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Semi-Markov chains

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SEMI-MARKOV CHAINS

by

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INTRODUCTION

Often problems in Markov chains can be extended in such a way that one may think of waiting times between changes of states. The purpose of this paper is to show how this can be done in some cases by redefining the states so that we still have a Markov chain and the calculation of unknowns can be done in terms of Markov chain theory of the original states and the probability function describing the distribution of waiting times in the various states.

In the first part of this paper we review the notions of Markov chains and prove the theorems we need. Following Feller [3] we define recurrent events and develop an expression for the probability that an event occurs in terms of the probability that it occurs for the first time in some trial (Theorem 1). Lemmas 2 and 3 are taken in part from Niven and Zuckerman [4]. Armed with Theorem 1 and the definition and classification of Markov chains, we give Feller's proof of Theorem 4 which characterizes irreducible, aperiodic Markov chains in terms of their transition probabilities and their mean recurrence times.

We then leave Feller and follow Anselone [2] who develops the notion of semi-Markov chains. First of all we extend the Markov chain by introducing the idea of waiting times in the states. We use the waiting times to define a semi-Markov chain. We then define the sub-state chain by pairing each state from the semi-Markov chain with the time the process will remain in that state. The substate chain is a Markov chain. It is this extension of the semi-Markov chain that yields the information we seek, that is, characterizes our problem. In a straightforward way we arrive at Anselone's results in Theorems 11 and 12, which are more or less analogous to Theorems 2 and 4 for Markov chains.

SOME LEMMAS FROM NUMBER THEORY

We begin by stating and proving three lemmas from number theory.

<u>Lemma 1</u>: From any set $\{a_i\}$ of positive integers with greatest common divisor one, it is possible to choose a finite subset with greatest common divisor one.

Proof: Choose a_1, the smallest element of $\{a_i\}$. If $a_1 \neq 1$, choose a_2, the smallest element of $\{a_i\}$ such that $a_1 \nmid a_2$ (i.e., a_1 does not divide a_2). Let $g_1 = (a_1, a_2)$. If $g_1 \neq 1$, choose a_3, the smallest element of $\{a_i\}$ such that $g_1 \nmid a_3$. Let $g_2 = (a_1, a_2, a_3)$. Then $g_2 \leq g_1$, but since $g_2 | a_3$ and $g_1 \nmid a_3$, $g_2 < g_1$. We continue in this fashion, obtaining a monotone decreasing sequence of positive integers. There exists, for some positive interger n, a $g_n = (a_1, a_2, \ldots, a_{n+1}) = 1$, for if $g_n > 1$ we can obtain a $g_{n+1} < g_n$.

<u>Lemma 2</u>: If a, b, and c are positive integers such that $(a,b) | c$, and $(a,b)c > ab$, then there exists at least one positive solution to $ax + by = c$.

Proof: There exist intergers x_o and y_o such that $ax_o + by_o = (a,b)$. All integral solutions r,s of $ax + by = c$ can be written in the form

$$r = cx_o/(a,b) + bt/(a,b)$$

and

$$s = cy_o/(a,b) - at/(a,b).$$

For a solution to be positive it is necessary and sufficient that

$$-\frac{cx_o}{b} < t < \frac{cy_o}{a} .$$

We have $cax_o + cby_o = (a,b)c > ab$ and, dividing by ab, we obtain

$$\frac{cx_o}{b} + \frac{cy_o}{a} > 1$$

or,

$$-\frac{cx_o}{b} < 1 - \frac{cx_o}{b} < \frac{cy_o}{a} .$$

Thus the length of the interval $(-\frac{cx_o}{b}, \frac{cy_o}{a})$ is greater than one so that there exists at least one integer t such that

$$-\frac{cx_o}{b} < t < \frac{cy_o}{a} .$$

<u>Lemma 3</u>: If $\{a_i\}$ is a finite sequence of distinct positive integers with greatest common divisor one, and k is an integer such that $k > \prod_{i=1}^{n} a_i$, then there exist positive integers x_i such that $k = \sum_{i=1}^{n} a_i x_i$.

Proof: From the equation $\sum_{i=1}^{n} a_i x_i = k$ we first derive the equation $a_1x_1 + b_1y_1 = k$ in such a way that $b_1 > 0$, $(a_1,b_1) = 1$, and $a_1b_1<k$. This will imply, by Lemma 2, that there exists at least one positive integral solution to $a_1x_1 + b_1y_1 = k$. Using an x_1 from one of these solutions, we construct the desired solution for the equation

$$\sum_{i=1}^{n} a_i x_i = k.$$

Suppose $\sum_{i=1}^{n} a_i x_i = k$. Let $\beta = -a_n/(a_{n-1}, a_n)$ and $\delta = a_{n-1}/(a_{n-1}, a_n)$.
Then $(\beta, \delta) = 1$ so that there exist α and γ such that $\alpha\delta - \beta\gamma = 1$.
Let $u = \delta x_{n-1} - \beta x_n$ and $v = -\gamma x_{n-1} + \alpha x_n$ so that $x_{n-1} = \alpha u + \beta v$ and $x_n = \gamma u + \delta v$. We claim that

$$\sum_{i=1}^{n} a_i x_i = \sum_{i=1}^{n-2} a_i x_i + (a_{n-1}\alpha + a_n \gamma)u.$$

Noting that

$$\begin{aligned}(a_{n-1}\alpha + a_n\gamma) &= \delta(a_{n-1}, a_n)\alpha - \beta(a_{n-1}, a_n)\gamma \\ &= (\delta\alpha - \beta\gamma)(a_{n-1}, a_n) \\ &= (a_{n-1}, a_n)\end{aligned}$$

we have $a_{n-1}\alpha + a_n\gamma > 0$. Since $(a_1, a_2, \ldots, a_{n-2}, (a_{n-1}, a_n)) = (a_1, a_2, \ldots, a_n)$ we have obtained an equation whose coefficients are positive integers with greatest common divisor one, and the number of coefficients is one less. We may continue in this manner obtaining $a_1x_1 + b_1y_1 = k$ where $b_1 = (a_2, a_3, \ldots, a_n) > 0$ and $(a, b) = (a_1, a_2, \ldots a_n) = 1$. Also

$$a_1 b_1 < \prod_{i=1}^{n} a_i < k.$$

Hence $a_1x_1 + b_1y_1 = k$ has at least one positive integral solution.

Now we wish to show that for suitable choice of x_1 $a_2x_2 + b_2y_2 = k - a_1x_1$ has a positive integral solution and, in general, that

$$a_r x_r + b_r y_r = k - \sum_{i=1}^{r-1} a_i x_i$$

where $r \geq 2$ and $b_r = (a_{r+1}, a_{r+2}, \ldots, a_n)$ has a positive integral solution.

Since the a_i's are all distinct we may suppose that a_n is the least a_i. Also, from the positive integral solutions of $a_1x_1 + b_1y_1 = k$, we shall choose the smallest x_1. This means that $x_1 \leq b_1$ since otherwise $a_1(x_1 - b_1) + b(y_1 + a_1) = k$ and $0 < x_1 - b_1 < x_1$.

