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A radiation-driven wind model for massive stars: The effects of non-isothermal and finite disk assumptions

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A RADIATION-DRIVEN WIND MODEL FOR MASSIVE STARS:
THE EFFECTS OF NON-ISOTHERMAL AND FINITE DISK ASSUMPTIONS

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A Radiation-Driven Wind Model for Massive Stars: The Effects of Non-Isothermal and Finite Disk Assumptions

Director: David B. Friend DBF

Wind models for massive stars are of particular interest because we are unable to have a complete picture of the evolution of these stars as well as the chemical composition of our own galaxy without understanding these mass loss mechanisms. For massive stars, the main force which drives the wind is the absorption and scattering of radiation from the star.

In order to model such a radiatively-driven stellar wind, I begin with the "standard" model of Castor, Abbott and Klein (CAK, 1975). Among the assumptions used in this model when determining the equation of motion for the wind are the following:

1) the wind does not change in time (steady-state)
2) the wind is spherically symmetric
3) the temperature of the wind does not change with distance from the star (isothermal)
4) the star is a point source of radiation
5) the star does not rotate
6) the star has no magnetic field.

The CAK predictions turn out to be too high for the mass loss rate of the star and too low for the velocity of the wind far from the star as compared to observations.

This work eliminates two of the simplifications in the CAK theory. First I treat the star as a disk with finite size (as previously done by Friend and Abbott, 1986). Treating the star as a finite disk reduces the radiation force near the star thereby bringing the mass loss rates closer to the observed rates.

Second I assume the temperature of the wind does change with distance from the star. I include a temperature distribution, which is a function of distance, in the equation of motion.

The questions I seek to answer are, "Is the isothermal assumption acceptable?" Does a temperature distribution complicate the equations unneccessarily? To answer these questions, the non-isothermal, finite disk results are compared to the finite disk results of Friend and Abbott as well as with observations to determine whether or not this isothermal assumption is reasonable.

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Chapter 1

Introduction

Modeling the mass loss mechanisms of massive stars is integral to the development of a complete picture of the evolution of massive stars as well as an understanding of the chemical composition and evolution of our entire galaxy. Radiation provides a mechanism by which a wind is driven when ions absorb the momentum of photons and are propelled away from the star, thereby producing mass loss. This work seeks to explain basic radiatively-driven wind theory, particularly in the context of the modification I have made to the theory—the assumption of a temperature distribution in the wind.

What are massive stars?

Stars are divided into spectral types, based on their temperatures, using the Harvard classification scheme below:

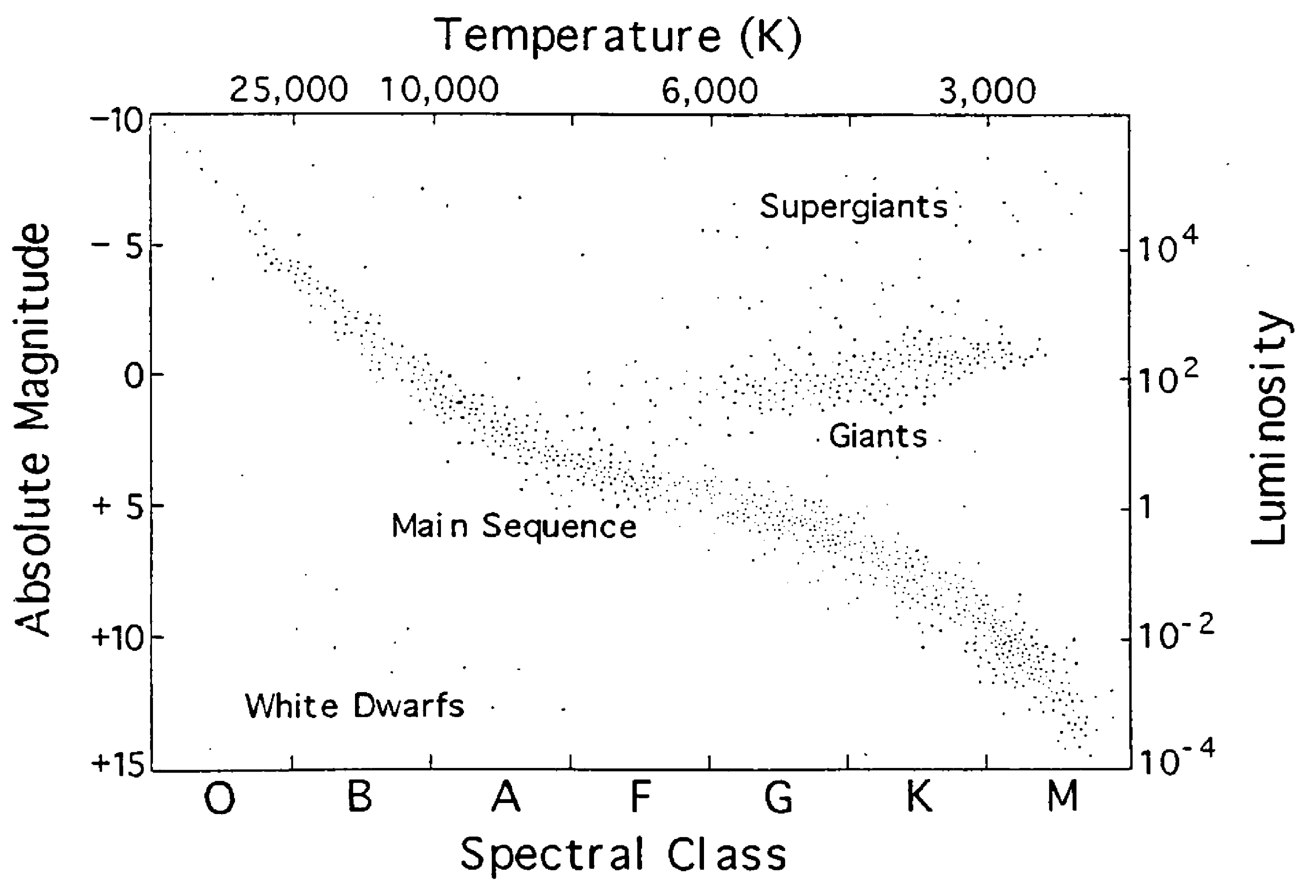
Spectral Type	*Characteristics*
O	Hottest blue-white stars with few spectral lines.
B	Hot blue-white stars.
A	White stars.
F	Yellow-white stars.
G	Yellow stars. (Our sun)
K	Cool orange stars.
M	Coolest red stars.

Certainly not all O type stars have the same temperature or radius, so each type is also scaled from 0 to 9 (hottest to coolest) and is given a luminosity class:

I Supergiant
II Bright Giant
III Giant
IV Subgiant
V Dwarf or Main Sequence

For example, the hottest B giant would be classified as B0III.

The Hertzsprung-Russell Diagram is a standard way to illustrate the relationships among spectral type, temperature, radius and luminosity.



http://imagine.gsfc.nasa.gov/docs/teachers/lifecycles/printable/LC_main_p8.html

Figure 1.1

Massive stars are considered to be O and B type stars. They range in mass from 10 to 60 $M_\odot$ where 1 $M_\odot = 1.998 \times 10^{33}$ g. Their effective temperatures (temperature at the surface or photosphere) are anywhere from 20,000 to 60,000 degrees Kelvin (compare to T_{eff} of the sun at ~5800 K).

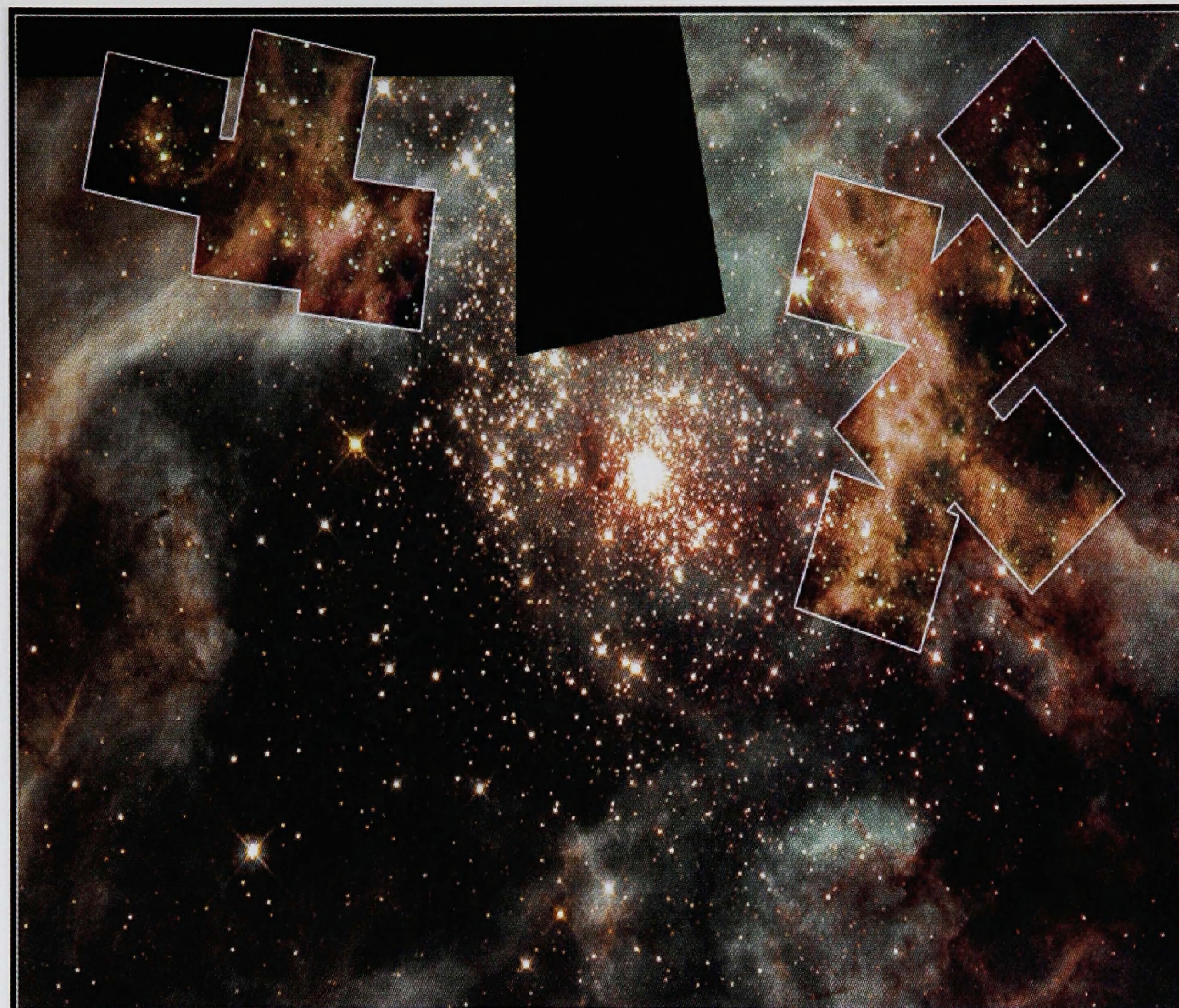
Where do we observe O and B stars? They tend to occur in open clusters and associations, or loosely grouped arrangements, which are closely confined to the plane of our galaxy. Rarely do we see any alone because their lifespans are relatively short (less than 10 million years), hence they do not have time to wander far from the nursery in which they were born. On the following pages are examples of places we find O and B stars. Figure 1.2 shows the formation of massive stars triggered by the winds of adult massive stars in the hub of the 30 Doradus Nebula in the Large Magellanic Cloud. N81, a massive star cluster in the Small Magellanic Cloud, is shown in Figure 1.3. The rapid and enormous mass loss from these young massive stars is evident in the sculpting of the nebula's gasses.

What are stellar winds and why should we study them?

Stellar winds are the expanding outer layers of a star's atmosphere. Massive stars may dump up to 50% of their initial mass back into the interstellar medium over only several million years. This substantial mass loss gives rise to the following important effects:

1) Possible induction of stellar formation. The star may lose so much mass that another star is able to form from this matter.

2) The evolution of the mass-losing star is significantly altered.



30 Doradus Nebula in the LMC **HST** • WFPC2 • NICMOS

PRC99-33a • STScI OPO • N. Walborn (STScI), R. Barbá (La Plata Observatory) and NASA

Figure 1.2

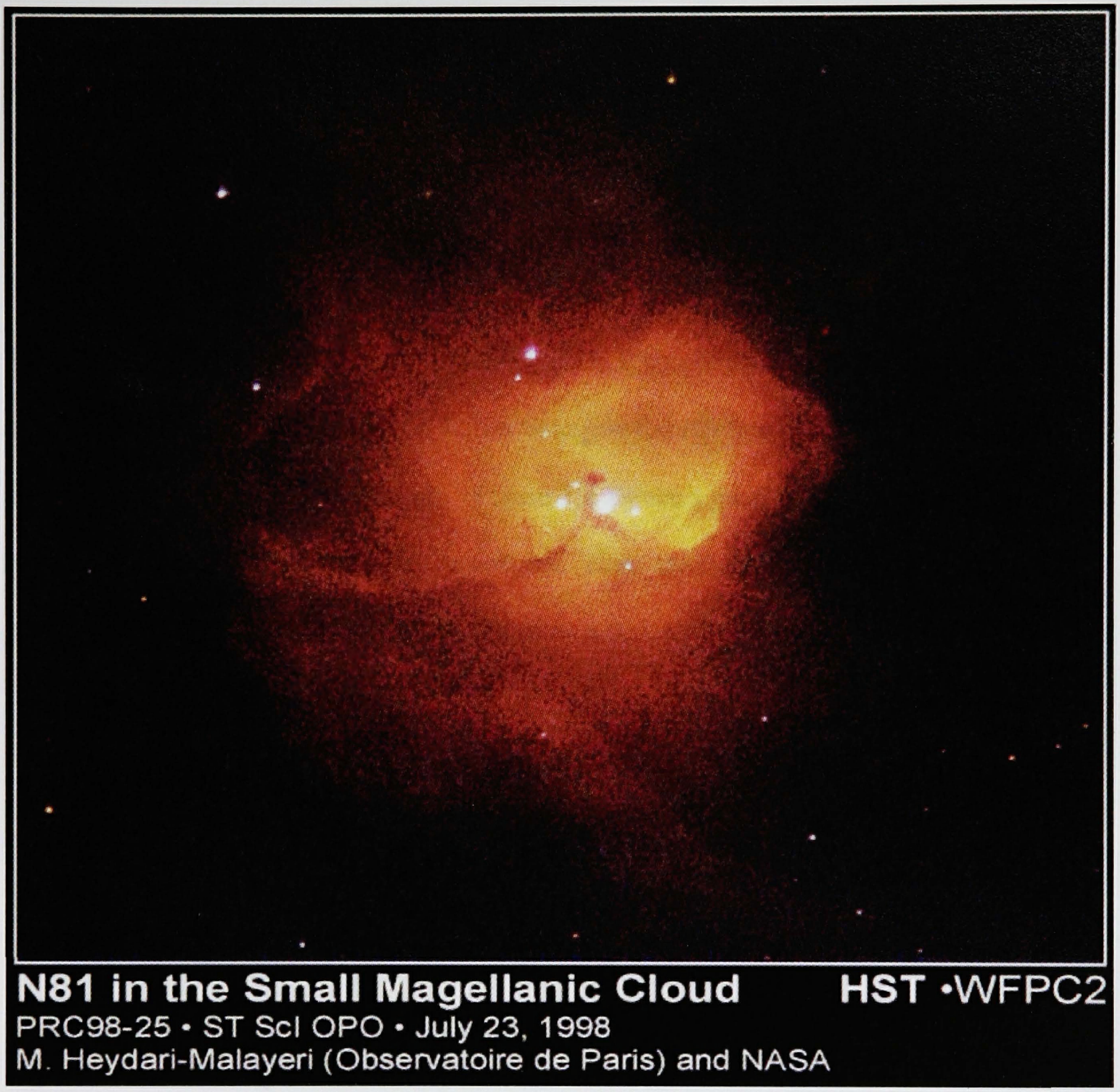

N81 in the Small Magellanic Cloud
HST •WFPC2
PRC98-25 • ST ScI OPO • July 23, 1998
M. Heydari-Malayeri (Observatoire de Paris) and NASA

Figure 1.3

3) Changes in the chemical makeup of the galaxy in which the star resides, hence influencing galactic structure, chemistry and evolution.

How do we know they exist?

The first evidence of stellar winds was discovered in the late 19th century. Astronomers, analyzing the spectrum of a blue luminous variable star known as P Cygni, discovered something very interesting. A P Cygni profile is characterized by a broad emission peak centered on λ_0, for example the Hβ line at 4860 angstroms, and a blueshifted absorption trough. Figure 1.4 shows a profile of P Cygni between the wavelengths of 4800 and 5050 angstroms. How are these spectral lines produced? A stellar wind can be thought of as a series of concentric shells surrounding the star.

