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Ballou (2005) provides a possibility for the introduction of the cubic into the eighth grade mathematics curriculum. As seen in her paper, the solution of the cubic essentially involves the use of a clever substitution in the general cubic in order to reduce it into a cubic without the square term, which in turn is factorable as a quadratic provided one makes another substitution. A natural question to ask ourselves, which mathematicians in the colorful theory of equations undoubtedly did as well, is how far can we push this technique of clever substitutions to solve higher degree equations. These substitutions are called Tschirnhaus transformations and have a pattern of the form $x = y - a/n$. In Ballou's (2005) paper the transformation used in the cubic was $x = y - a/3$ which allowed us to get rid of the so-called pesky x^2 term. For the general quartic the transformation is $x = y - a/4$ which in turn transforms the equation into a cubic solveable by Cardano's method. The question is what happens when we try our technique of Tschirnhaus transformations into the general quintic. To find out we first need to make a sidetrack into a little history packed with drama.

Shrouded in mystery and legend, the facts and fiction of the life of the great, but young mathematician, Evariste Galois, come together to create a captivating tale. Yet, it is unnecessary to embellish Galois' biography with intrigue and tragedy to have an interesting story. It is enough that by the age of twenty Galois discovered a method to determine the solvability of an equation and set the foundation for the branch of mathematics called Group theory. So many myths hover around Galois' contributions that not addressing these exaggerations but only looking at his mathematics is impossible. Therefore, this paper will discuss the important mathematical ideas Galois gave to the world as well as the myths surrounding the birth of those ideas.

Galois' contributions were at a level of mathematics most people don't encounter in their everyday lives. In fact, most high school math students won't have the opportunity to be exposed to Galois' ideas until they are in college and, most likely, only if they are mathematics majors. So why is it important for an elementary or secondary mathematics teacher to study Galois theory? To answer this let us look at the fundamental goal of beginning algebra: finding the solution(s) to an algebraic equation.

In high school algebra, solutions generally come easy. Occasionally a problem is presented where no solutions exist, but most often students are rewarded with "nice" answers. Quadratic equations pose some challenges such as having more than one answer and non-real solutions, but they *do have* solutions. As we move up degree by degree on the polynomial ladder from cubics to quartics etc, solutions to equations are no longer so "nice". Understanding the relationship between the behavior of polynomials and their solutions allows the mathematics teacher a vista of the polynomial landscape enabling the teacher to assist students in making connections about polynomials in general. Galois' work gives ultimate clarity about polynomials and their solutions.

The history of solving equations goes back more than 4000 years ago when a general solution to the quadratic equation was found. After finding the solution to the quadratic it follows that other solutions to higher degree polynomials were close behind. However, it wasn't until the 16th century that a mathematician named Cardano was finally able to find general solutions for cubic and quartic equations. It seemed that with each higher degree of a polynomial equation the complexity of the solutions also increased exponentially. This was the case with the quintic. But just as Cardano was successful in finding solutions to all quartic and quintic equations other mathematicians believed they could find general solutions to the quintic equation. After all, it is only to the 5th power.

Many brilliant mathematicians tangled with the quintic trying to make it behave like its juniors the cubic and quartic. Euler, Bezout, Malfatti, and Lagrange all tried but were unsuccessful in finding the solutions. These mathematicians were unsuccessful for one particular reason. In 1799, Ruffini, a mathematician, claimed that algebraic equations of 5th degree have no general solutions. A sufficient proof of this was not given until 1826 when Abel showed that this was true.[1] This development elicits the question, if quintics aren't solvable algebraically, which polynomials are solvable and which are not? Galois, barely a teenager at the time of Abel's proof, was on the way to discovering an answer to this question.

Although many myths surround Galois and his work, most historians agree about the details of Galois' childhood. Born in France on October 25, 1811, Galois was educated at home by his mother until the age of 12. He was considered an unremarkable student until he took his first mathematics class in 1827. His instructor, M. Vernier, wrote "It is the passion for mathematics which dominates him, I think it would be best for him if his parents would allow him to study nothing but this, he is wasting his time here and does nothing but torment his teachers and overwhelm himself with punishments,".[2] Despite this commendation Galois twice failed the exam to get into Ecole Polytechnique, the most prestigious university in Paris, and settled for enrollment in Ecole Normale where he continued his mathematical research. During this time is where we encounter the first Galois myth.

