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Abstract: Computer Algebra Systems (CAS) performs algorithmic processes quickly and correctly. Concern is commonly expressed that students using CAS will merely be pushing buttons but this paper indicates that, while CAS may assist students, this facility impacts on only one section of the mathematical modeling process: CAS may be used to help find mathematical solutions to mathematically formulated problems. Controlling and monitoring the use of CAS to perform the necessary routine processes requires the mathematical thinking referred to as algebraic insight. This paper sets out a framework of the aspects, and elements of algebraic insight and illustrates the importance of students developing each of the two key aspects: algebraic expectation and ability to link representations. This framework may be used for both planning teaching and monitoring students' progress.

CAS Support Mathematical Analysis in the Modeling Process

Mathematical analysis tools are now not only increasingly powerful but affordable and available. In particular, Computer Algebra Systems (CAS), available for PC's and hand held calculators, offer students support to allow them to work successfully through more complicated or time consuming mathematical manipulations and calculations. Heid (2003) describes clearly three key ways in which CAS can function as a cognitive technology:

- Students can use CAS for the repeated execution of routine symbolic procedures in rapid succession, without diminished accuracy and increased fatigue usually associated with the repetitive execution of by-hand routines…
- Students can assign rote symbolic tasks to the CAS so that they can concentrate on making 'executive' decisions…
- Students can use the CAS to apply routine symbolic algorithms to complicated algebraic expressions, without the confusion students sometimes experience when trying to apply a routine procedure to a complicated expression. (pp. 34-35).

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This capacity for CAS to be used by students to share cognitive load has obvious advantages for mathematical modeling. A CAS allows the user to work in numeric, graphic or symbolic modes and to move between these with mathematical precision and relative ease. For example, in modeling real world situations in order to solve estimation or optimization problems, it is common to begin by collecting and entering numeric data into a software package. CAS allows us to use the graphic mode to examine any pattern in this data; make use of the CAS's statistical capabilities to perform an appropriate regression on the data; and store the result in the graphic function editor ready for graphing or transfer to the symbolic mode. The model which has been created can be examined and refined for the particular case then the impact of changing the various parameters may be explored until a general model is developed or that notion discarded.

Monitoring CAS Work Requires Algebraic Insight

It must be clear though that CAS does not reduce the need for students to develop their skills in mathematical thinking. Figure 1, below, illustrates the typical process for mathematical modeling. Starting with a real world situation (top left) which must be formulated as a mathematical problem, the mathematician typically collects numeric data or moves immediately to a symbolic representation of the situation (top right). Using symbolic, graphic, numeric or geometric methods the mathematician works on the abstract version of the problem in order to progress towards some particular or general solution. Once a mathematical solution has been developed (bottom right) this abstract solution must be interpreted in terms of the real world (bottom left) and checked for applicability in the situation where this process began. If the solution is not adequate then the process must be repeated. This diagram highlights the fact that, currently, technologies like CAS only impact on one section of the modeling cycle, that is, the process of moving from the mathematically formulated problem to a symbolically formulated particular or general solution.

CAS assists with routines but does not take over the role of mathematical thinking. This is illustrated by Pierce and Stacey, (2001a) who report the following extract from a group interview conducted with first year undergraduate mathematics students working with CAS available for all aspects of learning and assessment:

> **Interviewer:** One of the other things that people argue about is whether or not people are really doing mathematics when working with a computer-algebra system. Are you doing it or is the machine doing it? Who's doing the maths?
>
> **Student A:** I reckon that we are actually doing it. The computer only spits out an answer to what you type into it
>
> **Student B:** It's just like with a calculator…it's just going a bit further, we're not just doing multiplication and division quickly, we're doing simple differentiations and stuff quickly.
>
> **Student C:** Also, you still have to interpret the answer or for that matter interpret the question so you can convert it into what the

> computer wants …you're still doing a lot of mathematics. (pp. 153-154).

