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A radiation-driven wind model for massive stars: The effects of non-isothermal and finite disk assumptions

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A RADIATION-DRIVEN WIND MODEL FOR MASSIVE STARS: THE EFFECTS OF NON-ISOTHERMAL AND FINITE DISK ASSUMPTIONS

by

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Approved by:

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Dean, Graduate School

Date
A Radiation-Driven Wind Model for Massive Stars: The Effects of Non-Isothermal and Finite Disk Assumptions

Director: David B. Friend DBF

Wind models for massive stars are of particular interest because we are unable to have a complete picture of the evolution of these stars as well as the chemical composition of our own galaxy without understanding these mass loss mechanisms. For massive stars, the main force which drives the wind is the absorption and scattering of radiation from the star.

In order to model such a radiatively-driven stellar wind, I begin with the "standard" model of Castor, Abbott and Klein (CAK, 1975). Among the assumptions used in this model when determining the equation of motion for the wind are the following:

1) the wind does not change in time (steady-state)
2) the wind is spherically symmetric
3) the temperature of the wind does not change with distance from the star (isothermal)
4) the star is a point source of radiation
5) the star does not rotate
6) the star has no magnetic field.

The CAK predictions turn out to be too high for the mass loss rate of the star and too low for the velocity of the wind far from the star as compared to observations.

This work eliminates two of the simplifications in the CAK theory. First I treat the star as a disk with finite size (as previously done by Friend and Abbott, 1986). Treating the star as a finite disk reduces the radiation force near the star thereby bringing the mass loss rates closer to the observed rates.

Second I assume the temperature of the wind does change with distance from the star. I include a temperature distribution, which is a function of distance, in the equation of motion.

The questions I seek to answer are, "Is the isothermal assumption acceptable?" Does a temperature distribution complicate the equations unnecessarily? To answer these questions, the non-isothermal, finite disk results are compared to the finite disk results of Friend and Abbott as well as with observations to determine whether or not this isothermal assumption is reasonable.
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Chapter 1

Introduction

Modeling the mass loss mechanisms of massive stars is integral to the development of a complete picture of the evolution of massive stars as well as an understanding of the chemical composition and evolution of our entire galaxy. Radiation provides a mechanism by which a wind is driven when ions absorb the momentum of photons and are propelled away from the star, thereby producing mass loss. This work seeks to explain basic radiatively-driven wind theory, particularly in the context of the modification I have made to the theory—the assumption of a temperature distribution in the wind.

*What are massive stars?*

Stars are divided into spectral types, based on their temperatures, using the Harvard classification scheme below:

<table>
<thead>
<tr>
<th>Spectral Type</th>
<th>Characteristics</th>
</tr>
</thead>
<tbody>
<tr>
<td>O</td>
<td>Hottest blue-white stars with few spectral lines.</td>
</tr>
<tr>
<td>B</td>
<td>Hot blue-white stars.</td>
</tr>
<tr>
<td>A</td>
<td>White stars.</td>
</tr>
<tr>
<td>F</td>
<td>Yellow-white stars.</td>
</tr>
<tr>
<td>G</td>
<td>Yellow stars. (Our sun)</td>
</tr>
<tr>
<td>K</td>
<td>Cool orange stars.</td>
</tr>
<tr>
<td>M</td>
<td>Coolest red stars.</td>
</tr>
</tbody>
</table>
Certainly not all O type stars have the same temperature or radius, so each type is also scaled from 0 to 9 (hottest to coolest) and is given a luminosity class:

- I Supergiant
- II Bright Giant
- III Giant
- IV Subgiant
- V Dwarf or Main Sequence

For example, the hottest B giant would be classified as B0III.

The Hertzsprung-Russell Diagram is a standard way to illustrate the relationships among spectral type, temperature, radius and luminosity.

http://imagine.gsfc.nasa.gov/docs/teachers/lifecycles/printable/LC_main_p8.html

Figure 1.1
Massive stars are considered to be O and B type stars. They range in mass from 10 to 60 $M_\odot$ where $1 M_\odot = 1.998 \times 10^{33}$ g. Their effective temperatures (temperature at the surface or photosphere) are anywhere from 20,000 to 60,000 degrees Kelvin (compare to $T_{\text{eff}}$ of the sun at $\sim 5800$ K).

Where do we observe O and B stars? They tend to occur in open clusters and associations, or loosely grouped arrangements, which are closely confined to the plane of our galaxy. Rarely do we see any alone because their lifespans are relatively short (less than 10 million years), hence they do not have time to wander far from the nursery in which they were born. On the following pages are examples of places we find O and B stars. Figure 1.2 shows the formation of massive stars triggered by the winds of adult massive stars in the hub of the 30 Doradus Nebula in the Large Magellanic Cloud. NGC 81, a massive star cluster in the Small Magellanic Cloud, is shown in Figure 1.3. The rapid and enormous mass loss from these young massive stars is evident in the sculpting of the nebula’s gasses.

*What are stellar winds and why should we study them?*

Stellar winds are the expanding outer layers of a star’s atmosphere. Massive stars may dump up to 50% of their initial mass back into the interstellar medium over only several million years. This substantial mass loss gives rise to the following important effects:

1) Possible induction of stellar formation. The star may lose so much mass that another star is able to form from this matter.

2) The evolution of the mass-losing star is significantly altered.
Figure 1.2
Figure 1.3

N81 in the Small Magellanic Cloud

HST • WFPC2

PRC98-25 • ST Sci OPO • July 23, 1998

M. Heydari-Malayeri (Observatoire de Paris) and NASA
3) Changes in the chemical makeup of the galaxy in which the star resides, hence influencing galactic structure, chemistry and evolution.

**How do we know they exist?**

The first evidence of stellar winds was discovered in the late 19th century. Astronomers, analyzing the spectrum of a blue luminous variable star known as P Cygni, discovered something very interesting. A P Cygni profile is characterized by a broad emission peak centered on $\lambda_0$, for example the H$\beta$ line at 4860 angstroms, and a blueshifted absorption trough. Figure 1.4 shows a profile of P Cygni between the wavelengths of 4800 and 5050 angstroms. How are these spectral lines produced? A stellar wind can be thought of as a series of concentric shells surrounding the star.

![Diagram of stellar wind](image)

Region 1 in Figures 1.4 and 1.5 represent the portion of the shell which is in our direct line of sight. The approaching matter is cooler than the layers below it, thus the photons coming from the star are absorbed and blue shifted. Region 2 is the front portion of the shell which is approaching us. We do not see the photons being absorbed in this region because they are scattered out of our line of sight. Instead we
Figure 1.4 P Cygni Profile

http://www.lsw.uni-heidelberg.de/%7Eostahl/PCyg.html
observe emission produced by the hot diffuse matter. Region 3 is the rear portion of the expanding shell. Similarly we observe emission here, except that it is redshifted, indicating that it is moving away from us. Not until 1929 did Beals interpret these effects as the ejection of high-velocity material from the star.

Soon after this discovery, astronomers found P Cygni-like profiles in the spectra of O, B, G, K, and M stars confirming the presence of winds (Cranmer, 1997). From these spectra, we can infer terminal velocities and mass loss rates for massive stars as well.

*What is “radiation-driving”?*

Radiative driving is thought to be the primary cause of continuous mass outflows for main sequence, giant and supergiant stars that are hotter than approximately B5 (Cranmer, 1997). In an atmosphere, ions will absorb the momentum and energy of photons that have come from below them (closer to the center of the star) moving the ions generally outward. Ions can absorb only in a limited frequency range and in a relatively short period of time would absorb all the photons possessing those frequencies. No wind could be sustained this way. However, since the ions are moving away from the star, the photon frequencies are red-shifted, and at each radius (r1, r2, etc.) a different frequency is absorbed. See figures 1.6 and 1.7.
Figure 1.6
Multiple Frequency Absorption

Figure 1.7
Cumulative Absorbed Frequencies
In this way, the ions in the stellar atmosphere may absorb many photons, acquiring their momentum, and drive the atmosphere outward in a wind.

**Overview of past research**

In 1971 Lucy and Solomon revived the theories of Milne (1924,1926) and Johnson (1925) which stated that the force on ions due to absorption could exceed gravity and thus matter could be ejected from a star. Lucy and Solomon (1971) applied this theory to explain the mass loss from O stars.

A few years later Castor, Abbot and Klein (CAK, 1975) produced the seminal work of radiation-driven stellar wind theory, the goal of which was to predict the phenomenon we observe, namely the mass loss rate and terminal velocity of the wind. The main details of this theory, which is the basis of this thesis, will be left for chapter 2. However, a brief discussion of the assumptions and shortfalls of CAK theory is merited.

Assumptions:

1.) Momentum is transferred from the radiation (photons) to the gas.

2.) The wind may be treated as an ideal fluid since it is highly ionized.

3.) The wind is isothermal (does not change in temperature).

4.) The wind is steady-state (does not change in time).

5.) The wind is spherically symmetric.

6.) The star is not rotating.

7.) The star has no magnetic field.

8.) The star is treated as a point source of radiation.
CAK theory predicts mass loss rates that are 100 times greater than those predicted by Lucy and Solomon. These rates compare much better with observation. The CAK terminal velocities are lower than observations by a factor of two or three for hotter stars and compare fairly well for the cooler O and B stars.

The shortfalls of CAK theory are found in the assumptions. Take for example the spherically symmetric assumption. Upon examination of this image of our own sun, it is apparent that the wind is not spherically symmetric.

Figure 1.8

http://umbra.nascom.nasa.gov/images/latest_eit_171.gif
The observations of our own sun indicate not only that the solar wind is not spherically symmetric, but that the temperature does change as a function of the radius, and the wind rotates. We also know that the sun has a magnetic field and is not a point source. Regardless of its shortcomings, CAK theory was an excellent beginning to our understanding of radiation-driven stellar winds.

Several modifications have been made to the CAK theory. These include:

1.) Abbott (1977, 1979) and Castor (1979) incorporated a much larger sample of spectral lines and a more correct treatment of the equations of radiation transfer to improve predictions.

