Advanced calculus in modern and theoretical physics

H. Bruce Early
The University of Montana

ADVANCED CALCULUS

IN

MODERN AND THEORETICAL PHYSICS

by

H. Bruce Early, Jr.

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A. S. Merrill
Chairman of Board of Examiners

W. G. Bateman
Chairman of Committee on Graduate Study

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I. THE PROBLEM

There has been felt a need of a course in Advanced Calculus which could fill a dual role. With very few exceptions, the students who are found in the Advanced Calculus classes are the same students who are enrolled in the classes of Theoretical Physics. A course in Advanced Calculus which retains its rigor from the standpoint of the mathematician and at the same time shows the practical applications of those principles to the problems of Theoretical and Modern Physics is justified.

II. THE PURPOSE

The purpose of this study is twofold: (1) to show the needs of those mathematical principles which are found in the problems of Theoretical Physics, (2) to indicate the perequisites and outline briefly a course of Advanced Calculus. The course in Advanced Calculus is intended to bridge the gap between the study of elementary calculus and the serious pursuit of the more advanced study of mathematics. It is of equal importance to give the student a brief introduction to the methods of analysis as well as to help the student to apply those mathematical topics in the study of physics.

III. TYPICAL PROBLEMS

The following pages contain typical problems which have been chosen from the fields of Modern and Theoretical Physics to show the mathematical principles necessary for the study of those courses in Physics. The selection has not been exhaustive but limited to problems which involve those mathematical principles which are usually treated in a course of Advanced Calculus. Problems which demand the use of differential equations have been omitted from this selected group.

The solutions are given and the comments following the solution indicate the mathematical principles necessary to solve the problem.

EXAMPLE From the development of Planck's law expressed in terms of the wave length of radiation instead of the frequency, and the development of Wien's displacement law, there occurs the following exponential equation[1] which requires a numerical solution in the evaluation of the Planck and Boltzmann constants.

$$\left(1-\frac{x}{5}\right)e^{x}=1 \qquad \text{where} \qquad x=\frac{ch}{k\lambda_m T}$$

By Newton's method

$$x = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$f(x) = \left(1-\frac{x}{5}\right)e^{x}-1 = \left(\frac{5-x}{5}\right)e^{x}-1$$

$$f'(x) = e^{x} - \frac{xe^{x}}{5} - \frac{e^{x}}{5} = e^{x}\left(\frac{4-x}{5}\right)$$

$x = 0$ is one solution but is not wanted

$x = 4 \qquad f'(x) = 0$

$x = 5 \qquad f(x) = 0$

From a table of exponentials choose $x_1 = 4.98$

$$x = 4.98 - \frac{\left(\frac{5-4.98}{5}\right)146-1}{\left(\frac{4-4.98}{5}\right)146}$$

$$x = 4.98 - \frac{-2.08}{-143} = 4.98 - .0145$$

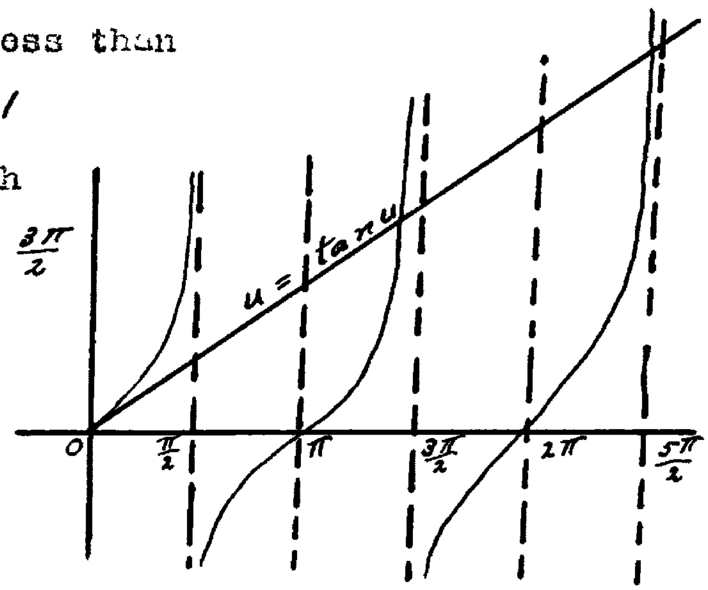
$$x = 4.965^{+}$$

1 C.R. Jeppesen, Modern Physics. (Unpublished manuscript 1941), p. 151.

EXAMPLE The equation[2] u - tan u = 0 gives the maxima of intensity of diffraction through a single slit. Plotting tan u against u it is seen that the solutions are: 0 and at values of u a little less than

$\left(\frac{2n+1}{2}\right)\pi$ where $n \geq 1$

Let u = 4.5 radians which is in the neighborhood of $\frac{3\pi}{2}$



$$u = u_1 - \frac{u_1 - \tan u_1}{1 - \sec^2 u_1}$$

$$u = 4.5 - \frac{4.5 - 4.6382}{1 - 22.67}$$

$$= 4.5 - \frac{-.1382}{-21.67} = 4.5 - .0063$$

$$u = 4.4937 \text{ rad.} = 1.4303\,\pi \text{ radians}$$

The solution in the neighborhood of $\frac{5\pi}{2}$

Let u = 7.7 radians

$$u_1 = 7.7 - \frac{7.7 - 6.4430}{1 - \left(\frac{1}{.15337}\right)^2} = 7.7 - \frac{1.257}{-41.09}$$

$$u = 7.7 + .035$$

$$u = 7.735^+$$

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2 Leigh Page, Introduction to Theoretical Physics. (New York: D.Van Nostrand Company, Inc. Second edition 1935) p. 590.

Let $u_2 = 7.74$ radians

$$u = 7.74 - \frac{7.74 - 8.732}{1 - \left(\frac{1}{.11375}\right)} = 7.74 - \frac{-.992}{-76.28}$$

$$= 7.74 - .013$$

$$u = 7.727$$

Let $u_3 = 7.727$ radians

Then

$$u = 7.727 - \frac{7.727 - 7.8321}{1 - \left(\frac{1}{.12665}\right)^2}$$

$$= 7.727 - \frac{-.1051}{-61}$$

$$= 7.727 - .0017$$

$u = 7.7253$ radians

$u = 2.459\,\pi$ radians

EXAMPLE The equation[3] $u - \sin u = .422$ occurs in the solution of the problem of eclipsing binaries.

Solve $u - \sin u = .422$ for a numerical solution by Newton's Method.

$f(u) = u - \sin u - .422$

$f'(u) = 1 - \cos u$

Choosing an approximate value from a table of radian measure, let $u = 1.4$ radians

Then

$$u = u_1 - \frac{u_1 - \sin u_1 - .422}{1 - \cos u_1}$$

$$u = 1.4 - \frac{1.4 - .98545 - .422}{1 - .16997}$$

$$u = 1.4 - \frac{-.00745}{.83003}$$

$$u = 1.4 + .0090$$

$$u = 1.409 \text{ radians}$$

$$u = 80^\circ\, 43'\, 13''$$

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3 Physics Staff University of Pittsburg, Atomic Physics. (New York: John Wiley and Sons, Inc 1933) Prob. 13 p. 299.

The preceding examples are of the type that show the need for facility in obtaining a numerical solution of a transcendental equation. The approximation reached by successive applications of Newton's Method is preferred in the Department of Physics to Horner's Method. The method discovered by Newton involves the finding of the derivative of a function and then substituting in the formula the last approximate value which was found. Horner's Method requires the translation of the axis by successive divisions until the desired result is reached. Horner's Method applies to the solution of algebraic equations but is not readily applicable to trigonometric, exponential, and logrithmic equations. Newton's Method applies to these equations and equally well to the algebraic equations.