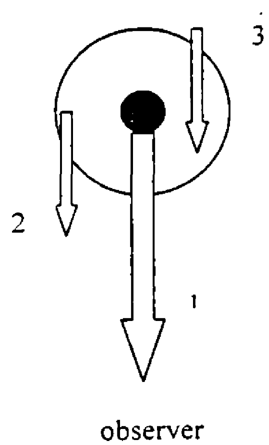


Figure 1.5

Region 1 in Figures 1.4 and 1.5 represent the portion of the shell which is in our direct line of sight. The approaching matter is cooler than the layers below it, thus the photons coming from the star are absorbed and blue shifted. Region 2 is the front portion of the shell which is approaching us. We do not see the photons being absorbed in this region because they are scattered out of our line of sight. Instead we

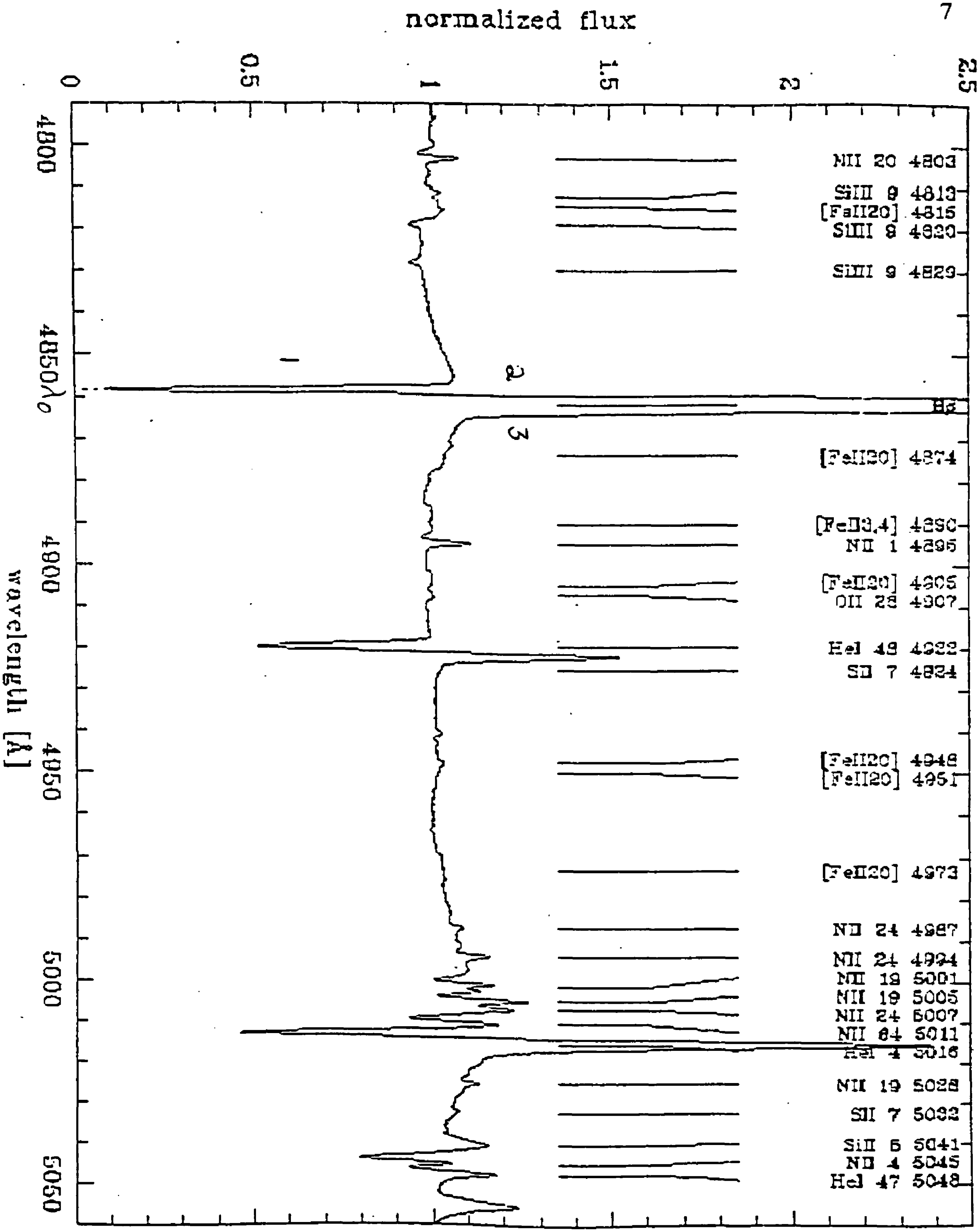


Figure 1.4 P Cygni Profile

http: www.lsw.uni-heidelberg.de %7Eostahl PCyg.html observe emission produced by the hot diffuse matter. Region 3 is the rear portion of the expanding shell. Similarly we observe emission here, except that it is redshifted, indicating that it is moving away from us. Not until 1929 did Beals interpret these effects as the ejection of high-velocity material from the star.

Soon after this discovery, astronomers found P Cygni-like profiles in the spectra of O, B, G, K, and M stars confirming the presence of winds (Cranmer, 1997). From these spectra, we can infer terminal velocities and mass loss rates for massive stars as well.

What is "radiation-driving"?

Radiative driving is thought to be the primary cause of continuous mass outflows for main sequence, giant and supergiant stars that are hotter than approximately B5 (Cranmer, 1997). In an atmosphere, ions will absorb the momentum and energy of photons that have come from below them (closer to the center of the star) moving the ions generally outward. Ions can absorb only in a limited frequency range and in a relatively short period of time would absorb all the photons possessing those frequencies. No wind could be sustained this way. However, since the ions are moving away from the star, the photon frequencies are red-shifted, and at each radius (r1, r2, etc.) a different frequency is absorbed. See figures 1.6 and 1.7.

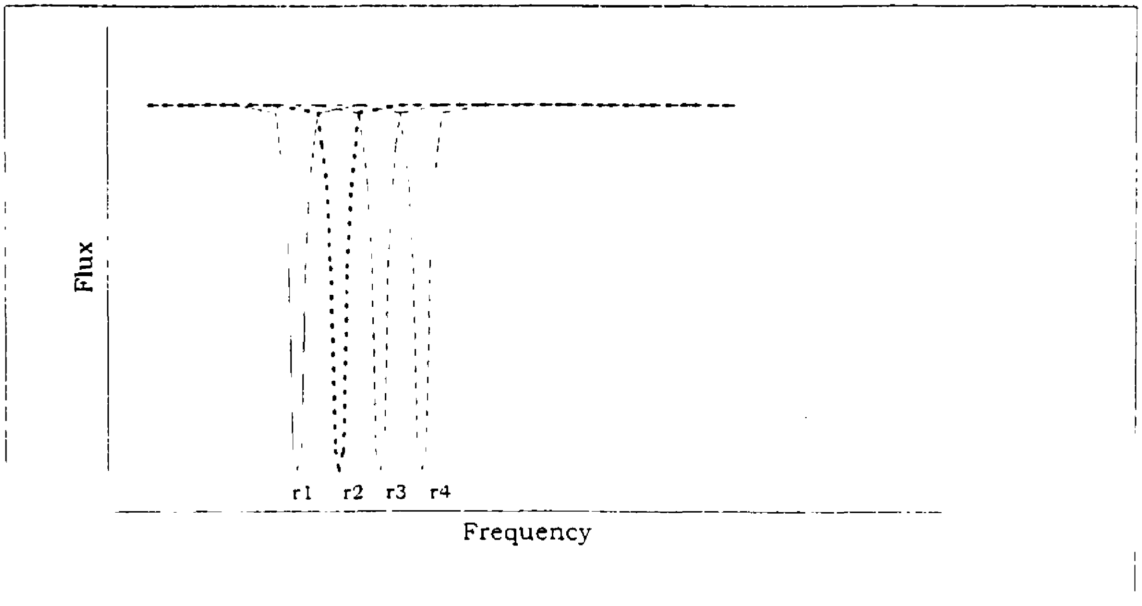


Figure 1.6
Multiple Frequency Absorption

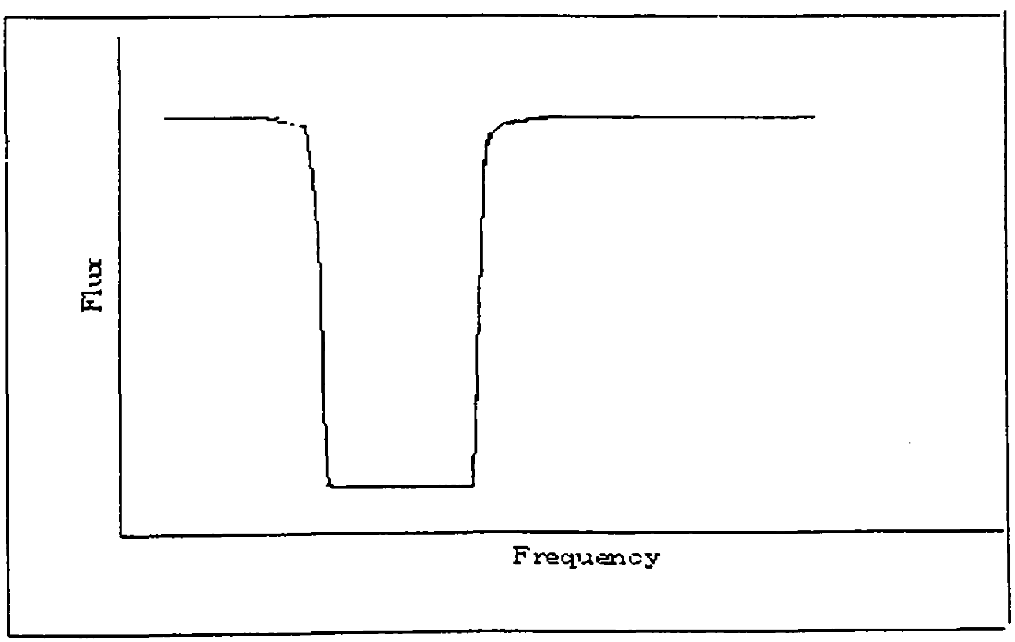


Figure 1.7
Cumulative Absorbed Frequencies

In this way, the ions in the stellar atmosphere may absorb many photons, acquiring their momentum, and drive the atmosphere outward in a wind.

Overview of past research

In 1971 Lucy and Solomon revived the theories of Milne (1924,1926) and Johnson (1925) which stated that the force on ions due to absorption could exceed gravity and thus matter could be ejected from a star. Lucy and Solomon (1971) applied this theory to explain the mass loss from O stars.

A few years later Castor, Abbot and Klein (CAK, 1975) produced the seminal work of radiation-driven stellar wind theory, the goal of which was to predict the phenomenon we observe, namely the mass loss rate and terminal velocity of the wind. The main details of this theory, which is the basis of this thesis, will be left for chapter 2. However, a brief discussion of the assumptions and shortfalls of CAK theory is merited.

Assumptions:

1.) Momentum is transferred from the radiation (photons) to the gas.

2.) The wind may be treated as an ideal fluid since it is highly ionized.

3.) The wind is isothermal (does not change in temperature).

4.) The wind is steady-state (does not change in time).

5.) The wind is spherically symmetric.

6.) The star is not rotating.

7.) The star has no magnetic field.

8.) The star is treated as a point source of radiation.

CAK theory predicts mass loss rates that are 100 times greater than those predicted by Lucy and Solomon. These rates compare much better with observation. The CAK terminal velocities are lower than observations by a factor of two or three for hotter stars and compare fairly well for the cooler O and B stars.

The shortfalls of CAK theory are found in the assumptions. Take for example the spherically symmetric assumption. Upon examination of this image of our own sun, it is apparent that the wind is not spherically symmetric.

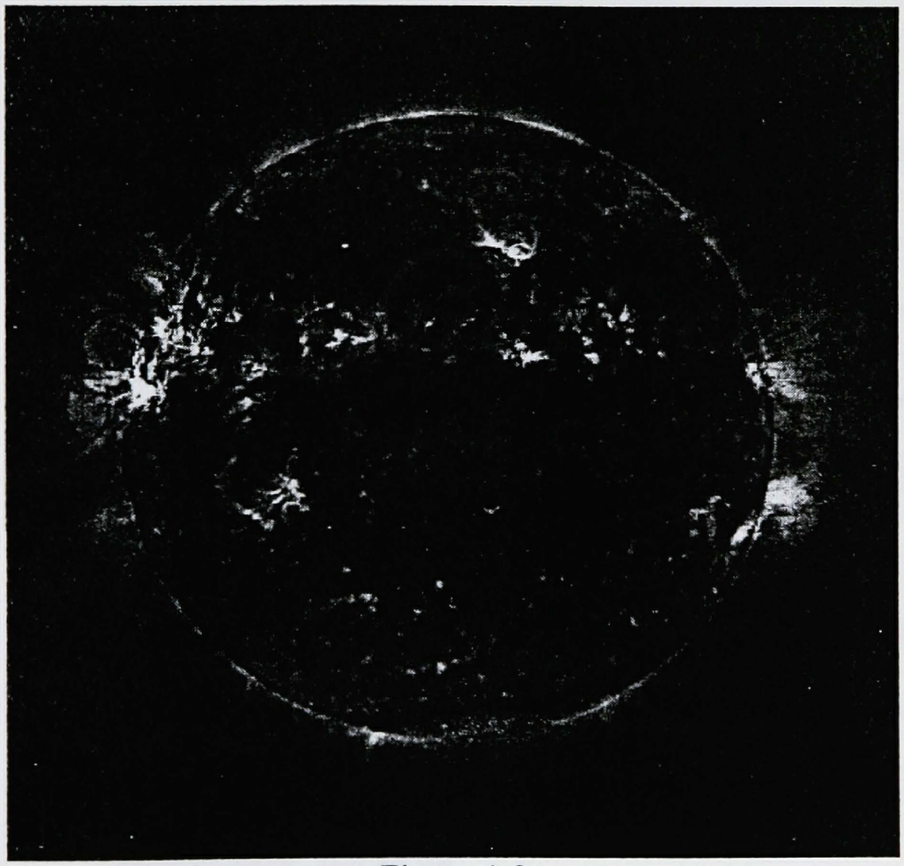


Figure 1.8

http://umbra.nascom.nasa.gov/images/latest_eit_171.gif

The observations of our own sun indicate not only that the solar wind is not spherically symmetric, but that the temperature does change as a function of the radius, and the wind rotates. We also know that the sun has a magnetic field and is not a point source. Regardless of its shortcomings, CAK theory was an excellent beginning to our understanding of radiation-driven stellar winds.

Several modifications have been made to the CAK theory. These include:

1.) Abbott (1977,1979) and Castor(1979) incorporated a much larger sample of spectral lines and a more correct treatment of the equations of radiation transfer to improve predictions.

2.) Friend and Abbott(1986) included the "finite disk factor" to compensate for the fact that the star is not a point source, but has some dimension. If the star is considered to be a point source, the line force is overestimated near the star and underestimated far from the star. This correction alone reduces the mass loss rate and increases the terminal velocity, bringing the results in better alignment with observations.

3.) Other additions or modifications to the theory have addressed many of the original assumptions and include rotation, magnetic fields, time-dependence, pulsations, oscillations, photospheric perturbations, asymmetries, and so on.

Overview of my research

In this work, I have used the original CAK model, with the assumptions that the wind is not isothermal and that the star is not a point source. Being that the solar wind is driven by thermal gradients, we may expect that a much hotter star would drive it's wind in the same fashion. I have chosen an adiabatic temperature distribution which I derive in Chapter 2:

$$T = T_o \left(\frac{r_{star}^2 a}{r^2 v} \right)^{\gamma - 1}$$

where T_o is the temperature where the velocity of the wind is equal to the sound speed, r_{star} is the radius of the star to the photosphere, and γ is c_p/c_v . In chapter 3, my results will be compared with those of the Friend and Abbott finite disk model and with observations of terminal velocity and mass loss rates for a sample of stars. And finally, Chapter 4 summarizes this work and discusses the effects of the non-isothermal assumption upon the theory.

Chapter 2

Derivation of Equations and Method of Solution

The basic equation of fluid motion is derived from the Navier-Stokes equation

$$\frac{\partial \bar{v}}{\partial t} + (\bar{v} \cdot \bar{\nabla})\bar{v} = -\frac{1}{\rho}\nabla P + \nu\nabla^2\bar{v} + \frac{\nu}{3}\bar{\nabla}(\bar{\nabla} \cdot \bar{v}) + \frac{\bar{F}}{m} \qquad (2.1)$$

where v is velocity, P is pressure, ρ is density, υ is viscosity, F is force and m is mass.

