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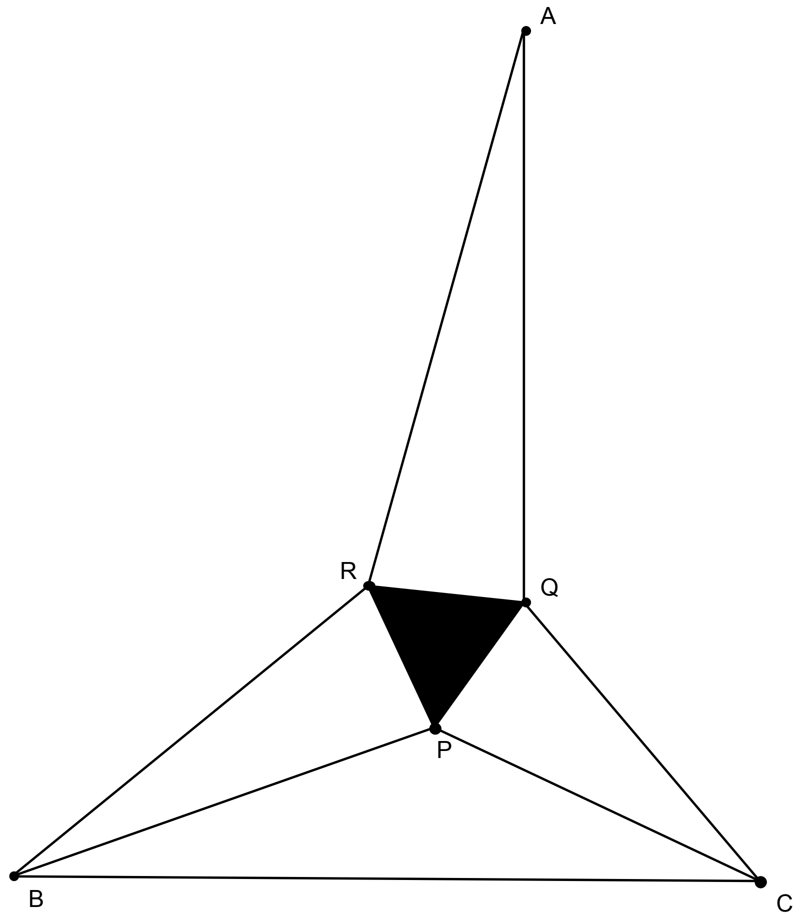
This paper deals with an unannounced theorem by Frank Morley that he originally published amid a collection of other, more general, theorems. Having intrigued mathematicians for the past century, it is now simply referred to as Morley's trisector theorem:

> *The three intersections of the angles of a triangle, lying near the three sides respectively, form an equilateral triangle.*

An example construction of this theorem can easily be made using Geometer's Sketchpad® (see Construction #1), but the purpose of this paper is to construct a formal proof of this theorem for any triangle. The importance of this problem is perhaps not so much in its applications to other areas of geometry, but in the ingenuity behind it. For thousands of years people had worked with compasses and straight edges in attempts to trisect the angle, but for some reason no one had ever noticed the equilateral triangle that forms when all of the trisectors of a triangle are constructed. It is also a great exercise in complex applications of simple geometry and truly reveals the power of just a few basic theorems. While some methods of proving this problem involve trigonometric techniques, the proof presented here is based solely on rudimentary high school geometry.

This problem first appeared in Morley's (1900) paper entitled "On the Metric Geometry of the Plane *n*-line" in the first issue of the *American Mathematical Society Translations* (Baker, 1978, p.737). However, Morley included it as a very specialized case of one of his more general theorems, so for quite a few years it was rather overlooked by much of the mathematical community. Morley's youngest son, Frank V. Morley believed his father's unwillingness to declare his result was due to a certain amount of uncertainty on Morley's part as