EXAMPLE[4] A body falls 360 feet from rest in 5 seconds. Find the limiting velocity, assuming the resistance to be proportional to the velocity.

u = initial velocity $\qquad u = 0$

s = 360 $\qquad w = \frac{g}{k}$ (limiting velocity)

t = 5

$$S = \omega t + \frac{u - \omega}{k}\left(1 - e^{-kt}\right)$$

$$360 = \frac{5g}{k} + \frac{0 - \frac{g}{k}}{k}\left(1 - e^{-5k}\right)$$

$$360k^2 - 5kg - ge^{-5k} + g = 0$$

Expanding $\quad e^{-5k} = 1 + \frac{(-5k)}{1} + \frac{(-5k)^2}{\underline{|2}} + \frac{(-5k)^3}{\underline{|3}} + \frac{(-5k)^4}{\underline{|4}} + \cdots$

Substituting

$$360k^2 - 5kg - g\left(1 - 5k + \frac{25k^2}{2} - \frac{125k^3}{6} + \frac{625k^4}{24}\right) + g = 0$$

$$-\frac{625k^2}{24} + \frac{125k}{6} + \left(\frac{360}{g} - 25\right) = 0$$

$$k^2 - .8k + .0507 = 0 \qquad k = .7306 \text{ or } \underline{.0694}$$

$$\omega = \frac{g}{k} = \frac{32.2}{.0694} = 463 \text{ ft./sec.}$$

Here is an example of arriving at the numerical solution of an exponential equation by expanding the expression involving the exponential into a power series and substituting the series in the equation.

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4 Leigh Page, Introduction to Theoretical Physics. New York: D.Van Nostrand Company, Inc. Second edition 1935. p. 70 prob. 25b.

EXAMPLE In the study of gravity waves in the hydrodynamics of perfect fluids the hyperbolic functions are the solutions of the differential equation:

$$\frac{d^2y}{dx^2} - y = 0.$$

The definitions of the hyperbolic functions rightly belong in the field of calculus. There should be no need for the defining of those functions in a Physics text such as occurs in "Introduction to Theoretical Physics" by Page (pp 245-6 paragraph 73), since an adequate definition is given in the Advanced Calculus texts. The circular functions are defined by use of the exponential series. The hyperbolic functions may be defined by the exponential series in an analogous manner such as follows.

Since $e^x = 1 + x + \frac{x^2}{\underline{|2}} + \frac{x^3}{\underline{|3}} + \text{--------} \frac{x^n}{\underline{|n}}$

and $e^{-x} = 1 - x + \frac{x^2}{\underline{|2}} - \frac{x^3}{\underline{|3}} + \text{-------}$

Then $\frac{e^x - e^{-x}}{2} = x + \frac{x^3}{\underline{|3}} + \frac{x^5}{\underline{|5}} + \text{--------}$

which defines sinh x

$\sinh x = \frac{e^x - e^{-x}}{2} = x + \frac{x^3}{\underline{|3}} + \frac{x^5}{\underline{|5}} + \text{------}$

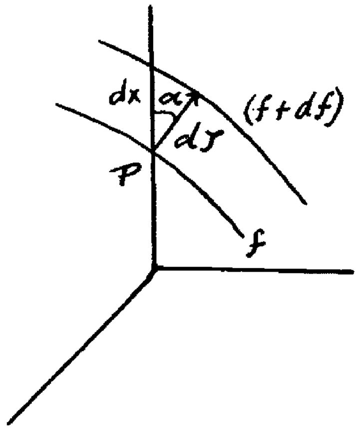
Also $\frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{\underline{|2}} + \frac{x^4}{\underline{|4}} + \frac{x^6}{\underline{|6}} + \text{--------}$

which defines cosh x

$\cosh x = \frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{\underline{|2}} + \frac{x^4}{\underline{|4}} + \frac{x^6}{\underline{|6}} + \text{---}$

EXAMPLE A treatment of the gradient in three variables suggested by Haas.[5]

All points at which the function has the same value as at the point P form a surface (called a field level in vectors).



Through P a perpendicular is drawn outwardly, called ζ and the increase of the function along this normal is called the gradient of the scalar field at P.

At P construct a coordinate system (ξ, η, ζ) such that the ξ and η axes lie in the tangent plane drawn at P to the surface and the direction of the positive ζ -axis is that in which the function increases in value. The following relations hold for components of the gradient:

(1) $\quad grad_{\xi} f = 0, \quad grad_{\eta} f = 0, \quad grad_{\zeta} f = \frac{df}{d\zeta}$

Besides the surface(field level) a neighboring one is drawn such as (f + df). Let $d\zeta$ be the intercept on the ζ-axis between the two surfaces. Let any straight line be drawn through P and dx the intercept on this axis made by the two surfaces. Then

(2) $$\cos \alpha = \frac{d\zeta}{dx}$$

Since the ζ -axis is perpendicular to both surfaces, a right triangle is formed. (The arc differs from the chord by

5 Arthur Haas, Introduction To Theoretical Physics. New York: D.Van Nostrand Company, 1926 Vol.I pp 33-34.

infinitesimals of higher order.)

The linear rate of increase of the function in the direction of x is

(3) $$\frac{\partial f}{\partial x} = \frac{df}{d\zeta} \cos\alpha \qquad \text{since} \quad \frac{df}{d\zeta} = \text{grad } f$$

Now transforming to an arbitrary system of coordinates the components of the gradient of the ξ, η, ζ system, since the ξ-and η-components are zero by equation (1); and letting α, β γ be the angles between the ζ-axis and the x-, y-, and z-axes respectively.

(4) $$\begin{aligned} \text{grad}_x f &= \text{grad } \zeta \cos\alpha \\ \text{grad}_y f &= \text{grad } \zeta \cos\beta \\ \text{grad}_z f &= \text{grad } \zeta \cos\gamma \end{aligned} \qquad \text{or} \qquad \begin{aligned} \frac{\partial f}{\partial x} &= \frac{df}{d\zeta} \cos\alpha \\ \frac{\partial f}{\partial y} &= \frac{df}{d\zeta} \cos\beta \\ \frac{\partial f}{\partial z} &= \frac{df}{d\zeta} \cos\gamma \end{aligned}$$

By substituting we get

(5) $$\begin{aligned} \text{grad}_x f &= \frac{\partial f}{\partial x} \\ \text{grad}_y f &= \frac{\partial f}{\partial y} \\ \text{grad}_z f &= \frac{\partial f}{\partial z} \end{aligned}$$

" The components of the gradient of a scalar are its partial derivatives with respect to the coordinates."

Continuing by squaring and adding,(4) and (5) become:

$$\text{grad } \zeta = \frac{df}{d\zeta} = \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + \left(\frac{\partial f}{\partial z}\right)^2}$$

(Since $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$)