2.) Friend and Abbott (1986) included the "finite disk factor" to compensate for the fact that the star is not a point source, but has some dimension. If the star is considered to be a point source, the line force is overestimated near the star and underestimated far from the star. This correction alone reduces the mass loss rate and increases the terminal velocity, bringing the results in better alignment with observations.

3.) Other additions or modifications to the theory have addressed many of the original assumptions and include rotation, magnetic fields, time-dependence, pulsations, oscillations, photospheric perturbations, asymmetries, and so on.

Overview of my research

In this work, I have used the original CAK model, with the assumptions that the wind is not isothermal and that the star is not a point source. Being that the solar wind is driven by thermal gradients, we may expect that a much hotter star would
drive it's wind in the same fashion. I have chosen an adiabatic temperature
distribution which I derive in Chapter 2:

\[ T = T_o \left( \frac{\frac{r^2 a}{r^2 v}}{r^2 v} \right)^{\gamma - 1} \]

where $T_o$ is the temperature where the velocity of the wind is equal to the sound
speed, $r_{\text{star}}$ is the radius of the star to the photosphere, and $\gamma$ is $c_p/c_v$. In chapter 3, my
results will be compared with those of the Friend and Abbott finite disk model and
with observations of terminal velocity and mass loss rates for a sample of stars. And
finally, Chapter 4 summarizes this work and discusses the effects of the non-
isothermal assumption upon the theory.
Chapter 2

Derivation of Equations and Method of Solution

The basic equation of fluid motion is derived from the Navier-Stokes equation

\[
\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\vec{v} = -\frac{1}{\rho} \nabla P + \nu \nabla^2 \vec{v} + \frac{\nu}{3} \nabla(\nabla \cdot \vec{v}) + \frac{\vec{F}}{m} \quad (2.1)
\]

where \( \vec{v} \) is velocity, \( P \) is pressure, \( \rho \) is density, \( \nu \) is viscosity, \( \vec{F} \) is force and \( m \) is mass.

The terms on the left-hand side of the equation represent the fluid acceleration, the last term on the right-hand side represents body forces on the fluid, such as gravity, and the remaining terms represent surface forces, such as pressure and viscous forces.

The first step in the derivation is to incorporate the assumptions mathematically. Recall the assumptions as introduced in Chapter 1:

1) Steady-state flow: \( \frac{\partial \vec{v}}{\partial t} = \frac{\partial \rho}{\partial t} = 0 \)

2) Inviscid flow: \( \nu = 0 \)

3) Irrotational flow: \( \nabla \times \vec{v} = 0 \)

4) Spherically symmetric flow

5) Isothermal flow

6) No magnetic field

The Navier-Stokes equation then becomes

\[
\frac{v}{r} \frac{dv}{dr} = -\frac{1}{\rho} \frac{dP}{dr} - \frac{GM(1 - \Gamma)}{r^2} + \frac{GM\Gamma}{r^2} M(t) \quad (2.2)
\]

where the following definitions apply:
1) M(t) is the force multiplier involved in the force per unit mass due to the radiation force provided by a number of spectral lines (Friend, 1982). Abbott (1977, 1982) calculated the force multiplier and found that it may be approximated by $M(t) = kt^\alpha$. In this work the values used for $k$ and $\alpha$ are 0.18 and 0.61 respectively, as found by Abbott. The function $t$ is the optical depth in the wind (CAK, 1975)

$$ t = \sigma_e \rho v_{th} \left( \frac{dv}{dr} \right)^{-1} \quad (2.3) $$

where $\sigma_e$ is electron scattering opacity (0.2 cm$^2$/g in this model) and $v_{th}$ is the mean thermal velocity of the protons in the wind with a temperature equal to the effective temperature of the star.

2) $P$ is the pressure.

3) $G$ is the universal gravitational constant.

4) $r$ is the radius.

5) $\Gamma$ is the ratio of the star's luminosity to the Eddington luminosity at which the radiation pressure force equals the force of gravity.

$$ \Gamma = \frac{\sigma_e L}{4\pi G Mc} \quad (2.4) $$

Substituting for $M(t)$ and $t$ in equation 2.2 we obtain

$$ \frac{dV}{dr} = -\frac{1}{\rho} \frac{dP}{dr} - \frac{GM(1-\Gamma)}{r^2} + \frac{GM\Gamma}{r^2} \left( \frac{dv}{dr} \right)^\alpha \left( \frac{1}{\sigma_e \rho v_{th}} \right) $$
We may combine this momentum equation with the mass continuity equation to eliminate \( \rho \), the density, from the last term. The continuity equation states that no material is created or destroyed in the static wind, so the same amount of mass flows per second through a sphere at any distance \( r \) from the center of the star.

\[
\frac{dM}{dt} = \dot{M} = 4\pi r^2 \rho v
\]

\[
\rho = \frac{M}{4\pi r^2 v}
\]

Substituting this in for \( \rho \) in the last term of the momentum equation, we obtain

\[
v \frac{dv}{dr} = -\frac{1}{\rho} \frac{dP}{dr} \frac{GM(1-\Gamma)}{r^2} + \frac{GM}{r^2} k \left( \frac{dv}{dr} \frac{4\pi r^2 v}{M \sigma_e v_{th}} \right)^\alpha
\]

Combining terms and allowing

\[
C = GM\Gamma k \left( \frac{4\pi}{\sigma_e v_{th} M} \right)^\alpha \quad (2.5)
\]

the general form of the CAK equation becomes

\[
v \frac{dv}{dr} = -\frac{1}{\rho} \frac{dP}{dr} \frac{GM(1-\Gamma)}{r^2} + C \left( r^2 \frac{dV}{dr} \right)^\alpha \quad (2.6)
\]
To progress further towards solving equation 2.6, we must write the pressure $P$ using an equation of state. CAK uses $P = p\alpha^2$ where $\alpha$ is the isothermal sound speed. I choose $P$ to have a temperature dependence by using the ideal gas equation of state:

$$P = \frac{k\rho T}{\mu m_H} \quad (2.7)$$

where $k$ is Boltzmann's constant, $T$ is temperature of the wind, $m_H$ is the mass of hydrogen, and $\mu$ is the mean molecular weight of gas particles in the wind which can be approximated as the following:

$$\frac{1}{\mu} = 2X + \frac{3}{4}Y + \frac{1}{2}Z \quad (2.8)$$

where $X$, $Y$, and $Z$ are the mass fractions of hydrogen, helium and heavy elements in the wind. For O and B star models, these fractions are assumed to be $X=0.73$, $Y=0.24$, $Z=0.03$ (Cranmer, 1996)

The temperature distribution for equation 2.7 is derived from the adiabatic relation as follows:

$$P = k_1 \rho^\gamma = k_2 \rho T \quad \text{where } k_1 \text{ and } k_2 \text{ are constants}$$

$$\rho^\gamma = k\rho T$$

$$T = k\rho^{\gamma-1} \quad (a)$$

Using the continuity equation

$$4\pi r_1^2 \rho_1 v_1 = 4\pi r_2^2 \rho_2 v_2$$
We get

\[ \frac{\rho_1}{\rho_2} = \frac{r_2^2 v_2}{r_1^2 v_1} \]  

(b)

If we write (a) as

\[ \frac{T_2}{T_1} = \frac{k \rho_2^{\gamma-1}}{k \rho_1^{\gamma-1}}. \]

substitute equation (b) into it, we get

\[ \frac{T_2}{T_1} = \left( \frac{r_1^2 v_1}{r_2^2 v_2} \right)^{\gamma-1} \]

Which can be rewritten as

\[ T = T_0 \left( \frac{r_*^2 a}{r^2 v} \right)^{\gamma-1} \]  

(2.9)

in which \( r_* \) is the radius to the star's photosphere, and \( T_0 \) is the temperature at the star's photosphere. Hence, I substitute

\[ P = k \rho T_0 \left( \frac{r_*^2 a}{r^2 v} \right)^{\gamma-1} (\mu m_H)^{-1} \]  

(2.10)

into equation 2.6 with the new variable \( A = k/\mu m_H \) and obtain

\[ -2AT_0 \gamma \left( \frac{r_*^2 a}{r^2 v} \right)^{\gamma-1} - AT_0 \gamma \left( \frac{r_*^2 a}{r^2 v} \right)^{\gamma-1} \frac{v'}{v} r^2 + r^2 v v' + GM(1 - \Gamma) - C(r^2 v v')^a = 0 \]  

(2.11)
where \( \frac{dv}{dr} = v' \). Combining the \( r \) terms and defining \( W = AT_0 \gamma (r^2 \alpha)^{-1} \), I obtain

\[
F = \frac{-2W}{r^{2\gamma-3}v^{\gamma-1}} - \frac{Wv'}{r^{2\gamma-4}v^{\gamma}} + r^2vv' + GM(1 - \Gamma) - Cf(r, v, v')(r^2vv')^\alpha = 0 \quad (2.12)
\]

where I have included the finite disk factor of Friend and Abbott (1986).

\[
f = \frac{r^n - \left( \frac{v}{rv'} \right)^\alpha (1 + \sigma \mu_c^2)^{1+\alpha}}{(1 + \alpha) \gamma \frac{r^2}{r^2}} \quad (2.13)
\]

where

\[
\sigma = \frac{rdv}{vdr} - 1 \quad \text{and} \quad \mu_c = \sqrt{1 - \frac{r^2}{r^2}}
\]

(all the required derivatives of the functions \( f \) and \( F \) may be found in appendix A).

Equation 2.12 may be written in a simplified form to assist in the graphical analysis to follow:

\[
N(r, v)v' + H(r, v) = Q(r, v)v'^\alpha \quad (2.12a)
\]

Equation 2.12 is nonlinear, so for a given \( r \) and \( v \) there is not a unique solution for \( v' \) and there is no analytic solution. In order to determine the solution numerically, I follow the graphical solution technique of CAK (see figure 2.1). For equation 2.12, we find that there can be zero, one or two solutions depending on the values (positive or negative) of \( N, H \) and \( Q \) in equation 2.12a (Friend, 1982).
Figure 2.1 Graphical Solution of the Equation of Motion
Region 1 in figure 2.1 represents the area where \( r < r_0 \) and \( v > a \). This region is the only area in which two solutions are possible—\( N \) and \( H \) are both positive. In region 1 is also the singular locus, defined by equation 2.15 below, which separates the area of no solutions from this area of two solutions.

