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Eddie M. Gray

Demetra Pitta-Pantazi

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Frames of Reference and Achievement in Elementary Arithmetic

Eddie M. Gray & **Demetra Pitta-Pantazi**
University of Warwick (UK) **University of Cyprus (Cyprus)**

Abstract: This paper considers the relationship between 8-11 years old students' numerical achievement and a possible disposition towards the construction of particular frames of reference. The paper uses the characteristics of a variety of *kinds* of images to focus upon frames of reference and explores a possible relationship between children's verbal descriptions of concrete and abstract nouns and the different ways they respond to aspects of elementary arithmetic. It seeks to establish whether or not children towards the extremes of arithmetical achievement (low and high achievement) have a disposition towards different kinds of frames of reference. The analysis suggests that at one extreme these may be largely descriptive, associated with episodes or specific characteristics that are derived from efforts to concretise the stimuli. At the other, these descriptive qualities are complimented by more relational characteristics that are indicative of greater flexibility in mathematical behaviour. The conclusions suggest that such differences may influence the interpretation that some children may make of the objects and actions that are the foundations of their numerical development and, as a consequence, this may affect the quality of the child's cognitive shift from concrete to abstract thought.

Key words: Frames of reference, arithmetic, mental representations

INTRODUCTION

In considering the different frames of reference that children project through their words this paper explores what may be a significant contributory factor in children's numerical development. It presents the possibility that the numerical growth may be influenced by a disposition towards articulating different frames of reference. We associate different kinds of frame with different kinds of a mental representation. The latter, seen as a mental reference that, divorced from the objects that give it place in the real world, is the product of imaging. Modality free, a mental representation may be seen as a presentation to the mind in the form of an idea or an image but unfortunately mental representations and mental processes are not directly observable. As a consequence, our guiding principle has not been to consider whether the form of a mental representation is verbal or visual but to recognise that human cognition requires different representational constructs to describe it. It is partly for this reason and partly because of the difficulties inherent in the study of mental representations that we identify the presentation made to the mind as a 'frame of reference'. Contextualised through an examination of selected literature on imagery, different frames of reference are identified and categorised from an analysis of what it is that a sample of children from the same elementary school choose to talk about when asked to respond to a variety of verbal stimuli.

Introduced by Minsky (1975) to indicate the way that an individual structures knowledge, frame of reference may be defined as:

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> The context, viewpoint or set of presuppositions or of evaluative criteria with which a person's perceptions and thinking seem always to occur and which constrains selectively the course and outcomes of these activities (Bullock & Stromley, 1988, p. 334).

We conjecture that the notion may provide us with the means to utilise theories associated with different kinds of imagery to provide additional insight and understanding of cognitive development in arithmetic and in particular our understanding of a divergence in numerical thinking.

The perspective for our discussion is placed within the theoretical context that embraces the notion that mathematical symbolism is open to ambiguous interpretation. Learners' interpretation of this symbolism has suggested that there is a bifurcation in approach that on the one hand may lead to flexible thinking but on the other to a procedural cul-de-sac (Gray & Tall, 1994).

Why should this happen? To gain a partial answer to this question we seek to identify whether or not different frames of reference appear to be dominant within 8-11 years old children at the extremes of achievement in elementary arithmetic. Our task is to establish what ideas are central to the thoughts of a child when, in the absence of perceptual items, there is an invitation to articulate their perceptions of a selection of concrete and abstract nouns. Our fundamental thesis is that there is a relationship between what a child chooses to talk about and the ways s/he may think about elementary arithmetic.

THEORETICAL BACKGROUND

The conceived cognitive development of numerical concepts would appear to be underpinned by the encapsulation of actions with perceptual items (Piaget, 1965; Steffe, von Glaserfeld, Richards, & Cobb, 1983; Kamii, 1985; Gray & Tall, 1994). The reconstruction of well-rehearsed procedures continually involves a qualitative change that enables the concept of number to be conceived of as a construct that can be manipulated in the mind. Such a change suggests that there is no longer the need to transform the object-like essence of this mental object into an action associated with perceptual or figural items. Fortunate are those who recognise this. However, for others the meaning may remain at an enactive level whereby elementary arithmetic remains a matter of performing or re-presenting an action with real or imagined items (Steffe et al., 1983; Gray & Pitta, 1999). We suggest that these different perceptions may represent a spectrum of thinking that is typified by qualitative differences in the kinds of things children communicate when responding to a request to articulate what they have in mind when they think about a range of verbally presented concrete and abstract nouns.

Substantial interest in the cognitive development of mathematics has focused on the relationship between actions and entities (Dienes, 1960; Davies, 1984; Dubinsky, 1992; Sfard, 1992, Gray & Tall, 1994). The qualitative change associated with actions becoming objects of thought has been linked with theories accounting for the transformation of processes into concepts. These have helped to shift attention from *doing* mathematics to *conceptualizing* mathematics. Although consensus recognizes that reconstruction as a result of process/object encapsulation or procedural reification does occur, we are still a long way from being able to describe how this is done although there is evidence to suggest that we may confirm whether or not it has been done. For example, Gray and Tall (1994) suggest that an analysis of a child's interpretation and use of numerical symbolism can provide such evidence in the field of elementary arithmetic. On the one hand the evidence may point to the substantive use of a procedure such as count-all, whilst on the other it may be exemplified by the use of known fact to establish unknown ones. The essence of the differences in the level of sophistication that may be determined from such an observation is captured within the notion of the proceptual divide (Gray, Pinto, Pitta, & Tall, 1999). This suggests that on the one hand we may see a cognitive style more in tune with the flexibility that is associated with an appreciation that a numerical symbol can represent a process or a concept, whilst on the other, the cognitive style may be more dominantly associated with a procedural interpretation of the task in hand.

Though the theories mentioned above have intrinsic differences, they share common ground in their attempts to account for the cognitive reconstruction that underscores the development of conceptual thinking in mathematics. However, it is our conjecture that an individual's perception and interpretation of the original objects and the actions that may be associated with them will influence the quality of this development. Objects have different facets and therefore are subject to different interpretations. For example, counting starts with objects perceived in the external world which have properties of their own; they may be round or square, red or green or both round and red. However, these properties need to be ignored (or at least temporarily disregarded) if the counting process is to be encapsulated into a new entity — a *number* that is *named* and given a *symbol* — that may then be associated with new classifications and new relationships. This foundation can provide a basis for growing sophistication in the nature of the entities operated upon and this is manifest in an increasing detachment from immediate experience. Actions with physical objects become the basis for mental operations with the number symbols which then act as if they were objects.

Qualitatively Different Ways of Thinking

The qualitative differences in the mathematical thinking of novice and experts have been extensively discussed over the years and they still receive considerable interest (Kruteskii, 1976; Bransford, Brown & Cocking, 1999; Baroody & Dowker, 2003). Gray (1991) revealed that the qualitatively different ways that young children approached tasks in elementary arithmetic reflected the consequence of divergent ways of thinking. Some children remained at a procedural level, which, in terms of information processing, could make things very difficult for them. Others operated at a conceptual level that appeared to provide greater flexibility. Later re-evaluation of the children's interpretations of symbolism enabled these differing levels of sophistication to be placed within the context of the 'proceptual divide' (Gray & Tall, 1994). Hypothesized to be a cognitive difference between those children who processed information in a flexible way and those who invoked the use of procedures, this term was derived from the notion of procept, a mathematical symbol that ambiguously represents process and concept (Gray & Tall, 1994). It provides a basis for articulating the potential divergence between those who see mathematics as a sequence of procedures and those who are able to utilize the flexibility associated with process/concept duality. However, although it clearly focuses on interpretations of symbolism it may also have a relationship with other dichotomies in mathematical thinking for example, instrumental and relational (Skemp, 1977), procedural and conceptual (Hiebert & Lefevre, 1986) or operational and structural (Sfard, 1991). For those demonstrating instrumental, procedural or operational thinking it is possible to achieve success but they are arguably on a spiral leading to increasing difficulty in their personal construction of mathematical concepts. In contrast, it is suggested that those demonstrating relational, instrumental and conceptual thinking have a cognitive advantage since they appear more able to associate procedures that enable them to perform arithmetical operations with number concepts.