In 1929, at the age of 17, Galois submitted to Cauchy at the Academie des Sciences his writings on the algebraic solution of equations. Cauchy was to review the papers and then present them to the Academie for possible publication. The story goes that Cauchy misplaced the articles and they were lost forever. It is unclear why this myth began. Demonstrating just how prevalent this myth still is today, Beachy and Blair (1996) write " . . . Galois presented two papers on the solution of algebraic equations to the Academie des Sciences des Paris. Both were sent to Cauchy who lost them (pg 316)"[3]. However it is Cauchy's own words that dispel this misconstruction. In a letter written January 18, 1830, Cauchy says he is unable to attend the session where he was to present Galois' work and requests another time to discuss the papers.[4] This verifies that Cauchy was in possession of Galois' work and was interested in submitting it. Cauchy, however, did not present the material at a future meeting but it is believed that he encouraged Galois to submit his research for the Grand Prize in mathematics. A few months later Galois submitted his manuscripts *On the condition that an equation be solvable by radicals* for the contest.[4] Unfortunately, this article did get lost or was not read and Galois is not considered for the prize.

One year later, Galois made a third attempt and submitted another version of his work to the Academy. The Academy was able to hold onto the article and Lacroix, Poisson, Lengendre and Poinsot, all esteemed mathematicians, finally reviewed the manuscript. Although these men held in their possession an incredible mathematical achievement, apparently they were not able to make heads nor tails of it and it was rejected. Poisson admits that he was unable to comprehend Galois' work and encouraged Galois to develop his theory further and in a larger mathematical context where it might be illuminated.[4] Due to misplaced papers and rejections, Galois began to believe that the Academy was against him in some way. Although the facts don't truly support this, future biographers exaggerated the circumstances surrounding these incidents and, consequently, myths began to develop supporting Galois' slight paranoia. It doesn't appear that anyone was trying to prevent Galois' work from coming out; it was just a series of unfortunate events and poor timing.[4]

Under normal circumstances Galois would have many more opportunities to develop and submit his work but, being a radical and antagonizing the French government, he was a political prisoner who somehow got himself into a duel with an adversary. The events leading up to the duel and Galois' subsequent death are unclear and only speculation exists about who this foe was and why they were dueling. On the eve of the fateful duel it has been written and widely believed that Galois, from prison, desperately constructed his great theorems in a letter to a friend--another myth, easily dispelled by the evidence that in 1830 Galois' papers "An analysis of a Memoir of the Algebraic Resolution of Equations", "Notes on the Resolution of Numerical Equations" and "On the theory of Numbers" were published and make up what is now called Galois theory. [4]

Despite these publications Galois' work was not recognized until 1846, 15 years after his death, when Louiville published it in his Journal commenting on Galois' solution, " . . . as correct as it is deep of this lovely problem: Given an irreducible equation of prime degree, decide whether or not it is soluble by radicals'[2] Louiville was commenting on Galois' insightful understanding of polynomial solutions and their relationship to what are now called groups, a term Galois used first, and the solvability of an equation in radicals. Galois had answered the essential question of what makes a polynomial solvable by radicals.

Understanding Galois theory is not an easy task. To have complete comprehension of the theory one must be familiar with mathematical concepts such as fields, rings, groups, isomorphisms, symmetry, permutations of roots, vector spaces and the list goes on. However it is possible to dabble close to the surface and have a general sense of what Galois was doing. For example let us look at the polynomial $f(x)=x^5 - 2x^3 - 8x - 2$. Using the fact that f (x) is irreducible using Eisenstein's criterion, and extending the field to adjoin roots of f (x), we get an extension of degree 5. Eventually, by calling upon the fundamental theorem of Galois theory, Cauchy's theorem, and by a proposition that every element of the Galois group of f (x) gives a permutation of those roots we determine that the Galois group of f (x) is isomorphic to S_5, a symmetric group. Since S_5 is not solvable, then f (x) is not solvable. Therefore, there exists a polynomial of degree 5 that is not solvable algebraically by radicals[6]. It would be beneficial for readers unfamiliar with these criteria to determine solvability to access several problems worked out by Beachy & Blair (see links 7 and 8 in Works Cited)

Not only did Galois theory provide a means to know about the solvability of an equation, what is so remarkable about Galois' discovery is he was working in an entirely new plane of mathematics, dealing with structures of polynomials and their solutions. The necessity of conceptualizing the notion of a group to determine solvability of equations produced modern mathematics. Galois' work with permutations of solutions as a structure (a Galois group) laid the foundation for investigations into other similar structures such as matrix groups.[5] This new world of mathematics evolved with the help of other mathematicians, who built upon Galois' innovations, and became known as Group theory.

While it will always be a mystery why Galois was the victim of a duel, it is the only mystery that remains concerning Galois. Galois' work is now understood and applied in numerous arenas, like a light illuminating the world of mathematics. Maybe, one day, this light will out shine the myths surrounding Galois and he will simply be a great man who, in his very short life, changed the world of mathematics.

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