[The Mathematics Enthusiast](https://scholarworks.umt.edu/tme)

Volume 4 | Number 1 Article 1

2-2007

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Recommended Citation

Bjuland, Raymond (2007) "Adult Students' Reasoning in Geometry: Teaching Mathematics through Collaborative Problem Solving in Teacher Education," *The Mathematics Enthusiast*: Vol. 4 : No. 1 , Article 1.

DOI: https://doi.org/10.54870/1551-3440.1056

Available at: https://scholarworks.umt.edu/tme/vol4/iss1/1

Adult Students' Reasoning in Geometry: Teaching Mathematics through Collaborative Problem Solving in Teacher Education

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Abstract: *This article reports research that is concerned with pre-service teachers*[2] *working collaboratively in a problem-solving context without teacher involvement. The aim is to focus on the students' heuristic strategies employed in the solution process while working on two problems in geometry. Two episodes from the dialogues in one group of students with limited mathematical backgrounds have been chosen to illustrate some mathematical movement throughout the group meetings, from working with the first problem to working with the second one. The findings reveal that three categories of strategies, visualising, monitoring, and questioning, play an important role in order to make progress with the problems. As a preparation for working on the two problems, metacognitive training in combination with cooperative learning was introduced to the students throughout a month. The study indicates that these critical components in the design of the instructional context stimulated the students with limited background in mathematics to improve their problem-solving skills. The analysis has particularly focused on the important role of the process writer that provokes the mathematical discussion by generating utterances categorised as looking-back questions. By recapitulating the solution process or the last idea introduced in the dialogue, the process writer stimulated the establishment of a common ground for the further discussion. The article also deals with issues of teacher involvement in students' mathematical discussions in collaborative working groups.*
Keywords: geometry; teacher education; mathematical reasoning; heuristic strategies; collaborative problem solving; dialogical approach

1. Introduction

This study focuses on observation, analysis and interpretation of the mathematical discussion of one group of students working on two geometry problems in a collaborative problem-solving context without teacher intervention. From a socio-cognitive perspective, I am particularly interested to illustrate how the students express elements of mathematical reasoning in group

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[2] Pre-service teachers are students who intend to pursue a teaching career. I will refer to the pre-service teachers as 'students'.

dialogue when they are in the process of solving the problems. Goos et al. (2002) emphasise that research on small-group learning in mathematics has revealed few insights into how students think and learn while interacting with peers. These authors suggest that research should focus on the potential for small-group work in order to develop students' mathematical thinking and problem solving skills. In Bjuland (2004), I have illustrated how one group of students reflected on their collaborative small-group experience as learners of mathematics and on their future role as teachers of mathematics. The outcome of the analysis revealed that reflections on their own learning processes stimulated reflections on their preparation for the profession of teaching.

In this article, two episodes from the dialogues of one group of students are analysed in order to illustrate important aspects from their process as learners of mathematics (the same group as reported from in Bjuland, 2004). The students in this group have limited mathematical backgrounds. In the first episode, chosen from the student discussion of the first geometry problem, the students attribute meaning to the concept of distance from a point to a line. The second episode, taken from the second problem, illustrates how the students are able to find a solution to a complex geometry problem. Based on the analysis, it is not possible to conclude that the students have developed in their reasoning process. However, there has been some mathematical movement throughout the group meetings, from working with the first problem to working with the second one. This study indicates how it is possible within a particular instructional context for students in teacher education with limited background in mathematics to improve their problem-solving skills.

Based on three models of problem solving (Polya, 1945/1957; Mason et al., 1982; Mason and Davis, 1991; Borgersen, 1994), reasoning can be defined as five interrelated processes of mathematical thinking, categorised as *sense-making*, *conjecturing*, *convincing*, *reflecting*, and *generalising* (Bjuland, 2002). Instead of focusing on these overall processes of reasoning, this article focuses on how the students express their reasoning through their ways of employing heuristic strategies in the solution process. One central aim of this study is therefore to contribute to the understanding of how students are able to use constructive heuristic strategies in their solution process. For instance, by identifying the strategies of visualising, monitoring, and questioning, we gain insight into the students' ways of approaching and making sense of the problems given, and into their different attempts at finding possible solutions.

