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Editorial: Social Justice, Taxicabs and Soap-Operatic Mathematics

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Welcome to the second volume of The Montana Mathematics Enthusiast. I am pleased to report that the journal is growing spectacularly in terms of its geographic reach, in the number of readers as well as the increase in the number of submissions for publication consideration. The external reviewing process has been beneficial to maintain a high standard in the published articles. A special thank you to those of you who selflessy gave of your time for the review process. The web-traffic statistics supplied by Tony Riehl indicate that TMME contributes to nearly half of the traffic to the MCTM webpage. Even more impressive is the fact that the journal pages were accessed from 32 different countries!

In this issue we present an eclectic collection of articles. The first feature article by Seth Braver and colleagues is a critical commentary on Eric Gutstein's (2003) paper published in the Journal for Research in Mathematics Education. Gutstein (2003) presented a compelling argument for the inclusion of social justice concepts in the mathematics curriculum, particularly in classes with minority students. In the spirit of true scholarship six mathematics and mathematics education graduate students (which included practicing public school teachers) debated the ideas presented by Gutstein in this JRME paper. The rich and sometimes controversial discussions led to the commentary presented in this issue. The satirical beginning sets the stage for an in-depth examination of the validity of Gutstein's ideas. Johnny Lott, past president of NCTM, was very kind to preface the commentary by outlining connections with the legislative plank of NCTM. Eric Gutstein expressed joyful surprise at the in-depth analysis of his ideas and contributed a reaction to the commentary.

The second article by Jerry Baty and Virginia McClendon presents an interesting classroom vignette, one that many practicing teachers may have encountered at some point in their careers. The vignette is used to present ideas from action research and its usefulness to conduct investigations into the processes of teaching and learning in the classroom.

Keeping with the mathematics tradition of the journal, the article by Chip Reinhardt focuses on the Taxicab (or the Manhattan) metric and the usefulness of this geometry to investigate real world problems. Reinhardt presents solutions to three real world problems using Taxicab Geometry. Readers are encouraged to judge whether the solutions presented

are in fact the „optimal" solutions and whether an improvement is possible on the solutions offered. The three problems used by Reinhardt are very accessible to high school students.

We conclude this issue with a colorful journey into mathematics history, particularly into the theory of equations which gave rise to modern Abstract Algebra. KaCee Ballou, a local middle school teacher presents the soap-operatic drama behind the solution of the cubic. The main point made by Ballou is for teachers to move beyond the quadratic as the pinnacle of middle school algebra and to expose students to the solution of the cubic via its colorful history. Catherine DeGrandpre, a high school mathematics teacher (at Martha's Vineyard) pushes Ballou's recommendations several steps further and completes the colorful history of the theory of equations by presenting a history and an understandable account of Evariste Galois' brilliant answer to the question of solvability of equations via radicals.

We hope you find the articles of interest. As a sneak preview for the Fall issue (Vol2, no.2), TMME will go international and feature articles from mathematicians and mathematics education researchers worldwide. In keeping with Montana traditions, the journal will strive to regularly feature articles from Montana teachers and teacher educators.

Last but not least, the journal expresses its appreciation to the authors in this issue for contributing thought provoking articles for the readers of the journal. As usual offers for reviewing manuscripts as well as book reviews are welcomed.
Thank you for your interest and continued support.

References

Gutstein, Eric. "Teaching and Learning Mathematics for Social Justice in an Urban, Latino School," Journal *of Research in Mathematics Education* 34 (January 2003): 37-73.

A Preface to "Gutstein Generalized—A Philosophical Debate"

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The University of Montana
Past President, National Council of Teachers of Mathematics

The National Council of Teachers of Mathematics (NCTM) has taken a very firm stand on the issue of equity in mathematics education. Not only is equity incorporated in the *Curriculum and Evaluation Standards for School Mathematics* (1989) and *Principles and Standards for School Mathematics* (2000), equity is the first plank of NCTM's legislative platform. That plank is listed below:

The National Council of Teachers of Mathematics—

- *Supports the right of every child to be taught by a highly qualified teacher of mathematics, one who is knowledgeable in content, who understands how students learn, and who uses appropriate instructional methods.*
- *Expects every child to have the opportunity to receive a strong mathematics education required for an economically secure future.*
- *Believes that no single test should limit future opportunities of students to learn mathematics.*
- *Expects every school to be a safe and supportive learning environment for students to learn mathematics* (NCTMc, 2004).

The plank is prefaced by "One who is mathematically literate can analyze data, reason, and solve problems by applying mathematical concepts and skills."

Because of the plank and its importance to the Council, two task forces were named, the Equity and Diversity Integration Task Force and the Achievement Gap Task Force, since 2004. With different goals, the first of how to incorporate equity and diversity more fully within the Council, and the second to make recommendations about how an organization of approximately 100,000 members can approach the achievement gap, NCTM continues to work toward a more equitable mathematics education for all students.

One of the recommended actions by the Achievement Gap Task Force is to identify a research agenda that includes a "focus on research that promotes improvement of curricular and instructional conditions that will close the achievement gap" (NCTMc, 2005). In particular was a recommendation for research that explores "characteristics of school curricula that empower students from underrepresented groups to learn," and "cultural factors that influence mathematics teaching and learning, including analyses of the function to teachers' *worldview* in the process of teaching and learning" (NCTMd 2005).

With the plank and task force reports mentioned, the Council has cast a wide net to incorporate differing views of mathematics and how it should be taught while continuing

to address equity issues. Certainly, mathematics taught from a social justice viewpoint is not opposed to the plank; nor is mathematics taught from a functional literacy viewpoint. The real goal is preparing students for their mathematical needs of the future. In both "Teaching and Learning Mathematics for Social Justice in an Urban, Latino School" (Gutstein 2003) and in "Gutstein Generalized—A Philosophical Debate" (Braver, et al., 2005), one sees both an approach and a debate based on the approach that have the real goal as an objective.

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Gutstein, Eric. "Teaching and Learning Mathematics for Social Justice in an Urban, Latino School," Journal *of Research in Mathematics Education* 34 (January 2003): 37-73.

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Gutstein Generalized - A Philosophical Debate

A Critical Commentary on Gutstein's (2003) thesis for the Incorporation of Social Justice in the Mathematics Curriculum

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Paper(s) submitted: 01. December 2004 Accepted with revisions: 02. February 2005.

The Scene: A Courtroom
The Year: 2004 (old style), 15 (new style - After Standards)

The Grand Inquisitor mounts the podium, and addresses the Debaters standing silently before him. A large crowd fills the hall.

Inquisitor:
Ladies and Gentlemen! You have been summoned here today to present the final arguments for and against these propositions which have so vexed our society in recent months. Each of you represents a vision of the future of mathematics education. Ere the sun sets we shall fix our resolve to one vision or the other. The victors, I doubt not, shall lead us into a glorious society of mathematically literate citizens whose ears shall be forever deaf to the cries of the vanquished.
(*Cheers from the crowd, impatient for the debate to be over with, so that they might learn what the new orthodoxy shall be*)

Inquisitor: (*Addressing the crowd*) Have we not acted thus in the past, my fellow citizens?
(*More cheers*)

Inquisitor:(*Addressing the crowd*) Do you recall the last time one stood in our midst and advocated abolishing graphing calculators from schools?

Chorus:(*A great simultaneous cry*) We are deaf to such voices!

Inquisitor: Do you recall he who last questioned the value of group work?

Chorus: (*As before*) Deaf we are, and deaf we shall be!
To such voices and thoughts we are deaf!

[1] The order of listed Author's names was chosen via a random draw. All authors have contributed equally to this commentary. This commentary was a result of a course assignment in Math 500: Contemporary Mathematics Curriculum, a graduate course offered by the Department of Mathematical Sciences, The University of Montana, taught by Bharath Sriraman, the editor of TMME in Fall 2004.

Inquisitor: Very good. I see that you have not forgotten your catechism. Still... How do you know it so well? I hope you have not been indulging in *rote memorization*?

(*Shocked expressions of outrage in the crowd*)

Inquisitor: Forgive my impertinent question, but we must be on guard against such abuses. Now then! We have read the reports that Eric Gutsein has provided, most notably his report "Teaching and Learning Mathematics for Social Justice in an Urban, Latino School". We have deemed the results outstanding. We believe that Gutstein's students have indeed learned the power of mathematics in a meaningful, relevant way. Yet one question remains: where one has succeeded, might not *every*one succeed? Perhaps if we, as a society, make a new commitment to teach elementary mathematics as a means of promoting social justice and equality, we will achieve the mathematically literate populace that we all dream of. It is for this reason that we shall hear arguments for and against the proposition, which follows.

(*He unrolls a small scroll, clears his throat, and reads*)

PROPOSITION 1: (Premise) In teaching Minority student groups[2], using[3] a standards-based math curriculum[4], such as MiC[5], in conjunction with[6] special projects[7] can support pedagogy[8] for social justice[9].

Arguments in favor of Proposition1:

The National Council of Teachers of Mathematics (henceforth NCTM) takes a strong position on providing equal opportunity for all math students. In fact this was one of the aims of the original 1989 NCTM Standards to reform mathematics curricula, that is, to create a mathematically literate population via systemic reform in schools. So using a

[2] Minority student groups: pertains to non-Caucasian students, in this case 98% Latino

[3] Using: means implementing the curriculum as intended.

[4] Standards-based math curriculum: Refers to a 5-8 middle grade curriculum in which emphasis is placed on conceptual discovery, pursuing open ended problems and extended problem solving projects, developing number sense, creating algorithms and procedures, and using appropriate technology for computation and exploration, and assessing learning as an integral part of the math class, de-emphasizing rote memory of algorithms and math facts.

[5] MiC: Refers to the Mathematics in Context curriculum, which comes out of the National Center for Research in Mathematical Sciences Education & Freudenthal Institute developed in 1997-1998. The curriculum contains 40 units, ten per grade level, in which students explore and connect the 4 strands of number, algebra, geometry, and probability and statistics.

[6] In conjunction with: Means that both of the two different sources of curriculum, the MiC and the special projects that Eric Gutstein developed for his particular group of students, were used throughout the two-year span of the study, not necessarily at the same time.

[7] Special Projects: As stated above were developed by the teacher, Eric Gutstein, particularly to help his students become aware social injustices and number distortions.

[8] Pedagogy: The science of teaching.

[9] Social justice: The concept of honoring ethnicity and diversity, and striving for equity among races, when considering social and cultural issues.

standards-based curriculum, like Mic, naturally helps in the goal of providing equal opportunity among students for math literacy by the following changes from a traditional curriculum. Standards-based curricula call for a variety of teaching methods and strategies, and therefore would be more likely to tap into the learning styles of most students. For example, Ornstein & Hunkins (2004)[10] , address multicultural education, and noted that, in studies done by Ramerez & Castenada, Hispanic students tended to be 'field sensitive' students. 'Field sensitive' children are described as being more influenced by personal relationships and by praise or disapproval from authority figures, including teachers, than are 'field independent' students." (Orstein and Hunkins, 2004, p. 382) Gutstein used a variety of teaching strategies, including encouraging discussion, and fostering personal relationships with his Hispanic students, especially encouraging them to believe in themselves, thereby giving them self-approval. Standards-based curricula also call for a variety of assessments, which meet the needs of the students. The use of appropriate technology is stressed, as is problem solving, communications, reasoning, and making connections. Standards-based refers to a broad rich curriculum available to all students. This is in keeping with teaching for social justice.

Gutstein (2003)[11] says, "Discussing these issues openly is a start. But equity is not here." (p. 39) Issues of social justice are controversial enough for teachers to bring up in classrooms in any subject, even subjects that seem more relative to the topic than math, like social studies or literature. Gutstein's special projects to promote social justice were ingeniously designed and bravely carried forth. He believed that the students should become a part of the solution by first becoming aware of their plight in the world. He wonders if students are questioning the 'powers that be', which shape their world. He leads them to this sociopolitical consciousness with activities such as the housing affordability project. He uses these mathematical statistics in a special project that might help students to critically ask why do so many kids in your neighborhood join gangs? He encourages discussions, which help make connections. He further advocates that his students believe in themselves and their abilities (he calls this a sense of agency), and in doing so promotes the development of pride in their language and culture of origin, a cultural identity. Gutstein (hopes that these goals of social awareness, a sense of agency, and cultural identity will motivate students to acquire more tools (like more math knowledge and technology) to investigate further their place in their world. He hopes an initial awareness will lead to further investigations, which can lead to taking action, to students actually doing something positive to overcome the situation. Gutstein quotes Ladson-Billings in saying, "Emancipatory education does not neglect discipline and knowledge."

Gutstein's goals were three fold. He wanted his students to be able to read the world around them using mathematical statistics, to develop math power, and to change their negative attitudes toward math. In describing his methodology, he tells about his students,

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[10] Ornstein, A.C. & Hunkins, F. P. (2004) *Curriculum: Foundations, Principles and Issues.* Allyn Bacon.
[11] Henceforth Gutstein

seventh and eighth graders, who were heading to a high school, which has a fifty percent dropout rate. He taught these students for two years in a row using a combination of the MiC curriculum and seventeen special projects. He was very thorough in keeping track of student progress using a wide variety of assessment methods. He used testing, and he kept journals of classroom observations and reflections. He collected samples of student work, including in-class work and homework, and the twelve unit tests. He saved the materials, results, and write ups of their in depth math projects. He videotaped in class presentations and conducted informal conversations and formal surveys.

Assessments of the two-year program were positive. Twenty-seven of the twenty-eight students had gained math power, but Gutstein couldn't say whether it was the MiC curriculum or the projects because they were so interrelated. All but one student became more adept at explaining math reasoning. Many different distinct ways of mathematical thinking from students resulted from these methodologies and curriculum. Gutstein observed that, "students invent and creatively apply mathematics", as seen from the cereal project, the cumulative years/birthday problem, the Mexico area estimation project, and the Shaquille O'Neal slam-dunk problem. Also students changed their attitudes. Gutstein believes that students learn more than what is explicitly taught, and that the classroom climate plays a role, as shown by the discussions that ensued in his class. He summed it up by saying that his students gradually developed an ability to understand the world around them. Evidence of this fact is the down town park project. Once students measured the distance and discovered how close their neighborhood actually was to the park, they took an interest in the issues. On a later survey / questionnaire twenty out of twenty-three students said they understood the world better now using math. He concludes by saying that his projects were the major site of learning to read the world, but that MiC contributed as well. Nearly all the students developed math power as evidenced by their creative and inventive use of mathematics. Most of the students changed their attitudes. One student's claim that math became an everyday thing must surely been viewed as a success story by Gutstein.

Arguments against Proposition 1:

> Lemma 1: MiC is the program used but Gutstein talks about honors classes. Does tracking[12] students into honors classes model equitable education? Does tracking indicate that some students deserve a better mathematics education than others?

NCTM standards call for math literacy for all students, but it isn't clear whether this functional literacy[13] or critical literacy[14] (In addition, curriculum standards may very well

[12] In the United States, the word tracking usually means the sieving of students into different classes based on performance on standardized tests, teacher recommendations and/or counselor recommendations.
[13] Functional literacy is defined as being able to read and do mathematics

worsen the existing inequalities because schools with more funding and resources stand to benefit more by being able to provide those things the standards call for, like more teacher training and time for developing and implementing the new programs. Also, studies show that classrooms with fewer student/teacher ratios result in more successes. The standards-based program, MiC, claims to have real world problems in it, but because of its designer's demographics, the so-called real world problems won't necessarily apply to all students who use this program in their math class. This program would need to be supplemented to include real-life problems. Along with supplementation comes a whole host of problems. Gutstein designed seventeen in-depth special problems that were real-life problems for his particular students. Not every teacher has the time, training, or inclination to supplement the mathematics classroom with real-life problems. Plus, not all classrooms are as homogeneous as Gutstein's 98% Latino classroom at Diego Rivera Middle School. In addition, this class was comprised of the honor students. This is definitely contradictory to the NCTM standards, which advocates non-tracking.

In order for these special projects to relate to promoting social justice, certain conditions must be in order. The projects must be specifically designed to use mathematics as a vehicle to become aware of racial, economical, gender, or other discrepancies and inequalities. That might require deconstructing the media and literature in order to use math to make a connection between one's personal life and the world at large. So the projects must personally connect the students to whatever the topic is, and this is easier said than done in a heterogeneous classroom. Then the question of "how can we use this data to promote social justice?" must be asked. The asking of this question in itself could easily provoke classroom conflict and tension. In order to use this mathematical data to ask these questions, the students who are being discriminated against are looking at a pretty bleak picture of themselves and their circumstances. It must seem overwhelming to some students at such a young age. It would take an extremely skilled teacher to navigate this classroom (which, no doubt Eric Gutstein is). Not every teacher is this skilled. The information revealed would have to be rewritten using positive cultural identity, and this doesn't metamorphose easily in the middle school students, especially when they are looking ahead to gang powered neighborhoods and devastating dropout rates. In order for action for social justice to occur, students must take the initiative and make the relationships themselves, otherwise social justice is over when the class is over. Gutstein admitted that at times he literally led his students to the trough, and the students themselves admitted that they would not otherwise have made the connections to their situations.

It was pointed out in the article that it was hard to imagine materials for teaching and learning mathematics for social justice. First of all, often this goes against the power

[14] Critical literacy is defined as approaching knowledge critically, seeing social events in the interrelationships of their historical and political contexts, and acting in one's own interest as a conscious agent in and on the world (Gutstein, 2003, p. 38).

structure. Going against the powers that be or notions that have been handed down for generations in families is no easy task. Teachers who create such materials may feel that their livelihoods or jobs could be at stake, and so why bother with all the extra work of creating special projects that may or may not promote social justice. Political pressure may be put on large companies who produce such materials, so why bother? Because of standardized testing, especially in light of the 'No Child Left Behind' act, so much functional mathematics must be covered in the curriculum in order for the school to be deemed successful, that many teachers must feel there isn't time to teach from material created for the 'math for social justice curriculum', the critical math literacy aspect. Also most math researchers focus on cognition more than any social issues that come up in math class. Besides, math is viewed as a scientific, neutral subject, not one that gets students all riled up about social and political issues.

Gutstein reported that student's understanding of social issues developed gradually over time. At first the issues were confusing and contradictory to family values and beliefs. Perhaps it would be better left up to the family to address such issues, rather than pursue them in math class. Gutstein couldn't say whether it was MiC or the projects, which gave the students their math abilities or math power, as he called it. His students may well have been successful anyway because they were the class of honor students. With non-tracked students this teacher may have had to spend more time with the standard -based curriculum, and less time with the projects in order to teach for math literacy. Finally Gutstein admitted that his students didn't spontaneously approach a real-world problem and use math unless he suggested it, so it is hard to determine if the projects themselves in combination with the MiC program actually fostered a pedagogy for social justice and changed attitudes, or if it was the teacher's suggestion, the need for approval, the need to pass the class, etc. In light of the above information it cannot be said that in teaching minority groups, using a standards-based program like MiC in conjunction with special projects, will support pedagogy for social justice.

Result 1

In the argument for this premise we described Gutstein‘s teaching mathematics for social justice in an urban Latino classroom. Gutstein discussed in detail the three components he feels are necessary for the teaching of social justice. They are the development of 1) sociopolitical awareness, 2) self confidence, and 3) a positive cultural identity. Gutstein had three goals for his students which were 1) attaining the ability to read the world using math, 2) developing math power, and 3) changing their negative attitudes toward math. The argument for the premise leads us to all the evidence that Gutstein compiled, which showed that his students indeed had attained his goals. His evaluation of this combined program was ever so thorough, with a multitude of assessments, work samples, journals, videos, etc. Also his evaluation of his own project was self critical, and some of these ideas we used in the argument against the premise, but in all actuality, we believe that more is learned from viewing things in an objective and critical manner. In other words, it

was an honest assessment, and his conclusion remained that the possibility for teaching and learning mathematics for social justice was alive and well. In fact he had just showed how.

In theory the NCTM standards-based curricula are intended to promote equity and social justice, while simultaneously teaching the appropriate grade level mathematics. I presented those specific components of the standard-based curricula, which help promote equal opportunity and discussed how, in theory, the MiC curriculum was nicely aligned with the standards. For example, the MiC program stresses the importance of multiple strategies for teaching and values the necessity of student interaction. In addition MiC emphasizes gradual development through personal experience. There is some truth to the fact that some of the supposed real-life problems would not necessarily apply to all students (in the against argument), but in conjunction with the special projects, and with a teacher who fosters self-confidence and social awareness in his students, this could be remedied. The claim that standards-based curricula could be of a benefit to the richer districts could be true, but not in theory, and not in this case. Preliminary evidence shows that if the MiC curriculum is implemented as intended, with teacher training, that MiC students do better than conventional math students in number, geometry, algebra, and probability/ statistical problems and in problem solving. Gutstein proved this to be true in light of the fact that many of his students were able to gain entrance to magnet and college prep high schools. In addition I pointed out that standard-based curricula are intended to develop critical thinking skills, like problem solving, reasoning, communicating, and connecting. The distinction between functional and critical math literacy is made and both bases are covered by the NCTM standards.

The pro-premise argument shows what awareness and self-assurance can lead to. Gutstein hoped his students would have the self-confidence, as human beings and as math students, to investigate some touchy sociopolitical issues and use math to make an argument. He, indeed, has a few follow up letters telling how his former students have been doing this, evidence against the point that students may have temporarily taken the posture of a free thinking, math-using, critical thinker just to gain approval, credit, or both.

The math abilities of Gutstein's students were brought up in the pro-argument, as well as in the con-argument, but the pros have it. We have discussed Gutstein's varied assessment techniques, and how his students had demonstrated a variety of inventive applications of mathematics, generating multiple solutions when appropriate, and the demonstrating the ability to communicate their findings to the teacher and to their classmates in a variety of ways. Gutstein could not tell if it was the MiC program, or the special projects, so I put emphasis on the 'in conjunction' aspect of the premise. Sure these were the honor students, and they may have been successful anyway, but Gutstein's records are compelling, almost impeccable.