The existence of a proceptual divide not only seems to indicate that some give a different interpretation to an arithmetical activity but it also seems possible that they do not perceive the potential of the activity they engage with. Their perception may be closely linked with identifying properties of objects (Pitta-Pantazi, Gray & Christou, 2002) or on remembering particular actions with these objects. However, even when teaching programs are designed to shift the child's focus from processes to thinking strategies (Thornton, 1990; Howat, in preparation), children who may be entering the procedural cul-de–sac appear to resist a change in the apparent security offered by their use of well established procedures.

Pitta and Gray, (1997) suggest that children who have difficulty with elementary arithmetic often appear to have difficulty in isolating numerical symbolism from the perceptual and figural objects and actions that give it meaning. The assumption that all children will extract from their experience with particular representations that which will enable them to become "experts" in a particular aspect of mathematics has been questioned by Cobb, Yackel, and Wood (1992). Pitta and Gray (1997) suggest that children can focus on qualitatively different aspects of such representations which may have consequences for the quality of the object that eventually dominates the child's thought. Sáenz-Ludlow (2002) suggests that:

> In the classroom, mathematical concepts are constructed and comprehended through an intentional process of interpretation, guided by the teacher, of mathematical notations/representations. In such an interpreting process, mathematical notations could represent mathematical objects for the teacher but for the learner it will entail a recursive representational process before he comes to clearly see that mathematical object." (p.37)

A tendency towards a recursive representational process may be short lived amongst some learners but longer-term amongst others. Why is this so? One response to this question is that presented within this paper. Notwithstanding the variety of variants that may influence the acquisition of knowledge, for example social and cultural, the classroom environment, and the quality of pedagogy, this paper examines whether or not there may be a relationship between children's level and quality of achievement in elementary arithmetic and their frame of reference. Though the child's developmental increase is frequently self-evident within the elementary school, the psychological mechanisms and the components that support or undermine it are not fully identified or understood (Baroody & Dowker, 2003). If we could recognize these components and understand better these mechanisms and their associations we could probably increase our ability to describe the qualitative difference between those on different arms of the proceptual divide.

Different Kinds of Mental Representation

The notion that more than one type of mental representation exists has received extensive comments (Presmeg, 1986; DeBeni & Pazzaglia, 1995; Drake, 1996; Cifarelli, 1998). Presmeg's distinction between concrete pictorial imagery, which maintains a focus on irrelevant detail, and pattern imagery, which disregards such detail and focuses on relationships, is significant to our discussion in the emphasis it places on the difference between 'description' and 'relationship'.

Such a distinction was a theme alluded to by Drake (1996) who considered types of imagery at three levels of sophistication. Level 1, common to all respondents within her study, was characterized by the reporting of very concrete visual images in which respondents were either observers or participants. Such images, primarily visual and from the subject's known physical world, were regarded as a tool or a program to achieve a particular goal. A child who reports the use of imagined counting units such as fingers or counters might be seen to project imagery at this level. Subjects operating within level 2 identified concrete and highly pictorial images which could act as a symbol. Some of the number forms reported by Thomas, Mulligan, and Goldin, (1995) and Seron, Presenti, and Noël (1992) may fit this category. Such images were primarily visual but could come from other modalities. Images classified as Level 3 were abstract and formed from all modalities. Such images are only arbitrarily related to the real word and in essence are symbols.

The relationship between understanding and imagery suggests that concrete and memory images appear to dominate amongst instrumental thinkers whilst abstract imagery appears to predominate amongst relational thinkers (Brown & Presmeg, 1993). More particularly, the development of mathematical thinking can be seriously influenced by strong early attachments to particular dominant images (Pirie & Kieran, 1994). Such observation may be particularly relevant in a child's development of concepts within elementary arithmetic. Those who predominantly use procedures appear to display less inclination to filter out information (Gray & Pitta, 1997). Relational thinkers on the other hand appear to reject information or, to put it another way, they are more able to select the information that is most relevant to a particular situation. This would suggest that images that reflect different levels of sophistication might have their roots in qualitatively different types of abstraction and the formation of qualitatively different frames of reference. The individual's active mental process of making sense of data through direct involvement may in turn reflect a predisposition towards a particular type of interpretation that reflects a personal frame of reference.

Attributes associated with different types of personal involvement with concrete nouns have featured in the classification of different kinds of imagery explored by Cornoldi, De Beni and Pra Baldi (1988) and De Beni and Pazzaglia (1995). Cornoldi et al. suggest that images spontaneously evoked from a single verbal cue may be identified as *general*, *specific* or *autobiographical* with the frequency of occurrence appearing in decreasing proportions. General images represent a concept without any reference to a particular example or to specific characteristics of an item. Reference to a single well-defined example of the concept without reference to a specific episode is identified as specific images. Autobiographic images are seen to be special cases of the 'specific' category that is enlarged to include the involvement of the self-schema. These involve either the subject without a precise episodic reference or objects belonging to the subject.

De Beni and Pazzaglia (1995) questioned the meaning that may be given to the autobiographic image category. They suggested that there is a distinction between images that refer to a single episode in the subject's life (*episodic–autobiographic*) and those that actually involve the subject without a precise episodic reference (*autobiographic images*).

We see these terms helpful in our attempts to classify of the frames of reference identified from children's articulation of what came to mind when they were presented with a series of verbal stimuli. The notion that different frames of reference may be related to children's mathematical understanding is not only represented within the literature but it is also suggested by direct evidence drawn from children's mathematical behavior (Gray & Pitta, 1999). Therefore, it is possible that a child's tendency to respond to the stimuli in consistent ways and their ability to use the process/concept ambiguity of mathematical symbolism are probably linked. The quality of the cognitive shift, which supports an individual's recognition of the latter, is associated with the 'choice' that is associated with identifying the characteristics that form the essence of their mental representation of the activity.

The consequences of this "personal selection" process have been displayed during several of our encounters in school. Although the relative merits of choice, interpretation and enculturation may be difficult to determine, the outcome of making unsophisticated choices can lead to disquieting observations (Gray & Pitta, 1999). There is an obvious tension between the interpretation of numerical activity as the external manipulation of physical objects and the internal manipulation of mental objects (Gray & Pitta, 1997).

RESEARCH CONSIDERATIONS

A mathematical object may be seen as a theoretical construct but at issue is whether or not the frame of reference a child associates with such an object differs from that which the child associates with concrete objects identified from their conceptual labels. We will argue that consistency in the quality of the interpretation has implications for the individual's tendency to transform arithmetical processes into numerical concepts. Two questions guide the development:

- What frames of reference may be associated with children's discourse on a series of presented verbal stimuli?
- Do particular kinds dominate amongst children who reflect different levels of achievement in elementary arithmetic?

A phenomenological approach seemed be the best way to understand the frames of reference that the individuals associated with the concrete and abstract nouns they were invited to elaborate on. The work of De Beni and Pazzaglia (1995) that used high image evoking nouns reflects such an approach and it is one that we shall follow. However, since we were attempting to establish links between frames of reference associated with arithmetical concepts and frames of reference associated with image evoking nouns our item bank contained examples of both.

We apply a qualitative approach since we feel that it is the initial identification of the surface details and their possible relationship with other aspects of knowledge that is important but our attempt to investigate frames of reference is not infallible. Although the method follows a psychological perspective and involves both immediate and more prolonged reflection on the items presented, it is possible that our subjects may not be providing personally established frames. They may also be providing frames associated with descriptions, interpretations and beliefs that have been acquired through habituation or instrumental/rote learning. However, even these pieces of information have the potential to be informative if a discernible pattern can emerge in the quality of the responses but in an attempt to minimise such influences we consider children from within one school.