Having reduced $\sum_{i=1}^{n} a_ix_i = k$ to $a_1x_1 + b_1y_1 = k$, we take as an induction hypothesis that

$$a_sx_s + b_sy_s = k - \sum_{i=1}^{s-1} a_ix_i$$

has positive integral solutions for $s = 1, 2, \ldots, r - 1 < n$. Each x_s is taken as the least positive solution. In order to find positive integral solutions to

$$a_rx_r + b_ry_r = k - \sum_{i=1}^{r-1} a_ix_i$$

we need to show that

$$a_rb_r < (a_r,b_r)(k - \sum_{i=1}^{r-1} a_ix_i).$$

For this it will suffice that

$$a_rb_r + \sum_{i=1}^{r-1} a_ix_i < k.$$

Since $x_i \leq b_i$ for $i = 1, 2, \ldots, r - 1$ we have

$$\sum_{i=1}^{r-1} a_ix_i \leq \sum_{i=1}^{r-1} a_ib_i.$$

Also, since $(a_{i+1}, a_{i+2}, \ldots, a_n) \leq (a_{i+2}, a_{i+3}, \ldots, a_n)$ -- i.e., $b_i \leq b_{i+1}$ -- we have

$$b_ra_r + \sum_{i=1}^{r-1} a_ix_i \leq \sum_{i=1}^{r} a_ib_i \leq b_r \sum_{i=1}^{r} a_i.$$

If any of the $a_i = 1$, it is a_n, and since $r < n$ we have

$$\sum_{i=1}^{r} a_i \leq \prod_{i=1}^{r} a_i.$$

Hence

$$b_r \sum_{i=1}^{r} a_i \leq b_r \prod_{i=1}^{r} a_i \leq \prod_{i=1}^{n} a_i < k.$$

So we have

$$a_r b_r + \sum_{i=1}^{r-1} a_i x_i < k.$$

By hypothesis we have

$$b_{r-1} y_{r-1} = k - \sum_{i=1}^{r-1} a_i x_i$$

so that b_{r-1} divides $k - \sum_{i=1}^{r-1} a_i x_i$. Since $b_{r-1} = (a_r, b_r)$, we have satisfied the conditions of Lemma 2. Therefore,

$$a_r x_r + b_r y_r = k - \sum_{i=1}^{r-1} a_i x_i$$

has at least one positive integral solution.

RECURRENT EVENTS AND MARKOV CHAINS

Suppose we have a sequence of experiments each with possible outcomes $E_1, E_2, \ldots, E_n, \ldots$. We speak of an attribute $\mathcal{E}$ of some finite sequence of trials. That is, any finite sequence of trials either possesses the attribute $\mathcal{E}$ or it does not. To say that $\mathcal{E}$ occurs at the n^{th} place of the sequence $E_{j_1}, E_{j_2}, \ldots, E_{j_n}$ means that this sequence possesses the attribute $\mathcal{E}$. For example, if the outcomes E_j are the positive integers, a particular attribute $\mathcal{E}$ might be "an even integer occurs on the fifth trial." In particular, we wish to speak of recurrent events.

Definition 1: The attribute $\mathcal{E}$ defines a recurrent event provided that:

1) $\mathcal{E}$ occurs in the n^{th} place and the $(n + m)^{th}$ place of the sequence $(E_{j_1}, E_{j_2}, \ldots, E_{j_{n+m}})$ means that $\mathcal{E}$ occurs in the last place of each of the two subsequences $(E_{j_1}, E_{j_2}, \ldots, E_{j_n})$ and $(E_{j_{n+1}}, E_{j_{n+2}}, \ldots, E_{j_{n+m}})$.

2) Whenever this happens

$$P\{E_{j_1}, E_{j_2}, \ldots, E_{j_{n+m}}\} =$$

$$P\{E_{j_1}, E_{j_2}, \ldots, E_{j_n}\} P\{E_{j_{n+1}}, E_{j_{n+2}}, \ldots, E_{j_{n+m}}\}.$$

We shall adopt the following notation:

$u_n = P\{\mathcal{E} \text{ occurs on the } n^{th} \text{ trial}\}$,

$u_0 = 1$,

$f_n = P\{\mathcal{E}$ occurs for the first time on the n^{th} trial$\}$,

$f_o = 0$, and

$$f = \sum_{n=1}^{\infty} f_n.$$

We note that

$$u_n = f_1 u_{n-1} + f_2 u_{n-2} + \ldots + f_n u_o, \quad n \geq 1.$$

<u>Definition 2</u>: A recurrent event $\mathcal{E}$ is called <u>persistent</u> if $f = 1$ and <u>transient</u> if $f < 1$.

<u>Theorem 1</u>: Suppose $0 \leq g_n \leq 1$, $\sum_{n=1}^{\infty} g_n = 1$, $g_o = 0$, and

$$x_n = g_1 x_{n-1} + g_2 x_{n-2} + \ldots + g_n x_o, \quad n = 1, 2, \ldots .$$

If g.c.d. $\{n \mid g_n > 0\} = 1$, then

$$\lim_{n\to\infty} x_n = \Big(\sum_{n=1}^{\infty} n g_n\Big)^{-1}$$

if $\sum_{n=1}^{\infty} n g_n$ is finite, and

$$\lim_{n\to\infty} x_n = 0$$

if $\sum_{n=1}^{\infty} n g_n$ diverges.

Proof: Let

$$r_n = \sum_{i=1}^{\infty} g_{n+i} = \sum_{i=n+1}^{\infty} g_i \text{ and } \mu = \sum_{n=1}^{\infty} n g_n$$

so that

$$\mu = \sum_{n=1}^{\infty} r_n. \text{ We have then, } r_n = r_o - \sum_{i=1}^{n} g_i$$

and therefore $r_{n-1} - r_n = g_n$. Substituting into

$$x_n = g_1 x_{n-1} + g_2 x_{n-2} + \ldots + g_n x_o,$$

we obtain

$$x_n = (r_o - r_1)x_{n-1} + (r_1 - r_2)x_{n-2} + \cdots + (r_{n-1} - r_n)x_o.$$

Thus

$$r_o x_n + r_1 x_{n-1} + \cdots + r_n x_o = r_o x_{n-1} + r_1 x_{n-2} + \cdots + r_{n-1} x_o.$$

This shows inductively that

$$1 = r_o x_o = \cdots = \sum_{i=o}^{n} r_i x_{n-i}$$

for all n. Now $x_1 = g_1 x_o \leq 1$ since $x_o = 1$ and $g_1 \leq 1$.

Suppose that $x_o, x_1, \ldots, x_k$ are all at most one. Then

$$x_{k+1} = g_1 x_k + \cdots + g_{k+1} x_o \leq g_1 + g_2 + \cdots + g_{k+1} \leq 1.$$

Therefore, there exists a $\lambda = \limsup x_n$, i.e., for every $\epsilon > 0$, there exists an M such that $n > M$ implies that $x_n < \lambda + \epsilon$. Also, there exists a sequence $\{n_v\}$ such that $\lim_{v\to\infty} x_{n_v} = \lambda$. Choose an integer $j > 0$ such that $g_j > 0$. Then we assert that $\lim_{v\to\infty} x_{n_v - j} = \lambda$.