The terms on the left-hand side of the equation represent the fluid acceleration, the last term on the right-hand side represents body forces on the fluid, such as gravity, and the remaining terms represent surface forces, such as pressure and viscous forces. The first step in the derivation is to incorporate the assumptions mathematically. Recall the assumptions as introduced in Chapter 1:

1) Steady-state flow: $\frac{\partial \bar{v}}{\partial t} = \frac{\partial \rho}{\partial t} = 0$

2) Inviscid flow: $\nu = 0$

3) Irrotational flow: $\bar{\nabla} \times \bar{v} = 0$

4) Spherically symmetric flow

5) Isothermal flow

6) No magnetic field

The Navier-Stokes equation then becomes

$$v\frac{dv}{dr} = -\frac{1}{\rho}\frac{dP}{dr} - \frac{GM(1-\Gamma)}{r^2} + \frac{GM\Gamma}{r^2}M(t) \qquad (2.2)$$

where the following definitions apply:

1) $M(t)$ is the force multiplier involved in the force per unit mass due to the radiation force provided by a number of spectral lines (Friend, 1982). Abbott(1977, 1982) calculated the force multiplier and found that it may be approximated by $M(t)=kt^{-\alpha}$. In this work the values used for k and α are 0.18 and 0.61 respectively, as found by Abbott. The function t is the optical depth in the wind (CAK, 1975)

$$t = \sigma_e \rho v_{th} \left(\frac{dv}{dr} \right)^{-1} \qquad (2.3)$$

where σ_e is electron scattering opacity (0.2 cm^2/g in this model) and v_{th} is the mean thermal velocity of the protons in the wind with a temperature equal to the effective temperature of the star.

2) P is the pressure.

3) G is the universal gravitational constant.

4) r is the radius.

5) Γ is the ratio of the star's luminosity to the Eddington luminosity at which the radiation pressure force equals the force of gravity.

$$\Gamma = \frac{\sigma_e L}{4\pi GMc} \qquad (2.4)$$

Substituting for $M(t)$ and t in equation 2.2 we obtain

$$v \frac{dv}{dr} = -\frac{1}{\rho} \frac{dP}{dr} - \frac{GM(1-\Gamma)}{r^2} + \frac{GM\Gamma}{r^2} k \left(\frac{dv}{dr} \frac{1}{\sigma_e \rho v_{th}} \right)^{\alpha}$$

We may combine this momentum equation with the mass continuity equation to eliminate ρ, the density, from the last term. The continuity equation states that no material is created or destroyed in the static wind, so the same amount of mass flows per second through a sphere at any distance r from the center of the star.

$$\frac{dM}{dt} \equiv \dot{M} = 4\pi r^2 \rho v$$
$$\rho = \frac{\dot{M}}{4\pi r^2 v}$$

Substituting this in for ρ in the last term of the momentum equation, we obtain

$$v\frac{dv}{dr} = -\frac{1}{\rho}\frac{dP}{dr} - \frac{GM(1-\Gamma)}{r^2} + \frac{GM\Gamma}{r^2}k\left(\frac{dv}{dr}\frac{4\pi r^2 v}{\dot{M}\sigma_e v_{th}}\right)^{\alpha}$$

Combining terms and allowing

$$C = GM\Gamma k\left(\frac{4\pi}{\sigma_e v_{th}\dot{M}}\right)^{\alpha} \qquad (2.5)$$

the general form of the CAK equation becomes

$$v\frac{dv}{dr} = -\frac{1}{\rho}\frac{dP}{dr} - \frac{GM(1-\Gamma)}{r^2} + \frac{C}{r^2}\left(r^2 v\frac{dv}{dr}\right)^{\alpha} \qquad (2.6)$$

To progress further towards solving equation 2.6, we must write the pressure P using an equation of state. CAK uses $P=\rho a^2$ where a is the isothermal sound speed. I choose P to have a temperature dependence by using the ideal gas equation of state:

$$P = \frac{k\rho T}{\mu m_H} \qquad (2.7)$$

where k is Boltzmann's constant, T is temperature of the wind, m_H is the mass of hydrogen, and μ is the mean molecular weight of gas particles in the wind which can be approximated as the following:

$$\frac{1}{\mu} = 2X + \frac{3}{4}Y + \frac{1}{2}Z \qquad (2.8)$$

where X, Y, and Z are the mass fractions of hydrogen, helium and heavy elements in the wind. For O and B star models, these fractions are assumed to be X=0.73, Y=0.24, Z=0.03 (Cranmer, 1996)

The temperature distribution for equation 2.7 is derived from the adiabatic relation as follows:

$$P = k_1\rho^\gamma = k_2\rho T \qquad \text{where } k_1 \text{ and } k_2 \text{ are constants}$$

$$\rho^\gamma = k\rho T$$

$$T = k\rho^{\gamma-1} \qquad \text{(a)}$$

Using the continuity equation

$$4\pi r_1^2\rho_1 v_1 = 4\pi r_2^2\rho_2 v_2$$

We get

$$\frac{\rho_1}{\rho_2} = \frac{r_2^2 v_2}{r_1^2 v_1} \qquad \text{(b)}$$

If we write (a) as

$$\frac{T_2}{T_1} = \frac{k\rho_2^{\gamma-1}}{k\rho_1^{\gamma-1}},$$

substitute equation (b) into it, we get

$$\frac{T_2}{T_1} = \left(\frac{r_1^2 v_1}{r_2^2 v_2^2}\right)^{\gamma-1}$$

Which can be rewritten as

$$T = T_o \left(\frac{r_*{}^2 a}{r^2 v}\right)^{\gamma-1} \qquad (2.9)$$

in which r_* is the radius to the star's photosphere, and T_o is the temperature at the star's photosphere. Hence, I substitute

$$P = k\rho T_o \left(\frac{r_*{}^2 a}{r^2 v}\right)^{\gamma-1} (\mu m_H)^{-1} \qquad (2.10)$$

into equation 2.6 with the new variable $A = k/\mu m_H$ and obtain

$$-2AT_o\gamma\left(\frac{r_*{}^2 a}{r^2 v}\right)^{\gamma-1} r - AT_o\gamma\left(\frac{r_*{}^2 a}{r^2 v}\right)^{\gamma-1} \frac{v'}{v} r^2 + r^2 vv' + GM(1-\Gamma) - C(r^2 vv')^{\alpha} = 0 \qquad (2.11)$$

where $dv/dr = v'$. Combining the r terms and defining W as $W = AT_0\gamma(r_*^2 a)^{\gamma-1}$, I obtain

$$F = \frac{-2W}{r^{2\gamma-3}v^{\gamma-1}} - \frac{Wv'}{r^{2\gamma-4}v^{\gamma}} + r^2vv' + GM(1-\Gamma) - Cf(r,v,v')(r^2vv')^{\alpha} = 0 \qquad (2.12)$$

where I have included the finite disk factor of Friend and Abbott(1986).

$$f = \frac{\frac{rv'}{v} - \left(\frac{v}{rv'}\right)^{\alpha}(1+\sigma\mu_c^{\ 2})^{1+\alpha}}{(1+\alpha)\sigma\frac{r_*^2}{r^2}} \qquad (2.13)$$

where

$$\sigma = \frac{rdv}{vdr} - 1 \qquad \text{and} \qquad \mu_c = \sqrt{1 - \frac{r_*^2}{r^2}}$$

(all the required derivatives of the functions f and F may be found in appendix A).

Equation 2.12 may be written in a simplified form to assist in the graphical analysis to follow:

$$N(r,v)v' + H(r,v) = Q(r,v)v'^{\alpha} \qquad (2.12a)$$

Equation 2.12 is nonlinear, so for a given r and v there is not a unique solution for v' and there is no analytic solution. In order to determine the solution numerically, I follow the graphical solution technique of CAK (see figure 2.1). For equation 2.12, we find that there can be zero, one or two solutions depending on the values (positive or negative) of N, H and Q in equation 2.12a (Friend, 1982).

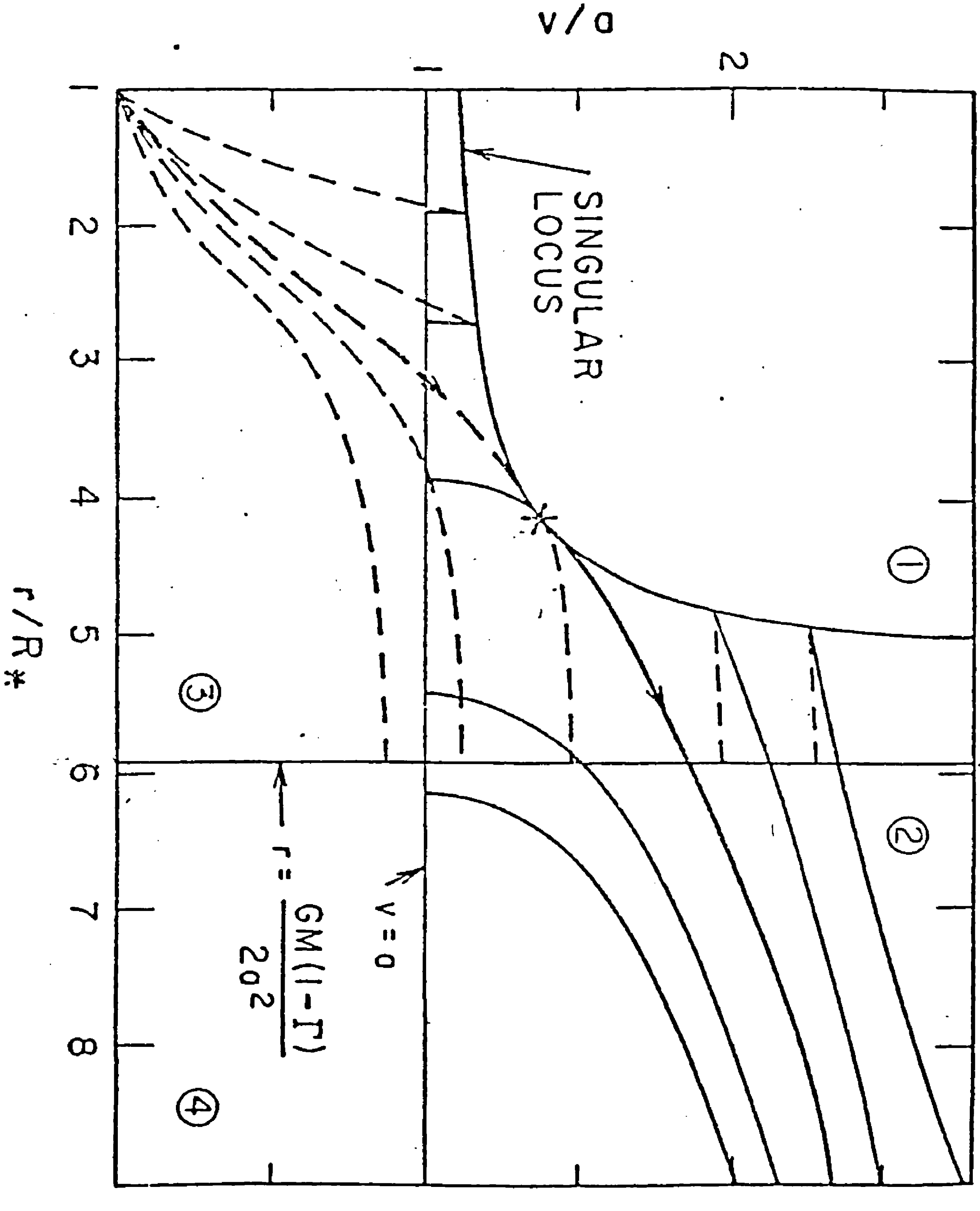


Figure 2.1 Graphical Solution of the Equation of Motion

Region 1 in figure 2.1 represents the area where $r<r_0$ and $v>a$. This region is the only area in which two solutions are possible—N and H are both positive. In region 1 is also the singular locus, defined by equation 2.15 below, which separates the area of no solutions from this area of two solutions.

Region 2 represents the area where $r>r_0$ and $v>a$, or where $H<0$ and $N>0$, and there is only one solution possible. Region 3 is the area where $r<r_0$ and $v<a$, $H>0$ and $N<0$, and once again there is only one solution possible. Region 4 has no solutions for subsonic velocities at radii far from the star ($r>r_0$ and $v<a$).

Examining figure 2.1 leads us to the conclusion that our unique wind solution must progress smoothly from subsonic to supersonic velocities and must graze the singular locus. How do we determine this solution analytically? The condition that the solution touch the singular locus in region 1 tells us that equation 2.12 has only one solution at this singular or *critical* point given by the singularity condition—equation 2.15. The condition that the velocity gradient be continuous at the critical point gives us the regularity condition—equation 2.16 (derived from dF/dr=0). Combining the momentum equation (2.12) with the singularity and regularity conditions we have

$$F = 0 \qquad (2.14)$$

$$\frac{\partial F}{\partial v'} = 0 \qquad (2.15)$$

$$\frac{\partial F}{\partial r} + \frac{\partial F}{\partial v} v' = 0 \qquad (2.16)$$

These equations enable us to solve for r, v, and v′ at the critical point. This is the only point through which the correct solution for the wind will pass (figure 2.1)

FINDING THE CRITICAL POINT

There are several steps necessary to solve for r, v, and v′ at the critical point:

(The Fortran code used to solve for the critical point and find the entire wind solution may be found in Appendix B)

1) Solve equation 2.12 for C.

2) Calculate the derivative of equation 2.15 and substitute in for C to get

$$f_1 = \frac{-W}{r^{2\gamma-4}v^{\gamma}} + r^2 v - \left(f_{v'}f^{-1} + \frac{\alpha}{v'}\right)N = 0 \qquad (2.17)$$

where

$$N = \frac{-2W}{r^{2\gamma-3}v^{\gamma-1}} - \frac{Wv'}{r^{2\gamma-4}v^{\gamma}} + r^2 vv' + GM(1-\Gamma)$$

3) Calculate the derivatives of equation 2.16 and substitute in for C to get

$$f_2 = \frac{-2W(-2\gamma+3)}{r^{2\gamma-2}v^{\gamma-1}} - \frac{W(-2\gamma+4)v'}{r^{2\gamma-3}v^{\gamma}} + 2rvv' + v'\left\{\frac{-2W(-\gamma+1)}{r^{2\gamma-3}v^{\gamma}} + \frac{\gamma Wv'}{r^{2\gamma-4}v^{\gamma+1}} + r^2 v'\right\}$$

$$- Nf^{-1}\left(v'f_v + \frac{\alpha f_{v'}}{v} + f_r + \frac{2\alpha}{r}f\right) = 0 \qquad (2.18)$$

4) Solve the two equations, f_1 and f_2, using coupled iterations as follows. The total derivatives of f_1 and f_2 will be equal to zero. With x=v and y=v′, we obtain

$$0 = f_1 + \frac{\partial f_1}{\partial x}dx + \frac{\partial f_1}{\partial y}dy$$

$$0 = f_2 + \frac{\partial f_2}{\partial x}dx + \frac{\partial f_2}{\partial y}dy$$

Let us rewrite these equations in matrix form with the definitions

We know f_1 and f_2, and we can calculate a, b, c, and d (partial derivatives of the finite disk factor). Consequently we can solve for dx and dy:

$$a = \frac{\partial f_1}{\partial x}, \quad b = \frac{\partial f_1}{\partial y}, \quad c = \frac{\partial f_2}{\partial x}, \quad d = \frac{\partial f_2}{\partial y}$$

$$0 = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} + \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix}$$

$$dx = \frac{1}{ad - bc}(bf_2 - df_1) \qquad dy = \frac{1}{ad - bc}(cf_1 - af_2)$$

5) To find x and y (or v and v′) at the critical point, we do the following:

a) Select a critical radius r_{crit}.

b) Make initial guesses for x and y (v_{crit} and v'_{crit})

c) Solve for dx and dy.

d) Calculate dx/x and dy/y. If one or both of these is greater than 1×10^{-5}, then increment x by dx and y by dy and solve for dx and dy again.

e) When *both* dx/x and dy/y are less than 1×10^{-5}, v_{crit} and v'_{crit} have been found for the given critical radius.

The critical point analysis also allows us to determine the mass loss rate by solving equation 2.5 (as long as the correct value for the critical radius was chosen. How we know this is discussed in the next section).

$$\dot{M} = \frac{4\pi}{\sigma_e v_{th}} \left(\frac{GM\Gamma k}{C} \right)^{1/\alpha} \qquad (2.19)$$

in which C is calculated from equation 2.12.