Region 2 represents the area where \( r > r_0 \) and \( v > a \), or where \( H < 0 \) and \( N > 0 \), and there is only one solution possible. Region 3 is the area where \( r < r_0 \) and \( v < a \), \( H > 0 \) and \( N < 0 \), and once again there is only one solution possible. Region 4 has no solutions for subsonic velocities at radii far from the star (\( r > r_0 \) and \( v < a \)).

Examining figure 2.1 leads us to the conclusion that our unique wind solution must progress smoothly from subsonic to supersonic velocities and must graze the singular locus. How do we determine this solution analytically? The condition that the solution touch the singular locus in region 1 tells us that equation 2.12 has only one solution at this singular or critical point given by the singularity condition—equation 2.15. The condition that the velocity gradient be continuous at the critical point gives us the regularity condition—equation 2.16 (derived from \( dF/dr = 0 \)). Combining the momentum equation (2.12) with the singularity and regularity conditions we have

\[
F = 0 \quad \text{(2.14)}
\]

\[
\frac{\partial F}{\partial v'} = 0 \quad \text{(2.15)}
\]

\[
\frac{\partial F}{\partial r} + \frac{\partial F}{\partial v'} v' = 0 \quad \text{(2.16)}
\]

These equations enable us to solve for \( r \), \( v \), and \( v' \) at the critical point. This is the only point through which the correct solution for the wind will pass (figure 2.1).
FINDING THE CRITICAL POINT

There are several steps necessary to solve for \( r, v, \) and \( v' \) at the critical point:

(The Fortran code used to solve for the critical point and find the entire wind solution may be found in Appendix B)

1) Solve equation 2.12 for \( C \).

2) Calculate the derivative of equation 2.15 and substitute in for \( C \) to get

\[
N = 0 \quad (2.17)
\]

where

\[
N = \frac{-2W}{r^{2\gamma-3}v^{\gamma-1}} - \frac{Wv'}{r^{2\gamma-4}v^{\gamma}} + r^2vv' + GM(1 - \Gamma)
\]

3) Calculate the derivatives of equation 2.16 and substitute in for \( C \) to get

\[
f_2 = \frac{-2W(-2\gamma + 3)}{r^{2\gamma-2}v^{\gamma-1}} - \frac{W(-2\gamma + 4)v'}{r^{2\gamma-3}v^{\gamma}} + 2rvv' + v' \left\{ \frac{-2W(-\gamma + 1)}{r^{2\gamma-3}v^{\gamma}} + \frac{\gamma Wv'}{r^{2\gamma-4}v^{\gamma+1}} + r^2v' \right\}
\]

\[
- Nf^{-1} \left( v'f_v + \frac{\alpha f_v'}{v} + f_r + \frac{2\alpha f}{r} \right) = 0 \quad (2.18)
\]

4) Solve the two equations, \( f_1 \) and \( f_2 \), using coupled iterations as follows.

The total derivatives of \( f_1 \) and \( f_2 \) will be equal to zero. With \( x=v \) and \( y=v' \), we obtain

\[
0 = f_1 + \frac{\partial f_1}{\partial x} \, dx + \frac{\partial f_1}{\partial y} \, dy
\]

\[
0 = f_2 + \frac{\partial f_2}{\partial x} \, dx + \frac{\partial f_2}{\partial y} \, dy
\]
Let us rewrite these equations in matrix form with the definitions

We know $f_1$ and $f_2$, and we can calculate $a$, $b$, $c$, and $d$ (partial derivatives of the finite disk factor). Consequently we can solve for $dx$ and $dy$:

$$
\begin{align*}
    a &= \frac{\partial f_1}{\partial x}, \\
    b &= \frac{\partial f_1}{\partial y}, \\
    c &= \frac{\partial f_2}{\partial x}, \\
    d &= \frac{\partial f_2}{\partial y}
\end{align*}
$$

$$
0 = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} + \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix}
$$

$$
\begin{align*}
    dx &= \frac{1}{ad - bc} (bf_2 - df_1) \\
    dy &= \frac{1}{ad - bc} (cf_1 - af_2)
\end{align*}
$$

5) To find $x$ and $y$ (or $v$ and $v'$) at the critical point, we do the following:

a) Select a critical radius $r_{\text{crit}}$.

b) Make initial guesses for $x$ and $y$ ($v_{\text{crit}}$ and $v'_{\text{crit}}$)

c) Solve for $dx$ and $dy$.

d) Calculate $dx/x$ and $dy/y$. If one or both of these is greater than $1 \times 10^{-5}$, then increment $x$ by $dx$ and $y$ by $dy$ and solve for $dx$ and $dy$ again.

e) When both $dx/x$ and $dy/y$ are less than $1 \times 10^{-5}$, $v_{\text{crit}}$ and $v'_{\text{crit}}$ have been found for the given critical radius.

The critical point analysis also allows us to determine the mass loss rate by solving equation 2.5 (as long as the correct value for the critical radius was chosen. How we know this is discussed in the next section).

$$
\dot{M} = \frac{4\pi}{\sigma_e v_{\text{th}}} \left( \frac{GM\Gamma k}{C} \right)^{\gamma/\alpha} 
$$

(2.19)
in which C is calculated from equation 2.12.

**FINDING THE COMPLETE WIND SOLUTION**

Upon examination of figure 2.1 we see that the correct wind solution passes through the critical radius. Our equation of motion (2.12) may now be numerically integrated outward and inward from this critical radius. In order to be certain we are on the correct solution branch, we begin with the velocity gradient calculated in the critical point subroutine and step outwards along a solution branch. If we do not step "steeply" enough, the integration will "crash." Initial step size adjustments must then be made in the computer code to ensure progression along the correct solution branch. The integration continues outwards to a specified radius (~40 stellar radii). The integration then returns to the critical point and continues inwards to the photosphere, or "surface" of the star. We define the photosphere (r=1 stellar radius) to be at the point where the optical depth is 2/3 (this is in fact how the photosphere is defined). If the calculated value for the radius does not equal unity when the optical depth is 2/3, we know that our critical radius was either too far from or too near the star. If this is the case, we simply adjust \( r_{\text{crit}} \) and run the integration again until \( r=1 \) when optical depth=2/3. We have then found a complete stellar wind solution.
Chapter 3

Results

The results of the numerical solution of the wind equation of motion will now be discussed for a variety of stellar parameters. Of primary interest is the comparison between this non-isothermal, finite-disk model and the finite-disk model of Friend and Abbott (1986). I am particularly curious whether or not the isothermal assumption is valid for these very hot stars.

What does a stellar wind solution look like? Figure 3.1 shows how the velocity changes with radius for an O9.5III star (the general shape of the curve remained the same for all stars I examined). Note that the velocity asymptotically approaches a value known as the terminal velocity.

Figure 3.1a displays a typical temperature vs. radius curve for the same star. Note that the temperature increases rapidly as we approach the star’s surface, but that we find the effective temperature to be very near the photosphere.
Wind Solution for O9.5III

Figure 3.1
Temperature vs Radius
09.5III ($T_o = 12000$ K)

Figure 3.1a
TEMPERATURE VARIATION FOR FIXED STELLAR PARAMETERS

As a first test of the non-isothermal code, I have looked at several different stellar types holding their parameters constant while varying the temperature for each case. The parameters used are given in table 3.1 (Drew, 1989). The wind solutions for two of these stars are illustrated in Figures 3.2 and 3.3. The temperature variation makes a difference in the terminal velocity of the star's wind on the order of 10%.
Table 3.1
Basic Stellar Parameters

<table>
<thead>
<tr>
<th>Spectral Type</th>
<th>Temp (K)</th>
<th>L/Lsun</th>
<th>R/Rsun</th>
<th>log g</th>
<th>dM/dt (M$_{\odot}$/yr)</th>
<th>$v_{\text{term}}$ (km/s)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Supergiants</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>O9.5 I</td>
<td>30000</td>
<td>3.0E+05</td>
<td>20</td>
<td>3.0</td>
<td>4.8E-07</td>
<td>1800</td>
</tr>
<tr>
<td>O9 I</td>
<td>32500</td>
<td>4.1E+05</td>
<td>20</td>
<td>3.3</td>
<td>8.3E-07</td>
<td>1800</td>
</tr>
<tr>
<td>O7.5 I</td>
<td>35000</td>
<td>6.3E+05</td>
<td>22</td>
<td>3.3</td>
<td>1.8E-06</td>
<td>1800</td>
</tr>
<tr>
<td>O6.5 I</td>
<td>37500</td>
<td>9.4E+05</td>
<td>23</td>
<td>3.5</td>
<td>3.5E-06</td>
<td>2300</td>
</tr>
<tr>
<td>O5 I</td>
<td>40000</td>
<td>1.2E+06</td>
<td>23</td>
<td>3.5</td>
<td>5.5E-06</td>
<td>2300</td>
</tr>
<tr>
<td>O4 I</td>
<td>45000</td>
<td>1.3E+06</td>
<td>19</td>
<td>4.0</td>
<td>6.0E-06</td>
<td>2900</td>
</tr>
<tr>
<td>O3 I</td>
<td>50000</td>
<td>2.0E+06</td>
<td>19</td>
<td>4.0</td>
<td>1.2E-06</td>
<td>2400</td>
</tr>
<tr>
<td><strong>Giants</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>O9.5 III</td>
<td>30000</td>
<td>1.2E+05</td>
<td>13</td>
<td>3.5</td>
<td>1.0E-07</td>
<td>2900</td>
</tr>
<tr>
<td>O9 III</td>
<td>32500</td>
<td>1.7E+05</td>
<td>13</td>
<td>3.8</td>
<td>1.8E-07</td>
<td>2900</td>
</tr>
<tr>
<td>O7.5 III</td>
<td>35000</td>
<td>2.3E+05</td>
<td>13</td>
<td>4.0</td>
<td>3.0E-07</td>
<td>2900</td>
</tr>
<tr>
<td>O6.5 III</td>
<td>37500</td>
<td>3.7E+05</td>
<td>14</td>
<td>4.0</td>
<td>7.0E-07</td>
<td>2900</td>
</tr>
<tr>
<td>O6 III</td>
<td>40000</td>
<td>4.8E+05</td>
<td>14</td>
<td>4.0</td>
<td>1.1E-06</td>
<td>2900</td>
</tr>
<tr>
<td><strong>Main Sequence Dwarfs</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>B0 V</td>
<td>30000</td>
<td>5.4E+04</td>
<td>8.5</td>
<td>3.5</td>
<td>2.6E-08</td>
<td>3200</td>
</tr>
<tr>
<td>O9 V</td>
<td>32500</td>
<td>7.5E+04</td>
<td>8.5</td>
<td>3.8</td>
<td>4.5E-08</td>
<td>3200</td>
</tr>
<tr>
<td>O7 V</td>
<td>35000</td>
<td>1.0E+05</td>
<td>8.5</td>
<td>4.0</td>
<td>7.5E-08</td>
<td>3200</td>
</tr>
<tr>
<td>O8 V</td>
<td>37500</td>
<td>1.8E+05</td>
<td>10</td>
<td>4.0</td>
<td>2.0E-07</td>
<td>3200</td>
</tr>
<tr>
<td>O6.5 V</td>
<td>40000</td>
<td>2.3E+05</td>
<td>10</td>
<td>4.0</td>
<td>3.2E-07</td>
<td>3200</td>
</tr>
<tr>
<td>O5 V</td>
<td>45000</td>
<td>4.9E+05</td>
<td>11</td>
<td>4.0</td>
<td>1.1E-06</td>
<td>3200</td>
</tr>
<tr>
<td>O3 V</td>
<td>50000</td>
<td>7.5E+05</td>
<td>11</td>
<td>4.0</td>
<td>2.3E-06</td>
<td>3000</td>
</tr>
</tbody>
</table>
Temperature Variation for Fixed Stellar Parameters (O9III)

