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Introduction

Through this research into the solution of the cubic, I hope to learn the thoughts of mathematicians during this period of history. This knowledge must be conveyed in a way that is historically and mathematically accurate, yet comprehensible and readily available to use in a class of eighth grade math students. It is my wish that as we begin to delve into the world of algebraic and graphical solutions to the problems in the eighth grade text, students will be able to make a connection with this information to that they are currently using to solve their problems. They should be made aware of the importance of math throughout history and able to enjoy parts of this problem-solving experience. It is commonly assumed that solving the quadratic equation is the algebraic pinnacle of the middle school curriculum. Through this paper and by using a historical approach I hope to present the cubic to future eigth graders. The motivation for doing this is follows.

As we look into the math classroom, we see students from diverse backgrounds struggling to make sense of information that they do not necessarily find relevant to their own lives. As a teacher, it is my duty to give students all of the information and opportunity to understand the value of these concepts. Through making mathematicians come to life, explaining how they struggled to arrive at these solutions and how they applied them to making a name for themselves, I hope that students will begin to see the important role that math once took, and that they are studying material that took lifetimes to create. Students should leave our classes with a respect for the subject and the authors of mathematics. It is my conviction that sharing some of the personal histories of those that created math will help them to develop this respect.

Background of the Problem

Looking into the solution for the cubic is not a new concept, nor was it in Cardano's day (the 16th century). The problem of the cubic had been troubling scholars since the early times of the Babylonians and Egyptians. Sometime between 2000 and 400 B.C., the Babylonians created the formula for solving the quadratic, an important step in Cardano's method for solving the cubic (Magnussen). And it has been found that the Babylonians in the 19th century B.C. found a table for solving cubic equations in which the problems would have integer solutions and be used in measuring the dimensions of an excavated room (Friberg). Mathematicians of the ancient Indian culture used their mathematical prowess for building elaborate sacrificial altars.

The Greeks, too, had a use for these complex math problems. It is said that their greatest challenge to prove their worthiness to the gods was to construct an altar that was in the shape of a cube twice the volume of the existing altar (Magnussen). This was a difficult problem brought to the people in a time of great wars and strife. As they had such difficulty managing the mathematics of this construction, many interpreted this charge one for the people to give up their wars and study mathematics as a people of peace.

The myth of the "Curse of the Cubic" came into being around this time. One source defines the curse as follows:
As he lay dying from the sword wounds inflicted by a Roman soldier, Achimedes uttered a curse to those who tried to solve algebraic problems:

> "*Lines and planes you may resolve;*
> *Cubes and others never solve!*" (Magnussen and Suzuki)

At this point mathematical advances in the struggle for the solution to the cubic took a rest. As empires changed, math was not forgotten, but development within the field came to a standstill until the Italians in the 16th century.
Work in the 16th century was, for the most part, secretive and well-guarded. There were many working in the world of mathematics as teachers and scholars, trying to make new discoveries, and struggling to come out on top in competitions of mathematical skill. During this time, high-stake public challenges of intellect occurred regularly. A challenger could ruin a reputation by putting forth 30 problems that his opponent would have around 40 days to solve before having to present his solutions in front of an audience. There would be problems provided to both men, and the winner would have solved more of the other man's problems accurately.

Due to his fear of a possible challenge, Scipione dal Ferro (1465-1526) kept his work a secret. Around 1515, dal Ferro was able to solve a cubic of the form $x^3 + mx = n$. On his deathbed in 1526, however, not wanting the progress to stop, he informed his student Antonio Fior of this great accomplishment. Fior also had the desire to keep this a secret until he could use it in a challenge to create a better name and reputation for himself. Upon challenging Niccolo of Brescia, known as Tartaglia, to a duel, Fior was joined in learning this solution. Tartaglia, during the days of his challenge, facing a set of compressed cubics in the form of $x^3 + mx = n$, derived the solution at the end of his time to solve these problems and saved his reputation. Due to the attention this contest drew, Cardano learned that there existed a solution and began appealing to Tartaglia to share his method to be published in Cardano's up and coming work on algebra, *Ars Magna*.

