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Some Explicit Solutions for a Class of One-Phase Stefan Problems

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Abstract. Salva and Tarzia, [N.N. Salva, D.A. Tarzia, J. Math. Anal. Appl. 379 (2011) 240 - 244], gave explicit solutions of a similarity type for a class of free boundary problem for a semi-infinite material. In this paper, through an elementary approach and less stringent assumption on data, we obtain more general results than those given by their central result, and thereby construct explicit solutions for a wider class of Stefan problems with a type of variable heat flux boundary conditions. Further, explicit solutions of certain forced one-phase Stefan problems are given.

Keywords: Explicit solution, Stefan problem, differential-difference equations
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INTRODUCTION

In this paper we give some explicit solutions to a class of one-phase Stefan problem driven by the partial differential equation (*PDE*)

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = \gamma \mathfrak{H}(x,t)\,, \quad x \in [0, \delta(t))\,, \quad t > 0\,, \tag{1}$$

with dimensionless spatial and temporal variables, respectively, $x \in \mathbb{R}$ and $t \in \mathbb{R}_0^+ \overset{\text{def}}{=} \mathbb{R}^+ \cup \{0\}$, and a moving front defined through a continuously differentiable δ_2, with nice enough forcing term $\gamma \mathfrak{H}(\delta(t),t)$, γ real, satisfying initial and boundary values of the form

$$\begin{gathered} u\Big|_{t=0} = e_0 x^p\,, u\Big|_{x=0} = e_1 \delta_2^p(t)\,, \frac{\partial u}{\partial x}\Big|_{x=0} = e_2 \delta_2^p(t) \frac{d\delta(t)}{dt}\,, \\ u\Big|_{x=\delta(t)} = e_4 \delta(t) + e_3 \delta_2^p(t)\,, \frac{\partial u}{\partial x}\Big|_{x=\delta(t)} = e_4 + e_5 \delta^p(t) \frac{d\delta(t)}{dt}\,, \end{gathered} \tag{2}$$

where $p, e_l \in \mathbb{R}$.

Eqs. $(1)-(2)$ is a class which contains the classical Stefan melting problem $(\gamma = 0, p = 0, e_1 = 1, e_2 \geq 0, e_3 = 0, e_4 = 0)$; Huppert's model [2] for hot turbulent flow over a cold surface $(\gamma = 0, p = 0, e_4 \neq 0)$ if the interval in Eq. (1) is replaced with $(\delta(t), +\infty)$; Salva and Tarzia [1] $(\gamma = 0, p = 1, e_4 = 0)$, to mention a few.

Several authors, for instance [1, 3, 4, 5], have obtained explicit solutions for one and multi-phase Stefan problems. A favored approach in the literature is a reduction of Eq. (1) to an ordinary differential equation (*ODE*) in a similarity variable proportional to $x/\sqrt{t}$.

Starting from such an elementary approach, which evolves into the study of an equivalent class of differential-difference equation (*DDE*) to Eq. (1), we obtain some pertinent explicit solutions.

MAIN RESULTS

In this section, we shall start by studying explicit solutions of Eq. (1) in the absence of forcing, and with a more general moving boundary $x \in (\omega(t), \delta(t))$. The approach taken involves that of the suggestion of a solution $u(x,t)$ which takes the form of an infinite series in a similarity variable $\xi \overset{\text{def}}{=} (x - \omega(t))/(\delta(t) - \omega(t))$. Subsequently, a

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constructive study of a class of exponential forcing term in spatio-temporal variables is given for the first time. Our first result is summarized in the following Lemma.