Suppose that this were not true. Then for any $\epsilon > 0$ and each N there exists an $n_v > N$ such that either $x_{n_v-j} < \lambda - \epsilon$ or $x_{n_v-j} > \lambda + \epsilon$. If $N > M$, then the latter is impossible so that there exists a λ' such that $x_{n_v-j} < \lambda' < \lambda$. Since $\lim_{v\to\infty} x_{n_v} = \lambda$, if we take N large enough we also have $x_{n_v} > \lambda - \epsilon$. For every $\delta > 0$ there exists an $R > j$ such that $r_n < \delta$ for all $n > R$, since $\sum_{n=1}^{\infty} g_n$ converges. Since $g_o = 0$ and $x_k \leq 1$, we take $\delta = \epsilon$ so that

$$x_{n_v} \leq g_o x_{n_v} + g_1 x_{n_v-1} + \cdots + g_R x_{n_v-R} + \epsilon$$

for $n > R$. Also $n > M + R$ implies that $x_n < \lambda + \epsilon$ so that

$$\begin{aligned} x_{n_v} &< (g_o + g_1 + \cdots + g_{j-1} + g_{j+1} + \cdots g_R)(\lambda + \epsilon) + g_j \lambda' + \epsilon \\ &\leq (1 - g_j)(\lambda + \epsilon) + g_j \lambda' + \epsilon \\ &< \lambda + 2\epsilon - g_j(\lambda - \lambda'). \end{aligned}$$

Choose ϵ such that $3\epsilon < g_j(\lambda - \lambda')$ so that $x_{n_v} < \lambda - \epsilon$, a contradiction of $x_{n_v} > \lambda - \epsilon$.

Similarly we see that $g_j > 0$ and $\lim_{n_v \to \infty} u_{n_v} = \lambda$ implies that $x_{n_v-2j} \to \lambda$, $x_{n_v-3j} \to \lambda$, ...

Consider, first, the case where $g_1 > 0$. We take $j = 1$ and conclude that $x_{n_v-k} \to \lambda$ for all k. Since

$$r_o x_n + r_1 x_{n-1} + \cdots + r_n x_o = 1,$$

we have

$$r_o x_{n_v} + r_1 x_{n_v-1} + \cdots + r_N x_{n_v-N} \leq 1$$

for $n = n_v$. For fixed N, $x_{n_v-k} \to \lambda$ for all $k \leq N$, so that

$$(r_o + r_1 + \cdots + r_N) \leq 1.$$

Since N is arbitrary, $\lambda\mu \leq 1$ or $\lambda \leq 1/\mu$. If $\mu = \sum_{n=0}^{\infty} r_n$ diverges, then $\lim_{n\to\infty} x_n = 0$. If $\mu < \infty$, let $\gamma = \liminf x_n$. The same argument as above shows that for every sequence $\{n_v\}$ for which $\lim_{n_v\to\infty} x_{n_v} = \gamma$ we have $\lim_{n_v\to\infty} x_{n_v-k} = \gamma$ for all k. If N is large enough so

$$\sum_{r=N}^{\infty} r_n < \epsilon \text{ , then}$$

$$1 \leq r_o x_{n_v} + \cdots + r_N x_{n_v-N} + \epsilon,$$

so that

$$1 \leq (r_o + r_1 + \cdots + r_N)\gamma + \epsilon \; .$$

Hence $1/\mu \leq \gamma$, so that $\lambda \leq 1/\mu \leq \gamma$. But $\liminf x_n \leq \limsup x_n$. Therefore, $\lim_{n\to\infty} x_n = 1/\mu$.

Consider, now, the case where $g_1 = 0$. By Lemma 1, we can choose from the set of integers j for which $g_j > 0$ a finite collection $\{a_i\}$, $i = 1, 2, \ldots, n$, such that g.c.d. $\{a_i\} = 1$. We know that when

$\lim_{n_v \to \infty} x_{n_v} = \lambda$, also $\lim_{n_v \to \infty} x_{n_v - a_i y_i} = \lambda$ for every fixed $y_i > 0$. By Lemma 3, $\sum_{i=1}^{n} a_i y_i = k$, where k is a positive integer, has positive integral solutions provided that $\prod_{i=1}^{n} a_i < k$. Hence $x_{n_v - k} \to \lambda$. The remainder of the proof follows as in the preceding case.

In the theory of Markov chains we consider outcomes whose probabilities depend only upon the outcome of the preceding trial. Hence, knowing the outcome of any particular trial, say E_k, we may neglect any further information about earlier states in making a probability statement about E_{k+1}.

Definition 3: A sequence of trials with possible outcomes (states) E_1, E_2, . . . is called a Markov chain provided that the probabilities of sample sequences are given by

$$P\{(E_{j_0}, E_{j_1}, \ldots E_{j_n})\} = a_{j_0} p_{j_0 j_1} p_{j_1 j_2} \cdots p_{j_{n-1} j_n}$$

in terms of an initial probability distribution $\{a_k\}$ for the states E_k at time zero and transition probabilities $p_{jk} = P\{E_k \mid E_j\}$ (i.e., the probability that E_k occurs, given that E_j occurred on the preceding trial).

Suppose we let $p_{jk}^{(n)}$ designate the probability that E_k occurs on the n^{th} trial after E_j occurred. Thus we see that $p_{jj}^{(n)} = u_n$ if a recurrent event occurs on the zeroth trial. ($\mathcal{E} = E_j$).

Definition 4: A Markov chain is irreducible provided that for all (j,k) there exists an n such that $p_{jk}^{(n)} > 0$. (Every state can be reached from every other state).

A state E_j is said to be periodic with period $t > 1$ if $p_{jj}^{(n)} = 0$ whenever n is not divisible by t and t is the smallest integer with this property.

Definition 5: An aperiodic Markov chain is a Markov chain in which no states are periodic.

Definition 6: A state E_j of an aperiodic Markov chain is a transient state provided that $\sum_{n=1}^{\infty} p_{jj}^{(n)} < \infty$.

Definition 7: A state E_j of an aperiodic Markov chain is a persistent null state provided that $\sum_{n=1}^{\infty} p_{jj}^{(n)} = \infty$ and $\lim_{n\to\infty} p_{jj}^{(n)} = 0$.

Definition 8: A state E_j of an aperiodic Markov chain is an ergodic state provided that it is neither transient nor null.

Theorem 2: If a state E_j is ergodic, then

$$\lim_{n\to\infty} p_{ij}^{(n)} = f_{ij} \Big/ \sum_{n=1}^{\infty} n f_j^{(n)}$$

where f_{ij} is the probability that, starting from state E_i, the system ever reaches state E_j, and $f_j^{(n)}$ is the probability that state E_j is reached for the first time on the n^{th} trial. The probability $f_j^{(n)}$ plays the role of f_n defined on page 9.