The classroom environment that lent itself to controversial, social and political discussions brought forth by one of Gutstein's special math activity was another topic in the pro-argument. Opinions and ideas of each student were valued, and together with the special projects, they investigated, from a mathematical point of view, topics such as wealth inequality or the conditions of immigrant farm laborers. Students felt safe enough to speak their minds. The confusion and uncomfortability felt by students is a natural outcome of any novel concept. Kearns and Harvey[15] substantiate this in their book, , by the following statement. "Genuine thought only begins with a disturbance that impinges on thought, with a perplexing and paradoxical question that forces thought." (Ornstein & Hunkins, 2004, p. 257) Gradually students began to question and understand their world better using math, as was reported in the questionnaire. It is true that in order to promote the teaching of social justice using math, certain conditions were met, but Gutstein showed that it can be accomplished, and that many students do continue to take the bull by the horns and address social injustices, and they use statistics and other math concepts to do this. In lieu of these facts, we must say that the following premise is a true statement: In teaching minority student groups, using a standard-based math curriculum, such as MiC, in conjunction with special projects can support pedagogy for social justice, and that Gutstein has demonstrated its possibility.

Gutstein has shared the results of his two-year study of teaching and learning mathematics for social justice. He has given us an example of how this can be accomplished, and he talked about the components he thinks are necessary in order for teaching and learning for social justice to occur. Those components sound familiar to us. For example, the first component of helping students to develop social and political consciousness, sounds suspiciously like the work of a reconceptualist curricular theorist. His ideas here are in alignment with that of social reconstructionism. Educators who consider themselves to be in this camp are interested in the relation of the curriculum to the social, political, and economic development of society, and they believe that through the curriculum educators will effect social change and ultimately create a more just society. His second component, that of developing a sense of agency in his students, a belief in themselves as individuals, smacks of humanism. Advocates of this type of education would push their students to the highest level of self-actualization, using terms like striving, enhancing, and experiencing, moving to take action, as well as independence, self-determination, integration, and positive self-concept. Humanist educators tend to form meaningful relationships with their students and promote acceptance of others, in addition to self-acceptance. A teacher who subscribed to a humanist philosophy would help learners overcome personal and psychological needs in order to facilitate self-understanding, which brings us to the third component, that of student development of positive social and cultural identities. To know oneself is not a new idea, and not just a humanistic concept. To have a positive outlook on ones cultural identity is in accordance with multicultural education, which acknowledges the dignity and relevance of diverse cultural backgrounds. Social goals of multicultural

[15] Kearns & Harvey. *A Legacy of Learning*. Quoted by Orstein & Hunkins (2004).

education are to reduce prejudice toward oppressed groups, to work toward equal opportunity and social justice for all groups and to effect an equitable distribution of power among members of different cultural groups. Multicultural educators have another goal of preparing future citizens to reconstruct society so that it better serves the interest of all groups. Clearly Gutstein had hopes that his students would someday be pressed into action, as self-advocates, and that they would use math as tool.

Gutstein used the MiC curriculum because he thought it would help develop his students' critical thinking skills that are necessary in the pursuit of equity and justice. He wanted his students to develop 'math power' and acquire the ability to read the world using math. The 'progressive formalization' approach of the MiC curriculum is in keeping with Gutstein's goals and outcomes. Gutstein's students invented and created personal solution methods to problems, gradually generalizing and formulating procedures in which to approach real-life problems. The use of real-life problems in the MiC' program was enriched with Gutstein's personally tailored special in-depth projects.

The combined use of these curriculum materials, the reform MiC curriculum and the special projects, along with a classroom atmosphere that allowed for a safe place in which to raise taboo topics, helped to promote teaching and learning for social justice in this case. Few educators have attempted this goal using a standards-based curriculum. Gutstein questions the relationship of standards-based curricula to the teaching of equity and justice, and he concludes that this is entirely possible under the conditions, which were the three components necessary in teaching and learning for social justice. In the future, Gutstein believes teachers can promote more equitable classrooms in many ways, but explicitly using math as a tool to understand and possibly remedy some of the inequities and injustices in our society. Gutstein also hopes that NCTM will consider a more aggressive approach toward the goal of a more just society.

PROPOSITION2: (Premise) Social justice[16] should be taught in the mathematics classroom.

Arguments for Proposition 2

This premise has far reaching effects for students, teachers, school districts, and the world in general. Gutstein is obviously a hard and fast believer of this premise. Gutstein, the teacher, kept papers, evaluations, and work done on the projects he created. He also tracked students and their education documenting factors that support learning in his students. The main objection raised in Lemma 1 raises its ugly head when constructing arguments in support of Proposition 2.

[16] See footnote 9 for definition under use

Idealistically speaking students experiencing this form of education can soon see themselves as agents of change. Many children grow up experiencing injustice and watching their parent and friends experiencing the same thing. Once they analyze the situation and understand that it can be changed, the next step for them will be to discover they can be one of the agents of that change. They will realize that they do not have to remain in a low-income class or dead-end jobs, and discover ways to change that part of their lives if that is what they choose to change. Also they realize they can work to instill respect for their culture. The understanding also helps them acquire pride in themselves and their diversity. Self-respect is a quality we all deserve. As mathematics empowers students to understand themselves and the world students gain a respect for the power that mathematics has in understanding. The students also practice looking at the world as an entire system. The world ceases to be disjointed.

Arguments against Proposition 2

The first argument that comes to mind is that this is a subject that fits better into a social studies curriculum. After all, don't we read about all of the injustices done to individuals because of their race or religion in history books? And isn't that where we read about the industrial revolution and discovery of new countries? Isn't that where all the reasons for wars between people who were different and didn't respect those differences are written down? Social studies is where ethnicity should be found as well as the entire history of Ellis Island and the immigrants who came to the U.S from Europe and Russia.

Mathematics teachers have high-stakes testing to worry about. There are no questions on AYP tests that evaluate how much self esteem a student has or how any student is going to make a change in how people of his ethnic background are treated in the United States or the world. There is nothing about evaluating the wealth of the world and how that wealth is distributed on any test that is endorsed to evaluate AYP. On the high-stakes tests there is no credit given for ideas that make students agents of change. The tests also do not address the individual research done by each student. And the makers of the test are very proud that the test does not recognize the diversity of the students.

Work will also dissuade teachers from teaching social injustice within their math classrooms. The inquiry-based NSF programs have met resistance from teachers because they take more effort to teach. They also require more input and energy from students and that takes motivation from the teacher until intrinsic rewards are apparent. Creating projects that are directly related to a teachers population is also more work. And of course grading and organizing all of the students' work takes even more time.

Some teachers will be afraid of teaching something that has not been adopted by the school board. They do not want to get in trouble.

Result 2

If we are looking for the best education for our youth social injustice needs to be part of the mathematics curriculum. If we are in favor of our students becoming solid citizens and making our world a better place we should be teaching social injustice in the mathematics classroom. If we want our students to be happy well-adjusted adults we should be teaching social injustice in the classroom.

Several of us, as practicing public school teachers have spent many years teaching STEM[17] materials and have seen how much investigation and real- world problems motivate students. Students also respond to learning about themselves and having their behavior and everything else about them validated. This will help them become happier people.

Most middle school models encourage interdisciplinary teaching. Social injustice fits into that framework perfectly. Middle schools boast of teaching the whole child. Social injustice is there. Most of us became teachers to make a difference in the world; to make a difference in the lives of our students. If this is true, than teaching social injustice fits. As teachers we know the need to assess our roles, whether it should be teaching social injustice within the mathematics classroom, as well as how any changes we make will affect our students.

PROPOSITION 3: (Premise) A problem-based curriculum[18] can empower students mathematically[19]

Arguments for Proposition 3

„I realize now how extremely important it is to have good mathematical skills so we can fully understand what is going on around us. Believe it or not, math can be incorporated into any situation."

These are the words of an eighth grade student that has been empowered by math, a student that recognizes math as a powerful tool and understands that our world is better understood thought mathematically literate eyes. This is not likely a comment made by an average eighth grader. This student has a confident and constructive outlook towards math because he participated in a class that used a curriculum based on real problems that were

[17] STEM: Six Through Eight Mathematics was an NSF funded program to reform middle school curricula. In its current mutation it is called MaThematics, an ongoing project.

[18] A problem-based curriculum is a curriculum that is centered around real-world problems that are meaningful to the students.

[19] Empowering students mathematically means

(a) students have a positive outlook towards math, they see it as a powerful tool, which they have the ability to use in their own lives.

(b) students are able to do math, they can perform traditional computations, but they also can use math in many different situations to think critically, solve problems, and communicate effectively.

meaningful to his life. As stated in the philosophy of *MiC*, a problem-based curriculum used by Gutstein,

> *"Mathematics is a tool to help students make sense of their world. Since mathematics originated from real life, so should mathematics learning."*

A problem-based curriculum can mathematically empower students by giving them a positive attitude towards math, while at the same time making them better at it. Gutstein's math class, which was based on real world problems from *MiC* and current social issues, offers proof of this.

First of all, students can't begin to be empowered by math unless they have a positive attitude towards it. This is very important and not always easy to promote since, as stated by Michaele Chappell of Middle Tennessee State University,

> *" Too much math unpopularity is advertised throughout society and the media."*

As she mentions, our students are perfect candidates for developing such harmful mindsets towards math.[20] Educators see this negative mindset often when math is not learned in context, and so students view math as isolated subject. As Gutstein puts it, "students in a traditional classroom believe that mathematics is a nonunderstandable collection of arbitrary rules and procedures," and this view is usually true whether or not students are conventionally successful. As a math teacher and member of society, this is not how I want the future leaders of our world to view mathematics. No one chooses to spend their time studying something seen as useless and arbitrary, so instead students should learn math within a realistic context that shows math as significant, and even crucial to their lives. If a student views the material she is studying as important to her life and worthy of her time, then she will be more interested in it therefore more likely to want to learn it and to utilize it.

A curriculum based on real world problems can also positively affect students' attitudes by showing many different mathematical perspectives, which give students an opportunity to develop their own mathematical methods. This is demonstrated in Gutstein's class, during a unit called "Cereal Numbers" for which students learn how to multiply and divide fractions. From this unit, Gustein gives many examples of student responses, each are correct and show distinct ways of thinking. He did not teach any of the methods used, "The students invented each solution, clearly communicated most of their work and thinking, and provided some of the rationale for their choices." This level of mathematical sophistication was typical of most of the students of his class.

[20] Chappel, M. F. (2003) Keeping Mathematics Front and Center: Reaction to Middle- Grades Curriculum Research Projects, In *Standards Based School Mathematics Curricula* (p. 295) Edited by S. Senk & D.Thompson. Lawrence Erlbaum and Associates.

This promotion of different ways of thinking opens up the field of mathematics to many students who may otherwise hate math, if it's portrayed as the memorization. From looking at math problems in a real world context, students realize that there are many different ways to approach any situation and are given authority to think in ways that make sense to them, and thus can internalize and take ownership of the math they are learning. This constructivist approach to math shows an important emphasis to, as Dewey put it, bringing together the child and the curriculum[21]. Looking at math from different perspectives and with multiple representations makes math more accessible to more students, and these students develop an attitude that they can do math, and use it in their lives. In fact in Gutstein's class, the problems he used were based on current social issues, so students not only saw math as useful and doable, but also as valuable to bettering their lives and improving the world; quite a powerful attitude.

In addition to improving students' attitudes about math, using a problem-based curriculum can also make students better mathematicians. As already stated, learning math in the context of meaningful issues makes students more motivated to study math, but are they actually learning it? Yes. *All* of Gutstein's students, who participated in such a curriculum, were successful in conventional measures (specially designed measures and district assessments given to all eighth graders) and most even graduated into magnet, college-prep high schools.

Gutstein's students were able to solve problems (real world problems based on current social issues as well as more traditional math problems) using many different strategies. They often used informal strategies that weren't explicitly presented to them and used their own invented methods to solve the problems. All were able to explain their reasoning in detail – showing a high level of conceptual understanding and mathematical maturity in their writing. These students showed that they learned more than just math facts; they learned how to apply their knowledge to many different realistic situations. In other words, they learned to THINK CRITICALLY.

A problem based curriculum encourages students to use different approaches, construct their own mathematical techniques, and be able to explain and discuss what they have learned, which means they are developing the ability to think and reason logically, which many educational theorists, including Dewey believed should be the main goal of any education. He and others believed that his was achieved when a student is given the opportunity to interact and reflect within a community of learners – the type of community formed when children work collaboratively on meaningful problems as they did in

[21] Putnam, R. T. (2003). Commentary on Four Elementary Mathematics Curricula, *Standards Based School Mathematics Curricula* (p. 163). Edited by S. Senk & D.Thompson. Lawrence Erlbaum and Associates

Gutstein's class[22]. His class also worked successfully because the problems he used were of interest to his students and built on their "natural development and curiosity," which was found to be an integral part of learning in the "Child Centered Movement" of Dewey, Judd and Parker at the turn of the 20th century[23].

In conclusion, a curriculum based on meaningful problems can empower students by giving them a positive attitude toward math and helping them to be successful mathematicians that are critical thinkers and problem solvers. For a young adult with mathematical power, the opportunities are endless. In fact, a positive attitude along with these acquired math and problem-solving skills can even be used to successfully advocate for justice in our society. So, by empowering students mathematically, this type of curriculum can even help to empower students in their own lives and in the world. This is best put by one of Gutstein's students, Frieda, who wrote that math is " sort of like a pass you could use to make the world a better place." Our goal, as math educators, should be that all students share this view.

Arguments against Proposition 3

Why do we teach children mathematics? Is it because we want them to be empowered by having a "positive attitude" towards it, or because we think it's an important subject that we want them to learn? We believe the latter. The main goal of a mathematics curriculum should be that students can do math, not that they like it, or even that they find it to be "powerful". A curriculum that's based on real world problems may be more "interesting," but I do not believe that it necessarily changes students' attitudes, and even so, this should not be a top priority. Nor does a problem-based curriculum make students better mathematicians, which in fact, should the *only* priority of any math curriculum.

First of all, we have no real proof that Gutstein's class really changed their attitude towards math. The only evidence is from surveys that Gutstein administered in his own class, even though he himself states, " self reporting is a problematic data source." He also admits that "finding evidence to support changed orientations is not easy, because they are internal." Therefore, the comments made by eighth grade students about their feelings towards math are not necessarily a reliable data source, which even Gutstein himself acknowledges.

Even if the students in Gutstein's class did improve their outlook towards math, these results would not necessarily be typical in an average American class. It was easy for Gutstein to find particular problems to which each individual student in class could relate because every student in his class comes from a similar background (99% Latino, 98% low

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[22] Senk, S.L. & Thompson, D. R. (2003) *Standards Based School Mathematics Curricula* (p. 6). Lawrence Erlbaum and Associates

[23] Ornstein, A.C. & Hunkins, F. P. (2004) *Curriculum: Foundations, Principles and Issues.* Allyn Bacon.

income, almost all speak Spanish). If a class were more diverse, having students from many, completely different backgrounds would it be as easy to find issues that would be meaningful to all students? Also, let us assume once again that students' attitudes did improve, how do we know that is was the curriculum that caused such change? It is likely that students answered the survey questions favorably because they felt their teacher's enthusiasm; Gutstein is obviously very passionate about his views of mathematics, so it's very possible that this passion affected students more than the "meaningful problems" they studied. For results to be legitimate, attitude surveys should have also been administered to another class of Gutstein's, one taught with his same enthusiasm, but without the problem-based curriculum. Only then could it be argued that the curriculum improved students' outlook.

Although it's easy to see that a curriculum based on realistic problems does not necessarily change students' outlook towards math, one needs to keep in mind that this should not be the main focus of any argument for or against a particular curriculum. The main goal of a mathematics program should be that children can do math, whether or not they enjoy it. If a curriculum is based on real world problems, such as Gutstein's, a great deal of valuable classroom time is lost on discussion of the problem as well as looking at the many different (often incorrect) ways that different students solved it. Gutstein gives an example of a unit within the algebra strand called "Comparing Quantities" for which students learn to solve systems of equations in "a variety of semiformal ways, including notebook notation," which is essentially what we know as Gaussian elimination, Of the 25 students that took to the assessment, only 21 answered the problem correctly, which means that 16% of the class could not perform the task. Is it necessary that students waste so much time studying different solutions? It seems that the most effective way to solve the problem would be to just use Gaussian elimination in the first place. It would benefit student more if teachers are given the opportunity to simply teach methods to the class and let them practice the method, and yes, of course apply it to other problems as well. With a more effective use of instructional time, the class can move on to other, equally important mathematics.

As stated on the website of the "Mathematically Correct Organization", advocates of "fuzzy" math programs such as the problem-based curriculum used by Gutstein, "Speak of higher-order thinking, conceptual understanding and solving problems, but they neglect the systematic mastery of the fundamental building blocks necessary for success in any of these areas. Their focus is on things like calculators, blocks, guesswork, and group activities and they shun things like algorithms and repeated practice. The new programs are shy on fundamentals and they also lack the mathematical depth and rigor that promotes greater achievement."[1] There is a lot of truth is this statement. . Maybe a problem-based curriculum gives students greater depth in group work and discussing situations from the real world, but it does not give them greater depth in doing *real math*, the traditional math that we did, and out parents did, and our grandparents did. Past generations have done well

in our society from learning and practicing the well-known mathematical algorithms, so why is this suddenly not good enough?

In addition to better attitudes, the advocates of such curricula also claim that it helps students become better problem solvers and critical thinkers. If this is what we want from our students, then why not have a class on problem solving and critical thinking? We do not have to sacrifice math in order to meet this goal, and as we've already declared, "problem solving" nor anything else, should not be main objective of a math program. The objective is that students can do math! When students leave middle school for example, they should be able to do operations with fractions, solve equations and factor expressions, They should be able to solve problems with this math, but do not need to apply them to current issues in the world. It is not necessary that students leave their math classes with a new way to solve problems and to think about math, the field of mathematics has worked the same for centuries, and will continue to do so.

In conclusion, we do not agree that a problem-based curriculum is needed in our schools because "it empowers students mathematically." Empowering students by giving them a positive attitude towards math, and by showing them that it's meaningful to their lives should not even be a goal of the curriculum. The purpose of a math curriculum is to show students how to do math. This means, unquestionably, that a curriculum needs to be based on only trusted mathematical facts and algorithms, Students need practice performing these algorithms, they do not need to spend their time inventing and discussing different ways of solving problems that are somehow supposed to make their lives more "mathematically meaningful." By doing and practicing math in the log-established way, students will gain mathematical skill, which is more useful and important than "mathematical power."[24]

Result 3

The arguments for and against the premise parallel the current debates in math education known as "the math wars." Many new curriculums have goals that go beyond students' ability to do basic computations in math, but there are some who feel that these new trends divert too much from the traditions of math education. Both sides believe they are doing what's best for students, and both sides have valid points and sound arguments.

So, which argument is sounder? Should we support a curriculum that is based on realistic meaningful problems because it can empower students mathematically, or should such a curriculum not be used because it considers more than what should be the one and only simple goal of a mathematics program: that students can do math?

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[24] http://www.mathematicallycorrect.com/

Some of us believe that a mathematics curriculum that is centered on real-world problems to which students can relate can indeed empower students mathematically therefore should be supported. Giving students mathematical power *should* be the goal of a school math program. Current curricula should not be dictated by traditions in how math is taught that haven't changed since the turn of the 20th century.

First of all, students' attitudes towards math *are* important. If their outlook towards math is not considered, than the goal of children becoming better mathematicians can never be accomplished. If a child sees math as an isolated subject of procedures they will not chose to study math further, and therefore never see math in all of its glory, as powerful tool in and of itself with many important applications, and as a way to think critically. We want children to become adults that are numerate, that look at the world through mathematically literate eyes so that they can analyze situations, solve problems and understand complex issues that are going on around them. A mathematician certainly does not see math as a completely isolated field that only involves computations, so why should we teach math in such a way? Ideally we want students leaving math classes as mathematicians that appreciate the power of their field, not simply human "computers" or "calculators."

It seems ridiculous that we would not consider the meaningfulness of the curriculum to be important. Shall we teach children math in a "traditional way" under the assumption that it's best simply because that's the way it's been done in our country for 100 years? I argue that since we're in the 21st century, the same reasons and ways of teaching math have changed. For example with new technology, not all procedural algorithms are still relevant for students to learn. Also, in an increasingly complex world and global economy, society does not need people that can simply perform mathematical algorithms. In the 21st century, people are needed who are critical thinkers, work collaboratively with others, solve problems and communicate effectively. These are traits that make someone mathematically powerful, traits that are gained by using a meaningful problem-based curriculum.

It is also important to note that teaching math in a way that empowers students does not imply that math content must be sacrificed. Just because a curriculum is based on realistic problems does not mean that students are learning less math. In fact, they're learning *more*. For example, as already stated, they're learning to think critically, and in addition, because the math is made meaningful to them through the context of the problems, they learn math more in depth. They explore different ways of doing problems, and are able to communicate their understanding effectively. Because they're learning about math in their own way (as well as learning the ways of their peers and the ways taught by their teacher) they are more likely to take ownership of the math, which means they are gaining a deep number sense, and are more likely to commit the mathematical thinking to long term memory. Also, they will more likely have the desire and ability to be able to apply the math knowledge to new situations and problems. Children are still learning the content considered important, but they're learning it a way that is more effective in the long term.

This way of approaching math curriculum is not simply a "trend." In fact, much research and many organizations such as The National Council of Teachers of Mathematics, the National Science Foundation, The U.S. Department of Education, and the National Research Council support it. Each of these organizations realized in the 1980's that mathematics education in the U.S. is not what it should be because the traditional way of teaching math was *not* working – students were not performing well on international standardized tests and were not successful mathematically in the workplace. As stated by the National Research Council in *Helping Children Learn Mathematics* (from their 1998 study that synthesized a huge amount of research done on math education) " Despite the dramatically increased role of mathematics in our society, mathematics classrooms the United States too often resemble their counterparts of a century ago." Therefore, new goals were established for math education[25], goals that include, but are not solely centered on computation. These new goals include ideas found to be important to learning, such as children's enjoyment of math, confidence in math, and engagement of math, which includes seeing math as a useful tool.

In conclusion, let us revisit Gutstein's class as an example. ALL of his students passed the eighth grade district assessment, but in addition to gaining to skills needed to do this, they also gained mathematical power. Due to the problem-based curriculum he used, students left his class being able to do computations, but more importantly, they also left able to solve problems and communicate effectively. They left his class looking at math in a new way, as a powerful tool that helped them and their peers tackle difficult complex issues in a logical way. Isn't that in fact exactly what mathematics is?