Lemma 1. *The PDE*

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0,\ x \in (\omega(t), \delta(t)), \tag{3}$$

with real valued $\omega(t)$ and $\delta(t)$, has a similarity-like solution

$$u(x,t) = \alpha_{-1}x + \sum_{j=0}^{\infty} \alpha_j(t)\,\xi^j, \tag{4}$$

if its coefficients $\alpha_j(t)$, α_{-1} known constant, are determined by the differential-difference equation

$$\begin{cases} \chi^2(t)\alpha_j'(t) - (j+2)(j+1)\alpha_{j+2}(t) + (j+1)\chi(t)\omega'(t)\alpha_{j+1}(t) - j\chi(t)\chi'(t)\alpha_j(t) = 0; \\ \chi(t) = \delta(t) - \omega(t). \end{cases} \tag{5}$$

Moreover, suppose the α_j are chosen in a parametrized form such that $\alpha_j(t) = \alpha_j(t;p) = a_j\delta^p(t)$, $\omega(t) = \kappa\delta(t)$, $\chi(0) = 0 = \delta(0)$, with κ, $a_j \in \mathbb{R}$. Then,

$$\begin{aligned} u(x,t) \equiv u(x,t;\kappa,p) = \alpha_{-1}x + \delta^p(t)\Big[C_1 + C_2\xi + \left(\frac{1}{2}\theta^2(-1+\kappa)(p(-1+\kappa)C_1 - \kappa C_2)\right)\xi^2 \\ + \left(\frac{1}{6}\theta^2(-1+\kappa)^2\left(-p\theta^2(-1+\kappa)\kappa C_1 + \left(-1+p+\theta^2\kappa^2\right)C_2\right)\right)\xi^3 + \dots\Big], \end{aligned} \tag{6}$$

where C_1 and C_2 are arbitrary constants.

The proof of Lemma 1, which relies on direct substitution and a systematic solution (albeit not closed) of the pertinent *DE*

$$\begin{cases} \theta^2(1-\kappa)\left(a_{j+1}\kappa(j+1) + a_j(1-\kappa)(p-j)\right) = 2a_{j+2}(j+2)(j+1), \\ \omega_2(t) = \theta\sqrt{t}, \end{cases} \tag{7}$$

$p \in \mathbb{R}$, $\kappa \in (-\infty, 1)$, and $\theta \in \mathbb{R}_0^+$, is direct. The solution to Eq. (3) in the case wherein $\kappa = 0$ and $p = 1$, which is the core of the results of [1], has a series expression which can be sieved from equation (6). The next theorem proffers a closed form solution for the more general case where $\kappa = 0$ and $p \in \mathbb{R}$.

Theorem 1. *Suppose that $\kappa = 0$ in Lemma 1. Then for $\gamma = 0$, PDE (1) has a class of solutions, which is parametrized by $p \in \mathbb{R}$ and expressible in terms of Kummer's confluent hypergeometric function ${}_1\mathsf{F}_1$ as*

$$u(x,t) = \alpha_{-1}x + \delta^p(t)\left[{}_1\mathsf{F}_1\left(-\frac{p}{2}, \frac{1}{2}, -\frac{x^2\theta^2}{4\delta^2(t)}\right)c_1 + {}_1\mathsf{F}_1\left(\frac{1-p}{2}, \frac{3}{2}, -\frac{x^2\theta^2}{4\delta^2(t)}\right)\frac{x}{\delta(t)}c_2\right], \tag{8}$$

where $\delta(t) = \theta\sqrt{t}$, c_l, $l \in \{0,1,2\}$ constants.

Furthermore, if initial and boundary value conditions

$$u\Big|_{x=0} = e_1\delta_2^p(t),\ u\Big|_{x=\delta(t)} = e_4\delta(t) + e_3\delta_2^p(t),\ \frac{\partial u}{\partial x}\Big|_{x=\delta(t)} = e_4 + e_5\delta^p(t)\frac{d\delta(t)}{dt}, \tag{9}$$

are imposed, the relations

$$\begin{aligned} &\left[-\frac{1}{6}(1-p)\theta^2\,{}_1F_1\left(\frac{3-p}{2}, \frac{5}{2}, -\frac{\theta^2}{4}\right) + {}_1F_1\left(\frac{1-p}{2}, \frac{3}{2}, -\frac{\theta^2}{4}\right)\right]c_2 + \frac{1}{p}\theta^2\,{}_1F_1\left(1-\frac{p}{2}, \frac{3}{2}, -\frac{\theta^2}{4}\right)e_1 = \theta^2\frac{e_5}{2}, \\ &{}_1F_1\left(\frac{1-p}{2}, \frac{3}{2}, -\frac{\theta^2}{4}\right)c_2 + {}_1F_1\left(\frac{-p}{2}, \frac{1}{2}, -\frac{\theta^2}{4}\right)e_1 = e_3,\ \alpha_{-1} = e_4 \end{aligned} \tag{10}$$

hold.