At a very general level, I am interested in what happens when you let a group of students in teacher education discuss mathematics in a collaborative problem-solving context without getting any input from their teacher. I am aware of the fact that collaborative work in a problem-solving context without teacher involvement is a radical position to take with respect to the teaching and learning of mathematics. In one respect it is natural to have a teacher visiting a group in order to stimulate the students' solution process. However, I want to suggest that students in teacher education can benefit from learning mathematics through collaborative problem solving without teacher involvement within a particular instructional context (see methodology section). This perspective is also exemplified by the study carried out by Borgersen (1994, 2004) with adult students working on problems in geometry in small groups. Throughout the analyses I am also concerned with the question: when is it appropriate for a teacher to become involved in the group discussion?

Based on these introductory considerations, the following research question has been formulated: Which heuristic strategies can be identified in the dialogues of a group of adult students when working on two geometry problems without teacher intervention? Based on this question it is natural to focus on whether these strategies stimulate mathematical progress in the students' solution process.

2. Theoretical background

Inspired by Barkatsas and Hunting (1996), the collaborative mathematical problem-solving process is defined as the cognitive, metacognitive, socio-cultural and affective process of figuring out how to solve a problem. My major concern for the analysis of the dialogues is the students' reasoning process, the cognitive and metacognitive component of the problem-solving process. In this article, elements of reasoning are specifically explored in the dialogues by the identification of the students' heuristic strategies employed in the solution process. More specifically, this study focuses on four groups of heuristic strategies categorised as *visualising*, *monitoring*, *questioning*, and *logical strategy*.

2.1 Heuristic strategies

In general, the concept of strategy is defined as 'a plan you use in order to achieve something' (Collins Cobuild Dictionary, 1993, p. 791). I am concerned with a particular branch of strategies well-known in problem-solving research based on the work of Polya (1945/1957). For the purpose of the analysis of the dialogues, the terms *problem-solving strategies* and *heuristic strategies* will be used interchangeably.

When students are translating a mathematical text into a visual representation by drawing an auxiliary figure or making a modification of a figure, they employ the strategy of visualising. Drawing a figure is widely accepted as a useful strategy in order to generate and manage the global picture of a problem situation (Duval, 1998; Mason and Davis, 1991; Polya, 1945/1957). According to Mason and Davis (op. cit), a figure provides structure, and encourages tying down thoughts and conjectures that buzz around in the pupils' minds. Duval (1998) claims that in a geometrical figure there are more constituent gestalts and more possible subconfigurations than the ones explicitly named in theorems.

In this study, the strategy of monitoring is related to the students' metacognitive activity throughout their problem solving, when they are concerned with monitoring their solution process and when they look back and consider a convincing argument. In the analysis, I am particularly concerned with the monitoring questions that stimulate the solution process. A monitoring question is identified both as a monitoring strategy and as a questioning strategy.

In the Socratic dialogues, one learns that it is more difficult to ask questions than to answer them. If a person is engaged in a dialogue only to prove himself right and not to gain insight, asking questions will seem to be easier than answering them (Gadamer, 1989). However, in order to be able to ask a question, one must wish to know, and that means knowing that one does not know. Asking mathematical questions is vital, both as part of the presentation of mathematics and in the context of problems for students (Mason, 2000). Mason claims that questions arise as pedagogic instruments in classrooms both for engaging students in and assessing students' grasp of ideas. It is the disturbance represented by the sudden shift of one's own attention that prompt a question

(Mason, op. cit.). When students work on problems in a small group without teacher intervention, they initiate the questions and ideas themselves. It is therefore interesting to categorise different types of questions (the strategy of questioning) that are constructive for the solution process.

In this study, the logical strategy is related to the students' attempts at building up a logical cause effect argument. In the analyses of the dialogues, I therefore focus on the students' use of *if-then structures* in the mathematical discussion.

2.2 A brief review of literature on problem solving

During the 1980s, research focused on case studies and interview research using thinking-aloud protocols to try to ascertain the distinctions in approaches to problem solving between successful and unsuccessful problem solvers, so-called experts and novices (Lester, 1994).

