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A classification of strategies employed by high school students in isomorphic combinatorial problems

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Abstract: *The aim of the paper is to discuss some aspects of the combinatorial thinking of high school students. We took one student - Jane and gave her 4 isomorphic testing problems. Then we tried to classify the different strategies the students took in their solutions.*

Keywords: combinatorics, high school students, strategies

1. Introduction

Combinatorics plays an important role in school mathematics. This theme has been recurrent in the mathematics education literature (Fischbein & Gazit, 1988; English, 2005; Lesh & Heger, 2001; Muter 1999; Sriraman & English, 2004), as well as numerous curricular documents worldwide (NCTM, 1991, 2000). Among the most influential is work of Kapur (1970) who called for incorporating enumerative combinatorics in the school curriculum. He elicited the following reasons to justify the teaching of elementary combinatorics in schools:

(1) The independence of combinatorics from Calculus facilitates the tailoring of suitable problems for different grades and usually very challenging problems can be discussed with pupils so that they discover the need for more "sophisticated" mathematics to be created.

(2) Combinatorics can be used to train pupils in enumeration, making conjectures, generalization and systematic thinking; it can help the development of many concepts, such as equivalence and order relations, function, sample, etc.

(3) Many applications in different fields can be presented.

All these reasons justify the interest in improving the teaching of the topic. Nevertheless, students' approaches to combinatorial problems are known for a high occurrence of mistakes (Batanero, Navarro-Pelayo & Godino, 1997; English, 1993, 1998, 1999). These studies suggest that teachers pay attention to the nature of mistakes made by students in combinatorial problems and facilitate students' overcoming these mistakes by providing alternative isomorphic problems. They therefore argue that the teachers' goal should include not only attending to students' mistakes but also helping students to arrive at correct solution. It has been argued that discerning the origin of mistakes can help the teacher to understand how to support students' further learning. To understand the thinking of students, it is important to answer following two questions:

(1) Which strategies[3] are chosen by the student?

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[3] "Strategies are goal-directed operations employed to facilitate task performance." (Bjorklund, 1990)

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(2) Why did the student chose a particular strategy?

The second question is deep and beyond the scope of this paper. However our intention here is to deal with the first question and create research-based implications for future research, which will enable us to answer the second question.

2. Theoretical Background

The research reported in this paper is grounded within the extant literature on this topic. In the massive literature review conducted by Sriraman & English (2004) on the state of research in the domain of combinatorial reasoning , they noted the following:

Piaget and Inhelder (1951) viewed combinatorial thinking as an aspect of the stage of formal operations. They characterized combinatorial reasoning as the capacity to determine all the possible ways in which one could link a given set of base associations with each other. Batanero, Navarro-Pelayo, and Godino (1997) provided a simple and highly illustrative account of Piaget and Inhelder's thesis on combinatorial reasoning: Given a problem where a set of objects are required to be arranged in all possible ways, children at the pre-operational stage use random listing procedures, without having an explicit systematic strategy. At the concrete operational level, children use trial and error strategies and are capable of devising "empirical procedures with a few elements" Finally at the stage of formal operations "adolescents discover systematic procedures of combinatorial construction, although for permutations, it is necessary to wait until children are 15 years old" (Batanero et. al, 1997). (Sriraman, B & English, L., 2004, p. 183)

Although Piaget's studies provided powerful insights into the development of combinatorial understanding, the materials that were used and the accompanying instructions were too scientific and abstract (Carey, 1985) for children. This would likely have masked the participating children's abilities in the combinatorial domain. Later research, which employed child-appropriate materials and meaningful task contexts, indicated that young children are able to link items from discrete sets in a systematic manner to form all possible combinations of items (e.g., English, 1991; 1992).

In one such study (English, 1991), 50 children aged between 4.5 years and 9.8 years were individually administered a series of 7 novel tasks that involved the dressing of cardboard toy bears (placed on stands) in all possible different outfits, with an outfit comprising a colored top and a colored pair of pants (or same-colored tops and skirts with different numbers of buttons, for two of the tasks). The findings indicate that, given an appropriate context, children are able to produce independently a systematic procedure for forming m x n combinations prior to the stage of formal operations postulated by Piaget and Inhelder. (Sriraman, B & English, L., 2004, p. 185-186)

