An exploratory study of undergraduate students' perceptions and understandings of indirect proofs

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AN EXPLORATORY STUDY OF UNDERGRADUATE STUDENTS' PERCEPTIONS AND UNDERSTANDINGS OF INDIRECT PROOFS

by

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The nature of mathematical proof, its components and different methods, are critical elements in understanding many mathematical activities. Yet, the constructing and understanding of proofs are not acquired spontaneously.

This study uses qualitative methods to explore and clarify undergraduate students' perceptions and understandings of various aspects of indirect methods in different mathematical contexts, and attempts to explain the reasons that may inhibit understanding of many of those aspects.

Data were gathered from 24 researcher-designed, task-based, semi-structured interviews with 12 post-Calculus II students majoring in sciences and/or mathematics, including pre-service high school teachers, and were analyzed using analytic-induction methods.

Some of the findings from this study are:

a) Most participants' thought processes were centered mainly on direct reasoning, and they demonstrated a strong tendency to view proofs as using direct methods. They were inclined to allocate the assumptions and the conclusion in a proof to the hypotheses and the conclusion in a statement, respectively. Thus, they did not recognize a proof of the contrapositive statement. In addition, some students did not correctly distinguish the hypothetically false assumption needed to start a proof by contradiction.

b) Most participants' understandings of the process of a proof were limited to its surface structure and/or its explicit semantics. They lacked an understanding of the deep structure of the method of contradiction. Some students' perception of contradiction was limited to the method's explicit procedural process, which was not sufficient for them to have a relational understanding of its indirect process in different settings or contexts, such as searching for counterexamples. Furthermore, the power of contradiction was not perceived and maintained as irrefutable by some students.

c) Students tended to use intuition in order to find counterexamples, or solve existence/non-existence problems. Indirect processes were not necessarily viewed as tools for exploring the truth of a statement or for gaining insight into a situation.

Some implications for the teaching of indirect proofs are discussed, and suggestions for future research on indirect methods are provided.
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[Proof by Contradiction] is one of a mathematician’s finest weapons. It is a far finer gambit than any chess gambit: a chess player may offer the sacrifice of a pawn or even a piece, but a mathematician offers the game.

G. H. Hardy (1877-1947) in *A Mathematician’s Apology* (p. 94)
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CHAPTER 1

INTRODUCTION

Proof is the idol before whom the pure mathematician tortures himself.  
N. Rose (1988)

The Problem

Over the past two decades, many mathematics departments around the country adopted introductory courses that teach basic proving techniques in mathematics (Smith, Eggen & St. Andre, 1983; Bloch, 2000). Despite all the efforts, many undergraduate students still do not or can not make the transition necessary to understand fully the characteristics of formal proofs and their intended roles in mathematics instruction (Moore, 1994). The purpose of constructing proofs for the students will not go beyond the expectations of their instructors and course requirement (Balacheff, 1991a). Since students develop a sense of mathematical proof and its value from their experiences in mathematics classrooms, Moore expresses concerns about the lack of integration of proofs into school and undergraduate curricula, which he believes is the source of many frustrations for undergraduates and teachers alike. Furthermore, findings in mathematics education research reveal that high school students either do not appreciate the roles and functions of formal proofs, or cannot see a distinction between empirical and formal deductive arguments in mathematics (Galbraith, 1981; Porteous, 1990; Chazan, 1993; Almeida, 2001). Other evidence from past research (Tall, 1992; Chazan, 1993; Moore, 1994; Healy & Hoyles, 1998) also shows that introduction of formal proof is a hard task for many reasons, one of which is that “the transition to proof is abrupt” (Moore, p. 249).

Consequently, many college students cannot distinguish, or find the distinction hard, between an informal argument and a formal proof and they do not realize that
inductive reasoning may not lead to generalization (Lewis, 1986; Martin & Harel, 1989; Goetting, 1995; Saeed, 1996). Moreover, most college undergraduate students (Almeida, 2000) and experienced in-service high school mathematics teachers (Knuth, 1999) do not see the role of proofs as the mathematics community intends.

Current National Council of Teachers of Mathematics curricular reform documents (NCTM, 2000) emphasize the importance of explanation and justification using deductive reasoning that demonstrate the nature and role of proofs in mathematics. Further, the goal of many college mathematics courses is to train students to acquire the cyclical process of mathematical thinking, reasoning and proving strategies similar to those of working mathematicians. This aspect of mathematics however is not acquired spontaneously because of the pedagogical processes and cognitive barriers (Tall, 1992; Tall 1998) involved in acquiring a full understanding of this cyclical process, and therefore appreciation of what constitutes a proof in mathematics is complex. Healy and Hoyles (2000) remark that:

The process of proving is undeniably complex, involving a range of student competencies—identifying assumptions, isolating given properties and structures, and organizing logical arguments—each of which is by no means trivial. (p. 396)

Past studies (Bittinger, 1969; Bell, 1976; Galbraith, 1981; Schoenfeld, 1985; Lewis, 1986; Balacheff, 1991a) indicate that “the ability to read abstract mathematics and do proofs depends on a complex constellation of beliefs, knowledge, and cognitive skills” (Moore, 1994, p. 250). Harel and Sowder (1998) express concern that some college students can never acquire the advanced proving skills that are necessary for them to appreciate the importance of evaluating a conjecture.
The nature of mathematical proof, its components and different methods, are critical elements in understanding most mathematical activity. Yet many students involved in college mathematics courses do not seem to recognize proofs as valuable and indispensable “internal activities” for promoting deeper understanding of mathematical concepts and theories, but as “external” activities (Almeida, 2000, p. 871). Furthermore, they think, “formal proof is irrelevant to processes of discovery or invention” (Schoenfeld, 1992, p. 359). Greeno (1994) points out that for students to learn mathematics without appreciating the role of proof would be an “impoverished” experience (p. 274).

Following Polya (1954, 1962), many research projects probing the nature of proofs have focused on students’ behaviors in discovering, conjecturing and convincing themselves of mathematical truths. In this same spirit, more recent studies (Chazan, 1993; Kynigos, 1993; Healy & Hoyles, 2000; Furinghetti, Olivero & Paolo, 2001) emphasize the context of the role of a dynamic geometric environment in which students explain and convince themselves of certain geometric facts as alternative ways to enhance understanding or construction of proofs. “They do not, however, address questions of understanding roles of proof in mathematical epistemology and discourse” (Greeno, 1994, p. 273), because many of those studies fall short of probing students’ understandings and beliefs about the role and characterization of valid proofs. For example, Chazan in his study used Supposer (geometry software) as a tool for explanation of facts but did not explore its influence on and role in the enhancement of students’ understandings of the characteristics of a proof. Students need to be aware of
what producing a mathematical proof means before they are engaged in its construction (Bell, 1976; Greeno, 1982).

Thus, it is essential that researchers and instructors understand what perceptions students have about the nature of proof and its role in a mathematical activity before they reshape curricula that teach students how to construct one. Students’ perceptions in mathematics influence the nature of their learning and practices, thus their proof perceptions will influence their understanding of proof processes (Moore, 1994). Therefore, in order to shed more light on the issue of students’ understanding of proofs, this research proposes to clarify some of the students’ perceptions of the nature and characteristics of indirect mathematical proofs, and the influence of those perceptions on their understandings of construction and use of indirect proofs.

The Research Questions

Proving, using indirect proofs, that is, proof by contradiction and contraposition is typically misunderstood by many undergraduates. (In this study, proof by contradiction will not only mean the method itself but also the process, e.g., finding a counterexample or lack thereof, through which one arrives at a conclusion by means of indirect arguments.) The process of indirect proofs has the potential to reveal many misconceptions and handicaps students may have about proofs in mathematics. Harel and Sowder (1998) contend that one of the cognitive forces influencing students’ thinking of proofs is their difficulty understanding proof by contradiction. The process of proof by contradiction “gives rise to feelings of frustration and bewilderment” and despite its apparent simplicity, to many students it is inaccessible when they first encounter it (Leron, 1985, p. 321).
Proof by contradiction in mathematics is a very useful tool and in some cases, it is indispensable. According to the duality principle in set theory, any statement can be “dualized”. Many mathematical concepts are defined as “being not something else” (G. St. George, personal communication, July 7, 2003). For example, irrational numbers are defined as being not rational. Rational numbers possess specific characteristics that irrational numbers do not possess. Thus, to prove a number rational, one needs to show its characterization that is specific for rational numbers. However, to prove a number irrational, one does not have any kind of characterization to work with. So, in this case indirect methods can be very effective, because one only needs to show why an irrational number cannot be a rational. Proof by contradiction is also an indispensable tool to prove the infinitude of prime numbers, because there is no known way of constructing them.

To prove a conjecture true with a direct argument in and of itself is not an easy task for average students; however indirect arguments sometimes can be easily found to help them understand why the conjecture is true. “To show that a statement is true, it is sometimes easier to show that it is impossible for the statement to be false.... This approach is useful when it is difficult to start a direct argument” (Billstein, Libeskind, & Lott, 2001, p. 32). While it is hard to argue against this statement, especially when the emphasis is on the word “sometimes,” past research (Lewis, 1986; Goetting, 1995; Saeed, 1996) that has dealt with students’ preferences and understanding of proofs in general, has indicated that most students find indirect arguments non-convincing. Also, if given the choice, they prefer direct proofs to indirect ones even when the indirect proofs presented to them are easier to construct and understand (Saeed). “Indeed, direct proofs are usually more informative and more intuitively understood” (Cupillari, 1989, p. 19).
The validity of indirect proof methods does not call for further investigation other than understanding the logic behind it, but the argument in a proof by contradiction leaves many doubts in students' minds about its proving power. Thus, students look for and find further justification in direct proofs when provided.

A proof by contradiction of an existence theorem shows that there is no counterexample by virtue of the logic embedded in the method. However, "mathematical learning of a proof is based on the learners' construction of a corresponding mental entity," which is not real in a proof by contradiction because it operates in a false world (Leron, 1985, p. 323). Since students find the absence of a counterexample hard to believe because of numerous possibilities (Balacheff, 1991b; Goetting, 1995), the method leaves many students in doubt about the existence of counterexamples. Tall (1995) contends that since proof by contradiction fails to construct a mathematical object that hypothetically exists, it creates a cognitive tension. And according to Balacheff, the experience of a contradiction in the developmental process through which students construct their knowledge is likely to provide a cognitive disequilibrium.

"Proof by contradiction is an essential element in formal mathematics and needs to be addressed, even though it involves significant cognitive difficulties. These difficulties are more subtle than is often assumed" (Tall, 1998, p. 8). To date, there are few studies or accounts in mathematics education literature that deal solely and deeply with undergraduate students' difficulties in understanding certain aspects of indirect processes. A systematic and comprehensive study is needed to investigate students' understandings of different aspects of indirect methods. Martin and Harel (1989) suggest that in order to fully explore and comprehend students' difficulties in understanding of
proofs, in-depth interviews need to be conducted. Therefore, the proposed study uses qualitative research methods to explore undergraduate students’ perceptions and understandings of different aspects of indirect processes in mathematical problem situations.

Finding out students’ indirect proof perceptions will not only provide an insight into their own way of thinking and understanding but also will help researchers and curriculum developers tackle the issues of misconceptions more aggressively and fruitfully. Such research also will help instructors to transmit effectively the intended nature of indirect proofs in classrooms. Since “the context [and environment] in which students meet proofs in mathematics may greatly influence their perceptions of the value of proof” (Alibert & Thomas, 1991, p. 230), instructors can review students’ convictions in a new setting that can result in better classroom presentations of the subject of proof in mathematics. Also, instructors can help make the transition from concrete to abstract mathematical thinking easier for students by identifying and removing inhibiting factors.

Thus, the focus of this study is to describe and explain some of the students’ perceptions of the nature and characteristics of indirect proofs and the effect of those perceptions on their understanding of the construction and use of indirect proofs in the context of mathematical problem situations. The following research questions are investigated.

1. What is the nature of undergraduate students’ perceptions about proofs in establishing the truth of a mathematical statement?

   a) What approaches or activities do they attempt in order to establish the truth of a mathematical statement and the validity of its proof?
b) What is the primary focus of their attention in proofs?

2. What are undergraduate students’ difficulties in understanding the aspects and characteristics of indirect proof processes?
   a) How do they make sense, if at all, of an assumption and its contradictory results in a proof by contradiction?
   b) How well can they judge the validity of an indirect argument?
   c) How can those difficulties in understanding indirect proof processes be explained in terms of their behaviors and perceptions of indirect proofs?

**Definition of mathematical terms**

*Converse* of the mathematical statement \( P \Rightarrow Q \) is \( Q \Rightarrow P \). The statements are not logically equivalent.

*Counterexample* is a case or an example that disproves the universal conditional statement \( P \Rightarrow Q \). (It refutes the truth of \( P \Rightarrow Q \) as general statement.) The only time the statement \( P \Rightarrow Q \) is false is when \( P \) is true and \( Q \) is false. Thus, a counterexample is a case that shows \( P \) true and \( Q \) false.

*Deductive reasoning* is a form of argumentation that uses a succession of established (true) assertions according to rules of logical inferences in a coherent manner to produce a new fact. The assertions may be mathematical definitions, assumptions, axioms (the truth of which can only be assumed) or preliminary statements that have been proved previously. In mathematics, deductive reasoning is one of the established methods of formal proof because it eliminates “the need for recourse to intuitive evidence and human judgement, both seen as potential sources of serious error” (Hanna, 1990, p. 6). “Intuition is fallible in principle; rigor is fallible only in practice” (Hersh, 1993, p. 395).
Direct reasoning is an argumentation strategy used to prove $P \Rightarrow Q$ true by assuming that $P$ is true and arriving, in a straightforward manner through manipulations on $P$ and other known facts, that $Q$ is true. If both $P$ and $Q$ are true then the implication is true.

Empirical or inductive reasoning uses inferences to generalize a phenomenon from sufficiently many non-discursive special cases, observations or experiences. In mathematics, this method at best may suggest a conjecture but does not establish its universal truth.

Indirect reasoning is an argumentation strategy commonly used when attempts to use direct reasoning fail to prove $P \Rightarrow Q$ true, possibly because of lack of information or resources (such as constructing infinite number of primes). The assumption that $P$ is true may not be sufficient to deduce $Q$ within the time or space limits available for direct reasoning. Indirect reasoning can also be used to search for or show existence or non-existence of counterexamples. Two commonly known methods of indirect reasoning are proof by contradiction and proof by contraposition.

Inverse of the mathematical statement $P \Rightarrow Q$ is $\sim P \Rightarrow \sim Q$, where $\sim P$ stands for “not $P$”. The statements are not logically equivalent.

Proof by contradiction is a method of indirect reasoning in mathematical proofs. Its main scheme is to assume that $P$, in a mathematical statement $P \Rightarrow Q$, is true and that the conclusion $Q$ is false, and then arrive at some contradictory or impossible result. This means that $P \Rightarrow Q$ is logically equivalent to $\sim(P \land \sim Q)$ where the symbol “$\land$” stands for conjunction “and.” The logic behind the method is that $Q$ can either be true or false. The strategy is to eliminate the possibility that $Q$ is false by arriving at a logical contradiction.
There is no foolproof method for knowing ahead of time where the contradiction arises, but if the assumption that $Q$ is false leads to a contradiction, then it must be an invalid or untenable assumption. The only other possibility is to conclude that $Q$ is true. As Sherlock Holmes (Doyle, 1960) would put it, “Eliminate all which is impossible, then what remains however improbable must be the truth” (p. 315).

Proof by contraposition is a method of indirect reasoning to prove $P \Rightarrow Q$. This proof strategy assumes $\sim Q$ and seeks a conclusion $\sim P$. Logically, $P \Rightarrow Q$ is equivalent to $\sim Q \Rightarrow \sim P$.

The process of proof by contradiction and that of contraposition share a common aspect and yet each has a different agenda. To prove $P \Rightarrow Q$ true using a proof by contradiction one assumes $P$ and $\sim Q$ and works towards obtaining a contradiction. However, as mentioned in the definition, there is no foolproof approach to knowing initially where or how a contradiction arises. On the other hand, to prove $P \Rightarrow Q$ true using a proof by contraposition one starts assuming $\sim Q$ and seeks the desired conclusion of $\sim P$, thus contradicting a known statement, $P$. Unlike in proof by contradiction, the contradiction in proof by contraposition is initially known. Put differently,

The contrapositive method can be thought of as a more “passive” form of contradiction in the sense that the assumption that $P$ is true passively provides the contradiction. In the contradiction method however, the assumption that $P$ is true is actively used to reach a contradiction. (Solow, 1982, p. 72)

Definition of terms related to learning and understanding

Compartmentalization (Vinner, Hershkowitz, & Bruckheimer, 1981; Vinner & Dreyfus, 1989) is a phenomenon that occurs when a person has two different, disconnected and potentially conflicting schemes in his/her cognitive structure. Different
situations stimulate different schemes. Due to this phenomenon, sometimes a given situation may not activate the scheme that is most relevant to the situation. “Even if students do have the required knowledge to check the result, frequently they do not use it or see its relevance to their present context” (Vinner, Hershkowitz, & Bruckheimer, p. 73). Compartamentalization is most likely to occur when a person is unaware of his/her thought process due to lack of reflection on that process.

*Concept Image* is defined (Tall, & Vinner, 1981; Vinner, 1983) as the set of all mental pictures or cognitive structures that a person associates with a certain concept and all of its properties and processes. The concept image plays an important role in the person's judgment of processes and cognitive tasks. There are many factors that determine the concept image. Some factors may contain seeds of future conflict. Thus, the concept image is subject to change with the new experiences of the person and his/her conceptualization of the external representations of the mental picture.

*Instrumental understanding* (Skemp, 1987) is a form of understanding possessed by someone who uses mathematical rules without knowing the reasons why they work. A person who has an instrumental understanding of a method (or a rule) is not aware of the overall relationship between successive stages of the method and the final goal. An instrumental understanding of a mathematical concept is not sufficient for a person to implement it in different and unfamiliar contexts or situations. “The mental structures acquired by instrumental learning have limited adaptability” (p. 169).

*Internalization* of a scheme is a process by which a person, after using a certain approach to a problem repeatedly, comes to the realization of its success and efficiency in
other contexts and situations. After a person has reflected upon this realization, he or she makes the scheme his or her own.

Relational understanding (Skemp, 1987) is a form of understanding possessed by a person who knows both what method to use in a situation and why it is used. A relational understanding of a mathematical concept is sufficient for a person to implement it in different and unfamiliar contexts or situations. This form of understanding “is more adaptable to new tasks” (p. 158). The possessor of a relational understanding can produce a number of plans necessary for getting results.
CHAPTER 2
LITERATURE REVIEW

One of the most remarkable gifts human civilization has inherited from ancient Greece is the notion of mathematical proof.

L. Babai (1992)

The purpose of this chapter is to present the existing research in the area of undergraduate students’ understandings of indirect proofs. Unfortunately, little research has been done in the past in this area. Further, only a few aspects of indirect methods were used to study students’ understandings and many of those studies were integrated within a study of understanding of proofs in general. Not much is known about students’ difficulties in understanding various aspects of indirect processes.

Before reviewing those studies, a brief historical perspective sets the stage for what led to proof by contradiction becoming a popular mathematical tool. Also, a look at some views about the functions of proofs held by the mathematical community provides the perspective underlying much of the later part of the literature and which led to the importance of proofs in mathematics education.

Mathematical proof: A brief historical perspective

During the time of Pythagoras (circa 500 BC), inductive reasoning was common among Greek scholars (Bell, 1940). Members of the Pythagorean School, who believed in whole numbers and their fractions and did not have any notion of irrational numbers, argued against the existence of such numbers as \( \sqrt{2} \) (Bell, 1940; Eves, 1990). The conflict created between their beliefs and the true nature of such numbers brought forth one of the most important contributions of Greek mathematics following the Pythagorean era, namely the concept of deductive reasoning featured in axiomatic methods (Bell,
1940; Eves, 1990; Kleiner, 1991). By the time (circa 300 BC) Euclid had published his *Elements* the trend of deductive reasoning had already overtaken that of inductive reasoning in mathematics. Consequently, rigor, which was one of the themes in the *Elements*, became common in geometry (Eves, 1990).

After Euclid’s *Elements*, there was little development in the concept of rigor in European mathematics; particularly during the middle ages (Bell, 1940). The style of presentation the *Elements* set forth did not transcend geometry, mainly because Grecian mathematics did not accept any notion of proof without a geometrical counterpart (Bell, 1940; Kleiner, 1991; Kaput, 1994). This lack of rigor in areas other than geometry also prevented the Greeks from further developments in mathematics (Bell, 1940; Kleiner, 1991; Kaput, 1994). Most of the western mathematics developed before the end of the eighteenth century was not based on as rigorous footing as Euclidean geometry was in the times of the Greeks. Leibniz’s methods of calculus that revolutionized mathematics, for example, lacked the rigor found in today’s methods of mathematical analysis (Cajori, 1991). Analogies rather than rigor were commonly used to explain or prove many mathematical concepts up until the eighteenth century.

Since the time of Euclid, the concept of proof in mathematics has also gone through many phases until finally, by the end of nineteenth century, through the efforts of Cauchy, MacLaurin, Lagrange and others (Bell, 1940; Kleiner, 1991), calculus was put on rigorous ground based on the deductive methods of the Greeks.

Along with deductive methods and rigor came a re-examination of the foundations of mathematics, and the realization that the heart of logic behind mathematical reasoning depended on the consequences of the axioms of arithmetic and
the internal consistency of the system they form. In an attempt to show the consistency of the axioms of arithmetic, David Hilbert developed a technique known as \textit{metamathematics} or \textit{proof theory} that used only concrete and direct reasoning (Bell, 1940; Eves, 1990). It excluded the use of proof by contradiction (Kleiner, 1991), which, incidentally, Euclid had used to prove the existence of infinitely many prime numbers (Heath, 1956). The method of contradiction was a serious problem in seventeenth century philosophy of mathematics. During that time, unsuccessful attempts were made by Cavalieri, Guldin and Arnauld to reformulate geometry or at least parts of it without any use of proof by contradiction. As it turned out, Hilbert's attempts were also deemed unsuccessful once Kurt Gödel proved his famous \textit{Incompleteness Theorem} (Eves), which showed that the consistency of any mathematical system can not be proved using mathematical logic because a consistent system has to be incomplete.

Therefore, with the new views of the axiomatic methods and the pursuit of consistency requirements in its structure, the method of proof by contradiction began to gain more ground and eventually became a standard mathematical tool in 20\textsuperscript{th} century mathematics. Before that, mathematics was intimately linked to the study of natural phenomena and thus it seemed impossible that contradictions or paradoxes in its methods could occur.

\textbf{Proofs in mathematics curriculum}

Early twentieth century endeavors by mathematicians towards a better understanding of their field did not go unnoticed by mathematics educators. With the advent of the “new math” reforms in the 1960s and early 1970s, there came much greater emphasis on proofs as the most important aspect of the mathematical activity in the
classroom. However, studies (Bittinger, 1968; Deer, 1969; Walter, 1972; Goldberg, 1973; Mueller, 1975) that measured the effects of such teaching practices showed no significant change in students' performances or their appreciation of proofs. The imminent demise of these reforms and the changes in the philosophy and practice of mathematics that were taking place at that time (Hanna, 1983) opened the door for mathematics educators to reassess the mathematics curriculum. As a result, emphasis was shifted towards intuition in classroom practices. This, as Hanna and Jahnke (1993) would put it, was a “shift to a pragmatic view of proof” that rejected “the normative attitude which prevailed until the seventies” (p.422). It was during this time that research studies such as Bell (1976) focused on behavior and understanding of students in proof situations. (An account of historical and philosophical perspectives of the events leading to this type of research is found in Hanna & Jahnke.) In particular, Bell found that in order for students to use proofs to resolve conflicts in mathematical discourse they needed to appreciate the purpose of proof.

A look at some views on the purposes of proof in mathematical discourse, which led to the importance of proofs in mathematics education, will provide the perspective underlying much of the research that followed later.

The purposes of proof in mathematics

Despite all the changes the concept of proof in mathematics has undergone throughout history, its definition is still a point of discussion. One of the ongoing disputes among mathematicians nowadays, for instance, is the utilitarian aspect of proof by contradiction. The two main competing mathematical philosophies that exist today are intuitionism or constructivism, founded by Brouwer, and formalism, founded by Hilbert.
(Hersh, 1997). Constructivists view the natural numbers as the basis of any fundamental notion in mathematics, so according to them a mathematical proof is valid only when it is ultimately constructed from natural numbers (Eves, 1990). This notion of proof is incompatible with that of the process of proof by contradiction, because no construction in that process is derived from natural numbers. Intuitionists, on the other hand, view a mathematical proof as an inherently meaningless game (void of context) that starts with undefined terms and axioms about those terms, and uses logical deduction to ascertain results (Eves). This notion of proof does not invalidate the process of proof by contradiction.

Despite disagreements, there seems to be some consensus on purposes of proofs among many mathematics educators as well as mathematicians. The most widely accepted purpose of a proof is to convince the reader and to verify the truth of a mathematical statement. However, many mathematicians would argue that proofs play a larger role than just convincing. For instance, de Villiers (1990) strongly criticizes this role as being "one sided." Instead, he provides a wider role for proofs in an analysis of five major functions of proofs in mathematics: verification or conviction, explanation or illumination, discovery, communication and systematization. Bell (1976) and Hersh (1993) also emphasize the first two of these functions. In addition, Bell also mentions the function of proof as a means of systematization.

Proofs verify

"The essence of mathematics lies in proofs [because] mathematical results become valid only after they have been carefully proved" (Ross, 1998, p.254). Unlike in any other scientific discipline where empirical justification through observations is good
enough, in mathematics proof plays the role of verifying a claim and has to be deductive in nature.

Proofs verify the truth of a statement but they are not "necessarily a prerequisite for conviction—to the contrary, conviction is probably far more frequently a prerequisite for the finding of proof" (de Villiers, 1990, p. 18). Polya (1954), throughout his book, also emphasizes this view that personal conviction (which depends on intuition, empirical verifications and lack of a counterexample) provides motivation for finding a proof. Similarly, Bell (1976) believes that personal conviction comes through different means that precede proof.

The function of proof as a tool for verification also transcends its other functions that are discussed next.

Proofs explain

Proofs do not just verify results; they also explain. Hanna (1990) remarks that for mathematicians a "proof is valued for bringing out essential mathematical relationships rather than for merely demonstrating the correctness of a result" (p. 8). Furthermore, since "quasi-empirical verification provides no explanation why results are true—it merely confirms [especially in cases] when the results concerned are intuitively self evident" (de Villiers, 1990, p. 20), therefore, proofs as means of explanation can provide a psychological sense of illumination, insight or understanding of relationships. For example, Kepler's laws of planetary motions that were based on numerous empirical observations and data were able to confirm future positions of the planets, but they did not answer the question of why those laws worked until Newton proved his laws of gravitation using deduction. Thus, Kepler had to check his laws against data for each
planet, but Newton on the other hand had more insight and had only to check his laws for Mars' orbit (Bell, 1940; Field, 2001).

In contrast, the growing trend of computer-assisted proofs in mathematical discourse created questions not only about their validity but also about whether they provide any insight as to why a theorem should be true (Kleiner, 1991; Horgan, 1993; Hersh, 1997). This in turn created issues about the importance of the use of proofs in the mathematics curriculum (Hanna, 1996). The trend of computer use for proofs in mathematics is known to have created a branch of mathematics called experimental mathematics. The proponents of this branch claim that its role is not to replace formal proofs but to help discover them because "experimentally inspired results that can be proved are more desirable than conjectured ones" (Epstein & Levy, 1995, p. 671). The ongoing counterargument against this claim is that experimentally produced proofs seem to utilize non-deductive methods that do not promote any understanding or illumination of relationships in their results that could help with other mathematical problems. On the other hand, if computers are trusted, mathematicians require the verification of the reasoning that motivated the program that in turn demands a mathematical proof of the algorithm utilized (Hersh).

**Proofs are tools for discovery and exploration**

Many educators, including Hanna, believe that "mathematical concepts and propositions are... conceived and formulated before proofs are put in place" (Hanna, 1983, p. 66). Therefore, she claims that deductive reasoning takes a backseat and does not necessarily contribute to the exploratory process of discovery in mathematics. On the stronger side of this argument, Lakatos (1976), drawing from Gödel's Incompleteness
Theorem, maintains that no proof is infallible and thus verification cannot rely on deductive methods. Therefore, he claims proofs are helpful tools but not central to mathematical discovery because attempted proofs can serve as a means for searching for mathematical truth.

Although there are many instances in the history of mathematics, such as Fermat's last theorem and Gauss' prime number theorem, where theorems were stated first and a proof found later, there are also instances where results were discovered only after the exploration of a proof of another result. De Villiers believes that "it is completely unlikely that some results (e.g., the non-Euclidean geometries) could ever have been chanced upon merely by intuition" (de Villiers, 1990, p. 21). Thus, in addition to building upon proven results, the proofs themselves can be used as tools for exploring and researching new horizons, which is why for many mathematicians results from computers are not sufficient.

**Proofs are tools to communicate results**

Formal proof is also an important component of communication in the mathematical community. The acceptance of the validity of a proof by mathematicians is believed to be a social process of verification through which the proven theorem becomes valid in a mathematical community (Hanna, 1983; Balacheff, 1991a; Hanna & Jahnke, 1993). This process "is based on the confidence of the mathematical community in the social systems that it has established for purposes of validation" (Kleiner, 1991, p. 311). In simpler terms, "proof is convincing argument, as judged by qualified judges" or experts (Hersh, 1993, p. 389). The reason this view is popularized is because a complete and formal proof is possible only in principle but not in practice (Hersh). A formal proof
produces the truth of a mathematical statement by attributing to it mathematicians’ epistemological value of reliability. The reliability does not come primarily from checking formal deductive arguments embedded in a proof. The requirement of logical deduction itself provides a guideline for mathematicians to communicate a result based on its proof’s “substantial” and not “formal” arguments (Thurston, 1994).

Therefore, proofs as a means of communication can transmit mathematical knowledge as well as create a forum for a critical debate not just within mathematical communities but also in classrooms (de Villiers, 1990). However, Hanna and Jahnke (1993) point out that this social aspect should be limited in schools because the process takes time and its key elements cannot be reproduced in a classroom setting. Besides, mathematicians can afford to assume many implicitly known issues about proofs, such as sparse language and logical leaps, that cannot be ignored in classrooms without hampering comprehensibility, learning and teaching process (Hanna & Jahnke).

**Proofs systematize results**

Proofs are indispensable tools to systematize or organize “various known results into a deductive system of axioms, definitions and theorems” (de Villiers, 1990, p. 20). This role of proof according to de Villiers helps to provide a global perspective of the subject of mathematics and its applications, and to identify relationships or inconsistencies across its branches. Hanna and Jahnke (1993) share this same view from the perspective that it reveals relationships and new dimensions for the justification of mathematical applications through the proven theorems, “which in the absence of proof would have remained separate” (p. 428).
Students' epistemological views on the nature and role of proofs

According to Hanna and Jahnke (1993), the teaching of proofs faces two major problems: “that of finding a proof and at the same time of conveying its meaning. Consequently, the epistemological context plays a much larger part in teaching than in the work of the research mathematician” (p. 433). The “conflict between the practice of mathematicians on the one hand, and their teaching methods on the other, creates problems for students” (Alibert & Thomas, 1991, p. 215). This view is supported by Reid’s (1995) observation that high school and undergraduate students used proofs primarily for exploration and explanation but not for verification of results. The role of proof as a means of communication and thus conviction viewed by mathematicians is not similar (Tall, 1979) to those of students in classrooms because students are convinced all too easily (Hersh, 1993). This latter aspect is clearly revealed in Balacheff (1991a). He observed that the social interactions among high-school students trying to convince each other of the validity of a solution were not the same as those of mathematicians engaged in a similar activity. For students, a solution or a proof may seem valid when there is no stronger convincing argument against it. Balacheff points out that argumentation is considered different in nature from mathematical proof: “The aim of argumentation is to obtain the agreement of the partner in the interaction, but not in the first place to establish the truth of some statement” (p. 188). In a similar vein, a classroom investigation using an open geometry problem as a tool for introducing proofs through peer collaboration did not result in any conclusive evidence of the use of proofs for communicating results or solutions among high-school students (Furinghetti, Olivero & Paolo, 2001). In another study (Tinto, 1990), it was found that high school geometry students considered proofs as
logical explanations and not as tools for discovering the truth of conjectures. Harel and Sowder (1998) contend that traditional classroom teaching of proofs imposes on the students a perception of proof that is "extraneous to what convinces them" (p. 237). Consequently, students think that proofs are not convincing arguments.

On the other hand, there is evidence that some students share mathematicians' views about the purposes of proof. In their extensive research on views of 14- and 15-year-old English students about proofs, Healy and Hoyles (2000) found that 50% of the students made references to the function of proof as verification or providing evidence, and that 35% of them also made references to its purpose as an explanation and communication tool. The researchers attributed this propensity of the students to make such references to the National Curriculum for England and Wales. However, they found that only 1% of their sample made any references to the function of proof as a tool for systematization or discovery. Leddy's (2001) study produced similar results. He observed that although high school students had difficulty constructing proofs and possessed many misconceptions about their nature, they did appreciate their value and power of explanation.

In contrast, Knuth's (1999) study of US in-service teachers' conception of proof found that they did not view proofs as a means for promoting understanding, and most of them "believed that a proof is a fallible construct" (p. 108) that is subject to a possible counterexample. Galbraith (1981) also found that adolescents did not view the role of a counterexample as a case to refute generality.
Students' understandings of proofs

Most recent research on proof focuses on students' perceptions of its roles for conviction and understanding. A large body of evidence from this empirical research indicates that most high school and college students have difficulties in following, understanding, as well as constructing deductive arguments.

Goetting (1995) interviewed 40 volunteer college students to determine their understanding of proofs. She investigated the types of arguments that students found convincing in relation to what they considered a valid proof. Three different understandings of proofs emerged from her data. One group of students considered proof to be that which verifies conclusively. This group rejected empirical evidence and considered the existence of a counterexample as valid proof. Another group, consisting mostly of secondary education students, understood proof as an explanation or a classroom exercise that verifies a range of cases and considered counterexamples or examples of existence as invalid proofs, because they lacked aspects of generality or formality as in two-column proofs. In a similar vein, Knuth (1999) found that high school in-service teachers had more difficulty recognizing and evaluating invalid arguments than they had with valid proofs. They were inclined to perceive invalid arguments as valid based on their surface structure as being general, deductive or using two-column proof. In their quantitative study, Martin and Harel (1989) also found that the deductive nature found in a proof could influence pre-service teachers to regard a wrong proof as valid.

The third group of students (Goetting, 1995), mostly elementary education students viewed proofs as personally convincing arguments that are not necessarily conclusive or general. The only conclusive evidence they accepted was the existence of a
counterexample to prove that a statement is false, which led them to believe that empirical evidence is valid proof. Martin and Harel (1989) also found that 65% of pre-service teachers accepted empirical evidence as valid proof. However, there is evidence that the number of students who believed in empirical evidence decreased when faced with an easy problem that they could prove (Lewis, 1986).

This latter view is in stark contrast with the findings of Healy and Hoyles (2000). Although empirical argument was the preferred mode of conviction, most English high school pupils were aware of its limitations and claimed that it was not considered valid proof in the eyes of their teachers. Again, the researchers claimed that these responses were shaped by the National Curriculum for England and Wales. Moreover, 62% were aware that no further verification for special cases was necessary once the generality had been proven (Healy & Hoyles). But on the other hand, Porteous (1990) found that more than 90% of adolescents would still try special cases even after they believed the general case had been proven.

Undergraduate students’ difficulties in understanding of indirect proofs

Very little research on the topic of proof tackles the subject of understanding of indirect proofs. Furthermore, many of those are integrated within the general context of understanding of proof. Only a few comprehensive studies or systematic accounts in the mathematics education literature deal solely and deeply with undergraduate students’ difficulties in understanding the indirect aspects of proving. Thus, very little is known about students’ perceptions and understandings of indirect processes.

In studying students’ preferences and understanding of different types of proof by contradiction, Tall (1979) gave two different proofs by contradiction for irrationality of
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\[ \sqrt{2} \] to 33 first year university students and asked them which they understood better on first reading and which caused confusion. One was a generic proof that assumed \( \frac{p^2}{q^2} = 2 \) and showed that the simplification of prime factorization for \( p^2 \) and \( q^2 \) in \( \frac{p^2}{q^2} \) can not lead to 2, and another was a standard proof (preferred by mathematical community) that assumed \( \frac{p^2}{q^2} = 2 \) in simplified form and then showed that both \( p \) and \( q \) must be even.

Tall’s results indicated no significant difference in the understanding or confusion between the two proofs. However, Tall attributed this result to the familiarity of the students with the standard proof. He later gave similar proofs that showed the irrationality of \( \sqrt{\frac{5}{8}} \) to 37 students and asked the same question. The results this time indicated that students significantly preferred the generic proof of the irrationality of \( \sqrt{\frac{5}{8}} \) to the standard proof and this preference was highly significant among students who had not seen either type of proof before. Tall attributes this significant preference to the use of an illustration of the simplification of the ratio \( \frac{p^2}{q^2} \) in the generic proof with an example, whereas the standard proof merely repeated the steps of that of showing irrationality of \( \sqrt{2} \) (without the even/odd terminology). In comparison of the two cases, Tall claims that students’ understanding was linked with familiarity and not the criterion of the logic in the proof. Thus, students’ claims to understand suggest an instrumental rather than relational understanding.
One of the difficulties students encounter in understanding the standard proof by contradiction for irrationality of $\sqrt{2}$ is found to be the proof of the statement, "if $p^2$ is even, then $p$ is even" that is nested within the former proof (Barnard & Tall, 1997). Another cognitive tension in understanding of the proof by contradiction for irrationality of $\sqrt{2}$ is found to be students' unfamiliarity with the prospect of proving something true by first assuming it to be false (Barnard & Tall).

Past research (Lewis, 1986; Goetting, 1995; Saeed, 1996) that has dealt with undergraduate students' preferences and understanding of proofs in general indicated that most students find indirect arguments non-convincing. Also, when given the choice, they preferred direct proofs to indirect ones even when the indirect proofs presented to them were easier to construct (Saeed). It seems that the argument in a proof by contradiction leaves many doubts in the minds of students about its proving power. Thus, students look for and find further justification in direct proofs when provided. In addition, they find direct arguments easy to understand. In agreement with the above, Knuth (1999) observed in his study that many in-service teachers found the aspects of indirect proofs in his research tool difficult to understand.

Lewis (1986) collected data from written responses and used quantitative methods to investigate advanced calculus students' perceptions of different aspects of proofs. In one situation, he investigated whether students accepted an indirect argument that showed $1 \neq 0$ (p. 164) and found that one-third of the 24 students did not accept an argument that started with a false premise. In a similar study, Saeed (1996) also used quantitative methods to investigate college students' understandings and preferences of proofs. One of the tools he used tried to find the extent to which they were convinced by a proof by
contradiction that showed the irrationality of $x - y$ when $x$ is rational and $y$ is irrational (p. 28). About 19% of the 101 students in the sample rejected the validity of a proof by contradiction, because they could not make sense of an argument with an assumption that needed to be proved in the first place. Another 34% of the students accepted proof by contradiction as a valid argument but could not explain its process with a high degree of proficiency. Similarly, 60% of the advanced calculus students in Lewis could not explain the steps of a proof by contradiction that showed $1 \neq 0$ and only 10% of those recognized the argument as indirect.

Other situations in both studies above examined the extent to which students understood a contrapositive argument $\neg P \Rightarrow \neg Q$ and its equivalence to $P \Rightarrow Q$. Four out of 24 advanced calculus students in Lewis (1986) and only 13% of the students in Saeed (1996) recognized contrapositive statements; the rest of the students could not validate the nature of the relationship. Furthermore, 34% of the students in Saeed preferred a direct argument that contained a mistake in it over an indirect one because they thought it was easier to understand. On the other hand, Lewis found that 74% of advanced calculus students preferred a direct argument to an indirect one and less than 22% showed no preference for correct proofs. One of the research questions in Goetting (1995) tried to determine how students evaluate proofs by contraposition and contradiction. She first asked the 40 participants in her study to find the truth of the statement: “For any integer $b$, if 2 is a factor of $b^2$, then 2 is a factor of $b$” (p. 190) and then gave a proof of its contrapositive. The students exhibited a strong tendency to give an argument for its converse even when some of them were aware of the distinction. When a proof of the contrapositive was shown to them, some rejected its validity because it did not prove the
converse. Others rejected it because they thought it was irrelevant to the given statement. None of the above mentioned studies however investigated why proof of the contrapositive was not recognized.

To determine how students evaluate a proof by contradiction, Goetting (1995) gave a proof by contradiction of the statement: “There does not exist any rational number \( r \) such that \( r^2 = 2 \)” (p. 190). She observed that some students had difficulty understanding the role or validity of choosing \( r \) in a rational form. Some of the other difficulties she found students had were either the algebraic steps in the presented proof or the fact that the initial assumption of letting \( r \) be a rational number was not contradicted directly. Harel and Sowder (1998) contend that one of the cognitive forces influencing students’ thinking of proofs is their difficulty in understanding proof by contradiction, “where they may believe the proof assumes what is to be proved” (p. 254). Students’ need for a step-by-step creation of a result induces a distrust of a proof by contradiction in them, which in turn unconsciously may limit their proof schemes (Harel and Sowder).

Goetting (1995) also observed that the participants in her study had more experience with proof by contradiction than contraposition, and they would refer to the proof by contraposition as a proof by contradiction. In particular, these results indicated that students did not know the difference between proof by contraposition and proof by contradiction.

Furthermore, Goetting (1995) found that students commonly approached a true/false statement by first looking for a counterexample. If they were successful, they would stop and claim the statement false. Otherwise, they would attempt some proof. However, most students in her study failed to produce proofs. She attributed a failure to
give a deductive argument to lack of motivation or lack of resources in mathematics. On the other hand, she, as well as Knuth (1999), also observed that many students find the absence of a counterexample hard to believe because of numerous possibilities. In this respect, since a proof by contradiction shows that there will be no counterexample by virtue of the logic embedded in the method, it is possible that the method leaves many students in doubt that somehow there should be some counterexample.

As seen above, past research about students’ perceptions and understandings of proofs found some sort of difficulty in understanding certain aspects of indirect proofs, but they did not investigate the difficulties students may have in understanding many other aspects of indirect methods. There were no attempts made in the above studies, beyond the context of the problems the students were probed in, to determine the reasons behind students’ difficulties in understanding of proof by contradiction, nor to determine why they did not recognize the contrapositive relationship. Also, those studies did not investigate students’ perceptions of the relationship between the processes of finding a counterexample and contradiction. Therefore, this study proposes to explore and find students’ perceptions and understandings of various aspects of indirect methods in different mathematical contexts as well as the reasons that may give rise to difficulties in understanding of those aspects.
CHAPTER 3

METHODOLOGY

This chapter describes the methodology used for answering the research questions listed at the end of chapter 1. It includes a description of the participants and the tool used to select them for the preplanned interviews. It also describes the tools used for those interviews and how they were implemented. Finally, it gives an account of the procedures used to analyze the data.

Overview and purpose of the study

The focus of this research was to investigate undergraduate students' perceptions and understandings of indirect proofs in mathematics. It was an exploratory case study that probed the students’ approaches to determining the truth of some mathematical statements as well as their understanding of the indirect proofs of those statements. In particular, researcher-designed, task-based, semi-structured interviews were conducted during the second half of the 2002 spring semester, to explore students’ approaches to proofs in problem situations as well as the effect their approaches might have on their understanding of indirect proofs of those situations.

The participants

The subjects of the study were students at The University of Montana, who at different stages of their studies were enrolled in different post-Calculus II mathematics courses, ranging from Linear Algebra and Discrete Mathematics, to Number Theory and Complex Analysis. Twelve students majoring in sciences and/or mathematics, including pre-service high school teachers, were selected from a group of 28 students who were willing to volunteer their time and participate in this study. All the participants, at some...
point in their academic careers, had been exposed to the methods of mathematical proofs in either Discrete Mathematics or Introduction to Abstract Mathematics courses, both beyond Calculus II level. Each participant was paid $30 for about three hours of participation time.

The 12 participants (Appendix A) formed three categories, with 4 students in each, based on the level of courses in which they had been enrolled. The first category was *Post Abstract-Math*, comprised of advanced undergraduate students who had already passed an Introduction to Abstract Mathematics class and had some experience with proofs before this research. The second category was *Abstract-Math*, comprised of students who were taking an Introduction to Abstract Mathematics course and were being exposed to formal proofs at the time of this research. The third category was *Discrete-Math*, comprised of students who were either enrolled in a Discrete Mathematics or an Introduction to Linear Algebra course. The students in this category were either being exposed or had been exposed to the methods of proofs in the Discrete Mathematics class at the time of this research.

**The participant selection tool**

The researcher devised a questionnaire about the students' mathematical backgrounds, as well as their beliefs about the nature and purpose of proofs in mathematics (Appendix B). It was distributed to 118 students from nine different mathematics classes beyond Calculus II.

Students were asked to supply certain information about their backgrounds and about their willingness to participate in upcoming interviews. They were also asked to pick on a five-point Likert scale the degree to which their opinions most likely matched a
structured set of belief statements about the nature and purposes of proof in mathematics. The statements in general reflected mainstream epistemological views on certain issues relating to mathematical proofs. These statements followed the structure and format commonly used in past research (Almeida, 2000; Ruthven & Coe, 1994) with modifications to make them more appropriate for the issues addressed in this study.

The participant selection process

There were 51 respondents: 15 of them were categorized as Post Abstract-Math students, another 15 as Abstract-Math students, and 21 were categorized as Discrete-Math students. (The smaller number of responses was due to many students’ simultaneous enrolment in different classes that the survey was conducted in, and they only took the survey once. Other reasons were due to insufficient attention paid to the survey by the instructors of those classes or simply because students were uninterested.) Out of the 51 students, only 28 respondents were willing to participate in the upcoming interviews. Of those 28 students, ten were Post Abstract-Math students, eight were Abstract-Math students and another ten were Discrete-Math students. Within each category, most of the willing participants had similar mathematical backgrounds.

From this group of 28 willing students, 12 participants, four from each category, (with as much difference as possible in subject opinions in each category) were selected according to the following procedure. First, each student’s response to a statement carried the item score according to the values for the Likert scale: 5 = strongly agree, 4 = agree, 3 = no opinion, 2 = disagree, 1 = strongly disagree. Then, in order to compare students’ responses, each of the statements in the selection tool was assigned an ideal response score by the researcher (Appendix C), against which the student item score was

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compared by calculating the absolute difference between the scores. For example, the response to the statement: “a valid proof in mathematics does not depend on other mathematical facts or axioms” by the researcher arguably would be “strongly disagree.” Therefore, this statement carried an ideal response score of 1. If a student’s response to this statement was “agree,” then an item-score of 4 was assigned to the responses of that statement. The absolute difference of the scores was then calculated to be $|4 - 1| = 3$. A total absolute difference score of zero was considered ideal.

The sum of the absolute differences for all the items in the questionnaire for each of the 51 respondents was calculated in this manner (Appendix D). The sums of the absolute differences were plotted on three frequency histograms—one for each category—to obtain three spectra of opinions. Then, four willing students, with different GPA’s, from different bars on each of the three histograms were purposefully selected, forming the 12 participants who were interviewed twice in this study.

The interviews

The 12 participants were individually contacted by e-mail in order to inform them of the nature of the research, what was expected of them, and to arrange appointments for the interviews. Each of the 12 participants was interviewed twice individually. Before the first interview, the researcher explained the purpose (Appendix E) of the interviews and expectations. They were given a chance to ask any questions to address their concerns and were asked to sign a consent form (Appendix E), required by the Institutional Review Board of The University of Montana.

The interviews were semi-structured to allow certain flexibility with the line of questioning. Each interview lasted from 55 to 80 minutes, and the researcher was the sole
interviewer and observer throughout those interviews. The interviews were conducted
during the second half of the 2002 spring semester and they were all audio- and
videotaped. The camera was focused only on their writing at all times, for student
anonymity purposes. During the interviews, the students had access to a pen, a calculator,
a measuring ruler and blank paper.

Between the two interviews, students were given a take-home task in preparation
for the second interview. The second interviews were conducted either on the day
immediately following the first interviews or the day after, except for one case (Dave),
when it was conducted on the fourth day following the first interview because of the
student’s unexpected work schedule.

The researcher

I earned a M.Sc. in applied mathematics in 1996. I have taken a variety of courses
ranging from differential equations, mathematical physics, analysis, dynamical systems,
probability and optimization to learning theories and curriculum studies. My teaching
experience includes four years of teaching intermediate and high school mathematics and
science, as well as seven years of teaching mathematics at the undergraduate level.

As a mathematics educator, I believe understanding mathematics starts with
intuition and its interplay with the analysis of simple and concrete problems. Learning
occurs best when it proceeds from the exploration of the concrete and simple to the
abstract and general. Also, a mathematical concept can be understood and learned when
students see its usefulness in different settings. Students retain information better if they
can relate what they learn to what they already know.
As a researcher, I believe students' perceptions and understandings of different aspect of proofs can be studied through exploration of their thought processes in proof-related situations. Students' approaches to (or mathematical behaviors in) proof-related situations reflect their own convictions about proofs. In this respect, interview settings can provide rich data that describe many subtle aspects of human thought process.

My choices to the participant selection survey (Appendix C: Ideal Responses) reflect my own viewpoint about mathematical proofs. In addition, I consider proof by contradiction to be an indispensable mathematical tool.

The research tools and their implementation

To probe the participants' use and understanding of indirect processes, the researcher designed two sets of tools and utilized them for collecting data through three stages: Interview 1, take-home Proof-checking task, and Interview 2. Each problem situation in the tools was designed to investigate students' perceptions of a certain concept of indirect proofs. In addition, the interview protocols were developed with the potential for eliciting responses that could demonstrate a level of understanding pertinent to the objectives of each task. Overall, they provided ample opportunity to assess consistency of each participant's understanding.

The first set of tools used a true/false format (Appendix F) and was utilized during the first round of interviews (Appendix G) to examine students' approaches to finding the truth of seven different mathematical statements. This tool served mostly for gathering data in order to investigate research question 1.

The second set of tools included the take-home proof-checking task (Appendix H), which was handed out to the participants right after the first interviews and was used
during the second round of interviews (Appendix I) to examine their competency in understanding and evaluating indirect proofs. It served to probe students' beliefs and their ability to understand and recognize different aspects, characteristics and structures of indirect mathematical arguments. It thus served mostly for gathering data in order to investigate research question 2.

The initial tools consisted of eleven situations and were first examined by the members of the dissertation committee. Based on their suggestions and comments, some changes were introduced which resulted in reducing the number of situations to seven. Then the seven situations were pilot-tested with a sample of three students (volunteers) not participating in the actual study. They had similar backgrounds to those in the actual study. Each student was interviewed twice. During the pilot testing, the problem situations underwent two revisions—one after each interview. The revisions and the appropriate changes were made based on the interviewees' performances and responses to the tasks until the scheme of each situation became apparent and workable. The purposes and implementation of those tools are described next.

**Interview 1: true/false statements**

The goal of this interview was to evoke students' responses to and elaboration on a set of seven true or false mathematical statements (Appendix F). The statements chosen for this purpose were elementary notions from different areas of mathematics. They were kept simple in order to minimize bias due to students' content knowledge of mathematics. In particular, use of quantifiers was avoided whenever the contexts of the statements were clear.
The data from the first interviews were used to evaluate students' approaches to different mathematical situations. In particular, the interview data helped the researcher to determine whether the students tried to use an indirect approach, and whether their approaches influenced them in evaluating the indirect proofs in the second interview. The type of inquiry in the first interviews served to elucidate students' immediate approaches or behavioral patterns when evaluating the truth of mathematical statements. It also helped the researcher to observe the reasons why certain approaches were used to find counterexamples or proofs. Thus, the main purpose of interview 1 was to construct a behavioral backdrop for analyzing students' understanding of indirect proofs in the proof-checking task and interview 2.

**Proof-checking task**

Greeno (1982) points out that mathematical proofs, unlike common empirical arguments, are stringent and "evidence that a student knows these requirements can be obtained in a task of checking proofs" (p. 85). Immediately following the first interviews, the participants were given a take-home proof-checking task to complete, before their next interviews. It consisted of questions about indirect proofs of seven statements similar to the true/false statements from the first interviews. So, the task was not new to the students. It was assigned as take-home in order to allow the participants to think about the questions in the task at their own comfort, and not be under any time restriction as they normally would during interviews. However, they were also asked not to use any outside resources while completing the take-home task. Thus, it was meant to help them prepare for the second interview and to avoid using interview time by thinking about the situations.
There were two distinct aspects of the take-home task. One aspect consisted of a statement with an indirect proof or an argument in support of the statement. The participants were asked to evaluate the validity of the argument and the degree to which they believed the statement was proved true or false (see situation 3, 5, 6 and 7 in Appendix H).

Another aspect of this task consisted of an indirect proof of an unstated conjecture, followed by four statements related to the context of the given proof. The students were asked to choose the statement they believed the proof showed, if there was one in the list of four (see situation 1, 2 and 4 in Appendix H).

**Interview 2: probing**

The second round of recorded interviews probed the students' understandings of indirect proof processes. It investigated their competency at recognizing, evaluating and judging different aspects, characteristics and elements of indirect mathematical arguments studied earlier in the take-home proof-checking task. As a follow-up on the earlier task, these interviews encouraged the participants to reflect on the nature of their perceptions as much as possible, and to reconstruct their understanding as they elaborated on the proofs. The data from these interviews served to determine whether students were able to distinguish indirect arguments and techniques used in the presented proofs. Therefore, the proof-checking task, in combination with the second interviews, probed students' understandings of indirect proof processes under available resources.

As mentioned, the statements in the task were similar to the true/false statements that the students had seen earlier in their first interviews. The second interviews probed the participants' perceptions and understandings of a similar situation but under a
different circumstance, such as being given an indirect proof. This was in contrast to the first interview situations, where they had tried to find the truth of a given statement with no resources but their own experiences with mathematical situations. Therefore, interview 1 helped to collect data about the students’ mathematical behavior (believed by this researcher to be a by-product of students' perceptions of proof) while interview 2 helped to collect data about the students’ perceptions and understandings of indirect methods.

The two sets of data were compared and contrasted according to the objectives (see chapter 4) of each situation. In this context, they helped identify the influencing factors, if any, in different perceptions and understandings among different categories of students, and when possible, identified circumstances under which the participants’ beliefs and behaviors showed changes. The following section describes the procedure used to analyze the data.

**Data analysis**

There were three sources of data—on-site notes taken by the researcher during the interviews, student written artifacts and interview transcripts. As mentioned, all 24 interviews were audio- and videotaped. First, verbatim transcriptions of the audiotapes were made. Then, the videotapes were watched to check the correctness of the transcripts as well as to include visible pointing of objects and non-verbal gestures made during the interviews. The corrected transcripts constituted the bulk of the qualitative data for this research. While the videotapes were watched, the researcher expanded his on-site notes by adding remarks about scenarios in the interviews. Once all the notes and transcripts had been prepared, the process of data analysis began, using analytic-induction methods.
(Patton, 1980; Taylor & Bogdan, 1984; Radnor, 2001). In addition to giving an in-depth description of the data (see chapter 4), this particular method of analysis for qualitative studies has the potential to increase the validity of the research. The procedure used to analyze the data is described next.

First, a case-by-case analysis of individual student’s responses to all the situations was documented in an unrestricted manner. This analysis was done by watching again the videotapes of each interview in order to obtain a general idea of the themes in the data and consequently a matrix of the students’ responses was constructed (see Appendix J). It also helped the researcher to form an idea of the characteristics of each participant.

Second, analysis across each problem situation for each of the two interviews documented the themes emerging from the previous analysis. This analysis was done using coding to compare and contrast answers and approaches to similar questions in the transcript while watching the segments of the videotapes for each situation across interviews. By so doing, the extent of each student’s understanding of a particular objective was determined. At first, the objectives of the tasks in each situation guided the scheme of the analysis. This analysis classified the common themes according to those objectives. Then, due to the exploratory nature of the research and in order to guarantee the illumination of the different aspects of the data, the scheme was either modified or expanded depending on the findings in the subsequent stages (three and four in the next paragraphs) of the ongoing analysis of the data.

Third, a further analysis across each problem situation was performed in order to devise the common features found in the above themes and to tie those features to the objectives of each situation and to the central issues in the research questions. As the data
analysis progressed, common aspects of difficulties in understanding indirect proofs among and across the categories of students began to surface, first within each problem situation then across the situations. Consequently, the interview transcripts were studied again to compare and contrast data and search for contradictory arguments against these common features.

Fourth, if an emergent theme was regarded as crucial to the objectives of the situation or the research question, it was reviewed further in order to search for inconsistencies or uncertainties regarding the conclusions, as well as to check and verify the conclusions. This was done across the situations depending on the common themes in the subsequent readings of transcripts and analysis of the interviews. In the ongoing analysis of the data, the researcher attempted to identify all the direct and indirect effects and attributes the perceptions of the students had on their understanding of indirect proofs.

The results of the analysis are described in chapter 4 and the common themes are highlighted and discussed in chapter 5.
CHAPTER 4

DESCRIPTION OF THE DATA

The data were collected from a series of interviews with 12 undergraduate students enrolled in different mathematics courses beyond Calculus II. The data, which were gathered from those interviews as well as from written responses to the researcher-designed tasks, were analyzed using analytic-induction methods. This chapter describes and explains the results of those findings in the context of each situation and its objectives. A matrix for the participants’ responses to each task is given in Appendix J.

In the following verbatim excerpts, pseudonyms were used for identifying the participants. Pseudonyms start either with the letter P, the letter A or the letter D, indicating Post Abstract-Math, Abstract-Math and Discrete-Math students, respectively. The researcher is identified with his initials; VB. The ellipsis symbol “…” in the excerpts indicates a break in the speech, and ellipsis inside brackets “[…]” indicates that a part of the speech that was deemed non-critical to the discussion was edited out for ease of reading. The phrases or the statements inside brackets are editorial comments. The symbol “----” indicates a jump to the part in the transcript that is relevant to the discussion.

Situation 1: Interview 1

The purpose of this interview was to investigate how students approach the problem of finding the truth of the statement:

\[ \text{Let } x \text{ be an integer. If } x^2 \text{ is even, then } x \text{ is even.} \]

An objective of this interview was to investigate whether students would invoke the contrapositive approach.
All 12 students claimed that the statement was true. Some started by saying that it was false, but after a few steps into their arguments, they convinced themselves otherwise.

**Post Abstract-Math students’ responses to situation 1 (interview 1)**

In an attempt to explain why they thought the statement was true, three (Pam, Perry and Paul) of the four students in this category showed some sort of evidence that appeared to be reasoning for the converse statement: “if \(x\) is even then \(x^2\) is even.”

Pam said, “if \(x\) is integer then it is either odd or even,” so she examined examples of odd and even perfect squares \((49 = 7^2, 64 = 8^2)\) to convince herself that the statement was true.

Pam: [...] I saw “if \(x\) is even,” so of the form \(2k\) so then \(x^2\) is of the form \(4k^2\), an even number times any other number is always even. But if I looked at if \(x\) was considered odd, it was \(2k+1\). So then \((2k+1)^2\) is always going to be odd, so... By looking at the two examples, I kind of saw that...[the statement is true].

During the interview, it was revealed that she focused on exhausting all the possible cases rather than approaching the problem indirectly through odd numbers using contraposition.

VB: So, what’s so special about this kind of example that renders the statement true?

Pam: Well, you look at all of the cases. You look at the cases that are even and the cases that are odd, which, I designated \(x\) to be. I said \(x\) was an integer so you have even integers and odd integers. And, um, and this pretty much; both those cases cover all integers.

Pam soon realized that she approached the problem backwards. When asked if she could think of any other method that could handle the problem, she mentioned induction and contradiction methods, but did away with those methods arguing that there would be
more cases and scenarios than just even and odd. She did not attempt indirect arguments despite the fact that she considered the odd cases of $x$ initially.

Pam:  Maybe, um, like an induction or a contradiction kind of proof. It would be a little, more not concrete, but...I think it would include more options and more scenarios.

VB:  So when you said 'contradiction', how would a proof by contradiction handle this?

Pam:  Um, you could assume that $x^2$ would be odd.... All right, which way does this go? (Pause). Oh, you shouldn’t. Okay, let me think about this one second (pause). No, you would go through; I believe you would assume like “if $x$ is even” then you would say then “$x$ is odd.” Or “if $x^2$ is even then $x$ is odd.” Through a proof of contradiction, I think you would go through and show that $x$ has to be even if $x^2$ is even, therefore contradicting the original statement...

- - -

[...] If you use the initial statement that “if $x^2$ is even then $x$ is odd,” you’ve proved with contradiction that $x$ has to be even.

Her understanding of the way proof by contradiction could handle this situation was to negate the conclusion of the statement (if $x^2$ is even then $x$ is odd) and then show some contradiction hence showing that “if $x^2$ is even then $x$ is even.”

The fact that Pam tried a proof of the contrapositive statement (if $x$ is odd then $x^2$ is odd) indicated that she was in some sense seeing a contradiction. However, she was unable to argue correctly about the way a contradiction could be found.

Similarly, Paul initially argued that “if $x$ was even then $x^2$ would be even,” but soon realized that his approach was the converse, and thus thought the statement was false.