To establish the frames of reference associated with the presentation of given items we chose to concentrate on what it was children focus upon when they talked about objects of the real world and objects of the abstract world. Words (associated with introspection) have formed the backbone for much of the research into children's strategies in elementary arithmetic (Steffe, et al., 1983; Carpenter & Moser, 1982; Seigler & Jenkins, 1989; Gray & Tall, 1994). It is therefore an approach we use to examine the existence of different frames of reference.

Method

In designing a methodology that seeks to explore the relationship between arithmetical achievement and mental representations several different theoretical frameworks are interwoven:

- Accounting for differences in children's arithmetical behavior may be seen to have direct links with the notion of actions interiorized as concepts (Piaget, 1965), different forms of mathematical understanding (Skemp, 1976) and qualitatively different forms of mathematical thinking (Gray & Tall, 1994). The dimension presented within this study takes forward the latter by considering frames of reference associated with extreme levels of arithmetical achievement. The paper considers whether or not a disposition towards different particular frames of reference may contribute towards qualitatively different thinking about arithmetic.

- The way in which children use their knowledge inevitably falls within an information-processing paradigm. However, an investigation that seeks to discover how this knowledge is constructed and the way in which a disposition towards particular frames of reference may contribute to it, draws upon a constructivist philosophy,

* The development draws in the work of cognitive psychologists particularly that of De Beni and Pazzaglia (1994, 1995).

Our approach does not attempt to build a general model established from *common* cognitive processes but instead attempts to consider differences in behavior exhibited by children who *already display differences* because they represent different ends of a spectrum in numerical achievement.

Item Selection

Two phases guided the preparation of item clusters within the study. The first focused on the quality of the children's achievement in elementary arithmetic, and the second on the range of items selected to obtain insight into each child's general frame of reference.

The numerical items included a range of number combinations presented verbally — the children were invited to respond to these using mental methods only — and in written format to which children could respond using a written format. The intention of both was to establish achievement and to identify the strategies children applied to obtain solutions. The problems were derived from Gray (1991) and from a range of past Standard Assessment Tasks (SATs) published by the Schools Curriculum and Assessment Authority (SCAA, 1995. 1996, 1997). This item bank also formed an integral part of a comparative study examining the influence of curriculum change on children's achievement (Gray, Howat & Pitta, 2002)The range of verbally presented items included:

- Combinations to 20 such as: 3+2; 9–7; 7+6; 4+7; 17–13; 12–8
- Two digit combinations such as: 14+8; 29–6; 64–26; 27+62; 73–32, 45+57
- Three digit combinations such as: 188+267; 396–157.

The written combinations included:

- Combinations to 20 such as: 6+3; 3+5, 9–8; 15–8, 13–5, 9–2
- Two and three digit combinations presented in the style seen within particular SAT's

30	47	274	5 + 135	11+696
+57	+15	+159		
21	82	293	438 – 21	687 – 47
–15	–24	–185		

As each item was completed the child was asked to indicate what was happening in their head as it was being done. If necessary, a supplementary question or questions clarified a particular response. Thus for each child an overall level of achievement and the quality of this achievement could be established. Classifying the quality of the verbal responses to the number combinations to 20 followed those of Gray (1991) and thus the use of counting procedures (count-all and count-on together with analogous strategies for subtraction could be identified) and the use of known and/or derived facts was noted. Classifying approaches to combination in excess of 20 drew upon Oliver, Murray and Human (1990) and Gray (1994). Thus approaches were identified as sequence counting, demonstrated at its most sophisticated when children used accumulation or iterative strategies, or transformation strategies, which involved a rearrangement of the combination before the combination was attempted. Most responses to the written combinations were algorithmic but distinctions between left to right or right to left approaches were classified together with errors that occurred, for example, smallest from largest.

For the purposes of this paper, the items presented in standard paper and pencil form served the general purpose of providing a basis for establishing a child's level of achievement. In stating this we are not making any observation about the quality of thinking established from success or failure in making a response to these items. We recognise that if only the level of achievement is used as a criterion for analysis it is hard to distinguish between those who have simply demonstrated a high degree of procedural competence and those who have a sophisticated level of relational understanding. However, by associating this level of achievement with the quality of the understanding that may be determined from an analysis of the strategies used to resolve the verbally presented items it is possible to identify general differences.

Previous exploratory studies (Pitta & Gray, 1996; Pitta & Gray, 1997a) had indicated that what children chose to communicate when responding to invitations to consider a range of concrete and abstract nouns provided indications of differences in the quality of their communication. Thus two principles guided the selection of the words from which frames of reference could be classified.

Firstly, there should be nouns that denote objects that could be easily identified by the children in the real world and had shown their tendency to evoke strong mental representations (Paivio, Yuille & Rogers, 1969). The previous study had also indicated that such words could evoke different levels of communication from children. The list contained the nouns:

dog, table, dots, football, animal, furniture, ball

Secondly, there should be abstract nouns identified as conceptual labels representing mathematical ideas. Such labels could be associated with the outcome of a numerical process or the cognitive reconstruction of this process to form a new conceptual entity. They may evoke mental qualities associated with concreteness but they cannot be perceived directly by the senses. Little psychological evidence is available to establish how such labels may be determined within a framework of concreteness and associated imagery. Indeed a list of 925 words constructed with such qualities (Paivio et al., 1969) contained no such item. Here, then, we were 'in the dark'. However, drawing upon the experience of the previous studies and consulting teachers about their perception of children's experience with them the following list was constructed:

five, ninety-nine, half, three-quarters, nought point seven five, number fraction.

The sample and arithmetical achievement

Initially, 32 children within one English elementary school, eight within each year group from Y3 to Y6 were selected for the study. The eight students from each year group were subdivided into two groups of four, each group of four representing the extremes of 'high achievers' and 'low achievers' within each year. The terms 'high achiever' and 'low achiever' are used for simplicity and do not reflect any opinions on the underlying ability of the children, nor do they imply that any longer-term prognosis is being made concerning their eventual levels of achievement in mathematics. Selection depended on the outcome of school and national assessment associated with the level each child achieved in a range of Standard Assessment Tasks (see for example SCAA, 1998) taken at the age of 7+, the end of Y2, and with predictions for performance in an associated test to be taken during Y6. The later was derived from an ongoing cycle of school based tests within Y4, Y5 and Y6. Such a method of selection identified children within the lowest and highest quartiles of achievement within each of the four year groups that held children aged 8, 9, 10 and 11- years-old.

However, even though 32 children participated in the subsequent interview process, for purposes of reliability and efficiency it was eventually decided to narrow this range to 16 children, two at the extreme of achievement within each year group (identified as either "high" or "low" within Table 1). This sample was established by considering the children with the highest and lowest levels of achievement in the numerical items presented during this study.

Table 1: Numerical achievement of the sample within the study (%)

	Year 3		Year 4		Year 5		Year 6	
Mean Age	7years 8 months		8 Years 8 months		9 Years 7 months		10 years 8 months	
Class Average	58%		71%		70%		81%	
Selected Children	High	Low	High	Low	High	Low	High	Low
Child 1	87	20	97	30	90	37	100	50
Child 2	77	27	100	47	90	63	97	60

The individual level of achievement of each child that contributed to the study is seen in Table 1. Marks are given as a percentage and an indication of the class average is also provided.

Qualitative differences between the two groups were identified through the analysis of the strategies that they used to obtain solutions. With only one exception, a child from Y3, all high achievers responded to the verbally presented combinations to twenty by either knowing solutions or deriving them. There was very little evidence of knowing or deriving amongst the low achievers. The dominant solution processes were either count-on or count-back. However, across the full sample the success rate for this section of the numerical component was high, only one child having a recorded error.