FINDING THE COMPLETE WIND SOLUTION

Upon examination of figure 2.1 we see that the correct wind solution passes through the critical radius. Our equation of motion (2.12) may now be numerically integrated outward and inward from this critical radius. In order to be certain we are on the correct solution branch, we begin with the velocity gradient calculated in the critical point subroutine and step outwards along a solution branch. If we do not step "steeply" enough, the integration will "crash." Initial step size adjustments must then be made in the computer code to ensure progression along the correct solution branch. The integration continues outwards to a specified radius (~40 stellar radii). The integration then returns to the critical point and continues inwards to the photosphere, or "surface" of the star. We define the photosphere (r=1 stellar radius) to be at the point where the optical depth is 2/3 (this is in fact how the photosphere is defined). If the calculated value for the radius does not equal unity when the optical depth is 2/3, we know that our critical radius was either too far from or too near the star. If this is the case, we simply adjust r_{crit} and run the integration again until r=1 when optical depth=2/3. We have then found a complete stellar wind solution.

Chapter 3

Results

The results of the numerical solution of the wind equation of motion will now be discussed for a variety of stellar parameters. Of primary interest is the comparison between this non-isothermal, finite-disk model and the finite-disk model of Friend and Abbott(1986). I am particularly curious whether or not the isothermal assumption is valid for these very hot stars.

What does a stellar wind solution look like? Figure 3.1 shows how the velocity changes with radius for an O9.5III star (the general shape of the curve remained the same for all stars I examined). Note that the velocity asymptotically approaches a value known as the terminal velocity.

Figure 3.1a displays a typical temperature vs. radius curve for the same star. Note that the temperature increases rapidly as we approach the star's surface, but that we find the effective temperature to be very near the photosphere.

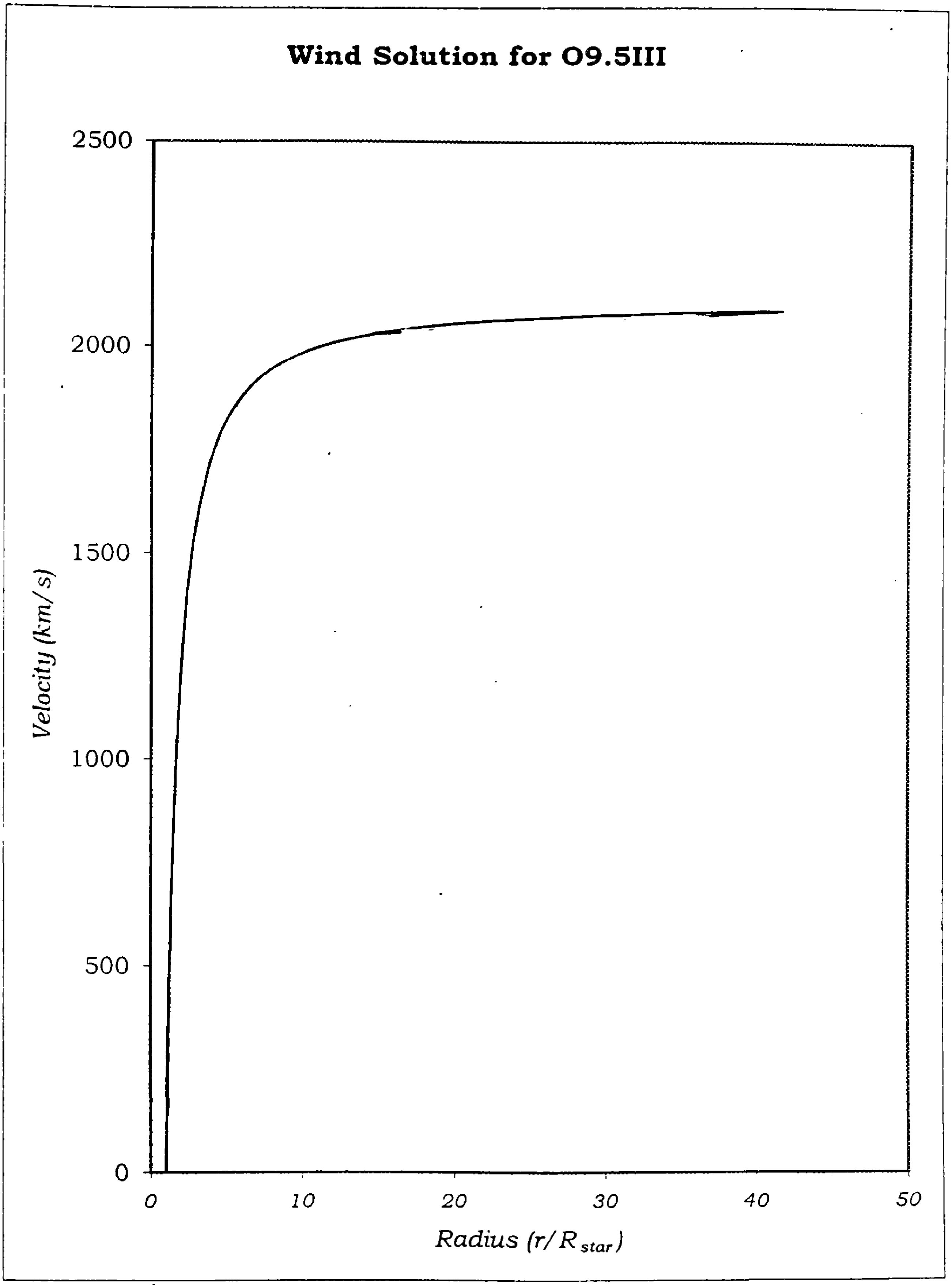

Wind Solution for O9.5III
Velocity (km/s)
2500
2000
1500
1000
500
0
0
10
20
30
40
50
Radius (r/R star)

Figure 3.1

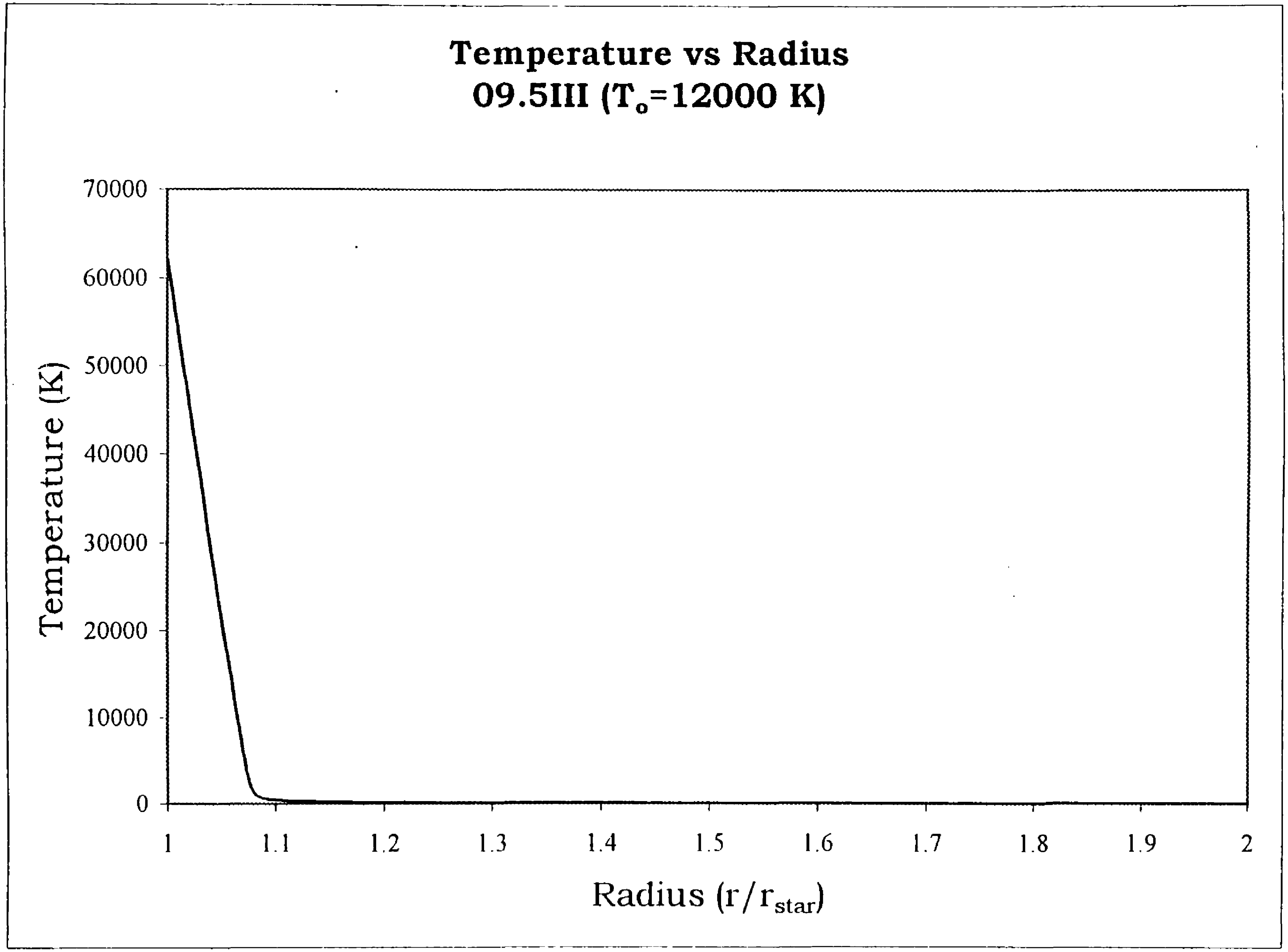

Temperature vs Radius
09.5III (T$_o$=12000 K)
Temperature (K)
70000
60000
50000
40000
30000
20000
10000
0
1
1.1
1.2
1.3
1.4
1.5
1.6
1.7
1.8
1.9
2
Radius (r/r$_{star}$)

Figure 3.1a

TEMPERATURE VARIATION FOR FIXED STELLAR PARAMETERS

As a first test of the non-isothermal code, I have looked at several different stellar types holding their parameters constant while varying the temperature for each case. The parameters used are given in table 3.1 (Drew, 1989). The wind solutions for two of these stars are illustrated in Figures 3.2 and 3.3. The temperature variation makes a difference in the terminal velocity of the star's wind on the order of 10%.

Table 3.1
Basic Stellar Parameters

Spectral Type	Temp (K)	L/Lsun	R/Rsun	log g	dM/dt (M_{sun}/yr)	v_{term} (km/s)
Supergiants						
O9.5 I	30000	3.0E+05	20	3.0	4.8E-07	1800
O9 I	32500	4.1E+05	20	3.3	8.3E-07	1800
O7.5 I	35000	6.3E+05	22	3.3	1.8E-06	1800
O6.5 I	37500	9.4E+05	23	3.5	3.5E-06	2300
O5 I	40000	1.2E+06	23	3.5	5.5E-06	2300
O4 I	45000	1.3E+06	19	4.0	6.0E-06	2900
O3 I	50000	2.0E+06	19	4.0	1.2E-06	2400
Giants						
O9.5 III	30000	1.2E+05	13	3.5	1.0E-07	2900
O9 III	32500	1.7E+05	13	3.8	1.8E-07	2900
O7.5 III	35000	2.3E+05	13	4.0	3.0E-07	2900
O6.5 III	37500	3.7E+05	14	4.0	7.0E-07	2900
O6 III	40000	4.8E+05	14	4.0	1.1E-06	2900
Main Sequence Dwarfs						
B0 V	30000	5.4E+04	8.5	3.5	2.6E-08	3200
O9 V	32500	7.5E+04	8.5	3.8	4.5E-08	3200
O7 V	35000	1.0E+05	8.5	4.0	7.5E-08	3200
O8 V	37500	1.8E+05	10	4.0	2.0E-07	3200
O6.5 V	40000	2.3E+05	10	4.0	3.2E-07	3200
O5 V	45000	4.9E+05	11	4.0	1.1E-06	3200
O3 V	50000	7.5E+05	11	4.0	2.3E-06	3000

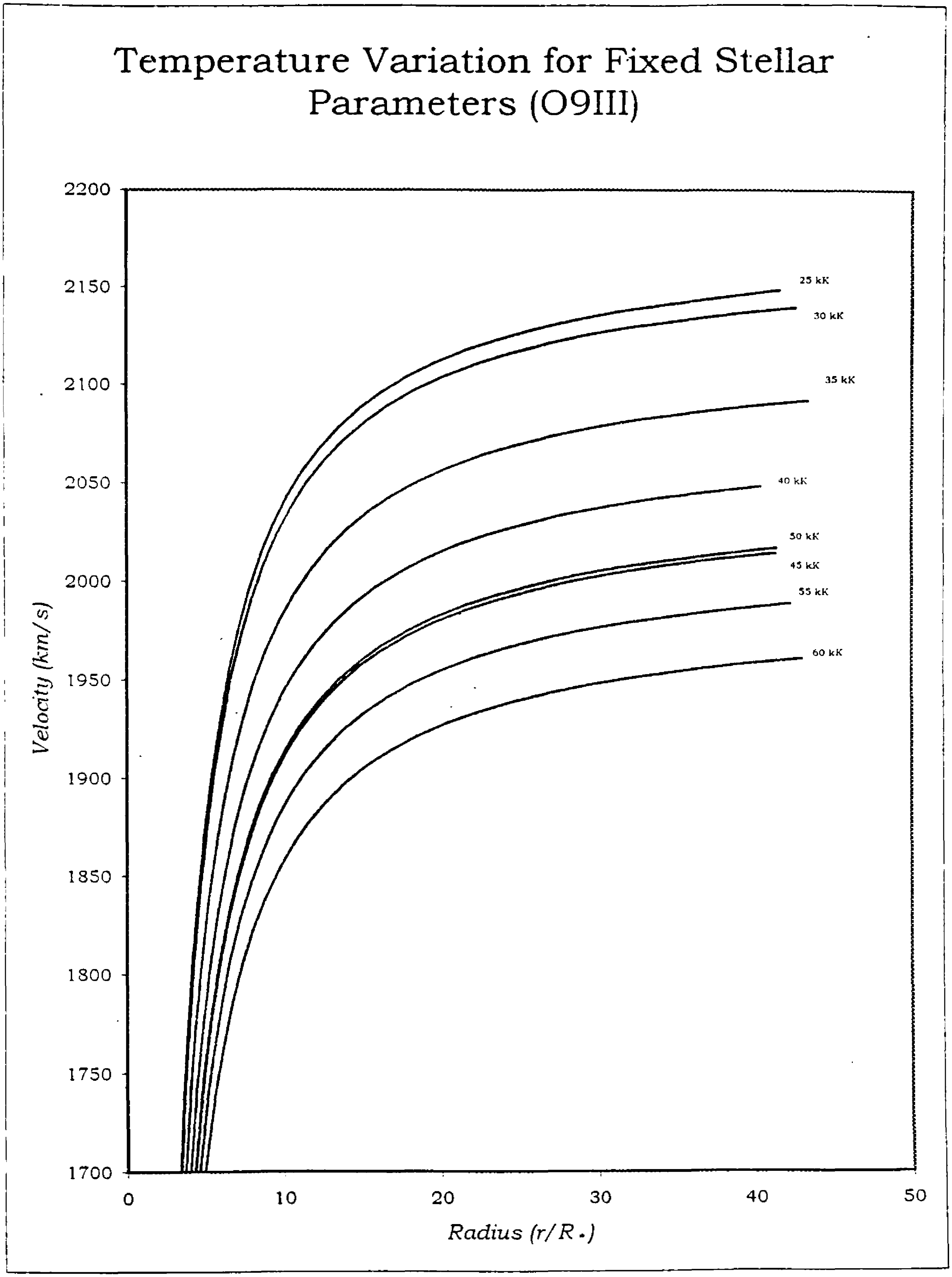

Temperature Variation for Fixed Stellar Parameters (O9III)
25 kK
30 kK
35 kK
40 kK
50 kK
45 kK
55 kK
60 kK
2200
2150
2100
2050
2000
1950
1900
1850
1800
1750
1700
Velocity (km/s)
0
10
20
30
40
50
Radius (r/R*)