Paul:  [...] if I define an integer, or I define even as $[\text{writes } 2n]$, such that $n$ exists in this, then this; okay, then this is certainly convincing for me. Then I’m sure that $4n^2$ would also be even, or $(2n)^2$ would also be even.

- - -

I think I read the proof wrong—backwards. I think I read the proof backwards. I’m sorry. I always do this kind of thing. Oh, I screwed up.

VB:  How backwards?
Paul: I was reading; I think I was reading "if \( x \) is even" (laughs) then this is what I was doing [writes \( x \) even \( \Rightarrow x^2 \) even]. Oh, I'm sorry.

For no apparent reason, the method of contradiction came to Paul's mind and after a few trials with numbers, he realized that if \( x \) was odd then there would be a contradiction.

VB: And how do you, what makes you to conclude that it's [the statement is] false?
Paul: Um, I would do by contradiction. Wait, uh (long pause).

- - -
Uh, let's see. I guess I'm thinking about a way to start and so I'm just, I guess testing \( 2^2, 4^2 \) ... All right, so then I'm thinking then that \( 3^2 \) ... So, these are all odd. Uh, \( 5^2 \) is odd. And these are even. So, it seems to suggest that I could look at it further, so... Let's see.

After using examples of odd and even integers, he was convinced that contradiction had occurred. Faced with this realization he thought the original statement was true.

VB: Well, do you think that there should be a proof of this?
Paul: Yes, certainly. I would say it's true, but I'm not sure why at this second. But I think I could show it.

Paul's mind seemed to wander as the interviewer tried many times to bring his attention back to the idea of contradiction that Paul initiated. The interviewer tried to have him explain how he could find a contradiction, but to no avail. Paul felt he should be able to prove such a simple statement and was frustrated when he could not keep track of all the things he tried. Nothing Paul tried seemed right to him, so he kept bouncing back and forth with his choice. In the end, he was convinced that the statement was true. It seemed that the reason for his frustration was the indirect nature of the situation. He did not have enough tools to deal with such situations.

Like Pam, Paul perceived that a contradiction was imminent and somewhat sensed that a different (indirect) approach was better suited for the situation. However, he...
was unable to find a good reason for a contradiction or to argue correctly about the way a contradiction could be found.

Similar to Paul’s and Pam’s approach, Perry wrote down an argument in terms of generic representations of even integers \( x = 2n \Rightarrow x^2 = 4n^2 \) that proved the converse of the given statement. However, unlike Paul, he did not realize he was showing the converse.

Perry: Um, basically I remembered from Math 305 [Introduction to Abstract Mathematics] that if a number is even it can be written as, some, like, assuming that \( n \) is even, it’s equal to \( 2k \), where \( k \) is another natural number. Therefore, I did some algebraic manipulation and decided that \( n \), so, assuming this is even, we’re going to get an \( n^2 = 4k^2 \). And \( 4k^2 \) is equal to \((2k)^2\), so this is also a natural number. Therefore, basically, what it’s saying, it’s taking advantage of the fact that when something is even there’s another natural number that it can be divided by evenly. And then, when it’s divided by 2 you’ll still get a natural number. And, so that’s how it works.

When asked whether he could think of other approaches for proving the statement he thought of using induction method to show the contrapositive statement.

Perry: […] you could do math induction on \( x \) and set it up to work. And I…you would do that by using the contrapositive to be, if it is not the case that “\( x \) is even,” then it is not the case that “\( x^2 \) is even.” And that would be a way to do it with math induction.

VB: Well, you said two things, now. Math induction, and…

Perry: Well, if you prove a contrapositive then you prove the original statement. It’s one of the laws of mathematics in logic.

Although you’d have to, you’d have to use the case \( n + 2 \) rather than \( n \), but it would work.

VB: But, your [initial] argument here doesn’t prove this \( [x \text{ not even } \Rightarrow x^2 \text{ not even}] \).

Perry: But it proves the same things.

In the above excerpt, Perry was trying to explain that the method of contraposition could work, but he thought one needed to use mathematical induction on alternate counting numbers in order to prove the contrapositive statement. In other words, he did not see the
use of contraposition alone as sufficient. He also did not realize that his initial proof did not accomplish the task of showing the given statement, indicating that he perceived the proof of the converse as a proof for the original statement.

It was clear that Perry was familiar with the concept of contraposition, but in his second interview on situation 1 he could not automatically recognize this particular method used in the proof (see later discussion).

Patty on the other hand, started by looking at the two possibilities for integers, even and odd. She then explained that those two cases could prove the given statement.

Patty: Um, well with \( x^2 \) you are going to take the same number times itself. So that number is either going to be odd or even. So, you look at the case of even times even, and for all evens, even times even is an even number. And then the same process for the odd numbers. For any odd number, or for all odd numbers an odd number times an odd number equals another odd number. So, if the square is even, then the \( x \) must have been even.

She then went on and proved that \((2k)^2\) is even and that \((2k + 1)^2\) is odd.

VB: So do you consider this argument to be a proof to say that this [given] statement is true?

Patty: Well, I would say that these \([2k]^2 \text{ is even and } (2k + 1)^2 \text{ is odd}\) statements prove these \([\text{even} \times \text{even} = \text{even}, \text{odd} \times \text{odd} = \text{odd}\]\) and so then, and then those statements can prove this \([\text{If } x^2 \text{ is even, then } x \text{ is even}]\). So, it's a multi-part proof [emphasis added]. Like, you can do this as say a lemma and then you'd use, and then you'd use that lemma to prove it.

- - -

[...] So, you’re looking at both cases, because an integer has to either be odd or even. So if you look at case one and case two...

Patty was trying to explain that the square of an integer could be either even or odd and since \((\text{even})^2 = \text{even} \text{ or } (\text{odd})^2 = \text{odd}\) then it must be the case that the root of a squared number was \text{even} or \text{odd} respectively. This very approach however started with assuming \( x \) was even which was the sought conclusion in the given statement. However, this attempt if carried out systematically could make one think about an indirect approach.
She realized this later to some degree when asked if there could be a different argument that could prove the statement.

Patty: You could look at probably the contrapositive, but that’s kind of what I did here. It would be to say \(x\) is some odd, which this would be an odd number and square it and show that it’s also odd. So, then by proving the contrapositive, then you’ve proved it.

Although she did not use the contrapositive statement explicitly, she explained that her reasoning was centered on its idea. This assertion however was the result of the questioning that prompted her to consider alternate approaches. So, her afterthoughts might not have occurred to her if not prompted. Patty was the only one among the Post Abstract-Math students who could actually come close to the concept of contraposition and explain in her argument why it could be used in this situation.

In this problem situation, it was common among Post Abstract-Math students to start working with the converse without paying too much attention to the given statement. The utilization of the converse was not necessarily a result of any misperception of the equivalency of the converse to a statement, but it was their unconscious or habitual reaction to simple situations such as this one. The simplicity of the statement amplified their unawareness of the correct direction in the statement and made them react in a way that seemed natural to them, and what seemed natural to them was to presume a direct implication in any given statement. Thus, considering the indirect nature involved in this problem, those natural attempts were interpreted as the concept image of students who mostly think in terms of direct proofs.

None of the students had actually invoked the indirect proof methods (contradiction or contrapositive) before they were asked for other approaches than their
own. Even then, only Patty was able to make the logical jump based solely on her own perception of the statement. Although Perry tried some sort of contraposition, he did not show a full understanding of the way the method worked in this situation.

**Abstract—Math students' responses to situation 1 (interview 1)**

In order to be convinced of the truth of the statement, initially the common approach that two (Amy and Art) of the students in this category used was to think of some even perfect square integers, and to try some odd integers to see if counterexamples exist. They did not think that examples were enough to prove the statement. The very nature of this approach however would force Amy to invoke the contrapositive of the statement on her own.

Amy: Well all the examples I can think of, first of all specific examples, are perfect squares. 16, 64, 36, are all squares with an even root.... Um... okay, I guess there's other examples to the contrary like 9, 25, 81 where squares have an odd root. All of them, those are just specific examples... and then okay if x was not even though, then x^2 would never be even.... Is that true? "If x is odd then x^2 is odd?"

[...] So the goal is that “x^2 is even” implies that “x is even” so here.... This is just using the contrapositive assuming that x is odd and proving that x^2 is also odd, which I could do by...

[...] It's really hard to go in this direction x^2 to x. It's easier x to x^2, especially since this was given.

Similarly, Art wrote “2 is even. 2^2 = 4 even. Therefore x^2 is even and x is even.”

VB: [...] Can you absolutely be sure that your approach tells that the statement is true?
Art: Um, I, no I can’t because I can’t prove anything by just example.

When asked for a method of proof, it occurred to Art that what he had observed in the process of his examples was contradiction.

VB: Can you think of any particular method of proving, or some kind of approach?
Art: Okay. You could, um, maybe by contradiction saying that $x$ is odd and you could find an example or find something that contradicts it and then your proof [statement] would be true.

He then proceeded to write the proof using contradiction as seen in Figure 4.1.

Art: Okay. Our given is $x^2$ is even and our goal is "$x$ is even." Okay, so what I did is I assumed, our goal, $x$ is not even, or $x$ is odd. And, then I went through and I found a contradiction. And it contradicts our already given fact that "$x^2$ is even.” Therefore, $x$ cannot be odd, so $x$ has to be even.

VB: Okay. So, what exactly did you contradict here? Which statement did you contradict here?

Art: We contradicted the given that $x^2$ is even, when I showed that if $x$ is, if $x$ is odd—say we have $x$ as 3 and $3^2$ is odd—um, that contradicts the fact that $3^2$ would be even.

Figure 4.1: Art’s proof from situation 1

In this non-generic proof, Art gave a counterexample to the statement, “if $x^2$ is even then $x$ is odd,” which showed that the negation of the conclusion of the statement, “if $x^2$ is even then $x$ is even,” would be false. This approach was similar to that of Pam’s, except that Pam could not complete her argument by showing a contradiction.

It was concluded that Art understood the method of contradiction and thus could initiate it if its indirect process was experienced through examples or counterexamples.

However, he needed to be prompted by the interviewer in order to initiate it.

Similarly, Alice used an indirect argument only after she was asked for methods that could overcome the hurdle that she was facing. She started using a direct approach to
the problem by assuming $x^2 = 2k$ but was bogged down when she could not write
explicitly its square root as an even integer.

Alice: [...] you’d have to set $x^2$ equal to like $2k$ to show that it was an even number and
prove it that way.
 - - -
   And I’m stuck I guess (laughs). I know this should be easier than I’m making it.
VB: And what do you want to know out of $x^2$ equals to $2k$ there?
Alice: I want to take the square root of both sides so I can get $x$ by itself (laughs). I don’t
know why this seems so hard.
 - - -
VB: [...] So, you cannot think of any particular method of proof that might handle this
kind of problem?
Alice: Oh, I guess you could do, uh, the contrapositive. Is that what you’re kind of
getting at, something like that?
VB: So, what is the contrapositive?
Alice: You could assume $x$ is odd, and try and prove that $x^2$ is odd. Oh, that would be a
really good method actually. If you assumed $x$ is odd, $x$ could be written as $2k + 1,$
and then.... Yeah, that’s probably how I should have done it.

This was a case where the student had read the statement correctly and attempted a proof
that was interpreted as direct approach starting from assuming $x^2 = 2k$ even. But when she
reached the point where she needed to calculate the square root of $2k$, she realized that
her attempt had failed. This failure did not prompt her automatically to consider an
indirect approach. As observed, she kept persisting and thinking that there was some sort
of manipulation that needed to be done in order to get the conclusion ($x$ is even) from the
direct approach, mainly because the problem seemed easy.

It appeared from the previous two cases (Art and Alice) that Abstract-Math
students were not always quick in considering an indirect proof when they perceive the
inadequacy of the direct approach. Only after they were questioned about proof methods
did they realize the power of indirect process in this situation.
On the other hand, the fourth student (Adam) in this category started processing the converse of the given statement.

Adam: Okay. My steps are thinking to be an even number. It's um arbitrary, so to be \(x\) even is, uh, 2 times \(a\) would be, would be so that \(x\) is even. So then \(x^2\) is \(4a^2\), which is two times two times \(a\) squared equals \(x\) squared \([2(2a^2) = x^2]\). So, then, that means \(x^2\) is even. So, then, that's my thought process on that.

Furthermore, although he produced that argument, he was not sure it supported the truth of the given statement. So, he volunteered to write down some examples although he did not believe they could be used instead of proof.

VB: Okay. Like try one example for me.

Adam: Okay. So, say, to go along with the question; so, um, let's do, um, 36. So, if 36 equals \(x^2\), then 6 equals \(x\), and 6 is even, and a 36 is even, so then...

VB: So, um, what exactly did you think of to get 36?

Adam: Um, I thought of. I thought of. I thought of numbers. I thought of an even number that could be square-rooted easily. So, like 100. And that's 10. So...

This was in contrast to the way he initially processed the statement. The fact that he thought of a perfect square number showed that he was aware of the direction the statement was presented but could not perceive, as his peers did, the indirect characteristics embedded in his example. In fact, he could only see the two numbers being even at the same time. He was not able to get as far as his peers in terms of considering indirect proof methods because based on his two conflicting processes, he did not have enough tools to see the incompatibility of his example with the (direct) proof that he attempted.

When asked if he considered his argument to be a valid proof for his answer he responded negatively because he thought that proofs could be learned through authority.

Adam: [...] I think for a valid proof, like, you would need to have someone else look at it and say what's wrong with it. One person's opinion can be wrong. So... I mean,
this was shown to me, I think, uh, this was... I'm having trouble recalling what
effectively was shown to me to be the proof for that. [...] 

[...] Maybe even another person who has experience with proofs and to look at it
and say "yeah, this looks good". I think then it would be a valid proof.

In conclusion, three (Amy, Alice and Art) out of the four students in this category
would invoke indirect methods after faltering with direct approaches. However, only
Amy invoked the method of contraposition on her own. The other students had to be
prompted before they used indirect approaches.

**Discrete-Math students' responses to situation 1 (interview 1)**

After a few indecisive arguments, Dan claimed that the statement was true
because his attempts had not contradicted the statement and because it intuitively seemed
to be true. To convince himself, he gave some examples of perfect square numbers.

Dan: Um, to prove that it's true for all cases, or, like, 'cause I know it's true for
particular cases like 64, 8, 4, 2, stuff like that, 4, 16. But why is it true for all
cases? Well, if you think about it like this way. Well, I guess I can't do it from the
back. Well, I guess I think it's true because, first of all, I haven't thought of
anything that contradicts it. So, that's part of it. The other thing is that it, for some
reason, seems just intuitively right to me. I don't know why.

When asked if he could think of any proof for the statement, he mentioned the
method of induction and exhaustion but he did not think they were helpful. While
thinking about exhausting the cases of integers, he mentioned the contrapositive of the
given statement.

VB: Okay. So, can you think of some argument that can be considered a proof?
Dan: [...] I could like try a formal proof by induction or something like that. But, I
think it's right without that. I don't know. I don't know if that helps at all.

Like an exhausted method, but that wouldn't work on this stuff. Like if it's, uh,
like if you could do all the cases, then.... Actually, you might be able to do that
because the integers are either going to be even or odd. So, you might be able to
do it by case. But I don’t know because this would be “x is even,” ’cause I know that if x is odd, \(x^2\) is going to be odd [emphasis added]. But I don’t know why.

Since he was caught in an intuitively easy situation and the only alternative he saw was the case of odd integers, it seemed to him that that would be a reasonable argument to show the statement true. However, as this next excerpt indicated, he did not know how to validate his argument until he was questioned further.

VB: So, what does that tell you about this situation: “if x is odd then \(x^2\) is odd?”

Dan: Well it would seem to kind of help to say that “if x is even then \(x^2\) is even,” but I’m not sure. I’m not sure why exactly though, because that doesn’t really prove it for every case.

VB: Okay. Do you know the relationship between these two statements: “if \(x^2\) is even then x is even,” and the other statement: “if x is odd then \(x^2\) is odd?” The statement that you just mentioned.

Dan: I guess they’re opposite of each other.

VB: What do you mean opposite?

Dan: Like, uh... (talks to self). Oh, yeah, because if you prove x is odd and \(x^2\) is odd then it proves this.

VB: Why?

Dan: Um, is it contraposition? Is that the right word? I think that’s it. I can’t. I can’t exactly remember the principle, but, um, oh, how does that work? I can’t remember. I’m sorry. I think that works I just don’t know why.

It was clear that Dan was experiencing some indirect thought process but he could not tell initially why or how that process worked. Only after further questioning and prompting about his thought process did he invoke the idea of contraposition. Although Dan was not sure why contraposition worked, it did not occur to him to use a truth table or check equivalence with the rules of logic.

Similarly, Dave gave examples of perfect square integers and claimed that the statement was true.

Dave: Well, um, I guess my reasoning behind this one was kind of proof by exhaustion. [...] You know, like, 6, square root of 6, or square root of, let’s say, oh, square root of 16 for example. Then, you know, 4 is even, \(4^2\) is 16. But, um, there’s probably a better way of proving that, probably by induction.
Again, proof by exhaustion was seen as a way to approach the problem. He also thought of proof by induction as a better alternative for showing the statement true, but he abandoned that method after he realized that it was not a fruitful way of showing it. His next approach for a proof was to split $x^2$ into a product of two $x$'s.

Dave: Um let me think about this for a second. Um, for example... (talking to self). Well, here's, here's maybe a different way of approaching it. Um, $x^2$ is $x$ times $x$. Um, if this $[x^2]$ is even, then this $[x]$ and this $[x]$ have to be even, right? So one of them have to be even.

VB: Okay, and why so?

Dave: Well, because, this “$x$ is even,” and $x^2$ is $x$ times $x$, so, this $x$ has to be even because it's the same $x$, as this one. I mean. That's kind of my reasoning behind it. You could say, um, yeah, I mean; it's the same $x$, so, twice.

The student's reasoning was based on the case where the product of even integers is an even integer. This argument alone could not be considered complete to show the statement true, because it either required the study of the alternative case for the product of two odd integers to eliminate contradictions, or it would require methods of number theory to complete it directly. Without any regard to this process, Dave later pondered other proof methods that he knew, including contradiction and contraposition.

Dave: Um or there’s contraposition, and contradiction. So, um, let’s see. I wonder if you can do it by contradiction. Um, let’s see. We could try. Say $x^2$ is not even [...] I mean, then we’re basically in the same situation again, but we’re just proving it. We’d be proving that if $x$ is not even, or, if $x^2$ is not even then $x$ is not even. So we’d be proving that “if $x^2$ is odd then $x$ is odd,” which doesn’t really, I don’t think that helps. Um... there’s contraposition which says A implies B, then not B implies not A. So, if $x$ is, if $x$ is not even, then A is not even. Oh, wait, yeah, I think that’s contraposition. I don’t know. I’m still thinking that this is... I don’t know if that’s the proper way of showing something like this, but that’s how I’ve convinced myself.

Dave’s interpretation of contradiction method turned out to be proving the inverse of the statement, which could explain why he abandoned that method. On the other hand,
although he was aware of the correct approach for contraposition he could not follow it through because he was unsure of himself.

Thus, it was observed that Dave had committed indirect proof methods of contradiction and contraposition to memory and that failed him in this situation because he did not see any connection between each method or fully understand the functionality of the methods. Moreover, it was also observed that he would rely on proof methods to work for the truth of the statement instead of letting his own reasoning or perception work through his arguments. This could be explained by the student’s reliance on proof methods by authority.

Dean on the other hand, tried examples that seemed to be examples of the converse statement: “if $x$ is even then, $x^2$ is even.” He also tried an odd number just to see perhaps if it could be contradicted.

Dean: So, I guess I’d probably take a couple numbers and try them. Okay. So, uh, 2, $2^2$ is 4, which is even, 2 is even. Uh, 3, uh, 3 times 3 is 9, is uh odd, so it doesn’t apply. Four squared is 16, which is even, and 4 is also even. Actually, I guess they’re true. And, uh...

VB: So can you absolutely be sure that your approach tells you that your statement is true?

Dean: No, um, because it doesn’t satisfy... I’ve just given a couple examples and said why I would think that. But I guess if I were to, I guess I’d have to do a proof to go about saying as to whether or not it, it would always be true. I’d have to prove it for something generically, which I’m not very good at.

Like his peers, he did not consider examples to be a valid proof. Because his examples were for the converse statement, it was concluded that, as with Post Abstract-Math students, his thought process for proofs was rather direct. Further data showed that he found it hard to go from $x^2$ to $x$ and that he did not have any other approach or argument to support his claim.
Similarly, Doug claimed that the statement was true because he thought the product of even integers is an even integer.

VB: So, why do you think it’s true?
Doug: [...] any even number times another even number is going to give us an even number. That about all I can say. But I think there’s another way. There’s a way of doing that generally. Like \(2m\ldots\). Any number times 2, any integer times 2 is going to give you a, an even number. So that squared is going to generate an even number. That’s all I can say.

Like Dean, the reasoning given by Doug above was interpreted as when the converse of the statement was processed. He did not attempt any indirect approach. Again, this indicated a thought process for proofs that seemed rather direct. Like his peers, Doug also thought that induction method could handle the proof but did not know how to set it up.

Doug: [...] I think you can use induction to prove it would be true in general for all cases, or all values of \(x\). [...] I’d say induction, but I, I don’t know how I would exactly set that up, mathematical induction.

There was no indication in the data that Doug would have approached this problem any differently.

In summary, only one (Dan) of the four students in this category was able to invoke the contrapositive of the given statement and even then, it was only after interviewer’s prompting. Dave also tried to use contraposition, not because he thought direct methods would fail, but because his habit seemed to be trying every method mechanically until he thought one would work. This approach by Dave was also observed in his other interviews.

An unsurprising finding was that the students who were involved in discrete mathematics courses had a favorite proof method (mathematical induction) and they would try to use it before anything else as an alternative method.
Situation 1: Interview 2

In the second round of interviews on the first situation, the proof of the contrapositive of the statement, “if \( x^2 \) is odd then \( x \) is odd” was given. The proof possessed the features of a direct proof for the statement: “if \( x \) is non-odd then \( x^2 \) is non-odd,” which indicated that \( \neg Q \Rightarrow \neg P \) was proved instead of \( P \Rightarrow Q \). Further, the statement: “if \( x \) is non-odd then \( x^2 \) is non-odd,” was missing from the four given statements that the participants were asked to choose from (Appendix H).

The purpose of this interview was to investigate students’ abilities to relate direct semantics in a proof to an indirect (contrapositive) situation. The probing intended to find out or confirm the concept image students had about proofs in such situations, especially with those students who tried to use direct arguments in their first interviews.

Furthermore, the purpose of the alternate statement (see interview 2 protocol for situation 1, Appendix I) and a proof of its contrapositive was to probe deeper into students’ perceptions of the method in the context of one statement and its proof, rather than having them to relate the proof to four different statements.

Post Abstract-Math students’ responses to situation 1 (interview 2)

All four students claimed that the proof did not show any of the given statements. They appeared to have been following the direct semantics of the given proof because they all thought that the proof showed that “if \( x \) is even, then \( x^2 \) is even.”

However, Patty mentioned that the proof came close to showing the second statement (if \( x^2 \) is odd then \( x \) is odd.)

Patty: Um, it almost proves number two [if \( x^2 \) is odd then \( x \) is odd], but you would just need one more step. You would just need to say because, because any even number when squared will be even, then it must be odd in order to produce an odd
number when squared. But since that step wasn’t written down as part of the proof then it doesn’t quite prove number two.

Patty was trying to argue that in order for the proof to be complete and show statement 2, “if $x^2$ is odd then $x$ is odd,” it also needed to show the (converse) statement: “if $x$ is odd then $x^2$ is odd.” Although this could be done, it was unnecessarily a redundant case. The student was thinking of covering the two cases—odd and even—before any conclusion could be reached about statement 2. In her first interview, she also referred to a similar connection between her own argument of cases and the idea of contraposition. As seen in the next excerpt, Patty was able to verbalize the logic embedded in the contraposition.

VB: So, what is the relationship of that statement [number 2] with the proof? Why would you want to put that statement [any even number when squared will be even] in? What’s the reason?
Patty: Um... Uh, I guess, just because it, it gives more information about what you found out, I guess. Um, I mean, from this information you can conclude that if $x^2$ is odd then $x$ must have been odd.

VB: So, what would be the, um, method of reasoning used in that case?
Patty: Um... Okay, because. Okay, $x$ has to be either odd or even, and if when $x$ is even you square it and you get an even number then it has to be your other choice, which would be $x$ being odd in order for $x^2$ to be odd. So, *it would just be taking the two cases* [emphasis added].

It’s, if you can’t get to a perfect square which is odd by squaring an even number, then you have to get there from squaring an odd number. [...] 

As observed in the above excerpt that although Patty could see the gap that joins both arguments of the two statements (contrapositive and the original), she did not explicitly mention in her interview that one was the contrapositive of the other.

Patty was able to see how a direct argument could be turned around to prove another statement indirectly. It was concluded that Patty had a good understanding of the concept of contraposition (her competency was also revealed in later situations), but
sometimes she might have trouble invoking the method on her own as observed during the first interview on situation 1. Furthermore, she seemed to be attached to the surface structure or the direct semantics of the given proof. Her criticism of the given proof being incomplete indicated that she was not at ease with the hidden potentials of the proof and she would demand it to be open to all possible conclusions. In this problem situation, this could be explained by the fact that she was reading too much into the cases of integers exclusively being either even or odd, thus losing the bigger more abstract picture of logical equivalence of the two cases leading to the idea of contraposition. In other words, she was able to discover the logical jump between the cases but didn’t seem to be able to make the connection between her own discovery and a fact or a method (contraposition) that she had learned from mathematics classes. This could also be explained by the lack of full internalization (in terms of making one’s own) needed to acquire the relational understanding of that learned fact.

Pam on the other hand, thought the proof could eventually show statement 3 (if \( x^2 \) is even, then \( x \) is even) which is the converse of the statement (if \( x \) is even then \( x^2 \) is even) she thought it proved in the first place.

Pam: […] if you went backwards through this proof starting with \( x^2 \) is even, you could break it down to eventually see that \( x \) is even. […]

In her first interview, she used this same process of showing the converse true in order to prove the true/false statement: “if \( x^2 \) is even then \( x \) is even.” Moreover, there she used the same steps of the given proof. So in this second interview, she realized that she had used similar steps to the given proof (of \( x^2 \) is odd \( \Rightarrow \) \( x \) is odd) to show the true/false statement.
"$x^2$ is even $\Rightarrow x$ is even," true. Thus, she explained her approach to the true/false problem from interview 1 as follows:

Pam: Oh, I went backwards. This one [true/false situation 1] over here, I did it backwards, so technically I don't think I showed this [if $x^2$ is even, then $x$ is even], but that's what I tried to do [by showing "if $x$ is even then, $x^2$ is even"].

The reason she thought her own proof did not directly show the true/false statement (if $x^2$ is even, then $x$ is even) was because, as she mentioned earlier, she thought the steps of the given proof could be retraced backwards to show "$x^2$ is even $\Rightarrow x$ is even." At this point in the interview, she was still unaware of the indirect nature of the problem.

She gave a similar explanation to the alternate statement: "if $x$ is even then $x^2$ is even." After examining the alternate statement and its proof of the contrapositive, she claimed that the proof showed that "if $x$ is odd then $x^2$ is odd" and circled the third choice (neither proves nor disproves the statement).

Pam: Oh, uh, three. It doesn't show anything about the statement.
VB: So do you think the proof is wrong?
Pam: No, I think the proof is great for showing that odd integers that if you have an odd integer squaring it will give you an odd, but if, it shows nothing about evens... to me.
VB: Uh-huh. What is the nature or the method of argumentation used in this proof?
Pam: Oh, let, me see, let me see. I would say it's a direct proof [that proves "if $x$ is odd, then $x^2$ is odd"].

Evidently, the concept image held by Pam was associated with the process of direct proofs. In order to investigate whether she could make any connection between contrapositive statements, the interviewer asked a direct question:

VB: What is the contrapositive of this [if $x$ is odd, then $x^2$ is odd] statement?
Pam: Oh, oh, it would be, say, oh, it would be if $x^2$ is even, well, I should say not odd, then $x$ is not odd.
She seemed to know the contrapositive but failed to realize the equivalent relationship between the alternate statement and the statement she thought its proof showed, at this point.

VB: So now, again, I want you to go back again, look at the proof, and tell me what you think.

Pam: (Pause). Um, Uh (talks to self and a long pause). Well, uh... with that proof I'd still say three [neither proves true nor false].

VB: Okay. Um, what is the relationship between a statement and its contrapositive?

Pam: Uh... What is the relationship between a statement and its contrapositive? Um, you start with, I don't know. Uh...(laughs). Um...

She needed a hint however before she could see how the pieces fell together.

VB: Okay. Let me say this. So, this proof showed this statement [if $x$ is odd, then $x^2$ is odd] and you said this...

Pam: (Interjecting) Oh, okay. It does work (laughs), okay, so they just assumed the opposite first and then showed the opposite so it does work. I just did not see it.

It was concluded that the knowledge base for contraposition was there but not used in this situation unless prompted by the interviewer, and even then, it was limited to the explicit process of the method.

Pam: Well, I kept, I was comparing these two [$x$ is odd $\Rightarrow x^2$ is odd and $x^2$ is even $\Rightarrow x$ not odd] I guess when you asked the question what is the relationship between the statement and the contrapositive. And as soon as you said something, I was like, oh, I'm supposed to relate these two together. And then I saw that, oh, they assumed the opposite of the "then", and we found the opposite of the "if", so via contrapositive (laughs) it works.

Similarly, Perry chose number 5 (none of the above) because he was looking at the "if" part of the given four statements and trying to match them up with the hypothesis (or assumption) of the proof.

VB: So why do you think that the presented proof in this situation shows, uh, the case number five there?

Perry: Well, to begin with, it's kind of through elimination. The first part of our proof is suppose $x$ is a non-odd, or even, integer. Therefore, that means, that our assumption has to be "if $x^2$" rather than "if $x$". So, we can immediately cross out
[... these [choices # 2, 3 and 4], then the other assumption.... And then just looking further on here [choice #1], "if $x$ is an odd integer", which contradicts our assumption that "$x$ is a non-odd integer". Therefore, none of these can be what we proved here. So, it has to be number five. [...] Basically, the assumption is, we have "if $x$ is an even integer"—that is our assumption. And so, all these [#1 through 4] have something other than "if $x$ is an even integer".

 VB: So, what is the consequence of that assumption?
 Perry: That $x$ is an even integer? Um, this proof would say that "if $x$ is an even integer then $x^2$ is even".

 Eliminating non-matching hypotheses in this manner indicated that the student was following a direct approach from the "if" part to the "then" part without any regard to other possible methods of proof. In other words, he was not open to the indirect approach of the proof and did not look at it from a different perspective.

 VB: What kind of argument they’re using in this proof?
 Perry: It’s a direct proof. They simply define $x$ to be $2k$ where $k$ is another integer, which is by definition of even integer. Then they simply directly square both sides and then show that $x^2$ is even be... by the same definition.

 The association of his concept image with direct proofs became even more apparent after he examined the alternate statement (if $x^2$ is even, then $x$ is even) and its proof of the contrapositive. He claimed that it showed, “if $x$ is odd then $x^2$ is odd,” and that it was not related to the statement, “if $x^2$ is even then $x$ is even.” He then circled the third choice (neither proves nor disproves the statement).

 Perry: [...] By all these, “suppose $x$ is an odd”. (Talking to self unintelligibly, pause). So, basically... So, it doesn’t really have anything to do with that [$x^2$ is even $\Rightarrow x$ is even]. Because the proof doesn’t link $x$ being odd and how that affecting things would affect $x$ being even or $x^2$ being even. So, basically it’s a proper proof, but it doesn’t have anything to do with the statement.

 VB: Hmm. And what is the method of proving used here?
 Perry: Um, it seems like it’s another direct proof. That it’s just taken this and directly squared it. But it doesn’t, there’s no linkage back to turn it into anything else.
Again, his reason for choosing number 3 was based mainly on his observations of the semantics in the proof, particularly in the assumption of the proof and its compatibility with the hypothesis of the given statement.

In order to investigate Perry's knowledge of contrapositive statements, the interviewer asked directly:

VB: All right. So, um, what is the contrapositive of this [if \( x^2 \) is even then \( x \) is even] statement?
Perry: If \( x \) is not even then \( x^2 \) is not even.
VB: Okay. And, um, what is the relation between any statement and its contrapositive?
Perry: Um, usually, if you prove a statement then its contrapositive is also true and vice-versa.
VB: Okay. So, do you think this proof uses the method of contrapositive at all?
Perry: Um, it—the thing is, it doesn't relate—it could do it except for the fact that it doesn't relate \( x \) not being odd to mean that \( x \) equals even; which is one of the things you would have to do in order for the proof to work. And it's usually a good idea to say that you're trying to prove the contrapositive. [...] If you don't say that you're proving the contrapositive, then there's nothing in the proof to link it to the statement. [...] Because it forces the reader to make, it forces the person who looks at the proof to make leaps that they shouldn't have to. In essence, it's not a complete proof. It's saying this, this, and this and we have to add these links in order for it to become a complete proof. So, it's missing these links, then it cannot be a complete proof.

Evidently, the student was familiar with the concept of contrapositive statements and seemed to be bothered by the missing link. So he tried to explain why he thought that the proof did not show the statement "\( x^2 \) is even \( \Rightarrow \) \( x \) is even." He thought the proof was incomplete because it did not explicitly mention the contraposition process. However, he did not explain why he did not think of making the contrapositive connection, like Patty did. The data indicated that his concept image associated with the process of the proof was that of direct methods. He did not turn the direct argument around to indirectly show an equivalent statement.
Like Perry, Paul also did not think that the proof showed that “if \(x^2\) is odd then \(x\) is odd,” because of his concept image associated with the direct process of the proof. He used an elimination process by matching the assumption in the proof to the hypotheses of the statements.

Paul: [...] none of the numbers here are, are odds. So, I made my choice up, like, like what they suppose I took to be, that “if” statement essentially, and then the “then” statement I took to be their conclusion. So, their conclusion involved a, uh, even integer and involved \(x^2\) being an even integer, and that’s none of these cases.

VB: So, what is the method of reasoning used here?
Paul: Umm, is it just directly? It just shows, it shows that this \([x \text{ is even}]\) situation leads to this \([x^2 \text{ is even}]\) one.

His attachment to the direct nature of the semantics used in the proof became even more apparent after he was asked to evaluate the alternate statement (if \(x\) is even, then \(x\) is even) and its proof by contraposition.

Paul: [...] I mean, if you just look at this statement [assumption in the proof] here. They suppose that it’s [\(x\) is] an odd integer. And at the end, their conclusion says that it’s [\(x^2\) is] odd, so I’m not sure if I understand why... It seems to me that there’s only one way to look at it [emphasis added], but I guess the steps could in turn show that this [if \(x^2\) is even, then \(x\) is even] is also true. Now, is that what you’re trying to get out of the statement?

Paul showed some doubts that somehow the given proof (by contraposition) could be used to show the statement: “if \(x^2\) is even then \(x\) is even.” However, the data indicated that he had no basis for his doubts because, in the end, he was not convinced that the proof showed the given statement. He claimed that the proof neither proved nor disproved the statement.

Further investigation as to whether Paul was familiar with the idea of contraposition revealed that he was unable to remember how contrapositive statements were constructed.
VB: So, I want to know what the contrapositive of this \( x \text{ is odd} \Rightarrow x^2 \text{ is odd} \) statement is.

Paul: I thought it was this \( x^2 \text{ is odd} \Rightarrow x \text{ is odd} \) statement.

VB: Okay.

Paul: Is that not what contrapositive is? I can't remember the definition of contrapositive. Is that not correct?

In conclusion, it was found that three (Pam, Perry and Paul) of the four students in this category did not relate a direct proof of a statement to an indirect proof of an equivalent statement. Put differently, they did not see the purpose of a proof beyond its surface structure. Although those students knew the validity of the relationship between a statement and its contrapositive, it was found that they did not adapt themselves to perceive that relationship in the context of this situation. This phenomenon was due to the compartmentalization of the concept of contraposition. The fourth student (Patty) was able to make the logical jump from one approach to another and could make sense of the relationship between them. However, she lacked the extent of the internalization of the concept of contraposition that was needed to acquire a full relational understanding of contrapositive statements.

Abstract-Math students' responses to situation 1 (interview 2)

Three (Amy, Alice, Art) out of the four students in this category had no trouble identifying the proof of the contrapositive of the second statement (if \( x^2 \text{ is odd} \) then \( x \text{ is odd} \)). The fourth student (Adam) initially thought that the proof did not show any of the given statements, because he was following the direct semantics embedded in the proof.

Adam: [...] the proof starts with \( x \), instead of \( x^2 \); so...I guess, I guess I'm stuck on where the proof starts with [...]  

 [...] You have two parts of a statement, two parts of a statement, and it says "if \( x \) is even, then \( x^2 \) is even" [...]
VB: Okay. So, are you saying that the proof actually proves "if $x$ is even, then $x^2$ is even?"

Adam: Yes.

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VB: And what method of proving used...?


His concept image associated with direct proofs was also observed in his explanation of the proof (by contraposition) of the alternate statement (if $x^2$ is even then $x$ is even.)

Adam: It doesn't have... It's the opposite of the statement [if $x^2$ is even, then $x$ is even], or it's not even... It's showing that, uh, it says "if $x$ is odd, then $x^2$ is odd", which it does show that. So, [...] um, well it shows some support that the statement's true, but just be... it shows again, it starts out with the odd, and not even

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VB: And again, what is the method used in this proof?

Adam: Method? It's just a direct, straightforward if—"suppose $x$ is odd, then $x^2$ is odd."

Only after the interviewer tried to engage him in the (direct) process that he was attached to did the student realize that the proof was using an indirect process.

VB: So if you were to actually, uh, prove say, uh, prove this [if $x^2$ is even, then $x$ is even] statement [...] how would you do it?

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Adam: With everything else I've been shown [...] what I would do is I would start with $x$ being squared, obviously [...] and it would have to be something where $x^2$ is even; so then you have to choose an $x$, an $x$ equal $2k$... Hmm (pause). Oh, I see the method on that one now.

VB: What's the method?

Adam: This one's contrapositive [...] because the contrapositive of that would be "if $x$ is odd, then $x^2$ is odd."

Later investigation also revealed that the student knew that the two statements were equivalent. Again, the knowledge base for the proof by contraposition existed but it was compartmentalized because it was not put to work without prompting. In contrast

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with his first interview however, Adam did not invoke this method, like his peers did, when questioned for a different approach.

As mentioned earlier, the other three students identified the given proof to be a proof of the contrapositive of statement 2 (if \(x^2\) is odd then \(x\) is odd.) Amy and Alice gave the following reasons why they picked statement 2.

Amy: Um...because what it proves is that “if \(x\) is even, then \(x^2\) is also even”... Um, and the only one that... and number 2 is the contrapositive of that statement... and that’s the only equivalent

Similarly,

Alice: Because, this here is a proof by contrapositive, and we’re trying to say that “if \(x^2\) is odd, then \(x\) will be odd.” So, we assume that “\(x\) is even”, and try and prove that “\(x^2\) is even,” which is a valid way of proving it. [...] This is just the only one that works.

It was observed earlier that Amy, in her first interview had no problem invoking the contrapositive statement, but Alice had to be prompted before she invoked it. It is worth noting here that Alice in her first interview of situation 1 seemed quite content when she discovered that invoking the contrapositive helped her to get the right answer. So, she might have committed to memory the fact that these types of statements could be looked at indirectly and that could have contributed to her success. Both Amy and Alice were able see how the direct semantics embedded in the given proof could be turned around to prove the contrapositive of a statement.

Similarly, Art had no trouble identifying the proof as a proof by contraposition of statement 2.

Art: Um, I believe that the statement 2 is the contrapositive of this proof; and the contrapositive and the original statement are equivalent so it proves statement 2.
Recall that in his first interview about situation 1, Art had actually used a contradiction to explain his choice. The interviewer tried to investigate the student’s understanding of the relationship between the two methods.

VB: Okay, so, are those two different, the contrapositive and contradiction? And how different are they?
Art: Um, with contradiction, you’re looking for something of a contradiction of a known fact. And, with the contrapositive, you can work, uh… you can take two equivalent statements and work towards your goal.

There was not enough evidence from the data to conclude whether or not the student could relate the two methods.

In summary, except for Adam, the students in this category were successful in identifying the given proof as a proof by contraposition. Unlike Post Abstract-Math students, they were not attached to the direct semantics of the proof. Adam, who at first was attached to the direct semantics, after prompting by the interviewer, realized the possibility of the indirect approach in the proof.

**Discrete-Math students’ responses to situation 1 (interview 2)**

One student (Dave) claimed that the proof in this situation showed statement 3 (if \(x^2\) is an even integer then \(x\) is even). Three (Dan, Dean, and Doug) students claimed that it did not show any of the given statements. They seemed to have been following the direct semantics of the proof because they all thought the proof showed that “if \(x\) is even then \(x^2\) is even.”

Dan gave the following explanation for his choice and why he thought other statements were not proved.

Dan: […] The first thing we’d suppose is that “\(x\) is an even integer”, and so we know that \(x\) is \(2k\); that’s fine, just using that fact. And then if we square both sides of that, we get \(x^2\) equals all this, meaning that that’s even. So, all that this is proving
is that "if you have an \( x \) that is even, that \( x^2 \) will also be even." That doesn't prove the same thing as saying that "if \( x \) is odd, then \( x^2 \) is odd." Says nothing about whether \( x \) is odd or not. It says nothing about \( x^2 \) actually. It only says something about \( x^2 \) as opposed to an even \( x \). So, it doesn't say that "if \( x^2 \) is odd, then \( x \) is odd." These could be true, but this proof doesn't prove it.

Based on his explanation, it was concluded that Dan's perception of the method of the proof was direct and it directly proved the statement: "if \( x \) is even then \( x^2 \) is even." He too was using elimination process by matching the assumptions of the proof to the hypotheses of the given statements. The following excerpt also confirmed that conclusion.

VB: So, what is the method of argumentation used in this proof?
Dan: Uh, well I guess the method itself is just like a direct proof using these facts; so there's no, uh, yeah, I don't know. See, you take an assumption that what you're trying to prove is true, and then you use these [given] facts to either show that it is contradictional or it works. Or actually you just take, you just take that this is an even integer and see what you get, it says, well, "\( x^2 \) is even," too. So, that's what we proved. Yeah, it came out kind of jumbled, but...

In his first interview on this situation, it was observed that Dan, after he was prompted, had actually realized that contraposition could work for this type of situation. Nevertheless, he did not see that connection between the statement he thought the proof showed and statement 2 in this interview. Thus, he was asked to examine the alternate statement and its proof by contraposition to see if the context of one statement and its proof could trigger his knowledge base for contraposition.

Dan: [...] So I'd say that it proves that the, that the statement--what's the statement?--is true. Wait. No, that doesn't prove it; hold on. Because we're talking about \( x^2 \) being even, then \( x \) is even. [In this proof] we're taking \( x \) odd and squaring it and getting another odd, right? (Pause). Well, if you use like this [first] proof and this [second] proof together, if you know both of these [proofs], then, then it does prove this [if \( x^2 \) is even then \( x \) is even]. But if you just know this [second proof], I don't think it proves this [if \( x^2 \) is even then \( x \) is even].

VB: How do you mean?
Dan: Well, you know now that “if x is odd, then \(x^2\) is odd” [from this second proof]. But that doesn’t necessarily mean that \(x^2\), “if \(x^2\) is even, then \(x\) is even,” unless you know that “if \(x\) is even then \(x^2\) is even” [which was shown by the first proof]. [...] Because we know “\(x\) is odd, \(x^2\) odd;” [and] “\(x\) is even, \(x^2\) even.” So, \(x\), “if \(x^2\) is even,” we have to know it’s even, because there it can’t be odd, because otherwise \(x^2\) would be odd.

Dan was trying to explain why he thought the second proof did not show the alternate statement: “if \(x^2\) is even then \(x\) is even.” His explanation was that the second proof which showed, “if \(x\) is odd then \(x^2\) is odd,” combined with the first proof which showed, “if \(x\) is even then \(x^2\) is even,” could be used to deduce the alternate statement, “if \(x^2\) is even, then \(x\) is even.” He did not believe that the second proof could by itself show that fact. He argued that both proofs were needed to show the alternate statement because they would cover both possibilities leaving no other choice but to deduce that “\(x\) must have been even if \(x^2\) was even.” In other words, he was trying to make the connection between a statement and its contrapositive without actually realizing or being aware of their prior connection through logical equivalence. The validity of his argument was a result of exclusion of other possibilities.

In this next excerpt, the interviewer tried to find out if Dan could connect his thoughts to the idea of contrapositive statements, he had seen earlier in the first interview.

VB: So, how are those things related?
Dan: I think there are two cases of this [if \(x^2\) is even then \(x\) is even], kind of. Well, they’re the only two cases. Either “\(x\) is even,” or “\(x\) is odd.” And so, if you know both these cases and one’s always going to be even and one’s going to be odd, then you can say stuff like this, but, otherwise, you can’t work backwards, I don’t think.

[...]

[...] Let’s just take an arbitrary \(x^2\), and this is either going to be odd or even, right? There’s no other possibility [other than “\(x\) is even \(\Rightarrow\) \(x^2\) is even” and “\(x\) is odd \(\Rightarrow\) \(x^2\) is odd”]. [...] If it’s \([x^2]\) odd, it \([x]\) has to be this [odd], and if it’s \([x^2]\) even, then it \([x]\) has to be this [even], because those are the only two possibilities, basically.
This case was strikingly similar to that of Patty. As in Patty's case, Dan wanted to have both even and odd cases proved in order to eliminate other outcomes, and thus to fill in the gap that connected both statements. Thus, it was concluded that Dan was able to turn a direct proof around and to make the logical jump to prove something else indirectly. He did not however recognize that the needed jump was already built into the logic of contrapositive statements that he had learned from mathematics classes. This again could be explained by the lack of full internalization of a learned fact necessary to acquire relational understanding.

In the next phase of the interview, the interviewer tried to engage Dan in a hands-on proving process, in order to see if he would think of an indirect proof.

VB: Okay, let me ask you this. Um, suppose now that you know how these things are manipulated in a proof, right? Using \( k, 2k + 1 \), and stuff, try to use that and come up with a proof for this [if \( x^2 \) is even then \( x \) is even].

Dan: Okay, so if we have \( x^2 \) and we know that that equals some...that means it's even, right? So then, I'm going to get... I don't know how to do that. You can't just do square roots; that's not going to do anything, I don't think, because the square root of \( 2k \), that doesn't do anything.

Above, he was trying a direct approach to taking the square root of \( x^2 = (2k) \), which he soon realized was not working. He also tried induction method and as before was unsuccessful.

VB: What other methods can you think of that can handle this situation?

Dan: I am not sure if I know enough methods to figure out one that would work, unfortunately. Let me think (pause). Well there is like, isn't it like the contrapositive of this? Which is basically what's this doing, I guess.

VB: You think so?

Dan: Yeah I think so. But I don't exactly remember why that works.

After looking at alternative ways of proving the statement, Dan was finally able to figure out the connection that was vaguely or intuitively obvious to him. It was finally revealed
that he knew how to write the contrapositive of a statement but he did not remember the logical connection between them.

Dan: [...] if contrapositives are indeed equivalent, then it would prove this. [...] I guess I am just forgetting the definition of contrapositives.

Thus, it was observed that Dan did not see the connection of the proofs to the statements because the relational understanding of contraposition was not fully internalized by him. Although he did not remember the connection, he did not try to use a truth table to check their equivalence.

Similarly, Doug claimed the proof showed none of the given statements because he could not match the proof’s conclusion with any given statement.

Doug: [...] You can maybe draw that conclusion if you went a little bit further, that $x$ is—you can maybe pick one [statement] out there that would satisfy it, but it wasn’t exactly what was said. [...] At least, as far as my perception...you couldn’t draw the conclusion, uh, you couldn’t draw the conclusion that were stated in the examples [statements] that were given.

Although Doug suspected that somehow the proof could be turned around to prove one of the given statements, he did not have any basis for that suspicion. Again, Doug’s perception of the proof was that it used direct methods to show the statement: “if $x$ is even then $x^2$ is even.”

VB: Would you say it’s a direct proof or indirect proof?
Doug: I’d say it’s a direct proof. Well, I’m not really sure about that, I’m not real sure about the distinction there.

The student’s remark above indicated that he could not distinguish between direct and indirect proofs. Doug had studied proofs three semesters before this interview, so it was possible that he did not remember the distinction. Thus, further probing tried to find out if he could find any (indirect) relationship between the alternate statement and its proof.
VB: So, here is another proof in support of this given statement. I want you to read it and let me know what you think.

Doug: (Pause). [...] One is proving something that’s, one’s asking or making a statement about something that’s even, and the other one’s therefore it’s saying that $x^2$ is, that $x^2$ is odd, for the final. It’s assuming.... So I would say no, it doesn’t prove it. [...] It doesn’t relate to this. It does not relate to the statement.

Since the concept image held by the student was based on direct methods, he did not see any relationship between the proof and its statement. However, he wondered if the proven statement, “if $x$ is odd then $x^2$ is odd,” in turn would prove “if $x$ is even then $x^2$ is even,” and thus it could be turned around and prove the given statement: “if $x^2$ is even, then $x$ is even.”

Doug: Well, I guess it shows that if $x$ is, uh, “if $x$ is odd then, uh, $x^2$ is odd.” I guess that from that you can maybe draw the conclusion that that, “if $x$ is even then $x^2$ is even.”

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“if $x$ is odd then $x^2$ is odd,” and from this, we can conclude that “if $x$ is even then $x^2$ is even.” So, I guess in a way indirectly [emphasis added] it does prove that [if $x^2$ is even then $x$ is even].

At this point Doug was suspecting that the proof might work, but it was just a speculation. In this next excerpt, he tried to explain how the indirect connection could be made.

VB: Now how, what is the reasoning that you can conclude the second statement from the first one?

Doug: It’s indirect. You just know that. It’s just that, it indirectly proves the statement; uh, how do I know that? Well, there’s only even and odd numbers. I mean, I don’t know how else to say it. I mean, as far as real numbers, I guess it’s (pause).

His words were not sufficient to conclude that he, like Dan, was actually aware of a valid relationship other than even and odd cases possibly leading to the alternate statement. He was using the term “indirect” not in a mathematical sense for a method of proof but rather
in a sense commonly used in the English language. Thus, the next phase of the interview investigated whether he could remember contrapositive relationship.

VB: Can you tell me what the contrapositive of this [if $x$ is odd then $x^2$ is odd] first statement is?
Doug: It would be, this is, “if $x^2$ is not odd, then $x$ is not odd.” I think that’s right.
VB: Okay. So now if you go back and check this [list of choices] thing again…?
Doug: It would be that it proves that the statement is true.

In his first interview on this situation, Doug did not invoke the contrapositive approach and in this second interview, he had to be prompted before he could see the process of contraposition in a proof. It was concluded that Doug’s concept image for proof methods was associated with direct methods and there was no indication that he would have invoked indirect methods even when it was forced upon him.

Dean also claimed that the proof did not show any of the given statements because he was attached to the direct semantics of the proof. His perception of the given proof was that it showed directly that “if $x$ was even then $x^2$ would be even.” In the next excerpt, he gave an explanation where he was trying to match the assumption in the proof to the hypotheses of the given statements.

Dean: […] It [the proof] was looking at non-odd integers, so I don’t know how you could get that [statement 1] from it. And “if $x^2$ is an odd integer, then $x$ is odd”—same thing. It was still looking at non-odds. Third statement, “if $x^2$ is an even integer then $x$ is even”—that’s kind of like the exact opposite of what it was doing. It was showing that “if $x$ is even then, $x^2$ is even.” […]

VB: And what can you say about, uh, the nature of argumentation? Like, is it direct, indirect…?
Dean: Oh, um, I think it’s direct. Yeah, I think it’s direct because it takes an even number and it proves that directly an even number has an even square. So, it’s addressing it directly.
Next, he was shown the alternate statement and its proof by contraposition and after studying it, he claimed that the proof neither proved nor disproved the statement.

VB: Okay. So, um, again, what do you think this proof is showing?
Dean: That "if $x$ is odd, then $x^2$ is odd."

Again, Dean's perception of the given process of the proof was direct. Further probing revealed that Dean was unable to remember how to write contrapositive statements that he studied the previous year. In the end however, he was fully convinced that the proof did not show the alternate statement because all he could relate the proof to was the statement it proved directly.

Further questioning tried to probe if Dean would use any kind of indirect argument with this type of situation. He was asked to prove the statement, "if $x^2$ is even, then $x$ is even." After trying a direct approach similar to taking square root of $x^2 = 2(2k^2)$, he said "it's a lot easier going from $x$ to $x^2$ than I think from $x^2$ to $x." This realization did not make him to use an indirect approach.

Dave, on the other hand, claimed that the proof showed statement 3: "if $x^2$ is an even integer then $x$ is even." This statement is the converse of what had been proved directly in this situation. In his explanation, he pointed out that the proof went from assuming "$x$ even" and concluding that "$x^2$ was even," but it was inadequate to conclude the reason why he chose statement 3. Nonetheless, it was concluded that he perceived the process of the proof as direct.

Dave: [...] So, when you square both sides of this it is essentially you know $2k$ times $2k$, and $x$ times $x$, and since this is even then the left side is even. [...]"...

VB: Yeah, can you, can you name the type of method the proof used?
Dave: To me this is kind of a direct approach.
Further probing tried to find out if he could see any indirect relationship between
the alternate statement, “if \(x^2\) is even, then \(x\) is even” and its proof by contraposition.

VB: Okay, here is um, here is a statement and its proof. It is similar to the one we’ve
seen here. Please read it and let me know what your answer would be here.

Dave: Okay. (Long pause, then he circles, 1. Proves that the statement is true).

VB: So, what is the nature of the reasoning used in this proof?

Dave: Um. I’m trying to think of a name, um...

VB: The method?

Dave: The method. It’s um contrapositive I think, because our statement is that we’re
trying to prove that \(x\) is even but yet we do that by proving that, um \(2k + 1\),
which is an odd integer. We do that; we prove this by proving that this is odd.
Huh, hold on a sec (laughs). I am confusing myself here... Actually, can I
change my answer? I think it might be number three actually. Because we’re not
really, we’re proving that if \(x\) is odd and \(2k + 1\). I mean we’re proving that this
\([x^2 = 4k^2 + 4k + 1]\) is odd but we’re not proving this [given] statement here. So, it
really doesn’t prove anything... to me. I mean it doesn’t prove this.

VB: So what do you think it proves?

Dave: Well it’s proving that (mumbling quietly)... We’re proving that “if \(x\) is odd then
\(x^2\) is odd.”

Dave’s initial perception of the proof was that it showed the alternate statement using
contraposition, but since it was not the obvious choice for him he changed his mind and
chose to indicate the more obvious (direct) statement it proved.

It was interesting to observe this student’s choices in this situation. In his first
interview, he sensed that contraposition could be used, but because he was unsure of
himself, he did not follow it through. There, it was observed that he had committed the
method to his memory without really understanding its functionality. Also, it was
observed that he would not rely on his own reasoning or perception to work through his
arguments because of his reliance on authority. In his second interview, after he
examined the original proof, he exhibited a concept image that fitted direct methods, and
since his initial perception (contraposition) of the second proof did not fit into that image,
he denied its indirect process. This, as in his first interview, could be explained by his
lack of confidence for the functionality of contrapositive statements and his reliance on
authority, which in this case was the direct semantics used in the proof.

In order to find out whether Dave could relate the process used in the proofs to
contrapositive statements, he was prompted with a direct question.

VB: So you said something about the contrapositive, alright? Um so, what is the
contrapositive of the statement \[\text{if } x \text{ is even then } x^2 \text{ is even}\] here?

Dave: Well, the contrapositive would be, I guess, it would be \"if \(x^2\) is odd, then \(x\) is
odd.\" So, I guess, maybe that's what (laughs) that would be a valid proof if you
were using the contrapositive method of proof, which, I mean, this to me is a valid
proof. So, I guess what I was thinking was this isn't a direct proof of this [given
statement], but it could be a proof by contraposition, I guess. So maybe [choice
number] one could be true, or this could be true if it was that approach.

Thus, it was finally revealed that Dave needed to be prompted in order to see clearly the
indirect relationship between the proof and the alternate statement.

In summary, all four students in this category exhibited a concept image that
showed their attachment to the direct semantics of the proof. Only Dan could make the
logical jump and show how the given (direct) proof could be turned around into an
indirect proof to show an equivalent statement. However, even then he lacked the full
relational understanding of contraposition needed to relate it to the process he described.

**Situation 2: Interview 1**

For integers \(a\) and \(b\), if \(a\) is not a multiple of 5 and \(b\) is not a multiple of 5,
then \(a + b\) is not a multiple of 5.

A counterexample can be found in order to disprove the statement in this
situation. However, in situations where one cannot find a counterexample, the best
alternative would be to use indirect arguments systematically. For instance, in this
situation, one can refute the conclusion of the statement (e.g., \(a + b = 5\)) and then
consider the partitions of the sum \(a + b\) (e.g., 2 + 3) until a contradiction is found.
Thus, the objectives of this situation were to observe the students’ approaches to finding a counterexample and to investigate the extent of their understanding of the concept of counterexamples. Combined with the second interview, the first interviews investigated students’ understanding of counterexamples in relation to an indirect process used in a proof by contradiction.

**Post Abstract-Math students’ responses to situation 2 (interview 1)**

All four students in this category successfully found different counterexamples. They all believed that one counterexample was enough to claim the statement false and that was the best and fastest way of disproving a statement.

Patty gave an argument which indicated that she viewed a counterexample as a particular case against the general.

Patty: Okay. So, 5 does not divide \(a\), 5 does not divide \(b\). So... does this imply 5 divides \(a + b\)? That's false because we can see a counterexample. Because 5 does not divide 7, 5 does not divide 8, but 5 does divide [...], um, 7 + 8, because 5 divides 15. So, that would be false.

VB: So, why... and what's so special about that one counterexample that renders the statement false?

Patty: For something to be true, it has to be true in all cases, and for it to be false, it only has to be false in one case. So, I mean, I could probably find another... [counterexample]. I could find another one, I mean, it’s pretty easy to do. Five doesn’t divide 12, 5 doesn’t divide 3, but again we have 15. We could even say 13, 5 does divide 25.

In order to validate her claim, Patty drew upon an external fact that she had learned in one of her mathematics classes in the past, but she did not indicate the exact source of the rule of inference.

VB: Okay. Well, do you consider this argument to be a valid proof?

Patty: Yes.

VB: Why?
Patty: Um, I guess according to rules of logic that we learn in 305 [Introduction to Abstract Mathematics], if you have one counterexample it’s enough to disprove the statement.

[...] To me, finding a counterexample is the best and fastest way to disprove something.

Patty seemed to have found the counterexample intuitively without much work.

When asked about different arguments to disproving the statement, Patty tried to generalize her argument but soon realized she was restating the same problem.

Patty: [...] Probably have to do something like $a = 5 + n$, where 5 does not divide $n$, and $b = 5 + m$, 5 does not divide $m$, and $n$ and $m$ are elements of the integers. And then show that, um [...] $10 + n + m$. [...] So, that would just say that 5 divides $n + m$. [...] It’s still coming down to finding a case. So, at the heart of it, it’s the same type of an argument.

Patty seemed to have internalized the process of disproving a statement by finding a counterexample, because in her response above she passively indicated that in her example (5 does not divide 7, 5 does not divide 8, but 5 does divide 7 + 8) the conclusion of the statement was negated. However, she did not reveal any signs indicating that she was aware of what she had internalized, especially when she had to refer to her mathematics class where she learned the rules of logic for justification. In later situations (6 and 7), it was observed that Patty was capable of using indirect arguments when searching for counterexamples, although in situation 6 she did not recognize her argument as indirect.

Perry also gave a similar type of argument for writing a generalized formula for generating counterexamples.

VB: So, what’s so special about this case [...] why it renders the statement false?

Perry: It is false. In this case, $a = 3$, not a multiple of 5, $b = 2$, not a multiple of 5, $a + b = 5$, which is a multiple of 5. But you could certainly pick other examples of $a$ and $b$ if you wanted. You actually could have $a$ equals $3 + 5$ to the $n$, and $b$
equals 2 + 5 to the $n$ [writes $a = 3 + 5n$ and $b = 2 + 5n$], and that would hold for all natural, for $n$ equal any natural number. So, if you wanted $a$ and $b$ that they'd hold for, [...] that general form gives you a bunch of counterexamples to it.

This latter reasoning came very close to using an indirect process similar to the one described earlier in the introduction of this situation, because Perry was thinking of two numbers that added up to a multiple of 5. He did not necessarily thought of two non-multiples of 5, such as 4 and 7, without regard to their sum. However, the student did not communicate it as indirect, because he was driven by its algorithmic approach that dictated a true hypothesis implying a false conclusion. When asked if he thought a different approach could have disproved the statement, he indicated his firm belief that in his experience, finding a counterexample was sufficient and did not acknowledge the use of indirect method for finding counterexamples.

Perry: Hmm... I think there's other ways of doing it, but I think showing one counterexample is the way that requires the least amount of effort. It's just I whipped out 3 and 2 off the top of my head, wrote them down because I knew that, after I thought about the problem for about a second, I just knew in my head that it would be easy to find a counterexample for this. And so, why bother putting more effort into it when a counterexample proves it false?