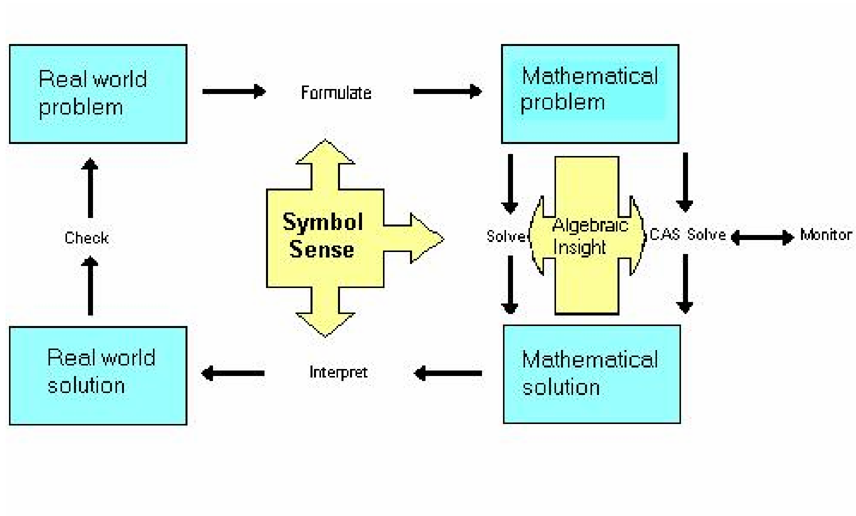

Figure 1. A model of problem solving showing the places of symbol sense and algebraic insight (Pierce & Stacey, 2002)

The processes of formulating and solving the mathematical problem then interpreting the solution *all* require what Fey (1990) and Arcavi (1994) call *symbol sense*. As Fey (1990) pointed out:

> Even if machines take over the bulk of computation, it remains important for users of those machines to plan correct operations and to interpret results intelligently. Planning calculations requires sound understanding of the meaning of operations – of the characteristics of actions that corresponds to various arithmetic operations. Interpretation of results requires judgement about the likelihood that the machine output is correct or that an error may have been made in data entry, choice of operations, or machine performance. (p.79)

Symbol sense is a broad concept encompassing a feel for the power of symbols; an ability to use symbols to express relationships; a sense of when to use symbols and when to use another approach; a sense for which symbolic manipulations will aid progress towards solution of a problem; an ability to recognise equivalent symbolic expressions; an ability to interpret the meaning of symbols in a given context and much more. In this paper we concentrate on the part of symbol sense required to monitor progress towards the solution of a mathematically formulated problem. This is the phase of the modelling process where a CAS may be able to perform the algorithmic tasks involved accurately and quickly. However, in order to direct and monitor this work the user needs the part of symbol sense we call *algebraic insight.*

Technology to date does not impact on the processes of formulation and interpretation; it does however offer alternative methods to progress between the mathematically formulated problem and a mathematical solution. Methods which were, in the past, considered too time consuming or tedious are now accessible. For mathematics teachers and students, limited by the constraints of class timetables and a crowded curriculum, CAS can offer the possibility of tackling interesting real problems which could not previously have been tackled in the time available. The support of CAS to correctly execute the algorithmic routines and manipulation required in a solution process may allow students to test their conjectures and develop their higher level mathematical thinking instead of setting their focus at the micro level of the steps involved in these routines. However, studying the value of the output from such a process of shared cognition will be dependent on correct input and the execution of appropriate commands.

Checking that mathematical expressions have been correctly entered in to CAS and that the output at each stage makes sense certainly requires *symbol sense*. As stated above, to draw specific attention to this part of symbol sense we refer to it as *algebraic insight*. Its place in the broader scheme of thinking required to work within and between the three mathematical representations typically afforded by CAS is illustrated in Figure 2 and the key aspects, elements and some common instances of this concept are outlined in Figure 3.

Figure 2 indicates that *algebraic insight* has two key aspects: first the thinking which allows us to monitor working within the symbolic mode of operating, that is *algebraic expectation;* and second the *ability to link representations*, in this case to link the symbolic with graphical or numeric representations. These two elements of algebraic insight will be discussed and illustrated in the following section.