Maher and her colleagues (Maher & Martino, 1996a, 1996b, 1997; Maher & Speiser, 1997; Martino & Maher, 1999; Muter & Maher, 1998; Muter, 1999; Speiser, 1997) conducted a series of longitudinal studies lasting up to ten years in which teaching experiments were set up to investigate the growth of mathematical knowledge via the use of combinatorial problems. The fascinating aspect about these studies was that the researchers focused on a group of students and studied the evolution of their mathematical representations, reasoning, argumentation and methods of proof, starting from grade five through grade twelve. The researchers in these studies typically used two or more related problems that were conducive to the formation of isomorphic mathematical structures. It was found that the problem solving strategies of the group of students *who worked on these problems evolved as they worked through these problems, on and off from 1993 to 2000. The representations used by these students became more and more abstract. As fourth-graders, these students discovered properties of combinations with reference to the given problems. The properties of combinations, for these students, grew from very concrete images, such as towers and pizzas (Maher, 1993; Maher & Speiser, 1997; Maher & Martino, 1996a & b; Maher & Kiczek, 2000). However, as tenth-graders, they were able to link these concrete notions to abstract notions of combinations and binomial coefficients found in Pascal's triangle. These findings do not simply confirm the findings of Piaget but also reveal how the development of combinatorial reasoning can "accelerate" from grades four to ten. The Piagetian model spans an eleven-year time period, whereas the findings of the longitudinal studies conducted by Maher and her colleagues indicate that with appropriate instructional scaffolding, students' combinatorial thinking can evolve into sophisticated structures in only seven to eight years! It should be noted that this rapid development is dependent on the use of appropriate tasks in order to facilitate this development in a much shorter time span.* (Sriraman, B & English, L., 2004, p. 184)

Another important finding of these studies was that there was a relationship between "carefully monitoring students' constructions leading to a problem solution" and teacher questioning at appropriate stages of problem solving, which challenged the students to pursue general solutions (Martino & Maher, 1999, p.53). The findings reported by Maher and her colleagues validate the Piagetian notion of how combinatorial reasoning evolves in problems requiring a set of objects to be arranged in all possible ways. These studies revealed that students' strategies evolved from random listing strategies and other trial and error or "empirical procedures" (Davydov, 1996) as fourth-graders, to systematic counting strategies as tenth-graders. This compares with the findings of English (1991, 1992), except in her studies, cited earlier, the children developed sophisticated strategies across a set of tasks within the period of task administration. Increasing notational sophistication, a disposition to think abstractly, the ability to generalize and an affinity for constructing proofs characterized the evolving strategies of the students (Maher, 1993; Maher & Martino, 1996a, 1996b, 1997; Maher & Speiser, 1997; Martino & Maher, 1999; Muter & Maher, 1998; Maher & Kiczek, 2000; Speiser, 1997).

3. The Present Study

Given the precedence of types of problems effective for research on combinatorial thinking, we used isomorphic testing problems in this research. However our attempt was not a mere replication of previous research. The research reported in this paper systematizes and synthesizes perspectives on combinatorial thinking to create an effective instrument for the comprehensive classification of combinatorial strategies employed by high school students in a new geographic location (namely the Slovak Republic). This furthers the aim of the mathematics education community to create research-based knowledge generalizable to age groups across geographic locations.

4. Method

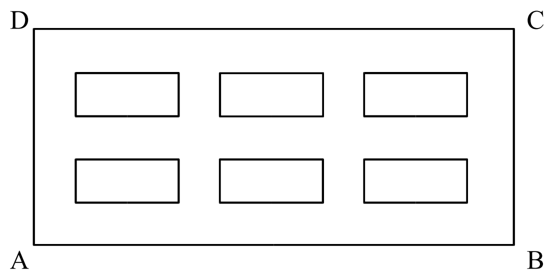
4.1. The Problems of the Study

Siegler (1977) defined the concept *isomorphic problem* (or *isomorphs*) as follows: "Isomorphs are problems that are formally identical but differ in their surface structure". If we expect that the solution of a problem is influenced by numerous parameters, it is necessary to keep all but one of the parameters invariant to establish the influence of the chosen parameter on the solution. In the Slovak Republic, high school students meet the phenomenon of isomorphism during the traditional teaching of combinatorics. They are often required to solve problems that are similar to standard "ground" types of the combinatorial problems.

As a starting point we used 4 isomorphic combinatorial problems:

The Town Problem

There are houses marked as rectangles on the figure. There are streets between them. By how many different ways can we get from the place A to the place C, if we move through the streets of the town only in the directions upwards and to the right?



The Ice Hockey Problem

An ice hockey match finished 2:3. What are the possible partial scores that could have led to the final score of this match? Find all different possibilities.

The Pigeonhole Problem

Write all possibilities in which 5 balls A, B, C, D, E can be placed into 2 pigeonholes u and v such that 2 balls are in the pigeonhole u and 3 balls are in the pigeonhole v.

The Line Problem

In how many ways is it possible to line up 3 ○ and 2 □?