Indeed, for practical reasons, one normally does not bother putting more effort into a situation like this when he or she knows when (instrumental understanding) a counterexample can be found. However, one needs to have an understanding of the relationship between indirect processes and that of searching for counterexamples in order to know how (relational understanding) a counterexample can be found in this situation. As will be seen later in the discussion of his second interview, Perry considered finding a counterexample and using indirect arguments (contradiction) as two different approaches with no common goals.
Similar conclusions were drawn from the data for Paul’s case. Paul, like his peers, could only draw upon the sufficiency of finding a counterexample in order to explain why he needed only one to show the statement false.

VB: Well, are you absolutely sure that the answer is false?
Paul: It’s certainly false in general if it’s false for this case [counterexample: \(a = 24\) and \(b = 1\)].

VB: Well, why... why does this render the statement false?
Paul: Oh, well here I’ve taken, I’ve taken their... Well, I’ve taken their two assumptions. I’ve taken two integers, so, I’ve taken two integers, which are not multiples of... [5] and I’ve taken their sum. Then they said here, if this statement was true then this should never be true, for any integers. And 24 and 1 are both integers... and so, and neither of them are multiples of five, and so, but their sum is a multiple of five [emphasis added].

Again, it seemed that Paul had internalized the process that arriving at a false conclusion through true premises constituted a counterexample, but he could not explain why that was enough to disprove a statement. Furthermore, his words (“I’ve taken their two assumptions, which are not multiples of 5 and I’ve taken their sum”) implied that he thought of the counterexample while using direct arguments.

On the other hand, Pam struggled with her understanding of the procedural approach for finding a counterexample when she tried to explain why a counterexample was enough.

VB: So, what’s so special about your counterexample that renders the statement false?
Pam: Well, I chose \(a\) of the integers and \(b\) of the integers and I assumed, I said like, I took the if, which is that \(a\) is not of the form \(5k\) and \(b\) is not of the form \(5k\), basically they’re not multiples of 5. So then if you find two different numbers when added together do make a multiple of 5 you’ve shown that this statement can’t be true because... I guess... I don’t know. Um, (pause).

VB: Well, do you consider this argument to be a valid proof?
Pam: Yeah. I would definitely say so.
VB: Why? And what aspects make it a proof?
Pam: Um, I would say it's a proof—uh, what makes it a valid proof? I took the if part of the statement to kind of be like my givens, which you usually do in a proof, and through those I would be able to make a counterexample, which is a valid proof...

Um, I think there could be a different way to prove it, I don’t know if it would be any better. Because the way I was taught this is a valid proof of the statement [emphasis added].

Again, she thought she just needed to find a counterexample because she believed in that process when she was taught. It was concluded that Pam had accepted and internalized the fact that showing a counterexample was sufficient to disprove a statement but the reason for why that fact worked was not internalized. In other words, she possessed only an instrumental understanding of counterexamples.

In conclusion, although the data showed that the students in this category knew the procedural steps (instrumental understanding) of finding a counterexample, there was no evidence that indicated an internal awareness for why (relational understanding) this procedure worked best. They did not elaborate on the rules of inference that the procedure was based on. They would draw upon the procedure itself in order to explain why a counterexample was enough. On the other hand, there was no evidence in the data that indicated acknowledgement of any relationship between finding counterexamples and using indirect processes.

Abstract-Math students’ responses to situation 2 (interview 1)

Only three (Amy, Art and Adam) of the four students in this category found counterexamples to disprove the statement. Two (Art and Adam) of them had no difficulty producing a counterexample. The fourth student (Alice) claimed the statement was true but could not give any reason.
At first, Amy had some difficulty finding a counterexample. Her strategy involved looking at the contrapositive of the given statement but struggled to get it right.

However, through her trial and error she learned what needed to be done.

Amy: Uh... and then I was thinking just intuitively if I could think about $a + b$, if it necessarily, if they ha... they’re divisible by 5. [...] Let’s try the same thing, so that $a + b$ can be written as $5n$ and $n$ is an integer. And so we have to show that 5 divides $a$ and 5 divides $b$. [...] Okay, I’m stuck.

VB: Well tell me about the characteristics [...] or the aspects of the proof you were looking for.

Amy: I was trying to show that if 5 does divide $a + b$, or $a + b$ is a multiple of 5, then $a$ would have to be a multiple of 5 and $b$ would have to be a multiple of 5... So, any multiple of 5 ends in a 5 or a zero and [...] If it doesn’t end in zero or five... Oh, okay this is, I think this is false because... Okay I was trying to think if there were examples of when either $a$ or $b$ or both were not multiples of 5 and $a + b$ was a multiple of 5 which would be really easy to find, you could have any... I mean you could have 2 and 3 [...] or there’s a lot of... Okay, so I guess I just found a counterexample.

She was successful in finding a counterexample through her indirect quest for checking for contradiction. Unlike students in the Post Abstract-Math category, her understanding of counterexamples was not procedural. It actively involved testing, discovering and checking other possible routes.

VB: [...] Do you think, uh, there should be a better argument or a proof that uses a different approach than what you did here?

Amy: Uh... well in this case, it was easy to find a counterexample. But I suppose that... like for this one, yeah I mean since it was easy it works and it’s probably the fastest way to do it, but there’s probably an... There’s definitely another way to do it without an example.

Well I think I might’ve been onto something with this business, with this contrapositive stuff though I got stuck at, here when I was trying to show [...] Although I think that finding a counterexample would be the best, the easiest way.

It was observed that, through her thorough analysis of the situation, Amy was able to relate the indirect process to that of finding counterexamples, even when she believed finding a counterexample was easiest in this case.
Although the procedural understanding of finding a counterexample did not keep her from seeing its connection to the indirect processes in this situation, she too could not elaborate on the reasons why she thought a counterexample was enough. Furthermore, she drew upon the procedure to explain its validity.

VB: So, what’s so special about one counterexample? Why does it render the statement false?
Amy: It show... well, it shows one example where $a$ and $b$... $a$ and $b$ are not multiples of 5 and $a + b$ is a multiple of 5. So, that means it’s not true for all integers.

Alice attempted a similar approach through the contrapositive, but she failed to find a valid argument for why she thought the statement was true. Her incentive for invoking the contrapositive however was not the same as Amy’s. In her first interview on situation 1, Alice showed signs of contentment after she realized the usefulness of the method of contraposition in that situation. It was her impression of the power of contraposition from the first situation and not from her personal judgement of the second situation at hand that made her decide to give it a try.

Alice: [...] I don’t know what to do with this question (pause). I’d do contrapositive now that I think about it. [...] I would say that, I would assume, I would say, $a + b$ is a multiple of 5 implies that $a$ is a multiple of 5 and $b$ is a multiple of 5.

... If I could prove the contrapositive, then this original statement here would be true.
...
I would, I don’t know, just, just by looking at this [her own incorrect interpretation of contrapositive], I would say that it’s a true statement. [...] 

For the rest of the interview on this situation, while she was thinking that the statement was true, Alice kept pursuing the contrapositive approach but to no avail. It did not occur to her that the statement might be false.
On the other hand, Art, who had no trouble finding a counterexample for this situation, also did not elaborate on the reasons for why he thought a counterexample was enough to disprove the statement.

VB: Why? What’s so special about this counterexample that renders the statement false?
Art: Um, well, to prove a statement is false you just need one counterexample, and there’s a counterexample. If it was true, you would have to go and prove it. I don’t have to prove—This just proves it’s false right there because I just found a counterexample.

When asked if there were other ways of finding the falsity of the statement in this situation, he reiterated what he thought was the best approach and did not acknowledge any relationship between the process of finding a counterexample and other methods.

Art: Um, no, I think if that—finding a counterexample to prove that something’s false is the easiest and probably the most efficient way, I mean, at least for this proof here [in my counterexample].

Again, after easily finding a counterexample, he dismissed other routes for disproving the statement.

Adam showed a similar belief about counterexamples but also added: “it’s easier to prove that it’s false than to prove that it’s true.” This assertion however might have been influenced by his experience with this (easy) case. Moreover, as in his interview on the first situation, his belief appeared to be influenced by some authoritative reasoning.

This was observed in the next excerpt when he was asked what aspects of the process of finding a counterexample made him think that the statement was proved false.

Adam: What makes it a proof? Well, um, from what I was told [emphasis added] that through, um, from my teachers that if you come up with one counterexample to a statement then that proves that this statement is false.
Adam's intuitive approach did not reveal much about his perception of counterexamples or how he thought of the counterexample. When asked whether he thought different approaches might have given the same result for the situation, he tried to examine the contrapositive approach but found it hard to finish his argument, which made him reiterate and confirm his belief.

Adam: Well, yes, um, but counterexample is the easiest. And usually in life, you try to make things as easy as possible (laughs).

VB: And what kind of approach or strategy do you think that proof would use?

Adam: Hmm, well you could maybe try, like, um, a strategy that I learned from a book that is, um, a contrapositive. If $a$ is a multiple of five and $b$ is a multiple—no, see that wouldn't work either (short pause). Well maybe that would work, because the statement, if the statement is false then, um... I don't know, I don't know of another... I just like to take things as easy as possible, so, as far as I know the counterexample is the easiest way to work.

It should be noted that Adam did not invoke this particular approach (contraposition) at all when he needed it most in situation 1.

As with other students, Adam too did not elaborate on the reasons for why he thought a counterexample was enough to disprove the statement.

VB: Well what is so special about this example that renders this statement false?

Adam: Um, because it's a counterexample and if you come up with a counterexample to the situation then the situation is false. [...] You only need one.

It was observed that those students (Art and Adam) who quickly and easily found counterexamples did not acknowledge the usefulness of other approaches in this situation. Amy, who did not immediately see that the statement was false, had to try different routes, including indirect arguments, before she realized that a counterexample could be found. It was concluded from the above cases that unless students experienced some difficulty finding a counterexample, they did not rethink their own processes and
were quick to dismiss any relationship between the process of finding a counterexample and other useful approaches such as indirect methods.

**Discrete-Math students’ responses to situation 2 (interview 1)**

All four students in this category found valid counterexamples to disprove the statement. They all believed that one counterexample was sufficient.

Dan found the counterexample $a = 2, b = 3$ without any difficulty. He explained that a counterexample could prove a statement false by contradicting the statement.

VB: So, what is so special about your example here that renders the statement false?
Dan: Well, because I have my $a$ is 2 and my $b$ is 3. They’re both not multiples of 3, and I do $a + b$ and it is a multiple of 5. Oh, I said 3 for 5. So, basically it contradicts. It agrees with all of this [hypothesis], which is like the proposition, and doesn’t agree with the conclusion so that’s what makes it false. I guess.

His explanation referred to the rule of inference for contradicting a statement. He inferred from the rule that this particular statement was false. This explanation indicated that he viewed counterexamples as particular cases against the general.

His intuitive approach produced a quick solution to the situation. When asked if he thought a different approach could have disproved the statement, he did not reveal much about his perception of how counterexamples could be found.

Dan: Hmm. I think in some cases, yes. But I think in this case, no. […] But in this case I think this is enough to say that it’s false. And there’s like so many other examples you can think of.

Similarly, Dean was quick to see a counterexample in this situation.

Dean: […] That’s false and I can, I can, I think I can think of a counterexample, because if I set this [$a$] equal to 3, $b$ equal to 2, $3 + 2 = 5$, which is 1 times 5, which is a multiple of 5.
Again, there was no indication of how this counterexample was constructed other than direct intuition. When asked of different explanation or argument to disprove the statement, his response did not acknowledge any.

VB: Why?
Dean: I think mine's good enough. Because mine, because, because I've been told I guess. I don't know—in classes you're always told that if you can show a counterexample that's all you need to show [emphasis added]. And it makes sense that if you can show a counterexample that's all you need to show, because how can it be true if it's not true for this. It's certainly can't be true for all cases if I can list a case where it's not true.

The emphasized words above indicated that the source of his belief for why he did not acknowledge any other processes for finding a counterexample was some authoritative reasoning and he was satisfied by what he was told because he found it logical. When asked why a counterexample rendered the statement false he reiterated his earlier answer that the statement did not work for the case he had found.

Dean: If I use $a$ and $b$, $a$ is not a multiple of 5 and $b$ is not a multiple of 5, but $a + b$ is a multiple of 5, so the statement doesn't work for my numbers.

Although the student showed some knowledge, that one case was sufficient to refute the statement in general, he did not indicate that that reasoning was a result of the rules of logical inferences. According to those rules, the only time a general statement is false when the hypothesis is true and the conclusion is false. This particular result of the logical inferences shows that there are no exceptions (as in most other sciences) to general statements in mathematics.

On the other hand, after silently thinking about the situation for a while, Dave also claimed that the statement was false and that all he needed was to find a counterexample, which he did without any difficulty.
Dave: I think, uh, the answer to this one is false?
VB: Why?
Dave: Um, well, I think all I have to do is just show one case where it’s not true...
   [...] I have to find a case where \( a + b \) is a multiple of 5, where \( a \) and \( b \) up here are not multiples of 5. So...

Dave’s words above indicated the thought process for his reasoning. It appeared that he was using an indirect approach to the problem, because he was thinking of a multiple of 5, a sum of two numbers first and then partitioning it into two non-multiples of 5.

When asked for the reason why a counterexample rendered the statement false, he tried to explain that (according to the rules of inference) his example made the hypothesis true but the conclusion false. However, his words fell short of conveying his discernment of the exact meaning of the rule.

Dave: [...] If \( a \) is not a multiple of 5, \( a \) is not a multiple of 5, and \( b \) is not a multiple of 5. Those are true, so this is true. This is true. Then, so \( a \), um, and \( b \)—right?—implies, and now this C [conclusion] statement, I guess I’m just seeing where \( a + b \) is not a multiple of 5.
   Um, I think I mean, I think, what a statement like this is saying for all \( a \) and for all \( b \). So, if you can find one \( a \) and \( b \) that makes the statement false, then the whole statement cannot be true.

On the other hand, Doug had a little trouble at first before he realized that the statement was false.

Doug: [...] After maybe thinking about it further I can actually see where \( a + b \) could be a multiple of 5.
VB: And how is that? [...] How could the sum of \( a + b \) be a multiple of 5?
Doug: Well, if it was like 1 + 4 is a multiple of 5. But, but initially my first response was wrong.

Again, there was no indication of how Doug thought of the counterexample. When asked about whether he thought a different explanation could have disproved the statement, his response, to some extent, was similar to that of his peers.
Doug: Not in this case, no. I think this is adequate. Otherwise, it’s kind of a waste. I’d say this is adequate.

VB: Why?

Doug: [...] If it’s true, there may be other ways of showing that it’s true, but, uh, why waste the time? (Laughs).

 [...] It would probably, uh, if there were another way to do it, and someone showed me that way, it would, it would probably give me more insight into the, maybe the real nature of the problem. [...] But, uh, the case of counterexamples, a counterexample is adequate. Uh, I mean, there’s nothing more that needs to be done as far as I can see [emphases added].

As before, the simplicity of the situation did not help him see any relationship with other processes. However, he acknowledged that the true nature of the problem was not revealed to him by a counterexample. This was an interesting distinction from the other responses to this particular question. Other participants were so sure of their responses that they did not doubt that there could be more insight into a way that a counterexample could be found than just using intuition.

When asked about why he thought a counterexample could disprove the statement, he tried to argue that (according to the rules of inference) when the hypothesis was true but the conclusion was false, the implication would become false. However, his lack of use of correct terminology did not convey the exact meaning of this rule.

VB: So, what’s so special about this example that renders the statement false?

Doug: It, you know, in a way it actually rewrites the conclusion of that hypothesis (pause). This [1 + 4 is a multiple of 5] statement, actually, with a few more words would make the hypothesis true. Whereas the hypothesis with this [a + b is not a multiple of 5] conclusion is false.

In the first interviews on situation 2, it was observed that 11 of the 12 students found valid counterexamples. However, only two (Amy and Dave) students explicitly showed the indirect approach they used before finding a counterexample. The other
students either used intuition or could not explicitly reveal their approach. When they were asked to give different explanations or arguments to their approach (intuition) they were unable to relate it to other (indirect) processes. They did not necessarily relate the process of finding a counterexample to that of indirect methods or contradiction. These observations however were made in this (relatively easy) situation, which in some cases might have influenced their perceptions of the relationship between finding a counterexample and indirect processes. Similar research using a more difficult problem might produce different results.

Moreover, although the participants knew that one counterexample was enough to refute a general statement, there was some indication in the data that some of them were not aware of the source of the reason why a counterexample was a case of contradiction according to the laws of logic. When asked for an explanation of their beliefs for why a counterexample was enough to disprove a statement some students resorted to an authoritative reasoning. Many participants drew upon the procedure of finding a counterexample itself (instrumental understanding) to validate their answers. They implicitly knew that to find a counterexample they needed to find a case that made the hypothesis of the statement true and the conclusion false. In some cases, students lacked the correct terminology, which made it harder to discern their thoughts from their understanding of the rules of logic. In other cases, students used the specific counterexample they found in order to validate their arguments as general.

**Situation 2: Interview 2**

In this second round of interviews on situation 2, students were given a proof by contradiction of the statement (statement 3 in the choices):
If $a$ is not a multiple of 5 and $b$ is a multiple of 5 then $a + b$ is not a multiple of 5.

The proof essentially showed the existence of a contradiction if the conclusion of the statement was negated. Thus, its process could be related to the process of finding a counterexample in the following manner. The conclusion of the statement could first be negated: "If $a$ is not a multiple of 5 and $b$ is a multiple of 5 then $a + b$ is a multiple of 5," and then a counterexample, such as $a = 3$ and $b = 10$, could be found. In fact, Dave in the first interview used this same approach in his argument.

It should be noted that the steps of the proof with some elimination and rearrangement could be used to prove directly the alternate statement:

If $a + b$ is a multiple of 5 and $b$ is a multiple of 5 then $a$ is a multiple of 5.

Thus, the objectives of this interview were to investigate the students' understanding of the indirect process in the given proof and how it related to finding counterexamples.

**Post Abstract-Math students' responses to situation 2 (interview 2)**

All four students in this category found the proof to be a proof by contradiction to statement 3. Patty, for instance, gave the following explanation for her answer.

Patty: [...] We showed that... the one we looked at is that $a$ is not a multiple of 5 and that $b$ is a multiple of 5, and then we assumed that then they, $a + b$ was a multiple of 5, but that created a contradiction. So, it shows that if $a$ is not a multiple of 5 and $b$ is a multiple of 5, $a + b$ can't be a multiple of 5.

Perry, as in his second interview on situation 1, came to the conclusion by the process of elimination through matching the assumptions in the proof with the hypotheses in the statement.

Perry: Because, okay, the first thing I did, was I went and looked at the assumptions, to make sure that the assumptions were the same. And so we have $a$ is not a multiple
of 5. From here we see that \(a\) is not a multiple of 5; therefore, it can’t be these (crosses out #1 and 4), because, those aren’t the assumption it uses. The second part is that \(b\) is a multiple of 5. So, we can cross this [#2] one out. And then, finally, we go down and look to, we get the assumption that, \(a + b\) equals \(5j\). With some substitution that leads to a contradiction; therefore \(a + b\) not equal to \(5j\). Therefore, \(a + b\) is not a multiple of 5. So, this [#3] fits it.

Pam in her take-home task wrote the following explanation for why she thought the proof showed statement 3.

I chose #3 because we were given \(a\) is not a multiple of 5 and \(b\) is a multiple of 5, and we went of [sic] to show that \(a + b\) is also a multiple of 5. At the end we got an impossibility so we showed that, if given \(a\) is not a multiple of 5, \(b\) is a multiple of 5 then \(a + b \neq 5j\).

Pam’s explanation above did not explicitly indicate how the process of contradiction showed the conclusion, so the interviewer attempted to confirm her interpretation of the contradiction process in the proof by asking specific questions.

VB: So, the last line here says that something is impossible, what exactly is that impossible case?
Pam: The impossible case is you can’t add a non-multiple of 5 and a multiple of 5 to get a multiple of 5 back out. Yeah, so, according to our assumptions \([a + b \text{ is a multiple of 5}]\) and how when we work it through we see \(a\) has to be a multiple of 5, this contradicts what we had up above. You can’t assume that it’s not a multiple of 5 and then find out that it is.

VB: Okay. So, what is the consequence of that contradiction?
Pam: That if \(a\) is a multiple of 5 and \(b\) is a multiple of 5 then \(a + b\) is a multiple of 5. Or any order therein.

Thus, Pam exhibited an understanding of how the process of contradiction in the given proof showed statement 3.

On the other hand, Paul was preoccupied with the idea of contrapositive from situation 1 that for a moment he wanted to change his choice to statement 4 (if \(a + b\) is a multiple of 5 and \(b\) is a multiple of 5 then \(a\) is not a multiple of 5.) He thought that statement 4 might have been proved by contraposition.
Paul: Well, if you use the contrapositive thing... Maybe I should stick to this [#3] side 'cause I know it better [emphasis added]. [...] I thought of this as like a, like a straightforward proof by contradiction. It, it assumed these things, it assumed these two points [hypotheses] and then it used that to say—then, then it assumed that this sum $a + b$ was a multiple of 5 and it found that to contradict its first two points. [...] So they can’t both be true at the same time so, so if we assume these two cases then this, this, then this later assumption that $a + b$ is a multiple of 5 is false. [...] 

One thing apparent in his words emphasized above was that he was more familiar with the concept of contradiction than contraposition. This could be attributed to his lack of understanding of the relationship between the two methods.

In order to clarify and understand Paul’s explanation better, the interviewer asked particular questions. His responses confirmed that Paul was interpreting the contradiction process of the proof correctly.

VB: [...] you said, it’s contradicting something, right? Can you tell me what fact is it contradicting?
Paul: Oh, it’s contradicting that $a$ is not a multiple of 5.

VB: Okay, so what is the consequence of that contradiction?
Paul: Uh, that, that $a + b$ is not a multiple of 5. It, it says here that $a + b$ is a multiple of 5, then $a$ also has to be a multiple of 5.

Only two students (Pam and Perry) were directly questioned again about the relationship between the contradiction process and finding a counterexample.

There was no evidence from Pam’s interview that she could validate any existing relationship between the two processes.

VB: [...] In order to prove this [true/false] statement [if $a$ is not a multiple of 5 and $b$ is not a multiple of 5, then $a + b$ is not a multiple of 5] or disprove this statement would you have used something like this [proof by contradiction]?
Pam: Um, yeah, you could I guess. [She pauses and then changes her answer]. Well, with this [true/false] statement [if $a$ is not a multiple of 5 and $b$ is not a multiple of 5, then $a + b$ is not a multiple of 5] if you assumed that this [$a$ is not a multiple of 5] and this [$b$ is a not a multiple of 5] is true, then I’m pretty sure all you need is a counterexample to disprove this statement. Um... this statement [#3] is worded
just a little bit differently just with the $b$ [is a multiple of 5], um... and you can use a more general case to prove or disprove this [#3] up here. Down here [in true/false case], I would say it’s really... I would still go with the counterexample in this one, so I wouldn’t do the same thing.

Since the student used other (unknown from the data) means to find a counterexample in her first interview, she did not believe that the indirect approach in the given proof could be associated with that of finding a counterexample. This belief however could have been induced by the (easy) nature of the true/false statement in this situation.

Similarly, the data from Perry’s interview did not indicate any acknowledgement of the relationship between finding a counterexample and using indirect methods.

VB: You gave me some argument here last time [in the first interview], [...] would you say your argument—the approach that your argument takes is similar to the approach that this proof [in the second interview] takes?

Perry: No, this [proof by contradiction] is a much more complicated formal proof of asserting something whereas this [counterexample] is just a proof that an assertion is wrong. So, this, and one of the ways you can do that is merely illustrate by counterexample. And so in essence, this [counterexample] isn’t asserting, it’s just asserting that something else is wrong by counterexample, whereas this [proof by contradiction] is asserting that something is right.

The student regarded the processes of finding a counterexample and finding a contradiction as separate approaches towards different goals, one for asserting something false and the other true. His emphasis on the end goals as being different prevented him from understanding the existing intimate relationship between the two processes. He showed surface understanding of the two processes but lacked the deep understanding of the relationship between the two approaches.

All four Post Abstract-Math students, either in their first or second interviews, dismissed the existence of any relationship between the two processes because of their “textual interpretation” of the concept of counterexamples as being the easiest and fastest
way of disproving statements. Perry and Pam did not acknowledge the use of indirect processes or contradiction as an alternate path for finding counterexamples in their second interviews, and Patty and Paul in their first interviews, because of their quick success with using intuition. They both showed surface understanding of the two processes but lacked the deep understanding of the relationship between the two approaches.

Abstract-Math students’ responses to situation 2 (interview 2)

Three (Amy, Art and Alice) of the four students in this category found the proof to be a proof by contradiction of statement 3. The fourth (Adam) student thought it did not show any of the given statements.

Adam: [...] Well, it doesn’t show any of these, because it shows that, um, if $a$ is a non-multiple and $b$ is a multiple [of 5] and $a$ and $b$, $a + b$ is a multiple [of 5], then it shows that $a$ is a multiple [of 5]. So...

In his interpretation, Adam treated the three given assumptions in the proof as the hypotheses of the statement it proved, and the result (i.e., the contradiction that “$a$ was a multiple of 5”) as the conclusion. This approach of allocating the assumptions and the conclusion in the order they appeared in a proof indicated a concept image associated with direct methods of proving.

In the next excerpt, although he acknowledged the contradiction of one of the assumptions ($a$ is not a multiple of 5), he did not follow the argument in the context of contradiction to show statement 3.

Adam: [...] So it, it, it just shows that, it contradicts that, um, it, $a$ would have to be a multiple of 5. So, it says that...hmm...well, okay. Basically, what it says, what it shows is that for $a + b$ to be a multiple of 5, $a$ has to be a multiple of 5 and $b$ has to be a multiple of 5. That’s what it shows. [...]
The statement he pointed out above was not true in general and it could be compared with the statement given in the first interview, for which he had easily found a counterexample. Thus, the interviewer tried to observe if Adam could see the relation between the two statements.

VB: Does that mean that it shows, um, that this [if \( a \) is not a multiple of 5 and \( b \) is not a multiple of 5, then \( a + b \) is not a multiple of 5] statement is true?

Adam: (Pause) well, in theory yes. Uh, if... but, I don’t know.

VB: But, do you believe this statement is true?

Adam: No, because I, I did a counterexample which showed that it wasn’t true. So, somewhere there is probably in my thinking [...] He was baffled by his own contradictory remarks. He found that the proof showed the statement given in the first interview, but he also found a counterexample to that statement a day before.

To try to make matters clearer, the interviewer asked particular questions. The data indicated that although Adam realized that a contradiction was found in the proof, he could not see its connection to statement 3.

VB: And what kind of method is this proof using?

Adam: Yeah, I’d say it’s a direct method. It doesn’t say, it shows.... Well, actually, it shows contradiction, because...

VB: It contradicts what?

Adam: Well, it contradicts that \( a \), uh, it contradicts one of the givens, that \( a \) is [not] a multiple of 5. And through the proof it shows that \( a \) is a multiple of 5. So it shows that it contradicts for \( a + b \), um, to be a multiple of 5 then \( a \) must be a multiple of 5. So, it contradicts the original given.

Well, actually it shows it right here. If \( a \) is a multiple of 5 and \( b \) is a multiple of 5 [then \( a + b \) is a multiple of 5, as in statement #1], that’s what this shows, right here.

The student now thought that the first statement, the inverse of statement 3, given in the choices was equivalent to what he thought the proof showed. Again, the student
recognized the semantics of contradiction embedded in the proof but could not make the connection between its process and statement 3. Therefore, there was no evidence to indicate that Adam could correctly relate the contradiction process to the statement it proved.

As was mentioned earlier, the other three students found the proof using contradiction to show statement 3. Alice, who had trouble finding the truth of the statement (if \( a \) is not a multiple of 5 and \( b \) is not a multiple of 5, then \( a + b \) is not a multiple of 5) in her first interview, showed a thorough understanding of the process in the given proof.

Alice: Because, I felt that it was a proof by contradiction, because... I say we’re trying to prove statement number three and they’re saying if \( a \) isn’t a multiple of 5 and \( b \) is, then \( a + b \) is not. So, then they assumed, assumed that \( a + b \) was a multiple of it, and ended up finding a contradiction to the original, to the givens, which was... And the contradiction they found was that \( a \) is a multiple of 5, but originally we said it wasn’t a multiple of 5.

To find out whether this student could now relate the use of contradiction to the process of finding the falsity of the statement in the first interview, she was asked to compare both cases.

VB: Okay. So, now that you know more about this statement [#3] and its proof, can you tell whether this statement [from interview 1] is true or false?

Alice: I would say this is false now... Oh, wait (pause). I would say this is false, yeah.

VB: Okay. So, would you use, uh, the same kind of argument?

Alice: Yeah, I would use, I would, um, assume that \( a + b \) is a multiple of 5 and try and find a contradiction, which is the same way they do it here, which is, yeah. But, so now saying this I see that contradiction works here, so I’d use it here.

Alice acknowledged the usefulness of the contradiction process in disproving the statement from the first interview. Unfortunately at this point of the interview, the student became bored and disinterested. Thus, it was not clear from the data how she would have
used the idea from the given proof to show the true/false statement false and whether she could have found a counterexample.

Amy and Art gave similar explanations for why they thought statement 3 was shown by the proof.

Art: I chose that because in the last part of the proof we have a contradiction where we needed... which by definition means \( a \) is a multiple of 5, which contradicts that our first given that \( a \) is a non-multiple of 5. And that’s why I chose number three, uh, statement number three.

Amy: Because... okay, because they proved a contradiction here. They assumed that \( a + b \) is a multiple of 5, but then the result for \( a \) was a contradiction and [...] by saying that \( a + b \) was a multiple of 5 they eventually came to the conclusion that \( a \) was a multiple of 5, but in this situation we were assuming that \( a \) was not a multiple of 5... So it has to be that \( a + b \) is not a multiple of 5.

Furthermore, in the first interview on this situation Art did not acknowledge any relationship between finding a counterexample and using indirect methods, but now was more open to the existence of the relationship between the two.

VB: Okay, so here in the true/false situation you have used some counterexample. Would you have used this kind of proof in order to prove that statement false? Or, would you change anything in this proof to disprove this statement?

Art: Yeah, you’d have to change stuff in this proof because you’d have to assume that your \( b \) value is a non-multiple of 5 and it would change... But, um, you could go through and I believe prove it, or, prove that the statement is false, I guess.

He then went on, used similar semantics to a proof by contradiction, and in the process generated a counterexample.

Art: I found, uh, an \( a \) and \( b \) value that does work. So, what I did is I assumed that \( a \) is not a multiple of 5 and \( b \) is not a multiple of 5 and that implies that \( a + b \) is not a multiple of 5. So, um, starting my proof I assumed that \( a \) is not a multiple of 5, \( b \) is not a multiple of 5, and \( a + b \) is a multiple of 5. And then I went through and let \( a \) equal integer \( k \), \( b \) equal integer \( k + 1 \), set that \( [a + b] \) equal to 5 and found \( a \) will equal 2, \( b \) will equal 3, and, which 2 + 3 is a multiple of 5. So, the statement is false.
As the procedure of the method of contradiction dictated, in addition to the assumptions in the hypothesis of the true/false statement he also assumed the negation \((a + b \text{ is a multiple of 5})\) of the outcome. He then assumed that there were two consecutive integers, \(a = k\) and \(b = k + 1\), each being a non-multiple of 5, that partitioned the sum \(a + b\) and added up to (multiple of) 5. This resulted in \(a + b = k + (k + 1) = 5\), which then gave the solution \(k = 2\), thus the counterexample \(a = k = 2\) and \(b = k + 1 = 3\).

Although this approach worked for a particular case of non-multiples of 5 being consecutive integers, it showed that the student was successful in integrating both processes (contradiction method and finding counterexamples) to reach a goal. Despite his initial belief that there was no point in using other approaches in this particular situation, the above case showed how these firm beliefs could be changed under favorable conditions. In later situations, Art also showed his newly acquired perception of indirect methods as a useful tool for finding counterexamples.

Earlier in the analysis of the first interviews on this situation, it was noted that Amy through her endeavors, which included indirect methods, acknowledged that indirect processes were useful in finding counterexamples. In her second interview on this same issue, she also indicated similar perception.

In conclusion, data from both interviews on the second situation showed that Abstract-Math students were open to the idea of studying indirect processes as a valuable tool for finding counterexamples, more so than the students in the other categories. The Abstract-Math students’ openness to newly learned proof processes might have been the immediate result of their experience at that learning stage.
Discrete-Math students' responses to situation 2 (interview 2)

Two (Dave and Dean) of the four students in this category claimed that the proof showed statement 3 (if $a$ is not a multiple of 5 and $b$ is a multiple of 5 then $a + b$ is not a multiple of 5.) The other two (Dan and Doug) claimed it showed statement 1 (If $a$ is a multiple of 5 and $b$ is a multiple of 5 then $a + b$ is a multiple of 5,) that is, the inverse of statement 3. Three (Dan, Doug and Dave) students claimed that the proof used the method of contradiction. Dave gave the following explanation for his choice.

Dave: Um... the reason I chose this [3] is because that's what we assume. The first part of this is what we assume, and what we arrive at is the second part of this. And it follows from the proof $a$ is not a multiple of 5, and $b$ is a multiple of 5. That's the first part. Then, you know, so that's our A [if part of the statement]. And then $a + b$ we're saying is a multiple of 5, but since we say it's impossible that it's not a multiple of five.

VB: Okay.
Dave: So really it's like $A$ implies not $B$.

Although his explanation was not clear, further questioning determined that Dave thoroughly understood how the contradiction process of the proof worked in this case to show statement 3.

The semantics of the proof rather than its process, motivated Dan and Doug to make their claims. Neither of them saw how the assumptions and conclusion of the method of contradiction were bound together.

VB: Okay, so why do you think that the presented argument in this situation shows statement number one and not the other statements?
Dan: [...] $a$ is not a multiple of 5 and $b$ is, and then we're supposing that $a + b$ is a multiple of 5 also, and we're seeing what that kind of yields. And in that, we get that it's impossible for $a$ to be a non-multiple of 5, if these two hold true. And, so I think that that proves that $a$ has to be a multiple of 5 if $b$ is a multiple of 5, and $a + b$ is a multiple of 5. That's what I'm saying. [...]

VB: Okay. This [b is a multiple of 5] statement and this [a + b is a multiple of 5] statement are true then...
Dan: That one \([a \text{ is a multiple of } 5]\) has to be true too, […] because that’s what we got right here \([a = 5(j - k)]\).

Dan perceived that the contradiction was due to what was initially assumed \((a \text{ is a non-multiple of } 5)\) but did not see exactly how the indirect process led to the conclusion \((a + b \text{ non-multiple of } 5)\), which was initially assumed false. Put differently, he thought the result of contradiction was to negate the very assumption that created the contradiction. That is, according to his perception, the assumption “\(a \text{ is not a multiple of } 5\)” was contradicted and the conclusion was that “\(a \text{ is a multiple of } 5.\)” In his written explanation on the take-home task, Dan also indicated that since the proof assumed that \(b\) and \(a + b\) were multiples of 5 then it proved that \(a\) had to be a multiple of 5. Both those observations indicated that Dan was following the direct implication (If \(a + b\) is a multiple of 5 and \(b\) is a multiple of 5 then \(a\) is a multiple of 5) made in the proof without regard to its hypothetical (false) conclusion’s impact that indirectly brought forth contradiction. Thus, Dan’s thought process was mainly centered on direct reasoning.

Although Dan stated that a contradiction was reached in the proof, his perception of its process seemed to have implied that statement number 1 fit best with the proof’s assumptions and conclusion. This perception however did not persist, as the next excerpt showed. He realized that statement 1 had a different conclusion than he perceived the proof showed.

VB: And what is the consequence of that impossibility?
Dan: The consequence of that impossibility is that [the sum of] a multiple of 5 and a non-multiple of 5 cannot ever equal a multiple of 5. The sum of them can never equal 5. So, I guess it’s proven this one [circles statement #4: If \(a + b\) is a multiple of 5 and \(b\) is a multiple of 5 then \(a\) is \textbf{not} a multiple of 5]. Yeah.
Above, he had actually spelled out the correct statement (If \(a\) is \textit{not} a multiple of 5 and \(b\) is a multiple of 5 then \(a + b\) is \textit{not} a multiple of 5) the proof had shown. Nevertheless, he did not pick that statement because the hypotheses that he thought the proof used fit better in statement 4. His strategy for matching the proof by contradiction with a statement was based solely on matching assumptions.

The association of Dan’s concept image with direct proofs became apparent when he believed the proof found a contradiction. His misunderstanding of the process of the proof hindered him from finding the correct statement in this situation because he tried to match the given statements with his perception of a direct proof. Thus, it was concluded that Dan was not totally consistent with his perception of proof by contradiction. He could not correctly distinguish the assumptions (hypotheses) in the given proof from the hypothetically false assumption that would bring forth a contradiction and ultimately the true conclusion.

Similarly, Doug could not explain how the pieces of the contradiction process were bound together in this situation.

VB: Okay. So, now why do you think the presented argument in this situation shows statement number 1?
Doug: Well, like I say, this was a proof by contradiction. It assumed not \(A\) and found a contradiction and then, from that, we can deduce that \(A\) is true, or, that was the structure I saw anyway.

VB: [...] What is the assumption?
Doug: Well, the, well, just as it states, that it’s not, that \(a\) is not a multiple and \(b\) is a multiple of 5. But, uh, what it actually shows is, is that, ultimately it shows that \(a\) and, or \(a + b\) is a multiple of 5, and therefore \(a\) must also be a multiple of 5. So, you know, it’s the fact that we’re trying to… I don’t understand. It’s just a logical structure that’s all I can say. And I don’t understand all of the nuances and intricacies of it, but there is some power in doing that sort of thing. In setting up certain things, certain ways are easier. Proving certain kinds of problems in
certain ways are easier than other ways. It’s just something I’ve learned a little bit about in the past couple years.

Doug’s perception of the proof was that it assumed two things, “a not a multiple of 5 and b a multiple of 5.” Then it showed that “a + b was a multiple of 5,” from which it was deduced that “a had to be a multiple of 5.” In other words, his perception of the process of contradiction was that one of the hypotheses (a is a multiple of 5) in statement 1 (if a is a multiple of 5 and b is a multiple of 5 then a + b is a multiple of 5) was negated and as a result created a contradiction, implying that the assumption had to be true and thus the statement was true. This misconception was a result of his misunderstanding of the process of contradiction in general. Moreover, it was observed that Doug’s words were rather driven by authority in his explanation of his perception of proofs by contradiction. He admitted that he did not understand the intricacies in the proof.

The following excerpt also confirmed Doug’s lack of understanding of the contradiction process. He thought the process was more about choosing a right assumption as if it was a hypothesis testing. In other words, he would negate an assumption and search for contradiction. If a contradiction were found then that assumption would not be true.

VB: Okay, so you said, you said, uh, it uses proof by contradiction [...] What is the contradiction?

Doug: The contradiction is that, that a is a multiple of 5, whereas we assumed that, we said it wasn’t. We assumed that it was not a multiple and we found a contradiction. The contradiction is that a is a multiple of five. Therefore we can assume, that gives us, uh information that we can, uh, assume that a is a multiple of 5.
It was concluded that Doug’s understanding of the contradiction process was insufficient, and when faced with a contradiction process he could not follow its argument correctly.

In conclusion, Dan and Doug had different perceptions of the process in the given proof when they claimed that it showed statement number 1. Neither of them could correctly distinguish the assumptions (hypotheses) in the given proof from the hypothetically false assumption that would bring forth a contradiction. Dan was unable to find the correct statement shown by the proof by contradiction, because of his perception of its process as direct or his mindset for direct methods. Doug was unable to find the correct statement because of his inadequate understanding of the contradiction’s indirect process in terms of its goals.

On the other hand, as mentioned earlier, Dean asserted that the proof showed statement 3. He gave the following explanation for his choice, which seemed like an interpretation of why he thought statement 3 was true rather than why the proof showed it true.

Dean: [...] Well, it just seemed like it was kind of saying that if \( a \) can’t be composed of 5 times, or \( a \) cannot be, and \( b \) can be, then you really can’t get something that can be out of that, if you add them together.

VB: Okay.
Dean: It’s like if you add an even and an odd, you’re going to get an odd, always.

Dean was trying to describe the process of contradiction in the proof, but he did not remember the name of the method used, possibly because the proof did not use the word “contradiction” explicitly.

VB: Okay. So, what is the type of approach or method used in this proof?
Dean: I guess this would be a... I don’t know what that would be called.

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It's where you assume something and then at the end you end up with something that isn't true because of your assumption, that zero equals one or something. [...] Um, since the case is impossible we can assume the assumption is false.

Evidently, Dean's perception of the process of the proof was that it reached a contradiction, but despite attempts by the interviewer, it was not clear from the data whether he had actually understood how the result of that contradiction proved statement 3 or his words were just motivated by the semantics of the proof. For instance, the following excerpts indicated how unspecific his answers were.

VB: So, what is the impossible case?
Dean: The impossible case? Um, \(a + 5k = 5j\), therefore \(a = 5(j - k)\). Five times anything means that \(a\) can be composed of 5 times something, and we know that \(a\) is a non-multiple of 5, so we know that's impossible.

VB: So can you explain the role of the assumption and its consequence in this proof?
Dean: Um, the assumption was kind of like assuming it was true...

VB: Okay.
Dean: ... then, the following would also be true. But since the following is not true, it can't be true.

It was observed in his first interview that Dean did not acknowledge the use of other methods in finding counterexamples. After seeing the proof, he admitted that he could have used similar arguments to disprove the statement given in the first interview, although he thought it would be inadequate.

VB: Now, would you have assumed or used an argument similar to the proof here in order to prove this [true/false statement] false?
Dean: Um. I would have thought it was adequate with just the example of it not working, but I could have used a proof like that to prove it false.

In summary, except in the case of Dave, the data did not indicate that the students had a clear and adequate understanding of the contradiction process. Dean was unable to communicate his understanding of proof by contradiction, which led to inconclusive
results. Dan and Doug could not correctly distinguish the hypothetical assumption that created the contradiction and the conclusion from the hypotheses of the proven statement.

**Situation 3: Interview 1**

Consider a triangle ABC. A straight line through the midpoint of the segment AB and parallel to segment AC bisects the third segment BC.

This problem was used to observe students' approaches to a simple geometric situation. One of the objectives was to check if students would consider such a situation with an insight into Euclid's parallel postulate. It may be unreasonable to expect such insight from students especially when they were unaided. However, the purpose of this problem was to use its results as a backdrop for triangulation of the results from the second round of interviews on this situation. The take-home proof-checking task drew upon Euclid's parallel postulate and used a proof by contradiction to show that the given statement above was true. The second interview aimed at finding students' perceptions of that particular aspect of proof in this geometric situation. Thus, it was necessary to observe the students' own approaches in this situation before drawing any conclusion about their understanding of the given proof.

**Post Abstract-Math students' responses to situation 3 (interview 1)**

Although all four students in this category agreed that the truth of this statement would be independent of the type of triangle used in their drawings, three (Pam, Patty and Paul) of the four students in this category found the statement to be true. Two of those (Patty and Paul) used the most popular argument of similar triangles. The fourth student (Perry) after about 13 minutes of experimenting with trigonometric relationships (sine and cosine rules) in a triangle failed to find any answer.
Pam gave a simple geometric argument for why she thought the statement was true, although she did not consider her argument to be a valid proof.

Pam: [...] If you bisect $AB$ and make a line parallel [to $AC$], you’re basically just moving this [$AC$ parallel] down this way [towards $B$]. I kind of just used that as judgement that it does bisect both of them—both of the segments $AB$ and $BC$.

VB: So what would it take [to make a valid argument]?
Pam: Um, I think it would take to show that, um... first you’d have to show the two line segments were indeed of equal length on each side when you bisect this [$BC$]. Um, you’d probably show this and draw your line. Um, you’ll probably want to show equal angles. Um, and stuff like that through using the laws of geometry and stuff. Um, that I don’t remember.

Although she did not find any valid argument for her answer, her argument seemed to indicate that she was trying to remember the argument of similar triangles.

Paul on the other hand, after a few trials with triangles, gave a proof of similar triangles and used their proportional length property to show that the statement was true.

When asked if he thought he gave a valid proof of the statement, he responded that he needed to prove the proportionality property as well.

VB: And do you consider the argument of the similar triangles to be a valid proof here?
Paul: Uh, it’s, yes it’s, but it leaves—it’s making that assumption that the ratio of the lengths is the same in similar triangles. [...] So the argument’s valid if that’s true, but I don’t think it has any authority without that statement being proven also.

VB: So all the facts that you use in a proof must be proved...
Paul: ...at some point.

Paul showed this belief throughout his interviews, which sometimes made him uneasy about his answers.

Similarly, Patty used the proportionality property of similar triangles to point out that the statement was true without actually showing why the triangles were similar at first. To absolutely be sure about her argument she had to try an obtuse triangle.
VB: Well, does your conclusion hold for this particular figure?

Patty: Um, I’m pretty sure that it is [a generalized proof]. I can draw... Let’s do a non... do an acute triangle [she draws a triangle with two small acute and one obtuse angles] and see if it still works. Okay, so we’ve got ABC if we bisect it and go parallel, and call this F and G. These two still have to be equal if it’s... if they’re actually equal. So this is still going to hold, and these are still going to be similar triangles, so, yeah, I’d say that it works in general.

It was observed earlier that Patty also showed a similar behavior in situation 2 when she had to try other counterexamples to be sure and later in situation 6 where she had to find a counterexample after she showed existence of counterexamples.

VB: Well, how many of these do you need to try in order to say it works in general?
Patty: Um, I think as long as it’s not dependent on what the triangle itself... Um, because these aren’t dependent on what the triangle looks like, I just was making sure that it... But because you’re, because it’s parallel, um, you’re going to be—you’re going to intersect the other side. [...] This was followed by a general proof that showed why the triangles were similar. Then she explained why she considered the similar triangles argument as valid.

Patty: [...] I guess, technically you’d have to verify that we’re talking about a tri—a Euclidean triangle, but—and we’re talking about Euclidean parallelism. But, as long as that’s assumed, which I’m guessing was the assumption, then that would be valid.

Here it became clear that the student was aware of the Euclidean context in the problem, but when asked if she could think of other arguments for solving the problem she mentioned analytical and trigonometric methods. Later questioning tried to expand on the Euclidean context to see if the student could have perceived contradiction due to that context. However, there was no evidence in the data to indicate that she had perceived the situation in the context of contradiction. This could be explained by the lack of relational understanding of that context, because a full internalization of relational understanding of
the Euclidean parallelism would be necessary for someone to understand its deep structure and to recognize its limitations in this situation.

**Abstract-Math students’ responses to situation 3 (interview 1)**

Two (Alice and Adam) of the four students in this category claimed that the statement was true. The other two (Amy and Art) claimed that it was false. They all based their answers on the figures they drew.

An interesting finding was that Amy tried to use similar triangles to tackle this problem but she was not convinced that the proportionality property could work for unequal side-lengths of a triangle.

Amy: Okay, well I can now divide these into similar triangles.

VB: Okay, how do you do that?

Amy: We know that we can... We have ratios that we can establish... Like okay... DE so DE is to AC [see Figure 4.2] as... oh I don’t know if I’m remembering this right, as DE is to AB? I don’t know if that’s right I just made that up... Yeah I’m going to say that’s right.

Although Amy had shown some doubts about the proportional side-lengths of similar triangles, her observation of different triangles led her to a false conclusion. She did not show any willingness to thoroughly check her answer against her initial observation.

Amy: I said it’s false only because I drew the picture [of a general triangle] and it looked like it was.

Based on her drawings, she claimed the statement was false in general but true only for specific triangles, such as isosceles and equilateral triangles.
Amy: Uh, yeah that was just like my first thought, but actually if I really think about it, it doesn't say what sort of triangle it is... Okay I guess that doesn't matter because you could find an example that works but you can find one that doesn't... So, you could say that it doesn't work for all of the triangles

VB: So, the statement is true for isosceles triangle but not for any other triangles?

Amy: Or for equilateral but just because we found two examples doesn't mean that this is true.

Data from this interview also indicated that Amy believed that geometric figures could be considered as counterexamples.

VB: Well, do you consider this kind of argument to be a proof for your answer?

Amy: Well, I consider... like these, if you can... these [my drawings] are counterexamples here, yeah, where a straight line with all these properties does not bisect the third segment... So since we found at least one [...] then we've proven that that's not true.

Moreover, Amy also believed that the similar triangles argument could also lead her to the same conclusion that the statement was false.

VB: Do you think there should be a better argument for a proof that uses a different approach than yours?

Amy: Well you could go... you could like... with all this similar triangle stuff spell it out and show that if none of the sides are equal then... then this can't be true. [...] As before, Amy did not show any willingness to check this claim for a general triangle because she was not sure of the proportionality property of similar triangles.

Similarly, Art claimed that the statement was false in general but true for certain triangles. Unlike Amy, he did not claim he found a counterexample through his drawings.

Art: There exists a triangle, um, that fulfills the requirements that a straight line through the midpoint of segment AB and parallel to segment AC that bisects the third segment BC, but not for every triangle.

VB: Do you have an idea of the kind of triangle, you think this is true for?

Art: Um, maybe an equilateral triangle.
One of the triangles he drew was an equilateral triangle in which the statement appeared to him to be true. However, he found it hard to believe that all types of triangles would be consistent with the statement.

VB: So, I want to know what made you […] say that the statement is false. Is that just a guess?
Art: Um, when I first drew my triangle, I was thinking, since uh […] the straight line through the midpoint AB and it has to be parallel to AC. I would… I kept thinking if the one side is longer than the other, how could it be parallel and still bisect it? That’s where my guess came from and that’s where I would start if I was doing this problem, but (pause)

Since he believed there were infinite possibilities, he doubted that the statement could be true, even when he failed to find a solid counterexample above.

Similar to Art’s but contrary to Amy’s belief, Alice did not think that diagrams prove or disprove anything. She started by applying the hypothesis of the statement on an equilateral triangle she drew, but she realized “no one ever said it was as equilateral triangle.” She then drew a non-particular triangle and gave the following explanation.

Alice: It still kind of looks, but this is just a picture, so I don’t really know how much this proves. I’ve got to think about a way of going about proving this all. […] Well, it kind of looks like it’s going to be, because it’s asking… trying—Oh, wait… (talking to self). Okay. We’re trying to figure out if it in fact does bisect BC. And it looks like it does…
   - - -
   Well, there’s an infinite number of triangles. I can’t just do it by drawing pictures. I know that. There’s got to be some logic behind that I should figure out.

   While struggling for a proof she briefly looked at the angles but soon abandoned the approach.

Alice: Well, I can go with, look at the angles, I guess.
VB: What about the angles?
Alice: Well, let’s see. No, I guess I can’t. Never mind. No, that doesn’t work (laughs). All I know is that these two [segments] here are the same and these two lines are parallel. I don’t know how I’d put that into a [proof].
Although she knew the limitations of using drawings for asserting geometric situations, her failure to give a generalized argument made her claim in the end that the statement was true based on her figure.

Adam also knew, through authority (teacher), the limitations of drawings in proofs. However, he claimed that the statement was true because he did not have any argument against his drawings and he was in a situation where he could only look at examples.

Adam: [...] Well, I guess one thing my teacher said to me was that proof by pictures was not about proof, so this isn’t about proof, but… She also said that examples give you a good idea of maybe what could happen. Um, and again I think through estimation and guessing, guess and check, and my trial and error I believe this to be true.

In summary, three students (Art, Alice and Adam) in this category based their findings on the diagrams they drew. Although they did not believe pictures could prove or disprove the statement, they could not find other alternatives. On the other hand, Amy tried to use the proportionality property of similar triangles but could not argue correctly. Furthermore, she believed diagrams were enough to draw conclusions from.

**Discrete-Math students’ responses to situation 3 (interview 1)**

All four students in this category claimed that the given statement was true. Two students (Dave and Doug) based their answers on the figures they drew and did not find any valid argument for their choice. Neither of them considered their figures as proofs but they both indicated that figures could give them a sense of the situation.

The other two students (Dean and Dan) claimed that the result followed from similar triangles. Dean drew a triangle and thought the statement was true, but did not think that a figure would be enough to make any conclusion. However, he soon realized
that the proportionality property of similar triangles could prove the result, although he admitted that he was not totally sure of his argument.

Dean: [...] Like if this side, if these two triangles were similar and this side is twice the other one, then all the rest of the sides follow that same proportion. I think that this side will be half of this one, and this will be half of this one. I don’t know really how to write that down.

VB: Well can you absolutely be sure that your approach or argument supports your answer?

Dean: I can’t absolutely be sure because I’m not completely sure that similar triangles follow that, but I think if I knew that for sure I think that I would, I would be absolutely sure, but I’d want something more formal, obviously.

Although, in his proof, Dean did not strive to show why the triangles were similar, he was able to explain how the given statement followed from the proportionality property of similar triangles.

Similarly, Dan was convinced that the statement was true because it seemed intuitively obvious to him from the triangles he drew. He thought similar triangles would show the result but did not exactly know why or how to make the similar triangles argument.

Dan: [...] Well, again, that seems intuitively obvious to me because I can think of a lot of triangles (he drew a slender triangle) and that works for them... Why exactly it works though? (Talking to self). Well, it’s definitely going to be, like, a similar triangle. Why is it the midpoint, though? (Talking to self and pause). Yeah, it still works. Why is that though?

Although he did not consider his approach to be a valid proof, he based it on drawing self-similar (Sierpinski) triangles as shown in Figure 4.3.

VB: Can you think of any particular approach for a proof? Or a method of proving?

Dan: Well, I’m thinking somehow if you do this [joining all three midpoints of the sides of a triangle], it does sort of create a similar triangle, and like no matter what there’s going to be like four of them [smaller similar triangles]. Does that make sense? [...]
Figure 4.3: Dan’s drawing from situation 3

Apparently, Dan had seen Sierpinski triangle before and he could relate that image to this situation. However, in his argument he did not indicate how or why the segment formed by joining the midpoints of the two sides would be parallel to the third side. There was no indication in the data that he could tell why the triangles were similar.

In conclusion, there was no evidence in the participants’ approaches to indicate that they would consider such a geometric situation with any insight into Euclid’s parallel postulate. It was pointed out earlier that expecting such an insight would be unreasonable especially when the students were unaided. However, the above conclusion was drawn because it was essential for triangulating the results from the second round of interviews on this situation. It was necessary to observe the students’ own approaches in this situation before drawing any conclusion about their understanding of the given proof from the second interviews.

Situation 3: Interview 2

In this second round of interviews on situation 3, the participants were given a proof of the true/false statement seen earlier:

Consider a triangle ABC. A straight-line \( k \) through the midpoint M of the segment AB and parallel to the segment AC bisects the segment BC.
The proof drew upon Euclid's parallel postulate and used contradiction to show the given statement true. It used implicit semantics by assuming a contradictory result only in Figure 2 (Appendix H, Situation 3) of the proof (reproduced here in Figure 4.4 and will be referred to as Figure 2 in the discussion of this situation). The interviews aimed at finding students' perceptions of that particular aspect of the proof in this geometric situation, as well as if their perceptions of the contradiction were limited to its explicit semantics.

![Figure 4.4: Figure 2 in the proof of situation 3](image)

One of the objectives of this interview was to investigate students' understanding of arguments using implicit contradiction. In contrast to situation 2 where the contradiction was obtained from one of the assumptions in the hypothesis, this one investigated students' understandings of the role of a fact (postulate) as the source of the contradiction to an implicit assumption made in a figure. Another objective of this interview was to investigate how students' perceptions of the given proof changed when the semantics of a proof by contradiction were made explicit in the alternate proof provided during the interviews (see Appendix I, Situation 3).

**Post Abstract-Math students' responses to situation 3 (interview 2)**

All four students in this category found the proof using a direct approach that followed from the given facts. None of them initially indicated that Figure 2, given in the proof, assumed the negation of the conclusion in the statement (as a first step to using contradiction), which was contradicted at the end of the proof.
Pam’s explanation indicated her understanding that the statement was proven as a direct consequence of the given facts. Her words did not bear on the figure as an implicit assumption.

VB: So, why do you think that the presented argument in this situation shows that the statement is true?

Pam: [...] But from [fact] two, if you join the midpoints of AB and BC, which makes the segment MN, they’re parallel to the third side. But according to Euclid’s Postulate there could only be one line going through that point. So, through looking at it, they kind of go through it and reading this [last line in the proof:] you see that line k—segment MN is on line k.

VB: What is the nature of argumentation or method used in this proof?

Pam: Um, I’m not sure. I would say it was fairly direct [emphasis added], but I don’t know.

Patty and Paul gave similar explanations for their understanding of the process of the proof, which were also interpreted as direct.

Patty: Um, well, I didn’t really think that the proof gave a whole lot. Basically, just from the prerequisite facts it’s, it makes it necessary that they be true. I mean, you put those two together and it’s true, but this [proof] just maybe illustrated it a little better. Um, I guess, because there can only be one line parallel to AC through M, but if we already know that the line joining the midpoints is parallel, then they obviously... they have to be the same two lines, or the same. Both have to be the same line, which is the argument here. [Emphasis added].

Also, Paul gave the following explanation.

Paul: Um, because logically it just follows, I guess. [...] I think all the steps are legitimate. And so, yeah, and so it shows that if this is true—the initial, they started with is true—that it leads to this conclusion.

On the other hand, Perry initially gave a circular reasoning for why he thought the given proof showed the statement true. However, later questioning in the data confirmed that, like his peers, he perceived the process of the proof as direct. His perception did not bear on the implicit assumption of the figure in the proof.

VB: So, what is the type of argument used in here?
Perry: It seems, it's a direct proof. When in doubt, default to direct. It uses the, the laws of geometry, which are cool.

The next questioning in this round of interviews investigated students’ perceptions of the assumptions used in the proof.

VB: Do you think that this proof used any assumption, explicit or implicit?
Pam: Well, um, it assumed Euclid’s Postulate. It really is cool (laughs).

Patty gave a similar answer to that question, but Perry referred to the hypotheses in the statement.

Patty: Well, it, it used the prerequisite facts.

Perry: Um, it uses the assumptions that $k$ [M] is the midpoint and that $k$ is parallel to AC. Those are the assumptions that it makes.

There was no indication in the data that they understood the role of Figure 2 in the proof. Their lack of understanding of the implicit assumption in Figure 2 was confirmed next.

VB: Okay. Other than those prerequisite facts and the hypothesis in this statement, is there any implicit assumption used in this proof?
Pam: Um, oh... No, I don’t think so. Let N be the midpoint (pause). No, I don’t think so. You kind of have to assume line $k$ is parallel to AC, but they tell you it is.

Similarly, Paul’s answer to that question was: “No I guess (pause). I don’t think so.”

The pauses throughout Paul’s interviews were signs of lack of confidence due to his incertitude about proofs.

On the other hand, Patty gave a rather interesting and insightful answer to that question.

Patty: Um, I don’t think so. I mean, I guess they assume that there’s only one midpoint, but I don’t think that’s a very hard assumption to make. I mean, there can only be one midpoint. Um, I guess just the fact that a midpoint is going to be, that a midpoint bisects a line, or line segment. But, I don’t know that that’s an assumption or just, I would say it’s obvious.
Although this was a valid observation, it was not sufficient to say that the student had seen the implicit assumption and contradiction in Figure 2 of the proof, at least not yet. Thus, the interviewer investigated whether she was aware of the contradiction process built in the proof.

VB: Well, do you think Euclid's parallel postulate contradicts anything in this problem?
Patty: If the figure is drawn accurately in both pictures, then there's obviously a contradiction but, um, they can't... If Euclid is considered to be correct, then they can't both be accurate, because you wouldn't have... They would be on the same line.

Here, the sentence “they can’t both be accurate...because they would be on the same line” indicated that the student was becoming aware of some contradictory process in the proof. However, she was unable to verbalize it because she did not see the connection of its argument to the figure. She lacked the relational understanding of the contradictory implications of Euclid’s postulate.

In order to confirm the above observation, she was asked to examine the alternate proof that made, in addition to Figure 2, the semantics of contradiction explicit.

Patty: [...] I would say that this proves it's true, and in this case it's proved by contrapositive. Because you're assuming that it's not true and then coming to a contradiction.

VB: So is the reasoning used the same or different? Do they use different reasoning?
Patty: I would say the reasoning is the same, but the method is different. You make an assumption here [in the alternate proof] and then use the same reasoning.

VB: So what method do they use?

Patty: [...] Um, [in the original proof] they're just, I think they're just going step-by-step finding things that are true based on the prerequisite facts that we know. I don't know...

VB: And the method here [in the original proof] is different than that [alternate] one?
Patty: Well, it's not a proof by... I guess it's kind of proof by contrapositive just because looking at the picture it's assuming that they're not the same line and then it proves that they are the same line. It doesn't explicitly say that we're assuming they're not the same, but maybe it implicitly assumes that. I don't know. [Emphases added].

Although she put it as contrapositive, this last paragraph indicated that she was becoming aware of an implicit contradiction in the original proof. Earlier, it was observed that Patty did not see this implicit assumption in the proof. Therefore, she was experiencing a change of perception of the original proof, by becoming aware of the figure's assumption, only after the specific questioning and her own reflection on the situation in light of the alternate proof. Since her last words above indicated that she was still not sure, she was going through a process of the relational understanding of the contradiction by the help of the alternate proof.

The data from the other interviews also indicated that although Pam, Perry and Paul correctly understood Euclid's parallel postulate and its relation to fact 2 in the proof, they did not think it contradicted Figure 2 in the proof. Perry, in particular, stated his observations in this next excerpt with confidence.