Figure 3.2
Temperature Variation for Fixed Stellar Parameters (O3I)

Figure 3.3
TERMINAL VELOCITY AND MASS LOSS VS EFFECTIVE TEMPERATURE

Figures 3.4 and 3.5 summarize the effects of temperature variation for fixed parameters on the terminal velocity and mass loss rate of the star. Mass loss was found to increase as the temperature increased, while the terminal velocity of the wind decreases as the temperature increases (the more mass the star is trying to expel, the slower the wind may go). The terminal velocity decreases on the order of 7% and the mass loss increases on the order of 5% over the range of temperatures in these samples.
Terminal Velocity vs Effective Temperature

$V_{\text{term}} = -4.26T_{\text{eff}} + 5231$ \textbf{O3I} \hspace{1cm} $V_{\text{term}} = -7.80T_{\text{eff}} + 3565$ \textbf{HD 149757}

$V_{\text{term}} = -6.6714x + 3965.2$ \textbf{O8V} \hspace{1cm} $V_{\text{term}} = -5.62T_{\text{eff}} + 2248$ \textbf{O9I}

$V_{\text{term}} = -5.85T_{\text{eff}} + 4755$ \textbf{O6III} \hspace{1cm} $V_{\text{term}} = -5.58T_{\text{eff}} + 2313$ \textbf{O9III}

Figure 3.4
MASS LOSS VS. LUMINOSITY

For the second exploration, I have calculated the mass loss rate versus luminosity for each spectral type listed in table 3.1. The calculations are made for both my model and the finite disk model and are presented graphically in figure 3.6. Friend and Abbott (1986) sought to show this important correlation between mass loss and luminosity. Observationally, mass loss rate increases with the luminosity, roughly as a power law. Both the Friend and Abbott model and my model show such a correlation. There are few discrepancies between the results of the two models, and the slopes of the linear fits are within 4%. 
Comparison with Finite Disk Model
Mass Loss vs Luminosity

Figure 3.6
Chapter 4

Conclusions

The field of radiatively-driven wind studies has grown immensely in its complexity since the days of Castor, Abbott and Klein. Most of today's current theories are based on the original CAK theory in which the researchers whittle away at its simplifying assumptions.

My graphical results are summarized as follows:

1) Increasing the effective temperature for a particular star decreases the wind's terminal velocity and increases the mass loss rate of the wind. Certainly increasing the temperature of the star's photosphere results in more mass being expelled. This increase in mass loss has the net effect of slowing down the wind, thus reducing the terminal velocity.

2) The mass loss rate is found to grow, nearly, as a power law with the luminosity of the star. My mass loss-luminosity results compare quite well with those of the Friend and Abbott model.

Comparisons of the non-isothermal, finite disk model and the finite disk model of Friend and Abbott clearly demonstrate, for the data analyzed, that the isothermal assumption, which many theories use, is a reasonable approximation. Initially, it seemed that the winds of these very hot stars should be affected by such extreme temperatures. I expected that the mass loss rates would rise and the terminal velocities drop in comparison with the isothermal, finite disk results. It turned out that the terms in the equation of motion (2.12) which are affected by the
temperature—the first two terms—were consistently several orders of magnitude smaller than the last term in equation 2.12, the radiation force term. After examining these terms, I was certainly not surprised that the temperature dependence had a negligible effect.

My results match some observations more closely than the Friend and Abbott model (see below), but our results rarely differ by more than a few percent.

The observational data in the chart below has been taken from Drew (1989).

<table>
<thead>
<tr>
<th>Star</th>
<th>$V_{\text{term}}$ (km/s) Non-Isothermal Model</th>
<th>$V_{\text{term}}$ (km/s) Finite Disk Model</th>
<th>$V_{\text{term}}$ (km/s) Observation</th>
</tr>
</thead>
<tbody>
<tr>
<td>O7.5 I</td>
<td>2120</td>
<td>2277</td>
<td>1800</td>
</tr>
<tr>
<td>O7.5 III</td>
<td>4334</td>
<td>4378</td>
<td>2900</td>
</tr>
<tr>
<td>O7 V</td>
<td>3705</td>
<td>3719</td>
<td>3200</td>
</tr>
<tr>
<td>HD 37043</td>
<td>2545</td>
<td>2440</td>
<td>2450</td>
</tr>
<tr>
<td>HD 37128</td>
<td>1867</td>
<td>1692</td>
<td>1500</td>
</tr>
</tbody>
</table>

My results indicate that the inclusion of a temperature dependence is not a necessary complication in order to improve the accuracy of results. Allowing the wind to be isothermal reduces complexity in the equation of momentum, thus easing (a bit) the computations of additional effects such as magnetic field, rotating winds, pulsations and a variety of other phenomena.
Appendix A
Derivatives of
F and f
\[ f_{vv} = \frac{1}{v} \left\{ 2 \left( \frac{rv'}{\sigma v} - 1 \right) + \frac{\alpha}{(1 + \alpha) \sigma v} \left( \frac{v}{rv'} \right)^{1 + \sigma \mu_c^2} - 1 \right\} \]

\[ f_{vv'} = \frac{r}{\sigma v^2} \left\{ \frac{\alpha \left( \frac{v}{rv'} \right)^{1 + \sigma \mu_c^2}}{(1 + \alpha)} \left( \frac{v}{rv'} - \frac{\mu_c^2 - \frac{\alpha v r_s^2}{rv'} r^2}{1 + \sigma \mu_c^2} \right) + v' f_{v'} - \frac{v^2 f_v}{rv'} \right\} \]

\[ f_r = \frac{v'}{\sigma v} \left\{ 3 - \left( \frac{v}{rv'} \right)^{1 + \sigma \mu_c^2} \left[ (1 + \alpha) \left( \frac{\mu_c^2 + 2 \sigma r_s^2 v}{rv'} \right) - \frac{(2 - \alpha) v}{rv'} (1 + \sigma \mu_c^2)^{1 + \sigma \mu_c^2} \right] \right\} \]
\[
\begin{align*}
\left\{ JZ + \left( \frac{\Lambda}{\Delta} + 1 \right)^2 JZ + \left( \frac{2\pi\Omega + 1}{\frac{\mu}{\pi} + 1} \right) \left( 1 - \alpha \right) + \left( \frac{\Lambda}{\Delta} \right) \right\} &= 0 \\
\left\{ \left( \frac{\Lambda}{\Delta} \left( \frac{2\pi\Omega + 1}{\frac{\mu}{\pi} + 1} \right) \right) \left( \frac{\Lambda}{\Delta} \right) \right\} &= \frac{\Lambda}{\Delta}
\end{align*}
\]
\[
\left( \frac{z}{(1 - n)\theta z} + \frac{\lambda}{\eta_{\bar{w}_{\bar{z}}}z} + \frac{\lambda}{\eta_{\bar{w}_{\bar{z}}}z} \right) n\left( \Lambda_{\bar{z}} \right) c - \frac{1 + i \Lambda_{\bar{z}}}{(1 + \lambda)\Lambda_{\bar{w}} - \frac{1 + i \Lambda_{\bar{z}}}{\Lambda_{\bar{w}}}} = \Lambda_{\bar{w}}
\]

\[
\left( \frac{\Lambda}{(\Lambda_{\bar{z}}^2)_{\eta}} - n(\Lambda_{\bar{z}}^2)^{\frac{1}{2}} - \frac{\lambda}{(\Lambda_{\bar{z}}^2)^{\frac{1}{2}}} + \frac{\eta_{\bar{w}_{\bar{z}}}^2}{\eta_{\bar{w}_{\bar{z}}}^2} \right) n(\Lambda_{\bar{z}}) c - \frac{1 + i \Lambda_{\bar{z}}}{(1 + \lambda)\Lambda_{\bar{w}} - \frac{1 + i \Lambda_{\bar{z}}}{\Lambda_{\bar{w}}}} = \Lambda_{\bar{w}}
\]