Proof: Let $h_{ij}^{(n)}$ represent the probability that, starting from state E_i, the system reaches state E_j for the first time on the n^{th} step.

Clearly, $f_{ij} = \sum_{n=1}^{\infty} h_{ij}^{(n)}$. Also,

$$p_{ij}^{(n)} = h_{ij}^{(n)} + (f_j^{(o)}p_{ij}^{(n)} + f_j^{(1)}p_{ij}^{(n-1)} + \ldots + f_j^{(n)}p_{ij}^{(o)}),$$

where we define $f_j^{(o)} = 0$, $h_{ij}^{(o)} = 1$. Thus

$$\sum_{n=0}^{\infty} p_{ij}^{(n)}s^n = \sum_{n=0}^{\infty} h_{ij}^{(n)}s^n + \sum_{n=0}^{\infty} f_j^{(n)}s^n \sum_{n=0}^{\infty} p_{ij}^{(n)}s^n.$$

We now define the following generating functions:

$$P_i(s) = \sum_{n=0}^{\infty} p_{ij}^{(n)}s^n,$$

$$F(s) = \sum_{n=0}^{\infty} f_j^{(n)}s^n, \text{ and}$$

$$H_i(s) = \sum_{n=0}^{\infty} h_{ij}^{(n)}s^n.$$

By comparison with the geometric series these series converge at least in the open interval (-1, 1). Also $|F(s)| < 1$ on the open interval (-1, 1).

$$P_i(s) = H_i(s) + F(s)P_i(s) \text{ or } P_i(s) = H_i(s)\left[1 - F(s)\right]^{-1}.$$

Since $F(s)$ has a power series expansion and $|F(s)| < 1$ on (-1, 1) we can write $\left[1 - F(s)\right]^{-1} = K(s) = \sum_{n=0}^{\infty} k^{(n)}s^n$. Rewriting what we have,

$$K(s) = 1 + F(s)K(s),$$

and in particular, equating coefficients of s^n,

$$k^{(n)} = f_j^{(1)}k^{(n-1)} + f_j^{(2)}k^{(n-2)} + \ldots + f_j^{(n)}k^{(o)}.$$

By Theorem 1 we have, then,

$$\lim_{n\to\infty} k^{(n)} = 1 \Big/ \sum_{n=1}^{\infty} nf_j^{(n)}.$$

Since

$$P(s) = H(s)\left[1 - F(s)\right]^{-1} = H(s)K(s),$$

we have

$$p_{ij}^{(n)} = k^{(n)}h_{ij}^{(o)} + k^{(n-1)}h_{ij}^{(1)} + \ldots + k^{(o)}h_{ij}^{(n)}.$$

For my fixed r,

$$\lim_{n\to\infty} k^{(n-r)}h_{ij}^{(r)} = h_{ij}^{(r)}/\sum_{n=1}^{\infty} nf_j^{(n)},$$

and

$$\lim_{n\to\infty} (k^{(n)}h_{ij}^{(o)} + \ldots + k^{(n-N)}h_{ij}^{(N)})$$
$$= \sum_{n=1}^{N} h_{ij}^{(n)}/\sum_{n=1}^{\infty} nf_j^{(n)}.$$

Therefore, given $\epsilon > 0$ there exists an N large enough so that

$$\left| k^{(n)}h_{ij}^{(o)} + \ldots + k^{(n-N)}h_{ij}^{(N)} - \sum_{n=1}^{N} h_{ij}^{(n)}/\sum_{n=1} nf_j^{(n)} \right| < \epsilon/3.$$

Further, since $\sum_{n=0}^{\infty} h_{ij}^{(n)} < \infty$ and the $k^{(n)}$ are bounded we have, for large

enough N,

$$\sum_{r=N+1}^{n} k^{(n-r)}h_{ij}^{(r)} < \epsilon/3,$$

or, in other words,

$$\left| p_{ij}^{(n)} - (k^{(n)}h_{ij}^{(o)} + \ldots + k^{(n-N)}h_{ij}^{(N)}) \right| < \epsilon/3.$$

If we further choose N sufficiently large that

$$\left| H(1)/\sum_{n=1}^{\infty} nf_j^{(n)} - \sum_{n=1}^{N} h_{ij}^{(n)}/\sum_{n=1}^{\infty} nf_j^{(n)} \right| < \epsilon/3,$$

we have, for $n > N$

$$\left| p_{ij}^{(n)} - H(1)/\sum_{n=1}^{\infty} nf_j^{(n)} \right| < \epsilon .$$

Therefore,

$$\lim_{n\to\infty} p_{ij}^{(n)} = f_{ij}/\sum_{n=1}^{\infty} nf_j^{(n)}.$$

Theorem 3: In an irreducible aperiodic Markov chain all states belong to the same one of the three classes defined above.

Proof: Let E_j be a fixed non-transient state and let E_k be some other state that can be reached from it in no less than N steps, and let $p_{jk}^{(N)} > 0$. A return from E_k to E_j must have positive probability since the chain is irreducible. That is, for some M, $p_{kj}^{(M)} > 0$. Clearly

$$p_{jk}^{(N)} p_{kk}^{(n)} p_{kj}^{(M)} \leq p_{jj}^{(N+n+M)}, \text{ and}$$

$$p_{kj}^{(M)} p_{jj}^{(n)} p_{jk}^{(N)} \leq p_{kk}^{(M+n+N)}$$

for all n.

Therefore, $\sum_{n=1}^{\infty} p_{jj}^{(n)} = \infty$ implies $\sum_{n=1}^{\infty} p_{kk}^{(n)} = \infty$, and $\lim_{n\to\infty} p_{jj}^{(n)} = 0$ implies $\lim_{n\to\infty} p_{kk}^{(n)} = 0$. Hence if E_j is persistent null, so is E_k.

Suppose E_j is a transient state. Then any other state, say E_k, that can be reached from it must be transient also, for if it were not, by the above, E_j would have to be non-transient also.

Since E_j persistent null implies any other state E_k is persistent null, and E_j transient implies any other state E_k is transient, it follows that E_j ergodic implies any other state E_k is ergodic.

Definition 9: A probability distribution $\{v_k\}$ is called stationary with respect to $\{p_{ij}\}$ provided that $v_j = \sum_{i=1}^{\infty} v_i p_{ij}$.

Theorem 4: An irreducible aperiodic Markov chain belongs to one of the following two classes:

1) The states are all transient or all null states, in which case there exists no stationary distribution.

2) The states are all ergodic, in which case $\{u_k\}$ is a unique stationary distribution where $u_k = \lim_{n \to \infty} p_{jk}^{(n)} > 0$.