VB: Do you think Euclid's postulate, uh... parallel postulate contradicts anything in this proof?
Perry: Um... no, it doesn't contradict our assumptions. It doesn't contradict any of the ideas.

Next, the interviewer investigated their perceptions of the contradiction method built in the proof by asking them to examine the alternate proof that used explicit semantics of the method of contradiction.

Pam: I like that [alternate] one better.
VB: Okay. Is the argumentation in both these two the same?
Pam: No, this [alternate] one uses, um, I think it's contradiction. I'm not extremely positive on—I'm pretty sure it is because they assume—they assume that \( k \) does
not bisect BC, but down below they show that it has to. So, it contradicts the assumption, so it means the assumption is—you have to flip the not over to a yes.

Clearly, the semantics used in the alternate proof were transparent to her. However, data from further probing indicated that her perception of contradiction was limited to the method's explicit semantics and its procedural process.

VB: So you don’t think both these proofs use the same method?
Pam: I don’t think so. No, this [alternate] one they start off saying that \( k \) does not bisect line BC, and in this [original] one I believe they kind of end up showing that line \( k \) does bisect BC. So, this [original] one you kind of end up with that. This [alternate] one they start off using that idea. I think this [original] one’s more direct, and this [alternate] one’s more contradictory.

Both Paul and Perry gave similar answers to that of Pam’s, except that they preferred the original proof to the alternate one because they thought it was easier to understand. In this respect, unfortunately Pam did not give any reason for her preference of the alternate proof in her interview.

VB: Okay. Well, can you explain the similarities and the differences between these two proofs?
Paul: Yeah, this [alternate] one used contradiction. [...] And this [original] one just proved it directly. [...] They essentially use the same facts to get to the answer.
VB: So, if you were to choose, if you... which one would you prefer?
Paul: I think this [original] one. [...] I don’t know. I just think it’s clearer. [...] I think it’s just easier to, to follow when it’s, when it’s directly than it is contradiction.

Perry also gave similar responses to those questions.

Perry: This [alternate] is a proof by contradiction. Saying that if we assume this and this, we get something that is contradictory. Therefore, one of our assumptions must be false. [...] They’re different in form, although they [...] prove the same thing.
VB: Are they using the same approach?
Perry: No they’re not. This [original] is direct, this [alternate] is proof by contradiction.
VB: Okay. Which one do you like?
Perry: I, I like the direct proof better. I think it’s a little bit quicker to follow, and it’s, \textit{direct proofs are just...a lot of times are easier to look at than the proof by contradiction} [emphasis added].
Clearly, the semantics used in the alternate proof were transparent to both Paul and Perry as well, and as with Pam, their perceptions of contradiction were limited to the method's explicit procedural process in this situation. These limited perceptions were also confirmed by their preference of the original proof because they thought it was straightforward and clearer.

In summary, it was concluded from the data that Patty, prompted by the interviewer's questions, reached a practical judgment of the contradiction process built in the original proof, but she lacked the relational understanding necessary to have observed that process on her own. Patty showed some degree of change in her initial perception of the proof in light of the alternate one. In contrast, the other students' (Pam, Paul and Perry) perceptions of the process of contradiction were limited to its semantics, which was not sufficient for them to reach a correct judgment of the method used in this particular situation. Their initial understanding of the original proof did not change in light of the alternate one that made the implicit assumptions explicit. They thought the first proof used direct methods and the alternate one used contradiction.

Abstract-Math students' responses to situation 3 (interview 2)

The distribution of students' answers in this situation was uneven. Two (Art and Adam) of the students in this category believed that the given proof showed the statement false and Alice believed that it proved the statement neither true nor false.

Only Amy believed that the proof showed the statement true, and she gave the following explanation for her belief.

Amy: Uh, well, it just... it just shows that this line $k$, if it's parallel... if it goes through M and is parallel to AC, then it must also go through N, which is the midpoint of
BC [...]. It just describes two different lines and then says that they are the same thing [emphasis added].

The emphasized sentence above came very close to describing the process of contradiction in the given proof, but data from the later part of the interview on this situation indicated that she initially did not perceive the implicit assumption used in the proof. They consistently indicated that her understanding of the process utilized in the proof did not go deeper than her initial words where she thought the proof used direct methods.

VB: So, can you name the method of proving used here?
Amy: Um... you mean direct or otherwise? [...] It's just direct.... It's just.... Well.... Yeah, it just takes these facts given and from there... reaches a conclusion about.... Yeah, it's just direct.

VB: Do you think Euclid’s parallel postulate contradicts anything in this problem?
Amy: (Long pause). Well, I guess they’re kind of saying that MN and $k$ are separate things, and they don’t... They’re not really explicit about the fact that MN is just a segment of the line $k$... which might imply that there are two lines through $M$ that are parallel [emphasis added]. But since they say that they coincide, that seems pretty clear.

Her words above indicated again that she could almost see how different parts of the proof were bound together, but could not make the ultimate transition to see the contradictory approach in the proof. This was also confirmed from the following excerpt:

VB: So, the parallel postulate does not contradict this situation [that you described], or some fact that is in this situation?
Amy: (Long pause) Um... I don’t think so.

Further probing, using the alternate proof, fell short of finding whether this student’s initial understanding would change in light of the explicit semantics used by the alternate proof. At the time of this interview, the researcher thought that her initial understanding had changed as expected, but upon further review and analysis of the following excerpt, no definite conclusion could be drawn.
Amy: I would say that it [alternate proof] proves it true and that it’s actually, *seems like a better proof, to me* [emphasis added].

VB: Are they the same at all?
Amy: Uh... a little, but this one starts off by assuming that it doesn’t bisect BC.
VB: And what does that mean?
Amy: Well, then you come to the conclusion that there’s two lines through M that are parallel to AC, but that they’re not the same line because one bisects BC and the other doesn’t. But that’s a contradiction to Euclid’s parallel postulate.

The semantics used in the alternate proof seemed transparent to her, and her perception of contradiction in the proof was evoked by the method’s explicit process. The fact that she liked the alternate proof better could confirm her perception of the overall process of contradiction when made explicit. But there was no distinctive data about her understanding of the original proof in light of the alternate one, to conclude if she could have identified whether both proofs used similar approaches. It should however be noted that Amy had come a long way from initially believing that the statement was false (see discussion of her interview 1 on this situation) to being convinced otherwise due to the given proofs.

An interesting finding was that unlike Amy, Art was able to perceive the contradiction in the original proof but he thought the statement was proved false because a counterexample was found by Figure 2 in the proof. In other words, he took Figure 2 to be the conclusion rather than the assumption in the proof.

Art: Um, by using the fact that only one parallel line can go through one point, that clearly shows that visually in Figure 2 that there’s two lines that will go through a point but they’re not going to be parallel. So it’s a counterexample, it proves the statement false.

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VB: Okay. So, what is the nature of argumentation or the method used in this proof?
Art: Contradiction.

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VB: Okay, so where did the contradiction come from?
Art: Umm, it contradicts the Euclid's parallel postulate that there's two parallel lines through a point.

According to Art's understanding, the proof found two separate lines that passed through a point M not on $\overline{AC}$. Thus, he concluded that neither of them could be parallel to $\overline{AC}$; otherwise the parallel postulate would be contradicted. His perception of the process of the proof led him to believe that Figure 2 was its result rather than the assumption, and its conclusion was that $\overline{MN}$ could not be parallel to $\overline{AC}$. In other words, he understood that the parallel postulate could not be contradicted but failed to also see the second fact's irrefutability as well. It should be noted that the second fact in the proof forced both line $k$ and $\overline{MN}$ to coincide and thus made the statement true.

The next question intended to clarify Art's understanding of the assumptions used in the proof. However, there was not enough in this next excerpt or in later data to conclude whether he fully understood the relationship between the hypothesis of the given statement and those facts.

VB: Okay, do you think that this proof used any assumption?
Art: It uses the assumptions via these prerequisite facts.
VB: Which are?
Art: Euclid's parallel postulate and the segment formed by joining the midpoints of two sides of a triangle is parallel to the third side

It should be noted that, in his first interview on this situation, Art believed that the statement was false because counterexamples could be found. Thus, in light of that observation, it was concluded that Art's perception of the contradiction process of the proof in this situation was induced by his belief of existence of counterexamples.

The above conclusion could raise a question whether Art might have interpreted the process of the proof differently had he not attempted a proof of his own, in the first
interview. In other words, his interpretation of the proof might have been different if he did not have any prior misperception about the given statement, or if his perception of this situation was rather abstract and free of any geometric context. In contrast, Amy in her first interview also believed that the statement was false but her interpretation of the proof was not influenced by her belief.

To find out more about why Art believed that the statement was proved false, and if he would change his perception of the contradiction, he was asked to compare the original proof with the alternate one.

VB: Do they use different methods?
Art: No, I think they are using the same method [contradiction], but this [alternate] proof seems more complete.

VB: Contradiction, okay (pause). So, what is the consequence of the contradiction?
Art: It proves the statement false. The consequence is that there are more than two parallel lines going through one point, and, that’s not possible.

VB: So, if you were to re-write this statement so that it’s true, how would you write it?

Art: Okay, consider triangle ABC, a straight line \( k \) through the midpoint \( M \) of the segment \( AB \) and is not parallel to the segment \( AC \) bisects the segment \( BC \).

The above excerpt indicated the same perception that Art had about the original proof except that he found the alternate proof to be “more complete.” Further, his last sentence where he rephrased the statement he thought was true, confirmed the earlier analysis about his mistreatment of fact 2 as false.

It was concluded from the above observations that Art had insufficient understanding of the relationship between the given facts and the proofs, which prevented him from perceiving its exact process in this situation. He was unable to interpret the contradiction process of the proof correctly, because it led him to refute a given fact rather than a hypothetical assumption.
Like Art, Adam also claimed that the proof showed the statement false because the triangle $ABC$ in Figure 2 ended up having two lines parallel to $\overline{AC}$. Again like Art, his perception of the process of the proof led him to believe that Figure 2 was its result rather than the assumption. This was also confirmed when he claimed that the proof did not use any implicit assumptions. Unlike Art however, Adam argued that there could only be one parallel line and thus line $k$ did not bisect $\overline{BC}$.

VB: Why, do you think that the presented argument in this situation shows that the statement is false?
Adam: Well, um, it's, it has two, two lines and there only should be one [emphasis added]. [...] Okay, it, it shows that, um, the line $k$ parallel to AC does not bisect segment $BC$, because the bisection [sic] of $BC$ is $N$, and $k$ does not go through $N$.

Yeah, because line $k$ does not pass through the point $N$ and the point $N$ is what it needs to pass through to make the statement true.

The above excerpt, where Adam claimed that the proof showed that the parallel line $k$ did not bisect $\overline{BC}$, as well as this next one indicated that Adam believed that fact 2 would have been untrue (contradicted) if $k$ had actually bisected $\overline{BC}$.

Adam: It says the segment formed by joining the midpoints of two sides of a triangle is parallel to the third side.[...] That's a fact. [...] But it coincides with the fact that, um, a straight line $k$ passes through $M$ is parallel to $AC$, as given. Therefore, [reads the last line in the proof] from the first fact above the straight line $k$ and $MN$ coincide.

The context in which the above words were uttered suggested that his overall perception of the proof was that it showed the straight line $k$ coinciding with $\overline{MN}$, but he believed that it was an incorrect conclusion because fact 2 was being contradicted. Thus, he thought Figure 2 was the correct conclusion and the given statement was proven false.

It should be noted that Adam did not explicitly use the word “contradiction” in his explanation, but that was the implication of his words, although he described the method
of the proof as direct. This observation was also confirmed from the next excerpt, where he was asked to explain his understanding of the role Euclid’s postulate played in the proof.

VB: So, what’s the role that fact [Euclid’s postulate] is playing in this situation?
Adam: Okay, so then, the role it has is that it’s saying that, um, the segment MN and the line $k$ should be the same line inside the triangle from points M, N. [...] 

VB: Do you think that Euclid’s parallel postulate contradicts anything in this situation... in this proof?
Adam: Well, it sets up a contradiction. It says, um, that like again segment MN and line $k$ are supposed to be the same line, but they’re not.

VB: But how does the parallel postulate come into play in this figure [2]?
Adam: (Pause). Comes into play in saying that, um, uh, segment MN should be the same line as $k$. That’s where it comes in. So it’s either, it’s saying that either line $k$ or line MN is not parallel to AC [emphasis added].

So far, the data consistently indicated that Adam understood the substance of the proof except that he mistook its conclusion because he could not make the right connection between the body of the proof and its implicit assumption in Figure 2.

To further probe Adam’s perception of the process of the proof and whether it would change under explicit semantics, he was asked to compare it with the alternate proof.

Adam: [...] It just, it [alternate proof] proves that the statement is false..

VB: [...] Is there a difference between these two proofs?
Adam: Well, very little. It’s just that it [alternate proof] says now assume the opposite that straight line $k$ does not bisect segment BC. [...] 

VB: And do they use the same reasoning?
Adam: Yeah, it’s pretty much straightforward, straightforward in saying that straight line $k$ and MN coincide. So, then, they are both saying that their initial assumption is incorrect [emphasis added].

VB: And what is that assumption?
Adam: Uh, uh, that straight line $k$ is parallel to AC [emphasis added]. Isn’t that right? That’s our first assumption. Or was it this [passes through midpoint M] one?
The initial assumption in the alternate proof was that the straight line $k$ did not bisect $BC$ and it was not explicitly stated in the original proof. The emphasized sentences above indicated that Adam misunderstood the assumptions made in both proofs. His misunderstanding led him to think that the straight line $k$ could not be parallel to $AC$.

It was concluded that although Adam understood the given facts and how they related to the essential parts of the process of the proof, he could not see how those parts fit into a whole coherent proof method. Not only was he unsuccessful in making the connection between the proof and its implicit assumption in the given figure, but also his perception of the original proof did not change in light of the alternate one.

Unlike her peers, Alice thought that the proof was wrong and that it failed to prove the statement either true or false. She gave the following explanation for her choice:

Alice: Because, we're trying to say that a straight line $k$ through the midpoint M and parallel to this $[AC]$ bisects BC, but the way they start out this proof is letting N - this be the midpoint of BC. So, they're assuming what they want to prove, which they can't do.

Alice was arguing that $N$, which happened to be the intersection point, could not be assumed as the midpoint but needed to be proved as midpoint. Thus, the role played by the assumptions in this proof did not make sense to her. Alice exhibited a similar perception of assumptions in situation 6.

After she had made it clear that she thought the proof was incorrect, she was asked if she could fix the mistake.

Alice: [...] What I would do is, since we're given that this point M is the midpoint, and we're given that it's [line $k$] parallel to AC, I would use that. I would then go up to this [fact 2] here that says the segment formed by joining the midpoints of two
of the sides of a triangle is parallel to the third side. And since we know that it goes through one midpoint and it's parallel to the third side, then it must be that it hits the midpoint [N]. And work in the way that you should, instead of how they assumed what they were trying to prove.

In other words, she would leave out the first line of the proof (let $N$ be the midpoint of $BC$ . Construct the segment $MN$ ) and instead she would add the hypothesis (let line $k$ parallel to $AC$ pass through the midpoint $M$ of $AB$ ) of the proven statement. Her reasoning here implied that she would be ignoring the construction steps of the proof because it was not yet clear in the proof's argument which way it should be constructed as Figure 2 implied. This showed that the student was trying to make a direct argument starting from the hypothesis and reaching the conclusion.

Although she indicated what the role of the second fact given in the proof was, she did not make it explicit that the conclusion followed from Euclid's postulate in her argument. So, the interviewer probed her understanding of Euclid's parallel postulate.

VB: Okay. Well, can you explain the role of Euclid's postulate in this proof?
Alice: What they say is, well... (Talking to self) what they're saying is if you take any point that's not on this line $[AC]$ right here, there's only one parallel line that can be drawn that goes through the point not on that line. But the part that we use is that if, if a line goes through a midpoint on a triangle, and it ends up being parallel to one of the sides then it must go through the midpoint of the other one [same as the given statement]. Or, you can say that if it goes through the midpoints of the two sides then it's parallel to the third [same as fact 2]. But, you just kind of use a different angle looking at that to solve this, to prove this.

The student was trying to argue that both the given statement and fact 2 were equivalent and that one followed the other, but still she did not see the role of Euclid's postulate as a sufficient condition for them to be equivalent. In other words, the binding postulate for eliminating a contradiction in the given statement was not perceived by Alice in this proof, which led her to claim that the proof neither proved nor disproved the statement.
To investigate the question of whether her understanding of this proof would change under explicit semantics, she was asked to compare it with the alternate proof.

VB: So, here's another argument in support of the statement. Please read it and let me know what you think.

Alice: (Pause). Well, uh... I would say the same thing—I think this is another wrong proof, because (pause).

VB: Well, are these two proofs similar at all, or they're just saying different things?

Alice: No, they're different proofs. [...] They're just both wrong, I think.

VB: So, well, what are the differences between those two?

Alice: Well, here [original proof] they started assuming that this was the, N was the midpoint of BC and, um, then here [alternate proof] they say that this one doesn't bisect BC, the line \(k\) doesn't bisect BC. And here [original proof] they said it went through the midpoint [N] meaning that it does bisect it. (Short pause) But, I think that this [alternate proof]'s wrong also.

Above, her initial perception of the original proof did not change. Thus far, there was no indication in the data that Alice had perceived the process of contradiction even when its explicit semantics were used in the alternate proof. In order to confirm this observation, the interviewer asked the following question:

VB: Well, do you think Euclid's Postulate contradicts anything in this [alternate] proof?

Alice: Well, yeah, because they end up saying that there's two, two lines that work, while this says there's only... They end up saying there's two parallel lines that go through the midpoints. Or I guess they end up saying there's two parallel lines to this side of the triangle, but this [Euclid's postulate] says there's only one that goes through a point not on the line, so then, that's a contradiction to it, yeah.

Although the student made a remarkable observation of the process of the proof here by actually seeing a contradictory result to the postulate, she did not make the transition necessary to change her initial choice because she still was not convinced that the alternate proof showed the statement true.

VB: It [still] didn't prove the statement true?

Alice: (Pause). No, I don't think it does.
In contrast to her interview on situation 2, where she understood the contradiction process used there, this student failed to understand the proof by contradiction used in situation 3. Thus, it was concluded that Alice’s understanding of proofs by contradiction was insufficient to help her see it in various settings. In other words, she lacked relational understanding of the method.

In summary, it was concluded that although Art perceived the contradiction in the original proof, he was unable to interpret its process correctly because he thought the proof created a counterexample in Figure 2. Adam also thought the proof refuted the given statement because he could not see the role of Figure 2 as the assumption that was refuted. On the other hand, Alice did not perceive the contradiction process in the proofs because the assumptions, including Figure 2, did not make sense to her. None of the students changed their perception of the original proof in light of the alternate one.

**Discrete-Math students’ responses to situation 3 (interview 2)**

Three (Dean, Dan and Doug) students claimed that the proof showed the statement true. Dave thought the proof used a counterexample to show the statement false.

Dean initially claimed that the proof neither proved nor disproved the statement, because he thought Figure 2 was not in agreement with the given proof. However, he later claimed that it showed the statement true.

Dean: Um... I guess I couldn’t really follow the proof that well. I guess I was kind of misled because k didn’t overlap MN [in Figure 2]. […] But I guess it kind of, it [the argument in the proof] makes so, it says that they do [overlap].

... I believe the argument... So I guess. I guess it shows the statement is true, but the picture doesn’t show that.

VB: Does it [the picture] have to?
Dean: Um, it would make it a lot easier for me to understand it if it did.

Dean’s confusion was due to his inability to see the connection between Figure 2 and the argument in the given proof. In the next excerpt, the interviewer tried to investigate if Dean understood the role of Figure 2 in the proof.

VB: Well, I think the picture is drawn in order to help understand the argument. So, what’s the idea of drawing an incorrect picture and making an argument about it in the proof?

Dean: Um, I guess they kind of state that therefore the straight line $k$ and segment MN must coincide.

VB: Okay. So, that implies...?

Dean: The picture’s wrong.

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VB: So what do you call this argument?

Dean: Um, I’m not sure. Um, I’m not sure.

Dean thought that the argument in the proof showed that Figure 2 was incorrect. This indicated that his perception of the process of the proof was in an appropriate context, but since he did not recognize the process as that of contradiction, it meant that he did not understand its role as an assumption at this point. This was also confirmed next.

VB: But, other than those [facts and hypotheses in the statement], is there any assumption used in this proof?

Dean: No.

VB: Why do you think so?

Dean: Because I think it wouldn’t be a valid proof if you were assuming more than you were given.

Only when the question of the role of Figure 2 was put directly, did Dean indicate that it was used as assumption. Nevertheless, he was not able to recognize the role of that assumption in the proof and thus the process of contradiction used, because he did not correctly explain the consequence of that assumption.

VB: So do you think the picture itself assumes something?

Dean: Uh, yeah, I guess so. It [incorrectly] assumes that $k$ does not coincide with MN.

VB: And now what is the consequence of that assumption?
Dean: Um, the consequence of that assumption is that MN is not parallel to AC, because only one line going through a point can be parallel to another line on the same plane [according to Euclid’s postulate].

The consequence mentioned above was invalid because it contradicted the second given fact in the proof. There was no indication in the data that he was aware of that contradiction. The correct consequence of the assumption made in Figure 2 would be a contradiction of Euclid’s postulate.

VB: Do you think Euclid’s Parallel Postulate contradicts anything?
Dean: Um, in Figure 2, if MN were parallel, then it would contradict it, because MN and $k$ do not coincide in that picture.

Now, Dean perceived Figure 2 as the conclusion or the consequence of the contradiction following from Euclid’s postulate, because he thought straight line $k$ and $\overline{MN}$ could not both be parallel to $\overline{AC}$. And since it was given that $k$ was parallel to $\overline{AC}$, then $\overline{MN}$ could not be parallel to $\overline{AC}$. This perception of the overall process of the proof indicated Dean’s lack of understanding of the method of contradiction used in the proof. This case was similar to those of Art and Adam.

It was concluded that in this situation, when students did not understand the exact purpose of the figure and thus the method of contradiction used in the proof, they would resort to violating a given fact instead of rejecting the assumptions of the figure.

The next phase of Dean’s interview investigated his perception of the alternate proof in relation to the original one.

VB: So, you said the second proof shows that the statement is true. […] Well, explain similarities and differences between these two proofs, if there are any.
Dean: Um, I think they’re very, very similar. They both deal with the fact that MN and $k$ must coincide. So it has to, so they have to be… So, MN has to be parallel to AC.
It was observed that the alternate proof had changed Dean's perception of the goal of the argument but not the process used in the original proof. He did not perceive both processes to be the same because he identified the original proof as direct.

VB: So why do you think this one [alternate proof] here is indirect?
Dean: Because we showed that if they weren’t, if they didn’t coincide, it doesn’t, it’s false. So, they have to coincide. So, I think that makes it indirect.
VB: Okay. So, you said these two proofs are very similar. You don’t see any difference between the two?
Dean: Well, I think now I think [the alternate] one’s indirect and [the original] one’s direct, but they both show the same thing and they use the same kind of logic.

In conclusion, the data indicated that Dean did not perceive the method used in the original proof as contradiction. His perception of the original proof did not change even after the alternate proof was examined. He could see both proofs aiming at the same goal for proving the coincidence of straight-line $k$ and $MN$, but he could not identify the processes used in the proofs as similar.

On the other hand, Dan claimed that the statement was proved true because it followed from the given two facts. The question of whether he understood the role of Figure 2 was investigated next and at first, he did not think it was used as an implicit assumption in the proof.

VB: All right. So, other than those that are in the statement is there any assumption in the proof?
VB: What about the figures?

VB: Do they consider any assumption?
Dan: Um, well they’re definitely not trying to, they’re not drawing it like parallel or not yet. Like, they’re not, they’re kind of assuming, they’re not drawing like this line and this line on the same exact plane. But that’s not really an assumption. That’s saying that we don’t really know yet [emphasis added]. I don’t know.
Although at first he was not sure what the role of Figure 2 was after he was directly questioned about it, he realized that the situation, which was not known initially, was the assumption in the figure.

Dan: [...] So we’re assuming that $k$ really, I guess we’re assuming that $k$ isn’t the same, isn’t on the same line as MN so far, because that’s what we’re trying to prove.

VB: So what is, what is the consequence of that assumption in that figure?

Dan: If $k$ is the same? Well, I guess since that’s what we’re trying to prove; it would be kind of dumb to assume—Well, yeah, I guess we assumed that’s true on a lot of other stuff. I guess I’m not really sure what the consequences of that assumption are. There’s always consequences to assumptions, though.

At this point, the process of assuming the opposite of what needed to be proved was not making sense to Dan. However, he soon remembered seeing the same kind of process used in the past and was quick to withdraw his comment. It seemed that he was actually going through the process of relational understanding at that juncture.

The next line of questioning tried to find out whether Dan could figure out the exact process used in the proof.

VB: So, what is the role of Euclid’s Parallel Postulate in this proof?

Dan: [...] So this $[\overline{AC}]$ is the line that everything’s parallel to. Right? So, from this one [fact 2] we know that since these are the two midpoints and this is a line segment $[\overline{MN}]$, then this $[\overline{MN}]$ has to be parallel to this $[\overline{AC}]$. Right? But we also, just through our supposition that this straight line $[k]$ through this midpoint $[M]$ is also parallel to this $[\overline{AC}]$. So, that’s where Euclid’s Parallel Postulate comes in because [...] there’s only one line that does that. So this line, if it’s parallel to this, going through this point, and this line that’s parallel to this, going through this point, means that they’re the same line, because there’s only one possibility.

It was clear from Dan’s explanation that he understood how different pieces of the proof led to the desired result in this proof. However, his explanation fell short of arguing how that process was related to Figure 2. In other words, there was no indication that he viewed Figure 2 as an assumption that was contradicted by Euclid’s parallel postulate.
So, further probing tried to find out whether Dan could tie together the overall process to the method of contradiction.

VB: So, do you think Euclid’s Parallel Postulate contradicts anything in this problem?  
Dan: It definitely contradicts this diagram [Figure 2].

Um, pretty messed up. I don’t know, like, because you can’t exactly—because that’s what you’re trying to prove, that these two lines coincide, right? So, they’re doing that because they’re saying we’re not really sure... I don’t know. It’s hard to say because I know these coincide. But since you’re proving it I guess you have to assume that they don’t [emphasis added], or that you’re—oh, I don’t know. I’m not sure how that is.

Through guided questioning above, Dan was able to interpret correctly the separate steps of the proof and how they led to the desired conclusion in the statement. However, he was not sure if the overall process could be associated with the method of contradiction learned in his current mathematics class. Thus, it was concluded that Dan did not have a thorough understanding of the way the method of contradiction worked even when his own interpretation of the steps of the proof led him to that process. In other words, he lacked the full internalization of the method necessary to make him realize its connection to this situation.

Next, the interviewer produced the alternate proof to see whether Dan’s perception of proof by contradiction was limited to its semantics.

VB: I want you to tell me if there’s any difference between this [alternate] proof and this [original] proof.  
Dan: Okay. (Pause). Um, instead of, instead of assuming that, um, $k$ does bisect BC, we’re assuming $k$ doesn’t bisect BC and we’re trying to prove, and we prove it wrong. [...] I guess this [alternate] proof’s better, I’d say.

VB: And what does this [alternate] proof do here?  
Dan: I guess it takes you more step-by-step through it, because it, um, tells you your assumptions instead of having you assume them, kind of. Like it illustrates it better, because we were assuming that it does not bisect BC just because of the diagram kind of. Do you know what I’m saying? But it never actually said it.
Dan’s explanation of the added steps in the alternate proof and the fact that he preferred it over the original one indicated that he was finally seeing the connection between Figure 2 and the given argument in the proof. That connection was not completely realized when he examined the original proof by its own. This indicated that his perception of proof by contradiction was limited to the semantics used in the alternate proof, and that he initially lacked the full internalization of the indirect process necessary to make the above mentioned connection from the original proof.

The following excerpt also confirmed the aforementioned conclusion that Dan made the ultimate transition to finally perceive the process of the method of contradiction through the alternate proof.

VB: What is the method of argumentation used in this [alternate] proof?
Dan: I guess it’s a little, I guess it’s more indirect because you’re assuming that what you’re trying to prove is false. And then you’re seeing how that helps you out. And if you’re assuming that it’s false and then you find that there’s some contradiction in there you know that it’s true.
VB: Is that, is that a valid argument?
Dan: I think so, because if it’s not false it has to be true. That’s like the only other possibility.

Doug also claimed that the statement was proved true because it followed from the given facts. Initially, he did not seem to perceive the role of Figure 2 as an assumption and his perception underwent some changes as the interview progressed until he could see somewhat the role of Figure 2 in the process of the proof. In fact, like Dan, he was going through the process of relational understanding with every question.

VB: Do you think Euclid’s Parallel Postulate contradicts anything in this problem?
Doug: Uh, no I didn’t actually. I don’t. I mean, the way they drew this one [Figure 2], it seems like it’s not parallel is the whole thing.

VB: So, what does that say about the problem or the situation?

Doug: Well, it shows that, it’s almost like it’s kind of misleading in a way, I don’t know. [...] The only way the thing holds is if they’re on the same line. It goes through MN, and the picture is actually not correct.

VB: So, is that a valid presentation for a proof...if the picture is not correct?

Doug: I guess it gives information, I mean, it gives a starting point. Uh, you know, it’s making some assumptions [emphasis added]. You know there’s assumptions...

At this point in the interview, Doug was starting to get a better perception of the process used in the proof. However, he was unable to verbalize the assumption in Figure 2 other than pointing out that Figure 2 was contradicting the real phenomenon.

VB: [...] is it [Figure 2] conveying anything when it’s wrong?

Doug: Yeah, it’s kind of like a, it’s a, well it wouldn’t necessarily be like a contradiction—well it might be a contradiction. It contradicts, you know, reality, or it contradicts the truth. So, it is an example. It’s not true. That is not true, k is not parallel with AC.

Unlike Dan, Doug was unable to explain the relationship between different parts or steps of the proof. He could not see how those steps led to the desired conclusion in the proof. He claimed the figure contradicted the truth but did not perceive that the contradiction was hypothetical and it was for argument’s sake (assumption) that caused a violation of a known fact.

The analysis of the data from the later parts of the interview, where Doug’s understanding of the alternate proof was probed, also confirmed the above result. It was observed that Doug did not see the process of contradiction used in the alternate proof, especially when he thought the statement was proved false this time.

VB: Okay. Here’s a proof of the same statement, a different proof. Um, please read it and let me know what you think.

Doug: (Pause). Yeah, I think that actually follows that... That fits the picture.

VB: So, what would be your choice?

Doug: I would say it proves that it’s [the statement is] false.
When asked whether a statement could be true and false at the same time he gave a confusing and inconsistent explanation.

Doug: [...] what throws you off is the picture to me. It is a true statement, that, uh [...] [...] Well I meant by saying this was false was, what I meant was, I wasn’t saying that this statement was false. I was saying that the conclusion drawn here was false, that the initial statement, the initial assumption was incorrect. Well, no, the initial assumption is correct, the reasoning here is false.

VB: So, the reasoning given in the second proof...
Doug: …is not consistent.
VB: Is not consistent, but the one given in this first proof…?
Doug: …is, is consistent. It seems to me (Laughs).

There was no indication that Doug had perceived the alternate proof as being similar to the original but using explicit semantics of the method of contradiction. Although he sensed passively the process of contradiction integrated in Figure 2 of the original proof, the alternate proof did not prompt him to consider that method. It was concluded that Doug did not perceive the method of contradiction employed in this situation, because as his words above implied, the alternate proof to him did not seem to be consistent with what he thought was shown by the original proof.

Unlike the previous students in this category, Dave claimed that the original proof showed the statement false, because he thought Figure 2 explicitly concluded that straight line $k$ did not coincide with $MN$, thus producing a counterexample. Again, it seemed that Dave misunderstood the role of Figure 2. The interviewer tried to investigate this matter further and as this next excerpt indicated, Dave did see Figure 2 making the implicit assumption that line $k$ did not go through the midpoint N.

VB: So, what is the assumption on the line $k$?
Dave: The assumption, oh, okay, the assumption here is that it's parallel to AC.
The next portion of the interview probed Dave’s understanding of the process used in the proof.

VB: Well, can you explain the role of Euclid’s Parallel Postulate in this proof?

Dave: Well, yes, AC is our line in the plane and like M for instance, is the point not on that line. And so they’re saying that you can draw this line $k$, and only one line $k$ that is parallel to this line here. So, we can’t have two parallel lines here. MN and $k$ cannot be parallel to AC, unless they are the same line.

VB: So, what is the consequence of that argument you just gave?

Dave: The consequence is either MN is parallel to AC, or $k$ is parallel to AC, or they both are because they’re the same line.

Although it seemed that Dave understood the steps of the proof, it was not clear at this point why he thought it showed the statement false or why he was not convinced that both MN and straight line $k$ were coincident. When asked about the method used in this situation, he replied that it was a direct proof. In contrast with situation 2, where the student knew about the workings of the method of contradiction used there, he did not realize that the reasoning in his words above was the concept underlying that method.

This indicated that he lacked a relational understanding of the method.

After Dave’s perception of the alternate proof and its relationship to the original proof was investigated, it was confirmed that, out of the three cases he mentioned above, Dave was considering the third case (that both MN and the straight line $k$ are parallel to AC) as proved impossible. In his emphasized words in the next excerpt, he mentioned that in fact, the alternate proof showed the coincidence of those two objects, something he thought was not proved in the original proof. This conclusion was also confirmed by the data from the later part of this interview.

VB: Okay, here is another proof in support of the statement. Um, please read it and let me know what you think.

Dave: (Pause). Now this proof proves to me that the statement is true, or that, to me it proves that MN and $k$ are the same line (emphasis added).
VB: Why?
Dave: Why? Because this to me is much more clearer. It uses more steps. We're also, we also show um-- we try to prove the opposite here. Or we assume the opposite, which is a proof by contradiction (emphasis added).

Evidently, the semantics of the proof played a role in recognizing that the method of contradiction was used. Thus, the interviewer tried to find if Dave's perception was limited to the explicit use of the semantics of the proof.

VB: So, what is contradicted in this proof?
Dave: Well, we assume that $k$ doesn't bisect BC. That it's. You know, not where N is basically. Um, and this here, this fact [1] here ends up showing...Yeah, this has proved our assumption that, um, $k$ doesn't bisect BC is disproved here in the last part of our proof. [...]

Dave's correct interpretation of the alternate proof above indicated that he understood the process of contradiction. However, his understanding was invoked by the semantics in the proof because he did not think the original proof used the same approach. As the excerpt below indicated, his understanding of the original proof did not improve in light of the alternate proof, implying that his perception of the method of contradiction was limited to its semantics.

VB: Okay. So which one do you think is correct, the statement is true or false?
Dave: I think it's true after reading this one, you know, the alternate proof. This [original] one here didn't give I think, enough information in the proof for me to be convinced that it was true. Or maybe it was because it used a different method that it convinced me. Maybe because it used the contradiction method.

VB: And the method used here [in the original proof]?
Dave: Direct approach, I guess. Or... (Talks to self) How many approaches are there? Direct, exhaustive, contrapositive, contradiction. Direct, I think, is what I would call this [original proof]. But direct doesn't always show you if there are counterexamples out there, I think (emphasis added).

The emphasized sentence above also confirmed Dave's reason, mentioned earlier in this analysis, for believing why the original proof showed the statement false.

Although Dave claimed that the original proof used direct methods, he had doubts
whether direct proofs were used to show counterexamples. As pointed out earlier, Dave thought that the original proof found a counterexample because it showed straight line $k$ and $MN$ were impossible to coincide. It should be reminded that Art from the Abstract-Math category also thought that the original proof found a counterexample in Figure 2, thus showed the statement false.

**Situation 4: Interview 1**

There exist three consecutive integers such that the cube of the middle integer is equal to the sum of the cubes of the outer integers.

[In other words: For some three consecutive integers $(n - 1), n, (n + 1)$ the relation $(n + 1)^3 + (n - 1)^3 = n^3$ holds.]

Next to guessing, the existence of the mathematical object $(n = 0)$ in this case could be found by algebraic manipulation. The main objectives of this interview were to investigate students' approaches as to finding a mathematical object and to see if they would proceed to a conclusion by reason or argument rather than intuition. This type of approach, that is, proceeding to a conclusion from general premises to a necessary and specific conclusion, by a reason or argument rather than intuition, will be referred to as an approach based on "dianoetic" interpretation of the equation. In other words, an *a priori* understanding that a sufficient condition for finding the existence of the mathematical object is to solve the equation will be considered as an understanding of the dianoetic implication of its role. In short, a perception that the solution of the equation would lead to an answer was considered a dianoetic interpretation of the equation. This interview was used to investigate whether students' approaches would be based on a dianoetic interpretation of the given equation, if they chose to solve the equation.
It should be noted that the main assumption in this situation and in its analysis was the following. Formulating the equation essentially presumed an indirect approach where the existence of the object was implicitly assumed, and then solving it algebraically purported either a discovery of the sought object or a search for contradiction. Thus, the dianoetic interpretation of the equation’s role implied an indirect approach, and approaching the problem solely by intuition could be interpreted otherwise. Although there seemed to be a thin line between these two approaches, leaning heavily on either approach could imply one’s level of comfort with using it in such existential problems. Thus, the observations from the first interviews were used to investigate students’ level of comfort with the indirect use of algebraic manipulation and their understanding of its dianoetic interpretation.

Combined with the second interviews on this situation, the results were triangulated in the following manner. The analysis of the first interviews tried to formulate a hypothesis for each student’s interpretation of the equation’s role whether it was dianoetic or not through their approaches. Then the analysis of the second interviews tried to validate or invalidate those hypotheses based on the students’ understandings of the given proof there. The overall analysis tried to find out about students’ understandings of indirect proofs in relation to finding existence of an object.

**Post Abstract-Math students’ responses to situation 4 (interview 1)**

Three of the four students in this category claimed that the statement was true; only Perry claimed it was false. Both Pam and Paul initially claimed the statement was false, but upon further reflection, they changed their answers. They all expanded the given equation correctly, but had different interpretations for their results.
Patty started expanding the given equation in order to get an idea of the situation. She did not have any good explanation or valid reason for why she adopted that route. She later had no trouble finding a conclusion to her reduced equation \( n^3 + 6n = 0 \). She had to try cases of \( n \) negative and positive before she realized that \( n = 0 \) worked.

Patty: [...] we're getting that \( 2n^3 + 6n = n^3 \), if there's a number \( n \) that this works for—assuming I did my math right which is possibly a big assumption—so that would give me \( n^3 + 6n = 0 \), which... So \( n^3 = -6n \). Uh... which I don't see happening because if \( n \) is negative, then \( n^3 \) is negative and, \( -6n \) is positive. So it doesn't work if \( n \) is negative. If \( n \) is positive, \( n^3 \) is positive and, \( -6n \) is negative (pause). Of course, in this case, zero works and probably... Okay, so if we tried zero we'd have \( 1^3 + (-1)^3 = 0 \), which gives us \( 1 - 1 \) is zero. So zero, \( n = 0 \) is our one answer that works. So, it's true. There does exist three...

Since Patty was seeking an answer that satisfied the reduced equation, it was concluded that she understood the diachronic interpretation of the equation's role.

Paul started looking at examples using trial and error, but since he failed to find a case, he thought that the statement would be false, at first.

Paul: Well one way you could do it is just guess a bunch of them.

Uh, let's see. Let's see. Actually the better way to do it is... I think the best way to do is just to expand this. And then if it's true it'll be explicitly true for whatever integer it is, but...

Although he realized a better approach for answering the question, he wanted to find an answer to the original statement by comparing both sides of the equation rather than carrying out the expansion.

Paul: Uh, it's not going to be equality. Whatever this \([(n + 1)^3 + (n - 1)^3]\) is, won't equal that \([n^3]\). [...] Like, yeah, it's going to give you two cubed terms, and then there's going to be some other stuff left over. [...] VB: And what does that say about the statement? Paul: Uh that this statement's just false.
Paul thought that the statement was false because the left-hand side of the equation could not be simplified into $n^3$—the right-hand side. He was relying more on his intuition rather than algebraic manipulation. Variations of this approach were also observed in other interviews with other students as well (see later analysis of this situation for Pam, Alice and Adam). Only when prompted by later questioning did Paul try to expand the equation, and upon further observations of the partially expanded equation, he stumbled upon a correct guess for the answer.

Paul: [...] well, okay so there it will work for $n$ equals zero. Okay, so... Okay, so, yeah, when you expand this you're going to see that. You'll see that. So, it'll work for $n$ equals zero, but it won't work for any other values of $n$. You can just...

Since Paul was hesitant to pursue a manipulation of the equation, he was not seeking a solution for the equation when he partially expanded its left-hand side. Rather he was trying to compare intuitively one side to the other, which implied an inadequate understanding of the dianoetic implication of the equation’s role.

Similarly, Pam also thought initially that the statement was false because she compared both sides of the given equation and thought they didn’t match.

Pam: [...] Basically I just foiled everything out on the left-hand side and right away, I should have seen this. You're going to get $2n^3$ on the left-hand side, and there's no other way to get rid of the other $n^3$. And $2n^3$ never equals $n^3$, except for zero [emphasis added].

VB: So when you did that what was your objective?

Pam: Um, well, I worked on it... I have to see things, so I usually work through an example. So, I just had to see these foiled out. And once I got down to where they were foiled out by and just did the initial addition of the two like the problem asks. Um, I just initially saw this $[2n^3 + 6n \neq n^3]$ and, yeah...

Evidently, her decision to expand the equation was not driven by its dianoetic interpretation in the problem. Her approach was more intuitive at this point because she wanted to see things “foiled out.” Although her algebra was correct and she mentioned...
the $n = 0$ case for the equation to hold, her interpretation fell short of the realization that zero was the only solution that produced the sought object.

Pam thought that the statement was false because $2n^3 + 6n \neq n^3$. It seemed that she thought the statement would be true if the sum, through direct manipulation, of the cubes of the outer integers (left-hand side) was equal to the cube of the inner integer (right-hand side). Put differently, it might have been the case that Pam was interpreting the left-hand side of the equation as producing the right-hand side result. However, upon further probing it was also revealed that she had considered the given statement universal instead of existential.

Pam: Oh, okay this $[2n^3 + 6n \neq n^3]$ would not work for all integers. There might be some integers it would work for. Whoa, very true. Actually, $-1$, $0$, and $1$ would work (laughs).

VB: So what does that tell you?

Pam: That tells me I was wrong. Because it doesn’t specify for all integers, it just says there exist three consecutive integers. I totally missed that, but these three would work.

Although it was not clear how she found that answer, thus whether she used a dianoetic interpretation of the equation, the data from her second interview indicated that she was not necessarily aware of the dianoetic implication of the equation’s role in this existential statement.

Perry, on the other hand, after expanding the equation and correctly reducing it to $n^3 + 6n = 0$, did not realize that zero was a solution because he made the mistake of dividing by $n$ (zero). Consequently, he thought the statement was false.

Perry: [...] So we get $n^3 + 6n = 0$. We can divide both sides by $n$, because zero divided by $n$ is still... squared plus six equals zero. And, that won’t hold, unless I can have $n$ be an imaginary number, which I can’t. So, this is a false statement, because for all $n$ of the integers, [...] $n^3 + 6 \neq 0$, $n^2 \neq -6$. - - -
VB: Okay. What were you trying to get out of all this algebra?
Perry: [...] what all the algebra is for is to reduce it down into one integer \([n]\) rather than three. And then look at what that one integer has to do--because the other integers are defined, can be defined in terms of that one integer.

Those words suggested that Perry was comfortable with using dianoetic approach rather than intuitive. If not for his algebraic miscalculation, he could have found the correct answer. It was concluded that Perry was implementing the dianoetic implication of the equation’s role.

**Abstract-Math students’ responses to situation 4 (interview 1)**

All four students initially claimed that the statement was false, but two of them (Amy and Art), upon more reflection, later changed their answers. They all expanded the given equation correctly, but as with the Post Abstract-Math students they had different interpretations of the results.

Amy’s first instinct was that she only had to find one case that could prove the statement true, so she started trying some examples but to no avail. This intuitive approach took her a while before she realized that the given equation would help her find \(n\).

Amy: So, it’s there exists so we only have to find one example to prove this. So, that’s the tricky part [...]. I’m trying to find an example but I feel like I’m not getting anywhere [...]

Okay, oh... well you can use this handy dandy little equation they’ve given us and solve for \(n\).

VB: And if you solve for \(n\) what do you get?
Amy: You’ll get a number that works, an example that proves it [emphasis added].

Well, intuitively I think it’s false (laughing) but I could... I could probably work it out and just solve for \(n\) and see what happens... I don’t really see it by just looking at it.

Clearly, Amy’s words above indicated that her intention was to carry out the equation’s dianoetic implication. Nevertheless, she felt uncomfortable expanding the expression. As
observed, she knew what the result of that expansion would lead her to and as found later, she also knew how to expand it. Only after the interviewer pushed her to actually perform the expansion did she find that -1, 0 and 1 would prove the statement true. Thus, it was concluded that although Amy started using an intuitive approach in this situation, she understood the dianoetic implication of the equation’s role.

Art took a little longer time before he realized what needed to be done to find the sought object. He manipulated the equation correctly but did not see the integer solution at first, but then upon further inspection he found the integer.

VB: Okay. So, again, explain to me just what happened here.
Art: All right, um, I went through and using this formula right here I went through and found my values \( n \). And I get, I found three values: two of which are imaginary, so, um, they’re not integers, which, those two cannot work then. And my third was zero, which was an integer, which is an integer. […] If \( n = 0 \), then we can use \( n - 1 \), \( n \) and \( n + 1 \). So, our numbers are -1, 0 and 1, which does hold true.

His approach to the problem suggested that he was implementing the dianoetic implication of the role of the equation. His perception was that the solution of the equation would lead him to an answer, although it took him a while to realize that. This next excerpt also confirmed that result.

VB: Do you think there should be a better argument or a proof that uses different approach than yours?
Art: Um, no. I think that this. I think directly proving it like that was the best way to go about it.

On the other hand, Alice thought that the statement was false because the left-hand side of the equation did not add up to the value on the right-hand side.

Alice: […] Unless I did something wrong back up here I’d say this is false, because if you solve this out, and solve this out, like I did here, and add them together [to get the sum of the left-hand side], they want you to get \( n^3 \), but…
She thought, in order for the statement to be true, the result from expanding the left-hand side of the equation should equal to its right-hand side. This interpretation of the equation was also confirmed throughout her interview on this situation.

Alice: Or if I add these together, all right, you get $2n^3 + 6n$, and that, well, that's what I get, and that doesn't equal $n^3$.
VB: And what does that tell you?
Alice: It tells me that there does not exist three consecutive integers such that this [condition] holds.

Again, Alice thought that the statement was false because she concluded that $2n^3 + 6n \neq n^3$. It should be noted that similar phenomena were also observed during earlier (Paul and Pam) interviews. Unlike Paul and Pam who later found the correct answer, Alice stated that the statement was false. However, she expressed some doubts about the non-existence of the object when there were infinite number of integers.

Alice: [...] there exist... I don't know, just the fact that it says there exist three consecutive integers, makes me think that maybe there is (pause). No, I think that this is right. I think that this is the way you need to prove it.

The fact that Alice did not pursue a solution for the reduced equation confirmed her lack of understanding of its role. Her understanding was that the truth of the equation needed to be checked rather than to use its solution indirectly to find the truth of the given statement.

Similarly, Adam expanded and simplified the left-hand side of the equation and compared it with the right-hand side, and since he did not think they matched, he claimed that the statement was false.

Adam: [...] So, what I did is I took the $n + 1$ and I cubed it and I got $n$ three, cubed plus $3n^2 + 3n + 1$. And then I took $n - 1$ and I cubed it and I got $n$ three, cubed minus $3n^2 + 3n - 1$. So, and then I added them up and I got 2, um, times $n$ cubed plus $6n$ equals $n^3$ [$2n^3 + 6n = n^3$]. Well, it does not equal $n$ cubed actually.
Like Alice, Adam also found it hard to believe in non-existence because of infinite number of integers. Then, he also tried a couple of examples with 6, 7 and 8 and with 1, 2 and 3, to verify non-existence.

Adam: Can I absolutely be sure? No, because I’m not very confident in my mathematical skills, but I, in doing it, my... I would say that it is false. But, then again the statement says there exist, so there could just be one [emphasis added]. There could be one set of three consecutive integers that this holds true.

Again like Alice, it appeared that Adam perceived that the truth of the equation needed to be checked rather than to use it indirectly as a tool for finding the sought object in the problem. Unlike Alice however, Adam also showed an intuitive approach through using examples due to lack of confidence in his mathematical skills.

As mentioned earlier in the introduction of this situation and its objectives, formulating an equation presumed an indirect approach where the existence of the sought object was implicitly assumed, and then solving it purported either a discovery of the object or a contradiction. Here, instead of assuming the existence of the object through the equation and searching for the object or a contradiction, the students (Alice and Adam) claimed the statement was false because they thought the left-hand side was not equal to the right-hand side. In other words, according to them the statement would be true only if the sum by direct manipulation of the cubes of the outer integers was equal to the cube of the inner integer. Pam and Paul from the Post Abstract-Math category also showed a similar perception.

The above observations could also be interpreted in terms of the students’ (Alice and Adam) lack of understanding of equations. According to them, the right-hand side of an equation might have always meant the answer to the result of the left-hand side.
manipulation. However, in the context of this situation, which called for a search for some mathematical object rather than an understanding of the concept of an equation, such a misconception hindered their search process only a little, as was observed with Pam and Paul. It was observed that although they both showed similar misconceptions, they eventually found the sought object through different means.

It should also be noted that this particular behavior could have been biased by the way the equation was presented to them in this situation. The results might have been different if the equation was given as \( n^3 = (n + 1)^3 + (n - 1)^3 \) instead of \((n + 1)^3 + (n - 1)^3 = n^3\). In the design of the tool, the researcher also considered not providing any equation for this situation to eliminate bias, but pilot testing indicated that some students might have trouble formulating or even starting the problem. It was noted earlier that one of the objectives of this situation was to observe whether students use the diaphetic implication of the equation’s role, so in the case of the equation’s absence the tool could have encouraged the intuitive aspect of approaching this problem.

The data from Alice’s interview did not indicate an understanding of the diaphetic implication of the equation’s role in this situation. Her interpretation of the equation was insufficient because she did not see its purpose beyond the goal for checking the result of the left-hand side to the right-hand side. Unlike the other cases (Pam, Paul, and Adam), it might have been the case that this misinterpretation of the equation hindered her from using it for finding the desired object. As to reasons for her misinterpretation, the data were inconclusive because it was not the goal of this interview or the tool to explore such misinterpretations. This phenomenon emerged only during this one interview and its reasons cannot be confirmed from the data.
However, Adam’s interpretation of the role of the equation was different and there were enough data in his case to draw conclusions. Through further probing that explored his perception, it was observed that Adam considered the equation as a tool for exploration of the truth of the given statement.

Adam: Why $n$? $n$ is nothing, $n$ is arbitrary. [...] You’re trying to model the statement into an abstract situation in which you can use mathematics as a key to finding, to proving if you’re right or wrong.

VB: So when you say [pointing to $2n^3 + 6n \neq n^3$] left-hand side cannot be equal to the right-hand side you are saying that there is no such $n$ that...

Adam: ...exists. So $n$ would be an arbitrary integer and the statement $n - 1$, $n$ and $n + 1$ indicates three consecutive integers.

These words could eliminate the possibility of interpretation of the right-hand side of the equation being the answer to the left-hand side. They however, as well as his engagement earlier with the equation, fell short of considering it as an assumption in terms of $n$ as the sought object for verifying the truth. Adam stopped short of assuming the existence of the object in the equation and working towards the search for contradiction or the desired object. Instead he thought the statement was false because $2n^3 + 6n \neq n^3$. Thus, it was concluded that Adam’s perception of the dianoetic interpretation of the equation was limited.

**Discrete-Math students’ responses to situation 4 (interview 1)**

Three (Dave, Dean and Dan) of the four students claimed that the statement was true. Doug did not find any reason to believe that the statement was true. He did not make use of the given equation other than substituting an example of three consecutive integers, which implied that he lacked an understanding of its dianoetic interpretation.

Doug: [...] I cannot say that that’s false. All you’re looking for is one [...]. My, what I did was just show that it was not true for one example and what the question asks is: is there an example where three consecutive integers, the sum of... And what
method do you use...? I don't, I'm not really sure how you do that. Really, I think it'd just be by, uh, brute force. *I don't know that there'd be a real elegant way of doing that proof other than just looking for one* [emphasis added].

Similarly, Dan tried a few examples because he knew he had to find one to show the statement true, but unlike Doug, he was able to guess -1, 0 and 1 because he realized negative integers played some part in the answer.

Dan: [...] So, if I could find one that works, then it would prove that true. It would be easy to do it with like a computer hopefully. And this is just an integer, so it can be a negative or whatnot, right? Let me think. Hmm... okay. Well, it would work for \( n = 0 \), I think, because that would be \(-1, 0, 1\). So that would be \((-1)^3 + 1^3 = 0^3\).

Since it was not hard for Dan to guess the answer in this case, he did not contemplate other approaches for proving the statement, even when he thought there might be some other way of proving it.

VB: Okay. And, anything [types of proof] comes to your mind about the approach such a proof might take?

Dan: Well, um, I just basically did like a direct proof. Like they said there should be one example that works with this and I said, okay, here's the one example. So, that's pretty much like a direct way to prove it.

The fact that Dan was inattentive to the equation's usefulness in this situation confirmed his extreme dependence on his intuition to solve this existence problem. Also, the fact that he could not think of a proof that made use of the equation indicated that he did not know the dianoetic implication of its role.

On the other hand, Dean approached the problem by trying to expand the left-hand side of the equation. He initially showed some understanding of its usefulness but he then withdrew from completing the expansion because he was not sure what he would do with the equation.

VB: So, what do you expect to get out of this [expanding] approach?
Dean: Um, hopefully $n$ equals something and then that number satisfies the condition. Hopefully I can reduce it down to that.

I'm trying to think of numbers that I, that I think would work, and I'm not really sure. I'm thinking, maybe...

VB: You're not following this [expanding] path anymore?

Dean: Um, I'm not sure if it's just because I'm nervous, or if it's just that I really don't know what I'm doing, but I can't see this [expanding] will make that into anything, so... So, I'm not really following that anywhere, no. Um... I'm thinking maybe 0 would work. [...] 

Dean did not feel comfortable using the equation, which indicated that he did not understand its usefulness for searching objects. He preferred to rely on his intuition for finding an answer. He did not solve the equation to get the answer, and when asked if he could think of a proof, just like his approach, his response did not substantiate the use of the equation.

VB: So, um, does anything come to your mind for a method, for a proof method that can handle this situation other than [guessing] examples?

Dean: Um, not really. [...] I think some proofs might but I think, once again, I think if it says there exist then you don’t even need to find [prove], you don’t need to do anything differently. So, I don’t know why you would.

Although initially Dean tried to use the equation, the above words indicated that he did not find it useful for solving the problem. Thus, his understanding of the dianoetic implication of the equation’s role was insufficient.

Dave tried a few examples as well, but did not believe he could reach an answer by looking at examples.

Dave: Well, because they are not really saying for all $n$, so it might be true that there exist three consecutive integers. So, it may not be false. Um...

Like Alice and Adam, Dave also had doubts that the statement could be false. His next attempt was to look at the equation and compare one side to the other, and for a moment he looked at -1, 0 and 1 but for some unknown reason he abandoned it. It was observed...
earlier that Pam also used the same approach and found this example but abandoned it, before she finally realized it was a solution.

Dave: Um, well, actually, now that I think about it more, um, I think it is false. Because, because, your \((n+1)^3\) is always going to be greater than your \(n^3\). So, and then when you add, uh, on top of that, I mean, without this \(3^3\) alone this \(5^3 + 3^3 = 4^3\) cannot be true for any \(n\), for any consecutive integers. I'm pretty sure, unless you're talking about maybe, something like this [writes \(-1, 0, 1\)], where, but this... When you cube something that's negative, \(-1\) time \(-1\) time \(-1\), that would be \(-1\), I'm pretty sure. But, yeah, this is, you know \((n + 1)\). I guess, in the general form, this would be this and you could probably simplify this \([(n + 1)^3 + (n - 1)^3 = n^3\] to show that it's not, um...

Dave then tried to simplify the equation but hesitated to continue because he was not sure what to do with it.

VB: Let me ask you this. Um, you're trying to expand the equation, right?
Dave: Right.
VB: And what do you expect to get?
Dave: I expect to get something that, uh, an equation that um... I'm not sure, I thought that might put it in a different light for me, but, um... I think I, I was trying to expand this [left-hand] side to see if it was equal to \(n^3\). Because they're saying that all of this stuff is equal to \(n\) cubed. […]

As his explanation indicated, his understanding of the usefulness of the equation was inadequate at this point, because just like the previous cases (Pam and Adam), his initial perception involved comparing sides and checking rather than solving the equation. However, he later showed a partial understanding of the equation’s role in the problem when he tried to compare the sides for different sets of values of \(n\). It was a passive approach for checking equality for some sets of values rather than an active manipulation in search of a solution.

In the end, Dave was convinced that the statement was false, because he reasoned that for \(n > 0\), \((n + 1)^3 > n^3\) and for \(n < 0\), \((n - 1)^3 < n^3\).

Dave: So, I think I convinced myself that it's false.
VB: Okay. So, you still believe that it’s false?
Dave: Yeah, um, you know, when I tried negative values, I convinced myself that the… whereas when we were talking about positive numbers, I knew that this one would always be greater than this one. So that when you cubed them, you know, they were growing proportionally, well not proportionally but, um… and then if you go backwards with negative values, then it’s um, the second term, the \((n - 1)\), that’s going to be bigger [smaller] than the \(n\) always, so. That’s how I reasoned it.

Of course his account was not completely correct because he did not consider the case for \(n = 0\). Had he considered the case \(n = 0\), he could have proved the existence. Dave’s approach was rather an intuitive one that used some dianoetic interpretation.

In conclusion, there was no evidence in the data that indicated full understanding by the Discrete-Math students of the role of the given equation. Although they all showed a complete understanding of what needed to be done to solve existence problems, that is, finding one case that satisfied the given condition, only Dave expressed some views about how that could be done using the equation. However, Dave’s argument did not make indirect use of the equation for finding the sought object but as a tool for checking sets of values directly. The other students felt more comfortable using guesswork based on their intuition. Since an approach based solely on intuition in this situation was regarded as a direct one, a reasonable explanation for their shortcomings could be their lack of perception for using indirect methods in existential problems.

It should be noted that the above findings were based on the researcher’s assumption that seeking a solution of the equation (not just expanding and simplifying it) implied an implicit \textit{a priori} understanding of its usefulness as a means for finding the sought object (dianoetic interpretation). In retrospect of the findings from both interviews on this situation, it might have been beneficial if a comprehensive examination of the students’ perceptions of the implicit presumptions that were built into the equation were
corroborated by the first interviews. That is, direct questioning about the equation's assumptions in the first interviews on this situation could have shed more light on the findings of the participants' awareness of its presumptions and their consequences before they chose to solve or manipulate the given equation algebraically. Further, the results might have been more interesting if they were also probed in the first interviews about their approach and whether they considered solving the equation a direct or indirect approach in this existential situation. Although both those aspects were to some extent incorporated in the second interviews in the context of the given proof there, the need for a comprehensive study of the students' awareness described above was due to the findings that emerged from the second interviews.

**Situation 4: Interview 2**

In this second round of interviews on situation 4, students were given an indirect proof of statement 4:

There do not exist three consecutive positive integers such that the cube of the middle integer is equal to the sum of the cubes of the outer integers.

This statement was similar to the true/false statement from the first interview except that it referred to nonexistence of positive integers due to the contradiction introduced by the constraints. The contradiction process of the given proof made the assumption explicit. Its contradictory conclusion was made explicit only in the statement of the choices, in order to leave some room for probing. The probing was intended to observe students' perceptions of the direct process used in the proof and how it indirectly related to determining the existence or non-existence of a mathematical object. In other words, one of its objective was to observe whether students could relate a proof that
directly showed "if \((n + 1)^3 + (n - 1)^3 = n^3\) then \(n = 0\)" to an indirect proof for the above existence problem. This situation investigated students' focus of attention in a proof that appeared to have the semantics of a direct proof and yet its purpose was to find a mathematical object indirectly. The statement \((if \ (n + 1)^3 + (n - 1)^3 = n^3 \ then \ n = 0)\) it proved directly was not provided in the choices.

The interviews were used to triangulate the results from the first interviews in terms of relating the solution of an equation to the process of finding an object. That is, they were used to observe how students' perceptions of the role of the equation for finding an object from the first interviews played out in a proof-checking situation. If in their true/false interviews they had chosen a process similar to the given proof, the probing was intended to find out how consistent their understanding of that process was with their own approach.

Post Abstract-Math students' responses to situation 4 (interview 2)

Only two students (Pam and Paul) in this category claimed that the proof could be used to show statement 4. The other two (Patty and Perry) chose statement 2, that is, "if \(n = 0\) then \((n + 1)^3 + (n - 1)^3 = n^3\)."

Pam's choice was based on consecutive integers being designated as \(n - 1, n, n + 1\) in the proof, so she chose statement 4 that specified consecutive integers over statement 2 that was only about \(n\). She gave the following explanation for her choice.