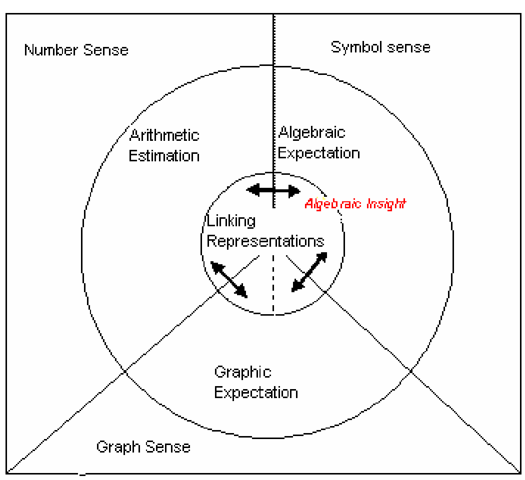

Figure 2. The place of algebraic insight and its components within the senses needed when working with CAS. (Pierce and Stacey, 2004)

Algebraic Insight

The framework set out in Figure 3, is designed to encourage reflection on the skills of algebraic insight and to serve as a basis for teachers in planning and assessing. The framework divides the first aspect of algebraic insight, algebraic expectation, into three elements relating to conventions and basic properties, structure and key features. The second aspect, ability to link representations, has elements which link the symbolic to graphic and numeric representations. The framework is not proposed as a catalogue of specific, itemized skills: the common instances chosen are merely illustrative and will, in practice, be age and stage appropriate.

The divisions within the framework are neither mutually exclusive nor exhaustive. Whilst these features would be desirable, the author does not believe they are fully attainable. The framework was developed in response to the literature and the author's experience of teaching with CAS. It is an attempt to analyze what it is that 'expert' mathematicians do when they look at a result to an algebraic problem and say 'there is a mistake here' or 'that looks all right'.

This is the thinking used in, what the problem solving literature, for example Schoenfeld (1985), calls 'monitoring' or 'control'. Examples of the application of the thinking summarized in the framework are described below.

Aspects	**Elements**	**Common Instances**
1. Algebraic Expectation	1.1 Recognition of conventions and basic properties	1.1.1 Know meaning of symbols
		1.1.2 Know order of operations
		1.1.3 Know properties of operations
	1.2 Identification of structure	1.2.1 Identify objects
		1.2.2 Identify strategic groups of components
		1.2.3 Recognise simple factors
	1.3 Identification of key features	1.3.1 Identify form
		1.3.2 Identify dominant term
		1.3.3 Link form with solution type
2. Ability to Link representations	2.1 Linking of symbolic and graphic representations	2.1.1 Link form with shape
		2.1.2 Link key features with likely position
		2.1.3 Link key features with

		intercepts and asymptotes
	2.2 Linking of symbolic and numeric representations	2.2.1 Link number patterns or type with form
		2.2.2 Link key features with suitable increment for table
		2.2.3 Link key features with critical intervals of table

Figure 3 A. Framework for algebraic insight (Pierce and Stacey, 2001b)

Algebraic Expectation

The term Algebraic Expectation is used to name the thinking process which takes place when an experienced mathematician considers the nature of the result they expect to obtain as the outcome of some algebraic process. First, *recognition of conventions and basic properties* of mathematics is a skill based on both knowledge and understanding of the meaning of symbols. At a basic level much of this knowledge will transfer from experience with numbers and arithmetic processes. In addition, to make mathematical meaning explicit our symbols must be arranged in a conventional manner, for example the meaning of , '2 $dx\, x$ sin $\int$ ' is quite unclear. In this case several alternatives such as $2\int \sin x dx$, or $\int \sin(2x) dx$ are possible and the correct sequence of symbols will rely on the user understanding both the context of the problem and role of each symbol, especially '2' in each of these expressions. Recognition of conventions and basic properties is demonstrated, for example, in three common instances: when students know the meaning of symbols; the appropriate order of operations; and the basic properties of operations.

The second element of *algebraic expectation* involves *identifying structure*. Consider, for example, $\dfrac{a(x+1)^5 + b(x+1)^2}{(x+1)}$. The vinculum indicates the first level of structure in this expression. The numerator can be seen as a strategic group of components consisting of two terms, while the denominator may be viewed as a single object. Considered at another level, $(x+1)$ can be identified as an object which is common to each of three terms which make up this expression. Common instances of identification of structure occur when students identify objects, strategic groups of components or simple factors.

Finally *identifying key features* forms the third element of *algebraic expectation*. Mathematical expressions can be scanned for key features: features that identify the form of the expression indicating whether it is, for example, trigonometric, exponential, or polynomial. Key features also provide information by which expectations may be formed. For functions, for example, these features may lead to expected number of solutions, solution type, number of maxima and minima, and domain and range.