The findings of Schoenfeld (1985, 1992), based on empirical material of more than one hundred videotapes of college and high-school students, working on unfamiliar problems, indicate that typical students spend the full 20 minutes allocated for the problem session in unstructured exploration. Roughly 60% of the solution attempts have a solution-profile in which the students read the problem and quickly choose an approach to it, and pursue it in that direction without reconsidering or reversing it. Schoenfeld (1992) shows a time-line graph of a solution process for a typical student and of a mathematician respectively, attempting to solve a non-standard problem. While the typical student spent most of the time in unstructured exploration or moving quickly into implementation of the problem, the expert spent more than half of his allotted time trying to make sense of the problem. The mathematician did not move into implementation until she/he was sure she/he was working in the right direction.

Another study carried out by Goos and Galbraith (1996) confirms the fact that students do not spend much time on making sense of an unfamiliar problem. These authors focused on the nature and quality of the interactions between sixteen-year-old secondary school students working collaboratively on application problems. The structure of the students' problem- solving attempts showed an immediate jump into implementation after an initial quick reading and analysis of the problem.

Carlson and Bloom (2005) used a multidimensional problem-solving framework with four phases (orientation, planning, executing, and checking) in order to describe the problem-solving behaviours of 12 mathematicians as they worked on four mathematical problems. The effectiveness of these experts in making intelligent decisions, leading to mathematical solutions stemmed from their ability to draw on their various problem-solving attributes (conceptual knowledge, affect, heuristics, and monitoring) throughout the problem-solving process.

These studies from individual problem solving of mathematicians (Carlson and Bloom, op. cit), from non-experienced problem solvers (Schoenfeld, op. cit), and from interactions between students working collaboratively (Goos and Galbraith, op. cit), show that metacognitive awareness is an important element in order to succeed in solving a problem. In his musings about mathematical problem-solving research, Lester (1994) also confirms that metacognition was seen as the driving force in problem solving. He claims that research is only just beginning to understand the degree to which metacognition influences problem-solving activity. However, Lester focuses on some results that have come to be generally accepted. One of these results shows that effective metacognitive activity during problem solving is quite difficult. It requires knowing not only what and when to monitor, but also how to monitor. The result shows that it is a difficult task to teach students how to monitor their behaviour.

During the 90s, research in mathematics education focused on peer interaction in small groups as an important issue (Cobb, 1995; Healy et al., 1995; Hoyles et al., 1991; Kieran and Dreyfus, 1998). Following Brodie (2000), I think that such a context could be crucial as an arena for learning since peer interaction is seen to provide support for the construction of mathematical meaning by students. It also allows more time for student talk and activity. According to Farr (1990), the dynamics of a three-person group changes dramatically compared to a two-person group, since it is possible to form coalitions in the former size of group, but not in the latter. As far as my five-person group is concerned, the dynamics of the group are quite complex since the perspective of every single student could be brought into the mathematical discussion.

Using Vygotskian terminology, Forman (1989) names three conditions needed for a Zone of Proximal Development, created by collaborating students, to be effective: Students must have mutual respect for each other's perspective on the task, there must be an equal distribution of knowledge, and there must be an equal distribution of power. According to Hiebert (1992), when students express themselves, they reveal different ways of thinking, ways that can be acquired by other members of the group. By expressing ideas and defending them in the face of others' questions, and by questioning others' ideas, the students are forced to deal with disagreements. I assert that this may stimulate the students to elaborate, clarify, and maybe reorganise their own thinking.

Goos et al. (2002) carried out a three year study concerning patterns of student-student social interaction that mediated metacognitive activity in senior secondary school mathematics classrooms. Analyses of dialogues of small group problem solving focused on how a collaborative zone of proximal development could be established through interaction between peers of comparable expertise. Unsuccessful problem solving was related to the students' poor metacognitive decisions during the problem-solving process and their lack of critically challenging each other's thinking. Successful outcomes were revealed if students challenged and rejected unhelpful ideas and actively stimulated constructive strategies.