PROPOSITION 4: An avenue into promoting the appreciation[26] of mathematics is by helping students use mathematics in justifying[27] their views of social justice.

Arguments for Proposition 4

A student's everyday life is constituted by the environment and daily trials/struggles that they are actually seeing and facing. We believe that Gutstein (2003) has found a way to promote the appreciation of mathematics in everyday life and the ability to change students' attitudes about math through the teaching of social justice. The concept of social justice relies on equity in behaviors, as defined previously. In this respect, students must learn to identify non-equitable behaviors and situations and learn to fight against them. Gutstein teaches this to his students' through the use of a series of real-world projects. These projects are designed to represent situations that students will actually see in their

[25] National Research Council. (2002). *Helping Children Learn Mathematics.* Kilpatrick, Jeremy & Swafford, Jane, editors.
[26] Appreciation is an understanding of the nature and quality of mathematics.
[27] Justifying is proving or showing their views to be right using mathematics.

own lives. For example, in one project students looked at the data gathered on traffic stops in regards to race and possible racial profiling. In this project, they analyzed the data for possible discrepancies in proportions of certain races being stopped. They used their math skills to analyze these data and compare the results. The students then were able to discuss what they learned and try to understand and explain any possible difference in proportions. This not only is an opportunity to learn about proportions and comparisons, but also to investigate data that directly concerns them. When students see data as relevant to their own lives, they are much more interested in the analysis of such data and the conclusions reached. In fact, they may even learn more in the long run. According to Senk and Thompson, in a study done in 1992 by The Netherlands National Institute for Educational Measurement, out of 29 mathematics scales; "Students who used realistic materials scored significantly higher on 12 scales, whereas students using the traditional texts were significantly higher on just 3 scales." This shows that at least in this study, students are learning and/or retaining more information when it is put into a real-world context.

The use of teaching in real-world contexts by Gutstein, however, was restricted to that which dealt with social justice (or injustice, as the case may be). I believe that this will lead to an even more engaged class, particularly if they are experiencing the injustice themselves. The class that Gutstein taught was made up of all Latino students, who experience many social injustices themselves. Therefore, they were very much interested in some of the projects on a personal level and concerned about themselves, their families, and their futures in terms of the information they were discussing. For example, one student, in response to discussion on world wealth and the distribution within this country, pointed out the injustice of his family making in one year of hard work what Michael Jordan makes in 40 minutes. In other responses to this same project, students reported feeling "shock", "unfairness", "anger", and "mad" (pg 51). These feelings elicit great conversation about social justice along with using math to explain why one might feel there is an injustice occurring. Without the analysis of the numbers, any arguments about unfair distribution of wealth in any country hold little to no weight.

Not only are students learning about math, they are also learning how to interpret and view the world around them. As we see above, students are seeing the wealth of our nation in a whole other light as they view their place in the world. They are also seeing how math plays a vital role in understanding the entire world around them. Students learn to use math skills and conclusions reached in mathematical computations as a tool for supporting their view that social injustices are occurring in their own area. An example of this comes from the project designed by Gutstein which looks at housing data in the local area. Students were given data on the highest median house price in the area at the time and then asked if they felt that racism had anything to do with housing prices in this highest priced area. They were asked open-ended questions not only on their views, but also on what other data they would look at and how it might prove or disprove that racism was involved. In this way, students were not only thinking critically about the world they are directly in,

but also about what other information is needed to determine if a problem is occurring. This is vitally important for them to remember in the future when getting information from the media. They will be able to think about what information may be missing, how it might impact the overall result, and what mathematical evaluations are truly necessary to make a determination of the situation.

Being a critically thinking member of society is not just one of my goals for students. Ornstein and Hunkins (2004) state that in 1990, former President Bush identified 6 goals of education. One of these was that "…every adult American will be literate and will possess the knowledge and skills necessary to compete in a global economy and exercise the rights and responsibilities of citizenship." They also stated that in 1944, the Educational Policies Commission listed ten aims of education. Three of which follow:

- *"Understanding of the rights and duties of citizens of a democratic society…"*
- *"Understanding of the methods of science, [and] the influence of science on human life…*
- *"Respect for other persons, insight into ethical values and principles, and the ability to live and work cooperatively with others."*

Overall, Gutstein had three main mathematical objectives. "Read the world using mathematics", "develop mathematics power" and "change dispositions toward mathematics." To read the world using math was, in his opinion, to "use mathematics as a tool to analyze social issues like racism and other forms of bias and to understand power relations and unequal resource allocation in society." I agree with him that one way to accomplish these goals is to introduce material using social justice as a tool.

Arguments against Proposition 4

While some agree with Gutstein, others may disagree. One note to make is that this response to real-world material using math is not always automatic. Even Gutstein admits, "I did not see students spontaneously approach a situation in the world and use mathematics to make sense of it unless I specifically asked them or suggested it." How can we expect that students will then go on and use math in this way on their own? Is there any proof that this will actually occur in time? The answer is that there is no concrete proof at this time, and any evidence of such can not be directly correlated to a learning environment set up as above. In this light, we must ensure that the classroom culture is perfectly set for the situation and responses we wish to evoke from students. Some teachers may not be able to do this, or may not be able to in their school. Discussions of the types of subjects that Gutstein addresses are highly controversial and could lead to parent and/or administrative negative feedback, which will deter even the most willing teacher.

Other problems with this method include the amount of time that is needed for actual discovery and discussion of these topics. Students are not just discussing directly the math

involved. In order to properly allow them to develop a sense of social justice, they must discuss the social issues at length as well, which takes considerable time away from the mathematics. Gutstein felt the pressures of time as well, he states that he found himself at times leading the students too heavily in the mathematics portion in order to be able to get through the material in the allotted time. This went against his teaching pedagogy and was an issue he addressed as one needing improvement. Also, if we are spending time on social discussions, where is the time for the real math? Are students actually learning what they need to learn? Gutstein says "I realized that the mathematics in the projects was often less challenging than that of [the text]." So, his method of introducing social justice was also a detractor of sorts from the math itself. In light of this, Gutstein supplements his materials with the MiC curriculum. He believes that this brings in the math needed for the students in realistic situations, but perhaps not ones that they are directly related to. This begs the question, is his method really worth it in the end? Or would MiC be enough? Or is traditional material just as good?

Another issue is that real-world applications do not always equal student interest. Not every subject is going to interest every student. In fact, some may not be relevant to any of the students. Significant teacher time must be spent learning about the local environment in which the students live. They must consider all of their students, who may account for several neighborhoods and cultures. They must be sensitive that their students may not wish to discuss some controversial subjects and that they may be affected negatively by the discussion from others. There are many related concerns to bringing these topics into the classroom at all and we need to take extra care here. Time involved in this background research and thought may be overwhelming for teachers and prohibitive to proper instruction. We must also consider that every community and school has a different makeup. If we truly want equity for all, we must take these ideas beyond the minorities. Gutstein asks "how might teaching for social justice in a white, middle-to-upper-income, suburban school be different?" He admits that he has no answer for this. This is a significant issue, since many schools fall into this category. We not only want to make our minorities and lower-income students aware of math in their world through social justice, but every child. This may be more difficult to bring into play in a world were students do not face such adversity as in Gutstein's class. What issues do we bring before white students? What social injustices do students in higher income levels face? I don't believe that these are as clear cut as those above.

Gutstein also warns, **"[using] Standards-based curricula can exacerbate differentials based on existing opportunity-to-learn inequities**." Issues involved here include teacher qualification, language issues, and funding issues for teacher time and materials. Some teachers may be unqualified to deal with the level of questions that arise from realistic math. The world does not come neatly packaged in a word problem, but rather is quite complicated and someone who is unfamiliar with higher level math may not be able to answer questions which will naturally come out in student discussion. They also may not

be able to communicate clearly with their students due to language barriers, which will leave those students experiencing their own social injustice in the very classroom that is designed to teach them to avoid it. Lastly, if we advocate the use of MiC materials as a proper supplement to teacher-designed projects in order to get the appropriate level of mathematics required, this may be an issue for funding in some schools. New textbooks and teaching materials are expensive. In school districts that are already on tight budgets, the acquirement of these materials may simply be impossible. This leaves students with only teacher-designed projects and old textbooks, which will either leave them short on the mathematical knowledge they need, or not address the use of math in the world at all (or in contrived situations at best), respectively.

Even taking into account the above concerns, we still believe that one way to promote the appreciation of mathematics in everyday life is through the teaching of social justice and helping students to use math in justifying their views of social justice. In response to the issue of the response of mathematizing situations not being automatic, I would say that even if it isn't automatic, students can learn to alter their view. There is evidence of a permanent change in viewpoints in Gutstein's students. One student took the lesson on map projections with her years later. She introduced to other students the discrepancies in the sizes of continents compared to others. In this argument, she needed to reuse what she learned about comparing ratios from actual calculations of landmass to the areas represented on the map. Obviously, this lesson impacted her and made a difference in her view of things. Another student years later in an essay wrote about the issue of being called a minority. While it was related to a project she had seen in her eighth grade class with Gutstein, she elaborated on her view of what minority really means and being mathematically correct when using such terms. These students demonstrated an ongoing understanding of math in the world around them.

In regards to issues with administration and possible problems here with implementation, we think that it is the job of the mathematics education world in general to open the minds of these persons and help them to see the benefits of change. As shown in this study, students still learned what was needed, passed their exams as required, and went on to excel in the future. While these were students labeled as being in a higher-track in school, they still have shown that this teaching technique did not detract from their learning.

The time needed for discovering topics and discussion by students is an issue, but with most other new teaching techniques this is also a problem. We need to remember that even though it doesn't look like a traditional classroom in many ways, this method is still working to educate students and in the long run, they learn the material needed. Gutstein states "Mathematics educators recognize that students in traditional mathematics classrooms believe that mathematics is a nonunderstandable collection of arbitrary rules and procedures, and this may be true whether or not these students achieve conventional school success." That is to say, while we may get through possibly more material in a

traditional classroom, students are not comprehending all of it and are unable to connect it's usefulness in their lives and therefore retain the information.

We agree that the supplementary materials alone are probably not enough. We do need to rely on a textbook as well, but Gutstein does not refute this fact. He sees the MiC curriculum as a support for his projects. They set the stage for discovery and talking about realistic situations. The MiC materials help to bring in the mathematics at a tougher level that may be difficult to attain with only local real situations. After all, not every situation is easily understandable at every grade level. We therefore need assistance from other curriculum to develop the skills necessary to handle more advanced topics.

We must also strive to make sure that the topics we choose to include in discussion are relevant to students. Even though not all topics will reach all students, if even one reaches a student, they may begin to see the world differently, so we must continue trying. If students do indeed care about the material and have strong opinions about the subject, they will begin to think even beyond what you request. One example of this is the student, Marisol, who extended the discussion of world wealth to considering what the average income for a country meant. She included in her response a discussion of how this average still does not give a full understanding of the whole picture, because we have no information about how the very wealthy affect this average. She was developing, on her own, an understanding of different measures of spread and how one measure of tendency may not be enough data to fully analyze a situation.

As for the issue of exacerbating the equity gap due to opportunity-to-learn and other such issues, these are not issues solely involved in this type of curriculum. If a curriculum can help students succeed and learn to be more productive and ethical citizens, it should be implemented. To address the issue of opportunity-to-learn, we must go beyond just one curriculum and address this regardless of curriculum. We cannot disregard a good curriculum and hold students back just to create supposed equality. This is a social injustice in and of itself. We would be punishing and discriminating against those more fortunate only for the purposes of keeping them lower on the playing field with others. While we cannot forget those in the lower-income and less opportunistic communities, we must also not forget those in all other levels.

Result 4

Overall, Gutstein has a valid teaching technique that we believe students are benefiting from. It may be that some students will be easier to implement this with, and others may have more opportunity to actually see this new curriculum idea, but all who see it, in my opinion, will benefit. It is of utmost importance that we teach students that math is valuable and they will use it in life. We need to show them exactly how to use it and do this in a way that they can relate to. Issues of social equity not only bring in lively discussion, opinions, and strong feelings from students, they also bring about the

opportunity for teachers to educate their students about life and how to treat others with respect. All levels of students need to learn to respect each other and recognize the need for social equity in this nation and in the world. The results of Gutstein's experimental class show how this type of curriculum can help students. His statements about students' learning, achievement and attitude best illustrate this.

"Overall, students not only communicated their findings well, they also developed confidence, represented mathematics in multiple ways, created their own solution methods, applied and extended others, and generally developed mathematical power."
Later he states that "[his] data show that perhaps all but three of the students changed their attitudes about mathematics." In my opinion, this is the single most important result of the entire 2-year study. If we can start to change attitudes about mathematics, students will learn more as they break the wall down in their mind, which they hold up against math. One student of Gutstein's said the following:

"Well, I thought of mathematics as another subject in school that I hated. And I didn't bother to think too much about world issues or everyday issues. Now I know it all relates. And I've learned how powerful math can be to help us explain our decisions and help us express ourselves, because like I said before, math makes things more clear." We couldn't have said it better myself, and won't continue to try.

A general Argument for the four results

The greatest argument for a social justice classroom is the anticipated positive feedback teachers can hope to receive from the students, as seen in Gutstein's class for two years he taught the students in seventh and eighth grades. One of the students, Marisol, wrote "…Throughout the two years of having him as a teacher, we always had math assignments that dealt with important issues in some way or another…"(Gutstein, 2003, pg. 69)

Another student wrote, "I thought math was just a subject they implanted on us jut because they felt like it, but now I realize that you could use math to defend your rights and realize the injustices around you." (Gutstein, 2003, p. 62). Another argument for the incorporation of social justice is the underlying motivation for teaching such a curriculum, namely "(a) to uncover and concretize components of teaching and learning mathematics for social justice and (b) to understand the relationship of a Standards-based curriculum to that process." (Gutstein, 2003, p.37).

In our opinion Gutstein's approach to social justice via mathematics could be considered pragmatic, i.e., The teaching beliefs underlying his approach is characterized by, "…[t]eaching method…concerned not so much with teaching the learner what to think as with teaching him or her to critically think." (Ornstein & Hunkins, 2004) in green book). We believe that pragmatism is a sound philosophical approach for those who choose to

implement Gutstein's approach in the classroom. It is evident that Gutstein's philosophy about social justice goes hand in hand with the pragmatic view and way of thinking. "An important principle of social justice pedagogy is that students themselves are ultimately part of the solution to injustice, both as youth and as they grow into adulthood." (Gutstein, 2003,p. 39) The NSF trends in recent funding has been to promote this type of learning. According to results of recent international standardized tests such as PISA, the United States is still well below the rest of the world, and reports analyzing the data indicate that this may well be due to the widening socio-economic gap between the rich and the poor.

The U.S. is a very diverse country. "Never in our history have we had such as culturally diverse student population in our schools as now" Ornstein and Hunkins, 2004, p.144). Gutstein took a class full of students dealing with **the same** social justice issues. In general it is very difficult to engage students in social justice issues when they are dealing with several different issues. What is social justice for one group may not be social justice for another group. If these issues are raised in class one runs the risk of pitting one social group against another. Instead of the class being a unified group that learns together the teacher has suddenly drawn a wedge between the students in the class. Another word of caution is the time commitment needed to implement this approach. Although Gutstein is a proponent of the social justice classroom where students are supposed to discover social inequities via mathematics, he indicates being pressed for time when implementing it. This is an issue that needs to be taken seriously when thinking of adopting the social justice approach in the curriculum.

In conclusion we think that the social justice classroom can create a real awareness in students. Knowledge is not only power, but confidence. The story of Frieda returning to the Dominican Republic and showing the "correct" maps of the world is an inspiration to students. She was confident in her knowledge, understood the situation, and took a stand with power. We think all students should have the right to own their education this way.

Response to *Gutstein Generalized – A Philosophical Debate*

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It is a pleasure, challenge, and an honor to respond to the thoughtful and innovative debate started by Braver, Micklus, Bradley, van Spronsen, Allen, & Campbell on teaching mathematics for social justice. They take seriously the issues in, and raise many interesting views about, my article, *Teaching and Learning Mathematics for Social Justice in an Urban Latino School* (JRME, January, 2003). I would like to respond to (connected) two points in particular: the relationship of functional to critical literacies, and the relationship of "critical thinking" in mathematics to learning mathematics for social justice. However, I would first like to clarify certain points about my article.

First, my notion of *agency* is about *social*, not personal, agency. By agency, I do not mean personal drive, sense of self-efficacy, nor motivation. Rather, I use agency in the sense in which Paulo Freire (1970/1998) spoke, of people becoming *subjects*, or individuals who see the possibility of being able to change the world and become part of historical processes. To have social agency means to see oneself as a potential agent of *social* change. Second, I taught multiple classes in Rivera's three programs: bilinguals honors, bilingual "general," and "general." The JRME article is about my almost-two years with a class in the bilingual honors program; that class was the only one I taught for more than one year, and I have much better data from that class. But my main point is that virtually *all* Rivera students, regardless of program, come from similar backgrounds: immigrant, bilingual, working-class and low-income families. Of the 28 students I taught in the class I discuss in the JRME article, only one did not qualify for free lunch—barely. All the parents were Spanish-dominant immigrants in that class, and all families lived in low-income, working-class Latino/a communities. Third, these same families strongly supported my curriculum and pedagogy of social justice. The evidence is that there was an outpouring of parental support when that curriculum/pedagogy came under attack from a replacement principal, after the original principal who had been for years strongly supportive of my work left Rivera (see Gutstein, forthcoming). And fourth, yes, I agree wholeheartedly with Braver et. al—the constraints, pressures, time limits, and high-stakes accountability tests and strictures under which I operated at Rivera are real and nontrivial. That is precisely why we need to educate the very students in front of us to the political realities of these repressive systems. We need them to join the struggles to fundamentally reconfigure the unjust, stratified educational and societal structures under which we live. This is the essence of teaching (mathematics) for social justice, as I understand it.

On functional and critical literacies—Michael Apple (1992), in his commentary on the NCTM Standards, drew out the distinction between functional and critical literacies by asking: "Whose definition of mathematical literacy is embedded in the Standards? Literacy

is a slippery term. Its meaning varies in accordance with its use by different groups with different agendas" (p. 423). Apple, citing Lankshear & Lawler (1987), contrasted a form of domesticating, functional literacy designed to make "less powerful groups…more moral, more obedient, more effective and efficient workers" versus a critical literacy that would "be part of larger social movements for a more democratic culture, economy, and polity" (p. 423). Critical literacy means to approach knowledge critically and skeptically, see relationships between ideas, look for underlying explanations for phenomena, and question whose interests are served and who benefits. Being critically literate also means to examine one's own and others' lives in relationship to sociopolitical and cultural-historical contexts. Although critical literacy includes acquiring or constructing knowledge of particular concepts, ideas, skills, and facts, it is also avowedly political—helping people recognize oppressive aspects of society so they can participate in creating a more just society (Macedo, 1994).

I believe it is a mistake to think that the NCTM Standards, by *themselves*, will bring us any closer to students having critical literacy. For example, one can have mathematical literacy (a goal of the Standards) without being critically literate. After all, Pentagon defense analysts and Wall Street stock forecasters certainly are mathematically literate, but the real question is: In whose interest do they use their mathematical competencies, those of capital and empire, or of the peoples of the world?

The question of "critical thinking" in mathematics versus learning mathematics for social justice is closely related. The argument is essentially the same as the one about functional and critical literacies. One can think critically in, and with, mathematics, but that in no way guarantees that one will be predisposed towards justice and equity. When one learns mathematics for social justice, the goal is that one not only seeks to deconstruct aspects of inequality, but also to change them. While critical thinking in mathematics is a necessary component of that process, it is not sufficient, as we can see in the Pentagon and Wall Street examples.

I appreciate the attention to my article that Braver et al. paid, and I agree with their conclusions. Teaching mathematics for social justice is not easy, nor uncomplicated. It is fraught with places where we can "go wrong," and make mistakes. But we fundamentally have to take chances and live bravely in times like this, where there is both danger and opportunity in front of us. As Freire (1994) said, "It is impossible to live, let alone exist, without risks. The important thing is to prepare ourselves to be able to run them well" (p. 79).

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Action: Tell me more Albert!

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The new school year opens with the instructor cheerfully announcing to the students on the first day, “Good morning, class! Welcome to the fun of a new language called *algebra*”. He continues, “Today we are going to have a conversation about your success in this classroom. First, is the area of mathematics called *algebra*, necessary for your future?"

None of the students responds.

Looking casually around the room, the instructor happens to notice a student in the last chair of the middle row and asks him, “Young man in the back of the room…what is your name, sir?”

The student responds, “My name is Albert…” and then promptly inquires, “What is your name? “

Taken back momentarily by the brassiness of the student’s tone, the instructor contemplates the best way to respond to the disrespectfully asked question. He determines he will give the student the benefit of the doubt and cordially replies, “Oh! I am sorry, let me introduce myself, my name is Mr. Baty”.

The student, feeling a need to share his opinion of the situation, quips, “Well! I do not like even being in this class. I do not like math!”

Not intimidated by the student’s outburst, the instructor rephrases the question he earlier had directed to the class and asks, “Albert, do you think *algebra* is necessary for your education?”

Pondering the question, the student retorts honestly, “I do not know that answer.” He then stands up and moves to leave the classroom.

Attempting to delay the student’s departure and desiring to avoid an incident, the instructor asks, “Oh, uh, Albert would you like to sit down and get your new book”.

Intrigued, Albert complies and sits down but immediately inquires, “Okay. Where is that new book? I only got an old book”.

Sensing he has the student’s interest, the instructor says,” Come get this newer addition.”

Albert obeys, and receives a brand new copy of the mathematics book.

“Albert, now sit down and I will continue our conversation!” asserts the teacher.

Acting in accordance with his teacher’s request, Albert sits down.

Relieved, the instructor proceeds with the lesson.

The next week, the scene replays itself. “Albert, sit down. Albert, sit down," the instructor loudly but wearily commands. Once he had finished saying that phrase, the one he already had repeated multiple times even though school began just a week ago, the determined instructor embarked to teach the lesson of the day. “As we know from previous lessons, our conversations indicated algebraic terms look and sound strange to us. Can anyone tell us the new terms that were included in last week's studies?” Choosing that particular moment to stand up and move around the room, Albert comes to the teacher’s attention once again.