Note that the problems in the instrument are robust because they yield the following isomorphic solutions. Each possibility which is a part of the solution of the preceding problems can be coded by the sequence of **0s** (three symbols) and **1s** (two symbols), where the symbol **0** means:

1. in the Town problem the move "to the right" (see forthcoming extract of the protocol)

2. in the Ice Hockey problem the goal scored by the opposing team

3. in the Pigeonhole problem the selection of the ball into the pigeonhole *v*

4. in the Line problem the symbol: ○

and the symbol **1** signifies:

1. in the Town problem the move "upwards" (see forthcoming extract of the protocol)

2. in the Ice Hockey problem the goal scored the "home" team

3. in the Pigeonhole problem the selection of the ball into the pigeonhole *u*

4. in the Line problem the symbol: □

Another important concept that emerged as will be revealed in the subsequent sections was that of “position”. By “position” we understand:

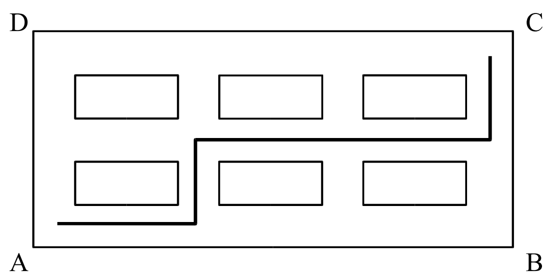
1. in the Town problem the serial number (or running count) of the move on the path from A to C

2. in the Ice Hockey problem the serial number of the goal

3. in the Pigeonhole problem the balls A, B, C, D, E (the ball A = 1st position, …)

4. in the Line problem the place in the line

The following example will help illustrate these nuances:



position	1.p.	2.p.	3.p.	4.p.	5.p.
path	**0**	**1**	**0**	**0**	**1**

4.2. Data Collection

These problems were assigned to 16-year-old Jane, a student in the 2nd year of high school. The student’s solution process was video recorded and later transcribed to create a time-stamped protocol. This protocol served as a basis for the detailed analysis that I will explain in the next section. As an illustration, I present a part of the solution protocol of the Town problem and Jane‘s solutions of the remaining problems.

Extract of the protocol

0.00 J. takes the red crayon and looks at the figure.

0.02 J. marks the path **00011** (a).

0.04 J. looks at the figure.

0.06 J. marks quickly in succession the paths **00110** (b), **00101** (c), **01010** (d).

0.12 J. looks at the figure.

0.14 J. marks quickly in succession the paths **01100** (e), **11000** (f), **10100** (g).

0.20 J. scratches her head and looks at the figure.

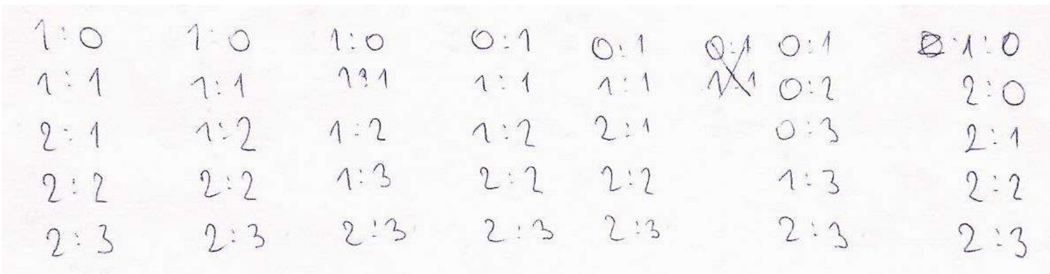
0.30 J. raises the crayon into the air and looks at the figure.

0.46 J. indicates by the crayon the paths **10001** (h), **10010** (i) in the air closely above the figure.

0.50 J. raises the crayon into the air, juggles with her hair and looks at the figure.

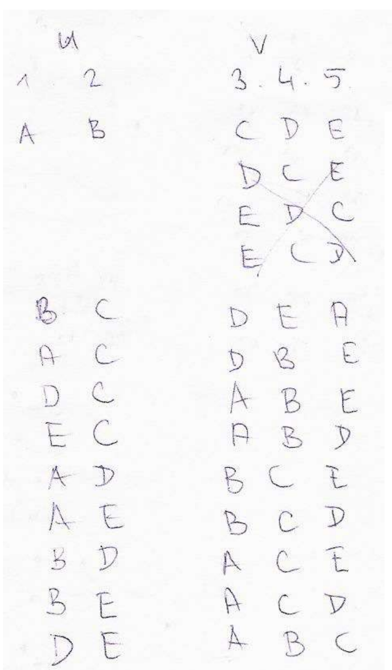
0.76 J.: “There are nine possibilities.”