It is necessary to ask, critically, whether it is sufficient to place students in collaborative groups in order to enhance mathematical reasoning. More specifically as Stacey (1992) puts the question: Are two heads better than one? In a study carried out by Stacey (op. cit) in which Year 9 students (average age 14 years) were given a written test of problem solving, groups of students did not acquire better results than individual student performance while solving the same problems. To investigate why this happened based on analyses from the dialogues of students solving problems in groups, Stacey (op. cit) observed that many ideas were brought into the discussions, but the students had difficulties in selecting those which would be effective. Constructive ideas were rejected in favor of simpler, but erroneous, ideas.

Other researchers have also put emphasis on the following question: Is learning mathematics through conversation as good as they say? (Sfard et al., 1998). A study carried out by Sfard and Kieran (2001), revealed that the students' communication was ineffective when two 13-year-old boys were learning algebra. Based on empirical material taken from a two-month-long series of group interactions, the findings indicated that the collaboration seemed unhelpful and it lacked the expected synergetic quality.

A study focused on the enhancement of mathematical reasoning in eighth-grade classrooms (384 students) by investigating the effects of four instructional methods on students' reasoning and metacognitive training (Kramarski and Mevarech, 2003). This study indicates that students need metacognitive training in combination with cooperative learning in order to enhance mathematical reasoning. These aspects are also critical elements in my design of the instructional context introduced in the methodology part.

3. METHOD

The data corpus of the study has been collected at a teacher-training college in Norway. In this particular year, 105 students attended the four-year teacher education programme in order to become teachers in primary (elementary) school or in lower secondary school. All the students had to participate in a problem-solving course in geometry in their first semester as a part of the mathematics programme. This course consisted of three parts: a first part of teaching over a month in September, a second part of small-group work without teacher involvement over three weeks in October, and a third part of teaching in which problems from the second part would be discussed and elaborated in some plenary lessons. During the research project I was a teacher in the first part of the problem-solving course and a researcher in the second one. In fact we were two teachers who carried out the teaching programme. During the first part it was important for us to reflect on the teaching at the end of each day. Another crucial aspect of these meetings was also to discuss and reflect on my two different roles during the project.

In the second part, I was only concerned with the observation of three groups of students, focusing on their problem-solving process with two geometry problems. The empirical material is based on the small-group work without teacher involvement from this period. The data comprises fieldnotes and audio transcripts of four group meetings (8 lessons) in each group, and 21 group reports from this collaborative small-group work. I chose not to use video in my data collection procedure since I believe that the pressure on the students working under video observation might influence the conversation more than audio recording.

3.1 Subjects

At the beginning of the semester, the students were divided into groups of five in alphabetical order by the administration at the college (21 groups). This means that I could not influence how the groups were arranged with regards to variables such as sex, mathematical attainment and so on. Three groups were randomly chosen for observation, and I am here concerned with one of these groups.

From table 1 below it can be seen that four of the students have only attended the compulsory course (1MA), and two of those students have low marks from this course. I do not know what the students have done in the period between upper secondary school (students graduate when they are 19) and their beginning at the teacher-training college. However, from the group reflection at the end of the fourth meeting, I know that Liv has not done any mathematics for five years (see Bjuland, 1997, p. 194).

Name	**Age**	**Gender**	**Mathematical background**
Unn	20	F	1MA (H)
Mia	21	F	1MA (M)
Gry	21	F	1MA (L)
Liv	22	F	1MA (L)
Roy	24	M	1MA (M), 2MN (M), 3MN (M)

Table 1: Background knowledge of the subjects

1MA: Compulsory course in the first year in upper secondary school.

2MN, 3MN: Voluntary courses in the second and third year in upper secondary school, preparing for further studies in natural sciences.

(L), (M), (H): Represents categories of grading: (L): Low marks, (M): Middle-average marks, (H): High marks

From the background knowledge about the subjects, it is possible to argue that Roy could play a dominant role in the group discussion. He is the only male student, he is the oldest, and he has also the best mathematical background. However, from the group reflection at the fourth meeting, Roy says that he had limited background knowledge in geometry when he started at the teacher-training college. Later in the same group reflection (see Bjuland, 1997, p. 201), he makes clear that he could hardly remember how to construct a perpendicular. Based on the background knowledge about the subjects, I have chosen to categorise this as a group of students with limited mathematical background.