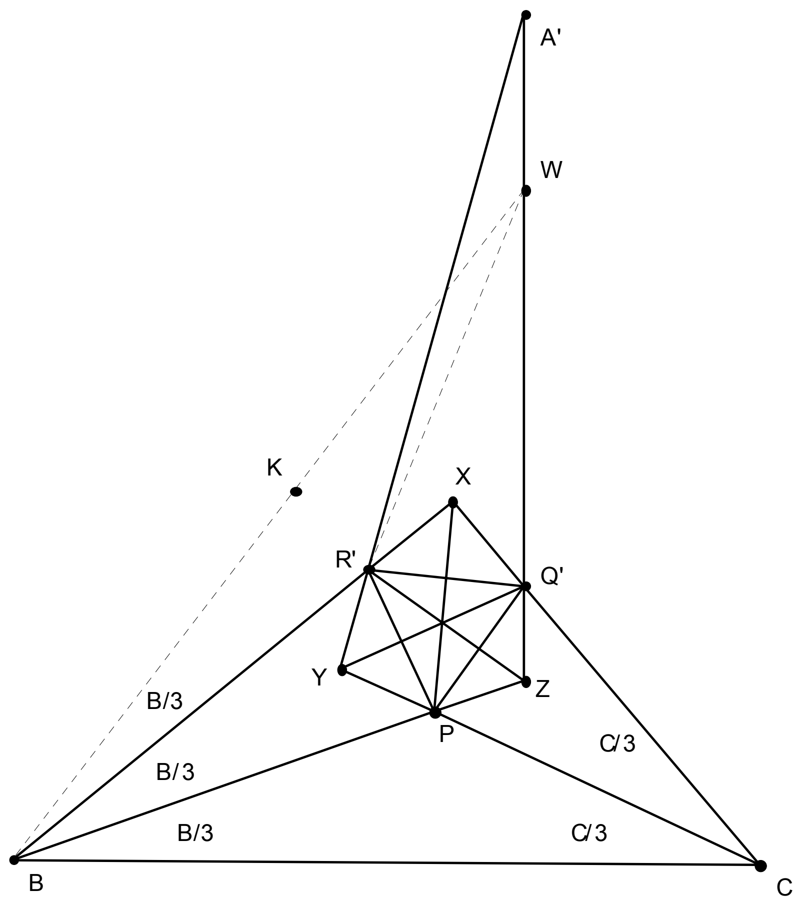


Construction #1 of Morley's Trisector Theorem

to his original discovery of the theorem. (740). Morley had trouble believing that such an eye-catching result could have been overlooked for the past 2,000 years and didn't want to be overly confident about his ingenuity until he was sure no one else had already observed it. For this reason, he used only "quiet, semi-private mentions" of the theorem to colleagues in the U.S.A, Britain, Europe, and the Far East in an effort to see if it had already been noticed by someone else before him (Baker, 1978, p.741). After no promising leads, he was eventually forced to put his name to the trisector theorem as its discoverer.

Despite his reluctance in advertising his "jewel," Morley did tell a few friends, "such as Richmond at Cambridge and Whittaker at Edinburgh," about his result "and by 1904 it had become public" (Baker, 1978, p.738). For the next decade the problem spread throughout Europe and intrigued mathematicians everywhere, but it was not until 1914 that a formal proof was offered up to the world (Kay, 2001, p.8). This first proof by Taylor and Marr would be followed by other more elegant proofs by Satyanarayana and Naraniengar, and by 1920 the problem had created enough interest to be set in St. John's group of Entrance Scholarships (738). To this day the problem is a challenging exercise for any mathematician, since there has been no overly simple proof of its claim. As with any good problem, a fair amount of creativity and work is needed to come up with the answer.

This proof of Morley's theorem is based on the figure in Construction #2, borrowed from David C. Kay (2001, p.223). It is based on ΔABC, which we assume is an existent triangle with points X', Q', and R' constructed as shown, where $m\angle XPQ' = m\angle XPR' = 30$ and $\Delta PQ'R'$ is equilateral. Note that P is the in-center of ΔBCX and $\overrightarrow{XP}$ bisects $\angle BXC$, and let Y and X be the intersections of the altitudes of $\Delta PQ'R'$ from Q' and R' with rays $\overrightarrow{CP}$ and $\overrightarrow{BP}$. From the angle measures to be obtained later in this proof, it can easily be seen that $m\angle YR'Q' + m\angle ZQ'R' > 180$, so rays $\overrightarrow{YR'}$ and $\overrightarrow{ZQ'}$ meet at some point A'.



Construction #2

First, observe that since $\overline{R'Z}$ is the perpendicular bisector of $\overline{PQ'}$, then $m\angle PQ'Z = m\angle Q'PZ = 180 - m\angle Q'PB = 180 - (m\angle XPB + 30)$, since $m\angle Q'PX = 30$. Now, look at ΔXPB. Since $\overrightarrow{XP}$ bisects $\angle BXC$, then

$$m\angle XPB = 180 - B/3 - \frac{1}{2}\left(180 - \frac{2}{3}B - \frac{2}{3}C\right)$$
$$m\angle XPB = 180 - B/3 - 90 + B/3 + C/3$$
$$m\angle XPB = 90 + C/3$$

So going back, $m\angle PQ'Z = 180 - (m\angle XPB + 30)$, now implies that

$$m\angle PQ'Z = 180 - (m\angle XPB + 30)$$
$$m\angle PQ'Z = 180 - (90 + C/3 + 30)$$
$$m\angle PQ'Z = 60 - C/3$$

and similarly, it can easily be seen that $m\angle PR'Y = 60 - B/3$.

We are now ready to establish the measure of angle A'. Using the concept of supplementary angles and the fact that $m\angle PQ'R' = 60$, we can see that $m\angle A'Q'R' = 180 - (60 + m\angle PQ'Z)$, and using what we have just found,

$$m\angle A'Q'R' = 180 - (60 + m\angle PQ'Z)$$
$$m\angle A'Q'R' = 180 - 60 - 60 + C/3 \quad .$$
$$m\angle A'Q'R' = 60 + C/3$$

Similarly, we find that

$$m\angle A'R'Q' = 180 - (60 + m\angle PR'Y)$$
$$m\angle A'R'Q' = 180 - 60 - 60 + B/3$$
$$m\angle A'R'Q' = 60 + B/3$$

and using the sum of the interior angles of triangle $\Delta A'Q'R'$, we see

$$m\angle A' = 180 - m\angle PQ'Z - m\angle PR'Y$$
$$m\angle A' = 180 - (60 + C/3) - (60 + B/3) \quad .$$
$$m\angle A' = 60 - C/3 - B/3$$

Now from our assumption that ΔABC exists we know that

$$m\angle A = 180 - m\angle B - m\angle C$$
$$\frac{m\angle A}{3} = \frac{180}{3} - \frac{m\angle B}{3} - \frac{m\angle C}{3},$$

and we write $A/3 = 60 - B/3 - C/3$, which should look familiar. Thus $m\angle A' = A/3$.