Figure 3.2

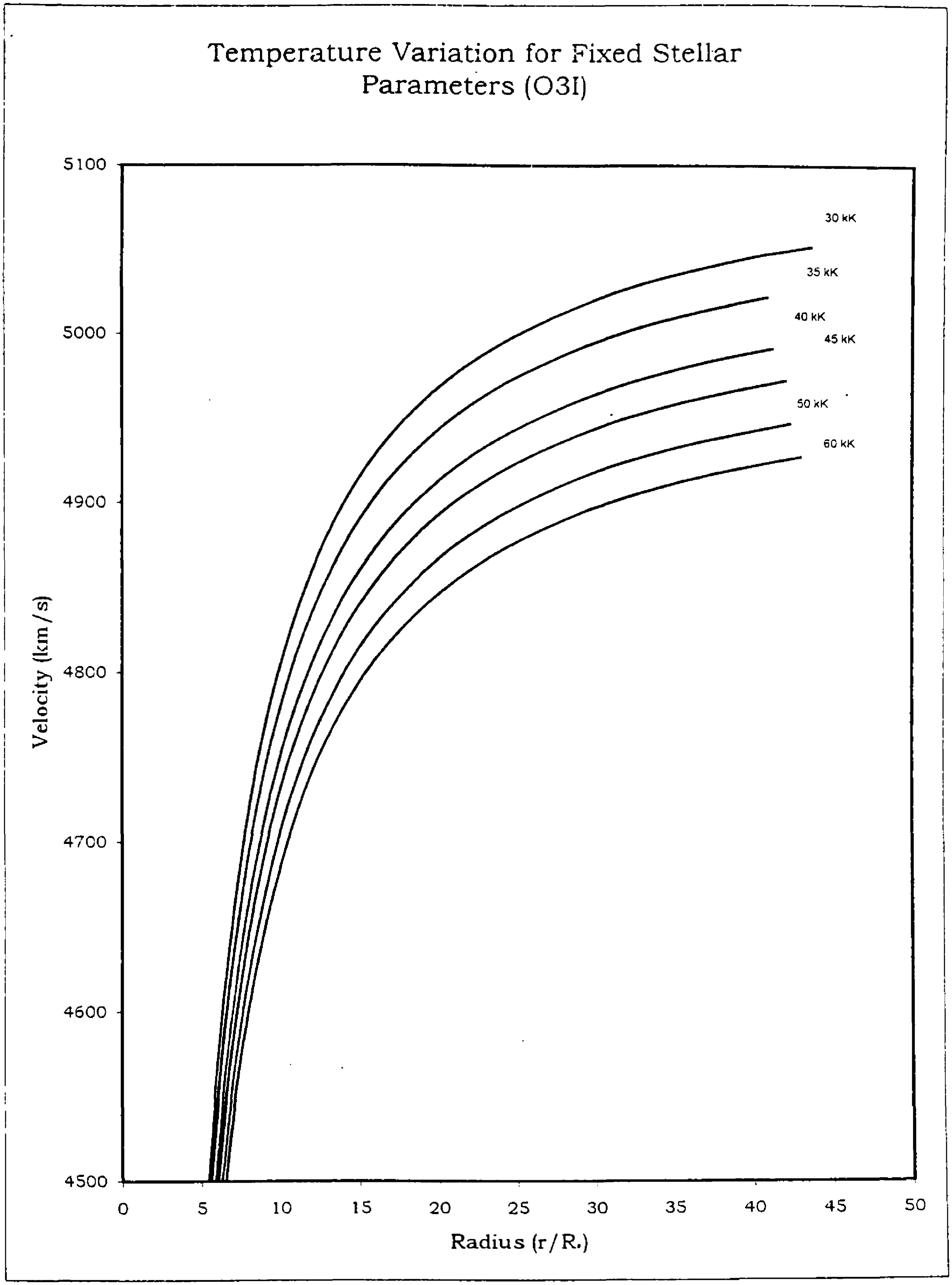
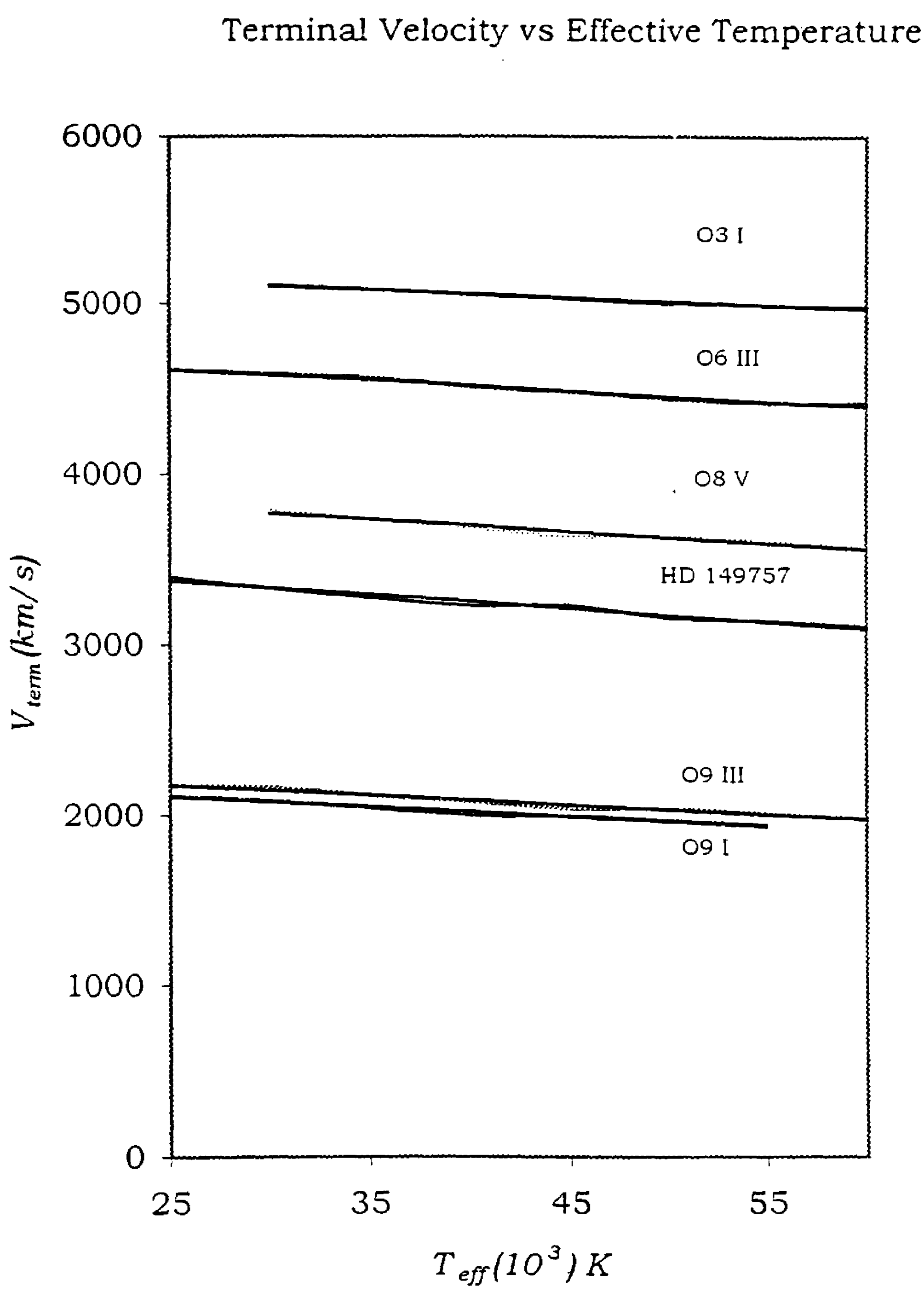

Temperature Variation for Fixed Stellar
Parameters (O3I)
5100
5000
4900
4800
4700
4600
4500
Velocity (km/s)
30 kK
35 kK
40 kK
45 kK
50 kK
60 kK
0
5
10
15
20
25
30
35
40
45
50
Radius (r/R.)

Figure 3.3

TERMINAL VELOCITY AND MASS LOSS VS EFFECTIVE TEMPERATURE

Figures 3.4 and 3.5 summarize the effects of temperature variation for fixed parameters on the terminal velocity and mass loss rate of the star. Mass loss was found to increase as the temperature increased, while the terminal velocity of the wind decreases as the temperature increases (the more mass the star is trying to expel, the slower the wind may go). The terminal velocity decreases on the order of 7% and the mass loss increases on the order of 5% over the range of temperatures in these samples.



Vterm = -4.26Teff + 5231 **O3I**
Vterm= -6.6714x + 3965.2 **O8V**
Vterm = -5.85Teff + 4755 **O6III**
Vterm = -7.80Teff + 3565 **HD 149757**
Vterm = -5.62Teff + 2248 **O9I**
Vterm = -5.58Teff + 2313 **O9III**

Figure 3.4

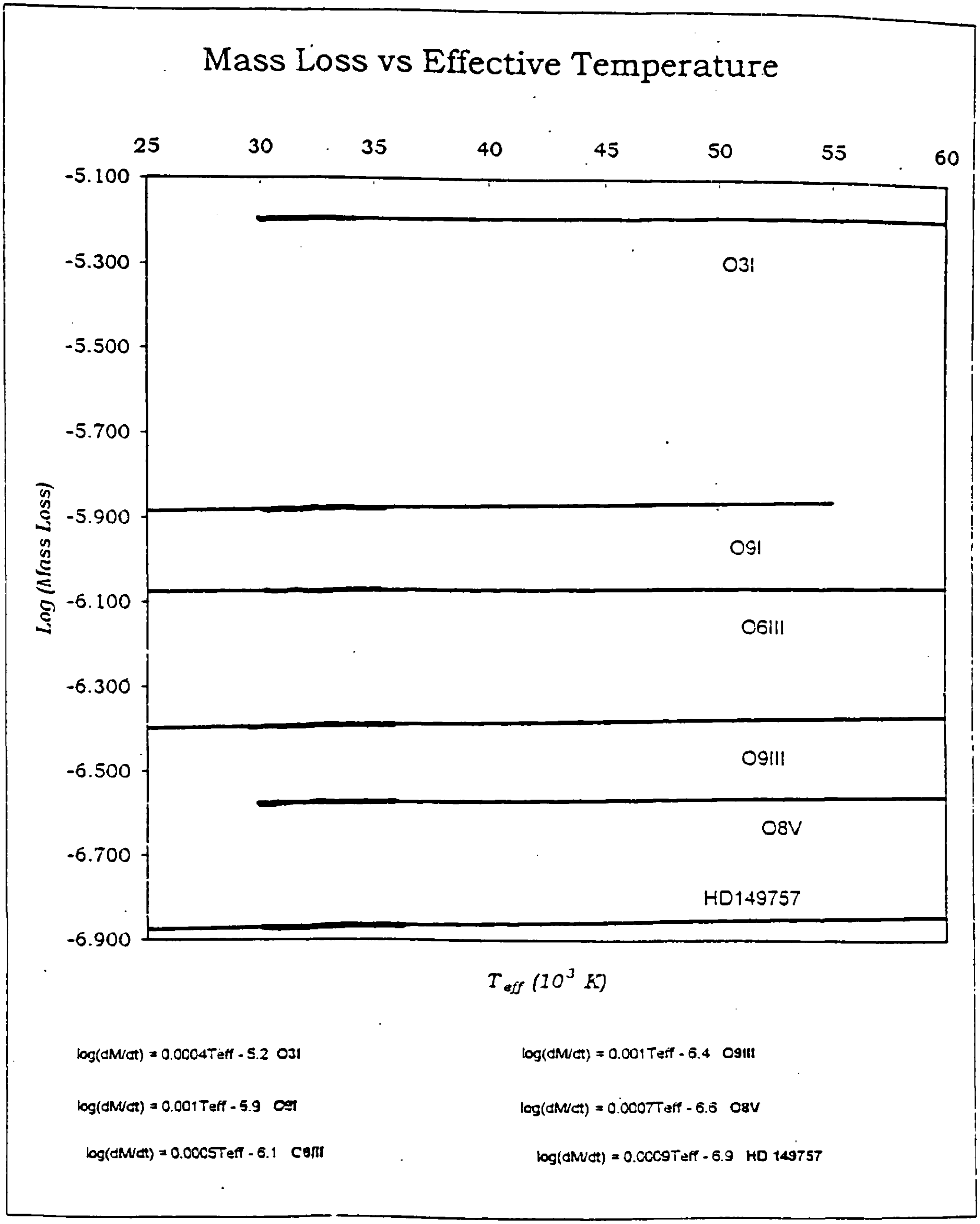

Mass Loss vs Effective Temperature
Log (Mass Loss)
-5.100
-5.300
-5.500
-5.700
-5.900
-6.100
-6.300
-6.500
-6.700
-6.900
25
30
35
40
45
50
55
60
O3I
O9I
O6III
O9III
O8V
HD149757
Teff (10³ K)
log(dM/dt) = 0.0004Teff - 5.2 O3I
log(dM/dt) = 0.001Teff - 5.9 O9I
log(dM/dt) = 0.0005Teff - 6.1 O6III
log(dM/dt) = 0.001Teff - 6.4 O9III
log(dM/dt) = 0.0007Teff - 6.6 O8V
log(dM/dt) = 0.0009Teff - 6.9 HD 149757

Figure 3.5

MASS LOSS VS. LUMINOSITY

For the second exploration, I have calculated the mass loss rate versus luminosity for each spectral type listed in table 3.1. The calculations are made for both my model and the finite disk model and are presented graphically in figure 3.6. Friend and Aboott (1986) sought to show this important correlation between mass loss and luminosity. Observationally, mass loss rate increases with the luminosity, roughly as a power law. Both the Friend and Abbott model and my model show such a correlation. There are few discrepancies between the results of the two models, and the slopes of the linear fits are within 4%.

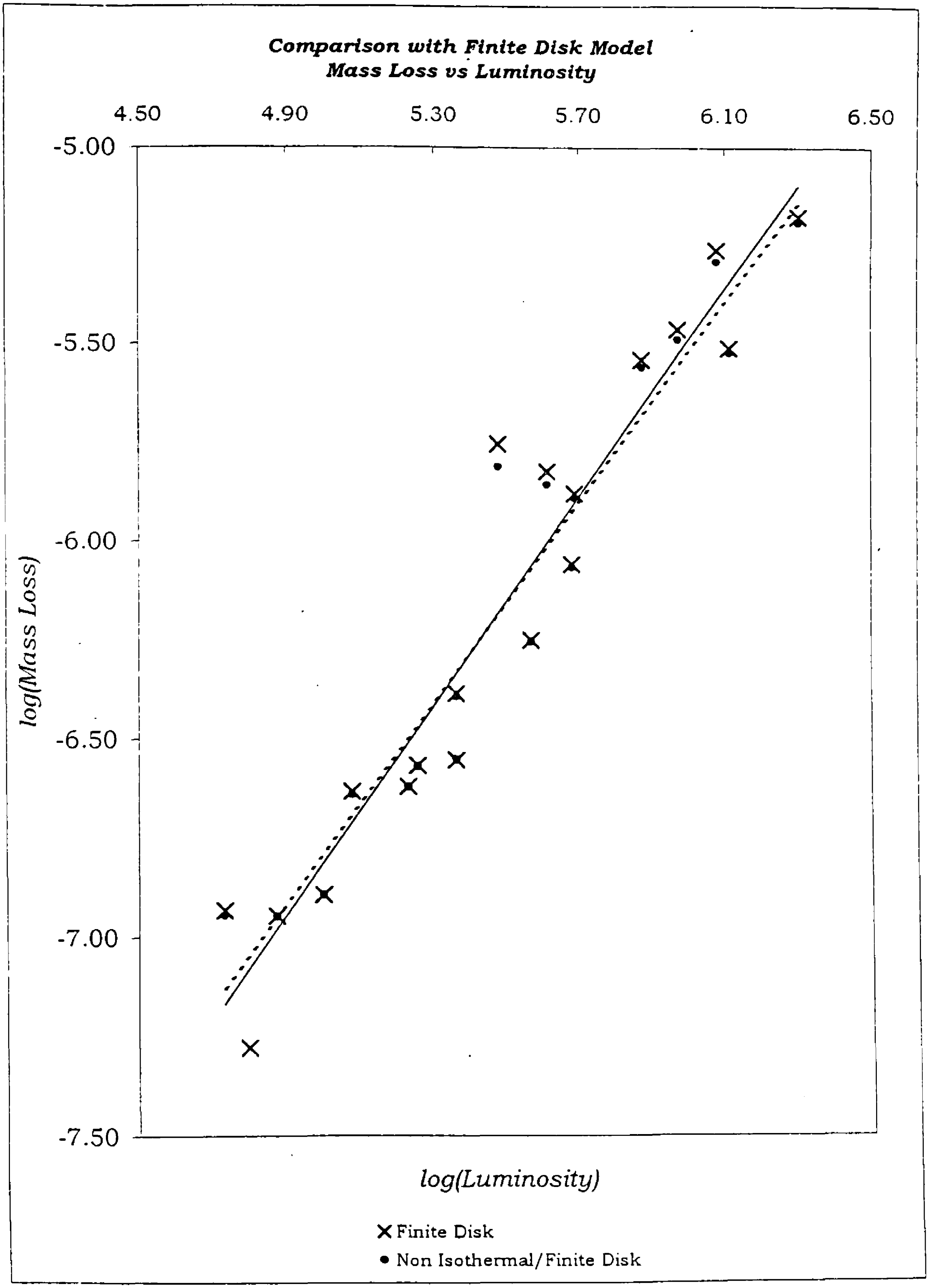

Comparison with Finite Disk Model
Mass Loss vs Luminosity
4.50
4.90
5.30
5.70
6.10
6.50
-5.00
-5.50
-6.00
-6.50
-7.00
-7.50
log(Mass Loss)
log(Luminosity)
Finite Disk
Non Isothermal/Finite Disk

Figure 3.6

Chapter 4

Conclusions

The field of radiatively-driven wind studies has grown immensely in its complexity since the days of Castor, Abbott and Klein. Most of today's current theories are based on the original CAK theory in which the researchers whittle away at its simplifying assumptions.