Proof: Suppose all states are ergodic. Then for fixed j and n,

$$\sum_{k=1}^{\infty} p_{jk}^{(n)} = 1 \text{ so that } \sum_{k=1}^{N} u_k \leq 1.$$

In the equation

$$p_{jk}^{(m+1)} = \sum_{v=1}^{\infty} p_{jv}^{(m)} p_{vk}$$

let m approach infinity. Then $\lim_{m \to \infty} p_{jk}^{(m+1)} = u_k$ so that

$$\lim_{m \to \infty} p_{jv}^{(m)} p_{vk} = p_{vk} \lim_{m \to \infty} p_{jv}^{(m)} = p_{vk} u_v.$$

For all finite t we have

$$p_{jk}^{(m+1)} = \sum_{v=1}^{\infty} p_{jv}^{(m)} p_{vk} \geq \sum_{v=1}^{t} p_{jv}^{(m)} p_{vk}.$$

Letting m approach ∞,

$$u_k \geq \sum_{v=1}^{t} u_v p_{vk},$$

and

$$u_k \geq \sum_{v=1}^{\infty} u_v p_{vk}.$$

We suppose that

$$u_k > \sum_{v=1}^{\infty} u_v p_{vk}$$

and sum both sides over k obtaining

$$\sum_{k=1}^{\infty} u_k > \sum_{k=1}^{\infty} \sum_{v=1}^{\infty} u_v p_{vk}.$$

Since we have absolute convergence we can interchange the order of summation so that

$$\sum_{k=1}^{\infty} u_k > \sum_{v=1}^{\infty} \sum_{k=1}^{\infty} p_{vk} u_v = \sum_{v=1}^{\infty} u_v,$$

a contradiction. Hence

$$u_k = \sum_{v=1}^{\infty} u_v p_{vk}.$$

Let

$$v_k = u_k \Big/ \sum_{j=1}^{\infty} u_j.$$

Then

$$v_k = \sum_{j=1}^{\infty} u_j p_{jk} \Big/ \sum_{j=1}^{\infty} u_j$$

$$= \sum_{j=1}^{\infty} \left(u_j \Big/ \sum_{j=1}^{\infty} u_j\right) p_{jk}$$

$$= \sum_{j=1}^{\infty} v_j p_{jk}$$

so that $\{v_k\}$ is a stationary distribution.

Let $\{v_k\}$ be any stationary distribution. Then,

$$v_j = \sum_{i=1}^{\infty} v_i p_{ij}$$

and

$$\sum_{j=1}^{\infty} p_{jk}^{(1)} v_j = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} p_{jk}^{(1)} v_i p_{ij}$$

so that

$$v_k = \sum_{i=1}^{\infty} v_i p_{ik}^{(2)}.$$

We proceed, inductively, by supposing

$$v_j = \sum_{i=1}^{\infty} v_i p_{ij}^{(m)}.$$

Then

$$\sum_{j=1}^{\infty} p_{jk}^{(1)} v_j = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} p_{jk}^{(1)} v_i p_{ij}^{(m)}.$$

Absolute convergence allows us to interchange limits so that we have

$$v_k = \sum_{i=1}^{\infty} v_i p_{ij}^{(m+1)}.$$

Letting n approach infinity in

$$v_r = \sum_{v=1}^{\infty} v_v u_r = u_r.$$

Hence the distribution is unique.

If the states are transient or null states and $\{v_k\}$ is a stationary distribution, then

$$v_r = \sum_{v=1}^{\infty} v_v p_{vr}^{(n)}$$

and $\lim_{n \to \infty} p_{vr}^{(n)} = 0$, so that no stationary distribution exists.

Example: Three chess players, Adams, Berlyov, and Schultz have a tournament in which the last player to win a game is champion. Adams and Schultz don't get on together so they never play one another. Therefore, whenever Adams or Schultz is champion he remains champion until dethroned by Berlyov. Berlyov, however, never wins two games in succession. Whenever Adams is champion, Berlyov has probability three-fourths in favor of defeating him. Whenever Berlyov has the choice of opponents he chooses to play Adams three-fourths of the time. Whenever Schultz is champion, Berlyov regains the championship only one-fourth of the time.

If we let the integers a, b, and s represent the states of Adams, Berlyov, and Schultz, respectively, being champion, we may write the transition matrix as follows:

$$\begin{array}{c} \\ a \\ b \\ s \end{array} \begin{array}{c} \begin{array}{ccc} a & b & s \end{array} \\ \begin{pmatrix} 1/4 & 3/4 & 0 \\ 3/4 & 0 & 1/4 \\ 0 & 1/4 & 3/4 \end{pmatrix} \end{array} = \begin{pmatrix} p_{aa} & p_{ab} & p_{as} \\ p_{ba} & p_{bb} & p_{bs} \\ p_{sa} & p_{sb} & p_{ss} \end{pmatrix}$$

This is an example of a Markov chain in which all states are ergodic. It is easy to verify that the vector whose entries are u_k is

$$\begin{array}{ccc} a & b & c \\ (1/3 & 1/3 & 1/3). \end{array}$$

SEMI-MARKOV CHAINS

We adopt the following notation:

$$T = \{0, 1, 2, \ldots\},$$

I represents some fixed set of intergers, and A denotes a Markov chain with random variables A_k and the A_k take on values in I. Initial probabilities will be denoted by $P\{A_o = i\}$ and transition probabilities by

$$p_{im} = P\{A_{k+1} = m \mid A_k = i\}.$$

We would like to introduce the notion of random waiting times in successive states.

We assume that the random variables (A_k, B_k), $k \in T$, define a Markov chain A' with transition probabilities

$$P\{A_{k+1} = m, B_{k+1} = n \mid A_k = i, B_k = j\} = p_{im}\, a_{mn}.$$

<u>Definition 10</u>: For any fixed sequence of events in A', we define t_k by $t_o = 0$ and recursively by $t_{k+1} - t_k = B_k$.

The B_k's are interpreted as waiting times in the states A_k. For any sequence of events we interpret the successive elements of T as "running time," and label the time of the k^{th} transition in A by t_k. That is, we label the time of the transition from A_{k-1} to A_k by t_k. Clearly

$$0 = t_o < t_1 < \ldots < t_k < t_{k+1} < \ldots.$$

We have the following conditions:

$$P\{B_k = j \mid A_k = i\} = a_{ij},$$

$$\sum_{j=1}^{\infty} a_{ij} = 1,$$

$0 \leq a_{ij}$, and

$a_{i0} = 0.$

The following theorem shows us that the introduction of waiting times still leaves us with a Markov chain.

<u>Theorem 5</u>: $P\{A_{k+1} = m \mid A_0, A_1, \ldots, A_k = i, B_0, B_1, \ldots, B_k = j\}$

$= \{P \ A_{k+1} = m \mid A_k = i\}$.

(Note that $A_0, A_1, \ldots, A_k = i$ is an abbreviation of $A_0 = i_0, A_1 = i_1, \ldots, A_k = i$).