Albert, where are you going? Albert ignored the question.

Albert, sit down.

Failing to detour Albert’s steady movement toward the door, the teacher adopts a new strategy and says, Okay, Albert, tell us how these terms might help us today. To the instructor’s surprise, Albert replies, “Well, Batty [Albert frequently called Mr. Baty, Batty], I copied all those things down and here is my copy. Would you like to see the terms?”

Pleased with the effort Albert had put into last weeks lesson, Mr. Baty still felt compelled to ask, “Albert, please tell me what are you doing in this class? It is third period and you have algebra first period”. [Mr. Baty already had assumed Albert has had another eviction notice from yet another English class and had selected the algebra class as a safe place to hide.]

Instead of responding, Albert sits down, although not for long. The unending battle for authority quickly resumed as Mr. Baty commanded, “Albert, sit down! Albert, please SIT DOWN!”

The command failing to catch Albert’s attention, Mr. Baty bellowed, “Albert, what are you doing now!”

Strangely, instead of ignoring Mr. Baty, Albert sat down without requiring any further directives and stayed in his seat the rest of the period. Class continued.

How many Albert’s are in your classroom? Join with Mr. Baty in this adventure of addressing the particular demands and pressing educational concerns of Albert and other similar students. Using action research as a tool, may throw some light on the Albert experience.

First, teacher’s action research conceptions depend on whether they put the accent on <u>action</u> or on <u>research</u>. When the accent is on <u>action</u>, there is an assumption the primary purpose is to modify their practice in some way (Fedman & Minstrell, 2000,). Second, when the accent is on research, there is an assumption the teachers purpose is merely observation of practices. With Albert, the accent was on action.

Albert influenced an idea [teaching through conversations] that broke teaching “out of the box”, leading to a classroom of students producing together. Classroom conversations with Albert influenced other student’s performances and perhaps enhanced their motivation by providing a level of understanding that connected mathematics to the student’s lives.

Overall, teaching Albert placed an emphasis on instructing students to learn the language of mathematics in a manner that exceeded merely learning the content. Albert forced the use of ‘action upon objects’ as a teaching strategy so understanding could emerge (Connell, 2003). Using activities as objects to act upon and thus promote increased learning, included having the students work in teams solving carefully selected problems at the dry erase boards. The teams shared knowledge with each other as they worked through the steps needed to solve the problem before presenting it to the class. The approach forced students to apply their knowledge and use the language of mathematics. By allowing only a brief amount of time [Albert’s focus was short-term] to complete the task, this kept the class moving forward. It did not matter if the teams finished the problem; the instructor finished them as a way to emphasize key concepts targeted for mastery.

The strategy of forcing the students and Albert to identify quickly what they knew, and discover what they did not know, expanded their skills for acting upon a mathematical problem. The students written steps for solving problems they completed at the board, their drawings that illustrated key components of the problem, and their oral explanations served as “records of action” that traced their learning. For Albert and the students in the numerous algebra classes he visited, requiring “records of action” opened the door for facilitating students mental processes needed to resolve the mathematical problems. A “record of action” that documents students thought processes serves as an informal assessment method. In particular, it can be a way of assessment that assists students understanding (Connell, 2001).

I recall the time Albert walked to the front board and provided several changes in the way I was presenting linear equations. His comments bridged the language gap between the cultural language that Albert and the students used, and the 'private language of mathematics’. Using his own language, Albert offered a unique piece of understanding that resolved the student’s mental confusion about standard equations. Albert drew pictures on the board to transform abstract equations into familiar concrete experiences. He translated terms and symbols into meaningful concepts as the students assisted him in making calculations to complete the drawing. After seeing the story told in pictures, the students made the mental connection between linear equations and the data needed to create an equation. It worked! Afterwards, the students routinely drew pictures to enable the creation of equations from word problems. Translating the private language of mathematics into meaningful symbols, and then into meaningful concepts, empowered the students. Albert’s

"record of action" opened their minds enough for them to recognize how much they already knew.

The transformation of both the teaching pedagogy and the organization of the activities, promoted the total engagement of the students in cognitively complex conversations about tasks as they worked. Albert's learning style created a need to cluster students together in small groups to produce specific mathematical products. Likewise, unique conversations with Albert aided designs for in-class cooperative group activities. The students' enhanced "verbal-intuitive exposition" skills, made it possible for them to understand, discuss, and draw mathematical procedures effectively. As Gallimore and Tharp (1995) wrote, language that accompanies joint productive activity is the major vehicle for the development of higher cognitive processes necessary for comprehension.

The story of Albert is a call for a modification in teaching pedagogy and organization of activities. Each of us has had an Albert. Do we fully understand how much Albert's learning depends on us? What effects, if any, did this teaching strategy have upon the many other students on a daily basis? How did it provide better learning for Albert?

Through the implementation of a 'record of action' as an instructional strategy with Albert and the other students, greater achievement gains on end-of-the course standardized tests of comprehension occurred. Passage rates on the end-of–course tests increased from 27% to 46% over a two-year period. High stakes test passage rates for the high school mathematics department increased from 17% to 87% passage rate over three years as more and more of the teachers implemented 'cognitively complex conversations' about the mathematical concepts as they were teaching their students.

Passage rates increased because the first conversation with Albert and the many other students that occurred on a daily basis, forced me to wonder and drive myself to make sense of my teaching experiences. I self-observed the changes Albert made in my style of interactions with him and with the other students. I developed an increased awareness of student's learning differences and how each one coped with the modern-day expectations of moving from their cultural experiences into the culture of mathematics.

After becoming familiar with Albert's learning style and frequently reviewing mentally our conversations, I gathered enough data to make conjectures about my new teaching and curriculum strategies. I can report <u>listening and attention</u> to each student's differences closes the achievement gap. All the Albert's in the schools across the nation need similar changes to occur in current teaching methods. As we observe and make the changes in ourselves, we become teachers! (Craig, 2003).

This short conversation about using action research demonstrates the progress Albert and I made. Using action research produced insights about what it takes to become a teacher

'who does make a difference'. Oddly enough, I now have Albert in my college mathematics classes. The research continues.

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Taxi Cab Geometry: History and Applications

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We will explore three real life situations proposed in Eugene F. Krause's book *Taxicab Geometry*. First a dispatcher for Ideal City Police Department receives a report of an accident at X = (-1,4). There are two police cars located in the area. Car C is at (2,1) and car D is at (-1,-1). Which car should be sent? Second there are three high schools in Ideal city. Roosevelt at (2,1), Franklin at (-3,-3) and Jefferson at (-6,-1). Draw in school district boundaries so that each student in Ideal City attends the school closet to them. For the third problem a telephone company wants to set up payphone booths so that everyone living with in twelve blocks of the center of town is with in four blocks of a payphone. Money is tight, the phone company wants to put in the least amount of payphones possible such that this is true.

> *What makes these problems interesting is that we want to solve them not as the "crow flies", but with the constraints that we have to stay on city streets. This means the distance formula that we are accustom to using in Euclidean geometry will not work. Fortunately there is a non Euclidean geometry set up for exactly this type of problem, called taxicab geometry. This system of geometry is modeled by taxicabs roaming a city whose streets form a lattice of unit square blocks (Gardner, p.160).*

Hermann Minkowski first seriously proposed taxicab geometry around the turn of the century. He was born in Russia and was young Albert Einstein's teacher in Zurich (Gardner, p160). Hermann published a whole family of "metrics", that is, examples of spaces in which a way of measuring distance is defined so that it fulfills the axioms of a metric space. Among these metrics is one that is referred to as taxicab metric. The distance formula in this metric is the same used today in taxicab geometry. (Reynolds, 77). It was not until the 1952, when Karl Menger established a geometry exhibit at the Museum of Science and Industry of Chicago that taxicab geometry actually got its name. Accompanying the exhibit was a booklet, entitled *You Will Like Geometry,* in which the term "taxicab" geometry was first used (Golland, 326).

Taxicab geometry is a non-Euclidean geometry that is accessible in a concrete form and is only one axiom away from being Euclidean in its basic structure. The points are the same, the lines are the same, and angles are measured the same way. Only the distance function is different. In Euclidean geometry distance between two points P and Q are defined by:

$$d(P,Q) = ((x_1,y_1),(x_2,y_2))$$
$$d(P,Q) = \sqrt{\ }((x_2-x_1)^2 + (y_2-y_1)^2\ )$$

The minimum distance between two points is a straight line in Euclidean geometry.. In taxicab geometry there may be many paths, all equally minimal, that join two points. Taxicab distance between two points P and Q is the length of a shortest path from P to Q composed of line segments parallel and perpendicular to the x-axis. We use the formula:

$d(P,Q) = [(x_1,y_1),(x_2,y_2)]$

$d(P,Q) = |x_2-x_1| + |y_2-y_1|$

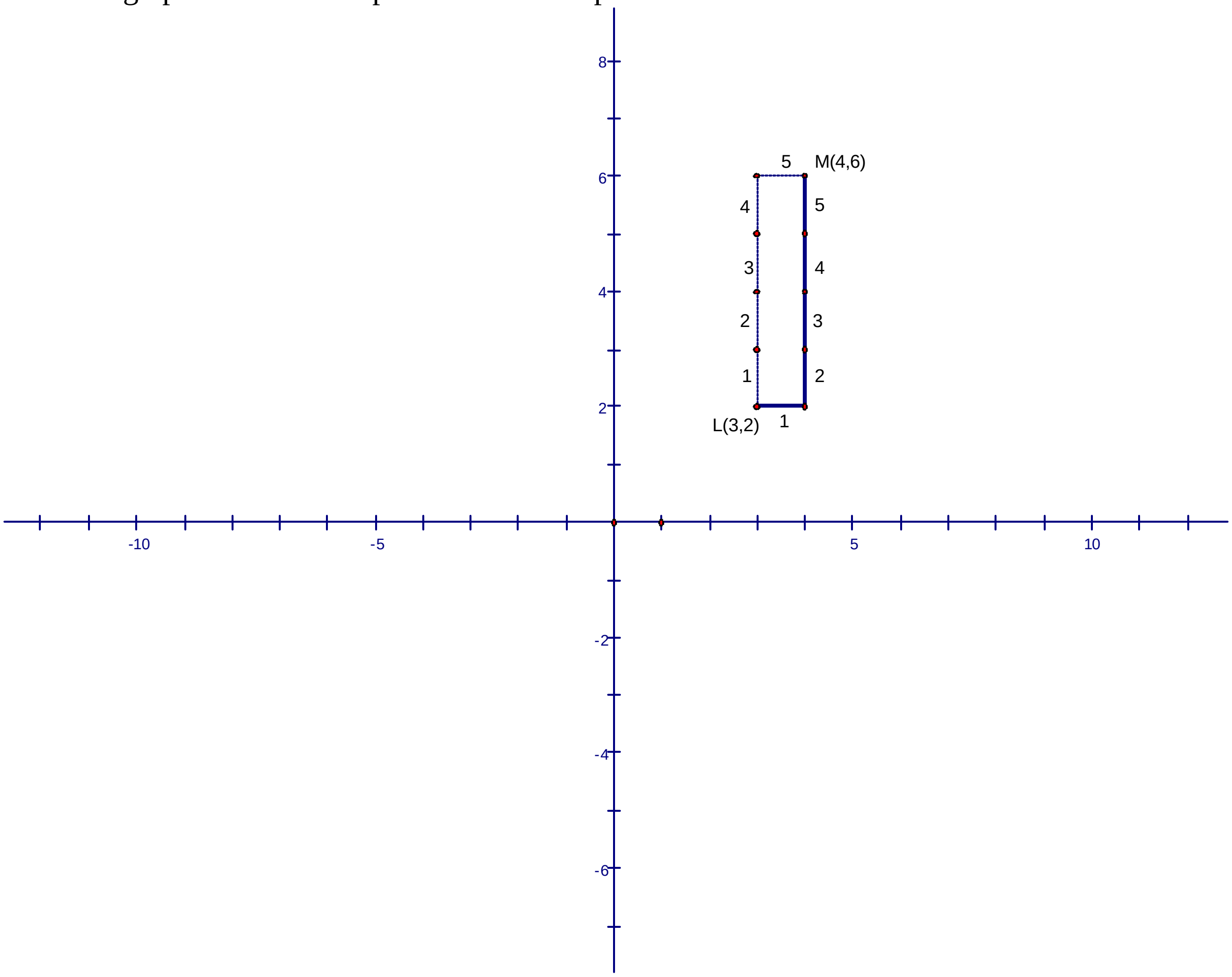
Lets look at the distance between points L(3,2) and M(4,6)

$d(L,M) = |4-3| + |6-2|$

$= |1| + |4|$

$= 5$

Here is a graph of two of the possible minimal paths from L to M.



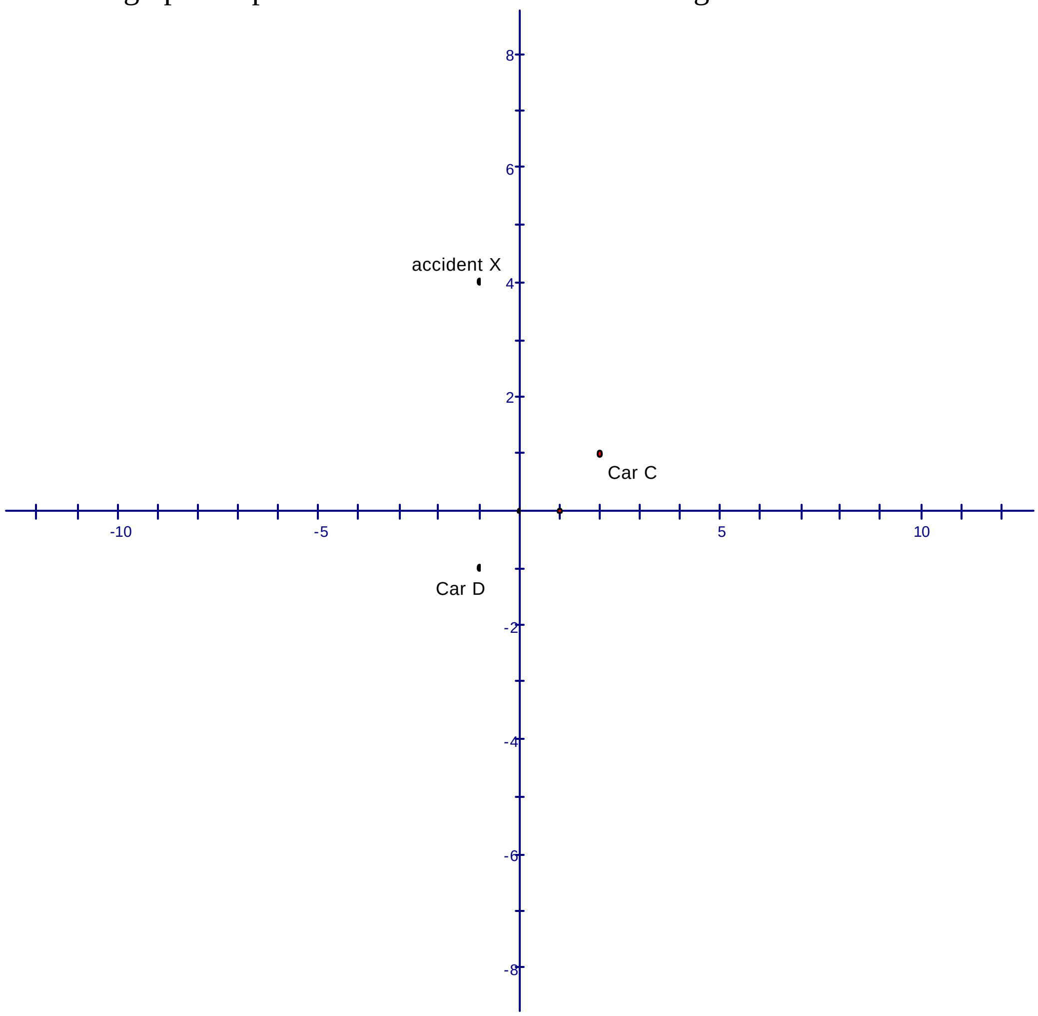
There are three more paths that are a distance of five. Can you find them?

Using our new distance formula, we will now set out to find the solutions to the problems proposed at the beginning of this paper.

Problem One

A dispatcher for Ideal City Police Department receives a report of an accident at X = (-1,4). There are two police cars located in the area. Car C is at (2,1) and car D is at (-1,-1). Which car should be sent?

First lets graph the problem to see what we are looking at:



The police cars cannot drive through peoples' houses. They have to stick to the streets. Taxicab geometry will be the best choice to solve this problem. One simply needs to compare the distance in taxicab geometry from the dispatcher to each patrol car.

The distance between accident X and car C is:

$$\begin{aligned} d(X,C) &= [(-1,4),(2,1)] \\ &= |2-(-1)| + |1-4| \\ &= |3| + |-3| \\ &= 3 + 3 \\ &= 6 \end{aligned}$$

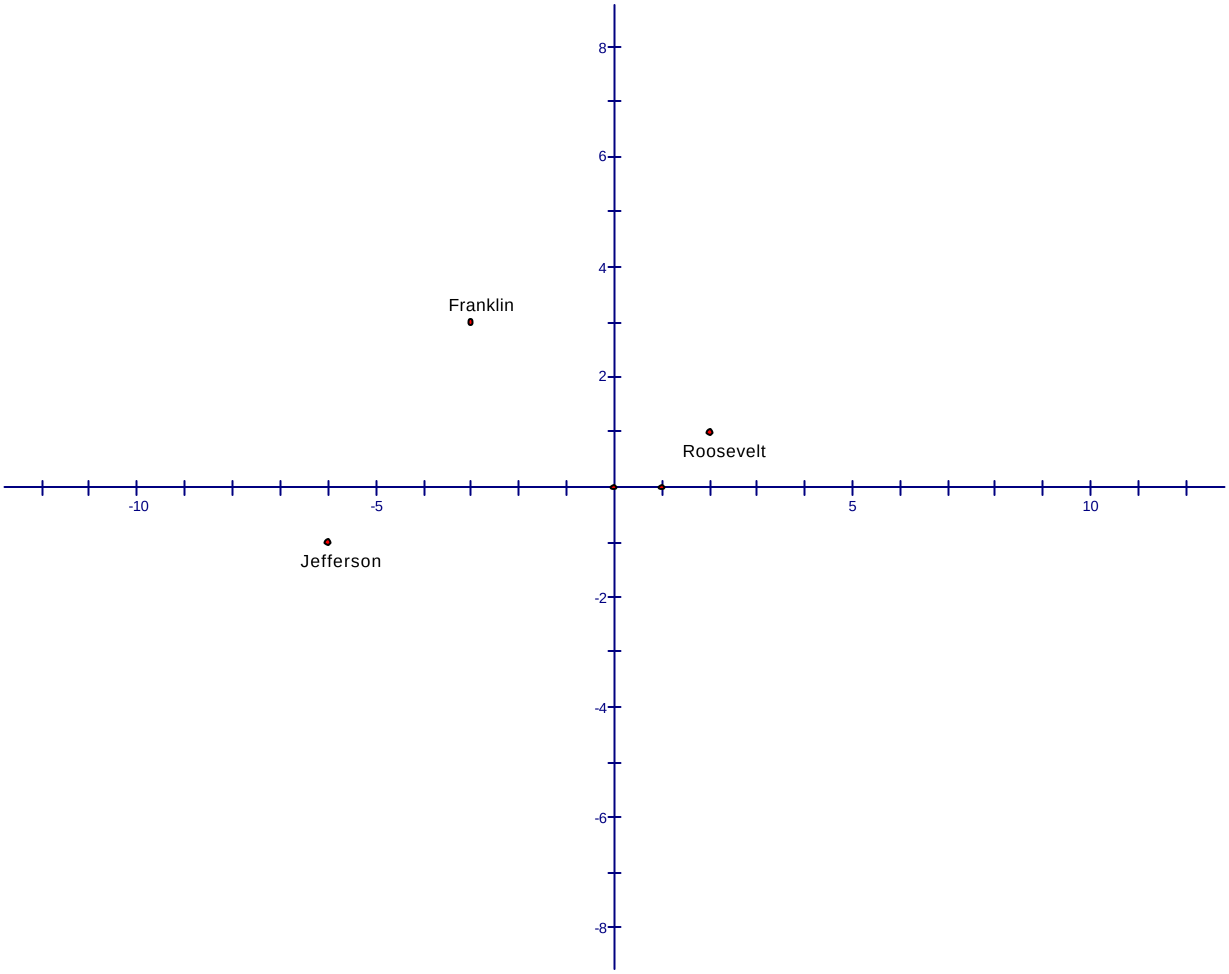
The distance between accident X and car D is:

$$\begin{aligned} d(X,D) &= [(-1,4),(-1,-1)] \\ &= |-1-(-1)| + |-1-4| \\ &= |0| + |-5| \\ &= 0+5 \\ &= 5 \end{aligned}$$

Thus we can clearly see that car D is one block closer to accident X.

Problem Two

We want to draw school district boundaries such that every student is going to the closest school. There are three schools: Jefferson at (-6, -1), Franklin at (-3, -3), and Roosevelt at (2,1).



Taxicab geometry will be a logical choice to find the solution because students will have to stick to the streets when traveling to school. The solution to this problem can not be found at once. The problem needs to be broken down into sections.

Section I

Lets first focus on a boundary between Franklin and Jefferson school. The boundary line needs to fall on the points were the distance between Jefferson and Franklin are the same. So we need:

$$d(\text{Jefferson}) = d(\text{Franklin})$$
$$d[(-6,-1),(x,y)] = d[(-3,3),(x,y)]$$
$$|x+6| + |y+1| = |x+3| + |y-3|$$

Solving for x and y becomes more difficult with the absolute values. We need to look at the cases were $x+6<0 \Rightarrow x<-6$, $y+1<0 \Rightarrow y<-1$, $x+3<0 \Rightarrow x<-3$, and $y-3<0 \Rightarrow y<3$. This translates into nine different cases:

	$-1 \le y \le 3$	$y < -1$	$y > 3$
$-6 \le x \le -3$	case I	case IV	case VII
$x < -6$	case II	case V	case IIX
$x > -3$	case III	case VI	case IX

We look at these cases because the absolute values will change the solutions.