3.2 The instructional context

In Bjuland (1997) there is a detailed description of the 28 lessons of mathematics that were designed for the first part of the problem-solving course as a preparation for the collaborative small-group work in the second part. It is important to give a brief outline of this period since this provides the background for the analysis of the students' dialogues.

The aim of this teaching part was to focus on basic classical geometry, prepare students to work on problems in small groups, and stimulate students to experience mathematics as a process, as described by Borgersen (1994). We focused particularly on geometrical concepts that were relevant for the small-group work without teacher involvement in the second part, for instance concepts related to circles, similarity, cyclic quadrilaterals, and the relationship between the angle at the centre and the angle at the circumference (Thales' theorem).

Some advice introduced by Johnson and Johnson (1990) on how cooperative learning can be used in mathematics was also presented in order to focus on the effect of group dynamics. These authors suggest the following basic elements in their standard for cooperative learning: 1) *Positive interdependence* (group members should 'sink and swim together' to reach a common goal); 2) *Promotive interaction* (the participants assist, help, support and encourage each other's effort to achieve); 3) *Individual accountability* (students cannot 'hitchhike' on the work of others, they are held responsible for their contribution to accomplishing the goal); 4) *Interpersonal and small group skills* (students should be taught in social skills which include leadership, decision-making, communication, trust-building and conflict-management skills); 5) *Group processing* (students reflect on their work and decide on ways to improve effectiveness).

We were particularly concerned with the fifth element since we used this in combination with metacognitive training. The students were asked to write about how the problem-solving process developed, how ideas or strategies emerged in the dialogue, and how every idea and suggestion was introduced or presented. These experiences were then brought into a plenary discussion afterwards. For instance, in the notes it was common that the students wrote something like: 'suddenly one of the group members came up with an idea and we solved the problem'. In the lectures, we stressed the importance of writing down explicitly which idea they had. We were aware of the fact that monitoring activity during problem solving is quite difficult (Lester, 1994). However by focusing on this process writing throughout one month, we aimed at giving the students the opportunity to focus on when to monitor and how to monitor during the problem-solving process.

The students were also introduced to two models that illustrate different stages in a problem-solving process. By introducing Polya's four-stage model (Polya 1945/1957), and an expansion of this model in seven main steps (Borgersen 1994), the students gained some insight into the various elements in a mathematical problem-solving process. An ongoing problem *Best place on Stadium*, formulated by Borgersen (1994) was used to illustrate the dynamic and cyclic stages of these models. The problem is adapted to a well-known everyday context for the students: 'As a student in Bergen, you would like to go to a football match in order to watch the local women's team Sandviken play against Trondheims-Ørn. If you have a ticket for the long side of the field, which place is the best for watching the goal scored by your home team?' (translation of the original Norwegian text, see Bjuland, 1997, p. 61).

In the plenary lectures we also presented dialogues from a mathematical classroom (Johnsen, 1996) similar to the classroom of Lampert (1990) in which the pupils learned mathematics through questions and answers in a conjecturing atmosphere. We analysed the nature of this classroom discourse and particularly focused on different types of questions emerging in the dialogue and how the teacher taught the mathematics through problem solving.

Throughout the group meetings of the second part, the students were expected to work on the problems without getting any help from their teacher. From an educational perspective, it can be argued that the students' collaborative problem solving of this part was not free from teacher intervention. The situation was within an instructional context and I, as an observer, was present throughout the group meetings. The students were also stimulated by certain objectives introduced on the sheet of paper with the mathematical problems, giving guidelines for a group report from the small-group work of the second part. This report was to consist mainly of three different elements: the solution of the problems, the process writing in which the students were to write down their ideas and strategies throughout the problem-solving process, and a reflection part in which the students were to reflect on their problem-solving process. Linked to this reflection, the report was also to include an evaluation of the totality of the small-group work, e.g. by answering questions like: What have we learned from this small-group work?

As far as the three project groups are concerned, my co-teacher was not involved in the students' mathematical discussion. He only visited the groups in order to give some general information or to check that the students were present since these meetings were a compulsory part of the course. The two geometry problems, given in the second part, would then be discussed and elaborated in some plenary lessons in the third part of the project.