Proof: $P\{A_{k+1} = m \mid A_0, A_1, \ldots, A_k = i, B_0, B_1, \ldots, B_k = j\}$

$$= \sum_{n=1}^{\infty} P\{A_{k+1} = m, B_{k+1} = n \mid A_0, A_1, \ldots, A_k = i, B_0, B_1, \ldots, B_k = j\}$$

$$= \sum_{n=1}^{\infty} P\{A_{k+1} = m, B_{k+1} = n \mid A_k = i, B_k = j\}$$

$$= \sum_{n=1}^{\infty} p_{im}\, a_{mn} = p_{im} = P\{A_{k+1} = m \mid A_k = i\}$$

<u>Theorem 6</u>: $P\{B_{k+1} = m \mid A_0, A_1, \ldots, A_{k+1} = i, B_0, B_1, \ldots, B_k = s\}$

$= P\{B_{k+1} = m \mid A_{k+1} = i\}$.

Proof: $P\{B_{k+1} = m \mid A_0, A_1, \ldots, A_{k+1} = i, B_0, B_1, \ldots, B_k = s\}$

$$= \frac{P\{B_{k+1} = m, A_{k+1} = i, (A_0, A_1, \ldots, A_k, B_0, B_1, \ldots, B_k = s)\}}{P\{A_{k+1}=i \ (A_0,A_1,\ldots,A_k=j,B_0,B_1,\ldots,B_k=s)\}\, P\{A_0,A_1,\ldots,A_k=j,B_0,B_1,\ldots,B_k=s\}}$$

$$= \frac{P\{B_{k+1} = m, A_{k+1} = i \mid (A_0, A_1, \ldots, A_k = j, B_0, B_1, \ldots, B_k = s)\}}{P\{A_{k+1} = i \mid A_k = j\}}$$

$$= \frac{P\{A_{k+1} = i, B_{k+1} = m \mid A_k = j, B_k = s\}}{P\{A_{k+1} = i \mid A_k = j\}}\,.$$

$= p_{ji}\, a_{im} / p_{ji} = a_{im} = P\{ B_k = m \mid A_k = i \}$.

Definition 11: For any fixed sequence of events in A, the sequence $\{x_t\}$, $t \in T$, defined by $x_t = A_k$ provided that $t_k \leq t < t_{k+1}$, is called a semi-Markov chain.

Theorem 7: For any subset I_o of I

$$\lim_{t \to \infty} P\{x_t \in I_o \mid x_o = i\} = \lim_{k \to \infty} P\{A_k \in I_o \mid A_o = i\} .$$

Proof: Let $\xi_t = P\{x_t \in I_o \mid x_o = i\}$ and $\alpha_k = P\{A_k \in I_o \mid A_o = i\}$, and note that $\{\alpha_k\}$ is a subsequence of $\{\xi_t\}$. Therefore, if $\{\xi_t\}$ converges to L, then $\{\alpha_k\}$ converges to L.

Suppose that $\{\alpha_k\}$ converges to L. That is, for all $\epsilon > 0$, there exists a positive integer N such that $N \leq k$ implies that $|\alpha_k - L| < \epsilon$. If $\xi_t = \alpha_k$, then $\min \alpha_j \leq \xi_i \leq \max \alpha_j$, whenever $k \leq j$ and $t \leq i$, so that $N \leq k \leq t \leq i$ implies that $|\xi_i - L| < \epsilon$. Hence, if $\{\alpha_k\}$ converges to L, then $\{\xi_t\}$ converges to L.

Therefore $\lim_{t \to \infty} \xi_t = \lim_{k \to \infty} \alpha_k$.

Definition 12: For a fixed sequence $\{A_k\}$ we define a substate chain $\{(x_t, y_t)\}$, $t \in T$, where y_t is given by:

1) $y_t = 0$ uf $t = t_k$, and

2) $y_t = t_{k+1} - t$ if $t_k < t < t_{k+1}$.

This means that if $y_t = 0$, then the semi-Markov process has just reached x_t. If $y_t = m > 0$, then m represents the time the process will remain in x_t.

The transition probabilities for the substate chain are given by

$$P\{x_{t+1} = m,\ y_{t+1} = n \mid x_t = i,\ y_t = j\} = q_{ijmn}$$

where $i, m \in I$ and $j, n \in T$. We note the following cases.

1. If $j = n = 0$, then $q_{ijmn} = a_{ij+1} p_{im}$. This is simply the probability of waiting in state i for one unit of time times the probability of going from i to m.

2. If $j = 1$ and $n = 0$, then $q_{ijmn} = p_{im}$. Knowing that this is the last unit of waiting time before a change of state, $j = 1$, we have the probability of going from i to m.

3. If $i = m$ and $n \geq 1$ and $j = 0$, then $q_{ijmn} = a_{i\ n+1}$. Given that the process is in state $i = m$, this is the probability that it waits here $n + 1$ units of time.

4. If $j \geq 2$ and $n = j - 1$ and $i = m$, then $q_{ijmn} = 1$. If the process is going to wait in state $i = m$ $j \geq 2$ times clearly the next state of the substate chain is $(i, j - 1)$.

5. Otherwise, $q_{ijmn} = 0$.

6. $P\{x_0 = i, y_0 = j\} = \left\{ \begin{array}{ll} P\{x_0 = i\} & \text{if } j = 0 \\ 0 & \text{if } j > 0 \end{array} \right\}.$

Since the transition probabilities for the substate chain are well defined in terms of probabilities that depend only upon the previous state, the substate chain is a Markov chain.

Example: Turning to our chess players again, suppose we introduce the following complication: Berlyov never plays more than one game in a day, and on any given day he is as likely to play as not.

A state in the substate chain might be the event that Schultz wins the championship by defeating Berlyov on the eighth day after the tournament began. This is represented by $(x_8, y_8) = (s, o)$. The probability that he will be champion the following day only is represented by

$$P\{(s,1)|(s,0)\} = q_{s0s1} = a_{s1}\, p_{ss} = (1/2)(3/4) = 3/8.$$

The probability that Berlyov regains the championship the day following that is represented by

$$P\{(b,0)|(s,1)\} = q_{s1b0} = p_{sb} = 1/4.$$

The probability that Berlyov will then retain the championship for three days is represented by

$$P\{(b,3)|(b,0)\} = q_{b0b3} = a_{b4} = (1/2)^4 = 1/16.$$

Letting $i \in I$ and $j \in T$ we adopt the following notation:

$$b_{ij} = \sum_{n>j} a_{in},$$

$$e_i = E\{B_k | A_k = i\} = \sum_{j=0}^{\infty} b_{ij} = \sum_{j=0}^{\infty} j\, a_{ij},$$

$$m_i = \lim_{k\to\infty} P\{A_k = i\},$$

$$m_{ij}(t) = P\{(x_t, y_t) = (i, j)\}$$

$$m_{ij} = \lim_{t\to\infty} P\{(x_t, y_t) = (i, j)\}, \text{ and}$$

$$I^* = \{(i, j) \mid i \in I,\ b_{ij} > 0\}.$$

Notice that

$$P\{(x_t, y_t) \in I^* \mid t \in T\} = 1.$$

Recall that A represents the Markov chain defined on page 21. We let A^* represent the substate chain where $(x_t, y_t) \in I^*$.