The solutions of the remaining problems



(a)	(b)	(c)	(d)	(e)	(f)	(g)	(h)
10100	**10010**	**10001**	**01010**	**01100**	**01**	**00011**	**11000**

Figure 1. The Ice Hockey Problem. Each column depicts one of Jane’s solutions for how the ice hockey match could have progressed. The chart beneath Jane’s solution depicts each match's progress as a sequence of zeros and ones, using the coding explained previously



(a) **11000**

(b) **11**

(c) **01100**

(d) **10100**

(e) **00110**

(f) **00101**

(g) **10010**

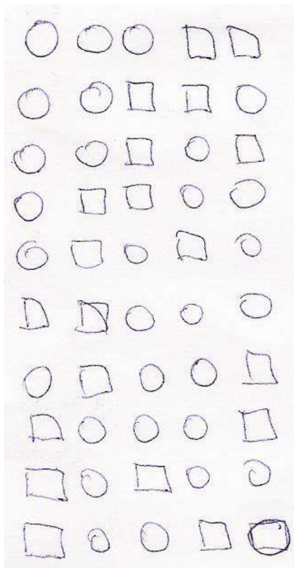
(h) **10001**

(i) **01010**

(j) **01001**

(k) **00011**

Figure 2. The Pigeonhole Problem



(a) **00011**
(b) **00110**
(c) **00101**
(d) **01100**
(e) **01010**
(f) **11000**
(g) **01001**
(h) **10001**
(i) **10100**
(j) **10010**

Figure 3. The Line Problem

4.3. Data Analysis

In order to compare the solutions of isomorphic problems presented in varying contexts with the goal of classifying strategies as well as understanding the influence of context on a particular strategy it was first necessary to identify and describe the strategies which the Jane used during the solution of a particular type of problem. To identify the strategies, we used the method of atomic analysis, introduced in work of Hejný (1992). This method consists of a thorough investigation of every detail – every "graphical atom" of the written work of a student. We examined nuanced details of Jane's solutions and characterized particular strategies that she followed when solving the problems. We then identified each strategy on the basis of the changes (i.e. permutations) occurring in the use of symbols within a solution. The next step consisted of describing these strategies. To make the description valid for each solution of the four problems, each permutation was converted to a sequence of zeros **0** and ones **1**. Each of the identified strategies was explained on the example of two associated succeeding permutations of three zeros and two ones. Then the strategy was generalized for any initial permutation.

We will explain a derivation of strategies on Jane's solutions of the Ice Hockey and Line problems.

Our basic assumption when analyzing students' solutions was that high school students create lists of possibilities in accordance to some guiding principle (i.e., not randomly). We also conjectured that the students would use the principle until they exhausted all the possibilities that it allowed them to identify.

When I compared associated running scores of matches (a) and (b) in the Ice Hockey problem, I found the only difference - in the third step: (a) *2:1* (b) *1:2*. If the team, that scored the third goal in (b), were identical to the team, that scored the third goal in (a), the two solutions for the scores would be identical. It looks like that Jane takes over the running scores from the last generated solution up to the point when the next step in the new solution must necessarily differ if the two solutions are not to be identical. If Jane used this principle systematically to guide her generation of different solutions, we should be able to identify it again in Jane's transition from the solution (b) to (c). Matches (b) and (c) differ in the fourth running score for the first time. It supports our hypothesis, because taking over the score *2:2* from (b) would result in generating a solution identical to (b).

Jane used this guiding principle in generating solutions of other problems as well. As an example, the ordering (c) in the problem "Line" generated in accordance to the same principle. Jane takes over the sequence of ○ and □ from the solution (b) up to the symbol in which the two sequences must differ in order to be different.

The following figure depicts the running scores of ice hockey matches (a), (b), (c) coded into the sequence of zeros and ones.

	1.p.	2.p.	3.p.	4.p.	5.p.
(a)	**1**	**0**	**1**	**0**	**0**
	↓	↓			
(b)	**1**	**0**	**0**	**1**	**0**
	↓	↓	↓		
(c)	**1**	**0**	**0**	**0**	**1**

This coding transcends the context of the original ice hockey problem and can be interpreted in the context of any of the isomorphic problems used with Jane. For example, in the Line problem the previous coded sequences would represent following solutions:

□ ○ □ ○ ○

□ ○ ○ □ ○

□ ○ ○ ○ □

The above-mentioned guiding solution principle that Jane used in generating solution (b) in the ice hockey problem can then be described as follows: *If the change from **1** to **0** doesn't occur at the position 3, we would not arrive at the solution different from the preceding solution (a).*

This guiding principle can be applied to any initial sequence to generate a new solution. We will call this generalized principle a *Strategy of a constant beginning.* This strategy can be executed in the following way

(1) Start copying the initial sequence from the left

(2) Identify the "critical" position, that is, the right-most position on which the new sequence can no longer be identical to the initial sequence if the two sequences are not to be identical.