\[
\left( \frac{\Lambda}{(\Lambda_{\bar{z}}^2)_{\eta}} - n(\Lambda_{\bar{z}}^2)^{\frac{1}{2}} - \frac{\lambda}{(\Lambda_{\bar{z}}^2)^{\frac{1}{2}}} + \frac{\eta_{\bar{w}_{\bar{z}}}^2}{\eta_{\bar{w}_{\bar{z}}}^2} \right) n(\Lambda_{\bar{z}}) c - \frac{1 + i \Lambda_{\bar{z}}}{(1 + \lambda)\Lambda_{\bar{w}} - \frac{1 + i \Lambda_{\bar{z}}}{\Lambda_{\bar{w}}}} = \Lambda_{\bar{w}}
\]

\[
\left( \frac{1 - \Lambda}{(1 - \eta)\Lambda z} + \frac{\eta_{\bar{w}_{\bar{z}}}^2}{\eta_{\bar{w}_{\bar{z}}}^2} + \frac{\lambda}{(\Lambda_{\bar{z}}^2)^{\frac{1}{2}}} + \frac{\eta_{\bar{w}_{\bar{z}}}^2}{\eta_{\bar{w}_{\bar{z}}}^2} \right) n(\Lambda_{\bar{z}}) c - \frac{1 + i \Lambda_{\bar{z}}}{(1 + \lambda)\Lambda_{\bar{w}} - \frac{1 + i \Lambda_{\bar{z}}}{\Lambda_{\bar{w}}}} = \Lambda_{\bar{w}}
\]

\[
\left( \frac{\Lambda}{(\Lambda_{\bar{z}}^2)_{\eta}} - n(\Lambda_{\bar{z}}^2)^{\frac{1}{2}} - \frac{\lambda}{(\Lambda_{\bar{z}}^2)^{\frac{1}{2}}} + \frac{\eta_{\bar{w}_{\bar{z}}}^2}{\eta_{\bar{w}_{\bar{z}}}^2} \right) n(\Lambda_{\bar{z}}) c - \frac{1 + i \Lambda_{\bar{z}}}{(1 + \lambda)\Lambda_{\bar{w}} - \frac{1 + i \Lambda_{\bar{z}}}{\Lambda_{\bar{w}}}} = \Lambda_{\bar{w}}
\]

\[
\left( \frac{\Lambda}{(\Lambda_{\bar{z}}^2)_{\eta}} - n(\Lambda_{\bar{z}}^2)^{\frac{1}{2}} - \frac{\lambda}{(\Lambda_{\bar{z}}^2)^{\frac{1}{2}}} + \frac{\eta_{\bar{w}_{\bar{z}}}^2}{\eta_{\bar{w}_{\bar{z}}}^2} \right) n(\Lambda_{\bar{z}}) c - \frac{1 + i \Lambda_{\bar{z}}}{(1 + \lambda)\Lambda_{\bar{w}} - \frac{1 + i \Lambda_{\bar{z}}}{\Lambda_{\bar{w}}}} = \Lambda_{\bar{w}}
\]

\[
\left( \frac{\Lambda}{(\Lambda_{\bar{z}}^2)_{\eta}} - n(\Lambda_{\bar{z}}^2)^{\frac{1}{2}} - \frac{\lambda}{(\Lambda_{\bar{z}}^2)^{\frac{1}{2}}} + \frac{\eta_{\bar{w}_{\bar{z}}}^2}{\eta_{\bar{w}_{\bar{z}}}^2} \right) n(\Lambda_{\bar{z}}) c - \frac{1 + i \Lambda_{\bar{z}}}{(1 + \lambda)\Lambda_{\bar{w}} - \frac{1 + i \Lambda_{\bar{z}}}{\Lambda_{\bar{w}}}} = \Lambda_{\bar{w}}
\]
\[
\left( \frac{z^\Lambda}{(1 - \varrho) \varphi} + \frac{z^\Lambda}{\lambda J z} + \frac{z^\Lambda}{\lambda J J} \right) \nu (^{\Lambda \Lambda} z \nu J) c = ^{\Lambda \Lambda} P
\]

\[
\left( \frac{z^\Lambda}{J z^\nu} + \frac{z^\Lambda}{J J} + \frac{z^\Lambda}{J J} + \frac{z^\Lambda}{\lambda J} \right) \nu (^{\Lambda \Lambda} z \nu J) c - z J + \frac{I + I - I - I}{\lambda M} = ^{\Lambda \Lambda} P
\]
Appendix B
The Model's
Fortran Code
C CAK MODEL WITH FINITE DISK FOR STAR
USE MSFLIB

COMMON/IDAT/DMDTD0, SIGE, ONEMAL, ALO1MA, TWOALP, VTH, GAMGMK, TWOMAL, GAM
2, GM, F5, GAMC
COMMON/DERM/FLAG, ALPHA, RALP, C, F7, F8, R2, ARP, ONEPAL, VPLASL, FC,
2RPORS, TNOT, WN, rccrit, vpcguess

DIMENSION V(2), VP(2), WORK(20), IWORK(5), RORP(100), VKMS(100),
2RDUM(50), VDUM(50)
EXTERNAL DER
LOGICAL FLAG
Real L

INTEGER(4) frequency, duration
frequency = 1000
duration = 333
A=3.6E6
A2=A*A
AKMS=A/1.65
VTH=A
ALPHA=.61
RALP=1./ALPHA
ONEPAL=1.+ALPHA
ONEMAL=1.-ALPHA
TWOMAL=2.-ALPHA
ALO1MA=ALPHA/ONEMAL
F7=-1./ONEMAL
F8=ALPHA*ONEMAL**(1./ALO1MA)
TWCALP=2.*ALPHA
XK=.18
OPEN(UNIT=7, FILE='finite2.dat', STATUS='REPLACE', ERR=110)
PRINT 9000
9000 FORMAT('INPUT DATA: rfac, rpors, mass')
READ(*,*) rfac

5200 vc=1.3
vpc=100.
L=1.*10**5.70
RPORS=11.
c GRAV=10**(4.0)
c XM=(GRAV/1.E2*(RPORS*6.95998E8)**2/6.67E-11)*1.6E3/1.986E33
xm=52.
c tnot=(L*3.9e33)/(4*3.14159*(rpors*6.96e10)**2*5.67e-5))**.25
rporsun=rpors
PFAC=.2
GAMC=1.6666667
TNOT=46400.
SIGE=.2
GM=XM*1.3273E26
GAM=SIGE*L*1.035E22/GM
GAMGMK=GAM*GM*XK
F5=GM*(gam-1.)
RP=RPORS*6.96E10
RP2=RP*RP
\[ A\text{ORP} = \frac{A}{RP} \]
\[ ARP = A \times RP \]
\[ ARP^2 = A \times RP^2 \]
\[ A^2RP = A^2 \times RP \]
\[ B0LT = 1.38 \times 10^{-16} \]
\[ HYDROGEN = 1.67 \times 10^{-24} \]
\[ WN = \frac{(B0LT/(16.65e-1 \times HYDROGEN)) \times TNOT \times GAMC \times (RPORS \times 6.96E10) / a2rp}{1} \]

\[ \text{ITER} = 1 \]

\[ DR = 1.1 \times 10^{-2} \times RCFAC \]
\[ HDR2 = 0.5 \times DR \times DR \]
\[ \text{CALL CRITPT(RCFAC, VC, VPC, V2)} \]
\[ \text{open(4, FILE='FIN33.DAT', status='replace', err=110)} \]
\[ \text{Write(4,*)'rc, vc, vpc', rcfac, vc, vpc} \]
\[ \text{DMDT = DMDTDO \times 1.9928E-25} \]
\[ \text{VCRC2 = VC \times R2} \]
\[ \text{IT = 1} \]
\[ \text{WRITE(7,2300)} \]

\[ \text{2300 FC, MAT(5X,'L/Lsun', 8X,'R/Rsun', 7X,'M/Msun', 7X,'PFAC', 9X,'DMDT')} \]

\[ \text{write(4,*)'This is CAK with finite disk for the following values:'} \]
\[ \text{write(4,*)1,rpors,xm,pfac,dmdt,tnot} \]
\[ \text{write(4,2301)} \]

\[ \text{2301 Format(6X,'r/R*', 8X,'V(KM/S)',' 6X,'VP(\auer/\$)', 6X,'3'RHO(G/CM3)', 6X,'TAU', 11X,'F')} \]

\[ \text{WRITE(7,2000) L, RPORS, XM, PFAC, DMDT} \]
\[ \text{RHO = DMDTDO/VCRC2/ARP2} \]
\[ \text{VKMS(1) = VC \times AKMS} \]
\[ \text{VPPS = VPC \times AORP} \]
\[ \text{V(2) = 0.} \]
\[ \text{RORP(1) = RCFAC} \]
\[ \text{rov = sqrt(r2)/vc} \]
\[ \text{RVPOV = ROV \times VPC} \]
\[ \text{SIGMA = RVPOV - 1.} \]
\[ \text{onepal = 1 + \alpha} \]
\[ \text{U2 = 1 - 1/R2} \]
\[ \text{Fc = r2 \times (RVPOV - (1 + SIGMA \times U2) \times ONEPAL / RVPOV \times \alpha) / ONEPAL / SIGMA} \]
\[ \text{TEMP = TNOT \times \left(1 / (RCFAC + 2 + VC)\right)^{(GAMC - 1.)}} \]
\[ \text{WRITE(4,2000) RCFAC, VKMS(1), VPPS, RHO, V(2), FC} \]