Only one high achiever gave an incorrect solution to the verbally presented addition and subtraction two digit combinations. This occurred because of a counting error. In contrast no low achiever gave correct responses to more than five of the six items. Errors amongst low achievers became particularly common when they attempted to establish differences that required some form of transformation of the numbers. At least one low achieving child within each year group gave no correct solutions to the subtraction combinations. However, proficiency in sequencing, particularly in tens, enabled all except one to give correct solutions to the addition combinations.

These differences can be accounted for by the different strategies that were applied. The younger low achievers continued to apply counting procedures and their efforts were frequently accompanied by the support of perceptual items such as fingers. As indicated, this form of procedure caused great difficulties with subtraction. The older low achievers displayed a tendency to ‘chunk’ 10s and apply a sequence illustrative of the use of an accumulation strategy that involved incrementing in ones and/or 10s. Only in four separate instances did high achievers apply a similar approach. Decrementing in ones was the only approach used by a Y3 child to resolve the subtraction combinations whilst three others also demonstrated the extensive use of accumulation strategies. The dominant approach of high achievers was to use a transformation strategy that usually involved rounding up or down to the nearest ten and, if necessary, applying a corrective element.

When the two groups attempted to obtain answers to the verbally presented three digit combinations, the differences that had emerged in resolving the two digit combinations signaled an even sharper division. The strategies used by the high achievers for the two digit combinations were successfully transferred to the three digit combinations although standard algorithms dominated resolution of the visually present items. However, there were differences between their occurrence in subtraction (four out of five of the instances) and in addition (one out of three instances). None of the low achievers successfully solved any of the verbally presented 3 digit subtraction combinations.

Overall, the sample selected not only reflected differences in achievement but also confirmed the existence of qualitative differences in thinking (Gray & Tall, 1994). Low achievers emphasised the routine application of counting procedures either sequencing in ones or in tens. Frequently such processes led to errors or an inability to generalise to the next stage of difficulty, hence their well below the class average mark. High achievers demonstrated flexibility that suggested that they had an element of choice available to them. At its most sophisticated, such choice was reflected within their ability to derive a solution using an alternative range of knowledge, transform given numbers to more manageable numbers and apply routine procedures if required. Everything else failing they could always count.

Presenting the numeric and non-numeric verbal items

Each verbal cue was presented with the following instructions:

- First Response: *What is the first thing that comes to mind when you hear the word...?*
- 30 second Response: *Talk for 30 seconds about what comes in your mind when you hear the word…*

The development of a two part questioning process, a 'first response' and a 30 second response, gave the interviewee an initial opportunity to provide a reference and then further opportunity to enrich the first response with greater detail provided from a network of other relationships. Drake (1996) summarises the issues associated with this type of item delivery:

> "The generation of an image promotes the development of a trace in the brain that integrates the separate components. Thus, accessing a part of the information encoded in the memory prompts the retrieval of all the other pieces of information contained in the image" (Drake, 1996, p.7)

Two interviewers carried out the video recorded interviews, sometimes together but more often independently. Joint interviews were common at the start of each phase of questions so that an agreed procedure could be identified, evaluated and, if necessary, modified. It was also felt that this approach would contribute to the strength of discussion associated with the classification of responses.

Classifying Responses

An exploratory study (Pitta & Gray, 1996, 1997b) had indicated that the verbal expansion that came from the associations that children derived from concrete and abstract nouns could be categorised using a phenomenological approach and therefore this approach was used in the current study. However, repeated analysis of the data indicated that the classifications that were arising had striking similarities to those of De Beni and Pazzaglia (1995). Therefore, a modified version of De Beni's and Pazzaglia's quadripartite classifications of general, specific, contextual and autobiographic formed a basis for the frames of reference to be identified. In the section below we explain some of the modifications that were carried out and the reasons for doing so.

Applying the 'contextualised' category as identified by De Beni and Pazzaglia (1995) would suggest that some frames of reference could have distinctive and relational characteristics. However, such a classification did not satisfy clear distinctions observed in the quality of the responses of some of the subjects within the current study. The notion of contextualised as used by De Beni and Pazzaglia could describe a scene or a sequence of scenes. From the analysis of the children's words a description of a scene or sequence of scenes could be best described as episodic and such descriptions were most often narrated in continuous speech. Thus the notion of episodic is used, but though there is no suggestion that a frame of reference identified as such is associated with the retrieval of a specific scene from the *remembered* past, it is associated with reference to a scene. In contrast however, characteristics that may be identified as 'autobiographic-episodic' are identified as of the outcome of autobiographic episodic memory and denote a specific scene that has occurred in an individual's past life. However, there were other responses that implied the existence of a context but their structure was fragmentary; more a collection of disconnected, seemingly arbitrary, general statements which, though they originated from a particular stimulus, appeared to digress in a coherent way. Thus we included the notion of 'multi-faceted'. Here the term is used to identify the fact that the presented conceptual label acts as a stimulus for associated yet diverging responses. Taken individually they may not appear to share a clear relationship but as a whole they are sourced from the same stimulus. In a sense we may identify multi-faceted as 'multi general'.

The influence of a similar divergence that stemmed from presentation of the numerical words led to the identification of a set of equivalencies in that there could be different ways of saying the same thing. The proceptual responses showed evidence of an understanding of the mathematical concept and procedures associated with the numerical item in question. For example the item "half" could have evoked the responses, "fifty per cent", "naught point five", "two quarters", "one divided into two equal parts" etc. Frames of reference that demonstrated such an occurrence of were identified as "proceptual".

As a result of this analysis the classifications of the frames of reference identified from the children's responses were identified as "general", "episodic", "specific", "episodic-autobiographic", "multi-faceted" and "proceptual". Each had the particular properties that are identified below.

A general fame of reference was identified when the subjects description did not specify any characteristics of the noun and nor did it suggest that the subject was talking about a specific item. For example, to the item 'table', the response "*A piece of furniture*" (Y4+. 'table')[1] was classified as 'general'.

In the case of numerical items, the most frequent general response was a reference to the mathematical symbol or a very general comment, for example, 'fraction' was identified as "*Part of*" (Y6+, 'fraction').

[1] When using direct quotes from the children we shall give an abbreviation to indicate the group and the age of the child. Y4+ indicates a child within the fourth year of schooling (nine-year-old) within the above average group. Y5- would indicate a child within the fifth year of school (ten-year-old) from the below average group. Words such as "table" will indicate the item responded to.

Utilising De Beni's and Pazzaglia's prescription, a specific frame of reference was characterized through a clear example (or multiple examples) illustrative of the presented stimuli. Note that no reference was made to the specific context where one can meet this illustrative example. For example, a low achiever's response to the word 'animal' included "*... a cheetah is one, a rabbit is one, a dog is one a cat, a Labrador, Dalmatian, owl, eagle, buzzard, etc.*" (Y3–, 'animal'). In the numerical context, children's specific responses most often arose when they were asked about "number" or "fraction", for example, "*like one, two, three, four, five, six, and ten are numbers*" (Y4–, 'number') or the word 'fraction' exemplified by "*A half*" (Y6+, 'fraction').

Included within this category was also an extended description of what the item looked like or the addition of a self-schema. For example, when a child was asked about number five she volunteered that *"It's like a circle on the bottom."* (Y3-, 5), whilst another responded to the word 'ball' by saying *"I see my ball with my name on it"* (Y3-, 'ball'). The addition of such autobiographic detail in this category followed Kosslyn's (1994) notion that the inclusion of such detail is a special case of exemplar (specific) which has been enlarged by the addition of the self-schema,

Classification of an episodic frame of reference was identified when the item was associated with an 'episode' (a scene or sequence of scenes) that occurred in a specific context. The classification does not refer to elements and neither is it associated with a specific event in the child's past life. One aspect of the qualities of this classification was as an active scene narrated or described in full detail, for example:

> "Boys can kick it around and sometimes it can get lost over the field."(Y3–, 'ball')
>
> "Number five. I think of a row of numbers and light shines on number five. A light goes along and stops over the number five." (Y5–, 'number')

A productive statement that was common to the general concept identified the notion of a multi-faceted frame of reference. It was not a description of a sequential event that had a clear beginning and end but it was identified as a collection of statements that seemed to have the potential to produce new ideas. Though they had a 'general' quality, the statements diverged to produce different ideas related to the item.