EXAMPLE[6] Show that $\nabla \cdot \nabla \left(\frac{1}{r}\right) = 0$

Let $r = \sqrt{x^2+y^2+z^2} = (x^2+y^2+z^2)^{\frac{1}{2}}$

$$\frac{1}{r} = (x^2+y^2+z^2)^{-\frac{1}{2}}$$

$$\nabla\left(\frac{1}{r}\right) = \frac{\partial\left(\frac{1}{r}\right)}{\partial x} + \frac{\partial\left(\frac{1}{r}\right)}{\partial y} + \frac{\partial\left(\frac{1}{r}\right)}{\partial z}$$

$$\nabla\cdot\nabla\left(\frac{1}{r}\right) = \frac{\partial^2\left(\frac{1}{r}\right)}{\partial x^2} + \frac{\partial^2\left(\frac{1}{r}\right)}{\partial y^2} + \frac{\partial^2\left(\frac{1}{r}\right)}{\partial z^2}$$

$$\frac{\partial\left(\frac{1}{r}\right)}{\partial x} = -\frac{1}{2}(x^2+y^2+z^2)^{-\frac{3}{2}}(2x) = -(x^2+y^2+z^2)^{-\frac{3}{2}}(x)$$

$$\frac{\partial^2\left(\frac{1}{r}\right)}{\partial x^2} = \frac{\partial}{\partial x}\left[-(x^2+y^2+z^2)^{-\frac{3}{2}}(x)\right]$$

$$= \frac{3}{2}(x^2+y^2+z^2)^{-\frac{5}{2}}(x)(2x) - (x^2+y^2+z^2)^{-\frac{3}{2}}$$

$$= 3(x^2+y^2+z^2)^{-5/2}(x^2) - (x^2+y^2+z^2)^{-\frac{3}{2}}$$

$$= 3\,r^{-5}x^2 - r^{-3}$$

By analogy:

$$\frac{\partial^2\left(\frac{1}{r}\right)}{\partial y^2} = 3r^{-5}y^2 - r^{-3}$$

$$\frac{\partial^2\left(\frac{1}{r}\right)}{\partial z^2} = 3r^{-5}z^2 - r^{-3}$$

$$\nabla\cdot\nabla\left(\frac{1}{r}\right) = 3\,r^{-5}(x^2+y^2+z^2) - 3r^{-3}$$

$$= 3r^{-5}r^2 - 3r^{-3} = \frac{3-3}{r^3} = 0$$

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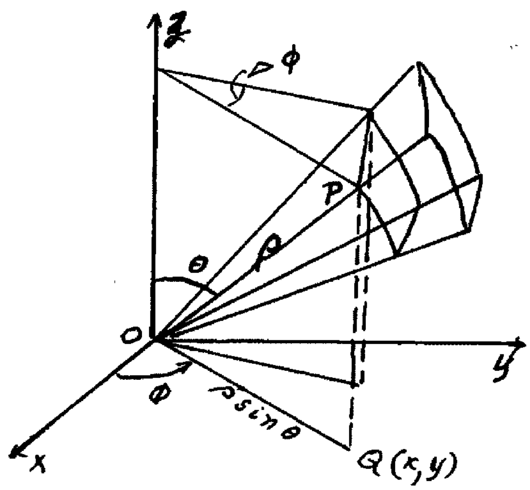
6 Leigh Page, Introduction to Theoretical Physics. New York: D.Van Nostrand Company, Inc, second edition 1935, p. 29 prob 14b.

The foregoing is an example which shows that the student of Physics should be familiar with the principles of partial differentiation. The problem also demands that the student understand the operator ∇ (del) and the use of the dot product in vector analysis.

The use of the operator ∇ in both the scalar and vector products, applied to vector functions leads to the definitions of the divergence and the curl of a vector function, respectively. The dot product which gives the divergence of a vector function makes use of Green's theorem. This theorem is sometimes called the divergence formula and when developed in Physics, particularly, in vector notation, is called Gauss' theorem. The vector product of the operator ∇ and a vector function is called the curl of a vector function and makes use of Stoke's theorem.

It is necessary that the student understand these theorems since they have important applications in the study of Physics; also, it is essential from the standpoint of mathematics that the student be able to develop and apply the above theorems in a purely mathematical manner.

EXAMPLE Quite frequently for simplicity's sake it is necessary to transform to a different system of coordinates. This change of variables gives rise to the use of the Jacobian or functional determinant. To illustrate in a simple case, the Jacobian will be used in transforming from rectangular to spherical coordinates.



Let P(x,y,z) be any point whose projection on the xy-plane is Q(x,y). The spherical coordinates of P are ρ, ϕ, θ; ρ is OP, ϕ the angle between OQ and the positive x-axis and θ is the angle between OP and the positive z-axis.

$x = OQ \cos \phi = OP \cos(90 - \theta) = \rho \sin \theta \cos \phi$

$y = OQ \sin \phi = \rho \sin \theta \sin \phi$

$z = \rho \cos \theta$

And the element of volume (Woods: Advanced Calculus page 159)

$$dV = dx\, dy\, dz = \left| J\left(\frac{x\ y\ z}{\rho\ \phi\ \theta}\right) \right| d\rho\, d\phi\, d\theta$$

$$J\left(\frac{x\ y\ z}{\rho\ \phi\ \theta}\right) = \begin{vmatrix} \sin\theta \cos\phi & \sin\theta \sin\phi & \cos\theta \\ -\rho \sin\theta \sin\phi, & \rho \sin\theta \cos\phi & 0 \\ \rho \cos\theta \cos\phi & \rho \cos\theta \sin\phi & -\rho \sin\theta \end{vmatrix}$$

$$J\left(\frac{x\,y\,z}{\rho\,\phi\,\theta}\right) = -\rho^2\sin^3\theta\cos^2\phi - \rho^2\sin\theta\cos^2\theta\sin^2\phi + 0$$
$$-\rho^2\cos^2\theta\sin\theta\cos^2\phi - 0 - \rho^2\sin^3\theta\sin^2\phi$$
$$= -\rho^2\sin\theta(\sin^2\theta\cos^2\phi + \cos^2\theta\sin^2\phi$$
$$+\cos^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi)$$
$$= -\rho^2\sin\theta$$

Then $dV = \rho^2 \sin\theta \, d\rho \, d\phi \, d\theta$

The above expression may also be derived by a geometrical method.

To apply the preceding to a practical problem: Find the x coordinate of the center of gravity of a solid of uniform density, σ, lying the first octant and bounded by the three coordinate planes and the sphere $x^2 + y^2 + z^2 = a^2$

$$\bar{X} = \frac{\sigma\int_R x\,dV}{\sigma\int_R dV} = \frac{\int_R x\,dV}{\frac{\pi a^3}{6}}$$

This can be evaluated by the iterated integral.

$$\int_R x\,dV = \int_0^a \int_0^{\sqrt{a^2-z^2}} \int_0^{\sqrt{a^2-y^2-z^2}} x\,dx\,dy\,dz$$

It is considerably easier to transform to spherical coordinates before integrating. Then,

$$\int_R x\,dV = \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^a \rho^3 \sin^2\theta\cos\phi\,d\rho\,d\theta\,d\phi$$
$$= \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \left[\frac{\rho^4}{4}\right]_0^a \sin^2\theta\cos\phi\,d\theta\,d\phi$$
$$= \frac{a^4}{4}\int_0^{\frac{\pi}{2}} \left[-\frac{1}{2}\sin\theta\cos\theta + \frac{1}{2}\theta\right]_0^{\frac{\pi}{2}} \cos\phi\,d\phi$$
$$= \frac{a^4\pi}{16}\int_0^{\frac{\pi}{2}} \cos\phi\,d\phi = \frac{a^4\pi}{16}$$

Therefore

$$\bar{X} = \frac{\frac{a^4 \pi}{16}}{\frac{\pi a^3}{6}} = \frac{3a}{8}$$

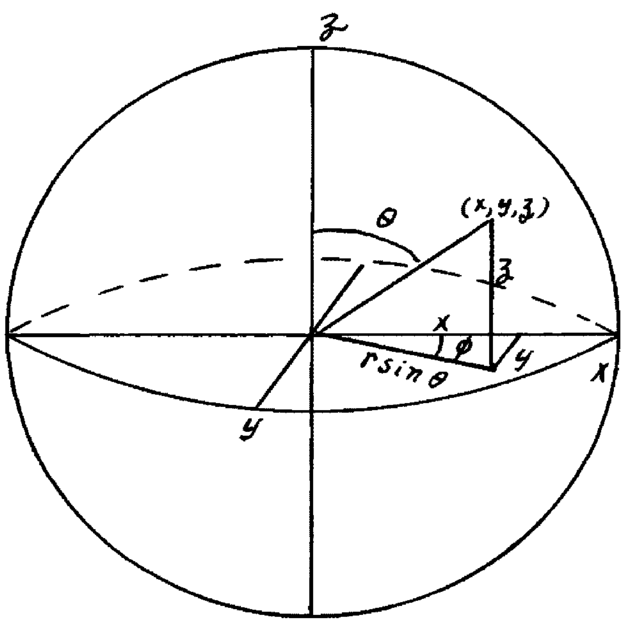
This example illustrates that it is often desirable to transform to a different system of coordinates, sometimes for ease of solution, other times to obtain a well known form which meets the demands of the problem. The student should, therefore, be familiar with the rectangular, the cylindrical, and the spherical systems of coordinates. The change from one system to another requires a change of variables. This change of variables requires the use of the functional determinant or Jacobian. The use of the Jacobian is quite frequently overlooked in the transformation from rectangular to spherical coordinates. It involves less effort to work out the relation between the variables by a geometrical method than to apply the Jacobian in such a simple case.