My graphical results are summarized as follows:

1) Increasing the effective temperature for a particular star decreases the wind's terminal velocity and increases the mass loss rate of the wind. Certainly increasing the temperature of the star's photosphere results in more mass being expelled. This increase in mass loss has the net effect of slowing down the wind, thus reducing the terminal velocity.
2) The mass loss rate is found to grow, nearly, as a power law with the luminosity of the star. My mass loss-luminosity results compare quite well with those of the Friend and Abbott model.

Comparisons of the non-isothermal, finite disk model and the finite disk model of Friend and Abbott clearly demonstrate, for the data analyzed, that the isothermal assumption, which many theories use, is a reasonable approximation. Initially, it seemed that the winds of these very hot stars should be affected by such extreme temperatures. I expected that the mass loss rates would rise and the terminal velocities drop in comparison with the isothermal, finite disk results. It turned out that the terms in the equation of motion (2.12) which are affected by the temperature—the first two terms—were consistently several orders of magnitude smaller than the last term in equation 2.12, the radiation force term. After examining these terms, I was certainly not surprised that the temperature dependence had a negligible effect.

My results match some observations more closely than the Friend and Abbott model (see below), but our results rarely differ by more than a few percent. The observational data in the chart below has been taken from Drew (1989).

Star	**V_{term} (km/s) Non-Isothermal Model**	**V_{term} (km/s) Finite Disk Model**	**V_{term} (km/s) Observation**
O7.5 I	2120	2277	1800
O7.5 III	4334	4378	2900
O7 V	3705	3719	3200
HD 37043	2545	2440	2450
HD 37128	1867	1692	1500

My results indicate that the inclusion of a temperature dependence is not a necessary complication in order to improve the accuracy of results. Allowing the wind to be isothermal reduces complexity in the equation of momentum, thus easing (a bit) the computations of additional effects such as magnetic field, rotating winds, pulsations and a variety of other phenomena.

Appendix A
Derivatives of
F and f

$$f = \frac{\frac{rv'}{v} - \left(\frac{v}{rv'}\right)^{\alpha} \left(1 + \sigma\mu c^2\right)^{1+\alpha}}{\left(1+\alpha\right)\sigma \frac{r_*^{\,2}}{r^2}}$$

$$f_{v'} = \frac{r}{v\sigma}\left\{\frac{1 - \left(\frac{v}{rv'}\right)^{\alpha}\left(1+\sigma\mu_c^2\right)^{\alpha}\left[\left(1+\alpha\right)\mu_c^2 - \frac{\alpha v}{rv'}\left(1+\sigma\mu_c^2\right)\right]}{\left(1+\alpha\right)\frac{r_*^2}{r^2}}\right\} - f$$

$$f_{v'v'} = \frac{-r}{v\sigma}\left\{\frac{\alpha}{v'}\left(\frac{v}{rv'}\right)^{\alpha}\left(1+\sigma\mu_c^2\right)^{\alpha}\left(\frac{v}{rv'} - \frac{\mu_c^2}{1+\sigma\mu_c^2}\right) + 2f_{v'}\right\}$$

$$f_v = \frac{-rv'}{\sigma v^2}\left\{\frac{1 - \left(\frac{v}{rv'}\right)^{\alpha}\left(1+\sigma\mu_c^2\right)^{\alpha}\left(\mu_c^2 - \frac{\alpha v}{rv'}\frac{r_*^2}{r^2}\right)}{\left(1+\alpha\right)\frac{r_*^2}{r^2}} - f\right\}$$

$$f_{vv}=\frac{1}{v}\left\{2\left(\frac{rv'}{\sigma v}-1\right)+\frac{\alpha}{(1+\alpha)\sigma v}\left(\frac{v}{rv'}\right)^{\alpha}\left(1+\sigma\mu_c^2\right)^{\alpha}\left[\frac{rv'}{v}\left(\frac{\mu_c^2-\frac{\alpha v}{rv'}\frac{r_*^2}{r^2}}{1+\sigma\mu_c^2}\right)-1\right]\right\}$$

$$f_{vv'}=\frac{r}{\sigma v^2}\left\{\frac{\alpha\left(\frac{v}{rv'}\right)^{\alpha}\left(1+\sigma\mu_c^2\right)^{\alpha}}{(1+\alpha)}\left(\frac{v}{rv'}-\frac{\mu_c^2-\frac{\alpha v}{rv'}\frac{r_*^2}{r^2}}{1+\sigma\mu_c^2}\right)+v'f_{v'}-\frac{v^2}{rv'}f_v\right\}$$

$$f_r=\frac{v'}{\sigma v}\left\{\frac{3-\left(\frac{v}{rv'}\right)^{\alpha}\left(1+\sigma\mu_c^2\right)^{\alpha}\left[(1+\alpha)\left(\mu_c^2+2\sigma\frac{r_*^2}{r^2}\frac{v}{rv'}\right)-\frac{(2-\alpha)v}{rv'}\left(1+\sigma\mu_c^2\right)^{\alpha}\right]}{(1+\alpha)\frac{r_*^2}{r^2}}-f\right\}$$

$$f_{rv} = \frac{-v'}{\sigma v^2}\left\{\left(\frac{v}{rv'}\right)^{\alpha}\left(1+\sigma\mu_c^2\right)^{\alpha}\left[\frac{(\alpha-4)v}{rv'} + \frac{\alpha\left(\mu_c^2 + 2\sigma\frac{r_*^2}{r^2}\frac{v}{rv'}\right)}{1+\sigma\mu_c^2}\right] + vf_v - rf_r\right\}$$

$$f_{rv'} = \frac{1}{\sigma v}\left\{\left(\frac{v}{rv'}\right)^{\alpha}\left(1+\sigma\mu_c^2\right)^{\alpha}\left[\frac{(\alpha-4)v}{rv'} + \frac{\alpha\left(\mu_c^2 + 2\sigma\frac{r_*^2}{r^2}\frac{v}{rv'}\right)}{1+\sigma\mu_c^2}\right] - \left(v'f_{v'} + \frac{v}{v'}f_r\right)\right\}$$

$$f_{rr} = \frac{-v'}{r\sigma v}\left\{\frac{\left(\frac{v}{rv'}\right)^{\alpha}\left(1+\sigma\mu_c^2\right)^{\alpha} v}{(1+\alpha)rv'}\left\langle\frac{4rv'}{v}(1+\alpha) - 2\sigma(2\sigma+5) + 2 - \alpha + \alpha(2\sigma-1)\left[\frac{(1+\alpha)\left(\mu_c^2\frac{rv'}{v} + 2\sigma\frac{r_*^2}{r^2}\right)}{1+\sigma\mu_c^2} - 2 + \alpha\right]\right\rangle + 2rf_r\left(1+\frac{\sigma v}{rv'}\right) + 2f\right\}$$

$$F=\frac{-2W}{r^{2\gamma-3}v^{\gamma-1}}-\frac{Wv'}{r^{2\gamma-4}v^{\gamma}}+r^2vv'+GM(1-\Gamma)-Cf\left(r^2vv'\right)^{\alpha}$$

$$F_r=\frac{-2W(-2\gamma+3)}{r^{2\gamma-2}v^{\gamma-1}}-\frac{W(-2\gamma+4)v'}{r^{2\gamma-3}v^{\gamma}}+2rvv'-Cf_r\left(r^2vv'\right)^{\alpha}-\frac{2Cf\alpha\left(r^2vv'\right)^{\alpha}}{r}$$

$$F_{rr}=\frac{-2W(-2\gamma+3)(-2\gamma+2)}{r^{2\gamma-1}v^{\gamma-1}}-\frac{W(-2\gamma+4)(-2\gamma+3)v'}{r^{2\gamma-2}v^{\gamma}}+2vv'-C\left\{f_{rr}\left(r^2vv'\right)^{\alpha}+\frac{4\alpha f_r\left(r^2vv'\right)^{\alpha}}{r}+2f\alpha\left(r^2vv'\right)^{\alpha}\left(\frac{2\alpha-1}{r^2}\right)\right\}$$

$$F_{rv}=\frac{-2W(-2\gamma+3)(-\gamma+1)}{r^{2\gamma-2}v^{\gamma}}+\frac{W(-2\gamma+4)\gamma v'}{r^{2\gamma-3}v^{\gamma+1}}+2rv'-C\left(r^2vv'\right)^{\alpha}\left(f_{rv}+\frac{\alpha f_v}{v}+\frac{2\alpha f_v}{r}+\frac{2\alpha^2 f}{v}\right)$$

$$F_{rv'}=\frac{-W(-2\gamma+4)}{r^{2\gamma-3}v^{\gamma}}+2rv-C\left(r^2vv'\right)^{\alpha}\left(f_{rv'}+\frac{\alpha f_r}{v'}+\frac{2\alpha f_{v'}}{r}+\frac{2\alpha^2 f}{rv'}\right)$$

$$F_v=\frac{-2W(-\gamma+1)}{r^{2\gamma-3}v^{\gamma}}+\frac{Wv'\gamma}{r^{2\gamma-4}v^{\gamma+1}}+r^2v'-Cf_v\left(r^2vv'\right)^{\alpha}-\frac{Cf\alpha\left(r^2vv'\right)^{\alpha}}{v}$$

$$F_{vv}=\frac{2W(-\gamma+1)\gamma}{r^{2\gamma-3}v^{\gamma+1}}-\frac{Wv'\gamma(\gamma+1)}{r^{2\gamma-4}v^{\gamma+1}}-C\left(r^2vv'\right)^{\alpha}\left(f_{vv}+\frac{2\alpha f_v}{v}+\frac{\alpha f(\alpha-1)}{v^2}\right)$$

$$F_{vv'} = \frac{W\gamma}{r^{2\gamma-4} v^{\gamma+1}} + r^2 - C\left(r^2 v v'\right)^{\alpha}\left(f_{vv'} + \frac{\alpha f_v}{v'} + \frac{\alpha f_{v'}}{v} + \frac{\alpha^2 f}{v v'}\right)$$

$$F_{v'v'} = -C\left(r^2 v v'\right)^{\alpha}\left(f_{v'v'} + \frac{2\alpha f_{v'}}{v'} + \frac{f\alpha(\alpha-1)}{v'^2}\right)$$