<u>Case I</u>: $-1 \le y \le 3$ and $-6 \le x \le -3$

Since $|x+6| \ge 0$ when $-6 \le x \le -3$, $|x+6| = x+6$

$|y+1| \ge 0$ when $-1 \le y \le 3$, $|y+1| = y+1$

$|x+3| \le 0$ when $-6 \le x \le -3$, $|x+3| = -x+3$

$|y-3| \le 0$ when $-1 \le y \le 3$, $|y-3| = -y+3$

Now under the conditions $-1 \le y \le 3$ and $-6 \le x \le -3$,

$$x+6+y+1 = -x-3-y+3$$
$$x+y+7 = -x-y$$
$$y = -x - 7/2$$

When we make our table of values we see:

x	y
-6	5/2
-5	3/2
-4	½
-3	-1/2

<u>Case II</u>: $-1 \le y \le 3$ and $x < -6$

Since $|x+6| < 0$ when $x < -6$, $|x+6| = -x-6$

$|y+1| \ge 0$ when $-1 \le y \le 3$, $|y+1| = y+1$

$|x+3| < 0$ when $x < -6$, $|x+3| = -x-3$

$|y-3| \le 0$ when $-1 \le y \le 3$, $|y-3| = -y+3$

Now under the conditions $-1 \le y \le 3$ and $x < -6$,

$$-x-6+y+1 = -x-3-y+3$$
$$-x+y-5 = -x-y$$
$$y = 5/2$$

When we make our table of values we see:

x	y
-7	5/2
-8	5/2
-9	5/2
-10	5/2
.	.
.	.
.	.

<u>Case III</u>: $-1 \le y \le 3$ and $x > -3$

Since $|x+6| > 0$ when $x > -3$, $|x+6| = x+6$

$|y+1| \ge 0$ when $-1 \le y \le 3$, $|y+1| = y+1$

$|x+3| \ge 0$ when $x > -3$, $|x+3| = x+3$

$|y–3| < 0$ when $-1 \le y \le 3$, $|y–3| = -y + 3$

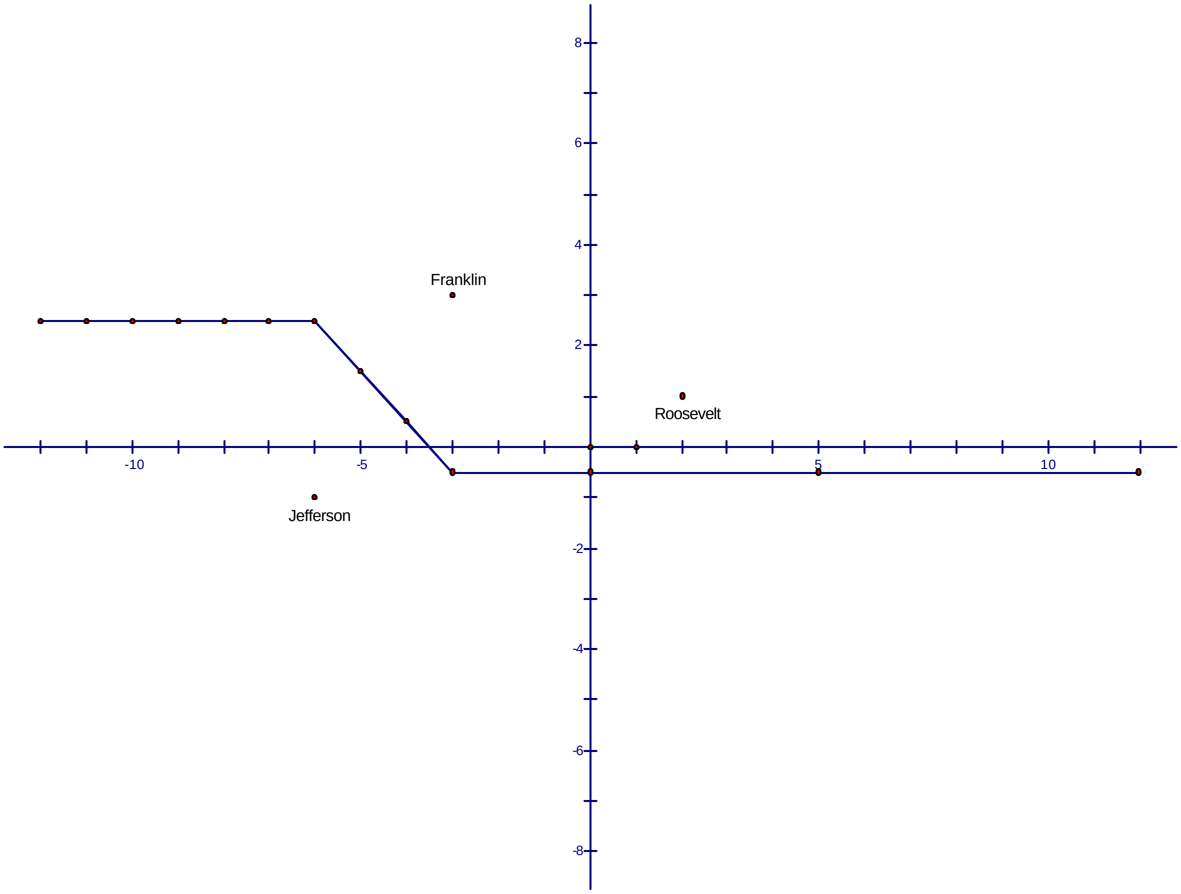
x	y
-3	-1/2
-2	-1/2
-1	-1/2
0	-1/2
.	.
.	.
.	.

Now under the conditions $-1 \le y \le 3$ and $x > -3$,

$$x + 6 + y + 1 = x + 3 – y + 3$$
$$x + y + 7 = x – y + 6$$
$$2y = -1$$
$$y = -1/2$$

When we make our table of values we see:

Below I have plotted the points and lines so far that follow d(Jefferson) =d(Franklin)



<u>Cases IV-IX</u>: No solutions exist when we look at these cases. You may do the algebra yourself to be convinced of this, but it is not necessary. By looking at the graph above, one can obviously see that there will not be any more solutions. If you look in quadrant III and IV, obviously any one living in the outer area will be closest to Jefferson school. Also

looking in quadrants I and II, anyone living in these outer boundaries will be closer to Franklin school.

Section 2:

We will now look at the boundary between Franklin and Roosevelt school. Again we want to find the points that are equal distance from the two schools to create our boundary. We are looking for :

$$d(\text{Franklin}) = d(\text{Roosevelt})$$
$$d[(-3,3),(x,y)] = d[(2,1),(x,y)]$$
$$|x+3| + |y-3| = |x-2| + |y-1|$$

Again there are going to be different cases we need to look at., these cases are:

	$1 \le y \le 3$	$y < 1$	$y > 3$
$-3 \le x \le 2$			
$x > 2$			
$x < -3$			

Case I: $1 \le y \le 3$ and $-3 \le x \le 2$

Since $|x+3| \ge 0$ when $-3 \le x \le 2$, $|x+3| = x+3$
$|y-3| \le 0$ when $1 \le y \le 3$, $|y-3| = -y+3$
$|x-2| \le 0$ when $-3 \le x \le 2$, $|x-2| = -x+2$
$|y-1| \ge 0$ when $1 \le y \le 3$, $|y-1| = y-1$

Under the conditions $1 \le y \le 3$ and $-3 \le x \le 2$

$$x + 3 - y + 3 = -x + 2 + y - 1$$
$$x - y + 6 = -x + y + 1$$
$$-2y = -2x - 5$$
$$y = x + 5/2$$

plug into a table of values:

x	y
0	5/2
-1	3/2

Case II: $x<-3$ and $y<1$

Since $|x+3| < 0$ when $x<-3$, $|x+3| = -x-3$
$|y-3| < 0$ when $y<1$, $|y-3| = -y+3$
$|x-2| < 0$ when $x<-3$, $|x-2| = -x+2$
$|y-1| < 0$ when $y<1$, $|y-1| = -y+1$

Under the conditions $x<-3$ and $y<1$,

$$-x - 3 - y + 3 = -x + 2 - y + 1$$
$$-x - y = -x - y + 3$$

but $0 \ne 3$ so there is no solution for $x<-3$ and $y<1$

Case III: $-3 \le x \le 2$ and $y<1$

Since $|x+3| \ge 0$ when $-3 \le x \le 2$, $|x+3| = x+3$

$|y-3| <0$ when $y<1$, $|y-3| = -y+3$

$|x-2| \le 0$ when $x<-3$, $|x-2| = -x+2$

$|y-1| <0$ when $y<1$, $|y-1| = -y+1$

Under the conditions $-3 \le x \le 2$ and $y<1$,

$$x + 3 - y + 3 = -x + 2 - y + 1$$
$$x - y + 6 = -x - y + 3$$
$$2x = -3$$
$$x = -3/2$$

Now plug into a table of values:

x	y
-3/2	1
-3/2	0
-3/2	-1
-3/2	-2
.	.
.	.
.	.

Case IV: $-3 \le x \le 2$ and $y \ge 3$

Since $|x+3| \ge 0$ when $-3 \le x \le 2$, $|x+3| = x+3$

$|y-3| \ge 0$ when $y \ge 3$, $|y-3| = y-3$

$|x-2| \le 0$ when $x<-3$, $|x-2| = -x+2$

$|y-1| >0$ when $y \ge 3$, $|y-1| = y-1$

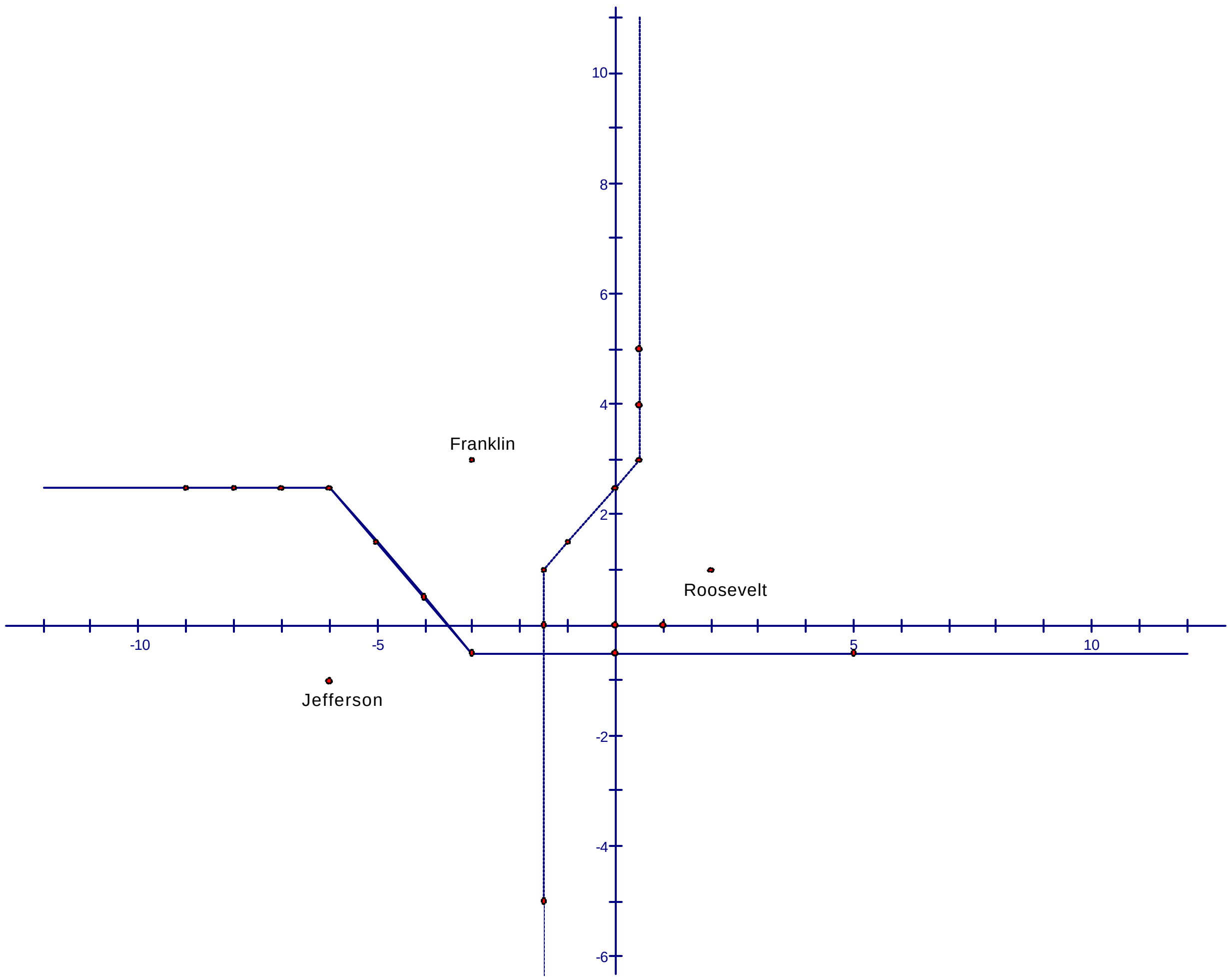
Under the conditions $-3 \le x \le 2$ and $y \ge 3$,

$$x + 3 + y - 3 = -x + 2 + y - 1$$
$$x + y = -x + y + 1$$
$$2x = 1$$
$$x = ½$$

plug into a table of values

x	y
½	3
½	4
½	5
.	.
.	.
.	.

Below I have plotted the points and lines so far that follow d(Roosevelt) = d(Franklin) with a dotted line.



<u>Cases V-IX</u>: No solutions exist when we look at these cases. Again you may do the algebra yourself to be convinced of this, but it is not necessary. By looking at the graph above, one can obviously see that there will not be any more solutions. If you look in quadrant I and IV, obviously any one living in the outer area will be closest to Roosevelt school. Also looking in quadrant II and III, anyone living in these outer boundaries will be closer to Franklin school.

Section 3

Lets now focus on a boundary between Roosevelt and Jefferson school. The boundary line needs to fall on the points were the distance between Jefferson and Roosevelt are the same. So we need :

$$d(\text{Jefferson}) = d(\text{Roosevelt})$$
$$d[(-6,-1),(x,y)] = d[(2,1),(x,y)]$$
$$|x+6| + |y+1| = |x-2| + |y-1|$$

Again there are going to be different cases we need to look at., these cases are:

	$-1 \le y \le 1$	$y < -1$	$y > 1$
$-6 \le x \le 2$			
$x > 2$			
$x < -6$			

Case I: $-6 \le x \le 2$ and $y < -1$

Since $|x+6| \ge 0$ when $-6 \le x \le 2$, $|x+6| = x+6$
$|y+1| < 0$ when $y < -1$, $|y+1| = -y +1$
$|x-2| \le 0$ when $-6 \le x \le 2$, $|x-2| = -x +2$
$|y-1| < 0$ when $y < -1$, $|y-1| = -y +1$

Now under the conditions $-6 \le x \le 2$ and $y < -1$,

$$x + 6 – y – 1 = - x + 2 – y + 1$$
$$x – y + 5 = - x – y + 3$$
$$2x = -2$$
$$x = -1$$

plug into a table of values:

x	y
-1	-2
-1	-3
-1	-4
.	.
.	.
.	.

Case II: $-6 \le x \le 2$ and $-1\le y \le 1$

Since $|x+6| \ge 0$ when $-6 \le x \le 2$, $|x+6| = x+6$
$|y+1| \ge 0$ when $-1< y \le -1$, $|y+1| = y+1$
$|x-2| \le 0$ when $-6 \le x \le 2$, $|x-2| = -x +2$
$|y-1| \le 0$ when $–1 \le y \le -1$, $|y-1| = -y +1$

Now under the conditions $-6 \le x \le 2$ and $–1 \le y \le -1$,

$$x + 6 +y +1 = -x + 2 –y +1$$
$$x + y + 7 = -x – y +3$$
$$x + y = -x – y – 4$$
$$y = -x - 2$$

plug into a table of values

x	y
-1	-1
-2	0
-3	1

Case III: $-6 \le x \le 2$ and $y > 1$

Since $|x+6| \ge 0$ when $-6 \le x \le 2$, $|x+6| = x+6$

$|y+1| > 0$ when $y > 1$, $|y+1| = y+1$
$|x-2| \leq 0$ when $-6 \leq x \leq 2$, $|x-2| = -x+2$
$|y-1| > 0$ when $y > 1$, $|y-1| = y-1$

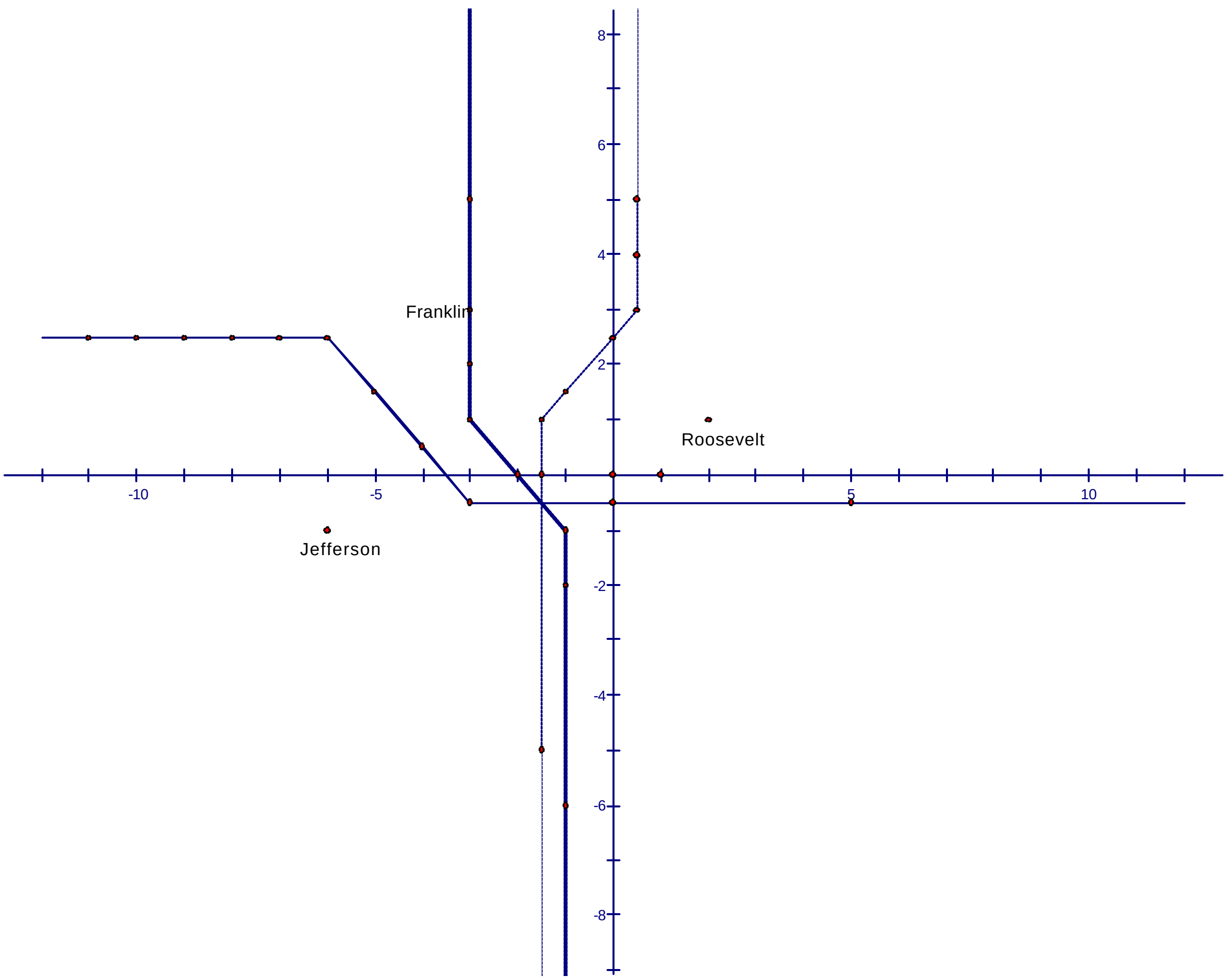
Now under the conditions $-6 \leq x \leq 2$ and $1 < y$,

$$x + 6 + y + 1 = -x + 2 + y - 1$$
$$x + y + 7 = -x + y + 1$$
$$2x = -6$$
$$x = -3$$

plug into a table of values:

x	y
-3	2
-3	3
-3	4
.	.
.	.
.	.

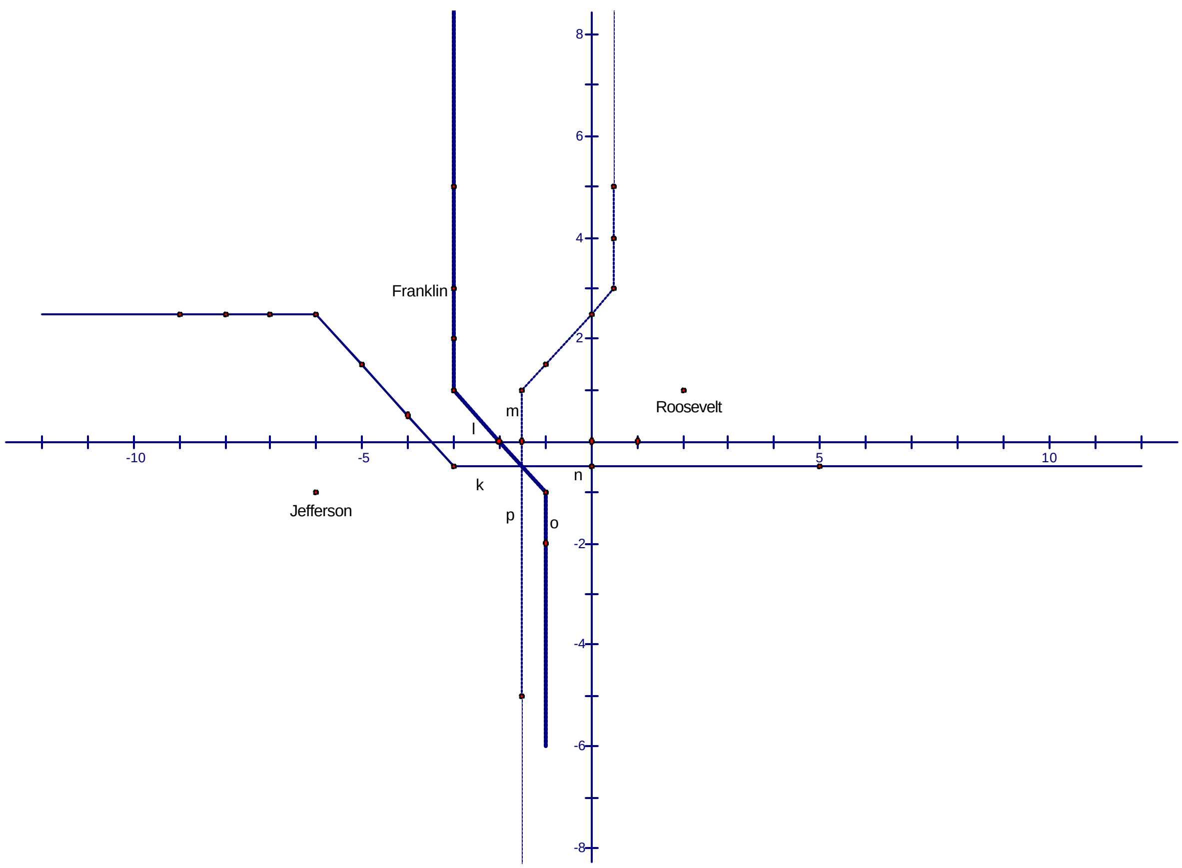
Below I have plotted the points and lines so far that follow d(Roosevelt) = d(Jefferson) with a thick line.