Algebraic expectation may be thought of as a parallel to the arithmetic skill of estimation. One of the most common examples of the need for algebraic expectation is seen when a mathematician looks at two expressions and decides, without doing any explicit calculations or manipulations, whether they are likely to be equivalent. This skill is particularly important for those working with CAS: checking the correct entry of mathematical expressions and matching CAS outputs with conventional by-hand presentation of various mathematical expressions.

The three elements of algebraic expectation may be thought of as three different lights illuminating the attributes of a mathematical expression and hence providing possible clues to inform our algebraic expectation. Students should be encouraged to consider any mathematical expression in the light of each of these three elements as part of their routine in making judgments about how best to progress the solution of a problem or in monitoring their working by-hand or by CAS.

Consider a rule to describe the surface area of a cylinder of given volume V:

$$A = 2\pi r^2 + \frac{2v}{r}.$$

Encouraging a*lgebraic expectation* means asking questions related to each of the elements outlined above. Initially we as teachers need to guide this process until it becomes a habit in our students' mathematical thinking. A 'checklist' of fundamental questions would include "What do each of the letters in this expression represent?" "What is the structure of this expression? Are there any simple factors? What are the key features that you notice and what do they tell us about the function and its possible solutions?

In the example given above:

Recognition of conventions and basic properties could involve: identifying *r, A* and *v* as variables; knowing the convention that the Greek letter π is used to represent a special irrational number; knowing the conventions of implicit multiplication and index notation so that evaluation of $2\pi r^2$ requires $2\times\pi\times r\times r$; knowing the convention for order of operations so that the multiplication and division precede the addition of the two terms.

Identification of structure means recognising that the two terms on the right hand side may be seen as two processes which could be treated as objects; there is a simple common factor of 2 on the right hand side; and the value of A depends on the value of *r*.

Identification of key features means recognising that the expression in r consists of the sum of a quadratic and a reciprocal function; the dominant term will be the term with r^2; key features such as the squared term mean that the equation may have none, one or two solutions; division by r means that there will be a restriction on the domain since $r \neq 0$.

In this section we have briefly outlined the elements of *algebraic expectation* and considered an illustration applying this thinking to a practical example. This analysis of the symbolic expression does not provide a solution for a problem but alerts the student to the attributes of the expression which may provide important insights for the process of monitoring the solution for a particular problem.
Further *algebraic insight* may be gained by linking the symbolic representation with graphic or numeric representation. In the example above, linking the symbolic form of the quadratic and reciprocal function to a parabola and hyperbola then visually adding the ordinates to gain an approximate image of the sum of these terms will give a visual impression of possible values for A. CAS can assist a student in examining how A varies with r and explore the effect of setting different values of the parameter V.
Next we will focus on the second aspect of *algebraic insight*: *ability to link representations.*

Ability to Link Representations

The process of progressing from working with a single data set to developing a general model will commonly start with collection of data and examination of this data set. A student with algebraic insight will be looking for patterns in the data which will be indicative of the form of a suitable symbolic model. For example, if for equally spaced values of the independent variable there is a very rapid increase in the size of the dependent variable this is likely to indicate exponential growth while a recurring pattern of values will indicate that a trigonometric function may provide the basic form of a suitable model. If the raw data has no obvious pattern then examination of first or second difference or ratios may quickly demonstrate whether the data is best modeled by a linear, quadratic or cubic polynomial or if an exponential function is the more appropriate choice. However, students commonly find using tables of values to identify patterns, and therefore algebraic form, quite difficult.

They commonly find the visual representation provided by a graph of the data more helpful. Ability to link symbolic and graphic representations and ability to link symbolic and numeric representation form the two elements of the second aspect of algebraic insight. We will now consider an example showing some ways in which algebraic insight may support the modeling process. Links to the algebraic insight framework, Figure 3, are included in parentheses.