This problem of finding the coordinate of the center of gravity involves the use of multiple integrals and the evaluation of such multiple integrals by repeated or iterated integration. Not the least part of the problem is choosing the best order of integration and adjusting the limits of the integral to correspond to that order.

EXAMPLE. Laplace's equation is of great importance in Theoretical Physics. It is encountered in the study of hydrodynamics of perfect fluids, electrostatics and magnetostatics. In order to solve Laplace's equation for the case where the function is a function of r, θ, ∅, the equation must be expressed in spherical coordinates.

Laplace's equation may be obtained in spherical coordinates be direct substitution.

For transforming $\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$ to spherical coordinates, let



$x = r \sin\theta \cos\phi$

$y = r \sin\theta \sin\phi$

$z = r \cos\theta$

also let

$\rho = r \sin\theta$

Then

$x = \rho \cos\phi$

$y = \rho \sin\phi$

$\rho^2 = x^2 + y^2$

$0 = \tan^{-1} \frac{y}{x}$

$$\frac{\partial \rho}{\partial x} = \frac{\partial\left(\sqrt{x^2+y^2}\right)}{\partial x} = \frac{x}{\sqrt{x^2+y^2}} = \cos\phi$$

$$\frac{\partial \phi}{\partial x} = \frac{-\frac{y}{x^2}}{1 + \frac{y^2}{x^2}} = -\frac{\rho \sin\phi}{\rho^2} = -\frac{\sin\phi}{\rho}$$

$$(1)\ \frac{\partial V}{\partial x} = \frac{\partial V}{\partial \rho}\frac{\partial \rho}{\partial x} + \frac{\partial V}{\partial \phi}\frac{\partial \phi}{\partial x} = \frac{\partial V}{\partial \rho}\cos\phi - \frac{\partial V}{\partial \phi}\frac{\sin\phi}{\rho}$$

Let $\frac{\partial V}{\partial x} = k$

$$(2)\ \frac{\partial^2 V}{\partial x^2} = \frac{\partial k}{\partial \rho}\cdot\frac{\partial \rho}{\partial x} + \frac{\partial k}{\partial \phi}\frac{\partial \phi}{\partial x}$$

$$(3)\ \frac{\partial^2 V}{\partial x^2} = \cos\phi\,\frac{\partial k}{\partial \rho} - \frac{\sin\phi}{\rho}\frac{\partial k}{\partial \phi}$$

$$\frac{\partial k}{\partial \rho} = \cos\phi\,\frac{\partial^2 V}{\partial \rho^2} + \frac{\sin\phi}{\rho^2}\frac{\partial V}{\partial \phi} - \frac{\sin\phi}{\rho}\frac{\partial^2 V}{\partial \rho\,\partial \phi}$$

$$\frac{\partial k}{\partial \phi} = -\sin\phi\,\frac{\partial V}{\partial \rho} + \cos\phi\,\frac{\partial^2 V}{\partial \rho\,\partial \phi} - \frac{\cos\phi}{\rho}\frac{\partial V}{\partial \phi} - \frac{\sin\phi}{\rho}\frac{\partial^2 V}{\partial \phi^2}$$

Substituting the above values in (3)

$$(4)\ \frac{\partial^2 V}{\partial x^2} = \cos^2\phi\,\frac{\partial^2 V}{\partial \rho^2} + \frac{2}{\rho}\sin\phi\cos\phi\left[\frac{1}{\rho}\frac{\partial V}{\partial \phi} - \frac{\partial^2 V}{\partial \rho\,\partial \phi}\right] + \frac{\sin^2\phi}{\rho}\frac{\partial V}{\partial \rho} + \frac{\sin^2\phi}{\rho^2}\frac{\partial^2 V}{\partial \phi^2}$$

By analogy from (4)

$$(5)\ \frac{\partial^2 V}{\partial y^2} = \sin^2\phi\,\frac{\partial^2 V}{\partial \rho^2} - \frac{2}{\rho}\sin\phi\cos\phi\left[\frac{1}{\rho}\frac{\partial V}{\partial \phi} - \frac{\partial^2 V}{\partial \rho\,\partial \phi}\right] + \frac{\cos^2\phi}{\rho}\frac{\partial V}{\partial \rho} + \frac{\cos^2\phi}{\rho^2}\frac{\partial^2 V}{\partial \phi^2}$$

Adding (4) and (5)

$$(6)\ \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = \frac{\partial^2 V}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial V}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial \phi^2}$$

Let $z = r\cos\theta$, $\rho = r\sin\theta$, by analogy from (6)

$$(7)\ \frac{\partial^2 V}{\partial z^2} + \frac{\partial^2 V}{\partial \rho^2} = \frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{1}{r}\frac{\partial^2 V}{\partial \theta^2}$$

Adding equations (6) and (7)

$$(8)\quad \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = \frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{1}{\rho}\frac{\partial V}{\partial \rho} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial \phi^2}$$

Since $z = r\cos\theta$ $\qquad r^2 = z^2 + \rho^2$

$\rho = r\sin\theta$ $\qquad \theta = \tan^{-1}\frac{\rho}{z}$

$$(9)\quad \frac{\partial r}{\partial \rho} = \frac{\rho}{r} = \frac{r\sin\theta}{r} = \sin\theta$$

$$(10)\quad \frac{\partial \theta}{\partial \rho} = \frac{1}{1+\frac{\rho^2}{z^2}}\cdot\frac{z}{z^2} = \frac{z}{z^2+\rho^2} = \frac{r\cos\theta}{r^2} = \frac{\cos\theta}{r}$$

$$(11)\quad \frac{\partial V}{\partial \rho} = \frac{\partial V}{\partial r}\frac{\partial r}{\partial \rho} + \frac{\partial V}{\partial \theta}\frac{\partial \theta}{\partial \rho}$$

Substituting (9) and (10) in (11)

$$(12)\quad \frac{\partial V}{\partial \rho} = \frac{\partial V}{\partial r}\sin\theta + \frac{\partial V}{\partial \theta}\frac{\cos\theta}{r}$$

Substituting for ρ in (8)

$$(13)\quad \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = \frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial \phi^2} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} + \frac{1}{r\sin\theta}\left[\sin\theta\frac{\partial V}{\partial r} + \frac{\cos\theta}{r}\frac{\partial V}{\partial \theta}\right]$$

$$(14)\quad \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = \frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial \phi^2} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{\cot\theta}{r^2}\frac{\partial V}{\partial \theta} = 0$$