At last, after several attempts to get Tartaglia to reveal his method, Tartaglia conceded and presented to Cardano the method of solving the cubic encrypted in a poem, making

Cardano promise not to publish the information. The poem however, translated below, Cardano thought rather obscure.

When the cube and its things near
Add to a new number, discrete,
Determine two new numbers different
By that one; this feat
Will be kept as a rule
Their product always equal, the same,
To the cube of a third
Of the number of things named.
Then, generally speaking,
The remaining amount
Of the cube roots subtracted
Will be our desired count. (Laubenbacher et al. 235-236)

To create a situation when we can use this poem to solve the cubic, we first must have a cubic of the correct form "when a cube and its things near / Add to a new number, discrete" means that we need an equation of the form $x^3 + mx = n$, or, by rearranging terms, $x^3 + mx - n = 0$. Say we are given the equation $x^3 + 3x - 9x - 27 = 0$. To obtain the correct form, we must "get rid" of the pesky x^2 term. We accomplish this by making a substitution in the form $x = y - \frac{a}{3}$, where a is the coefficient of the second term of our equation. In this case, $a = 3$, and our substitution will be
$x = y - 1$. The work behind this substitution could be arranged thus:

$(y-1)^3$:	y^3	$-3y^2$	$+3y$	-1
$3(y-1)^2$:		$3y^2$	$-6y$	$+3$
$-9(y-1)$:			$-9y$	$+9$
-27:				-27
	y^3	$+0y^2$	$-12y$	-16,

or $y^3 - 12y - 16 = 0$

Now that we have our desired form of the cubic, we will "Determine two new numbers different / By that one," which we will do by making another substitution. This substitution will be of the form $y = w + \frac{a}{w}$, where $a = \frac{-p}{3}$, where p is the coefficient of the y-term in our equation. In this case, $p = -12$, so $a = \frac{-(-12)}{3} = 4$, so we can make our substitution with the equation $y = w + \frac{4}{w}$:

$$(w+\frac{4}{w})^3: \qquad w^3 + 3w^2\left(\frac{w}{4}\right) + 3w\left(\frac{16}{w^2}\right) + \frac{64}{w^3}$$

$$-12\left(w+\frac{w}{4}\right): \qquad -12w \quad -\frac{48}{w}$$

$$-16: \qquad\qquad -16$$

$$w^3 - 16 + \frac{64}{w^3} = 0$$

Next, we can make this cubic look like a quadratic equation, something that we (as well as Tartaglia and Cardano) know how to solve. In order to do this, we must multiply by w^3:

$\left(w^3\right)^2 - 16\left(w^3\right) + \frac{64}{w^3}\left(w^3\right) = 0 \Rightarrow \left(w^3\right)^2 - 16\left(w^3\right) + 64 = 0$. Now that we have a quadratic equation in w^3, we can apply the quadratic formula. In this equation, $a = 1$, $b = -16$, and $c = 64$: $w^3 = \frac{16 \pm \sqrt{(-16)^2 - 4(1)(64)}}{2(1)} = 8 \pm \frac{\sqrt{0}}{2} = 8$. Now knowing that $w^3 = 8$, we can find that $w = \sqrt[3]{8} = 2$. Now that we have w, we must recall that we had a substitution earlier, in that $y = w + \frac{4}{w}$. But now we can use Tartaglia and Cardano's Method, revealed in the last lines of the poem, "Then, generally speaking, / The remaining amount / Of the cube roots subtracted / Will be our desired count." We can take $w = \sqrt[3]{8} = 2$ to be our values for a and b and make the computation necessary to gain our three solutions:

$$a + b = 2 + 2 = 4$$

$$aa + bb = 2\left(\frac{-1+\sqrt{-3}}{2}\right) + 2\left(\frac{-1-\sqrt{-3}}{2}\right) = -1 + \sqrt{-3} + -1 - \sqrt{-3} = -2$$

$$ba + ab = 2\left(\frac{-1-\sqrt{-3}}{2}\right) + 2\left(\frac{-1+\sqrt{-3}}{2}\right) = -1 - \sqrt{-3} + -1 + \sqrt{-3} = -2$$

But now we must recall that we made a substitution before this point ($x = y - 1$), as we had a cubic in x as our original equation. To find our roots in x, we must take these y-values we have just calculated as a result of Cardano's Formula, and subtract one from each, yielding: *3, –3*, and *–3* respectively. We can check this in the original equation $x^3 + 3x^2 - 9x - 27 = 0$, and we see that the values hold.