Now let us consider $m\angle A'ZB$. Since $\overline{R'Z}$ is the perpendicular bisector of $\overline{PQ'}$, then

$$m\angle A'ZB = 2 \cdot m\angle R'ZQ'$$
$$m\angle A'ZB = 2 \cdot (90 - mPQ'Z) \quad .$$
$$m\angle A'ZB = 2 \cdot (90 - (60 - C/3)) = 60 + \frac{2C}{3}$$

Similarly,

$$m\angle A'YC = 2 \cdot m\angle R'YQ'$$
$$m\angle A'YC = 2 \cdot (90 - m\angle PR'Y)$$
$$m\angle A'YC = 2 \cdot (90 - (60 - B/3)) = 60 + \frac{2B}{3}$$

Now construct ray $\overrightarrow{BK}$ such that $\overrightarrow{BC} - \overrightarrow{BX} - \overrightarrow{BK}$ and $m\angle KBC = B$. Consider that $m\angle A'ZB = 60 + \frac{2C}{3}$ and $m\angle KBZ = \frac{2B}{3}$, so $m\angle A'ZB + m\angle KBZ = 60 + \frac{2C}{3} + \frac{2B}{3}$. So now there are two cases: either $\overrightarrow{BK}$ and $\overrightarrow{ZA'}$ are parallel or they intersect at some point. However, if $\overrightarrow{BK}$ and $\overrightarrow{ZA'}$ are parallel, then, $m\angle A'ZB + m\angle KBZ = 60 + \frac{2C}{3} + \frac{2B}{3} = 180$, so

$$60 + \frac{2C}{3} + \frac{2B}{3} = 180$$
$$C + B = 180$$

which gives a contradiction since we assume that *ΔABC* exists. Therefore $\overrightarrow{BK}$ must intersect $\overrightarrow{ZA'}$ at some point which we will label *W*.

Now consider the line $\overleftrightarrow{RZ}$ were we to extend the segment indefinitely. We can easily see that ∠*WBR'* and ∠*R'BZ* slice off the same sections of $\overleftrightarrow{RZ}$ as ∠*BWR'* and ∠*R'WZ*, respectively. This implies that the ratio of $\frac{\angle WBR'}{\angle R'BZ}$ must be the same as the ratio of $\frac{\angle BWR'}{\angle R'WZ}$. However, we know that ∠*WBR'* = ∠*R'BZ* = *B*/3. Thus, ∠*BWR'* and ∠*R'WZ* must be equal as well. Since we know that their sum is equal to 2(*A*/3), then we can easily see that ∠*BWR'* = ∠*R'WZ* = *A*/3. This implies that $m\angle R'WZ' = m\angle R'A'Q' = A/3$ and therefore *W=A'*. Given this we must conclude that *m*∠*A'BC* = *B* as well.

Similarly, we can find *m*∠*A'CB* = *C*, which when combined with *m*∠*A'BC* = *B* implies that *m*∠*BA'C* = *A*. Thus *A'* = *A*, *Q'* = *Q*, *R'* = *R*, and therefore $\Delta PQR \equiv \Delta PQ'R'$, which by assumption is equilateral. So we see that for any triangle *ΔABC* we can use the angle trisectors to construct an equilateral triangle *ΔPQR*, and that Morley's Trisector Theorem is indeed true.

In conclusion, we have shown the validity of Morley's Theorem and that its proof need not involve any mathematical knowledge past that one might find in a high school geometry class. It is really just a matter of complexity and being able to look at the problem in new and creative ways. It would be interesting to look at the more general theorems under which Morley published this trisector theorem and to see what other generalizations of it he worked out. Another possible extension would be to see if an analogous construction might exist for other polygons with greater numbers of sides.

If nothing else has been learned from paper, we can at least taste the excitement and amazement Morley experienced whenever he first found this result. It seems preposterous that mathematicians could have missed such a simple and straightforward construction for so many years, but it is really quite reasonable when we consider the restraints Euclid imposed on the world of geometry. By not being able to construct angle trisectors with a compass and straightedge, people never really considered working with them too awfully much. Perhaps the genius in Morley's talent was not for the mathematics itself, but more in being able to see the world in new and different ways.

References

Baker, J. C.,and Cletus O. O. (1978). The Morley Trisector Theorem. *The American Mathematical Monthly*. Vol. 85, No. 9, *737-741*.

Kay, D. C. (2001). *College Geometry*. New York: Addison Wesley Longman, Inc.