Multiplying by r^2

$$(15)\quad r^2\frac{\partial^2 V}{\partial r^2} + 2r\frac{\partial V}{\partial r} + \frac{1}{\sin^2\theta}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial \theta^2} + \cot\theta\frac{\partial V}{\partial \theta} = 0$$

but $\left(r^2 \frac{\partial^2 V}{\partial r^2} + 2r \frac{\partial V}{\partial r}\right)$ can be written $\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right)$

and $\left(\frac{\partial^2 V}{\partial \theta^2} + \cot\theta \frac{\partial V}{\partial \theta}\right.$ can be written as

$$\left(\frac{\sin\theta}{\sin\theta}\frac{\partial^2 V}{\partial \theta^2} + \frac{\cos\theta}{\sin\theta}\frac{\partial V}{\partial \theta}\right) = \frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right)$$

Substituting in (15)

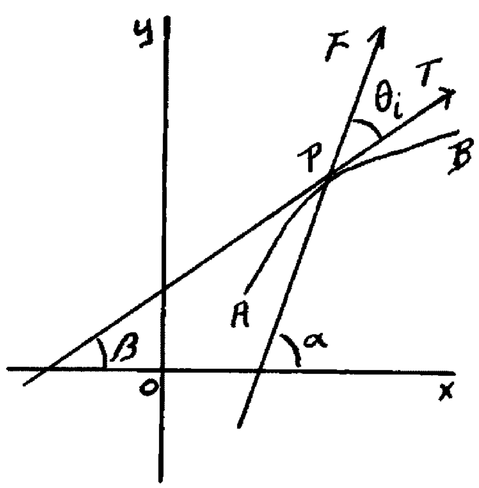
$$(16)\quad \frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial V}{\partial \phi^2} = 0$$

The form of the equation given in (15) is the form most frequently found in mathematical texts and agrees with the statement of the same equation found in Woods' Advanced Calculus, page 306, equation 3.

The form of the equation written in (16) is the usual polar statement found in Physics' texts, and agrees with Page: Introduction to Theoretical Physics, page 264 equation 79-6

EXAMPLE "The line integral of a mechanical force along a curve is termed the work performed along this path: according as it is positive or negative in sign, we speak of the work 'done by' or 'done against' the force." [7]

WORK. The force $F(x,y)$ acts at every point of the xy-plane. This force varies from point to point in magnitude and direction. An example of such conditions is the case of an electric field of force. The problem is to find the work done on a particle moving from point A to point B. The work done on a particle



moving through a distance Δs_i is $F(x_i, y_i)\cos\theta_i \Delta s_i$ (approximately). Therefore

$$W = \lim_{n\to\infty} \sum_{i=0}^{n-1} F(x_i, y_i)\cos\theta_i \,\Delta s_i$$

$$= \int_C F(x,y)\cos\theta \, ds$$

The tangent at P makes an angle $= \beta$ with the x-axis and the direction of F at P makes an angle $= \alpha$ with the x-axis.

Then $\theta = \alpha - \beta$

$$\cos\theta = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

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7 Arthur Haas, Introduction to Theoretical Physics. New York: D.Van Nostrand Company, Inc. 1926 Vol.I pp 36-37.

$$W = \int_c F(xy)\ (\cos\alpha \cos\beta + \sin\alpha \sin\beta)\ ds$$

But $F \cos\alpha$ = x-component of F

$= X$

$F \sin\alpha$ = y-component of F

$= Y$

and $\cos\beta = \dfrac{dx}{ds}$

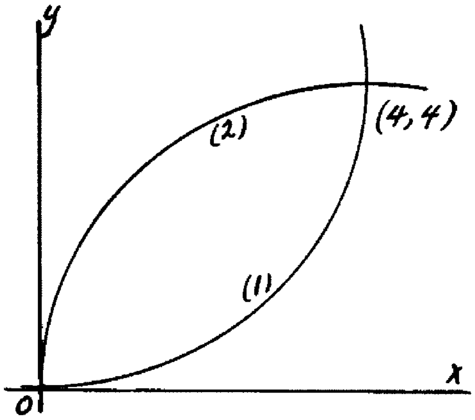
$\sin\beta = \dfrac{dy}{ds}$

Therefore

$$W = \int_c (X\,dx + Y\,dy)$$

The above is the most common form in which line integrals are found.

EXAMPLE. The area bounded by a closed curve such that no line parallel to either of the coordinate axes intersects the curve in more than two points can be expressed as the line integral.



$$A = \frac{1}{2}\int_{c}(-y\ dx + x\ dy)$$

To illustrate the application of the above statement, the area between (1) $x^2 = 4y$ and (2) $y^2 = 4x$ will be determined.

Then $A = \frac{1}{2}\int_{c}(-y\,dx + x\,dy) = \frac{1}{2}\int_{(1)}(-y\,dx + x\,dy) + \frac{1}{2}\int_{(2)}(-y\,dx + x\,dy)$

$$= \frac{1}{2}\int_{0}^{4}\left(-\frac{x^2}{4}dx + x\frac{x}{2}dx\right) + \frac{1}{2}\int_{4}^{0}\left(-y\frac{y}{2}dy + \frac{y^2}{4}dy\right)$$

$$= \left[\frac{x^3}{24}\right]_0^4 - \left[\frac{y^3}{24}\right]_4^0 = \frac{16}{3}$$

This application of the line integral requires the use of the definite integral and the choice of the correct limits. This example demands in the choice of limits that the student observe the positive direction in traversing the curve. The positive direction is that in which a person walking around the curve has the region on his left hand.

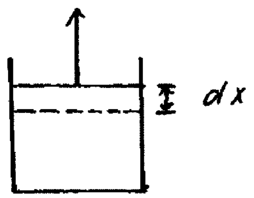
Another illustration of this particular application is

$$W = \frac{1}{4\pi}\int_{0} H\ dB$$

which is the work per unit volume per cycle and is equal to $\frac{1}{4\pi}$ times the area of the hysteresis loop.

EXAMPLE[8] The line integral applied to thermodynamics.

Let a unit mass of gas inside a cylinder of cross section A

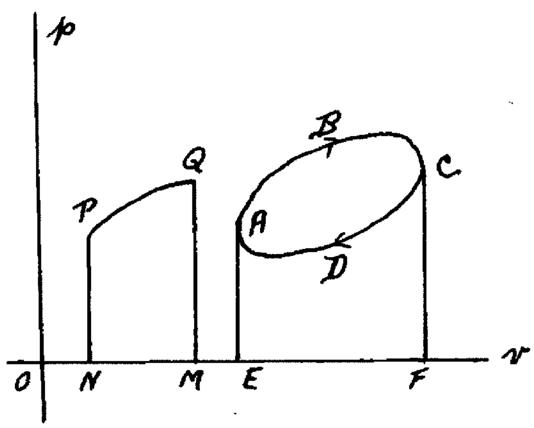


be subjected to a pressure p by means of a piston. The piston is moved upward a distance dx. Since the displacement is small the assumption is that the pressure is not affected appreciably.

The work done

$$dW = F \times \text{distance} = pA\ dx$$

$dW = p\ dv$ where dv is the volume increment.



The state of a gas can be represented by two variables, p and v. The state can then be represented by a point, and, as the variables p and v change slowly the point will describe a curve, arriving at Q. The external work done by the gas during the change is

$$\int_P^Q p\ dv = \text{[illegible]}$$

If the point representing the state of the gas is originally at A and if the volume and pressure of the gas are put through a succession of changes and finally return to

8 R. A. Houstoun, An Introduction to Mathematical Physics. Third impression. New York: Longmans, Green and Company, 1926 pp. 168-9-70.

their original values, the point will describe a closed curve such as ABCD, returning to the point of departure A. The gas is then said to be put through a cycle. The total external work done by the gas during the cycle is equal to the area of the closed curve ABCD, because the work done by the gas is moving from A to C is ABCFE, and the work done against the gas in moving from C to A is CDAEF. The area under the curve ABCD may be expressed in terms of a line integral taken around the boundary of the area when the value of the variables and the path of the curve are known. (Woods, pages 177-8)

Consider a unit mass of working substance (such as a perfect gas), suppose a quantity of heat dq to be supplied to it. According to the first principle of Thermodynamics:

it does external work

it increases the intrinsic energy of the substance.