Pam: Okay. Well I'm torn between [statements] two and four. I said four, um simply because they designate the three consecutive integers to be of this \([n - 1, n, n + 1]\) form. [...] So, that's why I chose this because positive integers... only two of them are positive one is not.

Well, [statement] two is just, I looked at it and I was like, well that's what it shows. But then I read and reread it and I saw that they designate the integers. So,
with these \([n - 1, n, n + 1]\) being the designated form of the integers and running through this [proof] there can't be any positive integers for which this \([(n + 1)^3 + (n - 1)^3 = n^3]\) is true. So, that's why I chose [statement] 4 and not 2 because 2... it's just like the example of this.

VB: But, if the word positive was not there, would you have chosen [statement] 2?
Pam: I would have chosen two and not four [...] because... if you delete that word positive [...] then this does not make, then this [statement 4] is not true, because there were three consecutive integers such that this [given equation \((n + 1)^3 + (n - 1)^3 = n^3\)] worked.

Contrary to Pam's choice, if the word "positive" was deleted from the statement the only possible choice left would have been number 5 (none of the above) and not statement 2 as she claimed. From the data above, it was concluded that the student's choice was not based on the contradiction process used in the proof to show statement 4, but rather its relevance to particulars such as designation for consecutive integers in the statements. In other words, she had picked the correct statement but did not give a reasonable answer for her choice. Further, despite the interviewer's efforts to bring her attention to the process of the proof, there was no indication in the data that she had transformed the indirect approach taken in the proof to show statement 4. The following excerpt, in which she was asked to compare statement 4 to the statement the proof showed directly, was an example of all that she had to say about how statement 4 followed from the proof.

VB: Okay. Now how is this... \([if (n + 1)^3 + (n - 1)^3 = n^3 then n = 0]\) statement different from statement number four?
Pam: Um, this one \([if (n + 1)^3 + (n - 1)^3 = n^3 then n = 0]\), just it kind of tells you what the integers are going to be. This one [statement 4] doesn't designate what the integers are it just designates kind of where they're at [i.e., consecutive and positive].

The above conclusion was also confirmed later in the interview when Pam was asked to compare her own proof of the true/false statement to that of number 4.
VB: Okay. So, do you see any connection between this [true/false] statement and number four?

Pam: [...] That word positive throws it off.

VB: So the thing that you did here [in the first interview], how is that related to this proof?

Pam: Actually I pretty much did about the same thing in this proof.

In comparing statements and proofs, she was only able to make the distinction between one statement being about integers and the other being about natural numbers. There was no elaboration beyond the superficial, or indication of the realization that the existence statement was contradicted because of the constraint introduced by positive integers.

In Pam’s first interview, it was observed that her approach to expanding the equation was not based on her anticipation of its results (non-dianoetic), but rather on her need for visualization of the expanded equation. That lack of anticipation was consistent with her lack of understanding of the process in the given proof, because as observed above she did not validate the indirect process built into both her own argument and the given proof.

Similarly, Paul’s explanation for his choice did not elaborate on the indirect method used in the proof.

Paul: I... Because it [the proof] found that the only one that was—the only three integers that weren’t... would be -1, 0, and 1. And, so, then, we know none of those would work so there are no three positive integers.

VB: Okay, one of your classmates said that the proof—this one here—uh, can be used to show that if this \((n + 1)^3 + (n - 1)^3 = n^3\) equation holds then \(n\) is zero. Do you agree?

Paul: Yes.

VB: So how is this statement different from your choice?

Paul: Well, uh... Well, this one says, um... well this one [statement 4] is just based on that statement is true. So, I guess it, it does show that one also. But that wasn’t a choice [...] that statement that \(n\) has to equal 0 is what proves that this [statement 4] is true [...]
Here, Paul was arguing that statement 4 was proved true as a consequence of $n = 0$ following necessarily from $(n + 1)^3 + (n - 1)^3 = n^3$, but he did not indicate how the process of the proof made one result follow from the other. The interviewer later tried to investigate his understanding of the indirect process of the given proof, but the student was frustrated with his superficial input, as his next words would indicate, to the point that the interviewer had to assure him of his positive input to this study.

Paul: Yeah, but I don’t know… If it only works for $n$ equals zero then it can’t work for any other integer, I guess. I’m sorry I’m not very good for your study. I don’t mean to waste your time.

Although the data was unclear whether the student understood the contradiction process used in the proof, there was no indication that he was going in the right direction either. Also, his lack of confidence in the second interview, compared with the results from his first interview, was inconsistent because there he showed some understanding for the equation’s role, which was the same in the given proof.

The other two students (Perry and Patty), who claimed that the proof in this situation could be used to show statement 2 (if $n = 0$ then $(n + 1)^3 + (n - 1)^3 = n^3$), gave similar explanations for their choice.

VB: So, why do you think that the presented argument in this situation shows statement number two?

Perry: Because in essence it’s saying that this $[(n + 1)^3 + (n - 1)^3 = n^3]$ holds for the case $n$ equals zero. And it doesn’t go further in saying that it doesn’t hold for other cases. It’s just a proof that gets it down to saying that it holds for $n$ equals zero.

Similarly, Patty’s answer to that question was:

Patty: Um, just algebraically it’s showing that this works for $n$ equals zero, so if $n$ equals zero then the statement [number 2] is true.
From those explanations it was apparent that they did not think much about the role of the explicit assumption at the start of the proof, so the next question probed their understanding of it.

VB: So does this proof assume anything?
Perry: Hmm, the proof... It just makes the, uh, it makes the assumptions that we have three consecutive positive integers. That’s, and then makes a statement about them.

VB: And what is the consequence of that assumption?
Perry: Um, that if \((n + l)^3 + (n - l)^3\) equals. Actually, if \(n = 0\), then \((n + l)^3 + (n - l)^3 = n^3\). So, it’s saying it holds for the zero case.

Perry appeared to have believed wholeheartedly that the process of the proof was to show statement 2 because he misunderstood the role of its assumption.

VB: Well, one of your classmates said that the, this proof here [...] shows that if this equation \([(n + l)^3 + (n - l)^3 = n^3]\) holds then \(n\) equals zero. Do you agree?
Perry: Um, yeah.

VB: So how is this statement different from yours?
Perry: Basically what happens is that, when I was looking at this [proof] I was thinking–this statement’s actually better than mine, because it’s taking these assumptions and boiling them down. But I was–when I was looking at this I was thinking this proof shows that this holds in the case \(n = 0\). Therefore, \(n = 0\), therefore, this [the assumption \((n + l)^3 + (n - l)^3 = n^3\)] is true when this \([n = 0]\) is true. Therefore, you can switch them around, because it’s a truth, it’s one of those–it’s always true. There’s not a causality. It’s just this is true when \(n = 0\). Thus it, therefore, they’re both ways of showing that causality. It’s using an if-then as causality, it’s just a way of relating, as a relational rather than a truth functional. [...] It proves this as a relationship and therefore I just made the assu–, and therefore I just took that [number 2] and had to be that, because I didn’t feel like putting down none of the above.

The student was trying to justify his choice for number 2 because he realized that the alternate statement was proved directly and that number 2 was not. He was arguing that the ‘if-then’ causality in this proof could be switched around in this case, because he was interpreting the proof in terms of relationships between equivalent equations, which can be necessary and sufficient. However, the proof only showed one direction, which was
not the direction of statement 2. Thus, he was not interpreting the proof in terms of three consecutive positive integers manipulated as required by the assumption in the proof that eventually contradicted statement 4. This could also be explained by the student’s attachment to the direct process of the proof, which again was the result of his concept image.

In his first interview on this situation, it was found that Perry was confident enough in using the equation as a tool for finding the sought object. In contrast, his dianoetic interpretation there was inconsistent with his perception of the given proof because the proof utilized the same process as his in the first interview. This dianoetic interpretation of the equation from the first interview did not help him unravel the indirect process used in the given proof.

On the other hand, Patty’s explanation for the role of the assumption used in the proof revealed that she was considering it as redundant and unnecessary, which could explain why she chose an incorrect answer.

VB: So, uh, does this proof assume anything?
Patty: Um... I don’t think so. I mean, I mean we’re using, I mean, it just assumes that we’re using integers, but... Oh, just a second. Yeah, I mean, it’s assuming three positive integers, which we come up with but it doesn’t result in three positive integers [emphasis added], but it still shows if $n = 0$ then, this $[(n+ 1)^3 + (n - 1)^3 = n^3]$ can be true.

Patty thought that the proof assumed three consecutive positive integers but it found one of them to be zero. Thus, she thought it must have been the case that $n$ was assumed zero in the hypothesis of the statement. The emphasized words above indicated that the student was aware of the result that contradicted the assumption of integers being
positive, but was unable to put the whole process together. As further probing confirmed, she did not see the purpose of the proof beyond the algebraic manipulations in it.

VB: And what is the um—the type of reasoning used in the proof?
Patty: It’s, I mean, this is just algebraic manipulation of the numbers.
VB: And is it a direct proof?
Patty: Um, if it were just to say three consecutive integers it could be direct. But when they’re saying positive integers, it’s not resulting in three positive integers, it’s resulting in a negative, zero, and one positive integer. So...
VB: So what does that mean? It wouldn’t, it’s not direct?
Patty: Well, I mean, to me it still seems direct. It’s just that (pause).

Her indecision indicated that she had some doubts about statement 2 following directly from the proof. Capitalizing on her doubts, the interviewer tried to find out if her attention would shift away from statement 2.

VB: Okay. Statement number two is a direct result of this proof?
Patty: Not quite, now that I look at it again, because, I mean, in our proof we’re assuming that \(n - 1, n, n + 1\) are positive integers, and they’re not. I mean when we get to a solution, they’re not positive integers. They’re just integers.

VB: So what does that tell you?
Patty: That, the existence of three positive integers such that this \([(n + 1)^3 + (n - 1)^3 = n^3]\) occurs is an incorrect assumption [emphasis added]. But, I mean, this proof still shows that this [statement #2] occurs. [...] But it also, see, I would also say that there do not exist three consecutive positive integers [statement #4], because when you plug in zero equal to \(n\), you don’t get three positive integers. See, I almost circled that one too, but then I decided not to. Because there exist three consecutive integers, but there don’t exist three consecutive positive integers such that this is the case.

As her words indicated, she came very close to identifying the correct statement for the proof, but her lack of attention to the indirect process of the proof and the role of its assumption kept her from making that jump. As pointed out, she was absorbed by the algebraic manipulation to solve the equation rather than the contradiction process in the proof. This also became apparent from later questioning as well, when she was asked to compare the proof to her argument from interview 1, which was similar to the proof.
VB: Okay. So, according to your argument [in the first interview], it proves that, um, there exist three consecutive integers [...] so that the equation holds. Now, how does that [same] argument prove statement number two?

Patty: Well, this [my] argument shows that it works for zero. Therefore, if \( n = 0 \), then this [statement 2] is true. I mean, um...(pause)

VB: So, if, if in the first interview I gave you this statement, number two, and asked you if that’s true or not, would you have used this [same] argument...?

Patty: No. This [given proof] isn’t how I would prove this [statement 2]. If I were going to prove this [#2], I would have just plugged in zero for \( n \) and see if it worked or not, because that would be the fastest way to prove this statement.

The questioning above tried to clarify why Patty chose statement 2 instead of 4, and Patty’s explanation revealed that she regarded the given argument as a proof for the existence of consecutive integers, which in turn proved statement 2. Nonetheless, she also argued that she would not have used the given proof to show statement 2. Although Patty argued that the proof showed statement 2, it seemed that the reason for her choice was not apparent to her. She gave different and conflicting answers. Thus, there was no indication in the data that she fully understood the indirect process of the given proof. In her first interview on this situation, Patty exhibited a perception of the dianoetic implication of the equation’s role but it did not help her unravel the indirect process used in the given proof.

In summary of the performances of the students in this category, Paul and Pam who showed a sketchy perception of the equation’s role during the first interviews consistently revealed a lack of understanding for their own processes throughout the second interviews when they faced a similar process in its proof. On the other hand, it was observed that Patty’s and Perry’s understanding of the dianoetic interpretation of the equation’s role did not necessarily persist when they faced a proof similar to their own. The students did not necessarily see the relationship between solving an equation and its
role in the indirect process for finding an object. It was possible that, in the first interviews when they were looking for the solutions of the equation, they took for granted its implicit assumption of the existence of the sought object in the first place. Hence, their intended process did not necessarily invoke indirect methods as was assumed earlier in the introduction of the analysis of the first interviews. It was noted in the analysis of the first interviews that one of the limitations of this situation was that the first interviews did not probe into students’ perceptions of the implicit assumption that was built into the equation. It should be noted that this limitation was due to the emergent phenomenon observed from comparing and contrasting the findings from both interviews on situation 4. More questioning in the first interviews could have found whether students were aware of the assumption of the equation before they chose to solve it, which could have shed more light on the overall results of this situation. In particular, it could have indicated the reasons why Patty and Perry did not see the relationship between solving an equation and its role in the indirect process for finding an object.

Therefore, when students expanded and simplified a representative model (equation) for the existence or nonexistence situation, there was no evidence to indicate that they did so because of full understanding of its indirect approach in terms of assuming existence in the equation and then eliminating contradictions. Rather, the data suggested that they did so instinctively because they sensed a direct process implicated by the equation. In other words, they were not necessarily aware of the implicit indirect argumentation built into the process of solving the equation.
Abstract-Math students' responses to situation 4 (interview 2)

All four students in this category claimed that the proof showed statement 4. Not all however, had a good explanation for its process.

Alice claimed without any hesitation that the given proof used contradiction to prove statement 4.

Alice: I think it was a valid proof, because it ends up being a proof by contradiction. I think because we go through and find that $n = 0$, which means that $n - 1$ was -1, but we said that they were three consecutive positive integers so that's a contradiction. So, that proved this statement true.

Further probing confirmed that she thoroughly understood the process of the given proof and how it related to the statement it showed directly.

VB: And how is that [if $(n + 1)^3 + (n - 1)^3 = n^3$ holds, then $n$ is zero.] different from this statement [#4]?

Alice: It's got a whole different goal, I guess. This [If $(n + 1)^3 + (n - 1)^3 = n^3$ then $n = 0$] one can be proved directly, where this [statement 4] one, we had to find the contradiction [emphasis added].

In her first interview on this situation, Alice lacked the understanding of dianoetic implication of the equation's role. That however had changed dramatically in this proof-checking task. Her approach, through expanding the equation in her first interview, indicated that she only used the equation to check its equality and there was no evidence that she utilized any indirect approach for finding existence. Yet in this second interview, she seemed to have overcome that obstacle. In order to understand why this happened the interviewer asked her to explain her approach to the true/false problem again.

VB: In your last interview, you said that this statement is false, and you gave some sort of argument here. Do, um, do you see any connection between these two statements here and there?
Alice: Yeah because this one [true/false statement] says there exists, and this [statement 4] one says that there does not exist. And it’s easier to do a contradiction of this [statement 4] one, because it’s easier to take a contradiction to there does not exist [statement], because you can show there does exist. Like right what they did here, but, on here [true/false statement] it’s harder to do a contradiction because it’s harder to show there does not exist. It’s more open-ended, I guess.

The student explained that it was easier, as in the given proof of statement 4, to assume existence and prove non-existence, using contradiction than the other way around, as in the true/false statement. The approach of the given proof, which she successfully interpreted as contradiction, and her approach to the true/false statement were actually similar, except that the result of the proof served different purposes in each case. In her explanation above, she tried to make that subtle connection between them, but failed because she did not think she used a similar approach and hence she just based her argument on their different goals. Thus, this piece of data confirmed that her intended approach to the true/false situation, as observed earlier, was rather intuitive and did not involve indirect processes.

The fact that the student could not validate her approach as similar, in light of the process of the given proof, suggested that it was not easy for someone like Alice to have a full relational understanding of the subtle workings of the method of contradiction in existence/non-existence problems. The method of contradiction was accessible to her only in explicit non-existence situations such as in this proof-checking task. A similar result was also found from her interpretation of the proof given in situation 6.

Adam also claimed that the given proof showed statement 4. He agreed that it showed directly the statement: “if \((n + 1)^3 + (n - 1)^3 = n^3\) then \(n = 0\).” But he did not seem to understand the exact method used to turn that argument around to prove statement 4.
VB: The proof shows this \([n (n + 1)^3 + (n - 1)^3 = n^3]\) if \(n = 0\)? Well, then, what type of reasoning does it use to show this?

Adam: Just a direct. It goes straight through and just says, so \(n = 0\), if \(n^3\) equals, so...

VB: So, would you say statement number four follows directly from [...] the proof?

Adam: Yeah, because it [statement 4] says there does not exist. It, it shows \(n = 0\); so then there’s not three consecutive positive integers that, so that you can’t do that, because if \(n = 0\), then that’s \(-1, 0\) and \(+1\).

In his first interview, Adam gave an incorrect answer for the statement. When questioned about his own argument from that first interview in light of the proof, he realized the mistake. However, he did not recognize that the approach was indirect.

VB: Is your approach the same as this [proof’s] approach?

Adam: Yeah, it, it, it expands it pretty much, though my mathematics must have been wrong.

VB: So, what went wrong in the end?

Adam: I don’t know, mathematics, mathematical error. I don’t know. Because you got \(n^3 + 6n \neq 0\) and I got \(2n^3 + 6n \neq n^3\). Mathematical error when you’re doing numbers.

He thought he made a mathematical mistake in his expansion and that led him to an incorrect answer but in fact, he did not make any algebraic error. Regardless of any algebra, his interpretation of the result from the equation was incorrect because of his insufficient understanding of the equation’s role, as found earlier and confirmed here.

Adam gave an incorrect answer to the true/false statement, because in his expansion of the equation he did not think any \(n\) would satisfy \(2n^3 + 6n = n^3\). In his second interview, he did not recognize the method of the proof but found it using the same approach as in his own proof. Thus, it was concluded from both interviews that Adam did not consider the use of the equation as an assumption of the existence that needed to be questioned or validated indirectly. This led him to think that the proof used direct methods.
On the other hand, it was observed in his first interview on this situation that Art showed a clear perception of the outcome of the equation's solution. Although he claimed that the proof showed statement 4, he did not see the use of an indirect method because he did not consider the assumption on the consecutive numbers as vital to the proof's assumption. Later data also confirmed that Art was unaware of the indirect method used in the proof.

VB: So, does this proof assume anything? And, if so, what is the consequence?
Art: Um (pause). It assumes the prerequisite facts. [...] And, these facts are useful in solving the equation. [...] If we didn't have that assumption, we couldn't solve the equation, if we didn't' assume that.

VB: [...] So what is the nature of argumentation used in this proof?
Art: Um, it's a direct proof where we find, uh, the value of n that does work.

Further questioning probed into the student's understanding of the relation of the proof to the statement: "if \((n + 1)^3 + (n - 1)^3 = n^3\) then \(n = 0\)." Although he agreed that the proof showed that statement, he did not see its different processes for the different goals of the two statements. All he could say was that the statements were equivalent without any regard to how the indirect method played out for statement 4.

Art: Well statement 4 is, this is-- since these [if \((n + 1)^3 + (n - 1)^3 = n^3\) then \(n = 0\) and its contrapositive] statements are equivalent then, they're equivalent to statement 4, which these show that there's not three consecutive positive integers such that the cube of the middle is equal to the sum of the cubes of the outers.

In other words, the only explanation for the relationship Art could give was that the proof showed the statement: "if \((n + 1)^3 + (n - 1)^3 = n^3\) then \(n = 0\)," which in turn proved statement 4. There was no indication in the data that he perceived the different approaches to the different goals in each statement.
In conclusion, although Art gave a dianoetic interpretation of the equation in his first interview, it was not the case that this interpretation was based on his understanding of the indirect method for finding the object. In other words, he did not consider the use of the equation as an assumption of the existence that needed to be validated indirectly.

In her first interview on this situation, Amy used a dianoetic interpretation of the equation’s role in showing the existence of the object. In her second interview, she found the given argument to be a proof of statement 4, as well as the statement: “if \((n + 1)^3 + (n - 1)^3 = n^3\) then \(n = 0\).” However, she did not show any awareness of its different processes of showing the different goals in each statement. When asked about the difference between the two statements and their relation to the given proof, her answer did not indicate any perception of the indirect approach of the proof to showing statement 4.

Amy: Uh, well that only... that \([(n + 1)^3 + (n - 1)^3 = n^3 \Rightarrow n = 0]\) only is like half of the proof. It only solves for \(n\) in this equation but then you have to take the \(n\) and remember where it came from and, then you end up with negative zero, -1, 0 and 1, because... So if this is true, then \(n\) has to be zero, \(n\) can’t be anything else. So, there’s no way for these all to be positive

Further questioning attempted to bring her attention to the indirect process of the proof but her answers were similar to or reiterations of the above explanation.

In summary, the results of this situation indicated that Abstract-Math students’ success at interpreting the solution of the equation as the sought object for the existence problem did not necessarily translate into seeing the indirect process in the proof-checking task. In particular, Amy and Art who used a dianoetic interpretation of the equation to find the sought object lacked the awareness of their own approaches. They
did not necessarily show any perception of indirect approach in this situation. Earlier, similar conclusions were drawn from the interviews of Patty and Perry.

**Discrete-Math students’ responses to situation 4 (interview 2)**

There were mismatches among the choices made by the students in this category. Most of the data from these interviews were inconsistent because the students kept changing their choices and giving contradictory answers to the questions.

Dean claimed that the proof showed statement 3 because he thought it showed the existence of consecutive integers. While trying to explain his choice, he realized that he misread the positive constraint on the integers, so he changed his answer to statement 2.

VB: Okay. Now, tell me how this statement [2], or how this proof shows this statement [2].
Dean: It, it solves down to here and then the only, the only option is \( n = 0 \), or \( n^2 = -6 \), and that’s not an integer. So it has to be \( n = 0 \).

Dean’s answers to the rest of the questions were incoherent. He gave different contradictory answers to similar questions. For instance, he thought the proof showed statement 2 but he also thought it was a direct proof that showed \( n = 0 \), without giving any valid reason for his contradictory results.

Dean: I guess, I just think it directly proves that \( n \) has to be equal to zero.

Thus, due to inconsistencies in the data, it was not clear why, after Dean chose existence (number 3) and after he realized the positive constraint on the integers, he did not pick statement 4 that claimed the non-existence. Instead, he chose statement 2. It seemed that he was able to recognize the direct manipulation of the proof as being used for searching existence, but why it did not serve him for seeking non-existence remained unanswered.
Like Dean, Doug claimed that the proof showed statement 3 because he disregarded the positive constraint on the integers. After he realized the constraint, he also changed his answer to statement 2.

Doug: It really doesn't prove--what it shows is that if \( n = 0 \), that statement holds.

There was no clear indication in the data for why he picked statement 2. This interview was going in the same direction as Dean’s interview until the interviewer decided to challenge him to think differently.

Doug: The fact that \( n \) equals— that \( n \) satisfies that equation, it does not prove that there’s three positive integers where \( n \) is the middle integer.

VB: Well, let me point out one thing for you here. Statement number 3 that you chose says that there exist three consecutive positive integers that satisfy some condition, [...] and now that you’ve realized one of the integers is not positive...

Doug: Right, right, that’s not correct.

VB: So, how can you fix statement number three?

Doug: Well, you just have to knock off the positive. If, if like, if positive wasn’t there it’d be a true statement.

Doug: (Pause). Well if \( n \) is the middle one, uh, you know I see the possibility that it actually shows that there does not exist three consecutive positive integers. Because if \( n \) is the middle one then, you know, you may be able to draw the conclusion that there’s not three positive ones.

Although now Doug picked the correct statement the proof showed, there was no evidence that he was aware of the indirect process of the proof. This result was not surprising because in his first interview on this situation, Doug did not see any usefulness for the role of the equation either.

Dan at first, claimed that statement 2 was proved, and that the proof “did not say anything about the other ones.” Then he sensed that statement 4 was also relevant.

Dan: [...] It’s only proving that if \( n = 0 \) then this \( [(n + 1)^3 + (n - 1)^3 = n^3] \) works. [...] And since this is true, then this [#4] also has to be true, because this is the only...
No, no, no. I think it's actually proving this [statement 2]. I think to actually prove this [#4] you'd have to say, so, you know, $n = 0$ and $n - 1 = -1$, and $n$ minus, $n + 1 = 1$. These are the only ones that work. [...] But I don't think this [proof] by itself shows this [#4]. I think this by itself just shows this [#2]. Ha, I'm probably totally wrong, but I'm going to stick by that for a little while.

VB: Okay, let's see. Well, one of your classmates said that the proof could be used to show that if the equation holds, then, $n$ is zero. Do you agree with that statement? Do you agree that the proof shows that statement?

Dan: Yeah that's probably true, actually. Let me think about it. Yeah, that's probably true because I keep messing this up. Because you can't say, A implies B so that means B implies A.

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Uh, I guess I might have to go with none of the above again. It might be this [#4] one, but I just don't think so. I change my answer to everything.

One of the characteristics of Dan was that he sometimes had difficulty seeing things clearly at first glance. Only when he was questioned further about his understandings, would he become attentive and then either correct or further confuse himself.

Dan: Uh, let's see. (Long pause). I need to put this... All right. So there aren't positive integers such that (pause). I guess I could say this [#4] is being proved. I don't know. I'm not exactly sure. I've thought too much about it now and I've confused myself.

It was observed that Dan was having trouble associating the direct proof of the statement: "if $(n + 1)^3 + (n - 1)^3 = n^3$ then $n = 0$" with an indirect approach for existence or non-existence problem. His remarks were inconsistent and the data remained inconclusive about the reasons for his indecision. It should be noted that Dan in his first interview was quick to guess the example -1, 0 and 1, and he was inattentive to the equation's role in finding the sought object. It was possible that his complete reliance on his intuition hindered his analytical thinking in this situation.

On the other hand, Dave claimed that the proof could be used to show statement 4 because it found the middle integer $n = 0$ and there was no other possibility for it.
Dave: [...] These [-1, 0, 1] are not three positive, consecutive integers. Um, so, this was the only one to me that held true with this \( n = 0 \) being the answer.

The above words indicated that Dave was able to relate the direct result of the proof to that of non-existence problem. However, due to time limitations, he was not questioned about his perception of the indirect process used in the proof and how it related to determining the non-existence of the object. In his first interview on this situation, Dave showed a partial understanding of the dianoetic interpretation of the equation’s role, and the data from this second interview could not indicate whether his understanding was based on indirect processes.

**Situation 5: Interview 1**

There does not exist any quadrilateral with sides of lengths 2, 3, 5, 11.

The objective of this problem was to investigate students’ approaches to an existence problem in a geometric situation and use them as a backdrop for analysis of second round of interviews on this situation.

**Post Abstract-Math students’ responses to situation 5 (interview 1)**

All four students in this category claimed that the statement was true because a side-length of 11 units exceeds the sum of the other three lengths. Their explanations suggested that their approach was through contradiction.

Paul initially said that it was false because he thought he had found a counterexample by physically constructing a quadrilateral.

VB: So, what’s so special about this quadrilateral that renders the statement false?
Paul: Well, the statement said that this couldn’t be done. And then, this shows that it can be done in at least one sense, in at least one way.

VB: Well, do you consider your approach to be a valid argument for your answer?
Paul: Yeah.

VB: Is it a proof?
Paul: Uh, yes, it could be a proof by contradicting the statement.

His words suggested that a geometric figure, if drawn accurately enough, could be considered in his view as a proof for existence of an object, at least until he was asked for alternate approaches.

VB: What kind of approach or strategy do you think it [a different argument] would use?
Paul: Um, I think it could, um... maybe show, like... uh, let's see... find if there was a limit to the ratio of the lengths. So, like, I don't think that you could have... This one was 11. Yeah, this length has to be bigger. Yeah, okay, so here's a better way, I guess. If... I think that if the sum of these lengths, this... I guess I'm wrong.
VB: Why do you think you are wrong?
Paul: I'm not sure just yet. Yeah, this can't be true, 'cause... these uh... So, the length of one side, the other two sides have to be at least bigger. I think, I mean otherwise, you could never (pause). Oh, wait a minute, no my thing's the right way.

He realized that there was some contradiction to his own picture if the sum of the shorter side-lengths was compared to the longest side-length. But he did not want to give up his initial claim yet because he was so sure of his construction. He found that contradictory fact hard to believe. But after struggling for a while he finally reconciled it.

Paul: [...] Okay, so yeah I guess my conclusion is that I think it's true. But I'd like to show that. I like to find out where I made a mistake on this. Maybe it's just not being careful, 'cause that's just not quite...
VB: Well, how reliable do you think your figure is?
Paul: My drawing, yeah, I guess it wasn't really very reliable. At the time, I didn't think it needed to be, so... Um, yeah, so just saying directly... I should find a better way of stating this. But, I mean, if you want to connect this point and this point with three segments that... that doesn't cross itself, well it has to be at least as long, the sum of the lengths of them have to be at least as long as the original segment.

On the other hand, Patty and Perry had no problem showing the geometric impossibility of drawing such a quadrilateral without even trying to draw any. Pam however had to draw a few quadrilaterals before realizing the impossibility. They all gave
similar arguments and were so satisfied with their answers that they kept reiterating the same argument throughout the interviews. The following are examples of their arguments.

Patty: Okay. This is going to be true because one side has to be length 11. I’ll do it on this just so we can have a visual argument. So, we’ll make it 11 centimeters. And then, that means that your other three sides have to add to more than that, which \(2 + 3 + 5\) is only 10. The fact that, you know, 10 is less than 11 makes it so you can’t get it to work. Because, even if you put your other three sides into a straight angle, which of course wouldn’t make it a quadrilateral, but we’ll pretend that they are such small angles that we can’t discern them, you can’t get three, using those three. So, there does not exist any [quadrilateral]. What I wrote isn’t a proof (laughs). I think what I said comes closer to a proof.

Perry: [...] And so one side is going to be larger than the other sides combined therefore there is no way that’s true, because this will never be able to reach this because eleven’s greater than \(2 + 3 + 5\).

Also, Pam gave the following argument.

Pam: [...] I would say true, um, simply because the longest side, 11, you can’t, you can’t add these lengths up to be 11. So, I think any arrangement of those three are never going to hit. There’s 2, 3, 5; 11 would always kind of kick past it if that makes any sense.

In her explanation for showing why a quadrilateral with the specified side-lengths cannot be drawn, Patty drew upon the fact that the shortest distance between two points is a straight line. Although she did not explicitly mention the word contradiction, her words implied that the “shortest distance rule” would be violated or contradicted.

Patty: Yes, because you have to be able to get from point A to point B with those three segments. You can’t, even if you were to just put them right on top, you wouldn’t get there. So, you’re certainly not going to get there if you go further because the shortest distance is the straight line, so...

Perry on the other hand, drew upon the Triangle Inequality in order to support his answer. However, he did not think he had a valid argument until he could prove the statement using the main logic behind Triangle Inequality.
VB: Okay. So now, can you absolutely be sure that this approach supports your answer?

Perry: Um, jeez, it doesn’t use the laws of mathematics very well, and basically what I’ve wanted to make sure—this is what I need to do is I need to find a way to create a quadrilateral— a quadrangle inequality, Quadrilateral Inequality from the Triangle Inequality. And then, if I was able to do that, then I could just plus these in and just do it. But I remember the Triangle Inequality, but I don’t remember the reasoning behind it.

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Yeah, I think the one that you... well, if I use the logic behind the, um, Triangle Inequality, and if I knew how that one works, then it would be based off of that, and that would be a better way to do it.

Similarly, during Paul’s second interview he pointed out that his argument in the first interview implicitly followed from the Triangle Inequality, although it took him a while to come to that realization.

VB: So are you saying that, um, your original argument comes from the Triangle Inequality?

Paul: Yeah, I think it can be considered at that, not directly though, but... Let’s see... Yeah, I guess, because, right, the distance between two points is always greater or equal to... then um... Okay, yeah, it makes sense. So these would have to, so the sum of these would have to be at least greater than that. And, yeah, it’s not really the same, but it’s, I think this is essentially the Triangle Inequality as far as the reasoning.

Pam however was not quite sure of the fact that she had used to explain her reasoning for non-existence of the quadrilateral.

VB: So what would it take to make it a proof then?

Pam: Um, I would definitely have to write it differently. And I’m pretty sure there’s a theorem, or postulate or something, a lemma, that would say the three shorter sides or whatever would have to equal the longer side, add up to equal the longer side. Um, so to find some other theorem, or some corollary, or whatever, to state along with this would help.

VB: And can you think of any particular method of such proof?

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Pam: Um, I guess you could also do a contradiction [emphasis added], like assume it was true and then show it doesn’t work through contradiction.
What she had thought to be the reason for impossibility of the quadrilateral’s construction seemed to describe an extension of the idea of Triangle Inequality to quadrilaterals, henceforth called “Quadrilateral Inequality” to mean $a + b + c > d$, where $a, b, c$ and $d$ represent the side-lengths of a quadrilateral. She also verbalized her process of showing the impossibility as an approach to a proof by contradiction.

In summary, all four Post Abstract-Math students approached this situation using arguments similar to the method of contradiction. Three students (Pam, Perry and Paul) claimed the construction of such quadrilaterals would be impossible because it would violate the Triangle Inequality, although only one (Pam) was explicit about the method of contradiction. The fourth student (Patty) claimed it would violate the shortest distance fact.

Abstract-Math students’ responses to situation 5 (interview 1)

Two students (Alice and Adam) found a valid reason for the impossibility of the construction and thus claimed that the statement was true. Art claimed that the construction of a quadrilateral was impossible but he could not think of a valid reason. The other student (Amy) felt that there was some sort of impossibility but could not tell definitely whether the statement was true or false.

Amy’s initial reaction was that she might be able to draw a quadrilateral as a counterexample, but soon realized the inefficiency of drawing figures.

Amy: Okay, there does not exist... 2, 3, 5 and 11. Okay, well let’s see. So I guess you could prove this wrong by finding a quadrilateral that does have leng... sides with these lengths. Okay so there’s only a few types of quadrilaterals, no that’s not true.

I could try to draw it but... or you could find some. [...] I don’t see why you couldn’t do it [...]. I’m not, that’s not gonna work for a proof because I might be
able to find it but I think it’s gonna take forever. I don’t think that’s [drawing] a very efficient way to do it.

After a few trials, she realized that there was some sort of impossibility in drawing a quadrilateral, so she tried to capitalize on that idea which came very close to the idea of Quadrilateral Inequality.

Amy: I was trying to find some relationship between the sides in a quadrilateral, some rule that... About how, something similar to the effect like with triangles where you have, you know that these two sides can’t equal this side or it wouldn’t be a triangle when you add them up. And I was thinking if there would be something like that for quadrilaterals. I’m not sure about that... and then you could say, you could add these numbers up [emphasis added] and somehow say that it didn’t work [implying that the statement is true].

Her train of thoughts suggested that she was trying to remember a fact similar to Triangle Inequality in a quadrilateral, rather than trying to reason or extend that fact out to a quadrilateral situation. This inclination towards a fact-based approach rather than discovery through reasoning was also observed in her other interviews. In addition, her search for a fact prevented her from reaching a conclusion, because she was trying to add and compare different side-lengths together.

Amy: Yeah... it would be just using known facts from geometry or known properties of quadrilaterals and then, then it... I guess you would assume and then showing that if you do have these lengths then it doesn’t, it’s not a quadrilateral, or there... yeah.

It should be noted that the semantics of the words she used suggested that her approach had she come up with that missing fact would have been a contradiction.

Similarly, Alice approached the problem by drawing a picture because she needed to visualize things. She didn’t take too long to realize what was going to happen.

Alice: I always have to draw a picture first, even though I know that doesn’t really [say much].
VB: So tell me again, why do you think the statement is true?

Alice: Okay, because it said there doesn't exist one and because, to form this quadrilateral, the sum of these three sides needs to be as much as this fourth side, and $2 + 3 + 5$ only adds up to 10, which is smaller than 11. […] Even if you lay them in a flat line, you'd have 2 here, and 3 there, and 5 there it's not even going to be 10 [11]. […]

VB: Okay, so, what would it take to make it a proof?

Alice: I would need to show that for any quadrilateral the sum of any three sides would have to be greater than the fourth side, or equal to it I guess. […] Because otherwise you can't fit that fourth side on there. I don't know, I guess may be another way you could do it is assume that you can draw a quadrilateral with these four sides, which would be a proof by contradiction [emphasis added]. I guess, [it] would be another way of doing it.

It was concluded that her reasoning was based on a proof by contradiction and for a proof, she would use arguments similar to the idea of Quadrilateral Inequality.

Similarly, Art attempted to draw a quadrilateral, which led him to realize that it was an impossible task.

Art: Um, what I was trying to do I was going to go ahead and draw a quadrilateral, and the way I attempted it, I couldn't. But it doesn’t mean that there can’t be (pause).

Why? Just from looking at it, the length of 11 is throwing everything off, since it's so much bigger. I'm just having a hard time picturing a quadrilateral in my head with a side that much longer, more than double any other side.

VB: So, what’s so special about those numbers? Why do they have to be, you know, more or less than the double?

Art: Well, since 11 is so big… Okay, like, for this example that I was going to draw if I go all the way out to 11, there’s no way from this, the other side, that I can get it to be 5. It's going to have to, it's going to cut me off short. And so that’s why I was saying that it's so much bigger that I was having a problem with it.

Like his peers, he felt that the construction was impossible and believed the statement was true. He did not consider his argument to be a valid proof because he could not give a definite mathematical reason for why it was impossible.

VB: Okay, can you think of some or any particular approach for a proof? Like some method?
Art: Uh, method of contradiction maybe. Just go through and, I don’t know, maybe it contradicts the definition of a quadrilateral. […] 

Like Alice, he believed that the method of contradiction could handle the situation he was in, but because he did not see a mathematical reason for the impossibility of a construction, he could not pinpoint the exact source of contradiction.

On the other hand, Adam’s initial reaction was that it should be possible to draw such a quadrilateral without regard to its side-lengths. However, he did not attempt to draw any until he was directed to do so.

Adam: It’s false because of course there exists a quadrilateral with sides 2, 3, 5, and 11. […] Um, I don’t think the lengths of the sides really matter […].

VB: But you didn’t really show me the counterexample. You just hand-waved it to me.

Adam: Okay, okay. I can draw one.

After a few trials, he realized that it was actually impossible to construct a quadrilateral and he gave the following explanation.

Adam: […] Hmm, maybe, maybe I’m mistaken. Well, I could not do it, because one side’s eleven, I can’t connect all four sides. I can’t connect the four sides of the quadrilateral from beginning to end.

VB: So, why can’t you…?

Adam: Why? Because, see I have to, for it to be a quadrilateral I have to connect the four points from beginning to end, so A, this is A, B, C and D. Now this [D] would have to go to point A and it does not. It does not connect. And there’s no way to manipulate this to, now, to make this a quadrilateral. So…

Further probing inquired if Adam had any valid mathematical reason for his answer to this situation.

VB: But why? What is the reason you cannot connect?

Adam: […] Because each time I found myself needing the fourth side to be longer, so… And I believe that the three sides if you add up three sides, they have to be longer than the fourth side, the longest side.
[...] I’m sure there’s some proof saying something about the four sides of a quadrilateral. And, I can’t think of it, but they have to, I think the three sides have to be more than the fourth side, something like that, maybe. [...] Those words indicated that his perception of the source for impossibility of construction was drawn from the idea of Quadrilateral Inequality. It should be noted that although Adam gave a valid argument, there was no explicit indication that he was knowingly utilizing a contradiction approach for his argumentation. Like Amy earlier, he was trying to base his argument on some (learned) fact that he thought he did not remember. It should also be noted that, so far in any of his interviews, there was no indication in the data that showed his thorough discernment of the semantics of the method of contradiction.

In summary, it was observed that, after a few trials with quadrilaterals, two (Alice and Adam) students were able to find a valid reason for non-existence of a quadrilateral based on Triangle Inequality. Amy argued in terms of contradiction and using Triangle Inequality but she could not reach a final decision because of her reliance on known facts. Art gave an argument for a contradiction, but he could not tell where the contradiction would come from.

**Discrete-Math students’ responses to situation 5 (interview 1)**

Three students (Dave, Dean and Dan), after a few trials of constructing a quadrilateral, claimed that the statement was true because one side of the sought quadrilateral needed to have a length greater than the sum of the other three side-lengths. The following are examples of their arguments, which they kept reiterating throughout their interviews.
Dave: Well, because, in geometry we learned that any four-sided object, um, with any
four-sided objects, the sum of any three sides cannot be greater than the fourth
side. Cannot be, let's see, how does that go. Basically, what I'm trying to say is,
this is 2, this is 3, this is 5, um, and that sums to 10, while the fourth side is 11.
So, if you drew these out in a straight line, you know, here's 10 and here's 11,
you couldn't get this end to fit this end.

Dean: There just isn't any quadrilateral with those sides. It just, because like I remember
in school I remember when you think about triangles, \( a + b \) had to be greater than
\( c \), or you couldn't have a \( c \) that was greater than \( a + b \), so, this quadrilateral can't
exist. [...]

Also, Dan gave the following argument.

Dan: So, I guess the way I'd prove this is like a diagram type of approach. I guess it
would have to be-- it's kind of intuitive, because I mean this is the biggest side,
but these sides have to add up to something greater than that for it to be a
quadrilateral. So, it has to be greater than 11 because if it were equal to 11 it
would just be like a straight line. [...]

Both Dave and Dean based their arguments on a fact similar to the Quadrilateral
Inequality that they thought they learned in school. Dan however based his argument on
his own intuition and discovery, which indicated self-confidence. It was observed earlier
in his interview on situation 4 that Dan also showed confidence in his intuitive approach.

Nonetheless, he later saw the relationship between his intuition and the Triangle
Inequality.

VB: So, what aspects of it, what aspects make it a proof?
Dan: [...] So I'm saying for all triangles, no matter what they are, or for all
quadrilaterals, no matter what they are, the [sum of the] smaller sides have to be
greater than the biggest side. That's what I'm saying. And since that is true, um,
then this has to be true that it can't exist. So, I guess I'm proving something else
to prove this.

On the other hand, based on his drawing of a slender quadrilateral, the fourth
student Doug thought the statement was false. He did not think a figure was a proof, but
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because he could not find any reason for otherwise, like Amy and Adam, he thought there
had to be some known fact about quadrilaterals that could deal with this situation.

Doug: [...] If I knew more about quadrilaterals, there’s probably just some kind of
general formulas stating whether that’s true or not. I don’t know.

The data from further probing of the students did not indicate any awareness of
the method of contradiction they utilized in this situation. When they were asked about
their method or a name for it, they just reiterated their arguments without validating the
method.

VB: Can you explain the approach or the method that you used here? [...] What
method do you call this proof used?
Dean: Yeah, yeah, I looked at this third side right off the bat and I saw that length 11.
And a quadrilateral, so I’ve got a four sided object, or whatever, in this case I’ve
got a long one and then, uh, three shorter ones. But you can’t get from both sides
of the long one given the three shorter ones, but, there is no quadrilateral.

Dan: I definitely think you could do this with a diagram. I don’t know if that’s a real
method or not. Maybe that has problems just in the actual method itself, just that
you can’t prove something [with diagrams]. But I think it could, because, I
mean... Yeah, I don’t know. I guess, I don’t know if I’m just using intuition or if
this is really true because like I know this is 11 and these have to be greater than
11 because otherwise it won’t work. But I don’t know if that’s a specific method
that uses diagrams, or if there’s a specific method, you know? I mean, for all I
know the definition of quadrilateral says that this doesn’t exist. Do you know
what I’m saying?

Similarly, Dave’s answer was:

Dave: Um... well, the way I kind of did it, I think would be called, like a direct proof
where I just um, you know, pulled my lengths out here into a straight line and said
you can never make this touch with this, you know? So, maybe something along
the lines of a direct proof might work for that. So, um, you can prove it
gometrically too, I’m sure. Which is kind of what I did.

The above excerpts, as well as the earlier case with Adam, indicated that the
students sometimes use indirect arguments without actually being aware of their use.
Situation 5: Interview 2

In this second round of interviews on situation 5, students were given a proof by contradiction to refute the statement:

There exists a quadrilateral with side-lengths 2, 3, 5 and 11 units.

There are three ways of joining the four corners of a quadrilateral with the following orientations: (2, 3, 5, 11), (2, 3, 11, 5) and (2, 5, 3, 11), where (2, 3, 5, 11) indicates the order in which the sides are joined. The proof essentially made the argument that if such a quadrilateral existed in one form then there would be some contradiction. It neither disguised the semantics of a proof by contradiction nor did it explicitly mention the word contradiction at the end. It simply concluded by showing some impossible algebraic relationship for the length of a diagonal. The impossible relationship found in the proof was obtained by manipulating the first of the following three possible systems of Triangle Inequalities derived from the three different orientations, where \( x, y \) and \( z \) represented the lengths of the diagonals.

\[
\begin{align*}
2 + 3 &> x \\
5 + x &> 11
\end{align*}
\begin{align*}
2 + 5 &> y \\
3 + y &> 11
\end{align*}
\begin{align*}
3 + 5 &> z \\
2 + z &> 11
\end{align*}
\]

The other two systems of inequalities if manipulated in the same way would result in a contradiction as well. By adding the two inequalities together (to get \( 2 + 3 + 5 > 11 \)) the given proof could be generalized to show the contradiction of the algebraic inequality \( 2 + 3 + 5 < 11 \). This generalization step was left out of the given proof in order not to bias the alternate proof (see objectives below) used during the interviews. In particular, it was kept from those students (e.g., Amy and Doug) who have not yet themselves discovered...
that general contradiction during their first interviews. As observed earlier in the first
interviews, ten students were able to discover that general conclusion by themselves.

This was a case of existence problem in a geometric situation. Normally, if one
cannot produce a mathematical object with a given set of properties the best alternative
would be to resort to a proof by contradiction to show the non-existence of such objects.
Thus, one of the objectives of this interview was to investigate students’ understandings
of an argument that used such contradiction process. The main objective of this interview
was to investigate if students’ perceptions of a proof by contradiction changed under a
similar but convincingly non-contradictory (and redundant) argument (see the alternate
proof for situation 5 in Appendix I). The generalization step of the original proof was not
made explicit in order to avoid biased answers based only on students’ perceptions of the
general inequality, rather than answers based on the general concept of irrefutability of
contradiction. Hence, this interview was intended to probe students’ understandings of
the power of refutation with a contradictory case for non-existence of a particular
quadrilateral.

Post Abstract-Math students’ responses to situation 5 (interview 2)

There was a mismatch among students’ answers to this situation. Perry said it
neither proved nor disproved the statement, while Patty said that the proof showed some
support that the given statement was false. The other students (Pam and Paul) claimed
that the original proof showed that the given statement was false.

Pam and Paul gave similar arguments for their choices. They argued that the value
of $x$ cannot both be less than 5 and greater than 6 simultaneously.
VB: Okay, so why do you think that the presented argument here shows that the statement is false?

Pam: Well, basically what they showed is kind of what I was going for. [...] They used, the inequalities should be going in the same direction. [...] The length of $x$ has to be at least 6, $x$ has to be just a little bit bigger than 6 for the bigger triangle to work. And it has to be a little bit less than 5 for the smaller triangle to work. You can’t be less than 5 and greater than 6 at the same time, so with that there cannot be a quadrilateral with sides of lengths 2, 3, 5, and 11.

Paul: Uh, ’cause it shows that by the triangular inequality that this [quadrilateral] situation is impossible. [...] On this side of it, it has, $x$ has to be greater than 6. But on this side of it, $x$ has to be less than 5. So... You can’t have both of those two statements. [...] One number can’t be both greater than 6 and less than 5.

Further probing investigated their understanding of the process of the proof. Both students exhibited a good understanding of the contradiction process used.

VB: Okay. Can you explain the role of the assumption and its consequence in this argument?

Pam: Um, the role of the assumption was to say, well, assume that there is a quadrilateral with those sides, and um, then just going through, using the Triangle Inequality stuff, you can show that your assumption was false so therefore there cannot be a quadrilateral with those sides.

VB: So, what kind of reasoning is used here?

Pam: I would say a contradiction.

Paul also gave a similar answer.

Paul: Uh, it was... Uh, let’s see. Yeah, this is contradiction.

VB: And what exactly is contradicted?

Paul: The Triangle Inequality. [...] If, if this was true it would contradict the Triangle Inequality. If, if this existed.

In the next portion of the interviews, their perception of contradiction in light of the alternate proof was probed. It was found that Pam’s perception of a contradiction was not something definitive or irrefutable. She gave an incorrect reason for the invalidity of the alternate proof.

VB: So, here is another argument in support of the original statement. Please read it carefully and let me know what you think.
Pam: I guess I don’t know enough, but I don’t agree with how they set up their inequalities, but that could just be...
VB: Did they set it up wrong?
Pam: That’s what I’m thinking. Let me see what they did here [in the original proof]. See with this one, what I noticed, this [11 units long] is kind of like the longest side, the base goes on this [less than] side of the inequality. And the same thing they did with the x. And here [the less than part in the alternate proof] they used one of the sides [of the triangle], I feel, they used a side, and then over here they used the [less than part as a] base.

VB: Does that make any difference as long as you are using the same idea of Triangle Inequality? And why would it make...why would it make any difference?
Pam: I don’t... Well, I think it would make a difference, because if I had triangle, like, 1, 2, and 3... Well, maybe it wouldn’t. Maybe it doesn’t make a difference. Maybe it doesn’t. Okay. I just think it’s wrong.
VB: Well, what’s wrong?
Pam: I just don’t think you could have a quadrilateral with sides 2, 3, 5...

The student was trying to explain that in the alternate proof the Triangle Inequality was set up wrong. From the way the original proof used the Triangle Inequality, she inferred that the longest side of a triangle is always compared with the sum of the other two shorter sides.

One explanation for this could be that she desperately wanted to find a mistake in the alternate proof because, as she indicated, such a quadrilateral was impossible to draw especially since she found, from her first interview, the reason why such a quadrilateral could not be drawn. Another explanation for that misconception could be that the Triangle Inequality was not well understood by the student, but this might be a less potent argument because as Pam realized later that it would not make a difference which side of the inequality a length was compared with. Data from later parts of the interview reinforced the former explanation and nullified the latter argument. Although she wanted the alternate proof to be wrong, she circled choice number 1 (i.e., the presented argument...
above proves that the statement is true), thus implying that she was somewhat convinced
by the alternate argument.

Pam: But with that proof, according to that it says [choice] one.

VB: Well, explain the best that you can, the difference between the two arguments.

Pam: Um, the two arguments, a big part of it is where they cut the quadrilateral into two
different triangles. I don’t think it should have mattered where you put the, where
you cut it. Um, because you still have two triangles when put together make the
quadrilateral. Um, but I, I want to say [emphasis added] there has to be
consistency on which sides you put behind the inequalities, but I don’t think it
matters. But the big difference is where they decided to bisect the triangle
[quadrilateral], or where they put the diagonal in. Uh...

VB: Can you have both? Can you make both arguments about the same statement?

Pam: I don’t think you can. I don’t want it to be. I really don’t want it to be. But it looks
like you can [emphasis added]. Uh...

Pam had no valid explanation for the inconsistent conclusions. She did not
understand why the original proof would show the statement false and the alternate proof
would show it true. She was unable to see the redundant manipulation of the Triangle
Inequality in light of the original proof. Furthermore, from the later part of her interview,
it was concluded that she lacked a concrete conceptual understanding of the irrefutability
of a proof by contradiction. When her perception of the contradiction in the original proof
was challenged, she did not insist that a contradiction should be inevitable under all
circumstances including the case in the alternate proof, especially when she had found the
source of the contradiction in her first interview.

Unlike Pam, Paul indicated that the alternate proof did not show the statement
true, nor did it show it false. But there was no indication in the data that he was fully
confident of his answer, even when he believed the statement was false.

VB: So why do you think it neither proves true nor false?

Paul: Well, I think it just shows that, that um, this length [BD] is okay. But that doesn’t
mean anything because this one [AC] still doesn’t work. [...] Yeah, well, just
'cause this length [BD] is okay, if this length [AC] doesn’t work then the quadrilateral still doesn’t work.

VB: So, uh, explain, the best that you can, the difference between these two proofs.

Paul: Um, let's see (Pause). Well, this [original proof] one assumed that it worked and showed that it was contradictory. This one [alternate proof] assumed that it worked and then—I don't know really what it—it just assumed that it worked essentially. [...] This [alternate] one just assumed the statement and then, and that’s all it really did, I think. I mean it showed this with the Triangle Inequality, but that doesn’t mean that it—that the quadrilateral works over all.

It was not clear from the data why he thought that the quadrilateral did not work over all in the alternate proof, despite attempts by the interviewer to have him elaborate more on his understanding of the proofs. During the interview, he showed signs of frustration for not having a good argument for his choice to the point that the interviewer had to reassure him of his positive input for this project.

VB: Okay, so you don’t think those two arguments are related?

Paul: No, I guess, I don’t know. Maybe I shouldn’t be doing this; I don’t think I’m very useful to you.

His lack of confidence, in addition to his lack of words, suggested that he did not have a concrete understanding of why the alternate proof failed to show the statement true or false. Since he did not draw upon irrefutability of the contradiction, it could be concluded that the power of contradiction was not perceived as definitive when his understanding of it was challenged, especially when he discovered the source of contradiction in his first interview.

On the other hand, Patty’s explanation for why she thought the original proof showed some evidence that the given statement was false revealed that she considered the proof incomplete, because it did not consider other possibilities for rearranging the sides of the quadrilateral.
VB: So, why do you think that the presented argument in this situation shows some support that the statement [there does not exist a quadrilateral with sides of lengths 2, 3, 5, 11 units] is true?

Patty: Um, they're showing that if this is the way the segments are set up, then it can't be true [that there exist a quadrilateral]. But I think that in order to, if you were going to use this argument to prove it, you would have to show that it doesn't matter if you, say, interchange the segments. Um, I mean, it, it doesn't work, regardless, but I think, um, if you just said this, someone could say, well, maybe you don't get this contradiction about the value of $x$ if you move some of them around. Like if, you had switched [side] 2 and 3 [around] or something. [Emphasis added]

Evidently, Patty understood the contradictory argument in the proof. She considered it incomplete because it did not take into account other possibilities for drawing a quadrilateral. Put another way, the proof left doubts in her mind about whether another reader would be convinced that it was not possible to draw a quadrilateral with those given side-lengths, even if the sides were switched around in another order so that triangles with different side-lengths were obtained.

Patty also gave a similar argument for the alternate proof when she was asked to comment on it.

Patty: (Long pause) Okay. I would again just say that it supports and not a proof because, I mean you could either say well, what about... I think you need to show that it doesn't matter what order these are in [emphasis added], in order for this statement to be [false...]

In case of the alternate proof, the redundant manipulation of the Triangle Inequality went undetected because her attention was concentrated on whether the alternate proof was similar to that of the original proof. Patty viewed both proofs as cases or examples but not as generally valid proofs. She wanted to see a proof that considered a generic form of a quadrilateral that entailed all possible ways of constructing it, or if particular quadrilaterals were constructed then a proof should have accounted for all the cases.

VB: So, what impact would that argument have on the statement?
Patty: It would just show that for all quadrilaterals with those sides that it can’t exist. I mean, you can’t make one with those four sides, which... I mean, it’s hard for me to say that this doesn’t prove it because you’re not showing all cases, when I know that, that it doesn’t exist one. But, I think to rigorously prove it you’d have to show that, you’d have to try all ways in order to use this argument (pause). You might come to the same conclusion with all of the ways that you set it up but I just don’t think that one example of not working means that there doesn’t exist an example where it works [emphasis added].

Patty believed that the contradiction reached in this particular proof would refute the existence of a quadrilateral because she found the source of that contradiction in her first interview. In other words, she knew by virtue of the given proof that the other cases would also result in contradiction. On the other hand, she claimed that showing a contradiction for one case would not prove the non-existence of a quadrilateral, especially when there were other ways of constructing it. It should be noted that Patty also showed a similar concern during her interview on situation 1 for checking all possible cases (even/odd) before a conclusion was reached.

The reason why she thought that other arrangements needed to be checked as well could be explained in terms of a procedural or algorithmic understanding of an existence/non-existence problem. As the procedure dictates, in an existence problem one needs to find only one case that makes the statement true. Thus, in a non-existence problem, failure to find the sought object with certain properties neither disproves the existence nor proves the non-existence.

In order to understand this phenomenon better, her argument needed to be contrasted against the perspective of the proof. The given proof did not show existence but it showed a contradiction if such a quadrilateral existed. In other words, the objective of the process of the given proof was to search and check for contradictions. A failure to
find any contradiction would not have proved the existence of the sought object, or made the statement true. In this case, the proof’s one counterexample of a quadrilateral’s construction was enough to refute its existence, because it drew upon the Triangle Inequality and implicitly implied the contradiction of the Quadrilateral Inequality \( 2 + 3 + 5 < 11 \), namely, the sum of the lengths of three sides was shorter than that of the fourth side. Most students were able to find this contradiction during their first interview, including Patty. Now, by virtue of that generalized contradiction in this situation, if one arrangement was not possible then no other arrangement could be possible. The geometric specificity of the sought quadrilateral plays no major role other than having the property \( 2 + 3 + 5 < 11 \) which might have just as well been represented as \( m + n + p < q \) or any other combination thereof. In other words, if the construction of a quadrilateral, satisfying the condition imposed by Quadrilateral Inequality, was possible in a certain way then its construction would also be possible through any rearrangement of its sides.

Therefore, it could be possible that Patty’s reliance on the procedural approach to the process of proving or disproving existence made her worry about whether another reader of the proof would be convinced. Although, in her first interview, she argued that the source of the contradiction was the inequality \( 2 + 3 + 5 > 11 \) in general, there was no indication that she realized that the inequality could also be deduced from the given proofs. Possibly, her preoccupation with the procedural approach made her overlook the generic nature of the given argument in this geometric situation. Nevertheless, her perception of the contradiction did not change in this situation when it was challenged.

Perry gave a similar explanation, but unlike Patty, he dismissed the given argument because he believed it did not prove nor disprove the statement.
Perry: Because it's showing this for a specific quadrilateral, where AD equals 11, you know, AB equals 2, BC equals 3, DC equals 5. And it's not generalizing that, I mean, what if BC equaled.... [...] Yeah, let's assume BC is equal to 5 and CD is equal to 3. Then we clearly have a different quadrilateral, because, we have different sides touching different sides, so it's not just a matter of rotating or labeling, it's actually a different quadrilateral [emphasis added]. And it doesn't, this proof doesn't account for the different quadrilaterals. So, it fails to generalize for those.

VB: Okay. So, um, this is not a general proof. Is that what you're saying?
Perry: Yes, and it doesn't set up to hold for any other case than this specific case. Even if it does hold in those cases, it still doesn't show that in this [proof].

Perry dismissed the proof altogether because, according to him, it failed to show all cases and thus he thought it was invalid in general. He was even more verbal about which quadrilaterals were considered equivalent or different. In the next excerpt, Perry reiterated his firm belief that the proof showed that a specific quadrilateral was impossible to draw and that there were other non-equivalent quadrilaterals the proof did not account for.

VB: Well, what then do you think this argument attempted to show?
Perry: I think it was showing it for a specific case, and it does a good job of that it just, it doesn't make, it doesn't deal with the fact that we could have the sides arranged in different orders from this order. [...] What this is saying is there exists some where it doesn't hold, but it's not dealing with the other cases where it may hold, or may not. [...] In the next portion of the interview, he was asked to compare his understanding of the alternate proof in light of the original one.

VB: Well, here's another argument in support of the given statement. I want you to read it and let me know what you think.
Perry: Okay. (Talking to self, unintelligibly, pause). Ah, in my—it doesn’t properly, see, its facts are incorrect in here, one and two. In part one it says \( x > 2 \) [\( 5 < x + 3 \)]. That is true, but \( x \) also has to be less than 8 [\( x < 5 + 3 \)], by the Triangle Inequality. And then, in part two, we have \( x < 2 + 11 \). Therefore, \( x < 13 \). But also, \( x > 9 \) [\( x + 2 > 11 \)], by the Triangle Inequality. Therefore...

VB: (Interjecting) Um is that part of ...[the proof], \( x > 9 \) you said?
Perry: I'm saying that that would also have to be true by the Triangle Inequality and it's [the alternate proof] ignoring it for these parts; its facts are incorrect.
The student was trying to explain that in the alternate proof the inequalities obtained from the Triangle Inequality were not correct. They did not tell the whole story because other inequalities should also be accounted for before any conclusion was drawn. Perry was the only student among the four in this category to have actually detected the correct reason why the proof failed to show anything. Thus, he again circled the third choice (neither proves nor disproves the statement). This next excerpt also confirmed the above conclusion where he elaborated more on how the proof went astray.

Perry: No, the algebra’s fine, but its facts are, like I said, its facts are incorrect. Because the Triangle Inequality has two parts, \( x \) must be great... One side plus \( x \) must be greater than the other side, but also, the two sides added to each other have to be greater than \( x \). And so, when it combines them, it doesn’t properly combine them, because if we...

His elaboration for how the proof went astray was not explicit about the consequence those missing inequalities would have had on the overall argument and on the truth of the statement. However, his words were consistent with the fact that the missing inequalities would have brought forth a mismatch, thus contradiction when compared with each other. Nevertheless, Perry made it clear in the later part of the interview that in order for the original proof to be complete and show the statement false, it had to consider all the possible combinations of quadrilaterals. A similar argument should be made on each one before a conclusion (contradiction) was reached.