Theorem 8: 1) A^* non-null implies that $\{m_{ij} \mid (i, j) \in A^*\}$ is the unique solution of the system

$$v_{ij} > 0, \quad \sum_{i,j=1}^{\infty} v_{ij} = 1, \quad \sum_{i,j=1}^{\infty} v_{ij}\, q_{ijmn} = v_{mn}.$$

2) A^* null implies that this system has no solution.

Proof: This is a consequence of Theorem 4.

<u>Lemma 4</u>: If $\lim_{t\to\infty} b_t = b$, and $\sum_{n=0}^{\infty} a_n = a$, then $\lim_{t\to\infty} \sum_{n=0}^{t} a_n b_{t-n} = ab$.

Proof: Given $\epsilon > 0$, we wish to show that if t is large enough then

$$\left| \sum_{n=0}^{t} a_n b_{t-n} - ab \right| < \epsilon .$$

We note that

$$\left|\sum_{n=0}^{t} a_n b_{t-n} - ab\right| = \left|\sum_{n=0}^{t} a_n b_{t-n} - b\sum_{n=0}^{\infty} a_n\right|$$

$$= \left|\sum_{n=0}^{t} a_n(b_{t-n} - b) - b\sum_{n=t+1}^{\infty} a_n\right|$$

$$\leq \sum_{n=0}^{t}|a_n|\ |b_{t-n} - b| + \left|b\sum_{n=t+1}^{\infty} a_n\right|$$

$$= \sum_{n=0}^{N}|a_n||b_{t-n} - b| + \sum_{n=N+1}^{t}|a_n||b_{t-n} - b| + \left|b\sum_{n=t+1}^{\infty} a_n\right|.$$

We choose N sufficiently large so that

$$\sum_{n=N+1}^{t}|a_n||b_{t-n} - b| < \epsilon/3 \text{ if } t > N.$$

This can be done since $|b_{t-n} - b|$ is bounded and $\sum_{n=0}^{\infty} |a_n|$ is convergent.

The choice of N is independent of t as long as $t > N$.

Next we choose $N_1 \geq N$ sufficiently large so that

$$\left| b\sum_{n=t+1}^{\infty} a_n \right| < \epsilon/3 \text{ if } t > N_1,$$

and also large enough so that t - N is sufficiently large to insure that

$$\sum_{n=0}^{N}|a_n|\ |b_{t-n} - b| < \epsilon/3.$$

This can be done since $|b_{t-n} - b|$ converges to zero for each fixed n. Therefore we have for the choice of t

$$\left| \sum_{n=0}^{t} a_n b_{t-n} - ab \right| < \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon.$$

<u>Theorem 9</u>: $m_{ij} = b_{ij} m_{i0}$.

Proof: Since

$$\begin{aligned} P\{(x_t,y_t) = (i,j)\} &= P\{(x_{t-1},y_{t-1})=(i,j+1)\}P\{(x_t,y_t)=(i,j)\,|\,(x_{t-1},y_{t-1})=(i,j+1)\} \\ &+ P\{(x_{t-1},y_{t-1}) = (i,0)\}P\{(x_t,y_t) = (i,j)\,|\,(x_{t-1},y_{t-1})=(i,0)\} \\ &= m_{i\ j+1}(t-1)q_{i\ j+1\ ij} + m_{i0}(t-1)q_{i0ij} \\ &= m_{i\ j+1}(t-1) + a_{i\ j+1}\, m_{i0}(t-1) \end{aligned}$$

for $1 \le j$ and $1 \le t$, we have

$$m_{ij}(t) = m_{i\ j+1}(t-1) + a_{i\ j+1}\, m_{i0}(t-1).$$

We note that

$$m_{ij}(1) = m_{i\ j+1}(0) + a_{i\ j+1}\, m_{i0}(0)$$

where

$$m_{i\ j+1}(0) = P\ (x_0,y_0) = (i,j+1)\ = 0.$$

Since

$$\lim_{t\to\infty} m_{i0}(t) = m_{i0},\ 0 \le m_{i0}(t) \le 1 \text{ for all } t \in T, \text{ and}$$

$$\sum_{n=1}^{\infty} a_{i\ j+n} = b_{ij} \le 1,$$

we have, by Lemma 4,

$$\lim_{t\to\infty} m_{ij}(t) = \lim_{t\to\infty} \sum_{n=1}^{t} a_{i\ j+n}\, m_{i0}(t-n) = b_{ij}\, m_{i0}.$$

<u>Theorem 10</u>: If A^* is non-null, then $\{m_{io} | i \in I\}$ is the unique solution of the system

$$v_i > 0, \sum_{i=1}^{\infty} e_i v_i = 1, \sum_{i=1}^{\infty} v_i \, p_{im} = v_m.$$

There is no solution of A^* is null.

Proof: Since $0 < m_{ij}$ and $1 \leq b_{ij}$, we have $0 < m_{io}$.

Also

$$\sum_{i=1}^{\infty} e_i v_i = \sum_{i=1}^{\infty} \sum_{j=0}^{\infty} b_{ij} v_i = \sum_{i=1}^{\infty} \sum_{j=0}^{\infty} m_{ij} = 1.$$

Suppose n = 0, then

$$\sum_{i=1}^{\infty} \sum_{j=0}^{\infty} m_{ij} \, q_{ijmn} = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} b_{ij} \, m_{i0} \, q_{ijm0}$$

$$= \sum_{i=1}^{\infty} b_{i0} \, m_{i0} \, a_{i1} \, p_{im} + \sum_{i=1}^{\infty} b_{i0} \, m_{i0} \, a_{i1} \, p_{im}$$

$$= \sum_{i=1}^{\infty} (a_{i1} + b_{i1}) \, m_{i0} \, p_{im}$$

$$= \sum_{i=1}^{\infty} b_{i0} \, m_{i0} \, p_{im}$$

$$= \sum_{i=1}^{\infty} m_{i0} \, p_{im} \, .$$

By Theorem 7 we have

$$\sum_{i=1}^{\infty} \sum_{j=0}^{\infty} m_{ij} \, q_{ijmn} = m_{m0}.$$

This solution is unique since another solution would contradict the uniqueness of

$$\sum_{i=1}^{\infty} \sum_{j=0}^{\infty} m_{ij} \, q_{ijmn}.$$

<u>Theorem 11</u>: Suppose A (and therefore A^*) is aperiodic.

1) A non-null and A^* non-null imply that

$$m_{ij} = b_{ij}\, m_i / \sum_{n=1}^{\infty} e_n\, m_n \;, \; (i, j) \in I^*.$$

2) A non-null and

$$\sum_{n=1}^{\infty} e_n\, m_n < \infty$$

imply that A^* is non-null.

3) A non-null and A^* null imply that

$$m_{ij} = \lim_{n \to \infty} b_{ij}\, m_i / \sum_{n=1}^{\infty} e_n\, m_n \;, \; (i, j) \in I^*.$$

4) A non-null and

$$\lim_{n \to \infty} \sum_{n=1}^{\infty} e_n\, m_n = \infty$$

imply that A^* is null.