This may be expressed mathematically by

$$dq = dU + dw$$

but $dw = p\,dv$ and since p and v are the independent variables

$$U = U(p,v)$$

then

$$dU = \frac{\partial u}{\partial p}dp + \frac{\partial u}{\partial v}dv$$

Then $dq = \frac{\partial u}{\partial p} dp + \frac{\partial u}{\partial v} dv + p\, dv$

$$= \frac{\partial u}{d p} dp + \left(\frac{\partial u}{\partial v} + p \right) dv$$

Integrating $$q = \int_c \left[\frac{\partial u}{\partial p} dp + \left(\frac{\partial u}{\partial v} + p \right) dv \right]$$

which is a line integral of the form

$$\int_c \left[M(p,v)\, dp + N(p,v)\, dv \right]$$

These line integrals which have been applied to work, area and thermodynamics may also be found in problems of the flow of fluids. The wide application of the line integral to problems of Physics shows the need to be thoroughly familiar with the line integral and the evaluation of the integral and substitution of definite limits.

EXAMPLE[9] Show by Gauss' and Stoke's theorems $\nabla \cdot \nabla \times \bar{V} = 0$ and $\nabla \times \nabla \varphi = 0$

(a) The divergence of the curl is $\nabla \cdot \nabla \times \bar{V}$

Let $\nabla \times \bar{V} = \bar{W}$

By Gauss' theorem: The volume integral of the divergence of a vector function of position $\bar{V}$ taken over any volume τ is equal to the surface integral of $\bar{V}$ taken over the closed surface surrounding the volume τ.

$$\int_\tau \nabla \cdot \bar{W}\, d\tau = \int_\sigma \bar{W} \cdot d\sigma = \int_\sigma \nabla \times \bar{V} \cdot d\sigma$$

By Stoke's theorem: The surface integral of the curl of a vector function of position $\bar{V}$ taken over any surface is equal to the line integral of $\bar{V}$ around the periphery of the surface. $\int_\sigma \nabla \times \bar{V} \cdot d\sigma = \int_0 \nabla \cdot d\bar{\lambda}$

Since the periphery is a closed curve and since the line integral of the gradient of any scalar function around a closed curve vanishes, the following is true,

$$\int_0 \nabla \cdot d\bar{\lambda} = 0$$

(b) The curl of the gradient $= \nabla \times \nabla \varphi$. Let $\bar{V} = \nabla \varphi$

By Stoke's Theorem:

$$\int_\sigma \nabla \times \bar{V} \cdot d\sigma = \int_0 \bar{V} \cdot d\bar{\lambda} = \int_0 \nabla \varphi \cdot d\bar{\lambda}$$

By definition, the line integral about a closed curve,

$$\int_0 \nabla \varphi \cdot d\bar{\lambda} = \int_0 d\varphi = 0$$

9 Leigh Page, Introduction to Theoretical Physics. New York: D.Van Nostrand Company, Inc. Second edition 1935. p. 39 prob. 17a.

The proceeding example shows the wide use of Gauss' (Green's) theorem or the divergence formula and Stoke's theorem. This is an application of changing a volume integral to an equivalent surface integral taken over the surface enclosing the volume and then changing a surface integral over into a line integral which traverses the rim of the surface in a positive direction. The integral normal to the surface is know as Gauss' theorem in Physics, in particular, when interpreted by vectors. In the mathematical field the same theorem is called Green's theorem. The same theorem is sometimes called the divergence formula in both the study of Physics and Mathematics.

This example also requires the use of vector notation and the meaning of the terms, gradient, curl and divergence.

CHAPTER II

ADVANCED CALCULUS

There are certain fundamental concepts and definitions which are prerequisite to the study of Advanced Calculus and at the same time some skill in the handling of them is necessary. The student should have not only a working knowledge of these fundamentals but a good portion of the theory for a background in order to pursue a successful study of this course. The essentials which are necessary are found in the following prerequisites:

College Algebra

Trigonometry

Analytic Geometry

Differential Calculus

Integral Calculus

Theory of Equations (Not necessarily a prerequisite but very desirable)

The course in Advanced Calculus proposes to cover that field as completely as possible during two quarters of study. There will be no attempt to make this course a formal study in analysis such as will be encountered in the Theory of Functions of a Real Variable, although some knowledge of theory is necessary if correct use is to be made of the science. From the standpoint of the mathematician the course will be an introduction to the methods of analysis. The scope of the course will be such that it will be inclusive of those mathematical principles vital to the study of Theoretical and Modern Physics. The treatment of these topics will give the skill in the applications of the principles as well as the theory. Although of major importance in the study of Physics and of equal importance in the study of Advanced Calculus, the topic of differential equations is not mentioned in the outline of the proposed course. The omission was intentional since it is suggested that a separate course in differential equations be offered to supplement the outlined course. The study of differential equations could be made prior to the study of advanced calculus or the study could be made concurrently with the first quarter of the calculus course.

The various texts considered as possible basic texts for the course were varied in the treatment and completeness of the topics usually found in Advanced Calculus. One author treated the course as an introductory course in analysis and gave very little consideration to the needs of those students who might desire to make practical applications of those principles in the field of Physics. One text offered a selection of those topics which are most frequently used in Physics and engineering, and gave a number of examples from those fields. From the standpoint of utility, the selection and treatment of those topics was good; but since this course must meet two requirements there was a noticable lack of rigorous presentation and of the number of topics selected.

The text suggested as basic for this course, "Advanced Calculus" by F.S.Woods, offers quite a complete selection of topics. They are presented in a rigorous and formal manner for the more simple cases but the more difficult are assumed to be true. To quote the author:

> "As an example of the method used, a proof of the existence of the definite integal in one variable has been given; for the multiple integral the proof has been omitted and simply the result stated. The student who has mastered the simpler case is in a position to read the more difficult case in easily accessible texts."

The following pages contain an outline of a proposed course which follows closely the suggested basic text by Woods. The chapter headings are those given by Woods, the numbers indicate the order in which they are to be studied. Several chapters and topics which occur in the Woods' text have been omitted in this outline, since there is no immediate need of them for the student of Physics. There have been some topics inserted which do not occur in the text but which are deemed necessary for the course.

Following the list of topics in each chapter are comments on the treatment given by Woods and some suggested elaborations or omissions.

CHAPTER I. PRELIMINARY

Functions

Continuity

The derivative

Composite functions

Rolle's theorem

Theorem of the Mean

Taylor's series with a remainder

Indeterminate forms

Infinitesimals

Fundamental theorems on infinitesimals

Geometric theorems on infinitesimals

The first differential

Higher differentials

Change of Variable

Numerical solution of equations

The purpose of this chapter is to review the fundamental definitions and theorems, and see to it that the definitions are more rigorous. The student should become more familiar with the epsilon-delta method in all definitions and statements of theorems and in the proofs of some of the theorems.

The definition of function is to be complete, clear and concise although no mention of classes or elements, such as will be found in the same definition in the study of Functions of a Real Variable, need be introduced here.

The so-called "more Cumbersome" definition of continuity:

" f(x) is continuous for x = a when if ϵ is any assigned positive quantity, no matter how small, it is possible to determine another positive quantity δ so that the difference in the absolute value between f(a + h) and f(a) shall be less than ϵ for all values of h numerically less than δ ; that is

$$|f(a+h) - f(a)| < \epsilon \qquad \text{when } |h| < \delta."$$

is much to be preferred for the students in this course. The illustrations of functions which are discontinuous at a point are clear but are not classified as to types or kinds of discontinuity. Special emphasis should be placed upon the types of discontinuity and their recognition. The four fundamental theorems on continuity are well stated but have no rigorous proofs. The proofs may be deferred until the study of more advanced works.