\[ \text{open(22, file='rvoutput.dat', status='replace', err=110)} \]
\[ \text{write(22,*)rcfac,vkms(1),TEMP} \]
\[ \text{2000 FORMAT(1X,6E13.5)} \]
\[ \text{FLAG = .TRUE.} \]
\[ \text{F1 = VC + HDR2 \times V2} \]
\[ \text{DRVPC = DR \times VPC} \]
\[ \text{F2 = -DMDTDO \times SIGE \times dr / (VCRC2) \times ARP} \]
\[ \text{R = RCFAC + DR} \]
\[ \text{V(1) = .995 \times (F1 + DRVPC)} \]
V(2) = F2
RND = R + PFAC/(VPC/VC + 2./R)
IFLAG = 1
VPLAST = VPC
eps = 1.0e-3
epsa = 1.0e-6
22 CALL RKFN(DER, 2, V, R, RND, eps, epsa, IFLAG, WORK, IWORK)
IF (IFLAG .GE. 3) WRITE(7, *) 'IFLAG=', IFLAG
R = RND
23 CALL DER(R, V, VP)
RHO = DMDTDO/V(1)/R**2/ARP2
IT = IT + 1
RORP(IT) = R
VKMS(IT) = V(1) * AKMS
VPPS = VP(1) * AORP
rov = r/v(1)
RVPOV = ROV*VP(1)
SIGMA = RVPOV - 1.
onepal = 1. + alpha
U2 = 1. - 1./R**2
Fc = R**2*(RVPOV-(1.+SIGMA)*U2)**ONEPAL/RVPOV**ALPHA)/ONEPAL/SIGMA
TEMP = TNOT*(1./((R**2*V(1))**2/(GAMC-1.)
WRITE(4, 2000) R, VMKS(IT), VPPS, RHO, V(2), FC
open(22, file='rvoutput.dat', status='replace', err=110)
write(22, *) r, vkms(it), TEMP
RND = R + PFAC/(VP(1)/V(1) + 2./R)
IF (R .LT. 40. AND. IT .LT. 75) GO TO 22
IMAX = IT
VINF = VMKS(IT)/SQRT(1. - 1./R)
WRITE(4, 4000) VINF
4000 FORMAT('/ TERMINAL VELOCITY = ', 6E13.5)
FLAG = .FALSE.
dr = 1.e-4*rcfac
drvpc = dr*vpc
hdr2 = .5*dr**2
fl = vc + hdr2*v2
R = RCFAC - DR
V(1) = .995*(F1-DRVPC)
V(2) = F2-V(2)
RND = R - PFAC/(VPC/VC + 2./R)
IFLAG = 1
IT = 0
12 CALL RKFN(DER, 2, V, R, RND, eps, epsa, IFLAG, WORK, IWORK)
IF (IFLAG .GE. 3) WRITE(7, *) 'IFLAG=', IFLAG
3000 FORMAT(X, 'IFLAG=', ', II)
R = RND
CALL DER(R, V, VP)
RHO = DMDTDO/V(1)/R**2/ARP2
IT = IT + 1
RORP(IT) = R
VDUM(IT) = V(1) * AKMS
VPPS = VP(1) * AORP
rov = r/v(1)
RVPOV = ROV*VP(1)
SIGMA = RVPOV - 1.
onepal = 1. + alpha
U2 = 1. - 1. / R2
Fc = R2*(RVPOV-(1.+SIGMA*U2)**ONEPAL/RVPOV**ALPHA)/ONEPAL/SIGMA
WRITE(4,2000) R, VDUM(IT), VPPS, RHO, V(2), FC
TEMP = TNOT*(1/(R**2*V(1)))**(GAMC-1.)
open(22, file='rvoutput.dat', status='replace', err=110)
write(22,*) R, VDUM(IT), TEMP

IF(V(2).GE..67) GO TO 20
IF(RHO.GE.1.E-8) GO TO 20
RND = R - PFAC/(VP(1)/V(1)+2./R)
GO TO 12
20
RCFACL = RCFAC
RCFAC = RCFAC - R + 1.
WRITE(4,5000) RCFACL, RCFAC
if(abs(rcfac1-rcfac)).gt.e-4) go to 5200
5000 FORMAT(X,'RCFAC= ',2E14.5)
open(unit=6, file='mdotvslumdata.txt', status='old', access='append' 2, err=110)
write(6,*) TNOT, I, XM, RPOAS, DMAS, VIN, RCFAC
CALL BEEPQQ(frequency, duration)
110 STOP
END
SUBROUTINE CRITPT(RC, VC, VPC, V2)

COMMON/IDAT/DMDTOO, SIGE, ONEMAL, ALO1MA, TWOALP, VTH, GAMGMK, TWOMAL, GAM 2, GM, FS, GAM
COMMON/DERM/FLAG, ALPHA, RALP, C, F7, F8, RC2, ARP, ONEPAL, VLPLAST, FC, 2RPORS, TNOT, WN, rcrit, vpcguess

real(4) n, m, nv, nvp
LOGICAL FLAG

onepal=1.+alpha
VC2=VC*VC
A=3.E6
A2RP=(3.e6)**2*rpors*6.96e10
GMC=1.6666667
BOLT=1.38E-16

HYDROGEN=1.67E-24
RC2=RC**2
UC2=1.-1./RC2
F1=ONEPAL/RC2

WRITE(7,2000)
ITER=0
WRITE(7,1000)ITER,VC,VPC
ONEOV2=1./VC/VC
RCOVC=RC/VC
R2VVP=RC2+VC*VPC
RVPOV=RCOVC*VPC
SIGMA=RVPOV-1.
FC=\nu*\nu-(1.+SIGMA*\nu)\nuONEPAL/RVPOV**ALPHA)/ONEPAL/SIGMA
F9=F1*FC
VSIGMA=VC*SIGMA
ROVSIG=RC/VSIGMA
F11=1.+SIGMA*\nu
F12=(F11/RVPOV)**ALPHA
FVPOF=RC/VSIGMA*\((1.-F12*(ONEPAL*\nu2-ALPHA/RVPOV*F11))/F9-1.\))
ALPOVP=ALPHA/VPC
FVPOF2=-ROVSIG*(ALPOVP*F12*(1./RVPOV-U2/F11)/FC+2.*FVPOF)
VPOVSIG=VPC/VSIGMA
R3VPOV=RC2*RVPOV
TWOSIG=2.*SIGMA
F14=UC2+TWOSIG/R3VPOV
F14F11=F14/F11
FRF=VPOVSIG*\((3.-F12*(ONEPAL*F14-TWOMAL/RVPOV*F11))/F9-1.\)
ALPOVC=ALPHA/VC
F15=UC2-ALPHA/R3VPOV
FVOF=VPOVSIG*RCOVC*\((1.-F12*F15)/F9-1.\)
F16=ALPOVC+FVOF
F17=2.*RVPOV*RC*ONEOV2
F18=F12*\((ALPHA-4.)/RVPOV+ALPHA*F14F11)/FC
FRVOF=-VPC/VSIGMA*ONEOV2*(F18+VC*FVOF-RC+FRF)
F19=1./RVPOV-F15/F11
FVVF=(2.\*(RVPOV/VSIGMA-1.)*FVOF-ALPHA/FC/ONEPAL/VSIGMA*F12*RVPOV 2*F19)/VC

RCVC=RC*VC
Beginning of my changes

WN=(BOLT/{16.65e-X*HYDROGEN})*TNOT*GAMC/a**2

C split up N into parts so that I can see where the floating point error is .

F5=GAMC*(gam-1.)

N=-2.*WN/(RC**(2.*GAMC-3.)*VC**(GAMC-1.))=WN*VPC/(RC**(2.*GAMC-2.*VC**GAMC)+R2VVP-F5/a2rp

P1=-WN/(RC**(2*GAMC-4)*VC**GAMC)+Rc2+vc-(FVPOF+(1/FC)+ALPHA/VPC)*N

C Since p2(f2) is so long, I am going to divide it up into parts labeled q

Q1=-2.*WN*-GAMC+1.} /{RC**(2*GAMC-3.)*VC**GAMC)+WN*VPC/(RC**(2.*GAMC-3.)*VC**GAMC)

Q2=WN*GAMC+4./RC**2.*GAMC-3.)*VC**GAMC)

Q3=2.*R2VVP/RC

Q4=-2.*WN*-GAMC+1.} /{RC**(2.*GAMC-3.)*VC**GAMC)

Q5=GAMC*WN*VC/(RC**(2.*GAMC-4.)*VC**GAMC+1.))

Q6=R2VVP/VC

Q7=VPC*FVPOF+ALPHA*fc*VPC/VC+FROF+2*ALPHA*FC/RC

P2=Q1-Q2+Q3+VPC*(Q4+Q5+Q6)-N*(1/FC)*Q7

C Now for a,b,c,d terms of the matrix.  a=flx, b=fly, c=f2x, d=f2y

C Each matrix term is broken down into parts labeled mata, etc

NV=-2*WN*-GAMC+1.)/{RC**(2*GAMC-3.)*VC**GAMC)+WN*VPC*

2 GAMC/RC**2.*GAMC-4.)*VC**GAMC+1.))=R2VVP/VC

NVP=-WN/(RC**(2.*GAMC-4.)*VC**GAMC+4.)+R2VVP/VPC

tmATA1=WN*GAMC/(RC**(2.*GAMC-4.)*VC**GAMC+1.))
tMATA3 = (NV/FC-N*FVOF/FC**2)*FVOF

tMATA4 = N*FVPVOF/FC

tMATA5 = ALPHA*(NV/VPC)

AMAT = tMATA1 + rc*rc - tMATA3 - tMATA4 - tMATA5

tMATBl = NVP/FC-N*FVOF/FC**2

tMATB2 = N*FVP2OF/FC

tMATB3 = ALPHA*(NVP/VPC-N/VPC**2)

BMAT = -tMATBl - tMATB2 - tMATB3

tMATC1 = -2*WN*(-2*GAMC+3)*(-GAMC+1)/(RC**((2*GAMC-2.)*VC**GAMC)

tMATC2 = WN*(-2*GAMC+4.)*GAMC*VPC/(RC**((2*GAMC-3.)*VC**GAMC+1))

tMATC3 = 2*RC*VPC

tMATC4 = 2*WN*(-GAMC+1.)*GAMC/((RC**((2*GAMC-3.)*VC**GAMC+1.))

tMATC5 = GAMC*WN*VPC*(-GAMC-1.)/(RC**((2*GAMC-4.)*VC**GAMC+2.))

tMATC6 = N/FC-N*FVOF/FC**2

tMATC7 = FROF+2*FC*ALPHA/RC+VPC*FVOF+FC*ALPHA*/VPC/VC

tMATC8 = FROF+2*FVOF*ALPHA/RC+VPC*FVOF+FVVOF+ALPHA*VPC*(FVOF/VC-

2FC/VC**2)

CMAT = tMATC1 + tMATC2 + tMATC3 + VPC*(tMATC4 + tMATC5) - tMATC6 - tMATC7 - (N/FC)

2*tMATC8

2*tMATD8

DETERMINANTS = AMAT*DMAT - BMAT*CMAT

DVCP = (BMAT*P2 - DMAT*P1)/DETERMINANTS

VPC = VPC + DVCP

DVPC = (CMAT*P1 - AMAT*P2)/DETERMINANTS

IF (VPC.LT.0) VPC = 1.