> "Keeps you fit. An exciting game. Millions of fans. Important in every nation. Children and adults play it. Different types of football and balls." (Y4+, 'football')
>
> "... maths and writing. Seven you could be doing some adding or times and the number seven might come up. Seven is also played in sport on the back of a shirt, has one digit, phone number." (Y5+, 'seven').

In the context of this study, the definition of 'autobiographic-episodic' has been taken directly from De Beni and Pazzaglia. Their definition allowed for the "occurrence of a single episode in the subject's life connected to the concept" (p. 1361). Therefore, examples such as the following were classified as 'autobiographic-episodic':

> "My friend wasn't good at fractions and she had to take extra work home."(Y4+, 'fraction')

> "We have recently done reflections and they had lots of halves in them. We had to put our mirror down the side and see the rest of it. I saw lots of those." (Y4+, 'half')

The inclusion of a proceptual mental representation is additional to classifications identified by De Beni and Pazzaglia (1995) and it is unique to the numerical items. References to mathematical relationships, processes, concepts, manipulations of mathematical symbols and/or indications of equivalence were classified as such. For example,

> "Can also be a decimal and like 0.75. Not a whole. Add one third of it and you get a whole" (Y6+, "three quarters"),

> "3 parts out of 4, fraction, 0.75, more than half." (Y6+, 'three quarters)

RESULTS

Comparative Case Study: Children of the same age.

To give a flavor of the differences between the children we will consider two children of the same age who are at extremes in their level of achievement in the numerical component. Sonia is a low achiever aged nine who, although she was able to obtain correct responses to all of the combinations up to 20, she achieved an overall score 30% in the numerical component.

To obtain solutions she used a counting process in 25 out of the 29 combinations that she tried. She attempted a variety of approaches that included count-on (for example solving 7+6, 14+.8), count-all (3+2) and count back (12-8). However, where differences were relatively small count-back caused temporary problems:

> Now I have lost track. 12,11,10,9,8,7,6,5,4,4. This is the easy way [proceeding to reapply count back on her fingers]. (12–8, verbal,)

As numbers became larger and differences greater, for example in 17–13 and 29–6, these difficulties were not overcome. Sonia extensively used her fingers to support her counting when dealing with the verbally presented items but there was evidence of the use of verbal counting by allowing number words to stand in for countable items when she dealt with the visually presented items.

Sonia's overall approach to the combinations to 20 illustrated that she had constructed an abstract sense of number (Steffe et al, 1983). This allowed her to make fairly extensive use of count-on. However, there was a distinction in the nature of the unit she used to support her counting. Although there was evidence of verbal counting, Sonia generally approached every combination by re-presenting each verbal or visual representation of a number with figural objects that could be counted. Each combination appeared to suggest that a counting sequence needed to be performed. The same counting episode re-occurred for each sum with the only difference being the numbers that the individual would begin and end his/her counting. Thus to put it more clearly, the counting procedures invoked the application of a counting scene, associated with a specific context and the presented combination. Within each episode we may also see specificity in that numbers were specifically associated with fingers. However, attempts to generalise the episodes and the specific nature of the unit used to support these episodes failed as the combinations required the use of larger numbers. Her approach did suggest that she had a frame of reference associated with number combinations that was largely episodic and specific.

Malcolm, from the same class, was also nine years of age. Within the numerical component he obtained 100% and the strategies applied to the orally presented number combination were identified as known facts (all combinations to ten), derived facts (extensively used with combinations to 20) or transformation strategies.

The differences in responses that Malcolm and Sonia gave to the range of words may be seen in Figure 1.

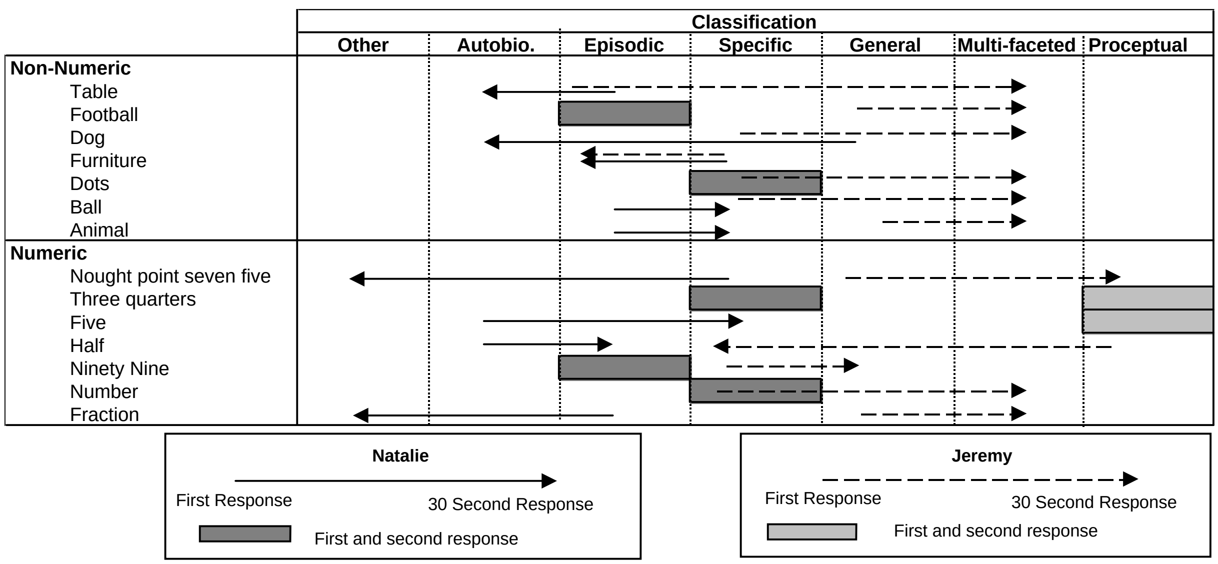

Figure1: First and second responses to the verbally presented items: Natalie and Jeremy

The order of the classifications within the figure has partly been guided by the literature (De Beni & Pazzaglia, 1995 in particular), the influence of the 30 second response and our analysis of the frequency of each classification. The articulation of a frame of reference identified as specific or episodic may start from the formation of a general frame of reference. The longer time span provided by the 30 seconds offers an opportunity for this frame to be enriched with more detail or with a network of relationships. Thus within Figure 1 we ascribe a somewhat pivotal role to the classification of responses identified as general. Those frames of reference identified from the specification of particular examples (specific) or particular episodes (episodic), and those more associated with relationships (multi-faceted) or relationship and equivalences (proceptual) are then arrayed to either side.

We also follow De Beni and Pazzaglia by taking note of their view and through analogy conjecture that episodic-autobiographic frame of reference have a different generation process:

> "[It is] not an enrichment of the general image, but a different process from the beginning. Given the verbal cue, a search takes place among the biographic memories associated with the cue leading to the choice of the one considered to be most representative"
> (DeBeni & Pazzaglia, 1995, pp.1361-1362)

It is for this reason that it is included in a position between the classification of 'other' responses — either the child's non-recognition of the item or an irrelevant comment that does not include an implicit reference to the item — and the classification 'episodic'. The relative positions of frames of reference identified as specific and episodic is solely due to the frequency with which each one occurs. Multi-faceted frames take precedence over proceptual ones not only because of their frequency but also because they may be identified from responses to either numeric or non-numeric items.

It can be seen from Figure 1 that Sonia's responses took little account of whether or not items were numerical or non-numerical. Not withstanding the items she did not respond to (nought point seven five, three quarters and fraction) her responses displayed qualities that were either general, specific or episodic.