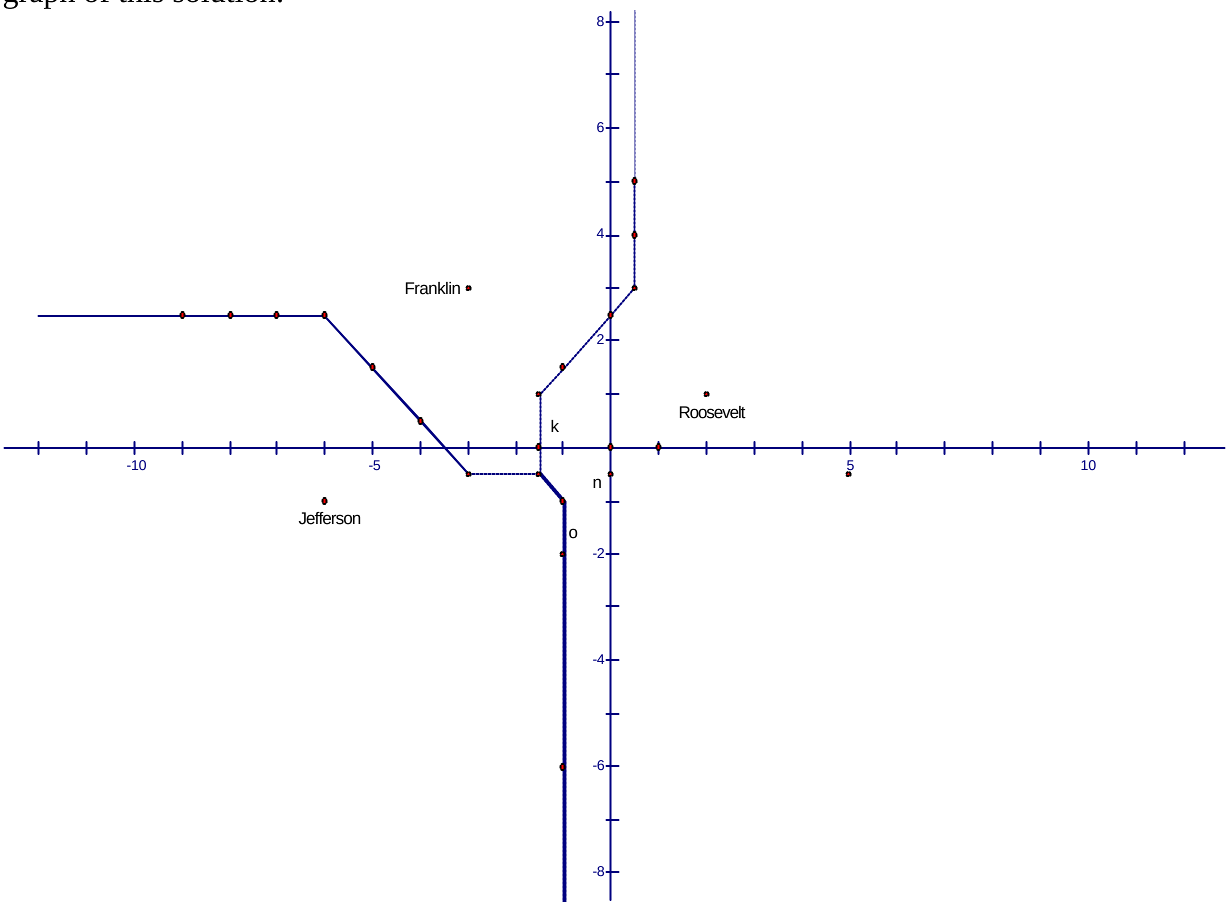
Cases IV-IX: No solutions exist when we look at these cases. Again you may do the algebra yourself to be convinced of this, but it is not necessary. By looking at the graph above, one can obviously see that there will not be any more solutions. If you look in quadrant I and IV, obviously any one living in the outer area will be closest to Roosevelt school. Also looking in quadrant II and III, anyone living in these outer boundaries will be closer to Jefferson school.

Section 4

Now we need to look at the information and use some basic logic to interpret the information we have. The previous figure has all the boundaries between two school districts on it. We must first find the point were the three boundaries intersect. This is at (-1/2,-1/2). Now we will look at each boundary from this point. I will label the boundaries for ease of explanation as follows:



First look at boundary *k*, it is obviously a correct boundary because there is no other boundary close that would properly divide Franklin and Jefferson. Boundary *l* was intended to divide Jefferson and Roosevelt, this boundary is not necessary because the points on this line are closest to Franklin. Since we removed *l*, *m* must stay to maintain a boundary between Roosevelt and Franklin. Now boundary *n* is not necessary because it is the boundary between Jefferson and Franklin. Roosevelt school is closer, so we can remove *n*. Our boundary *p* is for Franklin and Roosevelt. But we are closer to Jefferson and Roosevelt so we will disregard this boundary. Now *o* is left. We will keep this boundary since *p* was removed, *o* must stay to maintain the boundary between Jefferson and Roosevelt. After removing some of the boundaries, we have our final solution to a school district such that everyone is attending the school they live closest to. Below is the graph of this solution:



Problem Three

The telephone company wants to set up pay phone booths so that everyone living with in 12 blocks of the center of town is with in four blocks of a payphone booth. Money is tight, the telephone company wants to put in the least amount of booths possible such that this is true.

Taxicab geometry is the logical choice for solving this problem. For the people can not walk through backyards or jump over buildings to use the phone. They must stick to the street.

We must break this problem down into different sections.

Section 1:

We must first draw the boundary lines that are with in 12 blocks of the center of town. To do this we will need to find the areas that are ≤ 12 blocks. We need to look at the lines that are a distance of 12 from the center of town. To do this look at the equation:

$$d[(0,0),(x,y)] = 12$$
$$|x - 0| + |y - 0| = 12$$

As in the previous problem, we must evaluate this formula with different cases because of the absolute values. We must look at the following cases:

	$-12 = x \le 0$	$12 = x \ge 0$
$-12 = y \le 0$		
$12 = y \ge 0$		

<u>Case I</u>: $12 = x \ge 0$ and $12 = y \ge 0$

since $|x-0| \ge 0$ when $12 = x \ge 0$, $|x-0| = x-0$

$|y-0| = 0$ when $12 = y \ge 0$, $|y-0| = y-0$

when $12 = x \ge 0$ and $12 = y \ge 0$,

$$x-0 + y-0 = 12$$
$$y = -x + 12$$

<u>Case II</u>: $12 = x \ge 0$ and $-12 = y \le 0$

since $|x-0| \ge 0$ when $12 = x \ge 0$, $|x-0| = x-0$

$|y-0| = 0$ when $-12 = y \le 0$, $|y-0| = -y+0$

when $12 = x \geq 0$ and $-12 = y \leq 0$

$$x - 0 - y + 0 = 12$$
$$x - 12 = y$$

<u>Case III</u>: $-12 = x \leq 0$ and $-12 = y \leq 0$

since $|x-0| = 0$ when $-12 = x \leq 0$, $|x-0| = -x+0$

$|y-0| = 0$ when $-12 = y \leq 0$, $|y-0| = -y+0$

when $-12 = x \leq 0$ and $-12 = y \leq 0$,

$$-x + 0 - y + 0 = 12$$
$$-x - 12 = y$$

<u>Case IV:</u> $-12 = x \leq 0$ and $12 = y \geq 0$,

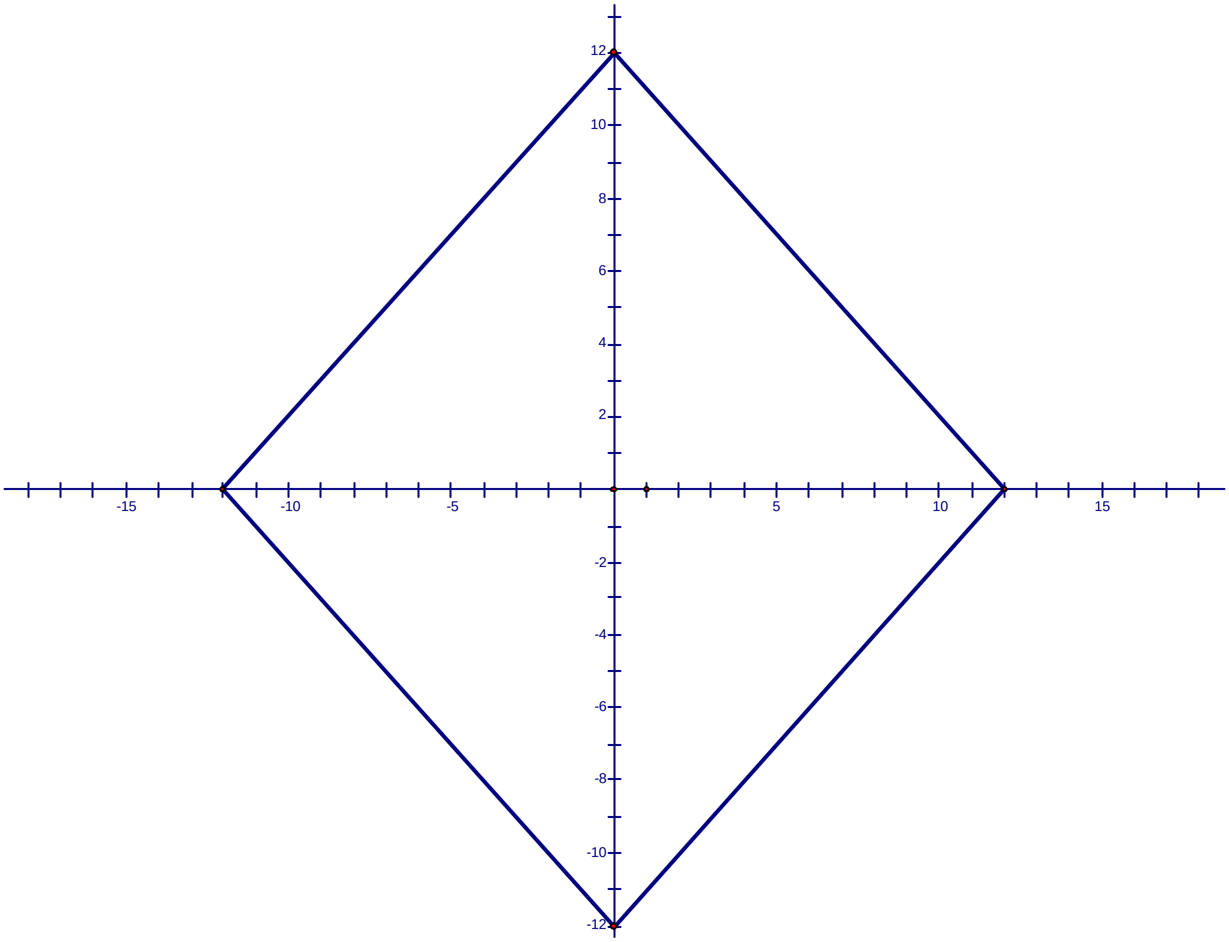
since $|x-0| = 0$ when $-12 = x \leq 0$, $|x-0| = -x+0$

$|y-0| = 0$ when $12 = y \geq 0$, $|y-0| = y-0$

when $-12 = x \leq 0$ and $12 = y \geq 0$,

$$-x + 0 + y - 0 = 12$$
$$y = x + 12$$

We have found four lines, each in a different quadrant, that have a distance of 12 to the origin. Below I have graphed our four cases to see the boundary were everyone with in these lines are within twelve blocks of down town.



What we have done here is graphed a circle of radius 12 in taxicab geometry.

<u>Section 2</u>

Now we need to find an equation for a line such that every one living on the boundary in quadrant I will be with in four blocks of. We can find this line by using the equation

$$d[(x,-x-12),(x_1,y_1)] = 4$$

$$| \; x - x_1 + (-x+12) - y_1 = 4$$

we only need to look at the cases were we are with in our original boundary. Therefore $x_1 = x$ and $y_1 = y$. Now $x - x_1 = x - x_1$ and $(-x+12) - y_1 = -x + 12 - y_1$

$$x - x_1 + -x + 12 - y_1 = 4$$

$$-x_1 + 8 = y_1$$

lets call this line *l*

The equation $-x_1 + 8 = y_1$ gives us a line to place possible payphones that is the optimal distance from our outer boundary in quadrant I. Everything between line *l* and our outer boundary will be = 4 blocks of our line *l*. We now need to find the boundary line for everyone that can use a phone located on line *l* that lives below it. To do this we will look at the equation :

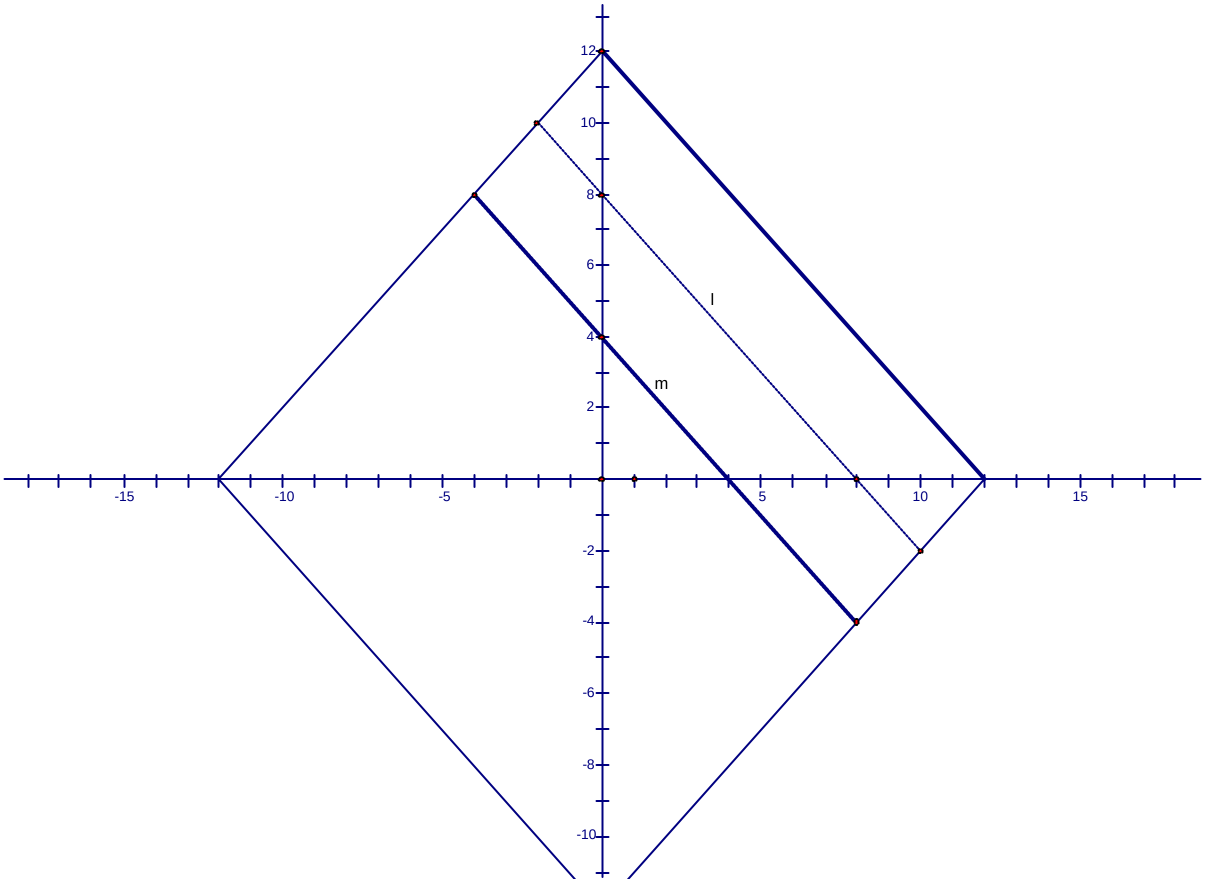
$$d[(x_1,x_1-8),(x_2,y_2)] = 4$$
$$|x_2-x_1| + |y_2-(x_1-8)| = 4$$

Again, we only need to look at the cases were we are with in our boundary. Therefore $x_2 = x_1$ and $y_2 = y_1$. Now $x_2 – x_1 = -x_2 +x_1$ and $y_2-(-x_1+8) = -y_2+(-x_1 +8)$
Now our equation becomes:

$$-x_2 +x_1-y_2+(-x_1 +8) = 4$$
$$-x_2 -y_2+8 = 4$$
$$-x_2 + 4 = y_2$$

lets call this line *m*

Now graph this equation to see the boundaries of our possible placement of phones. Everyone living between the two thick lines will be with in four blocks of phones located on the dotted line.



As we did before, we must find a line below line such that people living on or below this boundary (line *m*) will be with in four blocks. To do this we will use the equation:

$d[(x_2,-x_2+4),(x_3,y_3)] = 4$

Again, we only need to look at the cases were we are with in our boundary. Therefore $x_3 = x_2$ and $y_3 = y_2$. Now $x_3 - x_2 = -x_3 + x_2$ and $y_3-(-x_2+4) = -y_3+(-x_2+4)$
Now our equation becomes:

$$-x_3 + x_2 - y_3 + (-x_2 + 4) = 4$$
$$-x_3 = y_3$$

lets call this line *n*

If phones are placed along this line, everyone living between this line and the previous boundary will be with in four blocks of a phone. Now we need to find the boundary such

that people living below our new line will be with in four blocks of these phones. We use the equation:

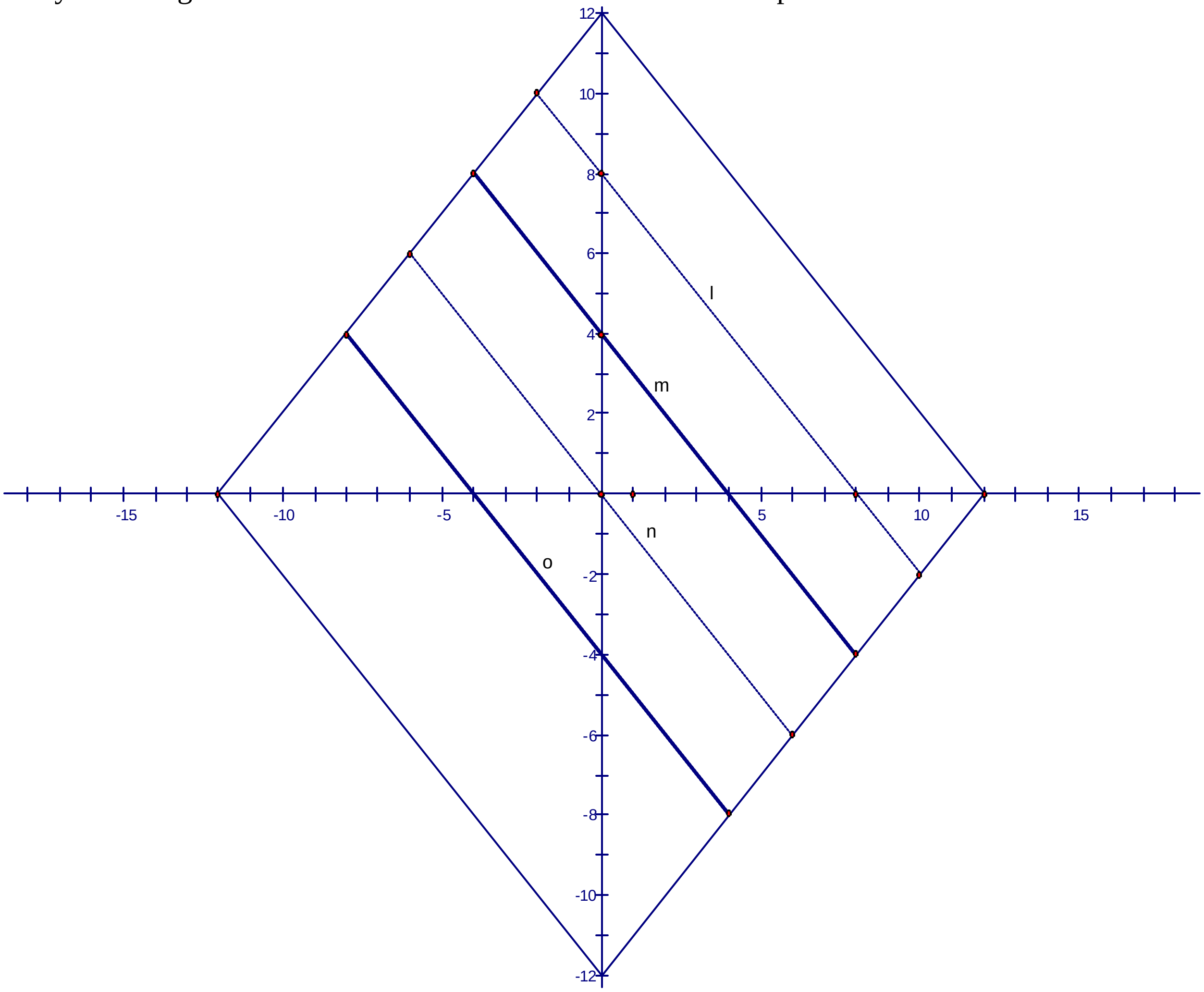
$$d[(x_3,-x_3),(x_4,\ y_4)] = 4$$

Again, we only need to look at the cases were we are with in our boundary. Therefore $x_4 = x_3$ and $y_4 = y_3$. Now $x_4 - x_3 = -x_4 + x_3$ and $y_4-(-x_3) = -y_4+(-x_3)$
Now our equation becomes:

$$-x_4 + x_3 - y_4 + (-x_3) = 4$$
$$-x_4 - 4 = y_4$$

lets call this line *o*

When we graph these two lines we will see the boundaries (lines *m* and *o*) such that everyone living between the two will be with in four blocks of phones located on line *n*.