ITER = ITER + 1

WRITE (3, 1000) ITER, VC, VPC, (DVPC/VPC), (DVCP/VC), RC

IF (ITER.GT.80) GO TO 1500
IF(ABS(DVPC/VPC).GT.1.E-5)GO TO 8
IF(ABS(DVC/VC).GT.1.E-5)GO TO 8

attempt to iterate on r,v,vp so i don't have to manually
change them to get critpt to converge

1500  iter=0
      it=it+1
      i1=it/350.
      i2=int(i1)
      vc=1+i2*.1
      iterate=iterate+1
      i3=mod(iterate,350)
      if(i3.eq.0)iterate=0.
      vpc=100.+1.*iterate
      if((vc.gt.6).AND.(vpc.gt.700))go to 60.
      if(it.gt.10000)go to 60.
      write(7,*)'vc,vpc,iter=',vc,vpc,iterate,+/**(dvc/vc),(dvpc/vpc)
      go to 8
12
      VC=VC+.01
      VPC=VPC+.2
      go to 8
9
      write(7,*)'Final convergence:vc,vpc ',',vc,vpc
      C1=-2*WN/(RC**(2*GAMC-3.)*VC**(GAMC-1))
      C2=-WN*VPC/(RC**(2*GAMC-4.)*VC**(GAMC)+R2VVP=FS/a2rp
      C3=(R2VVP)**(ALPHA)*FC
      C=(C1+C2)/C3
      DMDTDO=(GAMGMC*2RP**(ALPHA-1.)/C)**RALP/(SIGE*VTH)
      C
      Now doing v". Must first write Fv'v', Frr,Frv,Frv'Fvv'.
      C
      As before break each of these down, labeling them as dblvpvpl, etc
      DBLVPV=-(R2VVP)**ALPHA*(FVP2OF+2.*ALPHA*FVPOF/VPC+ALPHA*FC*
      2(ALPHA-1.)/VPC**2)
      DBLRR=-(2*WN*(-2*GAMC+3.)*(-2*GAMC+2.))/(RC**(2*GAMC-1.*VC**(GAMC-1
      2.)))
      DBLRR2=-2*WN*(-2*GAMC+4.)*(-2*GAMC+3.)*VPC/(RC**(2*GAMC-2.)*VC**GAMC)
      DBLRR3=2*VC*VPC
      DBLRR4=FRRF*V2V**ALPHA
      DBLRR5=FROF*2*ALPHA*R2VVP**ALPHA/RC
      DBLRR6=2.*FC*ALPHA*R2VVP**ALPHA*(2.*ALPHA-1.)/RC**2
      DBLRR=DBLRR1-DBLRR2+DBLRR3*C*(DBLRR4+2*DBLRR5+DBLRR6)
      DBLRV1=-2*WN*(-2*GAMC+3.)*(-GAMC+1.)/RC**(2*GAMC-2.)*VC**GAMC)
      DBLRV2=-2*GAMC+4.)*GAMC*VPC/(RC**(2*GAMC-3.)*VC**GAMC+1.)
      DBLRV3=2*RC*VPC
      DBLRV4=FROF+FROF**ALPHA/VC+2.*FC*ALPHA**2/VC
      DBLRV=DBLRV1+DBLRV2+DBLRV3*C*R2VVP**ALPHA*(DBLRV4)
      DBLRVPI=-2*(2.*GAMC-3.)*VC**GAMC)
DBLRVP2=2.*RC*VC
DBLRVP3=FRVPOF+ALPHA*FRV/VPC+2.*ALPHA*FVPOF/RC+
22.*ALPHA**2*FC/(RC*VPC)

DBLRVP=DBLRVP1+DBLRVP2-C*R2VVP**ALPHA*DBLRVP3

DBLVV1=2.*WN*GAMC*(-GAMC+1.)/(RC**2.*GAMC-3.)VC**2.*GAMC+1.))
DBLVV2=GAMC*(-GAMC+1.)*WN*VPC/(RC**2.*GAMC-3.)VC**2.*GAMC+1.))
DBLVV3=FNVOF+2.*ALPHA*FNVOF/VPC+ALPHA*FC*(ALPHA-1.)/VC**2

DBLVV=DLBVV1+DLBVV2-C*R2VVP**ALPHA*DBLVV3

DBLVVP1=WN*GAMC/(RC**2.*GAMC-4.)VC**2.*GAMC+1.))
DBLVVP2=RC*RC
DBLVVP3=FRVPOF+ALPHA*FRV/OFC+ALPHA*FVPOF/VPC+
2ALPHA**2*FC/(VC*VPC)

DBLVVP=DBLVVP1+DBLVVP2-C*R2VVP**ALPHA*DBLVVP3

DBLV1=2.*WN*(-GAMC+1.)/(RC**2.*GAMC-3.)VC**GAMC

DBLV2=GAMC*WN*VPC/(RC**2.*GAMC-3.)VC**2.*GAMC+1.))
DBLV3=RC*RC*VPC
DBLV4=FNVOF+ALPHA*FC/VPC

DBLV=DLBV1+DLBV2+DLBV3-C*R2VVP**ALPHA*DBLV4

C Now rewriting B and F6

B=DBLV+2*(DBLRVP+VPC*DBLVVP)
BSQRT=(B**2-4*DBLVPVP*(DBLRR+VPC*(2*DBLRV+VPC*DBLVV))))
if(bsqrt.lt.0)go to 70

V2=(-B+SQRT((B**2-4*DBLVPVP*(DBLRR+VPC*(2*DBLRV+VPC*DBLVV)))))/(2*
2DBLVPVP)

crit.rc=rc
ITER=0
IT=0
ITERATE=0

RETURN
60 IT=1
WRITE (7,4000) IT
STOP
70 IT=2
WRITE (7,4000) IT
1000 FORMAT(X,I2,6E12.4)
2000 FORMAT(’CMAK MODEL WITH TEMPERATURE DISTRIBUTION AND FINITE DISK
2FACT0R’/’CRITICAL POINT ITERATION:’/’#,5X,’VC’,11X,’VPC’)
4000 FORMAT(’V’,I1,’CALCULATION IS WRONG it too big’)
STOP
END

C FUNCTION F(RVPOV,SIGMA,R2,U2)
COMMON/DERM/FLAG, ALPHA, RALP, C, F7, F8, RC2, ARP, ONEPAL, VLAST, FC, 2RPORS, TNOT, WN, RC

C LOGICAL FLAG
F=R2*(RVPOV-(1.+SIGMA*U2)**ONEPAL/RVPOV**ALPHA)/ONEPAL/SIGMA
RETURN
END
SUBROUTINE DER(R,VA,VPA)
   THIS IS THE NEW DER SUBROUTINE.

DIMENSION VA(2), VPA(2)
COMMON/IDAT/DMDTDO, SIGE, ONEMAL, ALOIMA, TWOALP, VTH, GAMGMK, TWOMAL, GAM
   2, GM, F5, GAMC

COMMON/DERM/FLAG, ALPHA, RALP, C, F7, F8, R2, ARP, ONEPAL, VPLAST, FC,
   2RPORS, TNOT, WN, VPC1

LOGICAL FLAG
   if(r.gt.rc)flag=.true.

V=VA(1)

VPA(2)=-DMDTDO*SIGE/(V*R**2)/ARP
   a2rp=(3.e6)**2*rpors*6.96e10

HR=2*WN/(R**((2+GAMC-3)*V**(GAMC-1)))+F5/a2rp

U2=1.-1./R**2
   IF(U2.LT.0)U2=0.
   ITER=0

VP=VPLAST

QTEST=WN/(R**((2+GAMC-2.)*V**(GAMC+1.))
1
   IF((HR.GT.0).AND.(QTEST.GE.1))GO TO 99
   IF((HR.LE.0).AND.(QTEST.GE.1))GO TO 5
   IF((HR.LE.0).AND.(QTEST.LT.1))GO TO 7
   IF((HR.GT.0).AND.(QTEST.LT.1))GO TO 9
5
   R2VVP=R*R+V*VP
   R2=R*R
   GAMC=1.6667
   RVPOV=R/V*VP
   SIGMA=RVPOV-1.
   F1=ONEPAL/R2
   F11=1.+SIGMA*U2
   QTEST=WN/(R**((2+GAMC-2.)*V**(GAMC+1.))
   F12=(F11/RVPOV)**ALPHA
   FC=F(RVPOV, SIGMA, R2, U2)
   F9=F1*FC

VSIGMA=V*SIGMA

fvp=(r/v)/sigma*(((1.-F11/rvpo)**alpha*(onepal*U2-alpha/rvpo**
   2f11))/onepal*R2-FC)

   y1=c*fc*2vvp**alpha/abs(hr)
   y2=abs(1-qtest)*2vvp/abs(hr)

   bf=-y1-y2+1
   bfvp=-y2/vp-c*2vvp**alpha/abs(hr)*(fvp+alpha*fc/vp)

DVP=-bf/bfvp
VP = VP + DVP
ITER = ITER + 1

IF (ITER.GT.35) GO TO 99
IF (abs (DVP/VP).GT.1.E-4) GO TO 7
VP A(1) = VP
RETURN

7
rvpov = r/v*vp
sigma = rvpov - 1.
fc = f (rvpov, sigma, r2, u2)
ff = (abs (hr)/c)**ralp
ffovr2 = ff/v*r*r
b = ff*abs((1-qtest)/hr)

vp = ffovr2
if (flag) vp = (b/fc**ralp)**(-1./(1.-alpha))*ffovr2

R2VVP = R*R*V*VP
R2 = R*R
GAMC = 1.6667
RVPPOV = R/V*VP
SIGMA = RVPPOV - 1.
F1 = ONEPAL/R2
F11 = 1.+SIGMA*U2
QTEST = WN/(R**(2*GAMC-2.)*V**(GAMC+1.))
F12 = (F11/RVPPOV)**ALPHA
FC = F (RVPPOV, SIGMA, R2, U2)
F9 = F1*FC