(3) Change the symbol on the "critical" position and finish the new sequence accordingly

5. Results

In Jane's solutions we discovered 11 strategies. By (x)/(y) we mean that the model for creation of the permutation (y) was the permutation (x). By (x-y) we will denote all the permutations (x), (x+1), ..., (y).

1. Strategy of exhausted subset

We look for a new strategy because we have exhausted the subset of permutations with a common feature that we could have enumerated using the preceding strategies. We present several such examples:

I. All the permutations with a common prefix of a certain length have been found (i.e. all the permutations that are identical up to a certain position). For example: In case of the permutations (a), (b), (c) in the figure 1. it is about all the permutations, that begin with the symbols **10**.

II. All the permutations whose progress up to a certain position is based on the regular alternation of **1s** and **0s** have been found. For example: In case of the permutations (a)-(e) in the figure 1. it is about all the permutations, where **1**s and **0**s alternate in the first two positions in any order (i.e. **01** or **10**). Although one of such permutations is missing (**01001**), the set is considered to be exhausted, since it is not possible to obtain the missing permutation using the strategies 3[(a)/(b)] 3[(b)/(c)] 1[(a-c)/(…)] 2[(a-c)/(d-f)] 5[(a)/(d)] 5 [(b)/(e)] 5[(c)/(f)].

III. All the permutations whose progress begins with a sequence of one of the symbols (either **1** or **0**) and continues with a sequence of the other symbol have been found. For example: figure 1. – the permutations (g), (h).

IV. All the permutations having the symbol **1** in a certain position have been found (see the strategy of a constant element). For example: In case of the permutations (c)-(f) in the figure 2. it is about all the permutations that have the symbol **1** it the 3rd position.

V. All the permutations that contain all possible arrangements of two symbols **1** in given positions have been found. For example: In case of the permutations (a), (c), (d) in the figure 2. it is about all the permutations that have symbols **1** located in any two of three positions (1, 2 and 3).

This strategy is present in the solutions of all problems.

2. Group strategy

A preceding subset of permutations (with two elements at least) is used as a model for creating new permutations using some of the presented strategies. We will refer to this subset as to a *model group*.

For example:

	1.p.	2.p.	3.p.	4.p.	5.p.
(a)	**1**	**0**	**1**	**0**	**0**
	∇	∇	∇	∇	
(d)	**0**	**1**	**0**	**1**	**0**

	1.p.	2.p.	3.p.	4.p.	5.p.
(b)	**1**	**0**	**0**	**1**	**0**
	∇	∇	∇	∇	
(e)	**0**	**1**	**1**	**0**	**0**

	1.p.	2.p.	3.p.	4.p.	5.p.
(c)	**1**	**0**	**0**	**0**	**1**

∇- the change to the other symbol

The permutations (d) and (e) have been created from the permutations (a) and (b) in mentioned order by the symmetry strategy. The permutation (c) has not been used as a model for creation of an additional permutation because it would be identical to (e).

This strategy has been used in co-operation with other strategies in the solutions of the problems „Town“ (figure 4.; (d-e)/(f-g), see extract of the protocol), „Ice Hockey“ (figure 1.; (a-c)/(d-f), see the symmetry strategy) and „Line“ (figure 3.; (b-c)/(d-e), see the parallelism strategy).

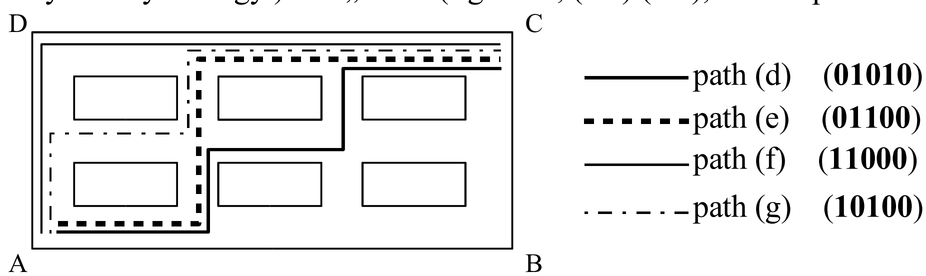


Figure 4. The Town Problem

3. Strategy of a constant beginning

The progress remains identical up to „the highest possible“ position (it is such a position that if the symbol in it is not changed, the entire permutation will have to be identical to the model).