Perry: You could generalize this [first proof] though. And it wouldn’t be... I mean, the thing is, there’s only, there’s not that many cases, so, you could just show this for every possible case of the quadrilaterals. [...] It wouldn’t take that long to whip, to enumerate all the cases and just use a similar proof for each of them. Or, you could set this up to show, you could find a way to generalize this, which would probably be more of a hassle than just enumerating the six cases [emphasis added], because you use the same proof only switch the numbers around.
As with Patty in this category, it was not enough for Perry to have one contradictory case to refute a statement. His reliance on the procedural approach for the process of proving or disproving existence made him to overlook the generic nature of the argument at hand, despite the fact that he discovered the contradiction of the generalized Quadrilateral Inequality in his first interview. The emphasized words above indicated that he wanted to find a way where the given proof could be generalized, yet he dismissed the possibility of doing so in favor of showing each case separately. His perception of contradiction did not help him to believe that the generalization could be deduced from the shown counterexample where the specificity of numbers disguised its generic nature.

Since the process in the given proof was mainly to contradict the existence of a quadrilateral with certain side-lengths, it had raised some doubts as to its generality and thus prompted a need for a generic justification. Nonetheless, there was no indication in the data that they (Patty and Perry) would try to implement a fact they discovered earlier in order to make the connection that would lead to the generalization of the given argument. In terms of a non-existence problem, it was not easy for the students to break away from the procedural process of showing the absence of a mathematical object where there were many other possibilities to consider. Their understanding of the procedure hindered them from seeing the implications of just one counterexample.

It should be noted that the above accounts were observed using specific numbers for the side-lengths of the quadrilateral, which created the need for generalization in the first place and accentuated the students' reliance on the procedural process. Similar research making use of notations rather than numbers might produce different results.
Abstract-Math students' responses to situation 5 (interview 2)

All four students in this category claimed that the given proof showed that the statement was false. However, three students (Amy, Art and Alice) changed their answer after studying the alternate proof.

Amy explained that a contradiction was reached because two different values of $x$ were found.

VB: Okay, so can you explain the role of the assumption and its consequence in the presented argument?
Amy: Uh, the assumption is that there is a quadrilateral with these given sides. And then, the proof shows that if that was true, then $x$ would have these weird... then $x$ would lie in these two disjoint intervals, which is impossible. So, it can't be that this assumption is correct.

[...] So that’s a contradiction, they used.

Clearly, the student exhibited a good understanding of the contradiction process used in the proof. Next, her conviction for non-existence was tested when she was asked to compare the given proof with the alternate proof.

Amy: Uh, I would say it provides some evidence that the [alternate] proof is true, or that the statement is true.

VB: Well, explain the best that you can the difference between the two arguments.
Amy: Uh, well they just draw, first they draw the diagonal in a different place, or they draw the other diagonal, and then they use different sides to compare them, which makes a difference.

VB: Okay, now you have two different things here. So, what can you say about the statement, the original statement? Is it true or is it false?
Amy: Ah, well... I'm not sure... I think it’s true that there exists at least one, but I don’t know why. I still have to think about why it matters what inequalities you use [emphasis added], because here if you had done $5 + 3 > x$, then you would get $x < 8$. Oh, you would, it would still work. So, in this [alternate] case no matter how you do it, it will work. But I don’t know why that is different from that [original proof].

VB: So, if you were to say anything about the original statement would you say it’s true or false?
Amy: I would say it’s true [that there exists a quadrilateral].

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The emphasized words above indicated that, like Pam, Amy was trying to make sense of the order of inequalities used in the Triangle Inequality. Each argument used different order of sides for the less than part of the inequality and she was trying to find out if that was a reason for inconsistent results. However, after a few checks with the inequalities she concluded, "it would still work" no matter how the inequalities were set up.

In the end, her inability to find a valid reason for inconsistent results made her change her mind about the existence of a quadrilateral. It should be noted that her lack of success to find a reason for non-existence of a quadrilateral in her first interview on this situation might have contributed to her decision. The above observation indicated that her perception of contradiction was something that could be refuted when challenged, instead of causing her to resolve the inconsistency by pursuing the matter further.

Similarly, Art said that the reason the statement was proved false was because two different values for the diagonal length were obtained thus refuting the statement. The following excerpt indicated his perception of the process of the proof.

VB: So can you explain the role of the assumption and its consequences in this argument?
Art: Um, yeah. We assume that there was a quadrilateral with those lengths, [...]... and we showed that the side AC, uh, would have two different values if we broke that triangle down, which can't happen.
VB: So, what exactly is the consequence of the original assumption?
Art: Um, if we assume that, we arrive at a contradiction. Therefore, it will show that the statement's false.

Evidently, Art understood the process of contradiction in the proof. Next, his perception of the contradiction was challenged when he was asked to study the alternate proof.

Art: Um, my impression is that the statement is still false, and from this proof, uh, if... it still doesn't give you an exact value of x.
In both excerpts above, it was observed that his explanation for the existence of such a quadrilateral hinged around finding a unique value for \( x \), probably because he viewed such a quadrilateral to be rigid if it existed. This was in contrast to the views of Patty and Perry from the Post Abstract-Math category who argued that other arrangements of sides were not considered by the proof. Nevertheless, that view neither persisted nor in the end, after he examined the alternate proof, did he find the statement proved false anymore.

VB: Well explain the best you can the differences between these two arguments.
Art: Between these two arguments? Um, they’re both breaking this quadrilateral down into triangles, and in this [original] situation it does not work and in this [alternate] situation it looks like it does work. And I would think that if there were a quadrilateral it would have to work in every situation [emphasis added].

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Well, in this statement [alternate proof] it does look like [emphasis added] there exists a quadrilateral with sides 2, 3, 5 and 11.

VB: So, does that mean that the original statement is...?
Art: Um, I guess it would mean it was true if the proof was correct, which I’m not exactly sure if it is or not. But if it is correct then the statement is true.

---

[...] there’s either going to exist a quadrilateral or there’s not going to exist a quadrilateral, and...

VB: So, does there exist or not?
Art: Um, I’m going to have to go... I’ll say there does exist a quadrilateral because of this [alternate] proof right here.

Since Art was not confident about his answer, it seemed that his perception of the original proof was unsteady. He did not maintain a definite view of the power of the contradiction or insist on its irrefutability. As with Amy, this indicated that his perception of contradiction was something that could be refuted when challenged, instead of causing him to resolve the matter by pursuing it further. The reason his perception of contradiction changed might be due to his failure to associate the validity of his reasoning
for the impossibility of constructing a quadrilateral in his first interview with the Triangle
Inequality.

On the other hand, Alice first claimed the argument proved the statement false, because it showed a counterexample to the existence of a quadrilateral.

Alice: Well, they assume that, um, there's a quadrilateral that does have these sides and don't cross, but, we assume that, we assume that there was and it showed that, that can't happen.

VB: Okay. So, what is the method of argumentation used here?

Alice: The counterexamples, I guess. Not really a counterexample, but... yeah, I'd say it's like a counterexample.

Next, her perception of the process of the proof was put to test when she was asked to comment on the alternate proof.

VB: Okay. So, here is another argument in support of the statement. Please read it and let me know what you think.

Alice: (Pause). Uh... yeah, this proves the statement's true.

Without exhibiting any doubt, she agreed that the statement was proved true and hence contrary to her initial perception, she believed that there exists a quadrilateral.

VB: So explain the best that you can the differences between the two arguments.

Alice: Well, now that I look back, I guess this [original proof] one was wrong because it just gave one, one example of how to position the sides and where to draw the line. Here [alternate proof] using the other diagonal, it came out, it worked. So, I guess, this [original proof] was not a correct proof, because, we're trying to say that there exists a quadrilateral like this, but giving one counterexample doesn't make it false [emphasis added], which I guess is what I did wrong. Because, here is an example where it does work.

After studying the alternate proof, Alice thought that her initial perception of the original proof was wrong because, as the procedure for finding existence of a mathematical object dictates, not finding an example would not prove non-existence. In other words, according to her, contradicting existence (by showing a counterexample) did not show non-existence.
In her first interview, it was observed that Alice, through her discovery of the Quadrilateral Inequality, believed that a quadrilateral did not exist. In this next excerpt from her second interview, she tried to invalidate her original approach. This again indicated that her perception of contradiction could be refuted and changed when challenged in such situations.

VB: So, again, last time you said that the statement is true [that there does not exist a quadrilateral]. Now that you know more about this situation, what would you say about this statement?

Alice: I'd say false, because here they're saying there does not exist one, but we've just seen here [in the alternate proof] that there does exist one. So, I'd say that it's false. And the reason I was wrong was because I was mixing up the Triangle Inequality with this [2,3,5,11]. I was trying to apply it to the four sides and really, it only has to do with the Triangle Inequality.

This explanation suggested that the reason she thought the alternate proof worked was because one had to check one side-length against the other two only, as with Triangle Inequality. That is, it was unlike checking one side-length against the other three as she did with a quadrilateral. In other words, she thought the argument of implying Quadrilateral Inequality from the Triangle Inequality was invalid.

Thus, it was observed that Alice, who in her first interview succeeded in discovering a valid reason for the impossibility of constructing a quadrilateral, was not convinced by her own discovery of a contradiction. Put another way, contradiction according to Alice was not something definitive to refute a statement but could be changed when challenged and faced with more information about the situation. In this case, her procedural understanding of a non-existence problem also contributed to her change of perception of contradiction in this problem.
So far, from Adam's previous interviews, there was no indication that he had a thorough understanding of the contradiction process. Here, he claimed that the proof showed the statement false and he gave the following explanation for his choice.

Adam: Why? Well, it says it right there in the last line. It says well, that's impossible because we cannot have \( x \) both greater than 6 and \( x \) less than 5 for a line segment that bisects the quadrilateral.

VB: And what type of argument is that?

Adam: It's a... It's basically just a counterexample. It's just saying, well, it's a direct, it's direct, and it uses... Yeah it's just a direct proof showing that the statement is false.

Adam's perception of the goal of the proof was finding a counterexample and he thought the process used for finding it was through direct methods. This also confirmed what was found so far about this particular student's lack of discernment of the semantics of indirect proofs.

To better probe Adam's perception of the given proof, he was given the alternate proof for comparison. Unlike any of his peers, however, Adam was not convinced by the alternate argument and claimed that it did not prove nor disprove the statement.

Adam: I don't know what the statement... I don't know what this proof shows. It shows, it just shows that, so that line segment can be anywhere from 2 to 13 units [of] length.

Despite efforts through further questioning, the student could not give any good reason for why he was not convinced. He simply reiterated the above words as well as what he thought was the correct fact.

Adam: I believe the statement's false [that there does not exist a quadrilateral].

VB: But the second proof doesn't really...

Adam: ...doesn't help me at all. [...] Because I don't understand why, I mean, I don't understand how, how the length \( x \) can be anywhere from 3 to 12 units. I mean, I know that, it just doesn't make sense to me that this segment could be (pause)

VB: Do you think that the \( x \) should be just one number, is that...?

Adam: No, no. No, it's just (pause).
Although Adam’s perception of the result of the original proof did not change in light of the alternate proof, he was unable to give a valid reason for his belief. In his first interview on this situation, Adam found a valid reason for why a quadrilateral could not be constructed. His discovery of impossibility of constructing a quadrilateral might be the reason why in this interview he was not convinced by the alternate proof.

In conclusion, the findings from this situation for the Abstract-Math students indicated that the students’ perception of contradiction was not definitive and irrefutable, that it could be challenged and changed when faced with more information about the situation even when they themselves discovered the contradiction in their first interview. Their own perception of the power of contradiction was not strong enough to help them invalidate an argument that was inconsistent to their own beliefs. Instead of arguing against a purported proof that was inconsistent to their beliefs, they changed their perception.

**Discrete-Math students’ responses to situation 5 (interview 2)**

All four students in this category claimed that the proof showed the statement false. The alternate proof did not alter the minds of the three students (Dan, Dave and Dean) who showed a thorough understanding of the contradiction in the original proof.

The following is a sample of what they said.

Dan: Um, sure. Since we’re assuming that it has these sides, um, what that means is, if we come to a contradiction then it’s false but if we see that that’s possible then it’s true. Whereas, if you were saying that this [given statement] isn’t true and if we arrived at a contradiction then it would mean it is true.
As before, the rest of the interviews probed the students' perception of the contradiction in light of the alternate proof. Dan gave the following reason for why he thought the alternate proof did not prove nor disprove the statement.

Dan: [...] Yeah, this doesn't really tell us anything, I don't think. Let me make sure—because it's not using it in the right way to show us anything. [...] Because if you do take, say \( x \) is 5, then it violates this [Triangle Inequality] rule in other cases. They're taking one case like, it's saying in a plane the sum of the lengths of two sides of a triangle has to be greater than that third side. They're only taking one case of that [Triangle Inequality], right? In this one, they're taking that third side is 5, whereas, they're going to have to take each case.

Dan's words above indicated that he had found a valid reason for not believing in the result of the proof. He explained that if \( x \) was assumed 5 units long which was within the proof's alleged interval, then the Triangle Inequality \( 2 + x > 11 \) would still be contradicted in the triangle ABD in Figure 2 (Appendix I, Situation 5) of the proof.

He was so sure of his conviction that he kept reiterating his words above. He also explained that if the quadrilateral were to work in this case then every possible combination of the Triangle Inequality needed to be checked to ensure absence of contradiction.

Dan: [...] this kind of quadrilateral isn't possible, because they're only taking one case of this [Triangle Inequality] rule. Like, \( x + 5 \) has to be greater than 3, but \( 3 + 5 \) has to be greater than \( x \) and \( 3 + x \) has to be greater than 5. It's all of that that makes this rule possible. So, you can't just take one case. So, even in this one I think you'd probably—well, see in this one [original proof] you wouldn't have to do that because you've found two cases that contradict it. This one [alternate proof], if you're going to prove, if you're going to assume this is right and show this is right, then you have to show that it's right in all cases. [...] 

During his first interview, Dan found a valid reason for the impossibility of constructing a quadrilateral and in this interview he showed a good understanding of the relationship between his own argument and the proof. Namely, he argued that his
reasoning was drawn from the Quadrilateral Inequality that could be inferred from the Triangle Inequality. This was a case where the student’s own discovery of the Quadrilateral Inequality made him firmly believe that the contradiction process was irrefutable.

Likewise, Dave gave a similar argument for why he thought the alternate proof did not prove the statement.

Dave: This one doesn’t prove anything to me. It’s saying that \( x \) can be between 13 and 2. But, that doesn’t work. This one...

VB: Why not?
Dave: Um... Well... because then you’ll end up with an, you’ll end with a situation where \( x \) could be anything in between these values. And, well, I suppose if we plugged in for \( x \) all the possibilities it would show that this quadrilateral didn’t exist. Because if we said for our example that \( x \) was [...] if \( x \) is 12, then this [ABD] could be a triangle, but this [BDC] couldn’t be a triangle, because, the sum of these two sides is 8. [...] 

Dave’s reasoning was strikingly similar to that of Dan’s. Dave explained that if \( x \) was assumed to be 12 units long, which was within the proof’s alleged interval, then the Triangle Inequality \( x < 5 + 3 \) in the triangle BDC would be contradicted.

As with Dan, it was also observed that Dave’s realization of irrefutability of a contradiction was also reinforced by his own observation and by making the connections from his first interview on this situation.

Similar to the previous cases with Dan and Dave, Dean also argued that the proof was invalid because it ignored the existence of a contradiction.

Dean: It, I think it’s trying to show some support that the statement is true. But I think that if you look at it, it kind of does show that it’s false, though, because it says, like, these, it’s showing support like \( x + 3 \) has to be greater than 5 and \( x \) has to be less than 2 + 11. But at the same time, \( x \) has to be, on this one [triangle ABD], \( x \) has to be greater than 9, because if it’s less than 9, \( x \) + 2 is less than 11. And on this [triangle BCD] one, uh, \( x \) has to be less than 8, because if it’s greater than
8 these sides don’t reach. But, it’s just kind of like I don’t know why you’d want to say these statements [1 and 2 in the proof], because they don’t really, they don’t help in the proof.

Like his peers, Dean found the contradiction of the Triangle Inequality $x < 5 + 3$, in the triangle $BCD$, because the alternate proof concluded that $x$ could be larger than 8. When asked for the reason why he was not convinced, his response was based on his previous observation from the first interview.

Dean: I just don’t think there is such a quadrilateral.
VB: Okay and why do you say so?
Dean: Because, when I looked at it earlier [in the first interview] I—the sum of these three sides is not equal to the fourth side, and it just, they can’t reach. You can’t get from A to B with only 10 units to work with when it’s 11 units away.

Again, the student’s own observation in his first interview was the main influence for his conviction about the nonexistence of the quadrilateral. His firm belief of irrefutability of the contradiction did not change.

On the other hand, Doug was not sure of the process used in the original proof. He sensed that it used contradictory arguments but to him the semantics of a proof by contradiction seemed to have indistinguishable attributes from direct proofs.

Doug: [...] the statement says there exists, so we’re assuming that it’s true, there exists a quadrilateral. [...] But then we find out in the process of that proof that it’s contradictory, that something’s contradictory. And I’m just trying to connect... It’s not true, there does not exist a quadrilateral with, the statement is false, but I don’t know if that’s contradiction or if that’s a direct proof. I think it’s more of a direct proof than a contradiction. But anyways... But it does find a contradiction in the fact that AC cannot have two different lengths.

Although Doug claimed that the original proof showed that a quadrilateral did not exist, he was not sure why the alternate proof showed a different result. After studying the alternate proof he showed doubts that a quadrilateral might be drawn.
Doug: Right, there's just an interval on which the statement is true. So, it does exist (pause). So I guess it, it puts the parameters on what x can be, whereas the other one, the first one [original proof], I'm not sure it did or it assumed that. I don't know why that came out like that.

It became clear that Doug was not totally convinced or impressed by the result of the original proof because he thought the alternate proof showed the existence of a quadrilateral. In his attempt to explain why the alternate proof found a quadrilateral, he first thought drawing a different diagonal segment might have played a role.

VB: Well, then, try to explain the difference and the similarities between these two proofs and why the inconsistency?

Doug: It looks like they're taking different line segments. I don't know why that would make a difference, but right off the top of my head it would. But apparently it depends on how you choose your line segments—how you construct. I would say that's the case, how you construct the triangle is important in this particular case—the order in which you connect the line segments.

This misconception led him to believe, as later data from the interview indicated, that Doug was convinced of the existence of the quadrilateral because of his understanding of the algorithmic process for finding an object. In other words, there is always a possibility that an object existed even when one could not find it.

Doug: [...] All we need to make that statement true, all we need is to find one example, and in the first case they just found one that wasn't true. But there's millions of other ones and the second one just verified that there is at least one where—So it is a true statement, there exists, there does exist a quadrilateral of those dimensions.

In his first interview on this situation, Doug was unable to find any reason for nonexistence of a quadrilateral. Thus, it was concluded that Doug's lack of success in seeing a contradiction, on his own in the first place, could have contributed to his inability to see the invalidity of the alternate proof. If he had experienced the impossibility of physically constructing the quadrilateral, like his peers, he might have
looked for reasons for inconsistency rather than just changing his perception of the original proof.

In conclusion, it was observed that the students who discovered or deduced the Quadrilateral Inequality stood firm on their belief in the contradictory result in this situation and when faced with an inconsistent argument tried everything to find the reason for the invalidity. It might be possible (e.g., Doug) that had they not had that experience the results could have been different.

**Situation 6: Interview 1**

There are no distinct positive integers $a$, $b$, $c$ and $d$ such that $a^2 + b^2 + c^2 = d^2$.

The objective of this problem was to investigate students' approaches to an existence problem in an algebraic situation, as well as to find the extent to which students were willing to test the equation or look for a proof. Finding a counterexample, a set of four integers (henceforth called “quadruples”) with a specific relationship, in this situation may not be an easy task. However, the context of this problem can be related to that of the Pythagorean relation (see proof of situation 6 in Appendix H). Therefore, this problem situation investigated whether students could think of a way for generalizing their search efforts, and if possible show the existence of such quadruples.

**Post Abstract-Math students' responses to situation 6 (interview 1)**

Two (Perry and Paul) of the four students in this category did not reach any conclusion about the statement and the third (Pam) guessed that the given statement was true. Only one (Patty) student was able to find a correct approach and thus an answer to the problem.
Patty’s initial approach to this problem was to try to find a counterexample that refuted the statement. However, her initial trials for finding a quadruple failed.

Patty: [...] I’m going to say it’s false and I’m going to try to find an example of where it does work. Look at a 3, 4, 5 right triangle and that works because, and so we know $3^2 + 4^2 = 5^2$. Oh, wait, this might not work either. [...] I don’t know if there exists such integers or not...

As it was pointed out in the introduction of this situation, Patty’s intuition about searching for Pythagorean triples prompted her to look for ways of generalization.

However, she was uneasy with using integers in her argument and the interviewer tried to facilitate her flow of thoughts.

VB: So you're saying you don’t know whether such integers exist or not but, um, what if I changed the word integer into real numbers?

Patty: Then it'll work because then you could use, I mean you could use anything you wanted. You could find some $b$ and $c$ such that $b^2 + c^2$, the square root of $b^2 + c^2$ equals 4 and you’d have $a$, $b$, $c$, and $d$.

VB: So how does that work?

Patty: You just, you’d say that $3^2$ plus the... All you’d have to solve for actually is, um, $4 = \sqrt{b^2 + c^2}$. So, you’d have, if you square both sides... So, um, I don’t think there are any integers that do it, though [emphasis added]. Because we’d have, 1 [squared] would give us 1, 2 [squared] would give us 4, 3 [squared] is 9, 4 [squared] is 16. None of these will add up to 16, but you could just, I mean, you could just take two things that add up to 16. You could just say 9 and 5, and then take the square root of... No, that doesn’t work.

The student’s argument started with $3^2 + 4^2 = 5^2$ and then stated that if $4 = \sqrt{b^2 + c^2}$, then by substitution $3^2 + (\sqrt{b^2 + c^2})^2 = 5^2$ it could be deduced that $a = 3$, $d = 5$, and $b$ and $c$ are any real numbers such that $b^2 + c^2 = 4^2$. This systematic approach for finding the real numbers would work and if manipulated differently, distinct positive integers could be found too. In fact, this approach was the same as the argument used in the second interviews, except the given argument also related the process to constructing triangles geometrically. The student did not pursue the matter of finding distinct positive integers...
until later because she did not believe the relation could hold for distinct positive
integers.

VB: You don’t have any answer right now, whether the statement’s true or false?
Patty: No, not from my work here. If I had to guess, if it were on a test or something and
I had to guess, I would say that it was true [meaning there does not exist a
quadruple]. Just because that’s the way I’m leaning, but I wouldn’t say that I had
proved that by any means. It’s just the way…

Although she could not find a counterexample yet, she was able to learn from her
systematic search for a counterexample and find an argument that showed the existence
of a counterexample.

VB: Okay. Here, here you were in a situation where you were looking for some
example or counterexample and you couldn’t find one. I mean, what would
normally be your next approach to handle this situation other than guessing?
Patty: I would, I might try a couple other trip—you know, a couple other Pythagorean
triples, but after that I would, um, try to look at properties of square numbers. It's
hard when there's four different ones [emphasis added] to deal with and they’re
not um, consecutive or anything, so it’s like they can be any positive integers.
Um, so I would, I would try to look at the general, look at them in general and see
if there was anything, anyway I could disprove it.

VB: Well, can you think of some approach that might handle a situation where you
couldn’t find a counterexample? In your experience, for example, can you…?
Patty: Well, if you, if you can’t find a counterexample, then your best--you have to try
and prove the--prove that it is true. Um, I mean, I would go about it, I would say
that there exists [emphasis added], okay, there exists some a and b in the positive
integers, or there exists some a, b, and c actually. […] So, okay, I’m changing my
mind to false, now that I do this argument, because that would say that then there
also exist some. [And after a long algebraic manipulation of symbols] So,
somewhere out there they do exist, so this is false because I found in general a
counterexample.

The above excerpt indicated that whenever her search for counterexamples failed she
would seek some alternative routes. From her words emphasized above, it could be
determined that she would follow an indirect process similar to that of contradiction to
actually prove or disprove the existence, although she did not explicitly mention, or
realize the method herself (see analysis of her second interview). This common approach
occurred after searching for a counterexample and finding out, through trial and error, the properties of a desired counterexample, such as its relationship to Pythagorean theorem. In fact, while she was explaining her approach she had actually written down a general argument where the existence of a counterexample to the statement was shown.

VB: So explain to me what you actually did, in words. How you approached the problem...

Patty: Okay. I tried to just look at it in general. We know that there, I mean, for any right triangle, where the two legs other than the hypotenuse are, um, integers, just because here we were using integers. If you square that one and that one, you’re going to get the third one, the square of the third one [hypotenuse]. So, um, so if we use this then, as the leg of our triangle, and then have another side of integer. Whatever. Then you have. This becomes your $d$. This becomes your $...$. If this were $a$ and this were $b$, then this gives you some answer. This would be $a^2$. No it’s not. I can’t deal with triangles. I can’t (laughs). The picture isn’t, isn’t working, but I’m, yeah--no it should be working.

Her argument was similar to the one given in the take-home task and used in the second interviews, except that she could not draw or relate hers to the construction of triangles as in the argument given in the take-home task. This observation was also confirmed by the data from her second interview on this situation (see analysis of her second interview). It should be stressed here that Patty’s success for finding an argument that showed the existence had come up in her search for generating a counterexample through algebraic manipulation and simplification, rather than fitting a combination of numbers in the given equation. In other words, her search for a counterexample culminated in showing a procedure for generating quadruples. Nevertheless, her existence proof for real numbers was not enough to show the existence of a set of integers that satisfied the relation. She had to make sure by finding a particular case where the existence of integers was shown.

VB: Well, do you consider this argument, the whole thing that you did here, as a valid proof to say that the statement is false?
Patty: [...] Okay, so one, so as a check here, I know that one of the Pythagorean triples is 5, 12, 13. So, if I’m saying that, that works, that $3^2 + 4^2$—so I would give values. So, I would be saying that $3^2 + 4^2 + 12^2 = 13^2$. So you get 9 + 16 plus one forty-four equals one sixty-nine, if it’s right, so we’ve got one sixty-nine. So, there’s an example. So, it’s obviously false.

In fact, since she constructed a general procedure for showing existence of counterexamples, she had no trouble reconstructing the steps with using specific numbers such as Pythagorean triples.

Pam also began looking for a counterexample in order to refute the non-existence and likewise she failed to find one. She started with a list of some perfect squares but too few to produce a counterexample.

Pam: There are no distinctive positive integers... Okay, this right away is telling me to look for a counterexample because it says no distinctive, so then if you find some distinctive you should be able to make it work [writes 1 + 4 +...]. It doesn’t work [writes 16 + 9] (long pause). Ooh, that’s not helping (laughs).

During the interview, Pam also wrote the Pythagorean relation $c^2 = a^2 + b^2$ and tried to draw a right triangle, but she did not systematically pursue this approach as Patty did.

Unlike Patty, who turned her endeavors into a procedure that showed existence, Pam’s failure however prompted her to believe that the statement was true, or integers satisfying the given relation did not exist.

Pam: Well, right now I would conclude this is true simply because I’ve gone through a few integers and tried it out and it’s not working. So...

VB: So, are you saying the statement is true because you could not find any counterexample?

Pam: Yeah, for the time being. Um, I suppose if I had more time, I might be able to find a way to prove or disprove this, um, but not this quickly.

[...] Um... if I can’t find a counterexample, then usually it means I’m going about this the completely wrong direction. And usually induction or contrapositive or one of those other methods... works better.
Despite some pushing by the interviewer, there was no indication in the data that Pam was able to pursue an argument as Patty did for this situation. It was observed that Pam’s search for finding a counterexample went unaccomplished, partly because of her failure to utilize systematically the Pythagorean relation.

Similar to Pam’s initial approach, Paul started to look at this situation by listing a few perfect squares in hopes of finding a sum of three perfect squares that resulted in another perfect square. Like Pam, he listed too few (only 1, 4, 9, 16, 25 and 36) to have any success with finding a perfect square sum as a counterexample.

Paul: Um, yeah, I was just going to write the squares of some integers, and then I was just going to see if any of them sum to a perfect square. If I could just tell off the top of my head (pause). I don’t know if this will work, but I figured I would just try (long pause). Yeah, I guess I’m not sure on this one. [...] I’m really not sure how to show this with just numbers.

VB: So why do you want to look at numbers?
Paul: There, I’m sorry. I don’t know how to show this algebraically, is what I was thinking.

VB: [...] Out of all these combinations of numbers, do you think you cannot find one set of numbers?
Paul: Well I know it’s true for two [as with Pythagorean triples], so I think it’d be true for three [quadruples].

Other than seeing similarities between this situation and that of Pythagorean theorem, there was no indication in the data that Paul revealed any alternate approach to finding a counterexample.

As mentioned earlier, Perry was also unable to reach a conclusion about the truth of the given statement. At first, he tried to relate this problem to Pythagorean theorem, but failed to make any use of that connection. He thought choosing three varying integers satisfying a relationship was not an easy task. Thus, Perry tried to reduce the equation into a fewer number of variables but to no avail.
Perry: Um, I’m thinking, I’m trying to take advantage of the Pythagorean theorem here, there’s plenty of integers for which $a^2 + b^2 = c^2$. So I’m trying to reduce it, reduce one of these away. [...] So, I don’t know if that’s the approach I want to take. I don’t think that approach is going to get me anywhere. [...] 

Three choices is just too many for anybody to work with. [...] It’s because I can’t think, for me it’s hard to think in three, with these three things changing at once. So, that’s what tripping me up basically. So, I’m trying to set them equal to something else, but I’m not succeeding very well.

It must be noted that Patty also found this varying relationship a difficult guessing game, which was why she strove for a rather systematic approach for searching a counterexample. In contrast to the other three students, Perry did not approach this problem by testing for counterexamples.

VB: Okay, and if you try some numbers, it won’t help?
Perry: Um, no, because it’s, because I’d have to prove it, proving it for one, proving that it doesn’t hold for one set of numbers doesn’t do anything [emphasis added]. I have to prove it for every set of numbers. [...] 

In this next excerpt, it occurred to Perry that this problem might have a geometric connection because of the Pythagorean theorem in a right triangle. This train of thoughts might have directed him to better manage the variables but he did not pursue it any further.

Perry: Hmm... I’m trying to play with this and with some manipulation. I was thinking geometrically, it might be, it might be interesting to play with that... If we had $a^2 + b^2$... I’m not entirely too sure how I would prove this either way. Like I said, if I can count zero as a positive integer then it’s really easy to prove. But it does say distinct positive integers, so...

The approach above came very close to that of Patty’s, but Perry did not pursue it with conviction. Perry’s approach could have resulted in a similar successful path as Patty’s, if he had built upon his realization of the geometric connection of the problem rather than just aimlessly narrowing down the number of variables.
In conclusion, it was observed that all four students were able to see similarities of this situation to Pythagorean triples. However, based on their own admissions they found it hard to extend or generalize the idea to quadruples. It was also observed that three students (Patty, Pam, and Paul) approached this situation by looking for a counterexample but to no avail. Only one (Patty) had actually extended the search, by building upon the properties of the desired counterexample and its connection to the concept of Pythagorean triples, and found a counterexample. Although Perry showed a similar but sketchy approach to that of Patty’s, he did not pursue it.

Abstract-Math students’ responses to situation 6 (interview 1)

None of the students was successful in finding the answer to this situation. They all initially tried to find a counterexample but were unsuccessful because they could not think of a strategy for narrowing down their search.

Amy gave up on finding a counterexample quickly because she did not think she could find one in a limited time.

Amy: (Long pause) no distinct positive integers $a$, $b$, $c$ and $d$... so they all have to be different... Okay, well my first thought is to find a counterexample just 'cause I like to play around with numbers. I would start with some $d$... 4... no 9... 0, 36, 1... Okay so that would take a long time but I think I could continue with that for a while and try to find a counterexample. [...] A counterexample would be finding, finding four distinct integers that satisfy this [relation].

VB: And you cannot think of any such combination of numbers?
Amy: Uh well I gave up before I tried very hard but...

Her approach was based on guessing rather than narrowing down choices. However, the next excerpt in which she briefly described her alternate strategy for finding a counterexample revealed behavioral attributes similar to the contradiction process.

VB: Suppose you cannot find a counterexample; what would you do?
Amy: Well then, I'd need a different approach because that doesn't tell me anything if I can't find a counterexample. That just means that, that doesn't mean anything... Yeah so then I'd have to go back to finding some relationship between these \([a, b, c\) and \(d]\), somehow show that they're... If this was true then one or more of these... they're not distinct or positive integers [emphasis added]. [...] So I can somehow get relations between \(a, b, c\) and \(d\) showing that they're not distinct positive integers and I think this is what I'm saying, and that would prove it true.

The emphasized words above indicated a process similar to a proof by contradiction.

Nonetheless, she thought the contradiction would result from the properties of the integers rather than the constraint the equation imposed on the positive integers. This also confirmed her thought process for guessing numbers rather than generating them through a systematic approach. In other words, she considered a combination of positive integers fitting into the equation and not the equation as generating a set of distinct positive integers, which would have led to a process of narrowing her choices down. It should be noted that Patty's success in this situation as described by the previous analysis was based on her narrowing down the properties that a counterexample should have.

Art also started looking for a counterexample to show the statement false. Like Perry, he did not try specific numbers. He approached it by assuming the statement false, and he thought if there were three consecutive integers \(a, b\) and \(c\), then the sum of their squares would produce the fourth integer \(d\).

Art: I approached it by setting values for \(a, b, c\) which were positive integers, with \(a = 2k, b = 2k + 1, c = 2k + 2\). I plugged that into the equation, and by, uh, using the quadratic formula I found my \(d\) squared. And ultimately we could find, uh, \(d\), but with my \(d\) squared right now \([d^2 = 12k^2 + 12k + 5]\), I have a value that is not an integer. So...

VB: Okay. So what's your conclusion then about the statement?

Art: Uh, it's true the way I have it done, but, with my values \(a, b, c\), I let them be, um, consecutive integers, and which doesn't take into account all integers, so I mean there... I'm going to say it's true, but, I mean, there is more work that I need to do, because I didn't prove all of it—I only proved for consecutive integers.
In his algebra Art made a conceptual error, because he tried to solve for $k$ using the quadratic formula (i.e., letting $d^2 = 0$ in $d^2 = 12k^2 + 12k + 5$) rather than checking if representation for $d^2$ was a perfect square. Nonetheless, Art's trial for consecutive integers indicated that he was trying to narrow down the possibilities by looking for a contradiction. Since he could not find a contradiction, he thought that the statement could be true for consecutive numbers and thus his final guess was that it could also be true for distinct positive integers.

Similarly, Alice's first instinct was that she needed to find a counterexample.

Alice: [...] Well, they have to be distinct. Oh, I guess the way I'd start out by doing this is just trying to think of a counterexample. Because just to me, it seems like there should be one [...] that there does exist positive integers $a$, $b$, $c$, and $d$ such that this holds, and that would tell me that this was false.

However, like Pam before, she changed her mind because she could not find a counterexample, and tried to think of a way the statement could be shown true but to no avail either.

VB: Can you come up with some counterexample then?
Alice: Let's see here (pause). Hmm... no, I guess I can't think of any. Well since, I guess, since I can't think of one, maybe try, and prove that, maybe I'd rethink my answer and try to prove this is true. But, I don't really know how to go about doing that. Gosh, the only way I can think of is trying to prove it false and doing a counterexample. I don't know how I'd go about trying to prove that it was true.

[...] If I had to, had to answer, I would say that it's true, because I can't think of any time when it wouldn't be true. [...] Yeah, I'd say it's true, but I need some method of proving that.

The next questioning tried to probe if she could think of any alternate approach or a proof that could handle an unsuccessful attempt for finding a counterexample.

VB: Okay. So, think about the method now...
Alice: I mean, I'd have to assume that there, however, assume that there are, four positive distinct integers such that this holds...

Although she did not finish her sentence, it seemed that she was going for a process that would use contradiction.

Like his peers, Adam also tried to find a counterexample because he thought that it was an easy alternative.

Adam: I am trying to find a counterexample, which is the easiest way to figure it out. So then I'm trying to figure out $a$, $b$, $c$. Um, I'm doing, figuring out numbers for $a$, $b$, $c$, squaring them and then adding them together and finding their, their total. And my object, my thinking is that if I can find that number that is square-rootable [perfect square], to be a positive integer, then... But it's a counterexample, but... I suppose a formal proof to prove that...

Like Pam and Paul before him, he listed a few perfect squares in hopes of finding a combination of three that added up to a perfect square. However, because of his short list he did not find any that fit into the equation and thus thought that the statement was true.

Adam: Um, my first assumption is [...] that there are no distinct positive integers $a$, $b$, $c$, and $d$ such that $a^2 + b^2 + c^2 = d^2$. But...

VB: Why?
Adam: Why? Just because I've done, like two examples and I haven't found one. But that's only two, you know. So then if you do [...] 3, 4, 5, 25, 16, and 9, 41, that's 50. Hmm, 50 doesn't have a square root. Yeah, now just because I'm doing this simply, hmm, I don't know how to prove that there are no distinct ones because...

Further probing tried to see if this student would consider alternate approaches to his failure for finding a counterexample.

Adam: To make it a valid proof? To go through every positive integer, which is pretty tough and not humanly possible. But you could maybe find an abstract way of doing it by maybe replacing numbers with letters.

VB: Can you think of the approach or method that an abstract proof might use?
Adam: Well, you could, um... you could make the statement negative and find a contradiction to it and say that $a$, um, $a^2$... So then there are positive integers such that $a$, $b$, $c$, $d$ such that, $a^2 + b^2 + c^2$ does not equal $d^2$...maybe. [...]

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According to his words, in a situation where Adam failed to find a counterexample he would choose an approach that uses the method of contradiction. 

In conclusion, it was observed that all four students in this category would approach a problem of non-existence by searching for a counterexample. Three (Amy, Alice and Adam) of them tried to guess a counterexample without any luck, and Art tried to use algebraic symbols and manipulations but lacked sufficient generalization. There was no indication in the data that showed that the students had seen any similarity of this situation with the Pythagorean relation. One of the interesting findings was that all four students described alternate approaches that seemed to use the contradiction process. However, none of them was able to show any contradiction, which led some to think that the non-existence of quadruples was true.

**Discrete-Math students' responses to situation 6 (interview 1)**

Again, all four students in this category tried to find a counterexample using guessing techniques and as before they were unsuccessful. Both Dean and Dan did so by listing some perfect squares, whereas Dave and Doug did so by substituting numbers in the equation. None of the students found a definite answer for the situation.

Dean thought that the statement was false based on his intuition for the existence of infinitely many combinations.

Dean: I think it’s false, and I think so because I think if I look at the formula I’ll be able to find a set of distinct integers that meets that criteria. Um…

... My goal would be to come up with three numbers that when squared and added together equal a fourth number that has a square root. I’m not really sure how to do that right now.
He then tried to list a few perfect squares but too few to actually find any
counterexample. When asked of alternate approaches for failing to find counterexamples,
he replied that he would look for a proof that would show the statement true. He did not
strive to show any explicit approach that searched for a proof.

Dan however, thought the statement was true because his search for
counterexamples from his list of perfect squares was fruitless.

VB: So, do you think there are no distinct positive integers such that this equation
holds?
Dan: I think so. See, I’m only saying that because I can’t think of any. Um…

[...] I think you could do two [as a Pythagorean triple] of course. I think you
could do 9 + 16 equals 25. But I don’t know if you could do three. Let me think
about it (pause). Well, I mean, I can’t think of anything just for these, which kind
of leads me to believe that it might be true. But the thing is you have to prove it
for like all positive integers. Let me just make sure. Yeah, I don’t think it works. I
think it’s true. [...]

Dan passively noticed the similarity of the situation to that of Pythagorean triples, but
according to his remarks above he did not think the relationship was possible for
quadruples. Nevertheless, he thought that the chances of finding a quadruple would
increase if more perfect squares were listed, but because there was no way to exhaust all
the integers he thought it might not be efficient.

Similarly, Dave after substituting some values for the variables argued that
searching for a counterexample in that manner was inefficient because of the infinite
combinations of quadruples. However, unlike Dan, he thought quadruples exist because
of infinite possibilities. Like other students, he found it hard to deal with four variables at
a time.

Dave: [...] I was trying to think of how you would do that mathematically, um, in a
general form so that you don’t have to plug and try things forever. And I just can’t
think of how I would approach doing that with um, maybe because of the amount of variables we’re dealing with is kind of throwing me, but um... yeah.

Doug on the other hand, thought that the statement would be true but there was no data to indicate why. He also admitted that the number of variables in the problem was too many for him to manage.

Doug: [...] It seems like when you try to, for me, anyway, when I try to figure out specific examples, it just blows up. I mean, the combinations blow up to a point where I can’t even think about the problem. [...] It just seems that there’s no way you could really brute force it and get anywhere. [...] I won’t, I just don’t know what to do with it. I wouldn’t even know what kind of method you would use to try to find something like that.

Evidently, this problem situation was overwhelmingly difficult for him to figure out a way to start. He thought it would be a “useless exercise” for him to try numbers.

In conclusion, it was observed that the students in this category did not employ any proof technique other than guessing. Neither did they engage in any systematic search for a counterexample, nor did they think it would be fruitful. The large number of variables combined with infinite number of their combinations seemed very unmanageable to them. They were quick to give up on a situation that involved infinite possibilities.

**Situation 6: Interview 2**

In this second round of interviews on situation 6, students were given an argument using contradiction to refute the statement:

There do not exist positive real numbers \(a, b, c\) and \(d\) such that:

\[
a^2 + b^2 + c^2 = d^2 \text{ or } a^2 + b^2 = d^2 - c^2.
\]

The statement is (trivially) false and the proof essentially made the argument that if such numbers did not exist, then there would be a contradiction, violating the existence
of a geometric figure (right triangles with side-lengths satisfying Pythagorean theorem). The argument did not disguise the semantics of contradiction. The word “contradiction” was intentionally made explicit in this proof because this situation, which was presented towards the end of the interviews, was used to probe students’ understandings of explicit contradiction. The data were used to triangulate the results with the other situations.

The problem in situation 5 was a case of existence in a geometric situation where its proof showed an algebraic contradiction. In contrast to situation 5, situation 6 was a case of existence problem in an algebraic situation and its argument showed a geometric contradiction. In the given argument, the existence was shown through the possibility of geometric construction of right triangles. It gave an algebraic argument that brought about a contradiction of its geometric counterpart.

Often times, if one cannot produce a counterexample (in this case a quadruple with a certain relationship), the best alternative would be to resort to arguments using contradiction. Such arguments could be made to either show the existence of a counterexample (when the statement is false) or to indicate how a counterexample could be found. The purpose of the given argument in situation 6 was to provide an insight into how a counterexample could be found. In fact, Patty in her first interview on this situation demonstrated this same phenomenon. Thus, one of the objectives of this interview was to investigate students’ understanding of the process of the given argument.

Post Abstract-Math students’ responses to situation 6 (interview 2)

All four students in this category claimed that the given argument showed the statement false. The data from the interviews of Patty and Paul indicated that the explicit semantics used in it was a deciding factor in their choice.
VB: Okay, and what do you say is the type of reasoning used here?
Patty: Um... I mean, part of it is substitution. You’re substituting $a^2 + b^2$ for $h$, and then, um, okay, let’s see here... There actually, the way they proved it is by contradiction, because they’re assuming that they don’t exist, which would mean that this doesn’t equal these, I’m saying it’s contradiction. That’s exactly what it says. It says that it leads to a contradiction [emphasis added]. So, making the assumption and working based on those assumptions and then coming to a conclusion which can’t be true, because it contradicts the facts that are given.

Despite this observation of her dependence on the word “contradiction” itself, the student was comfortable with the process of contradiction used in the argument, as the following excerpt would suggest.

VB: Okay. So what exactly is the contradiction at the end trying to contradict?
Patty: Well, it’s saying that there does not exist a $d^2$ such that $h^2 + c^2 = d^2$. But, um, that contradicts the fact that says there exists infinitely many $h$, $c$, and $d$ such that $h^2 + c^2 = d^2$ and because $d^2$ isn’t defined as any particular length, it’s just saying that it is $h^2 + c^2$. And we know that there has to be a $d^2$ such that $h^2 + c^2 = d^2$.

The former observation in Patty’s case could be explained in light of her own approach to the same situation during her first interview that was described earlier. In her first interview, she used substitution to find a procedure that showed the existence of numbers satisfying the given relationship, but she did not realize that her argument there used indirect processes. Thus, when she found similarities between the two arguments she thought it ought to be a substitution rather than contradiction. Nevertheless, the word “contradiction” in the given argument captured her attention and she argued that it used the method of contradiction and her own argument used some other approach.

VB: Well, last time, um, you said that the... you gave me some argument here and said that this statement is false. [...] Now, is there a difference between your argument...? Or explain to me, um, in terms of this proof, what your argument is doing there.
Patty: Um, they [in the given proof] assumed, they assumed that it was false—or assumed that the statement was true, that this didn’t exist. And then got to a point where that contradicted with the prerequisite facts. [Whereas] I looked at it and said, hmm, I think that there are distinct primes [positive integers], and so I
assumed, I assumed it was false. And um, reached a conclusion such that my assumption was true and therefore the statement was false, just by... But our, the steps of our argument are similar? [...] I mean, there’s some similarity. I mean, they’re letting $h^2 = a^2 + b^2$, and I let, $h^2 = f^2 + g^2$.

VB: I mean, other than algebraic manipulation, is the reasoning the same at all?

Patty: Uh-huh... I mean, our basic assumption at the beginning of the proof was, or at least stated assumption is different, and you could say assume that it doesn’t work even though you realize it’s probably not the case. But, I mean, it’s gone about in two different ways [emphasis added]. Um, they went through assuming it was true and then came to a contradiction. I went through assuming it was false and showed that, um, that the true statement was actually opposite of the statement. So, I don’t know that this would be the same method or not—the same reasoning.

The processes used in both arguments were similar. They both used indirect reasoning in terms of searching counterexamples or eliminating contradictions. However, Patty did not confirm the similarities. While working on the manipulation in her first interview, she was looking for a counterexample, and the given argument used contradiction to show how counterexamples could be found without actually looking for one.

It should be noted that throughout the interviews, Patty consistently showed similar behaviors. In particular, in situation 1 she used a two-case process (one case was redundant) that could eventually be considered equivalent to the contrapositive process, and yet she did not explicitly realize the similarities. In addition, in situation 3 while describing correctly the process used by the given proof, she did not explicitly connect it to contradiction. In other words, she was able to interpret the processes correctly without necessarily realizing the connection between a process and the learned method associated with it. It was concluded then that the relational understanding of indirect processes was not completely internalized. She was capable of making indirect arguments whenever
needed, but at times, she did not see the connection of her argument to the method she learned or put a label on her approach.

It was thus observed that, although Patty was able to find a valid (indirect) argument during her first interview, she lacked the overall perspective of the process. She did not rise above the specifics of her argument to see a larger picture of the method used. In terms of finding a counterexample to a statement, this phenomenon indicated that students sometimes unknowingly use indirect arguments in order to eliminate contradictions. Sometimes they do so without being aware of the relationship between their own process and indirect methods learned in mathematics courses.

Paul’s explanation indicated that he based his choice on the explicitly mentioned word “contradiction.”

VB: Well, can you explain the selection process you used?
Paul: Um, it shows... I, um, I read the proof and I thought that, that it led to a contradiction of this \[a^2 + b^2 \neq d^2 - c^2\] statement. [...] So then, I mean this was the only choice I had. I mean, since I thought, since I thought it was a contradiction of the statement then I just, I didn’t really spend much time considering these [other choices]. [Emphases added].

VB: The contradiction contradicts what exactly?
Paul: Um, this [there do not exist positive real numbers \(a, b, c,\) and \(d\) such that: \(a^2 + b^2 + c^2 = d^2\) or \(a^2 + b^2 = d^2 - c^2\)] statement would contradict the Pythagoras’ Theorem [fact 1, given]. Or vice-versa, I guess. [...] Or the Pythagoras’ Theorem would contradict that statement.

VB: What about fact number 2? Is that contradicted at all?
Paul: Um, (pause). I don’t see why that would have been contradicted. No, I don’t think that’s been contradicted.

Unlike Patty, his perception of the source of contradiction was incorrect because fact 2 was the source of contradiction in the given argument and not fact 1 as he described.

Thus evidently, Paul’s answer was primarily influenced by the word “contradiction” at
the end of the argument, because he did not exhibit a thorough understanding of the process.

It was concluded that Paul could sometimes have trouble interpreting the contradiction process correctly even when its semantics were made explicit. Furthermore, there was no indication that Paul saw the purpose of the contradiction as showing how counterexamples could be found.

On the other hand, there was no indication in the data from the other two (Pam and Perry) interviews that their choices were influenced by the word “contradiction.” However, they both seemed to have a clear understanding of how the argument used the contradiction process to show the statement false. The following excerpt indicated Pam’s understanding of its process for showing the statement false. Nevertheless, there were not enough data to conclude whether she had in fact understood it to be showing a process of finding counterexamples.

VB: So, what is the role of the assumption? [And] what’s the consequence of the assumption?
Pam: Oh, right here. They assume that it’s not equal... that $a$ plus $b$ does not equal $d$, the squares that does not equal. But, when you run through it, at least when I ran through it, if you use the assumption that this $[a^2 + b^2 = d^2 - c^2]$ is not equal, you find out that $h$ can replace these $[a^2 + b^2]$ two. So then, you find that $h^2 \neq d^2 - c^2$. But using this upper triangular and that [second] fact, you find that, you end up getting $h$ does not equal, $h^2$ does not equal $h^2$. So obviously this $[a^2 + b^2 \neq d^2 - c^2]$ right here has to be an equal sign because you contradicted this [fact 2].

The following excerpt from Perry’s interview however, indicated that he considered the proof to be showing a process of finding counterexamples in general.

VB: And what is its [the assumption’s] consequence?
Perry: Um, it leads to a contradiction. Therefore our assumption, our assumption must be incorrect. [...] Because it uses this generalized case, which is kind of nice. Plus it’s a non-existence proof, which is basically—or, it’s an existence proof, which

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means I don't even need the general case if you could show me one
[counterexample], then I'd be happy. So...

It was interesting to note Perry's last words about the proof being an existence
proof in general, which did not find any particular counterexample. Thus, in retrospect to
this situation and in light of the students’ difficulties observed during the first interviews,
进一步 studies might be beneficial if they probed students’ beliefs about an existence
proof that do not actually find a much-desired counterexample, but only show its
existence in general. In other words, follow-up research could cover another subtle
concept of contradiction by investigating students’ understandings of the power of
refutation without necessarily finding a counterexample.

In summary, in the first interviews it was observed that students did not
necessarily know which direction to turn once their attempts for finding a
counterexample failed. However, as observed from Patty’s second interview, even if they
had chanced upon a successful approach they might have done so without necessarily
being aware of its (indirect) process or method.

Abstract-Math students’ responses to situation 6 (interview 2)

Three of the students (Amy, Art and Adam) in this category claimed that the
given argument showed the statement false. The fourth (Alice) said it neither proved nor
disproved the statement because she thought it was making an assumption that needed to
be proved in the first place. It should be noted that Alice also made a similar remark
about the proof’s assumption in situation 3.

Alice: [...] We’re trying to prove that there does not exist positive real numbers a, b, c, d
such that this \[a^2 + b^2 = d^2 - c^2\], which is really saying this \[a^2 + b^2 \neq d^2 - c^2\] right
here. We’re saying there doesn’t exist these numbers, and that’s the same thing as
saying this right here. But that’s what they assumed to begin with. So they, they
didn’t assume the opposite, they assumed what they were trying to prove [emphasis added]. And so, that doesn’t, you can’t prove anything like that.

Yeah, I would say that assuming what you’re trying to prove is you’re first mistake. And then, anything after that doesn’t really matter, because you assumed your goal to begin it.

VB: Okay, well, what is the type of approach this proof here is using?
Alice: They’re trying to do contradiction but they didn’t do it right, I guess.

The following explanation from her take-home write-up could sum up and clarify her point even better.

We are assuming $Q$ here. If we were to look for a contradiction, we must assume not $Q$ and then find a contradiction. But the original statement was that there doesn’t exist $a, b, c, d$ such that... So, we must assume there does exist $a, b, c, d$ such that... and then find a contradiction.

Further inquiry, about her understanding of the source of contradiction, found that she understood it well, thus eliminated extraneous interpretation of her misperception of contradiction.

VB: Well, here at the end it says, uh, therefore, uh, the step leads to a contradiction. So what contradiction is that?
Alice: Uh, right here [fact 2] that there exists infinitely many numbers such that this holds.

Alice thought if the argument were to show $P \Rightarrow Q$ by contradiction, it should have started assuming $\neg Q$ and then reach a contradiction. Evidently, she was trying to turn the given existence statement into an “if-then” statement before using contradiction. Moreover, since the given argument ended by implying an explicit contradiction, she tried to assess its validity based on her general perception of that method, which would use a well-known scripted template that she described. Since the semantics of the argument did not follow that script, she claimed it did not contribute to the truth of the given statement. Her “template-like” perception of the contradiction process did not fit
into the process used in the argument. This suggested that her perception of the contradiction process was rigid, and could only be used to show true statements and not to explore the truth or falsity of uncertain statements such as the one under consideration.

Although that would be an ideal situation for the use of proofs by contradiction as in most textbooks, the given argument employed the method of contradiction to explore and then show a procedure for finding counterexamples, thus refuting the statement. Alice’s interpretation of the argument did not take into account the contradiction’s power of refutation for exploring the truth of the given statement. This (template-like) observation suggested that Alice did not perceive the use of contradiction process for searching counterexamples, although in the first interview she thought it was a useful tool for finding counterexamples.

It was concluded that Alice’s understanding of the use of contradiction in this situation was limited to its scripted process. Variations of this understanding were also observed in the cases of Pam and Perry from the other category and Amy in this category. Although, unlike Alice, the template-like perception of contradiction did not stop them to see the exact process of the proof, the following excerpts indicated how inconvenient the assumption at the beginning of the proof was to them.

Pam: Oh, because there, the assumption, or the statement said there do not exist real positive numbers, or real numbers. [...] So you assume for all positive reals $a$, $b$, $c$, and $d$ that it does work [emphasis added]. Oh... I misread it.

Perry: [...] It starts off with a negation, which can get awkward [emphasis added], and it just, I really had to think about what’s it really doing here and how’s it really creating the triangle? But...
Amy explained that the argument showed the statement false because it showed the existence of a triangle with one side being the hypotenuse of another. Further questioning tried to probe her perception of the contradiction process used.

Amy: Uh, it just... it’s a proof by contradiction. *It starts off assuming that what you’re trying to prove is wrong* [emphasis added], or... oh, assuming that it’s right... (Mumbling) yeah, assuming this \[a^2 + b^2 \neq d^2 - c^2\] is true, they’re not equal. And then showing that that can’t be true because these \(h\)’s, it’s possible for them to be equal in this situation.

VB: So, what exactly the contradiction here is trying to contradict?

Amy: It contradicts this fact [2], that there are infinite, infinitely many real numbers, so that you can draw this triangle. You can draw, \(c\) and \(d\) could be anything. You can always draw a triangle with this \(h\)... So, but when they say that \(h\) cannot equal \(d^2 - c^2\) they’re saying that there’s some \(d\) and \(c\) that it doesn’t work for this \(h\), which is not true.

It should be noted that the emphasized words above indicated a (template-like) phenomenon similar to the one described above. Here again its influence on Amy made her believe for a moment that a proof by contradiction ought to show a statement true using scripted steps. However, unlike Alice, she was not attached to that script.

Like Amy, Art also claimed that the argument showed the statement false because it showed the existence of a triangle with one side being the hypotenuse of another. He also explained that fact 2 was contradicted.

Art: The contradiction is that from Figure 2 [Appendix H, Situation 6], we can see that \(c^2 + h^2 = d^2\). And it [fact 2] contradicts here where it says that \(h^2 + c^2\) does not equal \(d^2\).

The data from both students’ (Amy and Art) interpretations of the given argument indicated that they were not influenced by the explicit use of the word “contradiction.” However, only the data from Amy’s interview indicated that she understood its indirect process for generating counterexamples.
VB: Okay. So last time again, uh, you couldn’t tell me whether the statement was true or false. Now that you know more about this situation, would you have used this proof to show whether the statement was true or false?

Amy: Yeah, definitely, I would have used it to find a counterexample. I didn’t even think of all this triangle business

In the case of Adam, there was no indication in the earlier data that he could fully describe the process of contradiction in a proof if it was not explicit. In fact, during all his previous interviews, he never showed certitude for how indirect processes worked in those situations. In this situation, he claimed the method of contradiction was used to show the statement false, but the data was not clear if his answer was based on the explicit use of the word “contradiction.”

Adam: Um, so it assumes, um, the opposite, contradiction, and then it says, um, through deduction (pause).

VB: So, tell me again. What is the role of the assumption?

Adam: The assumption is to... to by contradiction. So it’s, uh, the statement $a + b, a^2 + b^2 = d^2 - c^2$, so then, it, uh, does the opposite of that and tries to prove, um, uh, that $a^2 + b^2$ does equal $d^2 - c^2$, or, um... And then through manipulation it shows that $h$ plus $c$ squared does equal $d$ squared.

VB: And it contradicts what?

Adam: It contradicts the original statement. You, you assume this [$a^2 + b^2 = d^2 - c^2$] to be false and then prove it to be true, which... And the proof does do that.

It was not clear how Adam interpreted the argument because all he indicated was the final result of the contradiction and not its source. He did not explain the process clearly and seemed to lack a thorough understanding of the workings of the method of contradiction. There were no further data to reveal how the explicit use of the word “contradiction” influenced his thoughts, or whether he understood the argument of showing a procedure for finding counterexamples in general.

In summary, it was observed that Art and Amy had no difficulty understanding the process of contradiction used in the given argument. It was concluded that Alice’s
template-like understanding of contradiction hindered her from finding the exact purpose of the given argument in this situation. It was also concluded that Adam lacked a clear and good perspective of the workings of a proof by contradiction, even when its semantics were made explicit.

**Discrete-Math students’ responses to situation 6 (interview 2)**

Three students (Dan, Dave and Dean) claimed that the given argument showed the statement false and Doug thought it showed the statement to be true.

Doug did not exhibit a clear understanding of how the steps of the argument showed a contradiction.

VB: Why do you think that the presented argument in this situation shows that the statement is true?
Doug: Well this, structured logic again, basic structured logic, that if we show $A$ is true and then... not $B$, then we can find a contradiction and we can deduce that $A$ implies $B$.

VB: So which one of these prerequisite facts was contradicted?
Doug: Well, it wasn’t the prerequisite facts that were contradicted. It was our assumption that was... We found a contradiction in the assumption $[a^2 + b^2 ≠ d^2 - c^2]$.

VB: And what was the consequence of that contradiction?
Doug: That, we can deduce that the statement is true; there does not exist positive real numbers.

There was no indication in the data that Doug understood how the process of contradiction was set up. He seemed to have a vague and disconnected idea of how the process worked. He thought the process contradicted the assumption made at the beginning of the proof. His lack of understanding of this particular process was also confirmed from the other interviews.

Variations of a template-like perception of contradiction was also observed in all the other three cases as well, where they initially thought the argument showed the
statement true because it used contradiction. The following excerpts showed how uneasy they felt at first when they tried to explain the process of contradiction.

Dan: [...] Oh, yeah, that doesn’t work. Blah, hold on. Yeah, that doesn’t work. Hold on. Whew, I just had a brain freeze, whew. So let’s get back to it, okay. So, I said it proves it false. This again is a proof by contradiction. So, we’re assuming that for all this, what we’re trying to prove is wrong, actually. So, we’re assuming the statement is wrong [emphasis added]. Wait, if we’re assuming that this thing is wrong and we come up with a contradiction then this thing should be right. [...] 

Also, Dave’s uneasiness was revealed from the following excerpt.

Dave: The relation, um, here, has to be true. This relation \[h^2 + c^2 \neq d^2\].

VB: Now is that the same as saying that the original statement is false?

Dave: Maybe I answered this incorrectly. There does not exist, such that... no...

Similarly, the next excerpt showed Dean’s confusion.

Dean: I didn’t follow this proof very well, but I guess. [...] No, I guess it made sense. It’s just not. I wouldn’t have gone about it this way. - - -

VB: It proves that the statement is true?

Dean: Yes, because if the statement were false, it would lead to a contradiction. So, the statement must be true.

- - -

VB: So, what is the proof here assuming?

Dean: Uh, initially we assume that the statement is false.

However, this misperception did not prevail once they realized that the argument showed the existence of quadruples.

Dave: I mean, if the relationship holds true, then there has to be numbers that you can plug in that satisfy it.

The next excerpt also showed Dean’s realization of the correct result.

Dean: Wait, there do not exist. Oh, no this proves the statement’s false. I misread it. I really meant [choice number] five that whole time (laughs).

Therefore, it was concluded that in a situation where an argument explicitly used the word “contradiction,” students were quick to think of a template-like script and tried
to reevaluate the validity of the argument based on that script. It was observed that if the argument did not fit into their particular template, students would start to feel uneasy and sometimes confused.

This template-like perception of contradiction was not observed in other situations because in those situations the word “contradiction” was not explicitly mentioned in the proofs, even when they used the contradiction process. In other words, the word “contradiction” at the end of the argument in situation 6 prompted the observed behaviors above. It created various degrees of discomfort for some students (Pam, Perry, Dan, Dean and Dave) and triggered misinterpretation for others (Alice).