Sonia's most common 30 second responses were episodic:

> "A ball is round and you kick it, throw it, roll it, hit it." (Ball, 30 seconds response)

> "I love dogs. I have been asking for one since I was born"(Dog, 30 seconds response`)

> "Its an age or a number… when it is your birthday… you are four and the next year you are five"(Five, 30 seconds response,)

Some responses identified as episodic indicated Sonia's general difficulty with the arithmetical component.

> "It's a sort of… sort of… sort of work. I've done it in class. It is difficult to tell all about it."
> (Three quarters, 30 seconds response,)

Such a response was in contrast to that identified as proceptual given by Malcolm:

> "It's a fraction. If you have a half of a half you need three of them to make it. Its not a whole — it's a quarter less than a whole." (Three quarters, 30 seconds response,)

Malcolm's initial responses were largely general or specific:

> "A piece of furniture" (Table, general, initial response)

> "Fraction" (Half, general, initial response)

> "Football" (Ball, specific, initial response)

During the 30 seconds Malcolm expanded his response and projected multi-faceted mental representations for half of the non-numeric items and proceptual mental representations for numerical items for example:

> "Some are older than others and they different things, rabbits jump, goldfish swim, tigers run. Some animals are vegetarians, some are predators and some are cannibals. They all live in different climates and in different habitats."
>
> (Animal, generic, 30 seconds response)
>
> "It's an odd number. Its factors are 1 and 5. There are five dots on a dice and five fingers on a hand" (Five, proceptual, 30 seconds response)

For him, naught point seven five had meaning established from the separate features of the symbolism:

> "Its Three quarters… I represented the naught by one hundred and took the seventy-five because seventy-five is three quarters of 100". (Five, 30 seconds response)

Summing up the responses to the numerical and non-numerical items we could argue that frequently Sonia's initial responses to the presented items were associated with a particular item, for example initially "furniture" was associated with the chair she was sitting on and this was then elaborated to talk about what it was made of and what could be done with it "You can put books on it on it". The frames of reference that Sonia associated with the items presented to her were linked with a specific item or framed within a particular episode.

On the other hand, Malcolm projected mainly general and specific initial responses. Given the opportunity to expand his thoughts through the thirty seconds response enabled Malcolm to demonstrate the way each non-numerical concept could represent a core idea which could be associated with related ideas and the way numerical items could be linked to a set of equivalences or processes.

The differences in Malcolm and Sonia's mathematical achievement and the quality of this achievement, as identified within the presented numerical items, was reflected in differences in the quality of the frame of reference that they apply to the presented concrete and abstract nouns. What we see within these two the tendency for Sonia to report on the items in largely an episodic and/or specific way. Although the evidence suggests that Malcolm also demonstrates these characteristics, he does so with less frequency. Given the opportunity to expand his thoughts he raised them to a more sophisticated level of thinking in that he identified associations and relationships which would seem to signal that the presented item could be a "core" concept that has the potential to lead him towards other concepts.

Identifying trends in frames of reference.

Figure 2 represents the distribution of the sixteen children's responses to the numerical and non-numerical nouns. The order of construction is similar to that of Figure 1—the classification of general has the other classifications arranged above and below it.

Looking at a whole group can mask some of the qualities that we may see with individuals, however, as Figure 2 indicates, some general patterns emerge from the analysis of the full sample responding to all of the items.

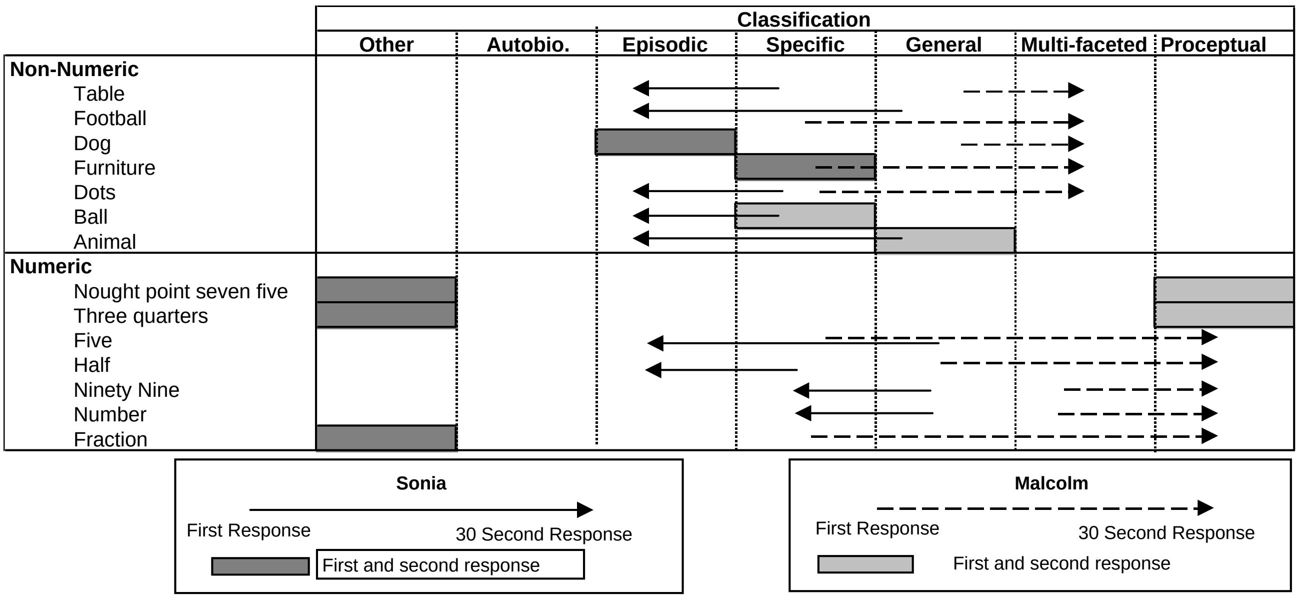

Figure 2: First and second responses to the verbally presented items: Sonia and Malcolm

It can be seen that low achievers provided a lower proportion of multi-faceted and proceptual responses than the high achievers. It is also evident that episodic and specific responses are common within both groups, the former to a much lesser extent amongst the high achievers than the latter. There are also indications that the general quality of the high achievers responses change between the initial response and the thirty seconds response.

Though Figure 2 provides us with an overall sense that some differences exist, a closer examination of the separated numerical and non-numerical nouns begins to present a clearer picture of these differences as shown within Figure 3 and 4. Interestingly the first responses of the two groups to the non-numerical items have a strong similarity particularly in the frequency of occurrence of episodic responses.

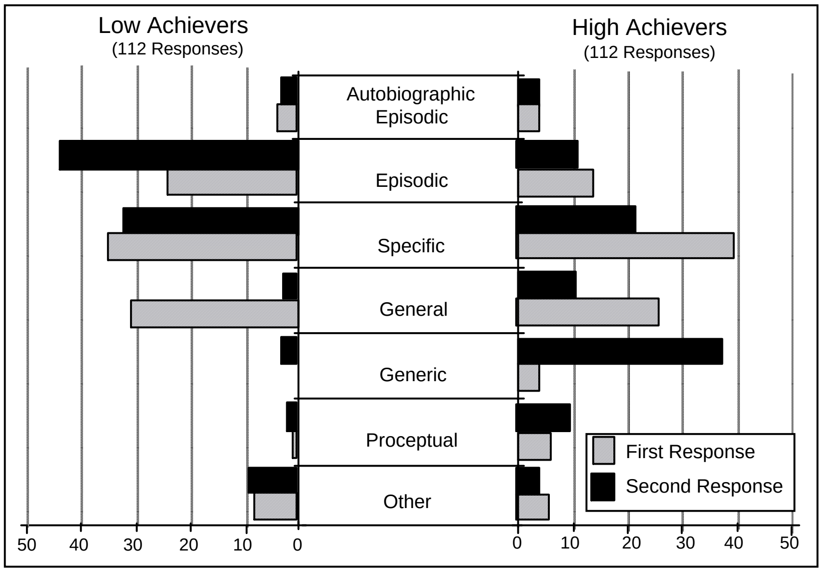

Figure 3: Distribution of frames of reference over the first and 30 second responses of high and low achievers.