VSIGMA = V*SIGMA

fvp = (r/v)/sigma*{(1.-fll/rvpov)**alpha*(onepal*u2-alpha/rvpov*2fll1)/onepal*r2-fc)}

y1 = c*fc*r2vvp**alpha/abs(hr)
y2 = abs(1-qtest)*r2vvp/abs(hr)

bf = -y1+y2+1
bfvp = y2/vp-c*r2vvp**alpha/abs(hr)*(fvp+alpha*fc/vp)

DVP = -bf/bfvp
VP = VP + DVP
ITER = ITER + 1

IF (ITER.GT.35) GO TO 99
IF (abs (DVP/VP).GT.1.E-3) GO TO 7
VP A(1) = VP
RETURN

9
R2VVP = R*R*V*VP
R2 = R*R
GAMC = 1.6667
RVPPOV = R/V*VP
SIGMA = RVPPOV - 1.
F1 = ONEPAL/R2
F11 = 1.0 + SIGMA * U2
QTEST = WN / (R** (2 * GAMC - 2.) * V** (GAMC + 1.))
F12 = (F11 / RVPOV)** ALPHA
FC = F(RVPOV, SIGMA, R2, U2)
F9 = F1 * FC

V_SIGMA = V * SIGMA

fvp = (r/v)/sigma*((1. - (fll/rvpo)*alpha*(onepal * u2 - alpha/rvpo*2fll))/onepal * r2 - fc)

y1 = c * fc * r2vvp**alpha/abs(hr)
y2 = abs(1 - qtest) * r2vvp/abs(hr)

bf = -y1 + y2 - 1
bfvp = y2/vp - c * r2vvp**alpha/abs(hr) * (fvp + alpha * fc/vp)

DVP = -bf/bfvp
VP = VP + DVP
ITER = ITER + 1

IF(ITER.GT.35) GO TO 99
IF(abs(DVP/VP).GT.1.E-4) GO TO 9
VPA(1) = VP
RETURN

99 WRITE(4,*) 'NO SOLUTION FOR VP--R,V= ', R, V
RETURN

END
SUBROUTINE RKFN(F, NEQN, Y, T, TFIN, EPSREL, EPSABS, IFLAG, WORK, IWORK)

IMPLICIT DOUBLE PRECISION(A-H, O-Z)
DIMENSION Y(NEQN), WORK(15), IWORK(5)
EXTERNAL F
DATA U26/1.5E-6/, REMIN/1.E-6/, ZERO/0.1
KIM=NEQN+1
K1=K1M+1
K2=K1+NEQN
K3=K2+NEQN
K4=K3+NEQN
K5=K4+NEQN
K6=K5+NEQN
IF(NEQN.LT.1)GO TO 10
IF((EPSREL.LT.0.) .OR. (EPSABS.LT.0.))GO TO 10
MFLAG=ABS(IFLAG)
IF((MFLAG.GE.1).AND.(MFLAG.LE.7))GO TO 20

10 IFLAG=7
RETURN

20 IF(MFLAG.EQ.1)GO TO 55
IF(T.EQ.TFIN)GO TO 10
IF(MFLAG.NE.2)GO TO 25
IF(IWORK(3).EQ.0)GO TO 45
IF(IWORK(5).EQ.3)GO TO 40
IF((IWORK(5).EQ.4).AND.(EPSABS.EQ.0.))GO TO 22
IF((IWORK(5).NE.5).OR.(EPSREL.GT.WORK(K6)).OR.(EPSABS.GT.WORK(K6+1 2)))GO TO 55

22 IFLAG=IWORK(5)
RETURN

25 IF((IFLAG.EQ.3).OR.(EPSABS.GT.0.))GO TO 45
40 IWORK(1)=0
IF(MFLAG.EQ.4).AND.(EPSABS.GT.0.)GO TO 55
45 IFLAG=IWORK(4)
55 IWORK(4)=IFLAG
WORK(K6)=EPSREL
WORK(K6+1)=EPSABS
RER=MAX(EPSREL, REMIN)
DX=TFIN-T
IF(MFLAG.EQ.1)GO TO 60
IF(IWORK(3).EQ.0)GO TO 65

60 IWORK(1)=1
IWORK(2)=0
IWORK(3)=0
A=T
CALL F(T, Y, WORK)
IF(T.NE.TFIN)GO TO 65
IFLAG=2
GO TO 100

65 IWORK(3)=1
YMAX=0.
YPN=0.
DO 70 K=1, NEQN
YPN=MAX(ABS(WORK(K)), YPN)
70 YMAX=MAX(ABS(Y(K)), YMAX)
ETN=RER*YMAX+EPSABS
H = ABS(DX)
IF(ETN.GE.YPN*H**5) GO TO 80
H = MAX((ETN/YPN)**.2, U26*MAX(ABS(T), H))
80 WORK(K1M) = SIGN(H, DX)
CALL RKFNS(F, NEQN, Y, T, TFIN, RER, EPSABS, IFLAG, WORK(1), WORK(K1),
2 WORK(K2), WORK(K3), WORK(K4), WORK(K5), IWORK(1))
100 IWORK(5) = IFLAG
RETURN
END

SUBROUTINE RKFNS(F, NEQN, Y, T, TFIN, EPSREL, EPSABS, IFLAG, YP,
2 H, F1, F2, F3, F4, F5, NFE)
LOGICAL HFAILD, OUTPUT

DIMENSION Y(NEQN), YP(NEQN), F1(NEQN), F2(NEQN), F3(NEQN),
2 F4(NEQN), F5(NEQN)

EXTERNAL F
DATA U26/1.5E-6/, ZERO/0./
DATA MAXNFE/6000/
DX = TFIN - T
IF(ABS(DX).GT.U26*ABS(T)) GO TO 50

DO 25 K = 1, NEQN
25 Y(K) = Y(K) + DX*YP(K)
A = TFIN
CALL F(A, Y, YP)
NFE = NFE + 1
GO TO 300
50 OUTPUT = .FALSE.
SCALE = 2./EPSREL
AE = SCALE*EPSABS
100 HFAILD = .FALSE.
Mmin = U26*ABS(T)
DX = TFIN - T
IF(ABS(DX).GE.2.*ABS(H)) GO TO 200
IF(ABS(DX).GT.ABS(H)) GO TO 150
OUTPUT = .TRUE.
H = DX
GO TO 200
150 H = .5*DX
200 IF(NFE.LE.MAXNFE) GO TO 220
IFLAG = 3
RETURN
220 CALL FEHLR(F, NEQN, Y, T, H, YP, F1, F2, F3, F4, F5, F1)
NFE = NFE + 5
EEOET = 0.
DO 250 K = 1, NEQN
ET = ABS(Y(K)) + ABS(F1(K)) + AE
IF(ET.GT.ZERO) GO TO 240
IFLAG = 4
RETURN
240 EE = ABS((-2090.*YP(K)+(21970.*F3(K)-15048.*F4(K)))+
2 (22528.*F2(K)-27360.*F5(K)))
250 EEOET = MAX(EEOET, EE/ET)
ESTTOL = ABS(H)*EEOET*SCALE/752400.
OPEN(10,FILE='TEST.DAT')
write (10, *) ESTTOL
IF(ESTTOL.LE.1.)GO TO 260

HFAILD=.TRUE.
OUTPUT=.FALSE.
S=1
IF(ESTTOL.LT.59049.)S=.9/ESTTOL**.2
H=S*H
IF(ABS(H).GT.HMIN)GO TO 200
IFLAG=5
RETURN

260 T=T+H
DO 270 K=1,NEQN
270 Y(K)=F1(K)
A=T
CALL F(A,Y,YP)
NFE=NFE+1
IF(HFAILD)GO TO 290
S=5.
IF(ESTTOL.GT.1.889568E-4)S=.9/ESTTOL**.2
H=SIGN(MAX(S*ABS(H),HMIN),H)
290 IF(OUTPUT)GO TO 300
IF(IFLAG.GT.0)GO TO 100
IFLAG=-2
RETURN

300 T=TFIN
IFLAG=2
RETURN
END

SUBROUTINE FEHLR(F,NEQ,Y,X,H,YP,F1,F2,F3,F4,F5,S)
DIMENSION Y(NEQ), YP(NEQ), F1(NEQ), F2(NEQ), F3(NEQ), F4(NEQ), F5(NEQ), 2S(NEQ)

CH=.25*H
DO 221 K=1,NEQ
221 F5(K)=Y(K)+CH*YP(K)
CALL F(X+.25*H,F5,F1)
CH=.09375*H
DO 222 K=1,NEQ
222 F5(K)=Y(K)+CH*(YP(K)+3.*F1(K))
CALL F(X+.375*H,F5,F2)
CH=H/2197.
DO 223 K=1,NEQ
223 F5(K)=Y(K)+CH*(1932.*YP(K)+(7296.*F2(K)-7200.*F1(K)))
CALL F(X+12./13.*H,F5,F3)
CH=H/4104.
DO 224 K=1,NEQ
224 F5(K)=Y(K)+CH*((8341.*YP(K)-845.*F3(K))+
2 (29440.*F2(K)-32832.*F1(K)))
CALL F(X+H,F5,F4)
CH=H/20520.
DO 225 K=1,NEQ

225  F1(K)=Y(K)+CH*(((6080.*YP(K)+(9295.*F3(K)-5643.*F4(K)))+
2 (41040.*F1(K)-28352.*F2(K)))
CALL F(X+.5*H,F1,F5)
CH=H/7618050.
DO 230 K=1,NEQ

230  S(K)=Y(K)+CH*((902880.*YP(K)+(3855735.*F3(K)-1371249.*F4(K)))+
2 (3953664.*F2(K)+277020.*F5(K)))
RETURN
END
Bibliography