This process needs to be repeated one more time. As we did before, we must find a line below line *o* such that people living on or below this boundary will be with in four blocks of a phone. To do this we will use the equation:

$$d[(x_4,-x_4-4),(x_5, y_5)] = 4$$

We only need to look at the cases were we are with in our boundary. Therefore $x_5 = x_4$ and $y_5 = y_4$. Now $x_5 - x_4 = -x_5 + x_4$ and $y_5-(-x_4-4) = -y_5+(-x_4-4)$
Now our equation becomes:

$$-x_5 + x_4 - y_5+(-x_4-4) = 4$$
$$-x_5-8 = y_5$$

Call this line *p*

If phones are placed along this line *p*, everyone living between this line and the previous boundary will be with in four blocks of a phone. Now we need to find the boundary such that people living below our new line will be with in four blocks of these phones. We use the equation:

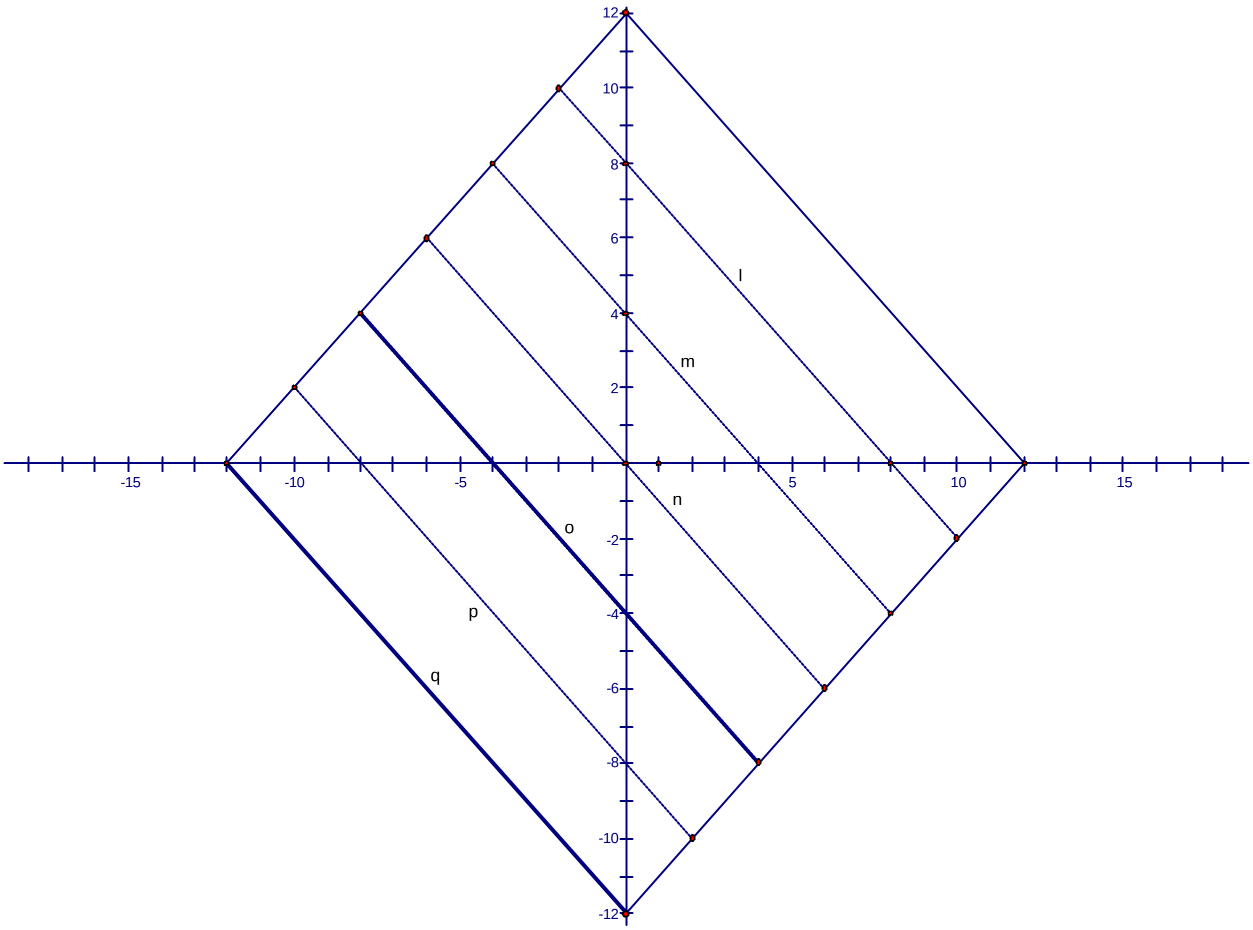
$$d[(x_5,-x_5-8),(x_6,y_6)] = 4$$

We only need to look at the cases were we are with in our boundary. Therefore $x_6 = x_5$ and $y_6 = y_5$. Now $x_6 - x_5 = -x_6 + x_5$ and $y_6-(-x_5-8) = -y_6+(-x_5-8)$
Now our equation becomes:

$$-x_6 + x_5 - y_6+(-x-8) = 4$$
$$-x_6-12 = y_6$$

lets call this line *q*

Line *q* is the same line as our outer boundary in quadrant III. Below I have graphed the boundaries and the line that the phones can be placed on.



We have found optimal lines to put our phone booths on, but we do not know were on these lines to place them. We will need to find out more information.

Section 3

We will repeat the exact same process as in section two. This time we will start with finding the line such that everyone living on the boundary in quadrant II is with in four blocks of. From section 1, we know the equation for our outer boundary is $y = x_7+12$
We need to solve the equation:

$$d[(x_7,x_7+12),(x_8,y_8)] = 4$$

For all the solutions to the upcoming problems in this section $x_n \geq x_{n+1}$ and $y_n \leq y_{n+1}$ because we only need to work within our stated boundaries. So the lines that we will be

solving will always have x values greater than the previous line and y values less then the previous line. Therefore all

$d[(x_n,y_n),(x_{n+1},y_{n+1})] = 4,$

$|x_{n+1}-x_n| \leq 0$ when $x_n \leq x_{n+1}$, so $|x_{n+1}-x_n| = x_{n+1}-x_n$

$|y_{n+1}-y_n| \geq 0$ when $y_n \geq y_{n+1}$, so $|y_{n+1}-y_n| = -y_{n+1}+y_n$

Back to solving $d[(x_7,x_7+12),(x_8,y_8)] = 4,$

$$x_8-x_7-y_8+(x_7+12) = 4$$

$$x_8+8 = y_8$$

call this line *r*

If phones are placed along this line *r*, everyone living between this line and the previous boundary will be with in four blocks of a phone. Now we need to find the boundary such that people living below our new line will be with in four blocks of these phones. We use the equation:

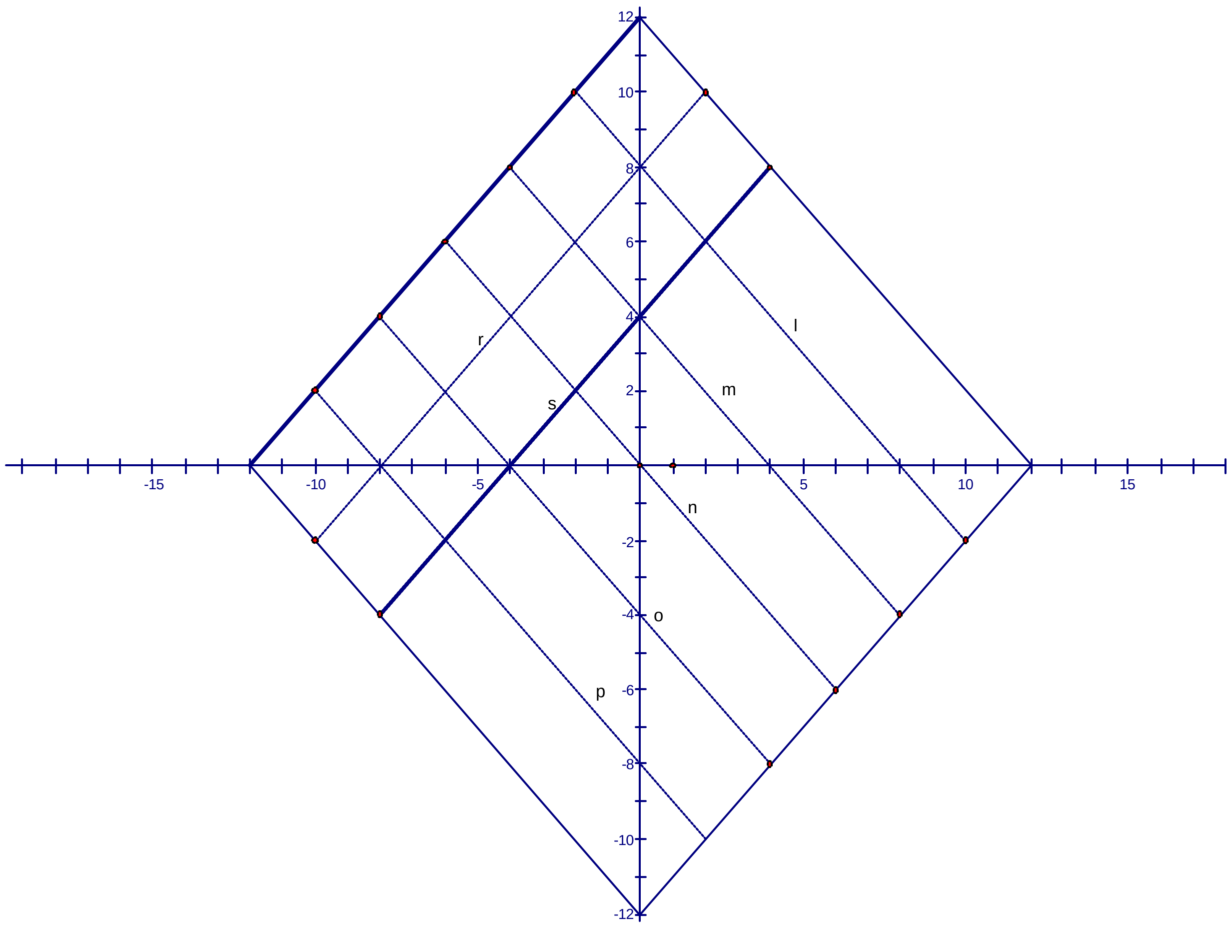
$$d[(x_8,x_8+8),(x_9,y_9)] = 4$$

$$x_9-x_8-y_9+(x_8+8) = 4$$

$$x_9+4 = y_9$$

call this line *s*

When we graph these two lines we will see that the lines *s* and $y = x_7+12$ (our outer boundary) are boundaries such that everyone living between the two will be with in four blocks of phones located on line *r*. I have graphed this below.



As we did before, we must find a line below line such that people living on or below this boundary (line *s*) will be with in four blocks. To do this we will use the equation:

$$d[(x_9,x_9+4),(x_{10},y_{10})] = 4$$
$$x_{10}-x_9-y_{10}+(x_9+4) = 4$$
$$x_{10} = y_{10}$$

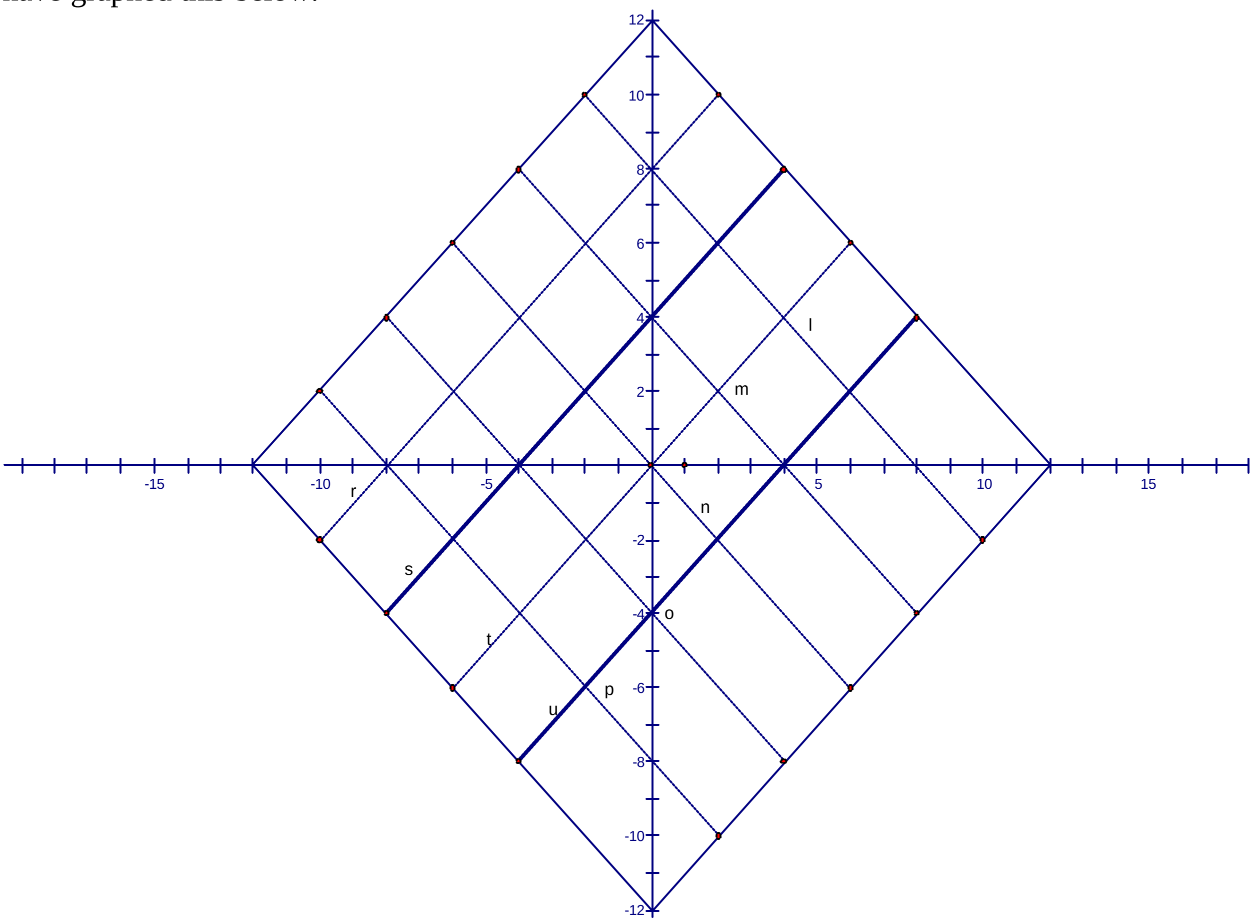
call this line *t*

If phones are placed along this line *t*, everyone living between this line and the previous boundary will be with in four blocks of a phone. Now we need to find the boundary such that people living below our new line will be with in four blocks of these phones. We use the equation:

$$d[(x_{10},x_{10}),(x_{11},y_{11})] = 4$$
$$x_{11}-x_{10}-y_{11}+x_{10} = 4$$
$$x_{11}-4 = y_{11}$$

call this line *u*

When we graph these two lines we will see that the lines *r* and *u* are boundaries such that everyone living between the two will be with in four blocks of phones located on line *t*. I have graphed this below.



This process needs to be repeated one more time. As we did before, we must find a line below line *u* such that people living on or below this boundary will be with in four blocks of a phone. To do this we will use the equation:

$$d[(x_{11},x_{11}-4),(x_{12},y_{12})] = 4$$
$$x_{12}-x_{11}-y_{12}+(x_{11}-4) = 4$$
$$x_{12}-8 = y_{12}$$

call this line *v*

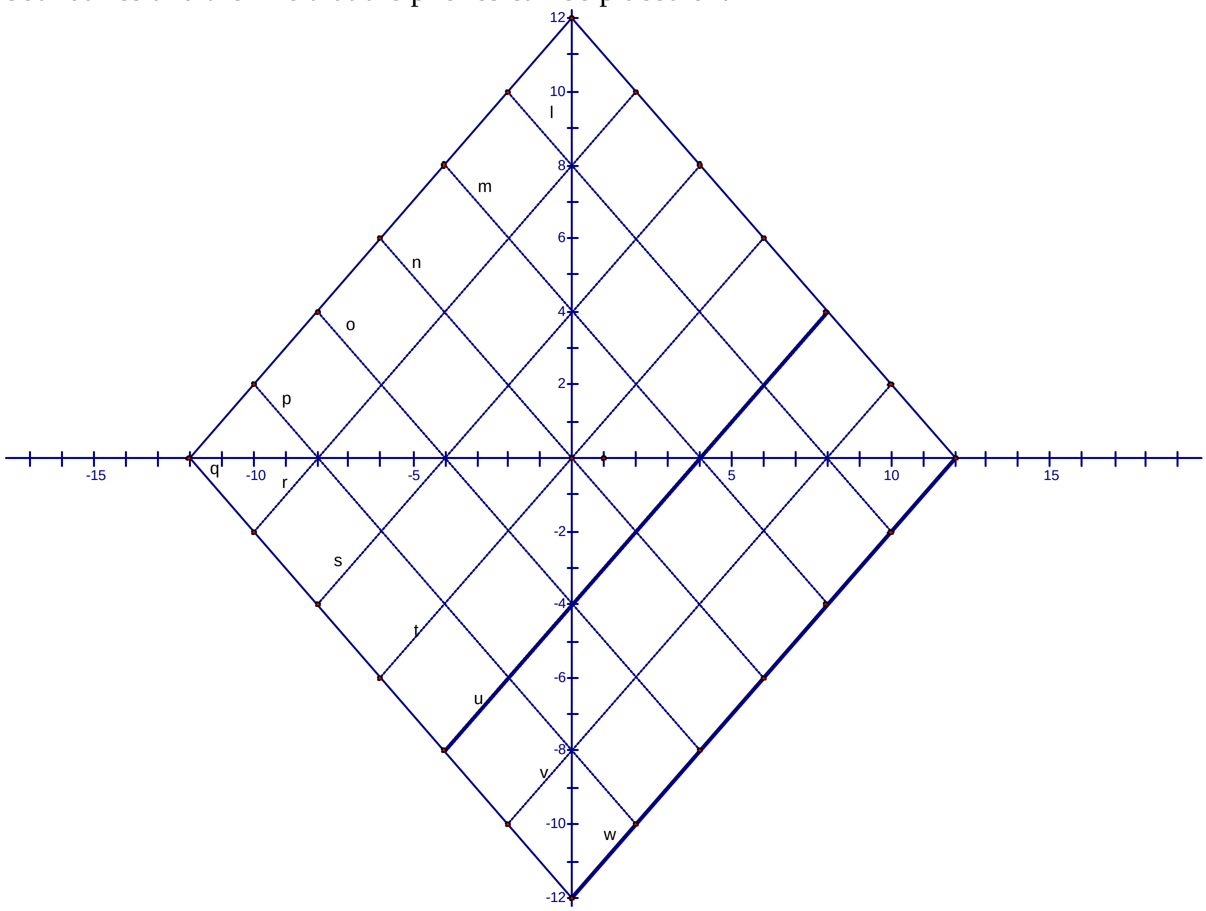
If phones are placed along this line *v*, everyone living between this line and the previous boundary will be with in four blocks of a phone. Now we need to find the boundary such

that people living below our new line will be with in four blocks of these phones. We use the equation:

$$d[(x_{12},x_{12}-4),(x_{13},y_{13})] = 4$$
$$x_{13}-x_{12}-y_{13}+(x_{12}-8) = 4$$
$$x_{13}-12 = y_{13}$$

lets call this line *w*

Line *w* is the same line as our outer boundary in quadrant IV. Below I have graphed the boundaries and the line that the phones can be placed on.



Section 4

Now we need to simply interpret our graph. The dotted lines in our graph above represent optimal places to put our phone booths. Until now we have not had a specific location, only a line for the booth to be placed on. Notice that the dotted lines intersect. These are the points were phone booths should be located. You can see that all the work we have done has divided our total area into 9 sections. Each one of these 9 sections as an intersection. By the way we constructed our graph, we know these intersections are the optimal places to put the least amount of phone booths such that everyone living with in 12 blocks of the center of town is with in 4 blocks of a pay phone.

We can see the taxicab geometry is a very useful model of urban geography. Only a pigeon would benefit from the knowledge that the distance between two buildings on opposite ends of a city is a strait line. For people, taxicab distance is the "real" distance. Taxicab geometry has many applications and is relatively easy to explore. I challenge the reader to explore other ideas in taxicab geometry. What do familiar geometric figures look like in taxicab geometry. We have already seen that circles in taxicab geometry look like squares. Are other figures transmuted?

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Cardano's Solution to the Cubic : A Mathematical Soap Opera

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Introduction

Through this research into the solution of the cubic, I hope to learn the thoughts of mathematicians during this period of history. This knowledge must be conveyed in a way that is historically and mathematically accurate, yet comprehensible and readily available to use in a class of eighth grade math students. It is my wish that as we begin to delve into the world of algebraic and graphical solutions to the problems in the eighth grade text, students will be able to make a connection with this information to that they are currently using to solve their problems. They should be made aware of the importance of math throughout history and able to enjoy parts of this problem-solving experience. It is commonly assumed that solving the quadratic equation is the algebraic pinnacle of the middle school curriculum. Through this paper and by using a historical approach I hope to present the cubic to future eigth graders. The motivation for doing this is follows.