Almost one half of each group's first responses are classified as specific but as we consider the second responses we see that the frequency of the high achievers episodic, specific and general responses are reduced by at least a half, whilst multi-faceted responses increase ten fold. Low achievers on the other hand maintain the frequency of specific responses and almost double the frequency of episodic responses.

Looking at Figure 4 we may see that all of the classifications are represented in different proportions by each of the two groups of children. What is noticeable is the frequency associated with the classification 'other'.

As indicated earlier, the younger children were not overtly familiar with decimal representations but also several of the low achievers did not know what to say about 'three-quarters' or 'fraction'. The general picture that emerged from the consideration of the non-numerical items also emerges here. General, specific and episodic classifications dominate the low achievers' responses. Also noticeable is that the frequency of the latter doubles on the second response.

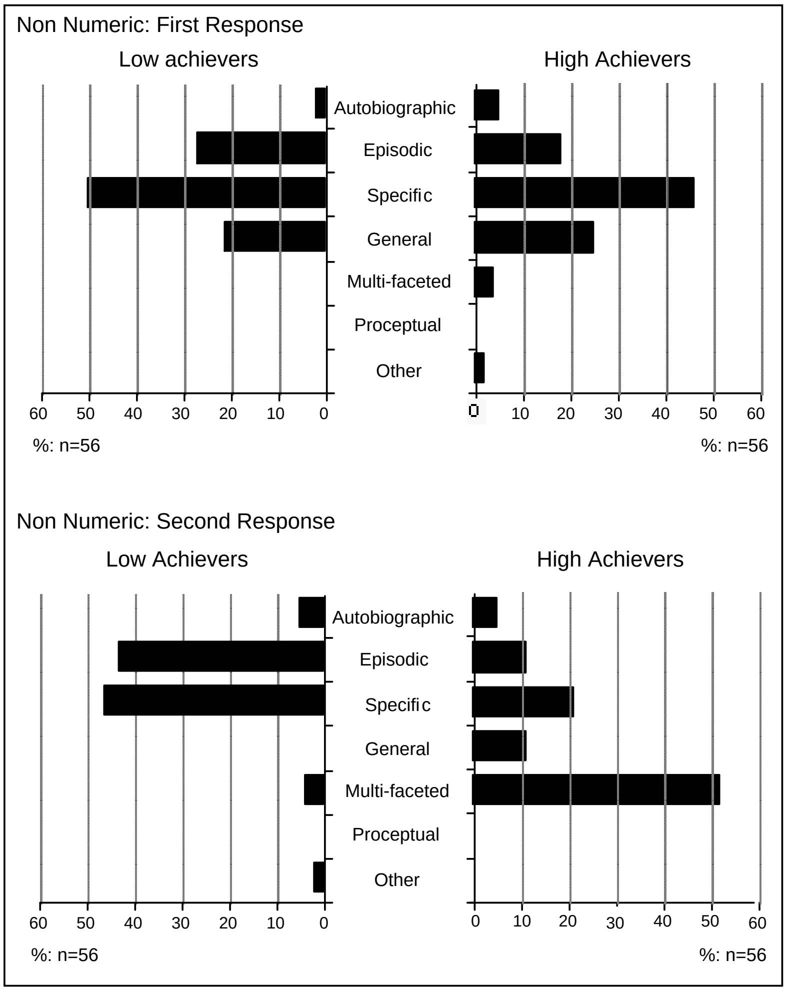

Figure 4: A comparison between the first and second responses of the high and low achievers to the non-numeric items.

Specific and episodic considerations also applied when we considered the strategies Sonia applied to obtain solutions to the number combinations.

Though specific and general responses account for almost two thirds of the high achievers' first responses their frequency declines to account for a third of the 30 seconds responses whilst multi faceted and proceptual responses increase almost three fold.

SOME GENERAL CONSIDERATIONS

Within both the numeric and the non-numeric sections children provided a one word 'general' frame of reference during the 30 seconds response. For example:

“Decimal.” (Y4+, ‘naught point seven five’)

“Game.” (Y6-, ‘football’)

However, given the opportunity to expand this frame, some, particularly the low achievers, frequently used the opportunity to either build around one idea or to provide multi-examples to give the sense of an item.

> “... a cheetah is one, a rabbit is one, a dog is one, a cat, a Labrador, a Dalmatian, owl, eagle, buzzard, etc.” (Y4-, specific, ‘animal’)

> “There are thirty three worlds or countries, thirty three bins, thirty three doors, thirty three houses and some people are thirty three.” (Y4-, specific, Thirty-three)

It may be seen from the examples above that this child provided responses to the non-numeric items and the numeric items which were qualitatively similar.

The additional time allowed for 30 seconds response also gave some, again particularly the low achievers, an opportunity to build very vivid stories around the words.

> “You sit at a table - tuck in your chair and you can have dinner at Christmas or a nice party and you can have all nice food and a table-cloth on it.”(Y3-, episodic, ‘table’)

An appraisal of Figure 3 and 4 can suggest some general conclusions. There is a strong similarity between the low and high achievers in presenting a first response to non-numeric items. However, the 30 seconds response provided the low achievers with an opportunity to expand on their initial thoughts by providing qualitatively similar examples, adding further description to the idea originally presented or through associating the item with an episode from past experience. The thirty seconds response was an opportunity for high achievers to associate the initial idea with other related ideas. However, we should not lose sight of the fact that amongst the high achievers approximately one third of the 30 second responses were episodic or descriptive. These characteristics were not the feature of any one particular age group and nor of any individual but were demonstrated within the responses of almost every child together with responses that were general and multi-faceted. This suggests that the frames of reference of the high achievers may have multi-faceted characteristics whereas those of the low achievers are more descriptive in that they focus on either expanding or describing a specific item or upon episodes associated with that specific.

The frequency of an episodic frame of reference amongst low achievers in the thirty seconds numeric phase contrasts sharply with the diversity of frames of reference of high achievers. A frame of reference for numeric items dominated by episodic characteristics could explain the low achievers’ tendency to place an over-reliance on counting.

Multi-faceted characteristics with less emphasis on the episodic were a feature of Malcolm responses, but it can also be seen as an overall feature of the response of those who were identified as high achievers. With the exceptions of one Y3 child and one Y5 child all of these provided evidence of a frame of reference that could be identified as proceptual. In contrast only one low achiever provided a response that could do the same.

The distribution of different frames of reference displayed by the two groups of children suggests that there are qualitative differences between them:

- The frames of reference of ‘low achievers’ are largely general, specific and episodic, the latter two characterised by descriptive attributes.

- ‘High achievers’ project a frame of reference which is multi-faceted and/or proceptual but they possess ‘general’ and ‘specific’ characteristics that resonate with the concrete or abstract nature of the item under discussion.

In essence, we see that the frame of reference of the low achievers does not show substantive change. No matter whether faced with numerical or non-numerical items, they consider them in the same way. Their ‘expansion’ of an initial general characteristic is epitomized by specific and/or ‘episodic’ frames. High achievers, on the other hand, appear more able to go either way. Whilst they can ‘expand’ a general frame in a specific or episodic way, they can also supply the multifaceted and proceptual qualities that indicate that their frames of reference are relational.

An examination of the general strategies used by the high and low achievers to solve the number combination to 20 illustrates the general differences that may be observed between the children. Table 2 provides a collective summary of the use of a variety of strategies that the children used to solve these problems.

Table 2: Comparison of the percentage frequency of the strategies used by high and low achieving children to solve the visually and verbally presented arithmetical combinations.

