$ identically onto itself then $B^r(K_2) = \tilde{f}_1^{\alpha} \tilde{f}_2^{\alpha} B^r(K_2) \subset \tilde{f}_1^{\alpha} B^r(K_1)$. Hence

(73) $\tilde{f}_1^{\alpha} B^r(K_1) = B^r(K_2)$.

Now the mapping $f_2^{\alpha} f_1^{\gamma}$ which maps K_1^{γ} into K_1^{α} is a simplicial approximation to $\varphi_2 \varphi_1$ by Lemma 92 and thus

(74) the kernel of the homomorphism $\tilde{f}_1^{\gamma}$ is zero.

Thus

(75) $\tilde{f}_2^{\alpha} B^r(K_2) = B^r(K_1)$.

Now by (72) and (75) clearly $\tilde{f}_2^{\alpha}$ is an isomorphism of $B^r(K_2)$ onto $B^r(K_1)$ and since $\tilde{f}_1^{\alpha} \tilde{f}_2^{\alpha}$ is the identity mapping then $\tilde{f}_1^{\alpha}$ and $\tilde{f}_2^{\alpha}$ are inverse isomorphisms:

(76) $\tilde{f}_1^{\alpha} = (\tilde{f}_2^{\alpha})^{-1}$.

In a similar manner clearly $\tilde{f}_2^{\alpha}$ and $\tilde{f}_1^{\gamma}$ are inverse isomorphisms:

(77) $\tilde{f}_1^{\gamma} = (\tilde{f}_2^{\alpha})^{-1}$.

Therefore

(78) $\tilde{f}_1^{\alpha} = \tilde{f}_1^{\gamma}$

because of (76) and (77). In a similar manner $\tilde{f}_1^{\beta} = \tilde{f}_1^{\gamma}$, since we may replace α by β and obtain valid relations. Therefore $\tilde{f}_1^{\beta} = \tilde{f}_1^{\alpha}$ by (77) and (78). This proves a). Since $\tilde{f}_1^{\alpha}$ is an isomorphism, $\tilde{\varphi}_1$ is clearly an isomorphism of $B^r(K_1)$ onto $B^r(K_2)$. This proves b). Now finally, if we denote the induced homomorphism of $B^r(K_2)$ into $B^r(K_1)$ by $\tilde{\varphi}_2$ then by relation (76) the isomorphisms $\tilde{\varphi}_1$ and $\tilde{\varphi}_2$ are inverses. This proves c) and concludes the proof of the theorem.

<u>COROLLARY</u>: If K_1 and K_2 are geometric complexes such that $|K_1|$ and $|K_2|$ are homeomorphic, then the Betti groups $B^r(K_1,G)$ and $B^r(K_2,G)$ are isomorphic for any coefficient group G.

<u>Proof</u>: By Theorem 16 a homeomorphism from $|K_1|$ onto $|K_2|$ defines exactly one isomorphism between $B^r(K_1)$ and $B^r(K_2)$.