**Situation 7: Interview 1**

For all real valued functions \(f, g \) and \(h\) on R (the set of real numbers),

\[ f(g(x)) = f(h(x)) \text{ then } g(x) = h(x), \text{ for all real values of } x. \]

The given statement in this situation is true only when \(f\) is a one-to-one function. Otherwise, it is false. A counterexample using any non one-to-one function \(f\) can be found. However, if the property of one-to-one functions built into the statement was not recognized, a trial and error process for testing the implication, when carried out systematically, might lead one to use indirect processes.

A search for a counterexample that starts by defining different functions for \(g\) and \(h\) is considered an indirect process, which tests the truth of the contrapositive statement. For instance, the proof given in the second interview provided a counterexample \(f(x) = x^2\) through the assumption that \(x = g(x) \neq h(x) = -x\), which indicated an indirect approach by finding a counterexample to the contrapositive of the given statement. In other words, the statement was proved false by feeding two different functions \(g(x) \neq h(x)\) into \(f(x)\) and
obtaining the same output. More on this process will be discussed in the introduction to the analysis of the second interviews.

The overall objective of this situation was to investigate whether students' thought process for searching for a counterexample would invoke indirect processes. Thus, the purposes of this interview were to investigate how students approach this problem and whether they find the statement false through a systematic search for a counterexample.

Post Abstract-Math students' responses to situation 7 (interview 1)

Two students (Patty and Paul) claimed that the statement was false. The other two argued that it was true. Both Patty and Paul found the same counterexample, $f(x) = x^2$, $g(x) = x$ and $h(x) = -x$, although Paul needed some encouragement in order to do so.

Patty's initial reaction was that the statement was true but upon further reflection on the problem, she found a counterexample.

Patty: [...] If it were for all $x$ (pause), um, I'm really sure that it's true. I don't have a proof for it yet, let me think about it for a minute (pause). I can't think of any two. Well, wait a second. What if, okay, $f(y) = y^2$, and $h(x) = x$ and $g(x) = -x$, [...] then $f(h(x))$ would be $x^2$ and $f(g(x))$ would also be $x^2$. I'm going to change my answer there, because I found a counterexample to say that's false.

The next excerpt indicated how Patty found that counterexample. Her words described what had taken place intuitively. She appeared to be using an indirect process.

VB: Can you explain to me the procedure, the approach you took in order to find this counterexample?

Patty: Um, intuitively it seemed to work fine. I mean if, if you get the same $f$ value then you'd think these $g$ and $h$ would be the same functions. But I thought well, I don't have a proof for that. So um, I tried to think of some things where you could... I wanted to look for something where you could put in, two different, um, values here [for $g$ and $h$] and come out with the same answer, or the same value [of $f$] here [emphasis added]. So I was thinking well, if you take the squared, that's something where you get to, from two separate inputs you get the same
answer. And so, I thought well, then, how can I get it so that I get the same answer with different inputs. And so, by doing the opposites, and then squaring them, then you end up with $f$ of the function equal...

It was interesting to see how she was able to verbalize her intuitive approach in this problem situation. The emphasized sentence above indicated that she had started assuming two different functions for $g$ and $h$, which in turn indicated an indirect approach (contradiction). Thus, it was observed that Patty's systematic search for a counterexample led her to use indirect arguments.

Paul first gave a general argument in terms of inverse functions and then tried to find a counterexample.

Paul: I think it's true... What's the rule? I think if there's an inverse of $f$ it's true. So, if this $f^{-1}$ exists, it's true. I'll say it's false. It's not true for all real value coordinates.

Yes. Well, for this statement I think all you would have to do is find a function that wasn't true. So it says for all real valued functions, so if I could find a real valued function that didn't have an inverse... Or, if I found one function that it didn't work for, and I think I'd look for functions without an inverse [emphasis added].

Evidently, Paul recognized the property of a function that made the statement true and that helped him look for a function (counterexample) that did not possess an inverse.

VB: Can you think of such functions that do not have an inverse?

Paul: An inverse? Yeah, I was just trying to think of that. Um (pause). Let's see. I'm pretty sure this $f(x) = x^2$ is, because its inverse would be multi-valued. [...] So if I took $x$, it would have to be $x^2$, and, let's see... I'm not sure how to do it, but it could be done like that, I think, because you could take...'cause either the positive or negative root would be okay for this. So you could find it to work and this could be equal to $-x$ or something. I think it would work, and this $f(g(x)) = f(h(x))$ would be true, but this $g(x) = h(x)$ wouldn't be true.

[...] This one, yeah. It's not really just guessing, 'cause I used the inverse to find it, but I'm sure there could be a better one [argument].
It was observed that Paul’s success was based on his ability to remember the inverse property of functions. Next, the interviewer probed his approach for finding a counterexample.

VB: See, I’m just trying to find out how... what brought you to the conclusion that you need an $f$ which does not have an inverse?

Paul: Oh I see. Uh, like, I needed to, I wanted to apply some function to both sides of the equation $[f(g(x)) = f(h(x))]$ to, uh, so that, um... I guess I was looking for something of the form, and I know that this would just be the inverse of $f$.

No, I just remembered that if you want to get rid of... if you essentially want to, you don’t... Like, I remember in algebra you can do a similar operation to both sides of the equation. [...] If I wasn’t interested in this $f$, I could do an operation to get out what the argument of $f$ was, and that would be the inverse of $f$, I guess.

Paul remembered that if he composed the inverse of a function with the function itself then he would get the identity function or in this case the argument of the function. This approach could be interpreted as direct, because he was looking for an $f$ that did not possess an inverse rather than looking specifically for different functions for $g$ and $h$. He was testing the implication directly based on his knowledge of manipulating functions, starting from the ‘if’ part of the statement. This conclusion was also confirmed from the following excerpt.

VB: So, you initially started thinking of the function $f$ and not $g$ and $h$, is that correct?

Paul: Yeah, I was more worried about $f$, that’s right.

Similar to Paul’s approach, Perry tried to remember how the inverse property of functions could be applied in this situation.

Perry: I was thinking about $f$ inverse, but then I realized there are real valued functions $f$ for which $f$ does not have an inverse function. So, that’s what I was hesitating on, because I was about to go with, um... Anyway, I was just thinking if $f$ has an inverse $f$ negative one $[f^{-1}]$, we’ll define that, we’ll compose the functions, we’ll get $g(x) = h(x)$. But that doesn’t hold, because $f$ can be a real valued function which doesn’t have an inverse.
From his words, it seemed for a moment that Perry was trying to disprove the statement since he realized that not all functions have inverses. However as the interview progressed, it became clear that he was trying to show the statement true, because he thought every function must have an inverse function.

VB: So, again, um, how, how did you reach the conclusion that the statement is true? 
Perry: Basically I was thinking that if \( f \) has an inverse. Let me show you, so we’ll assume \( f \) has an inverse. Then we take \( f^{-1} \) of \( fg \), a bunch of parentheses, equals \( f \) negative one \( f \) \( h \) \( x \) \( [f^{-1}(f(g(x))) = f^{-1}(f(h(x)))] \). We can cancel these \( f^{-1}(f) \) out, you get \( g(x) = h(x) \). And so that one I think, so now I’m wondering if \( f \)’s that don’t have an inverse if it works or not. But I’d imagine it would, it’s just... I’d need a different way of proving it [emphasis added].

That’s what I’m wondering. Um... well (talking to self). Actually every function \( f \) would have an inverse, it’s just sometimes we don’t know how to find that. But, I’m trying to, I’m just wondering if that’s a true statement or not. If \( f \) is a function with real values there’s some sort of manipulation [emphasis added] that makes it equal to one [identity]. Yeah, so, it is true, it’s just we can’t always find it.

I mean, there, this way doesn’t do it, but I still stand by the statement being true. Besides, I’d love to see the counterexample.

[...] Every function has an inverse, slash if it doesn’t have an inverse it still holds at the point, and we can show that it holds at the point. [...] 

Perry’s argument above disregarded the role of the domain on which inverse functions needed to be defined. In this situation, his focus was to remember whether every function had an inverse or not and he did not rethink the role of the domain in his claims. It did not occur to him to try particular cases to check his claims. It should be noted that throughout Perry’s interviews, it became apparent that when dealing with proofs he would always think in general terms and not attempt to try examples, unless he outright believed the statement was false.

Nonetheless, Perry’s initial manipulative approach to the functions starting from the “if” part of the statement indicated that he was trying to use direct methods. He failed
to reach the correct conclusion because he believed that every function could somehow be shown to have an inverse function.

Pam, on the other hand, started with what looked like an indirect approach for finding a counterexample, because she was looking for a case where \( g(x) \neq h(x) \).

VB: And what is your objective?
Pam: It's to see if you can find an \( h(x) \) that's not equal to \( g(x) \) that will work. So, this has to equal \( x \) squared plus two, \( f(x) \) of \( h(x) \) \([f(h(x)) = x^2 + 2]\). The only way it could work is if you have... I think this will work if \( g(x) = h(x) \). Um, and this is just an example, and I proved to myself why I believed this.

However, as observed above, she failed to carry out that argument because when she tried one example with \( h(x) = x^2 + 1 \) and \( f(x) = x + 1 \), not only did she pick a one-to-one function for \( f \), but she also assumed \( h(x) = g(x) \) was true, which was what was to be shown. Further, she was convinced that the statement was true in general, because she could not argue otherwise. Her argument was based on treating every function as a one-to-one function.

Pam: [...] You can't have two different things put through the same function and have the same thing come out on the other end. Does that make sense? Like I can't put...

Those words indicated a misconception that the student had about functions. Without being aware, she literally spelled out the definition of a one-to-one function. This phenomenon could be explained by her lack of identifying properties of functions, even when she was aware of the subtle property that functions sometimes possess, as this next excerpt suggested.

VB: Is that true for every function?
Pam: Uh, no I guess it depends on... Um, (pause). Can it be true for every function? I guess it depends on the \( f(x) \), the um... This \([f(x) = x + 1]\) function, if whether or not this \([g(x) = h(x)]\) would be true or not. Uh...
Abstract-Math students’ responses to situation 7 (interview 1)

Three (Amy, Alice and Art) of the four students in this category found counterexamples and claimed that the statement was false. The fourth student (Adam) contemplated both true and false outcomes and believed that the statement would be true.

Amy initially thought that the statement was true and drew a graph (having a parabolic shape similar to the graph of $x^2$) as she was looking for a case where the statement would be true. Then, as her words indicated in the following excerpt, she tried to process the contrapositive of the given statement and tried to relate the problem to the graph (parabola).

Amy: Um, I guess I'm sort of picturing a graph in my head as an example, because okay, says $f$ is a function for every $x$ in here—this is $x$ now.

Well since there's only one value of $f(x)$ for every $x$ then there can't be another $x$. If $g(x)$, and $h(x)$, are different then $f(x)$, they would have to be different because you can't get the same—wait that's not true... Yeah okay, for any one value of $x$ there's only one corresponding $f(x)$.

VB: So, are you thinking of this curve to be $f$?

Amy: Yeah. [...] Oh but what if you had... okay that's not true, what if $h(x)$ was here then $f(x)$, that's equal... Yeah, okay, they don't have to be equal, $f(g(x))$ and $f(h(x))$... Okay, now I think it's false now.

 [...] I could find a counterexample to prove that it's false.

In the end, Amy wrote the following counterexample: $f(x) = x^2$, $g(x) = 1$ and $h(x) = -1$.

Amy: Yeah well like this one. Let's say $f(x)$ is $x^2$ and you can say that this is one. So, here's 1 and here's -1, they're both points on $f(x)$. So, in this case here $f(1) = 1$ and here $f(-1) = 1$. So, then $g(x)$ can be equal to 1 and $h(x)$ could be equal to -1 but they still yield the same $f(x)$.

When asked about her approach in general, she replied that it was a contradiction.

Amy: That would be—you would prove a contradiction, because [...] if you show that this is, this is not equal to that [$g(x) \neq h(x)$] but this is [$f(g(x)) = f(h(x))$] true then that's a contradiction.
It was observed that Amy's success in finding a counterexample was based on her graphical visualization and her use of indirect (contrapositive) analysis of the situation. It was also observed that she was aware that her own approach employed contradiction (indirect). It should be noted here that Amy was the only student among the twelve participants who used a graph to visualize the mathematical situation.

Similarly, Art found the counterexample \( f(x) = x^2 \), \( g(x) = 2 \) and \( h(x) = -2 \) and claimed that the statement was false.

Art: Okay, um, I just chose, uh, a function \( f(x) = x^2 \), and then I let \( g(x) = 2 \) and \( h(x) = -2 \). And then \( f(g(x)) = 4 \) and \( f(h(x)) \) is also equals 4, but \( g(x) \) does not equal \( h(x) \). So, I just found a counterexample to the statement... proving it false.

Before he found this counterexample, there was a long break where Art was silently thinking. There was not enough data to determine how his train of thought led him to find a counterexample, other than Art pointing out that the true hypothesis of the statement led to a false conclusion in his approach.

Art: Uh, my counterexample is, we have \( g(x) = 2 \) and \( h(x) = -2 \), and it contradicts the \( g(x) = h(x) \), and they don't equal each other, because 2 doesn't equal minus -2.

In his second interview on this situation, the interviewer brought up this issue again. After Art examined the proof used in the second interview, he was asked to compare it with his own and as in the first interview, his words did not elaborate much on his approach.

VB: So, what's the difference between these two arguments?
Art: Um, it's not, there's not... not much actually. They both give a counterexample and this one is more arbitrary, and I just chose a number for this and they both show that the statement is false.

Thus, there was no indication in the data that Art used indirect arguments in finding his counterexample.
On the other hand, Alice first claimed that the statement was false and then found the counterexample $f(y) = y^2$, $g(x) = x$ and $h(x) = -x$.

Alice: Okay. Yeah. I would. Here again, I would do a counterexample, because let’s say the $f$ function was the squared function, so, I don’t know what I want to call it, $y^2$. And, let’s see, just a second here, $g(x)$ is... (talking to self, inaudible). Here, I guess I’m just trying to find a counterexample. Like assume that your $g(x)$ function was this $[g(x) = x]$ and your $h(x)$ function was this $[h(x) = -x]$.

VB: Okay.

Alice: And if you put them in there $[f(y)]$, you’re going to get [...] the same answer, but these weren’t the same function to begin with [emphasis added].

The words emphasized above indicated that Alice was approaching the problem indirectly. This was also confirmed when she considered her approach to be contradiction.

VB: Why, and what aspects make it a proof?

Alice: Because I, it’s a contradiction, I guess. A counterexample contradiction. And the contradiction is a valid reason for proving something false, or true, I guess.

Adam first claimed that the statement was true and tried to confirm his claim by looking at the example $f(x) = x^2$, $g(x) = x^3$.

Adam: [...] My first assumption would be yes, it would be true because let’s say you have, um, functions like $f$ to be $x^2$ and $g$ to be $x^3$ and $h$, well then, $f(g(x))$ would be $x$ cubed squared. Since they’re simple functions...

 [...] I was using an example to maybe show that the statement is true.

Upon further reflection on the problem, Adam thought that the statement might be false and tried to find a counterexample but to no avail.

VB: Are you absolutely sure that the statement is true?

Adam: No, that was my first assumption. Um... the more I think about it, the more it could definitely be false, because all I need to do is find a counterexample to say that. Define the fact that $f(g(x)) = f(h(x))$, but then $g(x)$ does not equal $h(x)$. But I cannot think of a counterexample to...
In the end however, his failure to find a counterexample made him to accept his initial instinct.

Adam: I don’t know. I guess my first assumption is true and so I will stick with that.

Despite the interviewer’s interventions to make him consider different avenues, such as considering different properties or classes of functions, Adam was unsuccessful. There was no indication in the data that he had approached the problem indirectly.

Although he studied the example of \( f(x) = x^2 \), he was unable to capitalize on it because he did not use indirect arguments like his peers. He did not try a graphical approach either.

**Discrete-Math students’ responses to situation 7 (interview 1)**

Two students (Dean and Dan) claimed that the statement was false and the other two claimed it was true. However, only Dean found a counterexample using \( f(x) = 0 \).

Dean: So here \([f(g(x))]\) I’d get zero and here \([f(h(x))]\) I get zero, but then \(g(x)\) and \(h(x)\) don’t necessarily, could be anything. \(g(x)\) could be, you know, \(g(x)\) equals \(x\) times \(5\) and \(h(x)\) equals \(x\) times \(3\). So, in that case, I think it’s false.

Although the data was not explicit about how Dean thought of this example, he was not asked direct questions about his method or approach, or whether he thought a one-to-one function \(f\) was needed to disprove the statement. However, this subject was brought up again during his second interview (see the analysis of his second interview).

On the other hand, Dan’s initial reaction was to use contradiction to disprove the statement.

Dan: […] Again, how I’d go about trying to do this is try to find a \(g(x)\) and \(h(x)\) that differ, but also satisfies this \([f(g(x)) = f(h(x))]\), is what I’d first try to do.

VB: And what do you call that approach?

Dan: Uh, contradiction, I guess. Because if you could find one where a \(g(x)\) doesn’t equal \(h(x)\), but it \([f(g(x)) = f(h(x))]\) still holds true that, whatever \(f\) you’re using, of \(g(x)\) equals \(f(h(x))\), because that’d be, like, a contradiction. […]
VB: And the process leads you to what in this situation? What, what does that say to you about the situation?

Dan: Right. If I could find a contradiction, that would mean that this is false.

Evidently, the process of contradiction he was referring to would lead him to a case of counterexample. However, he could not finalize the process by actually finding a counterexample despite some encouragement from the interviewer. All he could conclude was that $f$ needed to have a certain property, which seemed as if he was looking for a non one-to-one function $f$, as his words in this next excerpt indicated.

Dan: Let me think (pause). So, $f$ of $f(x)$ has to be some kind of function that would make the other two $[f(g(x))$ and $f(h(x))]$ the same.

Although he narrowed down the class of functions that he needed to look into, he did not realize that the common property was that of non one-to-one functions. In fact, he was so overwhelmed by the infinite number of functions that he felt it would be an impossible task to find one with the sought property.

Dan: So it really doesn’t help to think of like specific functions then. So there has to be some way of thinking of functions, that... Like, again, thinking of like the commonalties of functions. I don’t know.

Thus, it was not clear what actually motivated Dan to seek a counterexample, or what made him conclude that the statement was false.

Dave and Doug gave similar explanations as to why they thought the statement would be true.

Dave: Well, I'm trying to recall my properties of functions, um, there’s got to be some property that would show this is true. But it’s been a while since I took Calculus I. [...] In order for the output values to be equal, the input values would have to be equal to each other. [...] 

Similarly, Doug gave the following argument.
Doug: I mean, just really simply, I mean, if you have a function that has a certain output, and if that function equals another function with another input but they still equal the same output, then the inputs would have to be the same in the functions.

Both Dave and Doug seemed to harbor a misconception similar to that of Pam's, because they treated every function as a one-to-one function. Doug in particular recalled the definition of a one-to-one function and argued that the statement looked similar to that definition.

Doug: I think it has to do with one-to-one functions. [...] Well, it almost restates the same thing there if you have an $x_1$... Let $x_1$ and $x_2$ be elements of the real numbers and have some function that's defined on the real numbers and then if, if $f(x_1) = f(x_2)$ then $x_1 = x_2$. Something like that.

It was clear that Doug, through his recollection of one-to-one functions, was claiming that the statement was true. In fact, after reading the statement and comparing it with his definition of one-to-one functions, he concluded that $f$ must be one-to-one and that the statement was true. However, he did not indicate that the one-to-one condition made it true. He never indicated that the statement would be false for non one-to-one functions, even when he was asked directly.

VB: So, what you're saying, therefore is, if this statement is the case, then $f$ is one-to-one. [...] And, see, what I'm trying to get at here is that, you're assuming that the statement says $f$ is one-to-one, and according to your assumption your saying the statement is true.
Doug: Yes, yeah.
VB: Okay, but, do you have to make that assumption?
Doug: No, you don't. That it's one-to-one? Even before you start? No. You find out because of the property.
VB: So, do you still think this statement is true for non-one-to-one functions as well?
Doug: I still hold to that same thing because what it's saying is that for all real valued functions, if some input to $f$ is equal to, if $f$ is equal to $f$, then (pause). Yeah, I'd still say that's true, I'd still say that's true, I just can't...

Thus, Doug did not analyze the situation correctly because he failed to consider the case where the property of a function did not hold.

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On the other hand, as the interview with Dave progressed, he showed some doubts about his answer. Although for unexplained reasons he could not convince himself of the truth of the direct statement, as he tried to look at the converse and the contrapositive of the statement, he became convinced that the statement would be true.

Dave: [...] Because since we have to have the same function here in order for the composition of \( f(g(x)) \) and \( f(h(x)) \) to be equal. Um... I mean I know that if, if \( g(x) \) and \( h(x) \) were equal then this \( |f(g(x)) - f(h(x))| \) would be true. I know that. You know, what is that called the converse? [...] But, it doesn't seem like to me, that you could find two functions that have different input values but the same output values [emphasis added].

 [...] I mean, I've convinced myself of that, that if these \( g(x) = h(x) \) weren't true, then I don't think that it's possible that this \( f(g(x)) = f(h(x)) \) be true. [...] It seems like it has to be true, because when I'm thinking the other way [contrapositive] I'm not convinced for some reason.

Again, as with Pam and Doug, Dave did not exhibit a full understanding of the concept of one-to-one functions to grasp the consequences of their claims.

An interesting observation was that of all the 12 students only Amy used a graph (a parabola) of \( f(x) \) in order to associate its non one-to-one property with the situation. A graphical approach to this problem was conspicuously absent from all other attempts at finding a counterexample.

Situation 7: Interview 2

In this second round of interviews, students were given a proof of the true/false statement seen earlier in the first interviews. The given proof used the semantics of a proof by contraposition, and showed, by finding a counterexample, that the contrapositive statement \([if g(x) \neq h(x) then f(g(x)) \neq f(h(x))\] was false.

The argumentation used in the proof made the indirect thought process of searching for a counterexample explicit for the following reasons. The process of finding
a counterexample in this situation (see the analysis of Patty's first interview) had an inconspicuous phase which was made explicit in the given proof. A counterexample could be found when a true premise of the conditional statement implied the opposite of its conclusion. Thus, while searching for a counterexample one may look for different functions for $g$ and $h$ first, and then composes them with various $f$'s (a non one-to-one was required in this case) to see if $f(g(x)) = f(h(x))$ or not. The very nature of trying different $g$ and $h$ or having $g(x) = x$ and $h(x) = -x$ automatically presumes an indirect (contrapositive) approach which normally is taken for granted, otherwise the whole argument would be a circular one. This indirect approach was made explicit in the given proof.

Thus, one of the objectives of this interview was to investigate students' perceptions of the validity of the indirect process involving a combination of the two concepts described above (contrapositive and counterexample). Another objective was to investigate how well students related their own approaches to that process, if they had successfully found a counterexample in their first interviews.

Post Abstract-Math students' responses to situation 7 (interview 2)

All four students in this category claimed that the given argument proved the statement false. Three students (Patty, Perry, and Pam) gave similar explanations for how the steps of the proof indicated the contrapositive approach that ended up showing a counterexample. However, three of the students (Pam, Perry, and Paul) claimed that the contrapositive approach was unnecessarily redundant, which indicated that they did not appreciate or could not validate the indirect approach taken in the given proof.
In the following excerpt, Patty explained her understanding of the process of the proof.

VB: So, why is it [the assumption] followed by the statement that says we need to show that, you know, for every function \( f \) we have \( f(g(x)) \neq f(h(x)) \)?

Patty: I might be wrong again, but I think this is the contrapositive. You’re saying that, you know, because this is \( P \) then \( Q \), so you’re saying if, if this statement is true, then not \( Q \) will imply not \( P \). So they’re starting with not \( Q \) and then we need to show that not \( P \) is true. But you can’t show that because the statement isn’t true, so that...

VB: So, what happened after that?

Patty: Well, then they see a case where not \( Q \) is true, but \( P \) is also true. So, we’ve shown that not \( Q \) implied \( P \).

Patty in her first interview used an indirect method, similar to the one given here, to find a counterexample and she had no difficulty explaining the process used in the given proof. Furthermore, the data indicated that she regarded the use of contraposition as a necessary step in reaching the conclusion.

Pam gave a similar explanation for the process of the proof but she indicated that the contrapositive approach was redundant.

VB: Why the assumption is followed by that need to show...?

Pam: Contrapositive. I would say this works. They’re trying to, like, if this is our \( P \) up here and this is our \( Q \), they’re showing not \( Q \) so then not \( P \), and if that followed through then they would prove the above statement. However, with assuming not \( Q \) you can run it through—I used their thing and just said let \( f(x) \) just be \( x^2 \), and then let \( g(x) = x \) and \( h(x) = -x \). So, \( g(x) \neq h(x) \). But after you run both of them through \( f(x) \), they come out being equal to each other, which does not show not \( P \). So with us, we had not \( Q \), but it showed \( P \), which was not the goal with the contrapositive thing. [...] They show a counterexample for the contrapositive.

VB: And is that a correct argument?

Pam: Um, I’m not sure but you could... I’m not sure that would be a correct argument, but with the counterexample, I would probably try to use this as a counterexample for the statement, just as is [emphasis added]. Like, um, instead of assuming these things \( [g(x) \neq h(x)] \), I would say, well, if this \( [f(g(x)) = f(h(x))] \) is true, then let \( f(x) \) be \( x^2 \), and \( g(x) [be] x \), and then just show that you can have different \( h \) and \( g \) of \( x \)’s that are not equal, with a counterexample.
For reasons explained earlier in the introduction, this argument would be circular. In other words, letting \( g(x) = x \) and \( h(x) = -x \) and using them as a counterexample to show directly that if \( f(g(x)) = f(h(x)) \) then \( g(x) \neq h(x) \) is a circular argument. Pam understood that the given proof was using the contrapositive approach but disagreed on the necessity of that step because, as she explained, she would just use the counterexample directly for the statement.

Similarly, Perry explained that the proof found a counterexample while trying to show the contrapositive statement.

Perry: It seems like it’s going to be a proof of contrapositive. Because it’s taking our conclusion and it’s negating it.

Um, actually, it’s a counterexample to… See, it doesn’t prove this [given statement], because, it doesn’t properly prove that. But what it does do is it shows, is it gives us an example where this original statement is false.

VB: […] So, do you think those first two lines are redundant in this proof?
Perry: Um, actually they are because if anything, it doesn’t prove the contrapositive, but it does give this as a counterexample, which happens to contradict this [given statement].

But the way I tied it in, I didn’t properly tie this [conclusion of the proof] back to this [assumption of the proof], I tied this [counterexample] back to this [original statement].

Similar to Pam’s remarks, Perry also considered the contrapositive process of the proof redundant, because he saw the contraposition and counterexample as different strategies in this proof. He did not see them tied together for the same goal. It should be noted that Perry made a similar remark in his second interview on situation 2. There he claimed that the process of finding a contradiction and that of finding a counterexample were separate.
approaches for different goals, the former to show a statement true, and the latter to show it false.

Paul on the other hand, was not sure why or whether the contrapositive approach was used.

Paul: There... Is it the contrapositive? I think it's something like that. They're assuming the reverse of the statement is true. I'm having a problem remembering this stuff. Um, I haven't done this stuff in over a year. Yeah, I think it's a valid approach, but I think their conclusion... Let's see. [...]  

[...] I think that this is still correct, I think that statement is still false. And I think that the proof is okay, but I think it's in a format that I'm just not familiar with [emphasis added], like... So they're saying that...

- - -

VB: And why are they doing that?
Paul: Uh, I guess they thought it was the easier way to do it. I would have done the reverse of that [emphasis added]. [...]  

In his first interview, Paul used a direct approach indicating that the statement would be true only when \( f \) had an inverse. In this case, the indirect approach used in the proof did not make sense to him, although he passively agreed on the proof's validity because it found a counterexample.

It was concluded that three of the students (Perry, Pam and Paul) were able to follow the process of the given proof but were unable to see it as a whole entity for reaching a conclusion. They did not validate the use of contraposition as part of the indirect process for finding the counterexample. Patty was the only student in this category who found the use of contraposition necessary in the given proof.

Abstract-Math students' responses to situation 7 (interview 2)

All four students in this category claimed that the given argument showed the statement false.
Amy explained that the argument showed a counterexample to the contrapositive of the given statement.

Amy: Okay, this proof proves the counterexample, I mean the contrapositive, by finding a counterexample. So, it has to be that the assumption is wrong.

VB: Okay, so can you explain the role of the assumption in this proof?

Amy: Uh, the assumption is $g(x)$ not equal to $h(x)$, is what we want to contradict... or, it's a fact that we use to prove that it's not necessarily true that $f(g(x))$ is equal to, is not equal to $f(h(x))$.

In her first interview, Amy used similar indirect arguments to reach the same conclusion, and when she was asked to compare the two approaches, her explanation confirmed the earlier observation from her first interview. That is, she found it necessary to use an indirect approach to the situation.

VB: [...] Your argument [there] and the proof here, are they doing the same thing?

Amy: (Pause, mumbling) yeah, it's the same thing. I just used specific values for $x$, but still it was finding a counterexample.

VB: So you were trying to find a counterexample the same way the proof is trying to find a counterexample here?

Amy: Right. Here they used... we used the same $f(x), x^2$, but here instead, I used $g(x)$ as $1$. Or, yeah, and $h(x)$ as -1 and showed that $f(x)$ was the same for both of those.

Similarly, Art explained that the proof found a counterexample to the contrapositive of the given statement.

VB: Okay. So why is it [the assumption] followed by the need to show that $f(g(x))$ does not equal $f(h(x))$?

Art: Oh, okay. So, they need to show that $f(g(x))$ does not equal $f(h(x))$, they're going to go by contrapositive, which will flip those around and then negate them.

VB: So, what is attained in the end?

Art: [...] It gives us a counterexample. It implies that the contrapositive is false.

In the discussion of Art's first interview, it was mentioned that he did not elaborate on the issue of comparing his approach to that used in the proof when it was
brought up. Thus, no conclusion could be drawn as to whether Art found the indirect step necessary in the given argument.

Similarly, Adam explained that the proof contradicted the contrapositive of the given statement by showing a counterexample.

VB: Then what is attained in the end?
Adam: Oh, actually contradiction. Sorry, it's, it's the contrapositive. We took the contrapositive of the statement and then proved the contradiction of that.

Despite his failure to arrive at a correct conclusion in the first interview, Adam was able to identify the process used in the proof.

On the other hand, Alice explained that the proof used an approach similar to using the method of contradiction to find a counterexample to the given statement.

VB: Okay. So, can you explain the role of the assumption in this proof? [...] Why is it followed by the need to show that \( f(g(x)) \neq f(h(x)) \).

Alice: (Pause). We needed to assume that they weren't equal to each other, because here they said they would be equal to each other if this [hypothesis] held. But then, it's kind of a contradiction, we assumed that they \([g(x) \text{ and } h(x)]\) weren't equal and we still had this \([f(g(x)) = f(h(x))]\) hold, so that was our contradiction.

 [...] This \([f(g(x)) \neq f(h(x))]\) right here is just setting up our contradiction.

VB: Okay. And how is the contradiction reached here?
Alice: We prove that, we prove that even though \(g(x)\) and \(h(x)\) weren't the same thing we still got this \([f(g(x)) = f(h(x))])\) statement to hold, right here. So I guess, it is more of a counterexample [emphasis added], I guess.

VB: Is it a direct counterexample?
Alice: Well, not really, because you're assuming the opposite of the goal, I guess. So, yeah, it's more of a contradiction [emphasis added], I guess.

Alice was trying to explain that the role of the assumption was to set up a premise that would become the object of the contradiction. Moreover, after Alice examined the assumption in the proof she claimed it used contradiction, but when she examined the conclusion in the proof she claimed it found a counterexample. Her words did not convey
clearly whether the statement in the proof that required the need to show \( f(g(x)) \neq f(h(x)) \) seemed to her a necessary part of the contrapositive process of the proof. However, her perception of the process of the proof as contradiction could be viewed in light of her first interview on this situation. There, it was observed that Alice used an indirect process to find a counterexample and here she exhibited a perception of the connection between the two processes, finding a counterexample and a contradiction. Thus, when asked to compare both approaches she confirmed that her approach was the same as in the proof for finding a counterexample.

VB: Okay. So, in your argument last time you gave, um, this...
Alice: (Interjecting) which is the same thing. I just found a counterexample there too.

**Discrete-Math students’ responses to situation 7 (interview 2)**

Three students (Dean, Dan, and Dave) claimed that the proof found a counterexample to the statement. Doug claimed that the proof showed the statement true.

Dean explained that the proof started assuming that the conclusion \( g(x) = h(x) \) was false and as a result it tried to yield \( f(g(x)) \neq f(h(x)) \).

VB: So now why is it followed by the need to show that for every function \( f \) we must have \( f(g(x)) \neq f(h(x)) \)?
Dean: Well, I mean, we're trying to show that the, in order for this statement to be true if these \([g(x) \text{ and } h(x)]\) two are not equal, then these \([f(g(x)) \text{ and } f(h(x))]\) two must not yield equal solutions.

However, he was unsure how to explain the validity of his argument, because he was unable to put the contrapositive label on the process, although he thought finding a counterexample was a valid result.

VB: Is that a valid way of approaching...?
Dean: Yes.
VB: Why?
Dean: Because, um, I'm not sure, just, it seems valid.
Like, $g(x)$ and $h(x)$ are not the same function, but given their, we feed them into $f$ and we get the same answer. So this $[f(g(x)) = f(h(x))]$ holds true, but then this $[g(x) = h(x)]$ doesn't. [...] We can feed this function $[f]$ two unequal functions.

Furthermore, Dean also claimed that the proof gave an argument similar to his own in the first interview.

VB: Well, um, in your previous interview about the same situation you said that the statement is false because, um, you gave an example there... or counterexample.
Dean: Uh-huh, and I think that's the same kind of thing this [proof] did, because this just gives a different counterexample.

When asked about how the process of the proof yielded the counterexample, he reiterated his previous words without recognizing that an indirect process was used.

VB: But how is that counterexample found in this proof? Is it found directly or indirectly?
Dean: I'm not sure. Hmm... I'm not sure.
VB: How, how did you find your example here [in the first interview]?
Dean: I, uh, I just took two functions and then a third function. I knew this $[f(x) = 0]$ was always going to yield zero and these two $[g(x)$ and $h(x)]$ are different functions.

I thought of $f$ first, knowing that it could take in different values it could yield the same value.

It appeared that Dean's own thought process was indirect, because he wanted to identify an $f$ that could yield the same value for different inputs. However, the very essence of his thought process was identifying two things at the same time rather than $f$ by itself. This thought process involved an assumption $[g(x) \neq h(x)]$ that was made explicit by the process of the given proof, which he did not verbalize.

VB: Okay. But, this proof here—does it go in the same direction that you went about that problem?
Dean: I think it does, because it starts off by coming with $f(x) = x^2$, and I think they selected that one because they knew that if $x$ is negative and you square it, you'll still get $x$ positive squared. And if $x$ is positive and you square it you still get $x$...
squared. So, then they plugged in $g$ and $h$ where they knew that they'd get that outcome.

VB: But what about the initial assumption here?

Dean: Um... that's just life. I think they were just saying that we kind of, like want to, I'd like to prove this, initially this looks false. I'm going to look to prove it false. And I think that the easiest way to go about doing that would be to feed it two unequal functions [emphasis added].

The sentence emphasized above indicated that he would be assuming $g(x) \neq h(x)$ to start with, although he did not recognize that subtle phase of the indirect process. He appreciated its importance in the process of the proof but did not give a valid reason for it.

Dan on the other hand, gave the following explanation for his choice.

Dan: [...] So, again, we're assuming the opposite of the statement “then”. Right? So, what we want to do is see if we arrive at a contradiction. Because then if we do arrive at a contradiction, it would prove this [statement].

When he was asked why the assumption was followed by the need to show $f(g(x)) \neq f(h(x))$, he replied that the contrapositive statement was proven false.

Dan: Well, I guess because we're trying to prove the contrapositive of this. Because it gives us more ammunition for the proof. [...] I guess, it gives us room to arrive at a contradiction.

That's the contraposition. And so, that's what we're doing. And then we find that it's [contraposition] not possible, because there is a function $x^2$...

VB: So, what is attained in the end?

Dan: If the contraposition is false, then this one is also false.

The data indicated that Dan considered the contrapositive approach a necessary step ("ammunition" or "room") to find a contradiction to the given statement. Thus, he considered that phase of the proof as essential.
Similarly, Dave explained that the proof used the method of contraposition and showed a counterexample, but it was not clear whether he thought the counterexample was to the contrapositive statement.

VB: So what is the role of the assumption here, and why is it followed by the statement that we need to show \( f(g(x)) \neq f(h(x)) \)?

Dave: Well, because we’re saying every function, all we have to do is find one function that does not satisfy this \([f(g(x)) \neq f(h(x))]\) relationship. And since we do find that, then this \([g(x) \neq h(x)]\) is true. And since this is the, you know, not equal, then this \([g(x) = h(x)]\) in the conclusion of the statement] is false in here?

Unlike Dan however, Dave became confused about the overall process of the proof as the interview progressed.

VB: So does that mean one of the arguments is redundant? You understand the question?

Dave: Yeah. I mean in a contrapositive proof you don’t necessarily have to have a counterexample in there. In fact, we could have just said, we probably just could have, just went with our counterexample.

Although he found the counterexample to be the crux of the proof and the steps of the contraposition redundant, he was not totally convinced of his argument. He sensed that the role of the assumption in the proof was important in finding the counterexample.

Therefore, the interviewer asked him a direct question that pinpointed his uneasiness.

VB: Well, the counterexample found in the proof is a counterexample to what, which statement?

Dave: It’s actually. It’s a counterexample to this \([f(g(x)) \neq f(h(x))]\), to the assumption in the proof, because we’re assuming \(g(x) \neq h(x)\). So, it [contraposition] is necessary.

VB: But why is that assumed?

Dave: […] Um, it’s just the method that we’re trying to use to prove the statement. You have to, in a contrapositive proof [emphasis added]. […] You need to assume that not Q implies, and try to arrive at not P. […] So, we’re saying here not Q implies, you know, doesn’t imply not Q, which we proved.

It was observed that only after the interviewer’s prompting question, was Dave finally able to put the whole process of the given proof together. However, it was also
observed especially from his remarks in the last piece of the above excerpt that he made
due to progress because he did not question the validity of the use of contraposition. In
other words, he passively accepted the method as valid and worked around it to bind the
rest of the steps in the proof together. Thus, there was no definite indication in the data
whether this student fully understood the validity of the process of the given proof.

In contrast, Doug thought the proof used contradiction to show the statement true.

VB: Why do you think that the presented argument in this situation shows that the
statement is true?
Doug: Well, it's the same thing, like, we assume not B, that is we assume that, uh,
g(x)  h(x), and then by counterexample we showed that not B is not true.
Therefore, A does imply B.

Well if you assume something's not true and we found a counterexample that it is
ture, then, I guess that's a contradiction, and from that we can assume that it [the
statement] is true. That's what I was thinking. That was my line of thinking.

Doug's words did not convey the correct process used in the proof, and it seemed
he had misconceptions about the contradiction method, which became apparent later in
the interview. His explanation for contradiction conveyed a circular argument.

The data in the remaining parts of the interview indicated his misunderstanding of
the stated assumption in the proof. He did not see the role of the assumption in producing
the counterexample.

VB: And can you explain why this assumption is followed by the statement that says,
um, we need to show that f(g(x))  f(h(x))?
Doug: That would be consistent with the original statement.
VB: Do you understand why, or can you explain the reason why this proof starts with
those two lines?
Doug: Well, see, and it's equivalent to the, it's... If those hold, then, uh, the statement
holds [true].

In his last sentence, Doug was saying that if the contrapositive statement held true then
the original statement would be true. However, that was inconsistent with his later
remarks where he claimed the assumptions in the proof were shown false and thus according to him the original statement was proved true.

VB: So, what is attained at the end?
Doug: They give an example of, of... They give an example that contradicts the initial assumptions that \( g(x) \neq h(x) \) and \( f(g(x)) \neq f(h(x)) \). And from that therefore it's a false statement based on one example.

VB: And what does that say about the original [statement]?
Doug: It says that it's true. It says that the original statement is true.

From his explanations above, it was observed that Doug was unable to substantiate his claims. Thus, it was concluded that he was unable to connect the different parts of the proof together or see the validity of its process.

All the major findings that emerged from the overall goals of the seven situations in this chapter are summarized and discussed in chapter 5. The main conclusions drawn from this study are reported in chapter 6.
CHAPTER 5

DISCUSSION OF THE RESULTS

This chapter discusses the themes emerging from the main results of the interviews and the researcher's conclusions drawn from those results. The discussion is organized by the objectives of each situation used for collecting the data. However, the themes discussed do not necessarily emerge exclusively from one situation—they are substantiated by data from other situations as well. The point of these themes is to understand the complexities of the issues investigated in the context of the instruments and settings used in this study.

Contraposition

The objectives of situation 1 were to observe whether students would invoke indirect proof methods and whether they would associate a direct proof of a statement with its contrapositive. The following common themes emerged from situation 1.

There was evidence from the data that students' thought processes were mainly centered on direct reasoning. Most students attempted to use direct methods in order to find the truth of the statement: “if \( x^2 \) is even, then \( x \) is even.” They did not necessarily invoke the contrapositive statement even after their attempts failed. They sensed some sort of contradiction built into the statement when alternate possibilities (odd numbers) were checked, but they were unable to give valid arguments for how a contradiction could be found. They did not associate the contradiction in this case with contraposition.

The participants in this research study were familiar with the notion of contraposition, or they possessed the knowledge base for contrapositive statements. However, that knowledge for some students was incomplete and for others was
compartmentalized, which prevented them from putting it to work when needed. Only after they were prompted for proof methods that could handle the statement given in situation 1 did they think of the indirect method (contraposition) they had learned in mathematics classes. The absence of the use of indirect methods was a result of the students' compartmentalized mindset, which forced them to think in terms of direct proofs. This mindset can be described as an isolated part of the students' concept image (Vinner, 1983) associated with proof. Due to this concept image, they attempt most proof situations using direct methods.

Goetting (1995) and Barnard and Tall (1997) found similar results (reviewed in chapter 2). Barnard and Tall, in particular, claim that the cognitive units, (i.e., “a piece of cognitive structure that can be held in the focus of attention all at one time” (p. 41)), “$x^2$ is even” and “$x$ is even” coexist in some students’ minds that render the direction of implication irrelevant. And for other students the idea of “$x^2$ is even” is strongly linked to “$x$ is even” instead of the alternative “$x$ is odd.”

There was evidence in this study that sometimes students would give arguments that are considered redundant for such situations. They argued in terms of excluding other possibilities or in terms of covering all the possible cases (even and odd in this situation) without realizing that one of the cases is the contrapositive and is sufficient to prove the given statement. Such was the case when the student was able to make the logical jump from a direct argument to an indirect one, but lacked sufficient internalization of the concept of contraposition to acquire a relational understanding.
Another finding was that Discrete-Math students had a favorite proof method, i.e.,
mathematical induction, that they would try to use as an alternative to direct methods, in
situations where integers were involved.

Not only did most students attempt direct proofs but also, according to their
responses to the tasks in this study, they demonstrated a strong tendency to view proofs
as using direct methods. They were unable to disengage themselves from their concept
image for direct proofs. They did not look at a proof from a different perspective to see
its implications beyond its surface structure. This was attributed to their deep attachment
to the semantics of the direct nature of the proofs in situation 1, because the process of a
direct proof fit their concept image for proofs in general.

Most students' understandings of the process of proof were limited to its explicit
semantics. They did not automatically associate a direct argument with an indirect proof
of an equivalent statement. The most common approach they employed to associate a
proof with a statement was to match the hypothesis of the statement to the assumptions in
the proof. Although they knew the validity of the indirect relationship, they needed to be
prompted in order to think beyond the surface structure of a proof due to their
compartmentalization of the notion of contraposition.

On the other hand, there was evidence that most Abstract-Math students did not
seem to be as strongly attached to the direct semantics of a proof as Post Abstract-Math
students were. This phenomenon was attributed to Abstract-Math students' receptiveness
to proving methods at this stage in their studies because they did not exhibit
compartmentalization of a newly learned concept. In contrast, it seemed that once
students move on to higher-level mathematics courses, they don’t much contemplate proving methods they have learned in the past.

Counterexamples and the process of contradiction

The objectives of the second situation were to observe students’ approaches to finding a counterexample and to investigate their perceptions of counterexamples in relation to the method of contradiction. The following themes emerged from situation 2.

Although students were aware that one counterexample is enough to refute a general statement, there was indication in the data that some of them were not aware of the source of the reason why a counterexample is a case of contradiction according to the laws of logic. They knew implicitly that in order to find a counterexample they needed to find a false conclusion to a true hypothesis. They had a procedural understanding that the fastest way to disprove a statement is to find a counterexample, but they did not necessarily know the exact reason why that procedure works best. They could not explain the reasons for their beliefs or elaborate on the rules of inference that govern the procedure. At times they resorted to an authoritative reasoning but most of the time they drew upon the procedure itself or used the specific counterexample they found in order to validate their beliefs or arguments as general.

Often in mathematics, procedures such as algorithms are carried out by students who show little understanding of the logic in them (instrumental understanding). There is evidence that such may also be the case with proof processes as far as finding counterexamples is concerned.

Most students used intuitive approaches to find a counterexample to the statement given in situation 2. Only a few students who could not immediately see that a statement
was false would exhibit some indirect approach when searching for a counterexample. Even then, they might not necessarily realize that indirect arguments were being used.

Those students who quickly and easily found a counterexample to the situation did not acknowledge the usefulness of other approaches. When they were asked to give different explanations or arguments to their intuitive approach, they were unable to relate it to other (indirect) processes. They did not necessarily relate the process of finding a counterexample to that of indirect methods or contradiction even after a proof by contradiction was shown to them. A similar phenomenon was also observed in situation 7. There was even a case (Perry) where the student regarded the processes of finding a counterexample and searching for a contradiction as separate approaches towards different goals. This indicated the student’s instrumental understanding of the two processes and the lack of deep understanding of the possible relationship between the two approaches. In combination with the findings from situations 6 and 7, this perception also indicated that some students did not view indirect processes as necessary for discovering the truth of a statement. Therefore, unless students experienced some difficulty finding a counterexample, they did not rethink their own processes. They were quick to dismiss any relationship between the process of finding a counterexample and other useful approaches, such as indirect arguments, which may actually provide more insight into ways a counterexample can be found other than by intuition.

The general perception of the participants was that a counterexample is the best and fastest way to disprove a statement but they were not necessarily aware of concrete approaches that can best lead them to find a counterexample. This perception had led them to believe that counterexamples can be found only by intuition. Evidence from this
situation as well as situations 6 and 7 revealed this gap in their learning. Students showed a good understanding of the algorithmic process of finding counterexamples, but it seems that they had absorbed a superficial meaning of the concept of counterexamples too quickly too soon, which might have led them not to appreciate their role for gaining insight into a situation. It is possible that classroom learning might not have given them enough opportunities for making the connection to indirect proof processes, thus leaving them with a weak perception of the connection between finding a counterexample and indirect processes.

On the other hand, there was evidence from situation 2 as well as situation 6 that some Abstract-Math students were open to the idea of studying indirect processes as valuable tools for finding counterexamples, more so than the students in the other categories. There was a case in particular where the student (Art) successfully integrated the method of contradiction in order to reach a goal of finding a counterexample, after he was encouraged to do so. Despite his initial belief that there was no point in using other approaches to find a counterexample, this case indicated how those firm beliefs could be changed under favorable conditions. Further, the student exhibited his newly acquired perception of indirect methods as a useful tool for finding counterexamples in later situations as well.

The Abstract-Math students' openness to newly learned proof processes might have been the immediate result of their experience at that learning stage. Of course, this phenomenon might also be attributed to other factors, such as course instructor and textbook used, which were not considered in this study.
Some difficulties that students sometimes had with understanding proofs by contradiction were also found in situation 2. Most students recognized the implicit contradiction used in the semantics of a proof but some had difficulty associating the semantics with the indirect process of the method of contradiction.

One of the difficulties faced by some students was also due to their concept image of proofs associated with direct methods. Students with such a concept image were inclined to allocate the assumptions and the conclusion in a proof, in the order they appear, to the hypotheses and the conclusion in a statement respectively. Thus, some of them could not correctly distinguish the assumptions (hypotheses) in a proof from the hypothetically false assumption that is needed to start a proof by contradiction. They disregarded the hypothetical (false) conclusion’s impact that indirectly brought forth the contradiction. Consequently, they could not associate the indirect process of the proof with the statement it proved.

Another misinterpretation of the process of a proof by contradiction occurred when the student had an inadequate understanding of the contradiction’s indirect process in terms of its goals. Such was the case when the student (Doug) thought the process was more about choosing a right assumption. He would negate an assumption in a statement and search for a contradiction. If a contradiction were found then that assumption would not be true and the statement would be false.

**Implicit assumption in a proof by contradiction**

The objectives of situation 3 aimed at finding students’ perceptions of a proof by contradiction that used an implicit assumption in its figure as well as whether their
perceptions of the process of that proof were limited to its explicit semantics. The following common themes emerged from situation 3.

All participants except one acknowledged the limitations of diagrams for asserting geometric truths. The most common approach they used to show the statement given in this geometric situation was that of similar triangles. There was no evidence that students thought of the given geometric situation with any consideration for Euclid's parallel postulate, which was probably why the findings from the second interviews on situation 3 indicated the area where most students exhibited a misinterpretation of its proof by contradiction.

In this case, students mistook the proof by contradiction, where the conclusion of a statement was negated through an implicit assumption in the construction of its figure, for a direct proof. Most of the students, including those who otherwise exhibited a solid understanding of a proof by contradiction, could not associate or did not consider the construction of a figure in proof by contradiction as part of its assumption unless they were prompted. Thus, they thought the construction step of the proof was part of its direct argument.

There was evidence from situation 3 and others that the semantics of a proof by contradiction played a major role in many students' perception of the method. Their perception of contradiction was limited to the method's explicit procedural process, which was not sufficient for them to have a workable understanding of the method in different settings or contexts.

Most students lacked sufficient understanding of the deep structure of the contradiction process to be able to identify a proof as using contradiction even after its
process was made explicit. For instance, in situation 3 students’ initial perception (direct) or understanding of the original proof did not change or improve in light of the alternate one that used explicit semantics and assumptions in the contradiction process. They claimed the alternate proof used contradiction, but they did not acknowledge that the original proof used the same process.

On the other hand, there was some evidence that a small number of students (Patty and Dan) with minimal assistance were able to reach a practical judgement of the contradiction process in the proof that used an implicit geometric assumption. These students showed initial signs of a deeper level of understanding as their perception of the original proof improved.

The role of a constructed figure was the source of some students’ misunderstanding of the contradiction process. Some (Art and Dave) viewed the figure as a counterexample to the statement and others (Adam and Dean) as the result of a contradictory argument rather than its assumption. Furthermore, there was also evidence that when students perceived such roles for the figure they resorted to violating a given fact instead of rejecting the figure’s validity or its hypothetical assumption.

Another difficulty in understanding the contradiction process was due to a misunderstanding of the role of assuming the negation of the conclusion in a statement. Some students (Doug and Alice) did not see the point of making a hypothetical (false) assumption in a figure for the sake of argument that would eventually give rise to a contradiction. They argued that the figure (or its assumption) could not depict something that was needed to be proved or disproved in the first place. This misperception was again a result of the students’ concept image associated with direct proofs.
The role of an equation in an existence/non-existence problem

The objectives of situation 4 were to investigate students’ perceptions of the role of an equation in finding a mathematical object and to observe whether they could relate the purpose of manipulating it directly to that of determining the existence or non-existence of an object indirectly. The following themes emerged from situation 4.

As with searching for counterexamples in situation 2, students tended to use guesswork with intuition to tackle existence situations. This was true even when an equation, whose solution leads indirectly to the sought object, was given as in situation 4. Although all the participants knew what needed to be done to solve existence problems (i.e., finding one case that satisfied a given condition), only a few knew how that could be done using the equation. Nonetheless, only one student (Dan) relied solely on intuition and did not see any role for the equation. All the others used some sort of manipulation on the equation but most of them had little idea why they were making use of the equation. Some however realized the use of the equation only after they found a solution for it.

There was evidence that some students did not see the indirect role the equation played in searching for an object. They had different interpretations of its role. Some (Alice and Adam) thought that the truth of the equation should be checked (if left-hand side is equal to right-hand side) instead of finding its solution and using it indirectly to determine the existence of the sought object. Others did not even think it was useful or did not know what to do with the equation in an existence problem. Additionally, few of them thought that its role was to substitute numbers for the variable until one satisfied the
equation. Again, all these approaches were attributable to the lack of understanding of the indirect role of the equation as a model for existence/non-existence problem situations.

The findings from the second interviews on situation 4 indicated that some students could not associate the direct manipulation or solving of the equation in the given proof with an indirect argument that showed the non-existence of an object. One of the observed difficulties was that they could not tie in the proof’s explicit assumption to the algebraic manipulation for producing a solution that would invalidate that (hypothetical) assumption. This difficulty was again a result of their absorption in the direct process of the proof for solving the equation. Their attachment to its direct process was so strong that it effectively inhibited them from seeing beyond the surface structure of the proof even when some of them previously had used the equation to solve the existence problem.

Furthermore, there was some indication in the data that most students, including those who used the equation to solve an existence problem, were not necessarily aware or did not know that the process of solving the equation assumed implicitly the existence of a sought object, as in indirect methods. Also, they did not necessarily interpret the process of solving the equation as an indirect approach to validating existence/non-existence of an object. The observations of this situation suggested that when students expanded or solved an equation in search of an object, they did so instinctively because they perceived a direct process implicated by the equation. They did not necessarily perceive the subtle workings of indirect processes in the context of an existence/non-existence problem that used a model (equation) of a hypothetical situation.
Irrefutability of contradiction

The objectives of situation 5 were to investigate students’ understandings of the method of contradiction that showed the impossibility of existence of a certain quadrilateral, as well as whether their perceptions of contradiction changed under a convincing non-contradictory (and redundant) argument.

In their first interview on this situation, all but two students (Amy and Doug) were able to find a convincing argument in order to refute the existence of a quadrilateral. Almost every student discovered the extension of the idea of Triangle Inequality to quadrilaterals using contradictory arguments, although some might not have necessarily associated their arguments with indirect methods in this case. A deviation from this occurred when students (Doug and Amy) relied more on known facts or their knowledge base rather than on trying to investigate and discover the unknown.

Most students were able to interpret the process of the given proof in situation 5 as contradiction. However, a procedural understanding of the nature (existence/non-existence of a quadrilateral) of the problem in this situation caused some students to exhibit concerns about whether the process had covered all possible cases. Their concern was based on the procedure that in a non-existence situation not finding a case or showing a contradiction of one way of drawing a quadrilateral, does not guarantee its non-existence unless all possible cases are accounted for. Although such concerns are valid in most situations, students’ preoccupation with the procedural approach in this situation caused them to overlook the generic consequences of the contradictory \((2 + 3 + 5 > 11)\) argument and the context in which it was obtained in the proof. In other words, their perception of contradiction remained local to the particular case because of a
procedural understanding of non-existence problems. Instead of breaking away from the procedural process and looking beyond the immediate particulars to implicate a general contradiction that they had previously discovered (in interview 1), they considered each individual case separately as dictated by the procedure.

On the other hand, there was evidence that some students and most Abstract-Math students did not possess a practical understanding of the power of contradiction. When their perception of the contradiction in the original proof of situation 5 was challenged by an alternate proof, they changed their perception instead of trying to resolve inconsistent results by drawing upon the irrefutability of the contradiction found. They did not insist that a contradiction must be inevitable under all circumstances including the case in the alternate proof. The power of contradiction was not perceived and maintained as definitive even when they had previously (in interview 1) discovered a valid reason for the source of the contradiction.

A variation from the observed theme above was that most Discrete-Math students were not easily convinced by the purported proof that refuted their original perception of contradiction. They tried to investigate the matter more until they detected a gap in the purported proof. This phenomenon was attributed to their strong adherence to the fact (Quadrilateral Inequality) they discovered earlier in the first interviews. In contrast to Discrete-Math students, Abstract-Math students did not exhibit a strong perception of the irrefutable power of contradiction because they lacked confidence in their own judgement of proofs. Earlier in situation 2, it was observed that these students' openness to new ways of looking at proofs helped them to see a relationship between the processes of
finding a counterexample and contradiction but that openness here proved to make them vulnerable.

There was evidence that some (Alice and Doug) students' inability to perceive the irrefutability of contradiction was due to their adherence to the requirements of the procedure for non-existence situations. Their preoccupation with the procedure of the non-existence proof prevented them from seeing a redundant algebraic manipulation that missed its objective. One student (Alice) even invalidated a previously (interview 1) discovered fact (Quadrilateral Inequality) in order to refute a contradiction. Further evidence was provided by the failure of students (Amy, Art and Doug) to associate a discovered fact with some known fact, in their first interviews. For instance, if a student discovered the Quadrilateral Inequality and was unable to associate it with the well-known Triangle Inequality then he or she might refute the contradiction of the Quadrilateral Inequality when faced with more information about the situation.

**Use of contradiction to explore counterexamples**

The objectives of situation 6 were to investigate students' approaches to an existence problem of an algebraic relationship \(a^2 + b^2 + c^2 = d^2\) and their understanding of an explicit contradictory argument that explored and gave insight into how a counterexample could be found.

All 12 participants attempted to find a counterexample to the non-existence statement in this situation. Eleven of them did not actually try to look for a proof, not because they knew the statement was false or knew that integers that satisfy the relation exist, but because they did not know how to argue in non-existence problem situations. Instead, some thought that finding a counterexample was an easier task. However, only
one student (Patty) was successful in finding a counterexample. Her success was attributed to her systematic search for a counterexample and her ability to see the relationship of the situation to Pythagorean triples. Her systematic search involved algebraic simplification and manipulation that culminated in a useful pattern for generating counterexamples, rather than fitting a combination of numbers into the given relation.

As in situations 2 and 4, most students approached this type of existence problem by intuition and when their intuition failed, mainly because of the large number of variables involved, they did not necessarily know in which direction to turn in order to accomplish the task. Most of the time their failure to find counterexamples prompted them to conclude that the statement was true.

The approach of some students involved listing squares of integers in hopes of finding three that added up to a perfect square. However, they soon gave up on this approach because the number of combinations increased indefinitely. In particular, the large number of variables combined with the infinite number of their combinations seemed very unmanageable to Discrete-Math students.

Some students did see some connection between situation 6 and the Pythagorean relation but did not make use of that information to organize their search. They lacked the conviction that the Pythagorean triples could be extended to quadruples through a systematic search that narrowed down the number of variables involved. They thought the varying relationship of the quadruples was a difficult guessing game.

Another finding was that the participants from the Abstract-Math category did not see the Pythagorean connection to this situation. They described alternate approaches that
were similar to the process of contradiction but none was able to put that approach into practice to actually find a counterexample. The observations in this situation confirmed the earlier ones from situation 2 where these students acknowledged the usefulness of the contradiction process for finding counterexamples. However, this acknowledgement was not necessarily due to their beliefs but to their openness or receptiveness to different approaches at this learning stage. Some of them (Alice and Adam) still did not see the use of contradiction to explore situations.

There was evidence from this situation as well as from situations 3 and 4 that some students and most Discrete-Math students found it hard to believe in the non-existence of a mathematical object with certain properties that belonged to an infinite set.

On the other hand, the findings from this situation (and others) indicated that if their attempts at finding counterexamples or an indirect proof for that matter were successful they did not necessarily realize the process they used in their arguments because of a lack of relational understanding. For example, when students gave arguments that used indirect methods such as contradiction, they sometimes did not realize explicitly the connection their arguments had with proof methods learned in class. They might lack an overall perspective of the process in their argument, because they were unable to distance themselves from the specifics and see the larger picture. In terms of finding a counterexample to a statement, this indicated that students sometimes unknowingly used indirect arguments, in order to eliminate contradictions, but they did so without being aware of the relationship between their own processes and the indirect methods learned in mathematics courses.
There was evidence that the difficulty some students (Adam and Doug) encountered in understanding proofs by contradiction was not necessarily due to the implicit nature of this method. They had difficulty understanding a contradictory argument even when its semantics and process were made explicit, because they lacked a good perspective of the workings of the method. They exhibited a vague and disconnected perspective of how the process worked.

On the other hand, there was also evidence that explicit use of the word "contradiction" in arguments triggered certain aspects of its method, which in turn caused a certain uneasiness in some students' perception of it and misunderstanding in others. There was evidence that one such aspect that the word "contradiction" was associated with was an algorithmic understanding of its process. In particular, an argument that used the word "contradiction" explicitly but did not follow the script of a template associated with its method, created uneasiness and in some cases caused a misunderstanding of its process. After some students read the word "contradiction" in the proof of situation 6 and it did not fit in their template-like understanding of its process, they felt uneasy and sometimes confused. In contrast, a template-like understanding of contradiction was not revealed in other situations where the process of contradiction was used but the word "contradiction" was not made explicit.

Contraposition and Counterexample

The objectives of situation 7 were to observe whether students search for counterexamples to the given statement \[\text{if } f(g(x)) = f(h(x)) \text{ then } g(x) = h(x)\] by invoking indirect methods, and to investigate students’ perception of the validity of using
contraposition while searching for counterexamples. The following themes emerged from this situation.