With the knowledge that Cardano obtained from Tartaglia, he was able to apply this method to solving the cubic, but was confused by the numbers that it produced. He could not find a way to explain or describe these "complex numbers," and claimed they were rather useless. It was not until 1572 that Rafaele Bombelli was able to make sense of them.

Cardano's work and insights, however, led to a general solution for the cubic. As he searched for the roots of the equations, Cardano's trouble with imaginary numbers was not all that he had to accept. Cardano's work with the negative numbers was well-received only by bankers who had dealt with the gaining and spending of money. This new concept of having a solution that was less than nothing, in a time when the concept of the number zero was just becoming accepted was a very radical thought. That he developed an acceptance of the inevitability of these negative solutions and led the way to their acceptance as well as the acceptance of negative radicals was quite an impressive task.

In his publication of *Ars Magna* in 1545, Cardano brought forward the thinking of cubics and quartics with the acknowledged help of some of his contemporaries. He gives credit to himself throughout the book, as well as saying:

> *In our own days Scipione del Ferro of Bologna has solved the case of the cube and first power equal to a constant, a very elegant and admirable accomplishment. Since this art surpasses all human subtlety and the perspicuity of moral talent and is a truly celestial gift and a very clear test of the capacity of men's minds, whoever applies himself to it will believe that there is nothing that he cannot understand. In emulation of him, my friend Niccolo Tartaglia of Brescia, wanting not to be outdone, solved the same case when he got into a contest with his [Scipione's] pupil, Antonio Maria Fior, and, moved by my many entreaties, gave it to me. For I had been deceived by the world of Luca Paccioli, who denied that any more general rule could be discovered than his own. Notwithstanding the many things which I had already discovered, as is well known, I had despaired and had not attempted to look any further. Then, however, having received Tartaglia's solution and seeking for the proof of it, I came to understand that there were a great many other things that could also be had. Pursuing this thought and with increased confidence, I discovered these others, partly by myself and partly through Lodovico Ferrari, formerly my pupil.* (Cardano 8-9)

It is through this quote in his first chapter of *Ars Magna* or *The Great Art*, that Cardano makes clear that although he has derived solutions to the 12 other forms of the cubic, the solution to the form $x^3 + mx = n$ is an idea belonging to del Ferro and Tartaglia. He further makes it clear that knowing this solution gave him renewed confidence and vigor needed to find solutions to all of the forms. (It should be noted that at this time, due to restrictions in algebraic notation and the newly accepted notion of negative numbers, all 13 of these cases were necessary before a general solution could be derived (Cardano *xiii*).)

Conclusions and Implications

Although Cardano successfully published many works on math, science, medicine, and many other topics, his popularity did not last long. Perhaps by the "Curse of the Cubic," he was doomed to a life as an outcast. After the publication of *Ars Magna*, Tartaglia, once his friend, began a campaign to ruin Cardano's reputation. At this same time, Cardano's son, finding that his wife had been unfaithful, poisoned and killed her. After working hard to defend a son who was convicted and executed for murder, Cardano found himself to be a hated man. Father of a murderer and a prostitute, Cardano was also persecuted for his forward thinking in areas the church thought dangerous. Cardano was sentenced to jail in 1570 on the charge of heresy for supposedly casting the horoscope of Jesus Christ. After his release from prison to his death in 1576, Cardano was banned from teaching and from publishing any more of his work.

Cardano's life was an interesting one. Blame it on the "Curse of the Cubic," but he experienced tragedy and strife along with the successes of his discoveries in the field of math. His writings on the cubic solution brought about the gateway into complex numbers and the solution to the quartic in a general form. In fact, "Cardano's Solution is no mere textbook proof…it introduced the concept of imaginary numbers, forcing mathematicians to rethink the relationship between mathematics and nature, and to explore the idea of what is meant by the word 'real.' In this sense, Cardano's mathematics was more philosophical than practical" (Ashworth). His works in all subjects, and especially in math, during a time of secrecy in discoveries, made expansion in the world of algebra possible.

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