Appendix B
The Model's
Fortran Code

```
C     CAK MODEL WITH FINITE DISK FOR STAR
      USE MSFLIB

      COMMON/IDAT/DMDTDO,SIGE,ONEMAL,ALO1MA,TWOALP,VTH,GAMGMK,TWOMAL,GAM
     2,GM,F5,GAMC
      COMMON/DERM/FLAG,ALPHA,RALP,C,F7,F8,R2,ARP,ONEPAL,VPLAST,FC,
     2RPORS,TNOT,WN,rccrit,vpcguess


      DIMENSION V(2),VP(2),WORK(20),IWORK(5),RORP(100),VKMS(100),
     2RDUM(50),VDUM(50)
      EXTERNAL DER
      LOGICAL FLAG
      Real L


      INTEGER(4) frequency, duration
      frequency = 1000
      duration = 333
      A=3.E6
      A2=A*A
      AKMS=A/1.E5
      VTH=A
      ALPHA=.61
      RALP=1./ALPHA
      ONEPAL=1.+ALPHA
      ONEMAL=1.-ALPHA
      TWOMAL=2.-ALPHA
      ALO1MA=ALPHA/ONEMAL
      F7=-1./ONEMAL
      F8=ALPHA*ONEMAL**(1./ALO1MA)
      TWOALP=2.*ALPHA
      XK=.18
      OPEN(UNIT=7,FILE='finite2.dat',STATUS='REPLACE',ERR=110)
      PRINT 9000
 9000 FORMAT('INPUT DATA: rcfac,rpors,mass')
      READ(*,*)rcfac

 5200 vc=1.3
      vpc=100.
      L=1.*10**5.70
      RPORS=11.
c     GRAV=10**(4.0)
c     XM=(GRAV/1.E2*(RPORS*6.9599E8)**2/6.67E-11)*1.E3/1.989E33
      xm=52.
c     tnot=((L*3.9e33)/(4*3.14159*(rpors*6.96e10)**2*5.67e-5))**.25
      rporsun=rpors
      PFAC=.2
      GAMC=1.6666667
      TNOT=46400.
      SIGE=.2
      GM=XM*1.3273E26
      GAM=SIGE*L*1.035E22/GM
      GAMGMK=GAM*GM*XK
      F5=GM*(gam-1.)
      RP=RPORS*6.96E10
      RP2=RP*RP
```

```
      AORP=A/RP
      ARP=A*RP
      ARP2=A*RP2
      A2RP=A2*RP
      BOLT=1.38E-16
      HYDROGEN=1.67E-24
      WN=(BOLT/(16.65e-1*HYDROGEN))*TNOT*GAMC*(RPORS*6.96E10)/a2rp
      ITER=1
C
10    DR=1.E-2*RCFAC
      HDR2=.5*DR**2
      CALL CRITPT(RCFAC,VC,VPC,V2)
      open(4,FILE='FIN33.DAT',status='replace',err=110)

      Write(4,*)'rc, vc, vpc',rcfac, vc, vpc

      DMDT=DMDTDO*1.9928E-25
      VCRC2=VC*R2
      IT=1
      WRITE(7,2300)
 2300 FORMAT(5X,'L/Lsun',8X,'R/Rsun',7X,'M/Msun',7X,'PFAC',9X,'DMDT')


      write(4,*)'This is CAK with finite disk for the following values:'

      write(4,*)l,rpors,xm,pfac,dmdt,tnot

      write(4,2301)
 2301 Format(6X,'r/R*',8X,'V(KM/S)',6X,'VP(/S)',6X,
     3'RHO(G/CM3)',6X,'TAU',11X,'F')

      WRITE(7,2000)L,RPORS,XM,PFAC,DMDT

      RHO=DMDTDO/VCRC2/ARP2
      VKMS(1)=VC*AKMS
      VPPS=VPC*AORP
      V(2)=0.
      RORP(1)=RCFAC
      rov=sqrt(r2)/vc
      RVPOV=ROV*VPc
      SIGMA=RVPOV-1.
      onepal=1.+alpha
      U2=1.-1./R2
      Fc=r2*(RVPOV-(1.+SIGMA*U2)**ONEPAL/RVPOV**ALPHA)/ONEPAL/SIGMA

      TEMP=TNOT*(1./(RCFAC**2*VC))**(GAMC-1.)
      WRITE(4,2000)RCFAC,VKMS(1),VPPS,RHO,V(2),FC
C
      open(22,file='rvoutput.dat',status='replace',err=110)
      write(22,*)rcfac,vkms(1),TEMP
 2000 FORMAT(1X,6E13.5)
      FLAG=.TRUE.
      F1=VC+HDR2*V2
      DRVPC=DR*VPC
      F2=-DMDTDO*SIGE*dr/(VCRC2)/ARP
      R=RCFAC+DR
      V(1)=.995*(F1+DRVPC)
```

```
      V(2)=F2
      RND=R+PFAC/(VPC/VC+2./R)
      IFLAG=1
      VPLAST=VPC
      eps=1.0e-3
      epsa=1.0e-6
   22 CALL RKFN(DER,2,V,R,RND,eps,epsa,IFLAG,WORK,IWORK)
      IF(IFLAG.GE.3)WRITE(7,*)'IFLAG=',IFLAG
      R=RND

   23 CALL DER(R,V,VP)

      RHO=DMDTDO/V(1)/R**2/ARP2
      IT=IT+1
      RORP(IT)=R
      VKMS(IT)=V(1)*AKMS
      VPPS=VP(1)*AORP
      rov=r/v(1)
      RVPOV=ROV*VP(1)
      SIGMA=RVPOV-1.
      onepal=1.+alpha
      U2=1.-1./R**2
      Fc=R2*(RVPOV-(1.+SIGMA*U2)**ONEPAL/RVPOV**ALPHA)/ONEPAL/SIGMA
      TEMP=TNOT*(1./(R**2*V(1)))**(GAMC-1.)
      WRITE(4,2000)R,VKMS(IT),VPPS,RHO,V(2),FC
      open(22,file='rvoutput.dat',status='replace',err=110)
      write(22,*)r,vkms(it),TEMP

      RND=R+PFAC/(VP(1)/V(1)+2./R)

      IF(R.LT.40..AND.IT.LT.75)GO TO 22
      IMAX=IT
      VINF=VKMS(IT)/SQRT(1.-1./R)
      WRITE(4,4000)VINF
 4000 FORMAT(/' TERMINAL VELOCITY = ',6E13.5/)
      FLAG=.FALSE.
      dr=1.e-4*rcfac
      drvpc=dr*vpc
      hdr2=.5*dr**2
      f1=vc+hdr2*v2
      R=RCFAC-DR
      V(1)=.995*(F1-DRVPC)
      V(2)=F2-V(2)
      RND=R-PFAC/(VPC/VC+2./R)
      IFLAG=1
      IT=0
   12 CALL RKFN(DER,2,V,R,RND,eps,epsa,IFLAG,WORK,IWORK)
      IF(IFLAG.GE.3)WRITE(7,*)'IFLAG=',IFLAG
 3000 FORMAT(X,'IFLAG= ',I1)
      R=RND
      CALL DER(R,V,VP)
      RHO=DMDTDO/V(1)/R**2/ARP2
      IT=IT+1
      RDUM(IT)=R
      VDUM(IT)=V(1)*AKMS
      VPPS=VP(1)*AORP
      rov=r/v(1)
```

```
      RVPOV=ROV*VP(1)
      SIGMA=RVPOV-1.
      onepal=1.+alpha
      U2=1.-1./R2
      Fc=R2*(RVPOV-(1.+SIGMA*U2)**ONEPAL/RVPOV**ALPHA)/ONEPAL/SIGMA
      WRITE(4,2000)R,VDUM(IT),VPPS,RHO,V(2),FC
      TEMP=TNOT*(1/(R**2*V(1)))**(GAMC-1.)
      open(22,file='rvoutput.dat',status='replace',err=110)
      write(22,*)r,vdum(it),TEMP

      IF(V(2).GE..67)GO TO 20
      IF(RHO.GE.1.E-8)GO TO 20
      RND=R-PFAC/(VP(1)/V(1)+2./R)
      GO TO 12
20    RCFACL=RCFAC
      RCFAC=RCFAC-R+1.
      WRITE(4,5000)RCFACL,RCFAC
      if((abs(rcfacl-rcfac)).gt.1e-4)go to 5200

5000  FORMAT(X,'RCFAC= ',2E14.5)
      open(unit=6,file='mdotvslumdata.txt',status='old',access='append'
     2,err=110)
      write(6,*)tnot,l,xm,rpors,dmdt,vinf,rcfac
      CALL BEEPQQ(frequency, duration)

110   STOP
      END
```

```
      SUBROUTINE CRITPT(RC,VC,VPC,V2)

      COMMON/IDAT/DMDTDO,SIGE,ONEMAL,ALO1MA,TWOALP,VTH,GAMGMK,TWOMAL,GAM
     2,GM,F5,GAMC
      COMMON/DERM/FLAG,ALPHA,RALP,C,F7,F8,RC2,ARP,ONEPAL,VPLAST,FC,
     2RPORS,TNOT,WN,rccrit,vpcguess

      real(4) n,m,nv,nvp
      LOGICAL FLAG

      onepal=1.+alpha
      VC2=VC*VC
      A=3.E6
      A2RP=(3.e6)**2*rpors*6.96e10
      GAMC=1.6666667
      BOLT=1.38E-16

      HYDROGEN=1.67E-24
      RC2=RC**2
      UC2=1.-1./RC2
      F1=ONEPAL/RC2
c     WRITE(7,2000)
      ITER=0
c     WRITE(7,1000)ITER,VC,VPC
      ONEOV2=1./VC/VC
      RCOVC=RC/VC
      R2VVP=RC2*VC*VPC
      RVPOV=RCOVC*VPC
      SIGMA=RVPOV-1.
      FC=Rc2*(RVPOV-(1.+SIGMA*Uc2)**ONEPAL/RVPOV**ALPHA)/ONEPAL/SIGMA
      F9=F1*FC
      VSIGMA=VC*SIGMA
      ROVSIG=RC/VSIGMA
      F11=1.+SIGMA*UC2
      F12=(F11/RVPOV)**ALPHA
      FVPOF=RC/VSIGMA*((1.-F12*(ONEPAL*UC2-ALPHA/RVPOV*F11))/F9-1.)
      ALPOVP=ALPHA/VPC
      FVP2OF=-ROVSIG*(ALPOVP*F12*(1./RVPOV-UC2/F11)/FC+2.*FVPOF)
      VPOVSG=VPC/VSIGMA
      R3VPOV=RC2*RVPOV
      TWOSIG=2.*SIGMA
      F14=UC2+TWOSIG/R3VPOV
      F14F11=F14/F11
      FROF=VPOVSG*((3.-F12*(ONEPAL*F14-TWOMAL/RVPOV*F11))/F9-1.)
      ALPOVC=ALPHA/VC
      F15=UC2-ALPHA/R3VPOV
      FVOF=-VPOVSG*RCOVC*((1.-F12*F15)/F9-1.)
      F16=ALPOVC+FVOF
      F17=2.*RVPOV*RC*ONEOV2
      F18=F12*((ALPHA-4.)/RVPOV+ALPHA*F14F11)/FC
      FRVOF=-VPC/SIGMA*ONEOV2*(F18+VC*FVOF-RC*FROF)
      F19=1./RVPOV-F15/F11
      FVVOF=(2.*(RVPOV/SIGMA-1.)*FVOF-ALPHA/FC/ONEPAL/VSIGMA*F12*RVPOV
     2*F19)/VC

      RCVC=RC*VC
```

```
c       Defining fvpvof

        F66=RC/(VC*SIGMA)
        F67=(1+ALPHA)/RC**2
        F68=-(VC/(RC*VPC))**ALPHA*ALPHA*F11**(ALPHA-1.)*UC2*(-RC*
        2VPC/VC**2)
        F69=ONEPAL*UC2-ALPHA*VC/(RC*VPC)*F11
        F70=-(VC/(RC*VPC))**ALPHA*F11**ALPHA
        F71=ALPHA*VC/(RC*VPC**2)*F11-ALPHA*VC/(RC*VPC)*UC2*(-RC*
        2VPC/VC**2)
        F72=RC*(SIGMA+VC*(-RC*VPC**2/VC**2))/(VC*SIGMA)**2
        F73=1.-(VC/(RC*VPC))**ALPHA*F11**ALPHA*((1.+ALPHA)*UC2-
        2ALPHA*VC/(RC*VPC)*F11)
        FVPVOF=F66*(F67*(F68*F69+F70*F71)/F67**2-FVOF)-F72*F73/F67


        FVVPOF=ROVSIG/VC*(ALPHA/ONEPAL*F12*F19/FC+VPC*FVPOF-VC/RVPOV*FVOF)
        FRVPOF=(F18-VPC*FVPOF-VC/VPC*FROF)/VSIGMA
        F3=FVOF*FVPOF-FVVPOF
        FRROF=-VPOVSG/RC*(F12/FC/ONEPAL/RVPOV*(4.*RVPOV*ONEPAL-TWOSIG*(2.*
        2ALPHA+5.)+TWOMAL+ALPHA*(TWOSIG-1.)*(ONEPAL*RVPOV*F14F11-TWOMAL))+
        32.*RC*FROF*(1.+SIGMA/RVPOV)+2.)
C       Beginning of my changes

        WN=(BOLT/(16.65e-1*HYDROGEN))*TNOT*GAMC/a**2
C       split up N into parts so that I can see where the floating point error is
        F5=GM*(gam-1.)

        N=-2.*WN/(RC**(2.*GAMC-3.)*VC**(GAMC-1.))-WN*VPC/(RC**(2.*GAMC-
        2 4.)*VC**GAMC)+R2VVP-F5/a2rp


        P1=-WN/(RC**(2*GAMC-4)*VC**GAMC)+Rc2*vc-(FVPOF*(1/FC)+ALPHA/VPC)*N

C       Since p2(f2)is so long, I am going to divide it up into parts labeled q

        Q1=-2.*WN*(-2.*GAMC+3.)/(RC**(2.*GAMC-2.)*VC**(GAMC-1.))
        Q2=WN*(-2.*GAMC+4.)*VPC/(RC**(2.*GAMC-3.)*VC**GAMC)
        Q3=2.*R2VVP/RC
        Q4=-2.*WN*(-GAMC+1)/(RC**(2.*GAMC-3.)*VC**GAMC)
        Q5=GAMC*WN*VPC/(RC**(2.*GAMC-4.)*VC**(GAMC+1.))
        Q6=R2VVP/VC
        Q7=VPC*FVOF+ALPHA*fc*VPC/VC+FROF+2*ALPHA*FC/RC

        P2=Q1-Q2+Q3+VPC*(Q4+Q5+Q6)-N*(1/FC)*Q7

C       Now for a,b,c,d terms of the matrix.  a=f1x, b=f1y, c=f2x, d=f2y
C       Each matrix term is broken down into parts labeled matal, etc

        NV=-2*WN*(-GAMC+1.)/(RC**(2*GAMC-3.)*VC**GAMC)+WN*VPC*
       2 GAMC/(RC**(2*GAMC-4.)*VC**(GAMC+1))+R2VVP/VC
        NVP=-WN/(RC**(2*GAMC-4.)*VC**GAMC-4.)+R2VVP/VPC


        tMATA1=WN*GAMC/(RC**(2*GAMC-4.)*VC**(GAMC+1.))
```

```
      tMATA3=(NV/FC-N*FVOF/FC**2)*FVPOF

      tMATA4=N*FVPVOF/FC
      tMATA5=ALPHA*(NV/VPC)

      AMAT = tMATA1+rc*rc-tMATA3-tMATA4-tMATA5


      tMATB1=NVP/FC-N*FVPOF/FC**2
      tMATB2=N*FVP2OF/FC
      tMATB3=ALPHA*(NVP/VPC-N/VPC**2)


      BMAT = -tMATB1-tMATB2-tMATB3

      tMATC1=-2*WN*(-2*GAMC+3)*(-GAMC+1)/(RC**(2*GAMC-2.)*VC**GAMC)
      tMATC2=WN*(-2*GAMC+4.)*GAMC*VPC/(RC**(2*GAMC-3.)*VC**(GAMC+1))
      tMATC3=2*RC*VPC
      tMATC4=2*WN*(-GAMC+1.)*GAMC/(RC**(2*GAMC-3.)*VC**(GAMC+1.))
      tMATC5=GAMC*WN*VPC*(-GAMC-1.)/(RC**(2*GAMC-4.)*VC**(GAMC+2.))
      tMATC6=NV/FC-N*FVOF/FC**2
      tMATC7=FROF+2*FC*ALPHA/RC+VPC*FVOF+FC*ALPHA*VPC/VC
      tMATC8=FRVOF+2*FVOF*ALPHA/RC+VPC*FVVOF+ALPHA*VPC*(FVOF/VC-
     2FC/VC**2)


      CMAT=tMATC1+tMATC2+tMATC3+VPC*(tMATC4+tMATC5)-tMATC6*tMATC7-(N/FC)
     2*tMATC8

      tMATD1=-WN*(-2*GAMC+4)/(RC**(2*GAMC-3.)*VC**GAMC)
      tMATD2=2*RC*VC
      tMATD3=2*WN*(-GAMC+1.)/(RC**(2*GAMC-3.)*VC**GAMC)
      tMATD4=2*GAMC*WN*VPC/(RC**(2*GAMC-4.)*VC**(GAMC+1.))

      tMATD5=2*RC2*VPC
      tMATD6=NVP/FC-N*FVPOF/(FC*FC)
      tMATD7=FROF+2*FC*ALPHA/RC+VPC*FVOF+FC*ALPHA*VPC/VC
      tMATD8=FRVPOF+2*FVPOF*ALPHA/RC+FVOF+VPC*FVVPOF+(ALPHA/VC)*
     2(FC+FVPOF*VPC)

      DMAT=tMATD1+tMATD2-tMATD3+tMATD4+tMATD5-tMATD6*tMATD7-(N/FC)
     2*tMATD8

C     Now rewriting terms det, dvc, dvpc to get new dx and dy

      DET=AMAT*DMAT-BMAT*CMAT
      DVC=(BMAT*P2-DMAT*P1)/DET
      VC=VC+DVC
      DVPC=(CMAT*P1-AMAT*P2)/DET
      IF(VC.LT.0)VC=1.
      VPC=VPC+DVPC
      IF(VPC.LT.0)VPC=1.
      ITER=ITER+1
c     WRITE(3,1000)ITER,VC,VPC,(dvpc/vpc),(dvc/vc),rc

      IF(ITER.GT.80 )GO TO 1500
```

```
      IF(ABS(DVPC/VPC).GT.1.E-5)GO TO 8
      IF(ABS(DVC/VC).GT.1.e-5)GO TO 8
      IF((ABS(DVPC/VPC).lt.1.E-5).and.(ABS(DVC/VC).lt.1.e-5))GO TO 9
c            attempt to iterate on r,v,vp so i don't have to manually
c            change them to get critpt to converge
1500  iter=0
      it=it+1
      i1=it/350.
      i2=int(i1)
c     IF(((DVC/VC).LT.5.E-4).AND.((DVPC/VPC).LT.5.E-4))GO TO 12
      vc=1.3+i2*.1
      iterate=iterate+1
      i3=mod(iterate,350)