The derivative is defined and an example which shows that a function could be continuous and still not possess a derivative, is given. In the discussion of the derivative there occurs a very significant statement, "when a new function appears in analysis it is necessary to inquire first whether it is continuous and, secondly, whether it has a derivative".

The definition of composite function is in a satisfactory analytic form and the formation of the difference quotients is good.

The statements of Rolle's Theorem, the Theorem of the Mean, and Taylor's Theorem have been stated in a manner suitable for this course. There is not an explicit statement in the conditions that the derivative may be finite or infinite although that meaning is implied in saying that the function has a derivative in the closed interval (a,b). Too much emphasis cannot be placed on the study of these theorems. They are fundamental and most important in the proof of theorems which are found later in this course and in the study of the Functions of a Real Variable. It is of particular importance that the relation between the theorem of the Mean and the Extended Theorem of the Mean or Taylor's Series with a Remainder be shown.

Also it is important that the student realize the relation bewеen Taylor's and Maclaurin's series: that is, Maclaurin's series is a particular case of Taylor's series developed about the point zero. Taylor's formula has great theoretical value, and may also be used in calculation.

The importance of Rolle's Theorem is again shown in the proving of L'Hospital's rule for evaluating the indeterminate forms. This evaluation of indeterminate forms is of much importance in the study of infinitesimals. The concept of infinitesimals and their order is fundamental in the study of calculus and the importance cannot be stressed too much in this introductory chapter.

The discussion of the first and higher differentials should be elaborated upon. The definitions are good but should be explained more fully with examples.

The illustrations of a change of variable make use of the methods and formulas developed in the study of the differentials.

The topic Numerical Solutions of Equations need not include the algebraic solution of a cubic equation nor Horner's method. The importance of the solution of transcendental equations by Newton's method should be stressed and illustrated with applications. This method of Newton applies equally well to the algebraic, logrithmic, exponential, and trigonometric equations.

CHAPTER II. POWER SERIES

Definitions

Tests for convergence

Region of convergence

Uniform convergence

Function defined by a power series

Integral and derivative of a power series

Taylor's series

Operations with two power series

The exponential and trigonometric functions

Hyperbolic functions

Dominant functions

Conditionally convergent series

The definition of a power series and of convergence are good and well illustrated. The rigorous treatment may well be delayed until the study of Real Variables.

The tests for convergence, the ratio and comparison, are important. Since Power series are used extensively, their characteristics must be known, that is, if they are convergent or divergent and something of the region in which the series is convergent. From the standpoint of the mathematician a clear conception of uniform convergence is important.

The definition of a function by a power series and the proof of its continuity in the region of convergence is given in a satisfactory manner by Woods.

The integral and derivative of a power series and the three theorems which are given are valuable in both theoretical and practical work. The same may be said of the operations with two power series.

The study of the exponential, trigonometric and hyperbolic functions is a special example of the application of Taylor's theorem or to be explicit, Maclaurin's series. These functions and their expansion into a power series has an extensive application to the problems of Physics.

The treatment of the dominant functions and conditionally convergent series is quite clear.

The exercises given at the end of this chapter offer a great amount of practice in the application of the foregoing principles. Toward the end of the list of exercises occur some very significant expressions, such as:

$$\int_0^x e^{-x^2}dx \quad \text{and} \quad e^{\sin x}$$

CHAPTER III. PARTIAL DIFFERENTIATION

Functions of two or more variables

Partial derivatives

Order of differentiation

Differentiation of composite functions

Euler's theorem on homogeneous functions

Directional derivative

The first differential

Higher differentials

Taylor's series for f(x,y)

This entire chapter on partial differentiation is of the utmost importance in the study of both mathematics and Physics since a large portion of the functions are of more than one variable.* The study of the directional derivative or gradient** when taken in the direction of the normal is of particular importance in applied mathematics. A clear concept of the meaning of the gradient is necessary in the study of Physics and vector analysis which is taken up later in this course.

The importance of Taylor's series is again noted in its use with functions of two variables and partial derivatives.

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* See Examples pp. 9,11,13,16,20,&c
** See Example p. 9

CHAPTER IV. IMPLICIT FUNCTIONS

One equation, two variables

One equation, more than two variables

Two equations, four variables

Three equations, six variables

The general case

Jacobians

The treatment of implicit functions is satisfactory for students of this course and at the same time lays a solid foundation for the definition of the functional determinant or Jacobian. The theorems on the functional determinant are well stated and proved. The use of the Jacobian should be clearly understood by the students. This use should be well illustrated by such simple cases as the change of variables in changing to different systems of coordinates. The application of the Jacobian in Stoke's theorem should be emphasized.

CHAPTER V. APPLICATIONS TO GEOMETRY

Element of arc

Straight line

Surfaces

Planes

Behavior of a surface near a point

Maxima and minima

Curves

Curvature and torsion

Curvilinear Coordinates

This subject of Geometry is important to the student of applied mathematics because of its wide application in the field of Physics. The definitions developed here are applied immediately in the development of surface integrals and of Stoke's Theorem. Some authors have neglected the needs of the student of applied mathematics by the omission of essential topics of this chapter. One author makes no mention of such fundamental as curvilinear coordinates.

The definitions of this chapter in Advanced Calculus by Woods are developed, when possible, in the terms of calculus. The rigorous definitions of element of arc, direction cosines, lines, surfaces, normals and planes are of vital importance in the study of Physics as well as in the field of mathematics. The study of curvilinear coordinates should be made quite thorough in order to build a necessary background for the application of the two more common systems, the cylindrical and the polar or spherical coordinates.

CHAPTER VI THE DEFINITE INTEGRAL

Definition

Existence proof

Properties of definite integrals

Evaluation of a definite integral

Change of variables

Differentiation of a definite integral

Integration under the integral sign

Infinite limit

Differentiation and integration of an integral with an infinite limit

Infinite integrals

Certain definite integrals

Multiple integrals

The Gamma and Beta functions

The concept of the definite integral is developed in a satisfactory manner for the students of both mathematics and Physics, accompanied by a rigorous existence proof. The idea of the definite integral is one of the utmost importance to the students of applied and theoretical mathematics. The student of applied mathematics is interested in the evaluation of the definite integral, and the student in analysis is interested in the properties of the definite integral. The discussion of a function with points of discontinuity need not be elaborated upon since the essential points are adequately and concisely covered. A further discussion would necessitate a treatment of point sets, which is not advisable at this time.

The properties of definite integrals are well stated.

There is a rigorous development of the evaluation of the definite integral which leads to the statement of the fundamental theorem of integral calculus:

$$\int_a^b f(x)\,dx = F(b) - F(a) .$$

The study of Simpson's rule may be mentioned or entirely omitted, since it is felt that it has very little theoretical value; and the infrequent practical application does not justify its treatment in this course.

The discussion of differentiation and integration under the integral sign for integrals with finite or infinite limits is rigorous and lends itself to numerous examples. The preceding remarks are also applicable to the definite integral in which f(x) becomes infinite at one or more points, in particular at the upper limit $x = b$.

Certain definite integrals which are given as examples to show the application of certain special devices are quite important in the field of applied mathematics. Among these examples is found the following important integral:

$$\int_0^\infty e^{-x^2}\,dx$$

There is a clear and concise discussion of multiple integrals in which the importance of the Jacobian is shown in changing from rectangular to curvilinear coordinates.