The case of a non-zero forcing term is studied in a constructive manner. Suppose the forcing term is $\mathfrak{H}(\omega,\delta,x,t)$. In this instance, the series expression corresponding to Eq. (3) upon the assumption of a series expansion (4) is

$$\begin{aligned}\sum_{j=0}^{\infty}\left\{\frac{(j+2)(j+1)\alpha_{j+2}(t)}{(\delta(t)-\omega(t))^2}-\left[\alpha_j'(t)-(j+1)\alpha_{j+1}(t)\frac{\delta_1'}{\delta(t)-\omega(t)}-j\alpha_j(t)\frac{(\delta(t)-\omega(t))'}{(\delta(t)-\omega(t))}\right]\right\}\\ \times\left(\frac{x-\omega(t)}{\delta(t)-\omega(t)}\right)^j=\mathfrak{H}(\omega(t),\delta(t),x,t)\,.\end{aligned}\tag{11}$$

Motivated by the structure of Eq. (11), we shall amongst several possibilities consider a forcing term of the form

$$\mathfrak{H}(\omega(t),\delta(t),x,t;m)=m\sum_{j=0}^{\infty}\frac{(j+2)(j+1)a_{j+2}(t)}{(\delta(t)-\omega(t))^{(2-p)}}\left(\frac{x-\omega(t)}{\delta(t)-\omega(t)}\right)^j,\quad a_j,m\in\mathbb{R}\,,\tag{12}$$

which induces a system driven by the *DE*

$$(1-j)\delta(t)\delta'(t)a_j-(j+2)(j+1)(p-m)a_{j+2}=0\,.\tag{13}$$

In the special case of Eq. (13) where $p\in\{0,1\}$, $m\neq 0$ and a forcing term of type Eq. (12) imposed, we have the following result.

Theorem 2. *The forced one-phase Stefan problem*

$$\frac{\partial^2 u}{\partial x^2}-\frac{\partial u}{\partial t}=\mathfrak{H}_{[p]}\,,\quad x\in\Big[0,\delta(t)\Big)\,,\tag{14}$$

$p\in\{0,1\}$*, such that*

$$\mathfrak{H}_{[p]}=\begin{cases}\dfrac{-mr_{[0]}\delta(t)}{2\sqrt{t}(1-m)}\exp\left[-x^2/\left(4(1-m)t\right)\right] & p=0\\[2ex] \dfrac{-mr_{[1]}\delta(t)}{2t(1-m)}\exp\left[-x^2/\left(4(1-m)t\right)\right] & p=1\,,\end{cases}\tag{15}$$

$m<1$ *or* $m>1+\dfrac{1}{2}\,\delta(t)\dfrac{d\delta(t)}{dt}$*, has similarity solutions*

$$\begin{aligned}&\delta(t)=\theta_{[p]}\sqrt{t}\,;\\ &u(x,t)=\\ &\begin{cases}s_{[0]}+\left(r_{[0]}/\theta_{[0]}\right)\sqrt{\pi(1-m)}\,\mathrm{erf}\Big[x/\Big(2\sqrt{(1-m)t}\Big)\Big] & p=0\\ s_{[1]}x+\left(r_{[1]}\theta_{[1]}/2\right)\sqrt{\pi(1-m)^{-1}}\,x\,\mathrm{erf}\Big[x/\Big(2\sqrt{(1-m)t}\Big)\Big]+r_{[1]}\theta_{[1]}\sqrt{t}\exp\left[-x^2/\left(4(1-m)t\right)\right] & p=1\,,\end{cases}\end{aligned}\tag{16}$$

where r,s *are arbitrary constants.*
Moreover, if Eqs. (14), (15) *have boundary conditions*