For example:

	1.p.	2.p.	3.p.	4.p.	5.p.
(a)	**1**	**0**	**1**	**0**	**0**
	↓	↓			
(b)	**1**	**0**	**0**	**1**	**0**
	↓	↓	↓		
(c)	**1**	**0**	**0**	**0**	**1**

If the symbol in the position 3 (b) or in the position 4 (c) is not changed from **1** to **0** , the symbols in the higher positions will also have to remain unchanged, leading to the same permutation.

This strategy is present in the solutions of the problems „Town“ (figure 5.; for example (a)/(b), (b)/(c)), „Ice Hockey“ (figure 1.; (a)/(b), (b)/(c)) and „Line“(figure 3.; for example (a)/(b), (b)/(c)).

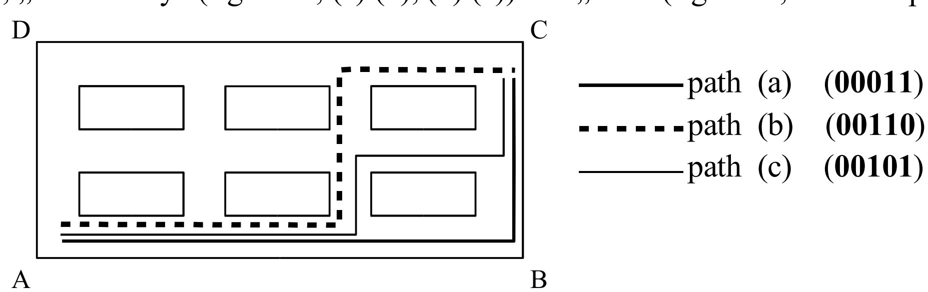


Figure 5. The Town Problem

4. Strategy of the same number of the permutations in groups

If a subset of permutations derived from a model group using the group strategy has less elements than the model group, other permutations are added to it to make the number of elements equal to the number of elements of the model group. These additional permutations are chosen so that all permutations in the resulting subset share a common feature that distinguishes them from the permutations of the model group.

For example:

	1.p.	2.p.	3.p.	4.p.	5.p.
(a)	**1**	**0**	**1**	**0**	**0**
(b)	**1**	**0**	**0**	**1**	**0**
(c)	**1**	**0**	**0**	**0**	**1**
(d)	**0**	**1**	**0**	**1**	**0**
(e)	**0**	**1**	**1**	**0**	**0**
(f)	**0**	**1**			
(g)	**0**	**0**	**0**	**1**	**1**

Because (c) cannot be used as a model for creation of (f) (see the group strategy example), the group of the permutations beginning with the sequence **01** is exhausted, but it has less elements (by one) than its model group (a-c). A random permutation beginning with the symbol **0**(just like (d) and (e)) is added to the group. The feature that distinguishes the new group from the model group is the symbol in the first position in this case.

This strategy is present in the solution of the problem „Ice Hockey"(figure 1.; (a-f)/(g)).

5. Strategy of symmetry

The symbols **1** are replaced with **0**s and the symbols **0** are replaced with **1**s in all positions up to the position where this kind of change is no longer possible because the exact number of **0**s and **1**s in the permutation is given. The remaining positions are filled with **0**s.

For example:

	1.p.	2.p.	3.p.	4.p.	5.p.
(a)	**1**	**0**	**1**	**0**	**0**
	∇	∇	∇	∇	
(b)	**0**	**1**	**0**	**1**	**0**

The symbol **0** in the 5th position cannot be replaced with the symbol **1**, because the two symbols **1** have already been used.

This strategy is present in the solution of the problem „Ice Hockey" in co-operation with the group strategy (figure 1.; (a)/(d), (b)/(e), (c)/(f)).

6. Strategy of parallelism

All symbols **0** move by one position to the right (we will denote this strategy as the strategy of parallelism **0**R), or to the left (strategy of parallelism **0**L), and the unoccupied positions are filled up with symbols **1**. If a symbol cannot be moved in the first step (because it is in the first or the last position respectively), it remains in its current position and only the other symbols move as described. We define the strategies of parallelism **1**R and **1**L for the movement of symbols **1** by analogy.

For example:

	1.p.	2.p.	3.p.	4.p.	5.p.
(a)	**0**	**1**	**0**	**1**	**0**
		↙		↙	
(b)	**1**	**0**	**1**	**0**	**0**
	↓		↙		
(c)	**1**	**1**	**0**	**0**	**0**

The symbols **1** move from the 2nd and the 4th position (a) to the 1st and the 3rd position (b). The remaining positions (2nd, 4th and 5th) are filled with with **0**s. Because the further movement of the symbol **1** in the first position to the left is not possible, it remains in its current position and the other symbol **1** moves from the 3rd position to the 2nd position.