As we look into the math classroom, we see students from diverse backgrounds struggling to make sense of information that they do not necessarily find relevant to their own lives. As a teacher, it is my duty to give students all of the information and opportunity to understand the value of these concepts. Through making mathematicians come to life, explaining how they struggled to arrive at these solutions and how they applied them to making a name for themselves, I hope that students will begin to see the important role that math once took, and that they are studying material that took lifetimes to create. Students should leave our classes with a respect for the subject and the authors of mathematics. It is my conviction that sharing some of the personal histories of those that created math will help them to develop this respect.

Background of the Problem

Looking into the solution for the cubic is not a new concept, nor was it in Cardano's day (the 16th century). The problem of the cubic had been troubling scholars since the early times of the Babylonians and Egyptians. Sometime between 2000 and 400 B.C., the Babylonians created the formula for solving the quadratic, an important step in Cardano's method for solving the cubic (Magnussen). And it has been found that the Babylonians in the 19th century B.C. found a table for solving cubic equations in which the problems would have integer solutions and be used in measuring the dimensions of an excavated room (Friberg). Mathematicians of the ancient Indian culture used their mathematical prowess for building elaborate sacrificial altars.

The Greeks, too, had a use for these complex math problems. It is said that their greatest challenge to prove their worthiness to the gods was to construct an altar that was in the shape of a cube twice the volume of the existing altar (Magnussen). This was a difficult problem brought to the people in a time of great wars and strife. As they had such difficulty managing the mathematics of this construction, many interpreted this charge one for the people to give up their wars and study mathematics as a people of peace.

The myth of the "Curse of the Cubic" came into being around this time. One source defines the curse as follows:
As he lay dying from the sword wounds inflicted by a Roman soldier, Achimedes uttered a curse to those who tried to solve algebraic problems:

> "*Lines and planes you may resolve;*
> *Cubes and others never solve!*" (Magnussen and Suzuki)

At this point mathematical advances in the struggle for the solution to the cubic took a rest. As empires changed, math was not forgotten, but development within the field came to a standstill until the Italians in the 16th century.
Work in the 16th century was, for the most part, secretive and well-guarded. There were many working in the world of mathematics as teachers and scholars, trying to make new discoveries, and struggling to come out on top in competitions of mathematical skill. During this time, high-stake public challenges of intellect occurred regularly. A challenger could ruin a reputation by putting forth 30 problems that his opponent would have around 40 days to solve before having to present his solutions in front of an audience. There would be problems provided to both men, and the winner would have solved more of the other man's problems accurately.

Due to his fear of a possible challenge, Scipione dal Ferro (1465-1526) kept his work a secret. Around 1515, dal Ferro was able to solve a cubic of the form $x^3 + mx = n$. On his deathbed in 1526, however, not wanting the progress to stop, he informed his student Antonio Fior of this great accomplishment. Fior also had the desire to keep this a secret until he could use it in a challenge to create a better name and reputation for himself. Upon challenging Niccolo of Brescia, known as Tartaglia, to a duel, Fior was joined in learning this solution. Tartaglia, during the days of his challenge, facing a set of compressed cubics in the form of $x^3 + mx = n$, derived the solution at the end of his time to solve these problems and saved his reputation. Due to the attention this contest drew, Cardano learned that there existed a solution and began appealing to Tartaglia to share his method to be published in Cardano's up and coming work on algebra, *Ars Magna*.

At last, after several attempts to get Tartaglia to reveal his method, Tartaglia conceded and presented to Cardano the method of solving the cubic encrypted in a poem, making

Cardano promise not to publish the information. The poem however, translated below, Cardano thought rather obscure.

When the cube and its things near
Add to a new number, discrete,
Determine two new numbers different
By that one; this feat
Will be kept as a rule
Their product always equal, the same,
To the cube of a third
Of the number of things named.
Then, generally speaking,
The remaining amount
Of the cube roots subtracted
Will be our desired count. (Laubenbacher et al. 235-236)

To create a situation when we can use this poem to solve the cubic, we first must have a cubic of the correct form "when a cube and its things near / Add to a new number, discrete" means that we need an equation of the form $x^3 + mx = n$, or, by rearranging terms, $x^3 + mx - n = 0$. Say we are given the equation $x^3 + 3x - 9x - 27 = 0$. To obtain the correct form, we must "get rid" of the pesky x^2 term. We accomplish this by making a substitution in the form $x = y - \frac{a}{3}$, where a is the coefficient of the second term of our equation. In this case, $a = 3$, and our substitution will be $x = y - 1$. The work behind this substitution could be arranged thus:

$(y-1)^3$:	y^3	$-3y^2$	$+3y$	-1
$3(y-1)^2$:		$3y^2$	$-6y$	$+3$
$-9(y-1)$:			$-9y$	$+9$
-27:				-27

$$y^3 \quad +0y^2 \quad -12y \quad -16, \quad \text{or} \quad y^3 - 12y - 16 = 0$$

Now that we have our desired form of the cubic, we will "Determine two new numbers different / By that one," which we will do by making another substitution. This substitution will be of the form $y = w + \frac{a}{w}$, where $a = \frac{-p}{3}$, where p is the coefficient of the y-term in our equation. In this case, $p = -12$, so $a = \frac{-(-12)}{3} = 4$, so we can make our substitution with the equation $y = w + \frac{4}{w}$:

$$(w+\frac{4}{w})^3: \qquad w^3 + 3w^2\left(\frac{w}{4}\right) + 3w\left(\frac{16}{w^2}\right) + \frac{64}{w^3}$$

$$-12\left(w+\frac{w}{4}\right): \qquad \text{-12w} \quad -\frac{48}{w}$$

$$\text{-16:} \qquad\qquad\qquad\qquad\qquad\qquad \text{-16}$$

$$w^3 - 16 + \frac{64}{w^3} = 0$$

Next, we can make this cubic look like a quadratic equation, something that we (as well as Tartaglia and Cardano) know how to solve. In order to do this, we must multiply by w^3:

$\left(w^3\right)^2 - 16\left(w^3\right) + \frac{64}{w^3}\left(w^3\right) = 0 \Rightarrow \left(w^3\right)^2 - 16\left(w^3\right) + 64 = 0$. Now that we have a quadratic equation in w^3, we can apply the quadratic formula. In this equation, $a = 1$, $b = -16$, and $c = 64$: $w^3 = \frac{16 \pm \sqrt{(-16)^2 - 4(1)(64)}}{2(1)} = 8 \pm \frac{\sqrt{0}}{2} = 8$. Now knowing that $w^3 = 8$, we can find that $w = \sqrt[3]{8} = 2$. Now that we have w, we must recall that we had a substitution earlier, in that $y = w + \frac{4}{w}$. But now we can use Tartaglia and Cardano's Method, revealed in the last lines of the poem, "Then, generally speaking, / The remaining amount / Of the cube roots subtracted / Will be our desired count." We can take $w = \sqrt[3]{8} = 2$ to be our values for a and b and make the computation necessary to gain our three solutions:

$$a + b = 2 + 2 = 4$$

$$a\alpha + b\beta = 2\left(\frac{-1+\sqrt{-3}}{2}\right) + 2\left(\frac{-1-\sqrt{-3}}{2}\right) = -1+\sqrt{-3} + -1-\sqrt{-3} = -2$$

$$b\alpha + a\beta = 2\left(\frac{-1-\sqrt{-3}}{2}\right) + 2\left(\frac{-1+\sqrt{-3}}{2}\right) = -1-\sqrt{-3} + -1+\sqrt{-3} = -2$$

But now we must recall that we made a substitution before this point ($x = y - 1$), as we had a cubic in x as our original equation. To find our roots in x, we must take these y-values we have just calculated as a result of Cardano's Formula, and subtract one from each, yielding: 3, -3, and -3 respectively. We can check this in the original equation $x^3 + 3x^2 - 9x - 27 = 0$, and we see that the values hold.

With the knowledge that Cardano obtained from Tartaglia, he was able to apply this method to solving the cubic, but was confused by the numbers that it produced. He could not find a way to explain or describe these "complex numbers," and claimed they were rather useless. It was not until 1572 that Rafaele Bombelli was able to make sense of them.

Cardano's work and insights, however, led to a general solution for the cubic. As he searched for the roots of the equations, Cardano's trouble with imaginary numbers was not all that he had to accept. Cardano's work with the negative numbers was well-received only by bankers who had dealt with the gaining and spending of money. This new concept of having a solution that was less than nothing, in a time when the concept of the number zero was just becoming accepted was a very radical thought. That he developed an acceptance of the inevitability of these negative solutions and led the way to their acceptance as well as the acceptance of negative radicals was quite an impressive task.

In his publication of *Ars Magna* in 1545, Cardano brought forward the thinking of cubics and quartics with the acknowledged help of some of his contemporaries. He gives credit to himself throughout the book, as well as saying:

> *In our own days Scipione del Ferro of Bologna has solved the case of the cube and first power equal to a constant, a very elegant and admirable accomplishment. Since this art surpasses all human subtlety and the perspicuity of moral talent and is a truly celestial gift and a very clear test of the capacity of men's minds, whoever applies himself to it will believe that there is nothing that he cannot understand. In emulation of him, my friend Niccolo Tartaglia of Brescia, wanting not to be outdone, solved the same case when he got into a contest with his [Scipione's] pupil, Antonio Maria Fior, and, moved by my many entreaties, gave it to me. For I had been deceived by the world of Luca Paccioli, who denied that any more general rule could be discovered than his own. Notwithstanding the many things which I had already discovered, as is well known, I had despaired and had not attempted to look any further. Then, however, having received Tartaglia's solution and seeking for the proof of it, I came to understand that there were a great many other things that could also be had. Pursuing this thought and with increased confidence, I discovered these others, partly by myself and partly through Lodovico Ferrari, formerly my pupil.* (Cardano 8-9)

It is through this quote in his first chapter of *Ars Magna* or *The Great Art*, that Cardano makes clear that although he has derived solutions to the 12 other forms of the cubic, the solution to the form $x^3 + mx = n$ is an idea belonging to del Ferro and Tartaglia. He further makes it clear that knowing this solution gave him renewed confidence and vigor needed to find solutions to all of the forms. (It should be noted that at this time, due to restrictions in algebraic notation and the newly accepted notion of negative numbers, all 13 of these cases were necessary before a general solution could be derived (Cardano *xiii*).)

Conclusions and Implications

Although Cardano successfully published many works on math, science, medicine, and many other topics, his popularity did not last long. Perhaps by the "Curse of the Cubic,"

he was doomed to a life as an outcast. After the publication of *Ars Magna*, Tartaglia, once his friend, began a campaign to ruin Cardano's reputation. At this same time, Cardano's son, finding that his wife had been unfaithful, poisoned and killed her. After working hard to defend a son who was convicted and executed for murder, Cardano found himself to be a hated man. Father of a murderer and a prostitute, Cardano was also persecuted for his forward thinking in areas the church thought dangerous. Cardano was sentenced to jail in 1570 on the charge of heresy for supposedly casting the horoscope of Jesus Christ. After his release from prison to his death in 1576, Cardano was banned from teaching and from publishing any more of his work.

Cardano's life was an interesting one. Blame it on the "Curse of the Cubic," but he experienced tragedy and strife along with the successes of his discoveries in the field of math. His writings on the cubic solution brought about the gateway into complex numbers and the solution to the quartic in a general form. In fact, "Cardano's Solution is no mere textbook proof…it introduced the concept of imaginary numbers, forcing mathematicians to rethink the relationship between mathematics and nature, and to explore the idea of what is meant by the word 'real.' In this sense, Cardano's mathematics was more philosophical than practical" (Ashworth). His works in all subjects, and especially in math, during a time of secrecy in discoveries, made expansion in the world of algebra possible.

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Commentary on Ballou's paper: Galois – The Myths and the Man

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Ballou (2005) provides a possibility for the introduction of the cubic into the eighth grade mathematics curriculum. As seen in her paper, the solution of the cubic essentially involves the use of a clever substitution in the general cubic in order to reduce it into a cubic without the square term, which in turn is factorable as a quadratic provided one makes another substitution. A natural question to ask ourselves, which mathematicians in the colorful theory of equations undoubtedly did as well, is how far can we push this technique of clever substitutions to solve higher degree equations. These substitutions are called Tschirnhaus transformations and have a pattern of the form x = y -a/n. In Ballou's (2005) paper the transformation used in the cubic was x = y - a/3 which allowed us to get rid of the so-called pesky x^2 term. For the general quartic the transformation is x = y –a/4 which in turn transforms the equation into a cubic solveable by Cardano's method. The question is what happens when we try our technique of Tschirnhaus transformations into the general quintic. To find out we first need to make a sidetrack into a little history packed with drama.

Shrouded in mystery and legend, the facts and fiction of the life of the great, but young mathematician, Evariste Galois, come together to create a captivating tale. Yet, it is unnecessary to embellish Galois' biography with intrigue and tragedy to have an interesting story. It is enough that by the age of twenty Galois discovered a method to determine the solvability of an equation and set the foundation for the branch of mathematics called Group theory. So many myths hover around Galois' contributions that not addressing these exaggerations but only looking at his mathematics is impossible. Therefore, this paper will discuss the important mathematical ideas Galois gave to the world as well as the myths surrounding the birth of those ideas.

Galois' contributions were at a level of mathematics most people don't encounter in their everyday lives. In fact, most high school math students won't have the opportunity to be exposed to Galois' ideas until they are in college and, most likely, only if they are mathematics majors. So why is it important for an elementary or secondary mathematics teacher to study Galois theory? To answer this let us look at the fundamental goal of beginning algebra: finding the solution(s) to an algebraic equation.

In high school algebra, solutions generally come easy. Occasionally a problem is presented where no solutions exist, but most often students are rewarded with "nice" answers. Quadratic equations pose some challenges such as having more than one answer

and non-real solutions, but they *do have* solutions. As we move up degree by degree on the polynomial ladder from cubics to quartics etc, solutions to equations are no longer so "nice". Understanding the relationship between the behavior of polynomials and their solutions allows the mathematics teacher a vista of the polynomial landscape enabling the teacher to assist students in making connections about polynomials in general. Galois' work gives ultimate clarity about polynomials and their solutions.

The history of solving equations goes back more than 4000 years ago when a general solution to the quadratic equation was found. After finding the solution to the quadratic it follows that other solutions to higher degree polynomials were close behind. However, it wasn't until the 16^{th} century that a mathematician named Cardano was finally able to find general solutions for cubic and quartic equations. It seemed that with each higher degree of a polynomial equation the complexity of the solutions also increased exponentially. This was the case with the quintic. But just as Cardano was successful in finding solutions to all quartic and quintic equations other mathematicians believed they could find general solutions to the quintic equation. After all, it is only to the 5^{th} power.

Many brilliant mathematicians tangled with the quintic trying to make it behave like its juniors the cubic and quartic. Euler, Bezout, Malfatti, and Lagrange all tried but were unsuccessful in finding the solutions. These mathematicians were unsuccessful for one particular reason. In 1799, Ruffini, a mathematician, claimed that algebraic equations of 5^{th} degree have no general solutions. A sufficient proof of this was not given until 1826 when Abel showed that this was true.[1] This development elicits the question, if quintics aren't solvable algebraically, which polynomials are solvable and which are not? Galois, barely a teenager at the time of Abel's proof, was on the way to discovering an answer to this question.

Although many myths surround Galois and his work, most historians agree about the details of Galois' childhood. Born in France on October 25, 1811, Galois was educated at home by his mother until the age of 12. He was considered an unremarkable student until he took his first mathematics class in 1827. His instructor, M. Vernier, wrote "It is the passion for mathematics which dominates him, I think it would be best for him if his parents would allow him to study nothing but this, he is wasting his time here and does nothing but torment his teachers and overwhelm himself with punishments,".[2] Despite this commendation Galois twice failed the exam to get into Ecole Polytechnique, the most prestigious university in Paris, and settled for enrollment in Ecole Normale where he continued his mathematical research. During this time is where we encounter the first Galois myth.

In 1929, at the age of 17, Galois submitted to Cauchy at the Academie des Sciences his writings on the algebraic solution of equations. Cauchy was to review the papers and then present them to the Academie for possible publication. The story goes that Cauchy

misplaced the articles and they were lost forever. It is unclear why this myth began. Demonstrating just how prevalent this myth still is today, Beachy and Blair (1996) write " . . . Galois presented two papers on the solution of algebraic equations to the Academie des Sciences des Paris. Both were sent to Cauchy who lost them (pg 316)"[3]. However it is Cauchy's own words that dispel this misconstruction. In a letter written January 18, 1830, Cauchy says he is unable to attend the session where he was to present Galois' work and requests another time to discuss the papers.[4] This verifies that Cauchy was in possession of Galois' work and was interested in submitting it. Cauchy, however, did not present the material at a future meeting but it is believed that he encouraged Galois to submit his research for the Grand Prize in mathematics. A few months later Galois submitted his manuscripts *On the condition that an equation be solvable by radicals* for the contest.[4] Unfortunately, this article did get lost or was not read and Galois is not considered for the prize.

One year later, Galois made a third attempt and submitted another version of his work to the Academy. The Academy was able to hold onto the article and Lacroix, Poisson, Lengendre and Poinsot, all esteemed mathematicians, finally reviewed the manuscript. Although these men held in their possession an incredible mathematical achievement, apparently they were not able to make heads nor tails of it and it was rejected. Poisson admits that he was unable to comprehend Galois' work and encouraged Galois to develop his theory further and in a larger mathematical context where it might be illuminated.[4] Due to misplaced papers and rejections, Galois began to believe that the Academy was against him in some way. Although the facts don't truly support this, future biographers exaggerated the circumstances surrounding these incidents and, consequently, myths began to develop supporting Galois' slight paranoia. It doesn't appear that anyone was trying to prevent Galois' work from coming out; it was just a series of unfortunate events and poor timing.[4]

Under normal circumstances Galois would have many more opportunities to develop and submit his work but, being a radical and antagonizing the French government, he was a political prisoner who somehow got himself into a duel with an adversary. The events leading up to the duel and Galois' subsequent death are unclear and only speculation exists about who this foe was and why they were dueling. On the eve of the fateful duel it has been written and widely believed that Galois, from prison, desperately constructed his great theorems in a letter to a friend--another myth, easily dispelled by the evidence that in 1830 Galois' papers "An analysis of a Memoir of the Algebraic Resolution of Equations", "Notes on the Resolution of Numerical Equations" and "On the theory of Numbers" were published and make up what is now called Galois theory. [4]

Despite these publications Galois' work was not recognized until 1846, 15 years after his death, when Louiville published it in his Journal commenting on Galois' solution, " . . . as correct as it is deep of this lovely problem: Given an irreducible equation of prime degree,

decide whether or not it is soluble by radicals"[2] Louiville was commenting on Galois' insightful understanding of polynomial solutions and their relationship to what are now called groups, a term Galois used first, and the solvability of an equation in radicals. Galois had answered the essential question of what makes a polynomial solvable by radicals.

Understanding Galois theory is not an easy task. To have complete comprehension of the theory one must be familiar with mathematical concepts such as fields, rings, groups, isomorphisms, symmetry, permutations of roots, vector spaces and the list goes on. However it is possible to dabble close to the surface and have a general sense of what Galois was doing. For example let us look at the polynomial $f(x)=x^5 - 2x^3 - 8x - 2$. Using the fact that f (x) is irreducible using Eisenstein's criterion, and extending the field to adjoin roots of f (x), we get an extension of degree 5. Eventually, by calling upon the fundamental theorem of Galois theory, Cauchy's theorem, and by a proposition that every element of the Galois group of f (x) gives a permutation of those roots we determine that the Galois group of f (x) is isomorphic to S_5, a symmetric group. Since S_5 is not solvable, then f (x) is not solvable. Therefore, there exists a polynomial of degree 5 that is not solvable algebraically by radicals[6]. It would be beneficial for readers unfamiliar with these criteria to determine solvability to access several problems worked out by Beachy & Blair (see links 7 and 8 in Works Cited)

Not only did Galois theory provide a means to know about the solvability of an equation, what is so remarkable about Galois' discovery is he was working in an entirely new plane of mathematics, dealing with structures of polynomials and their solutions. The necessity of conceptualizing the notion of a group to determine solvability of equations produced modern mathematics. Galois' work with permutations of solutions as a structure (a Galois group) laid the foundation for investigations into other similar structures such as matrix groups.[5] This new world of mathematics evolved with the help of other mathematicians, who built upon Galois' innovations, and became known as Group theory.

While it will always be a mystery why Galois was the victim of a duel, it is the only mystery that remains concerning Galois. Galois' work is now understood and applied in numerous arenas, like a light illuminating the world of mathematics. Maybe, one day, this light will out shine the myths surrounding Galois and he will simply be a great man who, in his very short life, changed the world of mathematics.

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> *"I am striving to gain insight and background knowledge in Algebra topics that will enrich my discussion and explanations in the classroom."*

Jerry T. Baty with 30+ years of experience in K-12 and the businesses world has allowed time to renew expressions for mathematical instructional practices. Moving from the high school to college, he has experienced the journey toward becoming a master teacher. Whether he presents *"The Logic Box', 'Ethnography Portrait of Action upon Objects', "Teaching to Learn: How we do that!"* and *'Alternatively, Cabri Geometry',* Baty uses his classroom for modeling teaching with mathematics as a linguistic language. Named the Governor/Texas Senate and House Outstanding Mathematician for the Aerospace Academy/NASA 2002 and later accepting an appointment to the framework review committee for teacher certification (SBEC) for Texas in 2002, he continues his journey with *Albert Tell Me More!*

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> *"I like to revisit Abstract Algebra and proofs to keep my skills sharp…so I can assist my students in making connections and facilitate their comprehension of algebra concepts. I'm hoping this will help me to be a better teacher of mathematics."*

Eric (Rico) Gutstein, PhD is an Associate Professor in the Curriculum and Instruction Department at the University of Illinois–Chicago. His research interests include mathematics education, teaching for social justice and critical literacy in an urban, multicultural context, and Chicago school policy. He has published numerous thought provoking articles in his listed areas of interest such as the one that provoked a commentary in this journal issue.

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Jane Micklus is an experienced public school teacher who combines her love of the arts (particularly music) with mathematics by presenting her students opportunities to integrate these two disciplines. She believes in a humanistic approach to education, one that allows for student creativity and is currently pursuing an advanced degree in mathematics.

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