		Known Facts	Derived Fac	Count-on	Count-all	Count-back	Count-up
Verbal n=56	High Achievers	68	27	5	0	0	0
	Low Achievers	11	11	55	5	18	0
Visual n=80	High Achievers	80	11	4	0	0	5
	Low Achievers	28	10	46	3	13	0
Overall Percentage	High Achievers	75	18	3	0	0	4
	Low Achievers	21	10	50	4	15	0

Table 2 shows that the high achievers can recall the solutions to the majority of the combinations and that those they cannot recall they derive from combinations they do know. In contrast, low achievers, particularly the younger ones, rely extensively upon counting and interestingly none apply a count-up approach to resolve subtraction combinations. Where they do use counting subtraction is seen as the inverse of addition, count-back is used extensively. The evidence of the use of derived facts is fragmentary and largely restricted to one Y6 child who made extensive use of combinations that made ten, to obtain solutions to, for example, 4+7 (7+3+1) and 7+6 (7+3+3). Without this contribution the overall percentage of derived facts used by low achievers would be halved.

Overall then, with the noted exception, we see that the low achievers relied extensively upon counting episodes supported by the use of a specific counting unit — fingers. In contrast the high achievers largely recalled number facts to 20. However, drawing immediate conclusions from such an approach is difficult. We are not easily able to identify whether or not the fact is the outcome of rote learned knowledge or the basis for relational thinking. The evidence to provide an answer to this question only comes when we consider the way the knowledge may be used to identify the solution to combinations that are not known or the way it may be used to solve more difficult combination. We suggest that such evidence came from the children's efforts to solve the two and three digit combinations, particularly those that were presented verbally since many of the visually presented combination were solved using a standard algorithm.

The major distinction between the high achievers and the low achievers when attempting the two and three digit combinations presented verbally was the continued emphasis on counting that remained amongst the low achievers. However, such were the difficulties associated with this approach that children within Y3 and Y4 were not required to attempt more than two combinations. Sonia relied on counting. She attempted to count back 6 from 29 and managed to reach 27 but then made a simple guess to give answer of 18. She seemed to recognise the difficulties that her reliance on counting and the use of fingers as a counting unit were causing. For example, when she tried 27+62 she indicated that she "didn't know how to do it on fingers because I had to go up to six tens".

Overall counting brought very little success for the children. A Y5 child attempted an accumulation strategy for 27+62 using his fingers to count in ones from 62. He gave the answer as 94. Only when children were relatively proficient with such a strategy or provided evidence that they were attempting a transformation strategy, did there seem to be a chance of success. However even here there were problems since frequently the children's efforts to sort out one aspect of the combination led them into some confusion in what they were trying to do with another part. A Y6 child indicated that the solution to 73–32 was 47:

> Thirty away from seventy is 60, 50, 40. Then I just took two out of three to make 47.

Whereas collectively the low achievers only provided nine correct solutions to the verbally presented combinations high achievers provide forty eight (out of a possible fifty six). Counting was applied in only two instance; these by a Y3 child attempting 29–3 and 27+62. In both instances the correct solution was give. Transformation strategies accompanying the use of derived facts in almost 70% of instances. Again however, particularly with the three digit combinations, there were instances where children actually forgot what they were doing.

CONCLUSIONS

Other theories that may be closely related to the findings within this study fall into two groups: those associated with cognitive development from the standpoint of mathematics education and those associated with psychological research into the use of mental representations. Within the former should be placed theories associated with notions of interiorisation, encapsulation, reification and the notion of the proceptual divide. In the latter we must consider those associated with imagery and different kinds of mental representation.

It is generally accepted that the development of early arithmetic evolves from interaction with the environment, new knowledge been constructed by the learner through active methods. It was Piaget's belief that reflective abstraction was the key to the process through which the actions associated with the active methods were projected to thought. This requires the ability to concentrate the mind and give careful thought to an object, an action and an idea. This may require filtering out or temporarily ignoring irrelevancies so that ideas may be separated from their context. Of course, the ability to filter or ignore would suggest that superfluous properties are recognized. It may be that some children are not easily able to recognize those properties and actions that are important and compare them to those that are unimportant.

Abstraction involves the construction of relationships between and amongst objects and reflection upon the interrelationships of the actions on them. The results presented here suggest that individuals have the potential to interpret quite differently the objects that are acted upon. We would suggest that high achievers seem more capable of looking at the objects, mentally put aside their general and specific characteristics to look through them and place an emphasis on their more intrinsic qualities and their relationship with other objects. Such a disposition, which temporarily subsumes the descriptive and focuses on more relational characteristics, would seem to better support the construction of number concepts.

In any context that involves an action on objects, the individual has the possibility of attending to different aspects of the situation. Indeed this is a theme that Cobb, Yackel and Wood (1992) see as one of the great problems in learning mathematics, particularly if learning and teaching are approached from a representational context. We suggest that in their search for substance and meaning based upon descriptive aspects characterised by specificity and episodes, children who have difficulty with elementary arithmetic are disadvantaged right at the start of their mathematical development. However, it may be a disadvantage that does not make itself apparent in the earlier stages of numerical development. The use of counting may illustrate this point. The greater majority of young children count (and so do many adults). For a young child counting can be seen as part of a stage in concept development. However, an older child's extensive reliance on counting may be the result of necessity. At issue is whether or not we may be able to distinguish which outcome is probable.

Frames of reference that are dominated by specificity and episodic activity may be more easily associated with empirical and pseudo-empirical abstraction. The former is more strongly associated with geometric development. The latter, may lead to a form of procedural competence that can bring success within a well-rehearsed area of content. However, it can also be associated with procedural misinterpretations.

This study is suggesting that individuals internalize different things that are manifest as a result of their different frames of reference. It follows that in some instances, active methods, though they are universally recognised as having the potential to lead to the encapsulation of numerical processes, may prove to be a potential source of longer term difficulty for some. For these individuals it may not be easy to abstract the results of actions because it may be difficult to step out of the frames of reference that are more strongly associated with specificity and episodic characteristics. It may be that a tendency towards frames of reference that are dominated by such characteristics may make the very notion of encapsulation difficult if not impossible. To encapsulate a numerical process into a numerical object there must be a recognition that an object, detached from the 'real' objects and the associated actions can exist and that it can be used as basis for more sophisticated actions.

Though high achievers recognize that the actions exist, can describe their purpose, and name them, their focus can be wider. They appear able to temporarily put aside fundamental actions to consider the more abstract qualities of objects interiorised from a co-ordination of these actions. They can think in terms of encapsulated process or reified procedures. However, they may not be just compressing and squeeze everything down but they may be actively taking out some components. It was 'actively taking out' and then attempting to enhance mathematical relationships that guided a small teaching experiment with a child whose mental representations were analogues of real objects (Pitta & Gray, 1997). Such an approach needs to be considered further if children are not to establish frames of reference in elementary arithmetic that may be the source of later difficulty.

Qualitative differences in the interpretation of numerical symbolism have led to the notion of the proceptual divide (Gray & Tall, 1994). The evidence within this study suggests that a closer look at the frames of reference that influence a learner's development may provide a clearer picture of the causes of this divide. The disposition of the low achievers within this study to articulate the descriptive aspects of a range of items would seem to be indicative of the tendency to rely upon procedures that are fundamentally episodic.

By its very nature the study seeks to identify common forms of behaviour exhibited by children at extremes of arithmetical achievement. This is not to be interpreted as suggesting that all children at these extremes will behave in the ways presented. In supporting the hypotheses in the study we do not wish to lose sight of the individual and neither do we wish to loose sight of children with the wide range of achievements that have not been directly investigated. Translating what may seem to be the extreme cases reported within this paper across the full range of the children that we meet is not easy. It is evident that within any spectrum it is the extreme cases that may make the most interesting ones. However, these extremes may also indicate that within the very large group that lie between the extremes there is the strong possibility of meeting less clear cut differences.

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