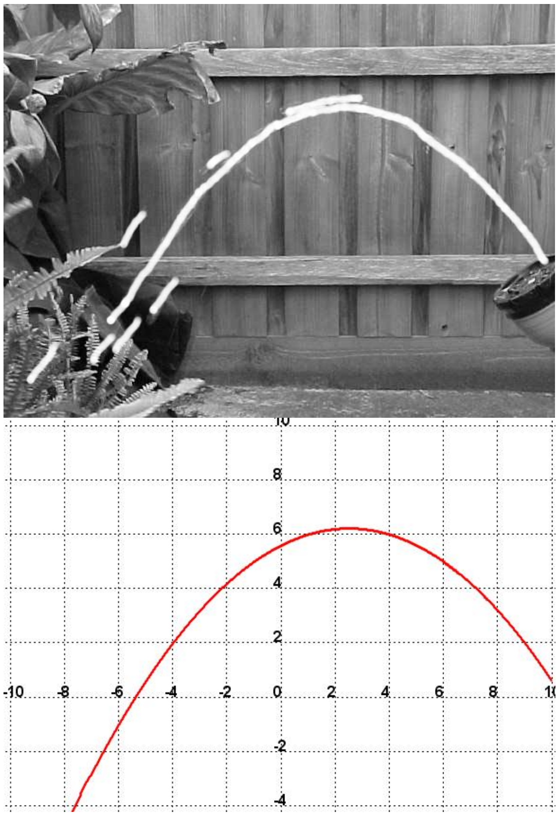

Figure 4. Garden Hose Spray and Graphic Representation

Algebraic Insight Supporting the Modeling Process

Consider the task of creating a mathematical model for the curve formed by a spray of a garden hose. First, working from a photo of a garden spray the student could aim to find a rule for a function whose graph would match this particular spray. In this case algebraic insight will be shown by the student who looks at the image formed by the spray from a garden hose, as shown in Figure 4, and recognizes that this is likely to be best modeled by a quadratic function (2.1). Further, key features such a the critical values of maximum, minimum or intercepts may be identified from a graph and in turn linked to values of various parameters of a function (2.1). A student who knew that a quadratic may be described by several equivalent expressions and that in this case the form $f(x) = a(x-h)^2 + k$ would prove easiest for finding a symbolic expression to describe the path of the water demonstrates a deeper level of algebraic insight (1.2, 1.3, 2.1). Algebraic insight allows the student to make such links between the numeric or graphic representation and their symbolic equivalent.

Recognising that the function rule which describes this graph, $f(x) = -0.1(x-2.5)^2 + 6.2$, will be equivalent to an algebraic expression which will also be a polynomial of degree 2, with a co-efficient of -0.l, a term in *x,* and a constant term with a value between 5 and 6 requires algebraic insight (2.1). Once a symbolic representation of the particular set of data has been achieved then the consequences of changing various parameters may be explored in a systematic manner (1.1, 1.3). Students may be encouraged to make conjectures and discover "what happens if….". This may be done as an abstract exercise without regard to the initial context but equally results obtained this way may also be interpreted in terms of the real life scenario and checked for reasonableness. In this way a student may move from the particular rule which matched this hose spray to a general rule which may be adapted, according to guidelines, to fit other sprays.

CAS Support Learning Algebra through Strategic Exploration

Developing students' algebraic expectation is important if they are to harness the power of CAS to support their working for iterative, complex or other time consuming manipulations where working by hand would take much longer or be open to simple errors. Students require a basic level of such understanding in order to even enter expressions correctly into a CAS (1.1), in particular to identify structure (1.2) and hence make appropriate use of parentheses. Once some very basic facility with the CAS is established it is also possible to use CAS to assist in the further development of students' algebraic expectation. For example, recognition of familiar patterns and relationships is the key to progressing work with symbols. This includes such strategies as identifying common factors, difference of two squares, perfect squares; coming to understand = as indicating the equality of the expressions linked by this symbol ; and later rules for derivatives and anti-derivatives. CAS may be used to explore strategic sets of examples which will give the student exposure to many correct simplifications, for example. Our experience is that as students start to see a pattern they may make conjectures which they test with CAS then progress to finding that working in their own head can be more efficient than using the CAS. At the same time, knowing that the support of CAS is available increases students' confidence to progress in mathematics.

Conclusions

CAS may be used to support and extend students' work in mathematics and it may also be used as a pedagogical tool. CAS may be used effectively to support students' work in mathematical modeling. The use of CAS does not preclude the need for mathematical thinking, it in fact highlights the need for symbol sense and in particular the two aspects of algebraic insight, namely algebraic expectation and ability to link representations. Mathematics teaching has, out of necessity, focused a great deal of time and attention on algorithmic routines. Since CAS does these effectively, attention may now be directed towards deliberately teaching these skills of algebraic insight.

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