5) A null implies that A^* is null.

Proof: 1) A non-null and $m_i = \lim_{k \to \infty} P\{A_k = i\}$, $i \in T$ imply that $m_i \neq 0$. A^* non-null and $m_{i0} = \lim_{t \to \infty} P\{(x_t, y_t) = (i, 0)\}$ imply that $m_{i0} \neq 0$. Hence there exists a $\lambda > 0$ such that $m_i = \lambda\, m_{i0}$. Since

$$\sum_{n=1}^{\infty} e_n\, m_{n0} = 1, \; \lambda = \lambda \sum_{n=1} e_n\, m_{n0}.$$

Hence

$$m_i = (\lambda \sum_{n=1}^{\infty} e_n\, m_{n0})\, m_{i0} = m_{ij}/b_{ij} \sum_{n=1}^{\infty} e_n (\lambda m_{n0}).$$

That is,

$$m_{ij} = b_{ij}\, m_i / \sum_{n=1}^{\infty} e_n\, m_n.$$

2) Since A non-null implies that $m_i \neq 0$ we have that

$$m_i / \sum_{n=1}^{\infty} e_n m_n > 0.$$

Also

$$\sum_{i=1}^{\infty} e_i \left(m_i / \sum_{n=1}^{\infty} e_n m_n\right) = \sum_{i=1}^{\infty} e_i m_i \Big/ \sum_{n=1}^{\infty} e_n m_n = 1,$$

and

$$\sum_{i=1}^{\infty} \left(m_i \Big/ \sum_{n=1}^{\infty} e_n m_n\right) p_i = m_n \Big/ \sum_{n=1}^{\infty} e_n m_n.$$

Hence A^* is non-null.

3) By 2) A^* null implies that

$$\lim_{n \to \infty} \sum_{n=1}^{\infty} e_n m_n = \infty .$$

Hence, since $b_{ij} \neq 0$ and $m_i \neq 0$, we have

$$\lim_{n \to \infty} b_{ij} m_i \Big/ \sum_{n=1}^{\infty} e_n m_n = 0.$$

Therefore,

$$m_{ij} = \lim_{n \to \infty} b_{ij} m_i \Big/ \sum_{n=1}^{\infty} e_n m_n = 0.$$

4) By 3) and 1) we have

$$m_{ij} = \lim_{n \to \infty} b_{ij} m_i \Big/ \sum_{n=1}^{\infty} e_n m_n = 0$$

so that A^* is null.

5) Suppose A^* is non-null. We have that $1 \leq e_i$ implies

$$0 < \sum_{n=1}^{\infty} m_{n0} \leq \sum_{n=1}^{\infty} e_n m_{n0} = 1.$$

Hence, for all i

$$0 < m_{i0} \Big/ \sum_{n=1}^{\infty} m_{n0} .$$

Also

$$\sum_{i=1}^{\infty} m_{i0} \Big/ \sum_{n=1}^{\infty} m_{n0} = \sum_{i=1}^{\infty} m_{i0} \Big/ \sum_{n=1}^{\infty} m_{n0} = 1,$$

and

$$\sum_{i=1}^{\infty} \left(m_{i0} \Big/ \sum_{n=1}^{\infty} m_{n0}\right) p_{in} = m_n \Big/ \sum_{n=1}^{\infty} m_{n0}.$$

Hence A is non-null.

Since $m_{ij}(t) = P\{(x_t, y_t) = (i, j)\}$,

$$P\{x_t = i\} = \sum_{j=0}^{\infty} m_{ij}(t).$$

Also, $b_{i0} = 1$ and

$$b_{ij} = \sum_{n > j} a_{in}$$

and

$$m_{ij}(t) = \sum_{n=1}^{t} a_{i\ j+n}\, m_{i0}(t-n), \quad 1 \leq j,\ 1 \leq t,$$

give us

$$\sum_{j=0}^{\infty} m_{ij}(t) = \sum_{n=0}^{t} b_{in}\, m_{i0}(t - n).$$

<u>Theorem 12</u>: Suppose A is aperiodic and $e_i < \infty$.

1) A null implies that

$$\lim_{t \to \infty} P\{x_t = i\} = 0 = \sum_{j=0}^{\infty} m_{ij}.$$

2) A non-null and

$$\lim_{n \to \infty} \sum_{n=1}^{\infty} e_n\, m_n = \infty$$

imply that

$$\lim_{t \to \infty} P\{x_t = i\} = 0 = \sum_{j=0}^{\infty} m_{ij}.$$

3) A non-null implies that

$$\lim_{t\to\infty} P\{x_t = i\} = e_i m_i \Big/ \sum_{n=1}^{\infty} e_n m_n = \sum_{j=0}^{\infty} m_{ij}.$$

Proof:

1) For all k, $0 \le m_{i0}(k) \le 1$ and $\lim_{k\to\infty} m_{i0}(k) = m_{i0}$ and $\sum_{n=0}^{\infty} b_{in} = e_i$

imply that

$$\lim_{t\to\infty} \sum_{n=0}^{\infty} b_{in} m_{i0}(t - n) = e_i m_{i0}$$

by Lemma 3. Also, A null implies A^* is null and $m_{ij} = 0$ for all $j \ge 0$

so that

$$\sum_{j=0}^{\infty} m_{ij} = 0.$$

Hence

$$\sum_{j=0}^{\infty} b_{ij} m_{i0} = 0$$

implies that $e_i m_{i0} = 0$. Therefore, we have

$$\lim_{t\to\infty} P\{x_t = i\} = \lim_{t\to\infty} \sum_{n=0}^{t} b_{in} m_{i0}(t - n) = e_i m_{i0} = 0.$$

2) A non-null and

$$\lim_{t\to\infty} \sum_{n=1}^{t} e_n m_n = \infty$$

imply that A^* is null and the above argument applies.

3) A non-null implies

$$m_{ij} = b_{ij} m_i \Big/ \sum_{n=1}^{\infty} e_n m_n$$

so that

$$\sum_{j=0}^{\infty} m_{ij} = \left(m_i \Big/ \sum_{n=1}^{\infty} e_n m_n\right) \sum_{j=0}^{\infty} b_{ij} = e_i m_i \Big/ \sum_{n=1}^{\infty} e_n m_n.$$

<u>Example</u>: Consider the example of the chess tournament. Suppose we wish to know the probability that Schultz is champion for three days. We make the following computations.

$$\lim_{t\to\infty} P\{(x_t, y_t) = (s, 2)\} = m_{s2},$$

$$e_i = \sum_{j=0}^{\infty} j(1/2^{j+1}) = 1,$$

$$b_{s2} = \sum_{n>2} 1/2^{n+1} = 1 - 1/2 - 1/4 - 1/8 = 1/8,$$

$$\sum_{n=1}^{\infty} e_n m_n = \sum_{n=1}^{\infty} m_n = 1/3 + 1/3 + 1/3 = 1,$$

so that

$$m_{s2} = b_{s2} m_s / \sum_{n=1}^{\infty} e_n m_n = 1/24.$$

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