This strategy is present in the solutions of the problems „Town“ (figure 4.; (d)/(g), (e)/(f))) and „Line“ (figure 3.; (b-c)/(d-e)) in co-operation with the group strategy.

7. Strategy of a constant element

One of the symbols **1** remains in its position, the other one takes a random position of the remaining ones. We shall think about this strategy only in the case when the subset of permutations having the symbol **1** in a certain position is exhausted in a continuous sequence of steps (see the strategy of exhausted subset, example IV). If we considered only two successive permutations regardless of the context, we would identify other strategies as well. For this reason we consider it necessary to introduce a requirement that, if a subset is exhausted (in the sense of the strategy of the exhausted subset, case IV), we will consider it to be exclusively according to the strategy of a constant element. As an exception, if there is a strategy leading to exhaustion of the same subset of permutations as the strategy of a constant element, we shall consider them both (or all of them if there are more such strategies).

For example:

	1.p.	2.p.	3.p.	4.p.	5.p.
(a)	**0**	**1**	**1**	**0**	**0**
			↓		
(b)	**1**	**0**	**1**	**0**	**0**
			↓		
(c)	**0**	**0**	**1**	**1**	**0**
			↓		
(d)	**0**	**0**	**1**	**0**	**1**

One of the symbols **1** remains in the 3rd position, while the other one progressively occupies all remaining positions. It is evident from the sequence of the individual permutations that the strategy of a constant element is used exclusively, although we could identify also the strategy of constant beginning between permutations (c) and (d).

This strategy is present in the solutions of the problems „Town“ (figure 6.; (e)-(h)), „Pigeonholes“ (figure 2.; (d)/(e), (e)/(f)) and „Line“ (figure 3.; (f)/(h), (h)/(i), (i)/(j)).

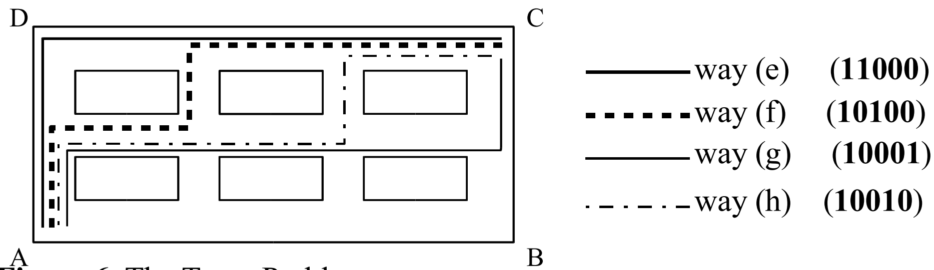


Figure 6. The Town Problem

8. Strategy of complement of all arrangements

A subset of permutations containing all except one possible arrangements of the two symbols **1** in given positions is completed with the missing permutation to form an exhausted subset in the sense of the strategy of exhausted subset, case V.

For example:

	1.p.	2.p.	3.p.	4.p.	5.p.
(a)	**1**	**1**	**0**	**0**	**0**
(b)	**0**	**1**	**1**	**0**	**0**
(c)	**1**	**0**	**1**	**0**	**0**

The permutations (a) and (b) represent two elements from the three element subset of the permutations that include all arrangements of two symbols **1** in the 1^{st}, 2^{nd} and the 3^{rd} position. The missing permutation is added.

This strategy is present in the solutions of the problems „Town“ (figure 5.; (a-b)/(c)) „Pigeonholes“ (figure 2.; (a-c)/(d)) and „Line“ (figure 3.; (a-b)/(c)).

9. Strategy of the odometer[4]

The principle of the odometer is already mentioned in the papers by L. D. English (1993): „This pattern is so named because of its similarity to the odometer in a vehicle.“ We have modified the description of this principle to correspond to the definitions of our problems because our problems and those in the cited papers differ. One of the symbols **1** remains in the position *x* (called constant element) while the other one progressively occupies all remaining positions from the lowest one to the highest one, without repeating previously discovered permutations. After exhausting all possibilities, next position (x+1) is chosen for the constant element and the process repeats. The strategy ends when all possibilities for the choice of the constant element position are exhausted. If we considered only two successive permutations regardless of the context, we would identify other strategies as well. For this reason we consider it necessary to introduce a requirement that, if a subset is exhausted according to this strategy, we will consider it to be exclusively according to this strategy, and we will not take the other possible strategies into account.

[4] We mean the distance counter in a vehicle.