At this point of the course it might be wise to introduce the study of the Gamma function and possibly the definition of the Beta function. Although of little immediate use to the student of Physics, the Gamma function is a special function defined by the definite integral. It is suggested that paragraphs Nos. 67 and 68 of Woods' Advanced Calculus be utilized for the treatment of this topic, or paragraph No. 79 of Higher Mathematics for Engineers and Physicists by Sokolnikoff be used as alternate material.

CHAPTER VII. LINE, SURFACE, AND SPACE INTEGRALS

Line integrals

Plane area as a line integral

Green's theorem in the plane

Dependence upon the path of integration

Exact differentials

Area of curved surface

Surface integrals

Green's theorem in space

Other forms of Green's theorem

Stoke's theorem

The line integral is as useful in theoretical and practical problems as the definite integral discussed in the preceding chapter. Woods defines the li e integral rigorously and gives the various forms in which the line integral is found, particularly the following:

$$\int_{a_1 b_1}^{a_2 b_2} F(x,y)\,dx \qquad \text{and} \qquad \int_{a_1 b_1}^{a_2 b_2} (P\,dx + Q\,dy)$$

The examples of work and of fluid flowing over a plane,

$$W = (X\,dx + Y\,dy) \qquad \text{and} \qquad V = h\int(-v\rho\,dx + u\rho\,dy)$$

v = velocity component parallel to OX

U = velocity component parallel to OY

are immediately applicable in the study of Physics.

The discussion of a plane area as a line integral with examples of application leaves little to be desired. Such an example would include the following: Find the area bounded by an ellipse and the chord connecting the ends of the major and minor axes by means of the line integral

$$A = \frac{1}{2}\int_{o}(x\,dy - y\,dx).$$

Green's theorem in the plane is clearly developed. The interpotation of the theorem is left to the student. It is essential that the student get the idea that Green's theorem in a plane establishes the relation between a line integral around a closed curve and a double integral taken over the region bounded by the closed curve.

The definition of a simply connected region and the discussion of the line integral in such a region being independent of the path of integration are clearly stated and well illustrated.

The line integral is used here to establish the necessary and sufficient condition of the exact differential independently from the method used in the discussion of partial differentiation.

The area of a curved surface and the surface integral are important for a background in the study of the following important theorems. This study of surface and the surface integral makes considerable use of the concepts found under the topic of curvilinear coordinates.

Green's theorem in space is clearly and thoroughly developed and is extended into other forms which find a ready application to the problems of Physics. Again it might be well to emphasize the meaning of the theorem. By means of this theorem a surface integral taken over a closed surface is related to a triple integral taken over the region of space enclosed by the surface. The direction of the normal is important, it being taken at each point of the surface in the outward direction. This particular theorem is known as Green's theorem in the study of mathematics but in the study of Physics it is known as Gauss' theorem or the divergence formula. This theorem is immediately applied to the flow of a fluid out across a surface and leads directly to the "equation of continuity" in hydromechanics.

Stoke's theorem is rigorously developed with the aid of definitions developed in the study of curvilinear coordinates. The fact that Stoke's theorem gives the relation between a line integral in space and the surface integral over the surface enclosed by the closed path of the line integral cannot be stressed too greatly.

Both Green's and Stoke's theorems have important applications in the following chapter on Vector Notation. The development of the expressions of divergence and curl respectively.

CHAPTER VIII. VECTOR NOTATION

Vectors

The scalar product

The vector product

Curves

Areas

The gradient

The divergence

The curl

A knowledge of vector notation and proficiency in the application of the principles of vector analysis is imperative for a successful study of theoretical Physics.

This chapter is an introduction to vector notation in which emphasis will be placed on the thorough understanding of the definitions and simple operations rather than a survey of the whole field of vector analysis.

The definitions are clear and complete, and the two products, the scalar and vector products, have been completely developed and interpreted for two factors.

The definitions of curves and areas in terms of vectors are concise and satisfactory for this course.

The gradient is developed again in this chapter but in vector notation. The definition of the gradient is extended to the define the operator del. The example of the development of the gradient as suggested by Haas which occurs in the forepart of this paper, begins with a surface and a normal to the surface. Haas proceeds to show that the components of the components of the gradient of a scalar are its partial derivatives with respect to the coordinates.

The operator del is applied to both scalar and vector functions. It is applied to a vector function in two ways: namely the scalar product and the vector product. The result of the application as a scalar or dot product gives by definition the divergence of a vector. The result is closely related to Green's theorem and this relation should be emphasized in the study of this particular result. The second application, as a vector or cross product, gives the expression known as the curl of a vector. This vector called the curl of a vector is closely associated with Stoke's theorem and too much emphasis cannot be placed on this fact.

CHAPTER IX. PARTIAL DIFFERENTIAL EQUATIONS

Introduction

Special forms of partial differential equations

The linear partial differential equation of
the first order

The Fourier series

The Fourier series with sines and cosines only

Laplace's equation in two variables

Application to the flow of heat

The Laplace equation in three variables

Application to potential

A partial differential equation is an equation which involves partial derivatives. There are but comparatively few cases which have an explicit solution. It is not the purpose here to study the theoretical background implied by partial differential equations, but merely to notice certain equations which are important in applied mathematics. There will be indicated certain methods for the solution of these particular equations. No mention will be made of the questions of the proof of existence of solutions, the convergence and the validity of operations upon such series.

The solution of partial differential equations involves arbitrary functions. In a practical application the problem is usually to determine a particular function which will satisfy the differential equation and at the same time meet the requirements of the practical problem.

Fourier series are introduced and applied to examples. Special consideration is given to the Fourier series with sines and cosines only.

The Laplace equation is considered in both two variables and three variables. In each case the equation is transformed from rectangular to polar coordinates. Laplace's equations are applied to the flow of heat and to potential.

CHAPTER X. FUNCTIONS OF A COMPLEX VARIABLE

Complex numbers

Graphical representation and trigonometric form

Powers and roots

The square root

Exponential and trigonometric functions

The hyperbolic functions

The logrithmic functions

The inverse hyperbolic and trigonometric functions

Functions of a complex variable in general

Conjugate functions

Conformal representation

Cauchy's theorem

Taylor's series

Poles and residues

Application to real integrals

The student who has had a course in the Theory of Equations will find a review of the fundamental concepts in the first portion of this chapter. The student who has not had the advantage of such previous study will find complete definitions and rules for manipulation of the complex number simply stated. Complex numbers are given graphical and trigonometric forms and the powers and roots are discussed. There is a simple treatment of the relation of complex numbers and the exponential, trigonometric, hyperbolic and logarithmic functions. In the discussion of functions of a complex variable, Woods gives a rigorous treatment of the topic. Conjugate functions are defined and it is shown that each of a pair of conjugate functions is a solution of the Laplace equation in two variables.

Conformal representation is too large a topic to be discussed thoroughly in such a course as this one. The integral of a complex function is introduced and a fundamental theorem stated. Cauchy's theorem and Taylor's series are well treated. The definitions of poles and residues are satisfactory for this course. The theorem on residues is used in several examples to evaluate integrals of real variables.

CHAPTER XI. ELLIPTIC INTEGRALS

Introduction

The functions sn u, cn u, dn u

Application to the pendulum

In the introduction to elliptic integrals there is a very good review of the various types of integrals which are encountered. Following the various types previously encountered there is introduced a new type called the elliptic integral and which requires a new function, the elliptic function, for its evaluation.

The three kinds of elliptic integrals are listed, a and the functions sn u, cn u, and dn u are defined.

The elliptic integral is then applied to the problem of the simple pendulum.

If there be sufficient time and the capabilities of the class warrant it, this course may be elaborated upon. The elaboration consisting of the addition of certain topics and the expansion of others already mentioned.

Concerning the latter the following topics are suggested:

The Gamma and Beta functions

Triple products in vector notation

The Fourier series

Elliptic integrals in the complex plane

The possible additional topics would include the following:

Harmonic functions

Bessel functions

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