      if(i3.eq.0)iterate=0.
      vpc=100.+.1*iterate
      if((vc.gt.6).and.(vpc.gt.700))go to 60
      if(it.gt.10000)go to 60
      write(7,*)'vc,vpc,iter=',vc,vpc,it,(dvc/vc),(dvpc/vpc)
      go to 8
12    VC=VC+.01
      VPC=VPC+.2
      go to 8

9     write(7,*)'Final convergence:vc,vpc  ',vc,vpc
      C1=-2*WN/(RC**(2*GAMC-3.)*VC**(GAMC-1))
      C2=-WN*VPC/(RC**(2*GAMC-4.)*VC**GAMC)+R2VVP-FS/a2rp
      C3=(R2VVP)**(ALPHA)*FC
      C=(C1+C2)/C3


      DMDTDO=(GAMGMK*A2RP**(ALPHA-1.)/C)**RALP/(SIGE*VTH)

C     Now doing v". Must first write Fv'v', Frr,Frv,Frv'Fvv'.
C     As before break each of these down, labeling them as dblvpvpl, etc

      DBLVPVP=-C*R2VVP**ALPHA*(FVP2OF+2.*ALPHA*FVPOF/VPC+ALPHA*FC*
      2(ALPHA-1.)/VPC**2)

      DBLRR1=-2*WN*(-2*GAMC+3.)*(-2*GAMC+2.)/(RC**(2*GAMC-1)*VC**(GAMC-1
      2.))
      DBLRR2=WN*(-2*GAMC+4.)*(-2*GAMC+3.)*VPC/(RC**(2*GAMC-2.)*VC**GAMC)
      DBLRR3=2*VC*VPC
      DBLRR4=FRROF*R2VVP**ALPHA
      DBLRR5=FROF*2*ALPHA*R2VVP**ALPHA/RC
      DBLRR6=2.*FC*ALPHA*R2VVP**ALPHA*((2.*ALPHA-1.)/RC**2)

      DBLRR=DBLRR1-DBLRR2+DBLRR3-C*(DBLRR4+2*DBLRR5+DBLRR6)

      DBLRV1=-2*WN*(-2*GAMC+3.)*(-GAMC+1)/(RC**(2*GAMC-2.)*VC**GAMC)
      DBLRV2=WN*(-2*GAMC+4.)*GAMC*VPC/(RC**(2*GAMC-3.)*VC**(GAMC+1.))
      DBLRV3=2*RC*VPC
      DBLRV4=FRVOF+FVOF*ALPHA/VC+2.*FVOF*ALPHA/RC+2.*FC*ALPHA**2/VC

      DBLRV=DBLRV1+DBLRV2+DBLRV3-C*R2VVP**ALPHA*(DBLRV4)

      DBLRVP1=-WN*(-2.*GAMC+4.)/(RC**(2.*GAMC-3.)*VC**GAMC)
```

```
      DBLRVP2=2.*RC*VC
      DBLRVP3=FRVPOF+ALPHA*FROF/VPC+2.*ALPHA*FVPOF/RC+
      22.*ALPHA**2*FC/(RC*VPC)

      DBLRVP=DBLRVP1+DBLRVP2-C*R2VVP**ALPHA*DBLRVP3

      DBLVV1=2.*WN*GAMC*(-GAMC+1.)/(RC**(2.*GAMC-3.)*VC**(GAMC+1.))
      DBLVV2=GAMC*(-GAMC-1.)*WN*VPC/(RC**(2.*GAMC-4.)*VC**(GAMC+2.))
      DBLVV3=FVVOF+2.*ALPHA*FVOF/VC+ALPHA*FC*(ALPHA-1.)/VC**2
      DBLVV=DBLVV1+DBLVV2-C*R2VVP**ALPHA*DBLVV3

      DBLVVP1=WN*GAMC/(RC**(2.*GAMC-4.)*VC**(GAMC+1.))
      DBLVVP2=RC*RC
      DBLVVP3=FVVPOF+ALPHA*FVOF/VPC+ALPHA*FVPOF/VC+
      2ALPHA**2*FC/(VC*VPC)

      DBLVVP=DBLVVP1+DBLVVP2-C*R2VVP**ALPHA*DBLVVP3

      DBLV1=-2*WN*(-GAMC+1.)/(RC**(2*GAMC-3.)*VC**GAMC)
      DBLV2=GAMC*WN*VPC/(RC**(2*GAMC-4.)*VC**(GAMC+1.))
      DBLV3=RC*RC*VPC
      DBLV4=FVOF+ALPHA*FC/VC

      DBLV=DBLV1+DBLV2+DBLV3-C*R2VVP**ALPHA*DBLV4

C     Now rewriting B and F6

      B=DBLV+2*(DBLRVP+VPC*DBLVVP)
      BSQRT=(B**2-4*DBLVPVP*(DBLRR+VPC*(2*DBLRV+VPC*DBLVV)))
      if(bsqrt.lt.0)go to 70


      V2=(-B+SQRT((B**2-4*DBLVPVP*(DBLRR+VPC*(2*DBLRV+VPC*DBLVV)))))/(2*
      2DBLVPVP)

      rccrit=rc
      ITER=0
      IT=0
      ITERATE=0

      RETURN
   60 IT=1
      WRITE (7,4000) IT
      STOP
   70 IT=2
      WRITE (7,4000) IT
 1000 FORMAT(X,I2,6E12.4)
 2000 FORMAT('CAK MODEL WITH TEMPERATURE DISTRIBUTION AND FINITE DISK
     2FACTOR'//'CRITICAL POINT ITERATION:'//' #',5X,'VC',11X,'VPC')
 4000 FORMAT(/' V',I1,'CALCULATION IS WRONG  it too big')
      STOP
      END


c     FUNCTION F(RVPOV,SIGMA,R2,U2)
c
```

```
c      COMMON/DERM/FLAG,ALPHA,RALP,C,F7,F8,RC2,ARP,ONEPAL,VPLAST,FC,
c      2RPORS,TNOT,WN,rc
c

C      LOGICAL FLAG
c      F=R2*(RVPOV-(1.+SIGMA*U2)**ONEPAL/RVPOV**ALPHA)/ONEPAL/SIGMA
c      RETURN
c      END
```

```
      SUBROUTINE DER(R,VA,VPA)
C     THIS IS THE NEW DER SUBROUTINE   .

      DIMENSION VA(2), VPA(2)
      COMMON/IDAT/DMDTDO,SIGE,ONEMAL,ALO1MA,TWOALP,VTH,GAMGMK,TWOMAL,GAM
      2,GM,F5,GAMC

      COMMON/DERM/FLAG,ALPHA,RALP,C,F7,F8,R2,ARP,ONEPAL,VPLAST,FC,
     2RPORS,TNOT,WN,VPC1

      LOGICAL FLAG
      if(r.gt.rc)flag=.true.


      V=VA(1)

      VPA(2)=-DMDTDO*SIGE/(V*R**2)/ARP
      a2rp=(3.e6)**2*rpors*6.96e10

      HR=2*WN/(R**(2*GAMC-3)*V**(GAMC-1))+F5/a2rp

      U2=1.-1./R**2
      IF(U2.LT.0)U2=0.
      ITER=0

      VP=VPLAST

      QTEST=WN/(R**(2*GAMC-2.)*V**(GAMC+1.))
1     IF((HR.GT.0).AND.(QTEST.GE.1))GO TO 99
      if((hr.le.0).and.(qtest.ge.1))go to 5
      if((hr.le.0).and.(qtest.lt.1))go to 7
      if((hr.gt.0).and.(qtest.lt.1))go to 9
5     R2VVP=R*R*V*VP
      R2=R*R
      GAMC=1.6667
      RVPOV=R/V*VP
      SIGMA=RVPOV-1.
      F1=ONEPAL/R2
      F11=1.+SIGMA*U2
      QTEST=WN/(R**(2*GAMC-2.)*V**(GAMC+1.))
      F12=(F11/RVPOV)**ALPHA
      FC=F(RVPOV,SIGMA,R2,U2)
      F9=F1*FC

      VSIGMA=V*SIGMA

      fvp=(r/v)/sigma*((1.-(f11/rvpov)**alpha*(onepal*u2-alpha/rvpov*
      2f11))/onepal*r2-fc)

      y1=c*fc*r2vvp**alpha/abs(hr)
      y2=abs(1-qtest)*r2vvp/abs(hr)

      bf=-y1-y2+1
      bfvp=-y2/vp-c*r2vvp**alpha/abs(hr)*(fvp+alpha*fc/vp)


      DVP=-bf/bfvp
```

```
      VP=VP+DVP
      ITER=ITER+1

      IF(ITER.GT.35)GO TO 99
      IF(abs(DVP/VP).GT.1.E-4)GO TO 5
      VPA(1)=VP
      RETURN

7     rvpov=r/v*vp
      sigma=rvpov-1.
      fc=f(rvpov,sigma,r2,u2)
      ff=(abs(hr)/c)**ralp
      ffovr2=ff/v*r*r
      b=ff*abs((1-qtest)/hr)

      vp=ffovr2
      if(flag)vp=(b/fc**ralp)**(-1./(1.-alpha))*ffovr2

      R2VVP=R*R*V*VP
      R2=R*R
      GAMC=1.6667
      RVPOV=R/V*VP
      SIGMA=RVPOV-1.
      F1=ONEPAL/R2
      F11=1.+SIGMA*U2
      QTEST=WN/(R**(2*GAMC-2.)*V**(GAMC+1.))
      F12=(F11/RVPOV)**ALPHA
      FC=F(RVPOV,SIGMA,R2,U2)
      F9=F1*FC

      VSIGMA=V*SIGMA

      fvp=(r/v)/sigma*((1.-(f11/rvpov)**alpha*(onepal*u2-alpha/rvpov*
      2f11))/onepal*r2-fc)

      y1=c*fc*r2vvp**alpha/abs(hr)
      y2=abs(1-qtest)*r2vvp/abs(hr)

      bf=-y1+y2+1
      bfvp=y2/vp-c*r2vvp**alpha/abs(hr)*(fvp+alpha*fc/vp)


      DVP=-bf/bfvp
      VP=VP+DVP
      ITER=ITER+1

      IF(ITER.GT.35)GO TO 99
      IF(abs(DVP/VP).GT.1.E-3)GO TO 7
      VPA(1)=VP
      RETURN

9     R2VVP=R*R*V*VP
      R2=R*R
      GAMC=1.6667
      RVPOV=R/V*VP
      SIGMA=RVPOV-1.
      F1=ONEPAL/R2
```

```
      F11=1.+SIGMA*U2
      QTEST=WN/(R**(2*GAMC-2.)*V**(GAMC+1.))
      F12=(F11/RVPOV)**ALPHA
      FC=F(RVPOV,SIGMA,R2,U2)
      F9=F1*FC

      VSIGMA=V*SIGMA

      fvp=(r/v)/sigma*((1.-(f11/rvpov)**alpha*(onepal*u2-alpha/rvpov*
     2f11))/onepal*r2-fc)

      y1=c*fc*r2vvp**alpha/abs(hr)
      y2=abs(1-qtest)*r2vvp/abs(hr)

      bf=-y1+y2-1
      bfvp=y2/vp-c*r2vvp**alpha/abs(hr)*(fvp+alpha*fc/vp)


      DVP=-bf/bfvp
      VP=VP+DVP
      ITER=ITER+1

      IF(ITER.GT.35)GO TO 99
      IF(abs(DVP/VP).GT.1.E-4)GO TO 9
      VPA(1)=VP
      RETURN

99    WRITE(4,*)'NO SOLUTION FOR VP--R,V=  ',R,V
      RETURN

      END
```

```
      SUBROUTINE RKFN(F,NEQN,Y,T,TFIN,EPSREL,EPSABS,IFLAG,WORK,IWORK)
c     IMPLICIT DOUBLE PRECISION(A-H,O-Z)
      DIMENSION Y(NEQN),WORK(15),IWORK(5)
      EXTERNAL F
      DATA U26/1.5E-6/,REMIN/1.E-6/,ZERO/0./
      K1M=NEQN+1
      K1=K1M+1
      K2=K1+NEQN
      K3=K2+NEQN
      K4=K3+NEQN
      K5=K4+NEQN
      K6=K5+NEQN
      IF(NEQN.LT.1)GO TO 10
      IF((EPSREL.LT.0.).OR.(EPSABS.LT.0.))GO TO 10
      MFLAG=IABS(IFLAG)
      IF((MFLAG.GE.1).AND.(MFLAG.LE.7))GO TO 20
10    IFLAG=7
      RETURN
20    IF(MFLAG.EQ.1)GO TO 55
      IF(T.EQ.TFIN)GO TO 10
      IF(MFLAG.NE.2)GO TO 25
      IF(IWORK(3).EQ.0)GO TO 45
      IF(IWORK(5).EQ.3)GO TO 40
      IF((IWORK(5).EQ.4).AND.(EPSABS.EQ.0.))GO TO 22
      IF((IWORK(5).NE.5).OR.(EPSREL.GT.WORK(K6)).OR.(EPSABS.GT.WORK(K6+1
      2)))GO TO 55
   22 IFLAG=IWORK(5)
      RETURN
25    IF(IFLAG.EQ.3)GO TO 40
      IF((IFLAG.EQ.4).AND.(EPSABS.GT.0.))GO TO 45
40    IWORK(1)=0
      IF(MFLAG.EQ.2)GO TO 55
45    IFLAG=IWORK(4)
55    IWORK(4)=IFLAG
      WORK(K6)=EPSREL
      WORK(K6+1)=EPSABS
      RER=MAX(EPSREL,REMIN)
      DX=TFIN-T
      IF(MFLAG.EQ.1)GO TO 60
      IF(IWORK(3).EQ.0)GO TO 65
       H=WORK(K1M)
       GO TO 80
60    IWORK(1)=1
      IWORK(2)=0
      IWORK(3)=0
      A=T
      CALL F(T,Y,WORK)
      IF(T.NE.TFIN)GO TO 65
      IFLAG=2
      GO TO 100
65    IWORK(3)=1
      YMAX=0.
      YPN=0.
      DO 70 K=1,NEQN
      YPN=MAX(ABS(WORK(K)),YPN)
70    YMAX=MAX(ABS(Y(K)),YMAX)
      ETN=RER*YMAX+EPSABS
```

```
      H=ABS(DX)
      IF(ETN.GE.YPN*H**5)GO TO 80
      H=MAX((ETN/YPN)**.2,U26*MAX(ABS(T),H))
80    WORK(K1M)=SIGN(H,DX)
      CALL RKFNS(F,NEQN,Y,T,TFIN,RER,EPSABS,IFLAG,WORK(1),WORK(K1M),
      2WORK(K1),WORK(K2),WORK(K3),WORK(K4),WORK(K5),IWORK(1))
  100 IWORK(5)=IFLAG
      RETURN
      END

      SUBROUTINE RKFNS(F,NEQN,Y,T,TFIN,EPSREL,EPSABS,IFLAG,YP,
      2H,F1,F2,F3,F4,F5,NFE)

      LOGICAL HFAILD,OUTPUT

      DIMENSION Y(NEQN),YP(NEQN),F1(NEQN),F2(NEQN),F3(NEQN),
      2F4(NEQN),F5(NEQN)

      EXTERNAL F
      DATA U26/1.5E-6/,ZERO/0./
      DATA MAXNFE/6000/
      DX=TFIN-T
      IF(ABS(DX).GT.U26*ABS(T))GO TO 50

      DO 25 K=1,NEQN
   25   Y(K)=Y(K)+DX*YP(K)
      A=TFIN
      CALL F(A,Y,YP)
      NFE=NFE+1
      GO TO 300
 50   OUTPUT=.FALSE.
      SCALE=2./EPSREL
      AE=SCALE*EPSABS
 100  HFAILD=.FALSE.
      HMIN=U26*ABS(T)
      DX=TFIN-T
      IF(ABS(DX).GE.2.*ABS(H))GO TO 200
      IF(ABS(DX).GT.ABS(H))GO TO 150
      OUTPUT=.TRUE.
      H=DX
      GO TO 200
 150  H=.5*DX
 200  IF(NFE.LE.MAXNFE)GO TO 220
      IFLAG=3
      RETURN
 220  CALL FEHLR(F,NEQN,Y,T,H,YP,F1,F2,F3,F4,F5,F1)
      NFE=NFE+5
      EEOET=0.
      DO 250 K=1,NEQN
      ET=ABS(Y(K))+ABS(F1(K))+AE
      IF(ET.GT.ZERO)GO TO 240
      IFLAG=4
       RETURN
 240  EE=ABS((-2090.*YP(K)+(21970.*F3(K)-15048.*F4(K)))+
      2  (22528.*F2(K)-27360.*F5(K)))
  250   EEOET=MAX(EEOET,EE/ET)
      ESTTOL=ABS(H)*EEOET*SCALE/752400.
```

```
      OPEN(10,FILE='TEST.DAT')
      write (10, *) ESTTOL
      IF(ESTTOL.LE.1.)GO TO 260


      HFAILD=.TRUE.
      OUTPUT=.FALSE.
      S=.1
      IF(ESTTOL.LT.59049.)S=.9/ESTTOL**.2
      H=S*H
      IF(ABS(H).GT.HMIN)GO TO 200
      IFLAG=5
      RETURN
260   T=T+H
      DO 270 K=1,NEQN
270   Y(K)=F1(K)
      A=T
      CALL F(A,Y,YP)
      NFE=NFE+1
      IF(HFAILD)GO TO 290
      S=5.
      IF(ESTTOL.GT.1.889568E-4)S=.9/ESTTOL**.2
      H=SIGN(MAX(S*ABS(H),HMIN),H)
290   IF(OUTPUT)GO TO 300
      IF(IFLAG.GT.0)GO TO 100
      IFLAG=-2
      RETURN
300   T=TFIN
      IFLAG=2
      RETURN
      END
      SUBROUTINE FEHLR(F,NEQ,Y,X,H,YP,F1,F2,F3,F4,F5,S)

      DIMENSION Y(NEQ),YP(NEQ),F1(NEQ),F2(NEQ),F3(NEQ),F4(NEQ),F5(NEQ),
      2S(NEQ)

      CH=.25*H
      DO 221 K=1,NEQ

221   F5(K)=Y(K)+CH*YP(K)
      CALL F(X+.25*H,F5,F1)
      CH=.09375*H
      DO 222 K=1,NEQ
222   F5(K)=Y(K)+CH*(YP(K)+3.*F1(K))
      CALL F(X+.375*H,F5,F2)
      CH=H/2197.
      DO 223 K=1,NEQ
223   F5(K)=Y(K)+CH*(1932.*YP(K)+(7296.*F2(K)-7200.*F1(K)))
      CALL F(X+12./13.*H,F5,F3)
      CH=H/4104.
      DO 224 K=1,NEQ
224   F5(K)=Y(K)+CH*((8341.*YP(K)-845.*F3(K))+
     2 (29440.*F2(K)-32832.*F1(K)))
      CALL F(X+H,F5,F4)
      CH=H/20520.
```

```
      DO 225 K=1,NEQ
225   F1(K)=Y(K)+CH*((-6080.*YP(K)+(9295.*F3(K)-5643.*F4(K)))+
     2  (41040.*F1(K)-28352.*F2(K)))
      CALL F(X+.5*H,F1,F5)
      CH=H/7618050.
      DO 230 K=1,NEQ
230   S(K)=Y(K)+CH*((902880.*YP(K)+(3855735.*F3(K)-1371249.*F4(K)))+
     2  (3953664.*F2(K)+277020.*F5(K)))
      RETURN
      END
```

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