For example:

	1.p.	2.p.	3.p.	4.p.	5.p.
(a)	**1**	**1**	**0**	**0**	**0**
(b)	**1**	**0**	**1**	**0**	**0**
(c)	**1**	**0**	**0**	**1**	**0**
(d)	**1**	**0**	**0**	**0**	**1**
(e)	**0**	**1**	**1**	**0**	**0**
(f)	**0**	**1**	**0**	**1**	**0**
(g)	**0**	**1**	**0**	**0**	**1**
(h)	**0**	**0**	**1**	**1**	**0**
(i)	**0**	**0**	**1**	**0**	**1**
(j)	**0**	**0**	**0**	**1**	**1**

The constant element remains in the 1st position, the second one of the symbols **1** progressively occupies the 2nd to 5th position. The constant element remains in the 2nd position, the second one of the symbols **1** progressively occupies the 3rd to 5th position. It cannot occupy the 1st position, because this permutation would be identical to (a). The strategy finishes by the occupation of the 4th position by the constant element, because there are no possible positions for the second symbol **1**, when the constant element occupies the 5th position, that would yield a new permutation.

This strategy is present in the solution of the problem „Pigeonholes" (figure 2.; (f)/(g), (g)/(h),(h)/(i), (i)/(j), (j)/(k)).

10. Strategy of rotation

The new permutation is created by rotating the preceding one by 180°.

For example:

	1.p.	2.p.	3.p.	4.p.	5.p.
(a)	**1**	**1**	**0**	**0**	**0**
(b)	**0**	**0**	**0**	**1**	**1**

A statement of the type:

Jane: „Here, when one (team) scored in a row." (The common description of both permutations (a) and (b) in the „Ice Hockey" problem) plays an important role in the identification of this strategy in a special case because if the order of the permutations was 00011, 11000, we could identify the used strategy as the strategy of symmetry. However, the statement indicates that it is not the progress of the model that is important for the respondent, but rather the fact that the result is to be a permutation rotated by 180°.

This strategy is present in the solution of the problem "Ice Hockey" (figure 1.; (g)/(h)).

11. Strategy of complement of the exhausted subset

While in the preceding steps the exhaustion of certain subset (the minimal number of the components is 5) of permutations with some common symbol occurred, the permutations are looked for with such common character which for it is valid:

Having exhausted a subset of permutations with a common feature (containing at least 5 elements) we look for a subset of permutations with another common feature such that:

1. the exhausted subset and the looked for subset of permutations are disjunct;
2. the exhausted subset and the looked for subset form a set of all permutations.

For example:

	1.p.	2.p.	3.p.	4.p.	5.p.
(a)	**0**	**0**	**0**	**1**	**1**
(b)	**0**	**0**	**1**	**1**	**0**
(c)	**0**	**0**	**1**	**0**	**1**
(d)	**0**	**1**	**0**	**1**	**0**
(e)	**0**	**1**	**1**	**0**	**0**
(f)	**1**	**1**	**0**	**0**	**0**
(g)	**1**	**0**	**1**	**0**	**0**
(h)	**1**	**0**	**0**	**0**	**1**
(i)	**1**	**0**	**0**	**1**	**0**

The permutations (a)- (g) form the exhausted subset of permutations with the difference of the positions occupied by the symbols **1** equal to 1 or 2. The permutations are looked for that have the difference of the positions occupied by the symbols **1** equal to 3 or 4.

This strategy is present in the solutions of the problems "Town" (see the extract of protocol (a-g)/ (h-i)) and "Line" (figure 3.; (a-e)/(f, h-j)).

Overview of strategies used in particular problems

permutation	(b)	(c)	(d)	(e)	(f)	(g)	(h)	(i)	(j)	(k)
"Ice Hockey"	3.	3., 1.I	2., 5.	5.	5., 1 I	4.	1.II, 10., 1.III			
"Pigeonholes"			8., 1.V	7.	7., 1.IV	9.	9.	9.	9.	9.
"Line"	3., 1.III	3., 1.I or 8.	3. or 2., 6.	6,1.I or 3.	2., 6. or 11. or 3.	3., 1.I	3. or 7. or 11.	7. or 11.	3., 1.I or 7., 1.IV or 11.	
"Town"	3., 1.III	3. or 8.	1. I	2., 6.	2., 6.	6.	3. or 7. or 11	3. or 7. or 11.		

6. Conclusions

Event though the presented problems are isomorphic, students have used different strategies to solve them. This observation correlates to the observations of other authors (Tőrner, 1987; Bauersfeld, 1985; English, 1999; Hefendehl-Hebeker&Törner, 1984; Hesse, 1985; …). It would be interesting in a future research to find which aspects in the problem context influence the strategy selection and the completion of the solution.

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