$$\begin{aligned}u\Big|_{x=0}=\varepsilon_{1[p]}\delta^p(t)\,,\;\frac{\partial u}{\partial x}\Big|_{x=0}=\varepsilon_{2[p]}\delta^p(t)\frac{d\delta(t)}{dt}\,,\\ u\Big|_{x=\delta(t)}=\varepsilon_{3[p]}\delta^p(t)\,,\;\frac{\partial u}{\partial x}\Big|_{x=\delta(t)}=\varepsilon_{4[p]}+\varepsilon_{5[p]}\delta^p(t)\frac{d\delta(t)}{dt}\,,\end{aligned}\tag{17}$$

then $s_{[p]},r_{[p]}$ *and* $\theta_{[p]}$ *are determined from the relations*

$$\begin{cases}\varepsilon_{1[0]}=s_{[0]}\,,\varepsilon_{2[0]}=2r_{[0]}\theta_{[0]}^2\,,\varepsilon_{3[0]}=s_{[0]}+r_{[0]}\sqrt{\pi(m-1)}\,\mathrm{erf}\left[\theta_{[0]}/\left(2\sqrt{1-m}\right)\right]\\ \varepsilon_{4[0]}=0,\;\varepsilon_{5[0]}=2r_{[0]}/\theta^2\exp[\theta_{[0]}^2/\left(4(m-1)\right)]\end{cases}\tag{18}$$

or

$$\left\{\begin{array}{l}\varepsilon_{1[1]} = r_{[1]}\,, \varepsilon_{2[1]} = 2s_{[1]}\theta_{[1]}^{-2}\,,\\ 2\varepsilon_{1[1]}\exp[\theta_{[1]}^{2}/\left(4(m-1)\right)] = \left(2\varepsilon_{3[1]}-\varepsilon_{4[1]}\right)-\left(\varepsilon_{2[1]}+\varepsilon_{5[1]}\right)\theta_{[1]}^{2}/2\,,\end{array}\right. \tag{19}$$

according as p is 0 *or* 1.

It is easy to see that if $\mathfrak{H}$ is taken such that

$$\mathfrak{H}\left(\omega(t),\delta(t),x,t;m\right) = \beta\sum_{j=0}^{\infty}\frac{\alpha_{j+1}\omega(t)}{\delta(t)-\omega(t)}\left(\frac{x-\omega(t)}{\delta(t)-\omega(t)}\right)^{j}, \tag{20}$$

our consideration is one which is synonymous to that of the study of a dual moving boundary *PDE*

$$\frac{\partial u}{\partial t}-\frac{\partial^2 u}{\partial x^2} = \beta\omega(t)\frac{\partial u}{\partial x}, \qquad x\in\left(\omega(t),\delta(t)\right). \tag{21}$$

with the corresponding governing *DDE*

$$\left\{\begin{array}{l}\chi^2(t)\alpha_j'(t)-(j+2)(j+1)\alpha_{j+2}(t)+(1-\beta)(j+1)\chi(t)\omega'(t)\alpha_{j+1}(t)-j\chi(t)\chi'(t)\alpha_j(t)=0;\\ \chi(t)=\delta(t)-\omega(t)\,.\end{array}\right. \tag{22}$$

In this vein, we give the following Corollary to Theorem 2, one in which $\beta = 1$, $\omega(t) = \kappa\delta(t)$. A special case of this, with $p = 0$, was considered by Yi et al. [6] in the study of a one-dimensional free boundary problem in an angular domain.

Corollary 1. *The PDE*

$$\frac{\partial u}{\partial t}-\frac{\partial^2 u}{\partial x^2} = \kappa\delta(t)\frac{\partial u}{\partial x}, \qquad x\in\left(\kappa\delta(t),\delta(t)\right) \tag{23}$$

with boundary conditions Eq. (17) *has solutions* (16)*, together with relevant conditions* (18) *or* (19)*, with*

$$\kappa = 1-(1-m)^{-1/2}, \quad 1 > m \in \mathbb{R}. \tag{24}$$

CONCLUSION

In this paper, we have given new explicit solutions to a class of one-phase Stefan problems both in the presence of a spatio-temporal exponential forcing term and in the absence of forcing. Our approach, one which relies on the study of an equivalent differential-difference equation, permitted a generalization of recent and standard results in the literature. Furthermore, this approach naturally admits the square root type evolution of moving fronts and its compatible fields while leaving the opportunity for future consideration of other forms of moving fronts.

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