As before, in this situation there was yet more evidence that students' concept image associated with direct methods caused them not to think in terms of indirect methods when searching for counterexamples. Some students tried to find a counterexample to the given statement using direct methods. However, only a few of them, who remembered that only one-to-one functions have inverses on the real numbers, succeeded in finding a counterexample.

On the other hand, although some students realized the usefulness of arguing in terms of contradiction in their search, they could not materialize its process because they had a tenuous perception of it. Some students even had difficulty working with functions, which, combined with their tenuous perception, made the problem of finding a counterexample in situation 7 even harder. In particular, a graphical approach to functions was conspicuously absent from their attempts to find a counterexample. Only one out of the twelve students used a graph.

There was evidence that most students were unable to make sense of an explicit indirect argument that finds a counterexample to the contrapositive statement. They could not validate or appreciate the use of contraposition when all that needed to be shown was a counterexample. They did not have any insight into or deep understanding of indirect approaches for finding a counterexample. They could follow the given process but could not necessarily see it as a whole entity for reaching a goal. Consequently, some students became confused and others thought that the contraposition process was redundant and unnecessary in such situations. One of the reasons for such misperceptions was that they
viewed contraposition and a search for a counterexample as incompatible strategies
towards a goal.

In contrast to the other situations, where indirect methods were used to find a
counterexample, sometimes students were able to see the connection between indirect
methods and the process of searching for a counterexample, but their perception of that
connection was vague and inconsistent depending on the situation at hand. At times they
exhibited dynamic thought processes that involved assuming and checking for
contradictions, but when their thought processes were made explicit (through proof-
checking tasks) they did not necessarily acknowledge the connections or verbalize their
thoughts.

**Contraposition and Contradiction**

There is evidence in the data, particularly in situations 1, 2 and 7, that some
students were more comfortable with the concept of contradiction than with
contraposition, possibly due to the use of the technical term "contraposition." The term
"contradiction," which is used in everyday English, seemed self-explanatory to the
students. Most of the time they were able to relate observed processes in the provided
proofs or within their own approaches to contradiction, but they seemed to have difficulty
remembering the process of contraposition. Since the method of contradiction is based on
exclusive "or" in the statement \( A \lor \neg A \), the logic made more sense to them than the logical
equivalence of \( P \Rightarrow Q \) and \( \neg Q \Rightarrow \neg P \). Thus, the logical workings of contradiction in the
semantics of an argument were recognized easily. Furthermore, students who were in
doubt about the equivalence of contrapositive statements did not make use of truth tables.
This study also found the above phenomenon to be due to a lack of understanding of the subtle similarity between the two processes. Although contraposition was viewed as an indirect approach, sometimes students had difficulty associating the contradiction process with indirect methods. There was also some evidence that some students had committed the methods of contraposition and contradiction to memory, which failed them in certain situations (such as situations 1 and 7).

Similar to the results in this study, Goetting (1995) found that the students had more experience with proof by contradiction than contraposition, and they would refer to a proof by contraposition as a proof by contradiction.

Sometimes students gave indirect arguments but they did not necessarily realize that their thought processes were indirect. For instance, Dan in situation 3 was not sure if the overall process of the given proof was based on the method of contradiction that he had learned in his mathematics class. He did not have a thorough perception of how the method of contradiction worked even when his own interpretation of the steps of the proof led him to that process. In other words, he lacked a full relational understanding of the method necessary to make him realize its connection to certain arguments and situations. Furthermore, Patty, in situations 1 and 6, gave valid indirect arguments, but in one case, she did not realize the similarities of her argument to that of contraposition until she was prompted. And in the other, she did not think of her argument as using contradiction even when the process was made explicit through a proof-checking task.

Variations among categories

There were not any noticeable differences of perceptions among the categories of the participants in this study, except in a few instances.
In situation 1, Abstract-Math students were quick to make the connection between the given proof and the contrapositive statement it showed because of their openness to proving methods and strategies at this stage of their studies, whereas students from the other categories had to be prompted in order to see that connection.

In situation 5, Abstract-Math students exhibited a lack of confidence in their perception of the irrefutability of proof by contradiction by not arguing against a purported proof that used redundant arguments. Instead, they were quick to be convinced and change their original perception. In contrast, Discrete-Math students were not convinced by the purported proof that gave an argument against their original perception of the contradiction. Rather, they insisted on finding faulty reasoning in the proof.

In situations 4 and 6, Discrete-Math students found it hard to believe in the non-existence of a mathematical object with certain properties that belonged to an infinite set. Also, in situation 6 they were quick to dismiss their efforts for finding a quadruple with a specific relationship, because of the large number of possible combinations.

The next chapter summarizes the conclusions of the results from this study that are pertinent to the research questions stated at the end of chapter 1.
CHAPTER 6

CONCLUSION

This final chapter provides answers to the research questions stated at the end of Chapter 1. It also gives the researcher’s views on the implications of this research for teaching of indirect proofs in mathematics education at universities. It concludes by pointing out the limitations of this study as well as suggestions for future studies.

Answers to the research questions

The research questions investigated in this study are presented one at a time in the next two subsections. Concluding answers that were found from this study follow each question. It is important to keep in mind that the following remarks are limited to the 12 participants in this study. However, as with any qualitative study, results from interview studies can, as Harel and Sowder (1998) summarize best, “offer indications of the state of affairs and a framework in which to interpret other work. [Thus] the following comments and speculations should be read in that spirit” (p. 277).

Answers to research question 1

What is the nature of undergraduate students’ perceptions about proofs in establishing the truth of a mathematical statement?

a) What approaches or activities do they attempt in order to establish the truth of a mathematical statement and the validity of its proof?

b) What is the primary focus of their attention in proofs?

Most of the participants’ thought processes appear to be centered primarily on direct reasoning and they attempt to use direct methods most of the time in order to find
the truth of a statement. In particular, they do not necessarily invoke the contrapositive statement or other indirect methods after their direct attempts fail.

Not only do most of them attempt direct proof most of the time but also, according to their responses to the tasks in this study, they demonstrate a strong tendency to view proofs as using direct methods. They do not look at proofs from a wider perspective to see the implications of a (direct) proof beyond its surface structure. They are unable to disengage themselves from their concept image associated with direct proofs and their semantics if not prompted.

Most participants' understandings of the process of a proof are limited to its explicit semantics. They do not automatically associate a direct argument with an indirect proof of an equivalent statement. The most common approach they employ to associate a proof with a statement is to match the hypothesis of the statement with the explicit assumptions in the proof.

The semantics of a proof play a major role in many participants' perception of its method. Thus, some students' perceptions of contradiction are limited to its explicit procedural process, which is not sufficient to provide them with a relational understanding of the method in different settings or contexts.

Most participants do not necessarily relate indirect processes to the process of finding a counterexample even after an indirect proof is presented to them for comparison. They focus on the end result of the proof rather than its underlying mechanism that illuminates the aforementioned relationship. They use intuition in order to find counterexamples. Only a few students who cannot immediately see that a statement is false exhibit an indirect approach when searching for a counterexample, but
sometimes they do so without necessarily recognizing their thought process as indirect. Indirect processes are not necessarily viewed as tools for exploring the truth of a statement or for gaining insight into a situation.

Students also tackle existence/non-existence problems using guesswork based on their intuition most of the time. When their intuition fails, they do not necessarily know in which direction to turn in order to accomplish a task. For instance, they do not know how to prove or disprove the non-existence of an object. Instead, some think that finding a counterexample is an easier task.

Most participants lack an understanding of the deep structure of the method of contradiction. In a proof by contradiction of a geometric situation, if the conclusion of a statement is negated through an implicit assumption in the construction of its figure, then they think the construction step is part of a direct argument. They do not consider the construction of a figure as part of its assumption unless they are prompted.

**Answers to research question 2**

What are undergraduate students’ difficulties in understanding the aspects and characteristics of indirect proof processes?

a) How do they make sense, if at all, of an assumption and its contradictory results in a proof by contradiction?

b) How well can they judge the validity of an indirect argument?

c) How can those difficulties in understanding indirect proof processes be explained in terms of their behaviors and perceptions of indirect proofs?

Due to their concept image, the thought process that students employ for direct methods restrains them from thinking in terms of indirect methods. They possess the
knowledge base for the methods of contraposition and contradiction but that knowledge, for some participants, is incomplete or tenuous and for others is compartmentalized. Although contraposition is viewed as an indirect approach, sometimes students have difficulty associating the contradiction process with indirect methods. Most participants recognize the semantics used in an implicit contradiction in a proof but some find it difficult associating the semantics with the indirect process of the method of contradiction.

However, the impediment in understanding proofs by contradiction is not limited to its implicit use. Some participants have difficulty understanding a contradictory argument even when its semantics and process were made explicit, because they possess partial instrumental understanding or lack a clear perspective on the workings of the method.

Sometimes the explicit use of the word “contradiction” in arguments triggers certain aspects of its method, which in turn causes uneasiness in some students and misunderstanding in others. One such aspect is the procedural or algorithmic understanding of the method of contradiction. If the process of a proof by contradiction does not fit their procedural understanding then it causes uneasiness.

Students have a procedural understanding of counterexamples but some lack altogether and others have only a vague relational understanding of indirect approaches to finding counterexamples. Some students realize the usefulness of arguing in terms of contradiction in their search for counterexamples but they cannot complete the application of this process because they possess only an instrumental understanding and a
tenuous perception of it. Others can not validate or appreciate the use of an explicit indirect argument that finds a counterexample to the contrapositive statement.

One of the difficulties some students face is also due to their concept image associated with direct proving methods. They are inclined to allocate the assumptions and the conclusion in a proof, in the order they appear, to the hypotheses and the conclusion respectively in a statement. Thus, some of them cannot correctly distinguish the assumptions (hypotheses) in a proof from the hypothetically false assumption that would bring forth the contradiction. They disregard the impact of the hypothetical (false) conclusion that indirectly brings forth the contradiction. Hence, they cannot associate the indirect process of the proof with the statement it proves.

Another impediment in understanding the contradiction process is due to misunderstanding of the role of assuming the negation of the conclusion in a statement. Some students do not see the point of making a hypothetical (false) assumption for argument’s sake that will eventually cause a contradiction. They think that the conclusion cannot be assumed because it is needed to be proved or disproved in the first place. This misperception is again a result of the students’ concept image associated with direct proofs.

The role of a hypothetical figure in a proof is another source of misunderstanding of the contradiction process. Some students view the figure as a counterexample because it negates the conclusion of the statement. Others view it as the result of a contradictory argument rather than its assumption. Also, when they perceive such roles for the figure they resort to violating a given fact instead of rejecting the validity of the figure or its hypothetical assumption.
Another source for misjudging proofs by contradiction is due to a procedural understanding of existence/non-existence situations. A procedural understanding prevents students from perceiving the irrefutable power of contradiction. Their preoccupation with the procedure of an existence proof keeps them from seeing redundant algebraic manipulations.

Implications for the teaching of indirect proofs

The findings from this study indicate that most students approach the problem of finding a counterexample intuitively. They do not necessarily relate this process to that of indirect methods even after an indirect proof is shown to them for comparison. Thus, mathematics curricula should emphasize the use of indirect processes as tools for finding counterexamples. There should be enough time for practicing problems on the integrated use of the three concepts, i.e., contraposition, contradiction and counterexample. Classroom learning should provide ample opportunities with more difficult problems to explore and discover the connections and differences among those processes until students gain an appreciation of their role for gaining insight into a situation.

The concept of contraposition can be studied in conjunction with contradiction. They can be taught in the same context rather than as separate methods with different goals. This can help students differentiate between the two methods. Textbooks need to make explicit the connection and the difference between the two processes and their relationship to the processes of searching for counterexamples. Examples using both methods of proving can be shown for the same statement so that the similarity and the difference between the processes of the two methods can be highlighted. For instance, different proofs of the statement, “If \( x \) is a real number and \( x^3 + x^2 + x + 1 = 0 \) then \( x = -1 \)”
can be shown as follows, in order for the students to see the difference between the two methods.

**Proof by contraposition:**
Suppose that $x \neq -1$. Then $x + 1 \neq 0$. Since $x$ is a real number then $x^2 + 1 \neq 0$.
Therefore, $(x + 1)(x^2 + 1) \neq 0$, and upon multiplication we get $x^3 + x^2 + x + 1 \neq 0$.
Thus, if $x^3 + x^2 + x + 1 = 0$ then $x = -1$.

**Proof by contradiction:**
Suppose $x^3 + x^2 + x + 1 = 0$ and $x \neq -1$. Then $(x + 1)(x^2 + 1) = 0$ and $x + 1 \neq 0$.
Therefore, we must have $x^2 + 1 = 0$, which is impossible because for every real number $x$, $x^2 + 1 > 0$. Thus, the assumption that $x \neq -1$ is untenable. So, $x = -1$.

Another aspect of the findings in this research is that students may have a template-like understanding of the process of contradiction that they think is rigid and which has limited use. A template-like understanding of contradiction may cause some students to think that the contradiction process is limited and is used to prove true statements only. In other words, they may think it is not used to explore, discover or refute the truth or falsehood of uncertain situations. They may not view this process as a tool for exploration, refutation and discovery of the truth of unknown statements. Thus, it is imperative that the treatment of the method of contradiction by textbooks accounts for its different uses, instead of merely demonstrating cases where its use is refined and finalized in a template form.

Furthermore, since the steps in a structured proof do not occur to the prover in the order presented, students would benefit if introductory and advanced mathematics textbooks routinely incorporated a discussion of the methods used in their proofs. Instead of just giving a proof of a theorem, they could include a detailed analysis of the thought process put into the final version of the proof, the reasons a particular technique is used...
and its advantages over the others. Strichartz's (1995) analysis textbook is one such example.

Limitations of the study

By their very nature, results from interview studies cannot be generalized. However, case studies, such as this one with a small number of subjects, can produce rich data and in-depth results that are good indicators of emerging trends within the population of the subjects studied.

This qualitative research studied students' perceptions and understandings of indirect proofs in researcher-designed problem situations, using clinical interview settings. The twelve subjects who were interviewed in this study were purposefully selected from a group of 28 willing participants and were paid for their services. They were grouped into three different categories—four in each. However, there have been no attempts to claim that each group of four students was representative of a whole population in their respective categories. This study did not attempt to relate the results to any particular curriculum or textbook that the participants have used in their proving classes, nor did it attempt to relate them to students' learning habits. Further, although the participants were asked not to use outside help for their take-home proof-checking tasks, there was no control over whether they were compliant or not.

The data consisted mainly of the interviews conducted and there was no attempt to gather any data from students' class-work or tests. Finally, all the conclusions in this study were drawn from and limited to students' responses to the researcher-designed problem situations. As with any qualitative research, different instruments using different protocols might produce different conclusions.
Suggestions for future research

One of the findings from situation 4 is that most participants, including those who use an equation to solve an existence problem, are not necessarily aware that or do not know that the process of solving the equation assumes implicitly the existence of a sought object, as in indirect methods. A comprehensive examination of the students' perceptions of the implicit presumptions—that are built into a hypothetical model (equation) to validate/invalidate an existence situation similar to situation 4—is needed to shed more light on students' awareness of those presumptions and their consequences.

In light of the students' difficulties observed during the first interviews of situation 6, follow-up studies could investigate a subtle concept of proofs by contradiction by studying students' understanding of the power of refutation without necessarily finding a counterexample. They could probe students' beliefs about an existence proof that does not actually find a much-desired counterexample, but only shows its existence in general. These types of studies could investigate the difficulty students have in constructing existence/non-existence proofs.

The findings of this research indicated that students' perceptions of the relationship between finding a counterexample and indirect processes are tenuous. Teaching experiments using more difficult problem situations than used in situation 2 could be conducted to investigate the type of intervention needed in order to materialize a perception of that relationship. Last but not least, teaching experiments could also explore the type of mediations needed for students to transform surface understanding into relational understanding and/or deep understanding of the structure of proof by contradiction.
REFERENCES


### Post Abstract-Math Participants

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<td>305, 341</td>
<td>221</td>
<td>4</td>
</tr>
<tr>
<td>Amy</td>
<td>Biology</td>
<td>Junior</td>
<td>4.0</td>
<td>305</td>
<td>251</td>
<td>3</td>
</tr>
<tr>
<td>Adam</td>
<td>Math Ed</td>
<td>Junior</td>
<td>2.5</td>
<td>221, 305</td>
<td>–</td>
<td>0</td>
</tr>
<tr>
<td>Alice</td>
<td>Math Ed</td>
<td>Sophomore</td>
<td>3.8</td>
<td>305</td>
<td>221</td>
<td>4</td>
</tr>
</tbody>
</table>

### Discrete-Math Participants

<table>
<thead>
<tr>
<th>Participants</th>
<th>Major</th>
<th>Status</th>
<th>GPA</th>
<th>Math Courses enrolled at time of research</th>
<th>Finished courses beyond Cal II</th>
<th>Math credits beyond Cal II</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dan</td>
<td>Comp. Sc./Math</td>
<td>Sophomore</td>
<td>4.0</td>
<td>225</td>
<td>–</td>
<td>0</td>
</tr>
<tr>
<td>Dave</td>
<td>Comp. Sc.</td>
<td>Junior</td>
<td>3.05</td>
<td>221, 225</td>
<td>–</td>
<td>0</td>
</tr>
<tr>
<td>Dean</td>
<td>Comp. Sc.</td>
<td>Junior</td>
<td>2.2</td>
<td>221</td>
<td>225</td>
<td>3</td>
</tr>
<tr>
<td>Doug</td>
<td>Comp. Sc.</td>
<td>Senior</td>
<td>3.6</td>
<td>221</td>
<td>225, 341</td>
<td>6</td>
</tr>
</tbody>
</table>

221: Linear Algebra  
225: Discrete Math  
225: Discrete Math  
251: Calculus III  
301: Math with Tech for Teachers  
305: Intro to Abstract Math  
311 & 312: Ord. & Partial DEq.  
326: Intro to Number Theory  
341: Intro to Prob. and Stats.  
381: Discrete Optimization  
396: Independent. Study  
406: History. of Math  
431: Eucl & non-Eucl. Geom  
452: Complex Analysis

Note: All the participants had H.S. geometry.
APPENDIX B

PARTICIPANT SELECTION TOOL
Participant Selection Survey

Dear Student;

You are being asked to participate in a study that will examine undergraduate students’ perceptions of mathematical proofs. Your participation in this preliminary survey is voluntary and it will be used for selection of a sample of undergraduate students involved in mathematics courses at the 200 level and above.

If you would like to participate in the upcoming interviews this semester, please write your

Name:

Phone or email:

GPA:

Your help with this research is greatly appreciated. You will be contacted if you are selected as participant. For your participation, you will be paid an amount of $30 for about 3 hours of your time.

Thank You.

For more details, please contact the following person.

Varoujan Bedros.
Tel. 243-4485
Email: vjbedros@yahoo.com
PART 1: Student Background

Please, answer the following questions either by circling the appropriate answer or filling in the space.

1. Did you take geometry in high school?  
   YES  NO
   If yes, when? ________
   If yes, did you have to write proofs in geometry?  
   YES  NO

2. Have you had Math 225: Discrete Mathematics, or its equivalent?  
   YES  NO
   If yes, when? ________

3. Have you had Math 305: Intro. to Abstract Math, or its equivalent?  
   YES  NO
   If yes, when? ________

4. I am a:
   freshman  sophomore  junior  senior  other (specify) ________

5. My major is: ___________________

6. I have completed a total of ____ credits of college mathematics courses beyond Math 153: Calculus II.

7. Please write down all the numbers of math courses beyond Calculus II, indicating the semester / year, you have taken or are taking them at UM.

<table>
<thead>
<tr>
<th>Course Number</th>
<th>Semester / Year</th>
<th>Course Number</th>
<th>Semester / Year</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
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</tbody>
</table>

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### PART 2: Please, circle your best response to the following statements.

<table>
<thead>
<tr>
<th>Statement</th>
<th>SA</th>
<th>A</th>
<th>NO</th>
<th>D</th>
<th>SD</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. A proof in mathematics is different from proof in other areas.</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2. A valid mathematical proof verifies the truth of a mathematical relationship.</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3. A mathematical relationship becomes valid after it has been proved.</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4. Proofs in mathematics are not necessary.</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5. Evidence from examples is enough to prove what is true in mathematics.</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6. Some proving strategies in mathematics are not obvious.</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7. A statement in mathematics is proved true by using irrefutable rules of logic.</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>8. A valid proof in mathematics does not depend on other mathematical facts or axioms.</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>9. There is no gain in proving a mathematical situation false.</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10. Proofs contribute to the growth of mathematical knowledge.</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>11. A mathematical statement becomes valid only when two or more people can prove it.</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>12. In mathematics, one cannot prove that something is impossible.</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>13. An argument in a proof can either be direct or indirect.</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>14. There is no point in understanding someone else's proof.</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>15. The same mathematical statement can be proved using different methods of proof.</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>16. In mathematics, one counterexample is not enough to disprove something.</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>17. My proof writing skills increased directly with my mathematical experience.</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
APPENDIX C

IDEAL RESPONSES
### Researcher's choices for part 2 of the participant selection-survey

<p>| | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>A proof in mathematics is different from proof in other areas.</td>
<td>(SA)</td>
<td>A</td>
<td>NO</td>
</tr>
<tr>
<td>2.</td>
<td>A valid mathematical proof verifies the truth of a mathematical relationship.</td>
<td>(SA)</td>
<td>A</td>
<td>NO</td>
</tr>
<tr>
<td>3.</td>
<td>A mathematical relationship becomes valid after it has been proved.</td>
<td>(SA)</td>
<td>A</td>
<td>NO</td>
</tr>
<tr>
<td>4.</td>
<td>Proofs in mathematics are not necessary.</td>
<td>SA</td>
<td>A</td>
<td>NO</td>
</tr>
<tr>
<td>5.</td>
<td>Evidence from examples is enough to prove what is true in mathematics.</td>
<td>SA</td>
<td>A</td>
<td>NO</td>
</tr>
<tr>
<td>6.</td>
<td>Some proving strategies in mathematics are not obvious.</td>
<td>SA</td>
<td>(A)</td>
<td>NO</td>
</tr>
<tr>
<td>7.</td>
<td>A statement in mathematics is proved true by using irrefutable rules of logic.</td>
<td>(SA)</td>
<td>A</td>
<td>NO</td>
</tr>
<tr>
<td>8.</td>
<td>A valid proof in mathematics does not depend on other mathematical facts or axioms.</td>
<td>SA</td>
<td>A</td>
<td>NO</td>
</tr>
<tr>
<td>9.</td>
<td>There is no gain in proving a mathematical situation false.</td>
<td>SA</td>
<td>A</td>
<td>NO</td>
</tr>
<tr>
<td>10.</td>
<td>Proofs contribute to the growth of mathematical knowledge.</td>
<td>(SA)</td>
<td>A</td>
<td>NO</td>
</tr>
<tr>
<td>11.</td>
<td>A mathematical statement becomes valid only when two or more people can prove it.</td>
<td>SA</td>
<td>A</td>
<td>NO</td>
</tr>
<tr>
<td>12.</td>
<td>In mathematics, one cannot prove that something is impossible.</td>
<td>SA</td>
<td>A</td>
<td>NO</td>
</tr>
<tr>
<td>13.</td>
<td>An argument in a proof can either be direct or indirect.</td>
<td>(SA)</td>
<td>A</td>
<td>NO</td>
</tr>
<tr>
<td>14.</td>
<td>There is no point in understanding someone else's proof.</td>
<td>SA</td>
<td>A</td>
<td>NO</td>
</tr>
<tr>
<td>15.</td>
<td>The same mathematical statement can be proved using different methods of proof.</td>
<td>SA</td>
<td>A</td>
<td>(NO)</td>
</tr>
<tr>
<td>16.</td>
<td>In mathematics, one counterexample is not enough to disprove something.</td>
<td>SA</td>
<td>A</td>
<td>NO</td>
</tr>
<tr>
<td>17.</td>
<td>My proof writing skills increased directly with my mathematical experience.</td>
<td>(SA)</td>
<td>A</td>
<td>NO</td>
</tr>
</tbody>
</table>

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### Post Abstract-Math Students

<table>
<thead>
<tr>
<th>Participant</th>
<th>Total Absolute Difference Score</th>
</tr>
</thead>
<tbody>
<tr>
<td>Patty</td>
<td>7</td>
</tr>
<tr>
<td>Perry</td>
<td>10</td>
</tr>
<tr>
<td>Pam</td>
<td>20</td>
</tr>
<tr>
<td>Paul</td>
<td>14</td>
</tr>
</tbody>
</table>

### Abstract-Math Students

<table>
<thead>
<tr>
<th>Participant</th>
<th>Total Absolute Difference Score</th>
</tr>
</thead>
<tbody>
<tr>
<td>Amy</td>
<td>16</td>
</tr>
<tr>
<td>Alice</td>
<td>9</td>
</tr>
<tr>
<td>Adam</td>
<td>25</td>
</tr>
<tr>
<td>Art</td>
<td>20</td>
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</tbody>
</table>

### Discrete-Math Students

<table>
<thead>
<tr>
<th>Participant</th>
<th>Total Absolute Difference Score</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dan</td>
<td>25</td>
</tr>
<tr>
<td>Dean</td>
<td>10</td>
</tr>
<tr>
<td>Dave</td>
<td>19</td>
</tr>
<tr>
<td>Doug</td>
<td>16</td>
</tr>
</tbody>
</table>
APPENDIX E

INTERVIEW INSTRUCTIONS

AND

PARTICIPANT CONSENT FORM
Instructions to the interviewees

After getting acquainted with the student, recite these before interviews.

• I believe you have some idea about the purpose of these interviews as being part of the research for my Ph.D. dissertation. I hope you are OK with the presence of the camera and the taping of these interviews. The reason these interviews are taped is because I want to capture as completely as possible all the responses to the questions, so the taping can help speed up the interview by eliminating the need for taking notes. As you can see from the way the camera is pointed, it will only capture the letters and writings on the papers and not your identity. If at any point during the interview you feel that you’d rather not have your responses tape-recorded just let me know and I will stop the tapes.

• Please, understand that these interviews are not meant in any way to test your performance in mathematics. Their purpose is to gather data about your thought processes as you answer my questions. So, I would like you to speak your thoughts or think loud. Your responses are not meant to be classified as right or wrong but rather as how you personally approach or relate to the presented situations. I want you to be relaxed and comfortable, and not feel any pressure if you do not know how to answer a question. You can also let me know if you do not understand a question and I try my best to rephrase it for you.

• Furthermore, do not let the trivial or repetitious nature of my questions bother you. They are not meant to indicate at all whether you answered the questions satisfactorily or not. In other words, I may sometimes ask questions that you might feel that you have already answered correctly. Like I mentioned, they are not meant to be judgmental of the correctness of your answers but rather to make sure that every aspect of the interview was covered as planned, leaving no room to faulty interpretations of the context of the data.

Before we begin, I like you to read the contents of these pages (hand a copy of Interviewee Consent form required by the Institutional Review Board) and sign it if you agree. Let me know if anything is unclear or if you have any questions regarding this research.
SUBJECT INFORMATION AND CONSENT FORM

Title: Mathematics Education Research

Researcher:
Varoujan Bedros
The University of Montana
Dept. of Mathematical Sciences
32 Campus Drive
Missoula, MT 59812
Phone: (406) 243 4485
Email: vjbedros@yahoo.com

Study Director:
Libby Krussel, Ph.D. (Associate Professor)
The University of Montana
Dept. of Mathematical Sciences
32 Campus Drive
Missoula, MT 59812
Phone: (406) 243 4818, Fax: (406) 243 2674
Email: Krussel@mso.umt.edu

Purpose
You are being asked to take part in research interviews that will study students’ understanding of a mathematical concept. The purpose of these interviews is to learn how students verify the truth of mathematical statements and how they check proofs. The interviews are centered around why, how and what students think about certain mathematical situations. It is meant to capture your thought process while you reach conclusions without making you feel pressured if you do not have any answers. It is neither designed nor intended to investigate your competency in the field.

Procedures
• If you agree to take part in this research study you will be interviewed twice. Each interview will take about an hour.
• The interviews will take place within the premises of The Dept. of Mathematical Sciences.
• There is no formal preparation except before the second interview you will be required to work on a take-home task that takes about an hour. This task will help you to collect your thoughts before the second interview.
• The whole process will take about three hours of your time distributed over 2 or 3 days.

Payment for Participation
You will receive $30 at the end of the second interview for your entire participation.

Voluntary Participation/Withdrawal
• Your decision to take part in this research study is entirely voluntary.
• You may withdraw from the study at any time for any reason.

Confidentiality
• Your records will be kept private and will not be released without your consent except as required by law.
• Only the researcher and his faculty supervisor will have access to the data.
• Your identity will be kept confidential.
• If the results of this study are written in a scientific journal or presented at a scientific meeting, your name will not be used.
• The data will be stored in a locked file cabinet.
• Your signed consent form will be stored in a cabinet separate from the data.
• The audiotape and videotape will be transcribed without any information that could identify you. The tapes will be erased or destroyed after a year.
Discomforts
There are no apparent risks associated with this study, however it is known that for some students answering mathematical questions can cause them “mathematical anxiety”.

Benefits
Although you may not directly benefit from taking part in this study, your help will greatly be appreciated and your participation may reveal new ways of understanding students’ difficulties in mathematical proofs.

Compensation for Injury
Although we do not foresee any risk in taking part in this study, the following liability statement is required in all University of Montana consent forms.

In the event that you are injured as a result of this research you should individually seek appropriate medical treatment. If the injury is caused by the negligence of the University or any of its employees, you may be entitled to reimbursement or compensation pursuant to the Comprehensive State Insurance Plan established by the Department of Administration under the authority of M.C.A., Title2, Chapter 9. In the event of a claim for such injury, further information may be obtained from the University’s Claims representative or University Legal Counsel.
(Reviewed by University Legal Counsel, July 6, 1993)

Questions
• You may wish to discuss this with others before you agree to take part in this study.
• This consent form may contain words that are new to you. If you read any words that are not clear to you, please ask the person who gave you this form to explain them to you.
• If you have any questions about the research now or during the study contact the investigator or the study director whose names appear above.
• If you have any questions regarding your rights as a research subject, you may contact the Chair of the IRB through the Research Office at the University of Montana at 243-6670.

Subject’s Statement of Consent
I have read the above description of this research study. I have been informed of the risks and benefits involved, and all my questions have been answered to my satisfaction. Furthermore, I have been assured that any future questions I may have will also be answered by a member of the research team. I voluntarily agree to take part in this study. I understand I will receive a copy of this consent form.

Name of Subject _________________________

Subject’s Signature _________________________

Date _________________________

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APPENDIX F

TRUE / FALSE SITUATIONS
Interview 1
True / False Questionnaire

SITUATION 1

Instructions: Please, take a few minutes to think about the truth of the mathematical statement given below. Then answer if you think the statement is true or false by circling one.

STATEMENT:

Let $x$ be an integer. If $x^2$ is even, then $x$ is even.

a. TRUE  b. FALSE

Please, write down any argument that you think would convince you, as well as anyone interested in your argument and answer.
Interview 1
True / False Questionnaire

SITUATION 2

Instructions: Please, take a few minutes to think about the truth of the mathematical statement given below. Then answer if you think the statement is true or false by circling one.

STATEMENT:

For integers $a$ and $b$, if $a$ is not a multiple of 5 and $b$ is not a multiple of 5, then $a + b$ is not a multiple of 5.

a. TRUE  b. FALSE

Please, write down any argument that you think would convince you, as well as anyone interested in your argument and answer.
Interview 1
True / False Questionnaire
SITUATION 3

Instructions: Please, take a few minutes to think about the truth of the mathematical statement given below. Then answer if you think the statement is true or false by circling one.

STATEMENT:

Consider a triangle ABC. A straight line through the midpoint of the segment AB and parallel to segment AC bisects the third segment BC.

a. TRUE          b. FALSE

Please, write down any argument that you think would convince you, as well as anyone interested in your argument and answer.
**Interview 1**

**True / False Questionnaire**

**SITUATION 4**

**Instructions:** Please, take a few minutes to think about the truth of the mathematical statement given below. Then answer if you think the statement is true or false by circling one.

**STATEMENT:**

There exist three consecutive integers such that the cube of the middle integer is equal to the sum of the cubes of the outer integers

[In other words: For some three consecutive integers \((n - 1), n, (n + 1)\) the relation \((n + 1)^3 + (n - 1)^3 = n^3\) holds.]

a. TRUE       b. FALSE

Please, write down any argument that you think would convince you, as well as anyone interested in your argument and answer.
Interview 1
True / False Questionnaire

SITUATION 5

Instructions: Please, take a few minutes to think about the truth of the mathematical statement given below. Then answer if you think the statement is true or false by circling one.

STATEMENT:

There does not exist any quadrilateral with sides of lengths 2, 3, 5, 11

a. TRUE           b. FALSE

Please, write down any argument that you think would convince you, as well as anyone interested in your argument and answer.
Instructions: Please, take a few minutes to think about the truth of the mathematical statement given below. Then answer if you think the statement is true or false by circling one.

STATEMENT:

There are no distinct positive integers $a$, $b$, $c$ and $d$ such that $a^2 + b^2 + c^2 = d^2$.

a. TRUE  b. FALSE

Please, write down any argument that you think would convince you, as well as anyone interested in your argument and answer.
Interview 1
True / False Questionnaire

SITUATION 7

**Instructions:** Please, take a few minutes to think about the truth of the mathematical statement given below. Then answer if you think the statement is **true** or **false** by circling one.

**STATEMENT:**

For all real valued functions \( f, g \) and \( h \) on \( \mathbb{R} \) (the set of real numbers), if \( f(g(x)) = f(h(x)) \) then \( g(x) = h(x) \), for all real values of \( x \).

a. TRUE  
b. FALSE

Please, write down any argument that you think would convince you, as well as anyone interested in your argument and answer.
APPENDIX G

TRUE / FALSE
INTERVIEW 1 PROTOCOL
Interview 1 protocol
(True / False)
for
SITUATIONS 1, 2, 3, 6, 7

1. I would like you to explain how you arrived at your choice.

    *Probe*: What exactly do you want to know in this problem? What is your OBJECTIVE? Explain HOW you can reach that objective and thus your conclusion.

2. Can you absolutely be sure your approach or argument supports your answer?

    *If examples / counterexample / figures were used*:
    a) How many such examples/figures are enough to say the statement is true/false?
    b) What’s so special about your example/figure? Why does it render the statement [True/False]?
    c) *If stuck* How would you normally proceed in a situation when you could not find a counterexample / example?

3. Do you consider your argument to be a valid proof for your answer?

    *If YES…*
    a) Why? What (aspect) makes it a proof? Is it a generalized argument?
    b) Do you think there could be other argument or a proof that uses different approach than yours?

        *If YES…* What kind of approach (strategy or method) do you think it could use?
        *If NO…* Why?

    *If NO…*
    a) What would it take to make it a proof? Mention some aspects.
    b) Can you think of any particular approach (method or strategy) for proof?
Interview 1 protocol
(True / False)

SITUATION 4

1. I would like you to explain how you arrived at your choice.

Probe:
What exactly do you want to know in this problem? What is your OBJECTIVE? Explain HOW you can reach that objective and thus your conclusion.

2. [If the student expands / uses the equation]
What do you expect to get out of this approach?

3. Can you absolutely be sure your approach or argument supports your answer?

[If examples / counterexample were used]:
What's so special about your example? Why does it render the statement [True/False]?

4. Do you consider your argument to be a valid proof for your answer?

[If YES…]
a) Why? What (aspect) makes it a proof? Is it a generalized argument?

b) Do you think there could be a better argument or a proof that uses different approach than yours?

[If YES…] What kind of approach (or method of proof) do you think it would use?
[If NO… ] Why?

[If NO… ]
a) What would it take to make it a proof? Mention some of its aspects.

b) Can you think of any particular approach (method or strategy) for proof?
Interview 1 protocol
(True / False)

SITUATION 5

1. I would like you to explain how you arrived at your choice.

   **Probe:**
   What exactly do you want to know in this problem? What is your OBJECTIVE? Explain HOW you can reach that objective and thus your conclusion.

2. Can you absolutely be sure your approach or argument supports your answer?

   **[If false by drawing a quadrilateral]:**
   a) Can you absolutely be sure that such a quadrilateral can be drawn to scale?
   b) What's so special about this quadrilateral?

   **[If true by lack of counterexample]**
   Are you saying that nobody can draw such a quadrilateral? Why?

3. Do you consider your argument to be a valid proof for your answer?

   **[If YES…]**
   a) Why? What (aspect) makes it a proof?

   b) Do you think there could be a better argument or a proof that uses different approach than yours?

      **[If YES…]** What kind of approach (strategy or method) do you think it could use?
      **[If NO… ]** Why?

   **[If NO… ]**
   a) What would it take to make it a proof? Mention some aspects.

   b) Can you think of any particular approach (method or strategy) for proof?
Proof - Checking
SITUATION 1

Instructions: The proof below can be used to prove one of the five statements following it. Please read the proof very carefully and identify the statement you think it proves. Also, be prepared to elaborate on your choice during your next interview.

Pre-requisite fact used:
An even integer \( x \) can be written as \( x = 2k \), for some integer \( k \).

Proof:
Suppose \( x \) is non-odd (or even) integer then \( x = 2k \) for some integer \( k \).
Squaring both sides of this equation we get: \( x^2 = (2k)^2 = 2(2k^2) \)
Now \( 2(2k^2) \) is even. So, \( x^2 \) is non-odd (or even).

Thus; \textit{we have proved that: [please circle only one]}
1. If \( x \) is an odd integer then \( x^2 \) is odd.
2. If \( x^2 \) is an odd integer then \( x \) is odd.
3. If \( x^2 \) is an even integer then \( x \) is even.
4. If \( x^2 \) is an odd integer then \( x \) is non-odd.
5. None of the above.

Question 1: Please write a brief explanation for your choice.

Question 2: Discuss why the above proof does not prove the other choices.
Instructions: The proof below can be used to prove one of the five statements following it. Please read the proof very carefully and identify the statement you think it proves. Also, be prepared to elaborate on your choice during your next interview.

Prerequisite fact used:
Definition: An integer \( m \) is a multiple of another integer \( n \) if there exists an integer \( k \) such that \( m = k \cdot n \).

Proof:
Let \( a \) and \( b \) be integers such that:

- \( a \) is a non-multiple of 5 and \( b \) is a multiple of 5, then (by definition) there exists some integer \( k \) such that \( b = 5k \) ....... equation (1)
- Suppose now that \( a + b \) is a multiple of 5 then (by definition) there exists some integer \( j \) such that \( a + b = 5j \) .......equation (2)

Substituting equation (1) in (2) we get: \( a + 5k = 5j \Rightarrow a = 5(j - k) \) which by definition means \( a \) is a multiple of 5. But according to the assumption in the second line, this is impossible.

Thus; we have proved that: [circle only one]
1. If \( a \) is a multiple of 5 and \( b \) is a multiple of 5 then \( a + b \) is a multiple of 5.
2. If \( a \) is not a multiple of 5 and \( b \) is not a multiple of 5 then \( a + b \) is not a multiple of 5.
3. If \( a \) is not a multiple of 5 and \( b \) is a multiple of 5 then \( a + b \) is not a multiple of 5.
4. If \( a + b \) is a multiple of 5 and \( b \) is a multiple of 5 then \( a \) is not a multiple of 5.
5. None of the above.

Question 1: Please write a brief explanation for your choice.

Question 2: Discuss why the above proof does not prove the other choices.

Remember to be prepared to elaborate on your choice during your next interview.
Instructions: The following mathematical statement is followed by an argument, which is not necessarily, a complete or a correct proof. Please, read it carefully and answer the questions below.

Pre-requisite facts used:
1. A version of Euclid’s parallel postulate: In a plane, one and only one parallel line can be drawn through a point not on that line.
2. The segment formed by joining the midpoints of the two sides of a triangle is parallel to the third side.

STATEMENT:
Consider a triangle ABC. A straight-line \( k \) through the midpoint \( M \) of the segment \( AB \) and parallel to the segment \( AC \) (see Figure 1) bisects the segment \( BC \).

\[
\begin{align*}
\text{Figure 1} & \quad \text{Figure 2} \\
\begin{array}{c}
A \quad B \quad M \quad C \\
\end{array}
\end{align*}
\]

Proof:
Let \( N \) be the midpoint of \( BC \). Construct the segment \( MN \) (see Figure 2)
The second fact above implies that \( MN \) is parallel to \( AC \).
But the straight-line \( k \) also passes through \( M \) and is parallel to \( AC \) (as given), therefore from the first fact above the straight-line \( k \) and segment \( MN \) coincide.

Question 1: Please, circle the choice that you believe applies best here
The presented argument above:
1. Proves that the statement is true,
2. Shows some support that the statement is true,
3. Neither proves nor disproves the statement,
4. Shows some support that the statement is false, or
5. Proves that the statement is false.

Question 2: Please, write down a brief explanation for your choice and be prepared to elaborate on it during your next interview.
Instructions: The proof below can be used to prove one of the five statements following it. Please read the proof very carefully and identify the statement you think it proves. Also, be prepared to elaborate on your choice during your next interview.

Pre-requisite facts used:
\[(n + 1)^3 = n^3 + 3n^2 + 3n + 1 \quad \text{or} \quad (n - 1)^3 = n^3 - 3n^2 + 3n - 1\]

Proof:
Suppose there exist three consecutive positive integers \((n - 1), n, (n + 1)\), such that;
\[n^3 = (n + 1)^3 + (n - 1)^3\] [i.e., the cube of the middle integer is equal to the sum of the cubes of the outer integers.]
Expanding this equation (given facts), it becomes:
\[n^3 = (n^3 + 3n^2 + 3n + 1) + (n^3 - 3n^2 + 3n - 1)\] which simplifies into
\[n^3 + 6n = 0 \Rightarrow n(n^2 + 6) = 0 \Rightarrow n = 0, \text{(i.e., the middle integer is zero).}\]

Thus; we can use this proof to show that: [circle only one]
1. If \(n\) is an integer then \((n + 1)^3 + (n - 1)^3 \neq n^3\).
2. If \(n = 0\) then \((n + 1)^3 + (n - 1)^3 = n^3\).
3. There exist three consecutive positive integers such that the cube of the middle integer is equal to the sum of the cubes of the outer integers.
4. There do not exist three consecutive positive integers such that the cube of the middle integer is equal to the sum of the cubes of the outer integers.
5. None of the above.

Question 1: Please, write a brief explanation for your choice.

Question 2: Discuss why the above proof does not prove the other statements.

Remember to be prepared to elaborate on your choice during your next interview.
Instructions: The following mathematical statement is followed by an argument, which is not necessarily a complete or a correct proof. Please, read it carefully and answer the questions below.

Pre-requisite fact used:
Triangle Inequality: In a plane, the sum of the lengths of two sides of a triangle is greater than the length of the third side.

STATEMENT:
There exists a quadrilateral with side lengths 2, 3, 5 and 11 units.

Proof:
Assume there is such a quadrilateral ABCD with lengths of sides:
AB = 2 units, BC = 3 units, CD = 5 units and DA = 11 units as shown in Figure 1.

Figure 1

Draw the diagonal AC to form two triangles ABC and CDA (see Figure 2). If the length of diagonal AC is $x$ units, then from the triangle inequality (above fact): (1) $x + 5 > 11$ in triangle CDA, i.e., $x > 6$ and (2) $x < 2 + 3$ in triangle ABC, i.e., $x < 5$.
But that's impossible because we cannot have both $x > 6$ (case 1) and $x < 5$ (case 2).

Question 1: Please, circle the choice that you believe applies best here.
The presented argument above:
1. Proves that the statement is true,
2. Shows some support that the statement is true,
3. Neither proves nor disproves the statement,
4. Shows some support that the statement is false, or
5. Proves that the statement is false.

Question 2: Please, write down a brief explanation for your choice and be prepared to elaborate on it during your next interview.
Instructions: The following mathematical statement is followed by an argument, which is not necessarily a complete or correct proof. Please, read it carefully and answer the questions below.

Pre-requisite facts used:
1. Pythagorean Theorem: In any right triangle (Figure 1) \( a^2 + b^2 = h^2 \).
2. For all positive real numbers \( h \) and \( c \), there exist a positive real number \( d \), such that \( h^2 + c^2 = d^2 \).

STATEMENT:
There do not exist positive real numbers \( a \), \( b \), \( c \) and \( d \) such that:
\( a^2 + b^2 + c^2 = d^2 \) or \( a^2 + b^2 = d^2 - c^2 \).

Proof:
Assume for all positive real numbers \( a \), \( b \), \( c \) and \( d \) that \( a^2 + b^2 \neq d^2 - c^2 \).
Construct a right-angled triangle with sides \( a \) and \( b \), and hypotenuse \( h \) (Figure 1).
By Pythagorean theorem \( h^2 = a^2 + b^2 \), and from our assumption we deduce that:
\[ h^2 \neq d^2 - c^2 \Rightarrow h^2 + c^2 \neq d^2 \]
But this last relation cannot be true because by fact 2 above one can always construct a right-angled triangle with hypotenuse \( d \) and sides \( h \) and \( c \). (Figure 2)
This therefore leads to a contradiction.

Question 1: Please, circle the choice that you believe applies best here.
The presented argument above:
1. Proves that the statement is true,
2. Shows some support that the statement is true,
3. Neither proves nor disproves the statement,
4. Shows some support that the statement is false, or
6. Proves that the statement is false.

Question 2: Please, write down a brief explanation for your choice and be prepared to elaborate on it during your next interview.
Proof - Checking

SITUATION 7

Instructions: The following mathematical statement is followed by an argument, which is not necessarily a complete or a correct proof. Please, read it carefully and answer the questions below.

STATEMENT:
For all real valued functions $f$, $g$ and $h$ on $\mathbb{R}$ (the set of real numbers), if $f(g(x)) = f(h(x))$ then $g(x) = h(x)$, for all real values of $x$.

Proof:
Assume that we have $g(x) \neq h(x)$ for some real values of $x$.
We need to show that for every function $f$ we must have $f(g(x)) \neq f(h(x))$.
However, this is not possible because there exists a function $f(x) = x^2$ such that if $g(x) = x$ and $h(x) = -x$ then $f(x) = f(-x)$; because $x^2 = f(x) = f(-x) = (-x)^2$ for all real values of $x$.

Question 1: Please, circle the choice that you believe applies best here.
The presented argument above:
1. Proves that the statement is true,
2. Shows some support that the statement is true,
3. Neither proves nor disproves the statement,
4. Shows some support that the statement is false, or
5. Proves that the statement is false.

Question 2: Please, give a brief explanation for your choice and be prepared to elaborate on it during your next interview.
APPENDIX I

PROOF-CHECKING
INTERVIEW 2 PROTOCOL
Interview 2 protocol
Proof-checking

SITUATION 1

1. Did you find the steps of the proof clear? Did they make sense?
   [If NO....] Are there any parts that did not make sense to you? Why?
   [If YES...] Good!

2. Why do you think that the given argument shows statement [student's choice of statement] and not the other statements?
   Probe: I'd like you to explain the criteria you used to arrive at your choice by taking me through the process of each step in the proof.

3. [If the student's answer is 5] You said the given proof does not prove any of the given statements. What then do you think the given proof is showing? Why?
   [Look for a possible answer as: If x is even then x^2 is even].
   Probe: to see if the student could recognize the contrapositive approach used.
   So you don’t think any of the other four statements says the same thing (equivalent) as your statement?

4. Can you explain the role of the assumption and its consequence in the given proof?
   Probe: What is the method (or type) of argumentation used in this proof?

5. [If answer is not 2, produce the proof in the next page]
   a) I want you to look at this statement and its proof and let me know what you think.
   b) What is the type of (proof) argumentation used in this proof?
   c) [If 3, neither T/F] Does this proof show any of the previous 4 statements?
   d) [If all attempts fail] What is the contrapositive of the given statement?

6. [Compare and contrast arguments] In your previous interview about a similar situation, you answered [True / false, choose student's answer].
   • [If a different argument was used]
     Would you have used a similar argument with k's as in this (given) proof to show: If x^2 is an even integer then x is even?
     [If YES...] How? Can you write a proof? Would you change anything?
     [If NO...] Why?
   • [If a similar argument using k's was used]
     You have used a similar argument to prove: If x^2 is an even integer then x is even. How do you explain your choices and the proofs in both these cases?
SITUATION 1
Alternate statement

STATEMENT:

Let \( x \) be an integer. If \( x^2 \) is even, then \( x \) is even.

Pre-requisite facts used:
1. An odd integer \( x \) can be written as the sum of an even integer \( 2k \) and 1, i.e., \( x = 2k + 1 \), for some integer \( k \).
2. \( (a + b)^2 = a^2 + 2ab + b^2 \).
3. The sum of even integers is even.

Proof:
Suppose \( x \) is an odd (non-even) integer then \( x = 2k + 1 \) for some integer \( k \) (fact 1).
Squaring both sides, we get: \( x^2 = (2k + 1)^2 = 4k^2 + 4k + 1 \) (fact 2).
Now \( 4k^2 \) and \( 4k \) are even \( \Rightarrow \) \( 4k^2 + 4k \) is even (fact 3)
Therefore, \( x^2 = 4k^2 + 4k + 1 \) is odd (or non-even).

The presented argument above:
1. Proves that the statement is true,
2. Shows some support that the statement is true,
3. Neither proves nor disproves the statement,
4. Shows some support that the statement is false, or
5. Proves that the statement is false.
Interview 2 protocol
Proof-checking

SITUATION 2

1. Did you find the steps of the proof clear? Did they make sense?
   [If NO... ] Are there any parts that did not make sense to you? Why?
   [if YES... ] Good!

2. Why do you think that the given argument shows statement [student's choice of statement] and not the other statements?
   Probe: I'd like you to explain the criteria you used to arrive at your choice by taking me through the process of each step in the proof.

3. [If the student's answer is 5]
   You said the given proof does not prove any of the given statements. What then do you think it is showing? Why?
   [Look for a possible answer as: If \( a + b \) is a multiple of 5 and \( b \) is a multiple of 5 then \( a \) is a multiple of 5]
   Probe: So you don't think any of the other four statements says the same thing (equivalent) as your statement?

4. [Probe to see if the student's reasoning is based on a direct proof]
   One of your classmates said that the above proof shows that:
   \[
   \text{If } a + b \text{ is a multiple of 5 and } b \text{ is a multiple of 5 then } a \text{ is a multiple of 5.}
   \]
   Do you agree? [If YES... ] How does the proof show this statement?
   [If NO... ] Why?

5. Can you explain the role of the assumption and its consequence in the given proof?
   Probe: What is the method (or type) of argumentation used in this proof?

6. The last line in this proof refers to some impossible case, what is it and what is its consequence?

7. [Probe to see if the student can relate the process of finding a counterexample to that of showing contradiction]
   In your previous interview about a similar situation you answered [True / false, choose student's answer] using [counterexample/examples or some proof]
   a) Do you see any connection between the true/false statement there and your choice of statement here in this situation?
      [If YES ... ] What is it? Explain.
      [If NO ... ] Why?
   b) Would you have used the given proof or something similar to prove/disprove the true/false statement? How?
Interview 2 Protocol
Proof-checking

SITUATION 3

1. Did you find the steps of the proof clear? Did they make sense?
   [If NO...] Are there any parts that did not make sense to you? Why?
   [If YES...] Good!

2. Were you familiar with the idea of Euclid’s parallel postulate, or did you just learn about it here?

3. Why do you think that the given argument shows that [student's choice of answer] and not the other cases?
   Probe: I’d like you to explain the criteria you used to arrive at your choice by taking me through the process of each step in the proof.

4. [If the student's answer is 3]
   Do you think that the proof is wrong?
   [If YES...] Do you know how to fix it?
   [If NO...] What then did this argument attempt to show?
   Probe: What it might take to validate the statement as either true or false?
   Would you introduce any changes in the proof to make it work?

5. Do you think this proof used any (implicit/explicit) assumption?
   [If YES...] What is it and what are its role and consequence?
   [If NO...] Why do you think so?

6. Can you explain the role of Euclid’s parallel postulate in this proof?
   Probe:
   a) What is the type or the method of argumentation used in this proof?
   b) Do you think Euclid’s parallel postulate contradicts anything in this problem?

7. Here is another argument in support of the statement [produce the proof in the next page]. Please, read it and let me know what you think.
   Probe: a) Is this proof the same as the original one?
   b) Do they use the same approach / method?
   c) Can you explain the similarities or differences.

8. In your previous interview about this situation, you answered [True / false, choose student's answer]
   • [If student's answers were inconsistent]
     How do you explain your different choices for both these situations?
   • [If student's answers were consistent]
     Is there a difference between your argument there and the one presented here?
Pre-requisite facts used:
1) A version of Euclid's parallel postulate: In a plane, one and only one parallel line can be drawn through a point not on that line.
2) The segment formed by joining the midpoints of the two sides of a triangle is parallel to the third side.

STATEMENT:
Consider a triangle ABC. A straight-line $k$ through the midpoint M of the segment AB and parallel to the segment AC (Figure 1) bisects the segment BC.

Proof:
Let the straight-line $k$ be parallel to $AC$ and pass through the midpoint M of $AB$ (see Figure 1).
Now, assume the opposite that the straight-line $k$ does not bisect $BC$.
Let N be the midpoint of $BC$. Construct the segment MN (Figure 2).
The second fact above implies that $MN$ is parallel to $AC$.
But the straight-line $k$ also passes through M and is parallel to $AC$ (as given), therefore from the first fact above the straight-line $k$ and segment MN coincide. Thus, our initial assumption is incorrect.

Question: Please, answer by circling the choice that you believe applies best here
The presented argument above:
1. Proves that the statement is true,
2. Shows some support that the statement is true,
3. Neither proves nor disproves the statement,
4. Shows some support that the statement is false, or
5. Proves that the statement is false.
Interview 2 protocol
Proof-checking

SITUATION 4

1. Did you find the steps of the proof clear? Did they make sense?
   [If NO:] Are there any parts that did not make sense to you? Why?
   [If YES:] Good!

2. Why do you think that the given argument shows statement number [student's choice of answer] and not the other statements?
   Probe: I'd like you to explain the selection process you used to reach your conclusion by taking me through each step in the proof.

3. Does this proof assume anything and if so why? What is its consequence in the proof?
   Probe: What is the type (or method) of argumentation used in this proof?

4. [If the student's answer is 5]
   You said the given proof couldn't be used to show any of the above statements, what then did you think it was trying to show? Why?
   [Look for a possible answer as: If \((n + 1)^3 + (n - 1)^3 = n^3\) then \(n = 0\)].
   Probe: So you don't think any of the other four statements follows from your statement?

5. a) One of your classmates said that this proof could be used to show that:
   \(If (n + 1)^3 + (n - 1)^3 = n^3\) then \(n = 0\). Do you agree?
   [If YES:] How is this statement different from your choice?
   What would be the method of proof that shows this statement?
   [If NO:] Why?

   b) Another student said that the above proof could be used to show that:
   \(If n \text{ is a non-zero number} then (n + 1)^3 + (n - 1)^3 \neq n^3\). Do you agree?
   [If YES:] How is this statement different from your choice?
   What would be the method of proof that shows this statement?
   [If NO:] Why?

6. In your previous interview about a similar situation you answered [True / false, choose student's answer] using [counterexample/examples or some proof]
   Would you have used the given proof or something similar to prove/disprove the true/false statement? Would you change anything? What?
Interview 2 protocol
Proof-checking

SITUATION 5

1. Did you find the steps of the presented argument clear? Did they make sense?
   [If NO...] Are there any parts that did not make sense to you? Why?
   [If YES...] Good!

2. Were you familiar with the idea of triangle inequality, or did you learn about it here?

3. Why do you think that the given argument shows [student's choice of answer] and not the other cases?
   \textit{Probe:} I'd like you to explain the criteria you used to arrive at your choice by taking me through the process of each step in the proof.

4. [If the student's answer is 3]
   When you answered that the presented argument neither proved nor disproved the statement, was it because it contained a mistake?
   [If YES...] Do you know how to fix it?
   [If NO...] What then did this argument attempt to show?
   \textit{Probe:} What it might take to validate the statement as either true or false?
   Would you introduce any changes in the proof to make it work?

5. Can you explain the role of the assumption and its consequence in the presented argument?
   \textit{Probe:} What is the type or method of argumentation used here?

6. Here is another argument in support of the given statement, \textit{[produce the proof in the next page]}, please read it and let me know what you think.
   \textit{Probe:} Do you still think the original argument showed your initial choice?
   Explain the best that you can the difference between the two arguments.
   Do you now think there exist a quadrilateral or not?

7. In your previous interview about this situation, you answered \textit{[True / false, choose student's answer]},
   \begin{itemize}
   \item [If student's answers to true/false and this task were \textit{inconsistent}] How do you explain your different choices for both these situations?
   \item [If student's answers to true/false and this task were \textit{consistent}] Is there a difference between your argument there and the one presented here? Would you have used a similar argument there? Why?
   \end{itemize}
Pre-requisite fact used:
Triangle Inequality: In a plane, the sum of the lengths of two sides of a triangle is greater than the length of the third side.

STATEMENT:
There exists a quadrilateral with side lengths 2, 3, 5 and 11 units.

Proof:
We show the existence by drawing such a quadrilateral ABCD with lengths of sides: AB = 2 units, BC = 3 units, CD = 5 units and DA = 11 units (Figure 1), and by finding possible bounds for the length of its diagonal BD.

[Diagram of quadrilateral with labels A, B, C, D and side lengths 2, 3, 5, 11]

Draw the diagonal BD to form two triangles ABD and BCD (see Figure 2). If the length of diagonal BD is x units, then from the triangle inequality (above fact):
1. $5 < x + 3$ in triangle BCD, i.e., $2 < x$; and
2. $x < 2 + 11$ in triangle ABD, i.e., $x < 13$.

Combining cases (1) and (2) we get: $2 < x < 13$ which shows that the diagonal BD can have any length between 2 and 13 units.

Question: Please, circle the choice that you now believe applies best for the given statement.
The presented argument above:
1. Proves that the statement is true,
2. Shows some support that the statement is true,
3. Neither proves nor disproves the statement,
4. Shows some support that the statement is false, or
5. Proves that the statement is false.
Interview 2 Protocol
Proof-checking

SITUATION 6

1. Did you find the steps of the presented argument clear? Did they make sense?
   [If NO...] Are there any parts that did not make sense to you? Why?
   [If YES...] Good!

2. Why do you think that the given argument shows that [student's choice of answer]
   and not the other cases?
   Probe: I'd like you to explain the selection process you used by taking me
   through each step in the argument.

3. [If the student's answer is 3]
   When you answered that the presented argument neither proved nor disproved the
   statement, was it because you found a mistake in it?
   [If YES...] Do you know how to fix it?
   [If NO...] What then did this argument attempt to show?
   Probe: What it might take to validate the statement as either true or false?
   Would you introduce any changes in the proof to make it work?

4. Can you explain the role of the assumption and its consequence in this argument?
   Probe: What is the type or method of argumentation used here?

5. What is the contradiction at the end trying to contradict?
   Probe: What does the result of this contradiction show? What is its purpose?
   Where does the contradiction come from?

6. In your previous interview about the same situation, you answered [True / false, 
choose student's answer]
   • [If student's answers were inconsistent]
     How do you explain your different choices for both these situations?
   • [If student's answers were consistent]
     Is there a difference between your argument there and the one presented here?
     Would you have used a similar argument there? Why?
Interview 2 Protocol
Proof-checking
SITUATION 7

1. Did you find the steps of the presented argument clear? Did they make sense?
   [If NO...] Are there any parts that did not make sense to you? Why?
   [If YES...] Good!

2. Why do you think that the given argument shows that [student's choice of answer] and not the other cases?

   Probe: I'd like you to explain the selection criteria you used by taking me through each step in the argument.

3. [If the student's answer is 3]
   When you answered that the presented argument neither proved nor disproved the statement, was it because you found a mistake in it?
   [If YES...] Where? Do you know how to fix it?
   [If NO...] What then did it attempt to show?
   Probe: What it might take to validate the statement as either true or false?
   Would you introduce any changes in the proof to make it work?

4. Can you explain the role of the assumption in this proof?

   Probe: Why is it followed by the need to show that \( f( g(x) ) \neq f( h(x) ) \).

5. What is attained (the conclusion) in the end?

   Probe: How does the conclusion relate to the first part of the argument?

6. In your previous interview about the same situation, you answered [True/false, choose student's answer]
   • [If student's answers were inconsistent]
     How do you explain your different choices for both these situations?

   • [If student's answers were consistent]
     Is there a difference between your argument there and the one presented here? Would you have used a similar argument? Why?
### RESPONSES TO INTERVIEW 1

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<td>Sit 6</td>
<td>F</td>
<td>?</td>
</tr>
<tr>
<td>Sit 7</td>
<td>F</td>
<td>F</td>
</tr>
</tbody>
</table>

**Notes:**
- Question mark "?" Indicates "no sure answer".
- T? or F? indicate "a guess".

### RESPONSES TO INTERVIEW 2

<table>
<thead>
<tr>
<th>Post Abstract-Math</th>
<th>Abstract-Math</th>
<th>Discrete-Math</th>
</tr>
</thead>
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<tr>
<td>Patty</td>
<td>Paul</td>
<td>Perry</td>
</tr>
<tr>
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</tr>
<tr>
<td>Alt1</td>
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<td>3</td>
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<td>Alt3</td>
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<tr>
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<tr>
<td>Alt5</td>
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</tr>
<tr>
<td>Sit 7</td>
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<td>5</td>
</tr>
</tbody>
</table>

**Notes:**
- 3(1) indicates "initial response 3 then changed to 1".
- Dash mark "—" indicates "no response required".

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