3.3 Problem selection

The group members worked on the following problems:

Problem 1

A. Choose a point P in the plane. Construct an equilateral triangle such that P is an interior point and such that the distance from P to the sides of the triangle is 3, 5 and 7 cm respectively.

B. Choose an arbitrary equilateral triangle ΔABC. Let P be an interior point. Let d_a, d_b, d_c be the distances from P to the sides of the triangle (d_a is the distance from P to the side opposite of A, etc.)

- a) Choose different positions for P and measure d_a, d_b, d_c each time. Make a table and look for patterns. Try to formulate a conjecture.
- b) Try to prove the conjecture in a).
- c) Try to generalise the problem above.

Problem 3

Given a right-angled triangle ΔABC $(\angle B = 90°)$ and a semicircle Ω, with centre O and diameter AQ, where Q is a point on AB. The points P $(P \neq A)$ and R on Ω are given so that P is on AC and OR is perpendicular to AB.

- a) Find $\angle APR$ and $\angle QPC$.
- b) Prove that $\angle BQC = \angle BPC$.
- c) Prove that if B, P, R are collinear (are points on a line), then BC and BQ are of equal lengths.
- d) Formulate the converse of the theorem in c). Investigate whether this formulation is a theorem.

The experiences from observations of one student group working on these problems at another teaching-training college helped me to obtain a thorough understanding of the mathematics. For a group of students in their first semester at a teacher training college, I therefore believe that these problems are sufficiently difficult to ensure that the participants are dependent on one another in order to succeed in finding a solution. The students are challenged to experience the cyclic structure of the problem-solving process, as described by Borgersen (1994; 2004). These problems also have embedded in them some important geometrical concepts to be learned.

Problem 1A was meant to be an introductory problem, focusing on important constructions using a compass and straight edge. Even though this is a bit difficult introductory problem, it is possible to come up with a solution based on the lessons designed for the first part of the problem-solving course.

Problem 1B stimulates the students to experience the cyclic structure of the problem-solving process, from finding the distance sum based on drawings, measurements, and constructions of conjectures via attempts at proving the conjectures to generalisations and formulation of new problems. Based on observation from students working on this problem, I suggest that most student groups of the problem-solving course would manage to solve problem 1Ba. Even though problem 1Bb and 1Bc are quite difficult, especially for some of these student groups, most of these groups are able to deal with a diversity of geometrical concepts. These discussions can then function as a starter for an elaborated dialogue in a plenary session in the third part.

Problem 3 recapitulates many geometrical concepts that were focused on in the teaching part. It could be argued that this problem does not offer the same rich setting for exploration as that of problem 1. Even though we have to take into consideration that these geometry problems are different in nature, I want to emphasise that problem 3 is not an easy problem for these students. First of all the students have to read and analyse carefully the introduction part. A crucial component of approaching and making sense of the text is to draw an auxiliary figure in order to visualise the problem. The text consists of mathematical concepts and symbols compressed within less than three lines. The students must interpret several pieces of information in order to make the transition from the presentation in the text into a visual representation. It is also possible to find two different solutions for angle $\angle APR$ (problem 3a). In order to solve problem 3b, the students must combine knowledge of cyclic quadrilaterals and apply Thales' theorem.

3.4 Unit of analysis

I have chosen the dialogical approach (Marková and Foppa, 1990; Linell, 1998) to the data analysis since this stance 'allows one to analyse the co-construction of formal language among participants in a defined situation' (Cestari, 1997, p. 41). More specifically, this approach permits me to identify interactional processes, which, in the analyses of these particular episodes, are the students' utterances expressing their heuristic strategies used in the solution process.

Following Wells (1999), I have divided the two episodes at three levels, indicating a gradually more detailed analysis. First, the episodes have been divided into thematic segments (sequences). Then each segment has been divided into exchanges. Finally, each exchange has been divided into utterances (moves), either initiating, responding or follow-up utterances. The exchange level constitutes the most appropriate unit for the analysis for me in order to capture the dynamic characteristics of the dialogues. This is quite in correspondence to the dialogical approach and Marková's (1990) three-step process as the unit of analysis. This means that in the sequence of conversation each utterance is interrelated to the previous utterance as well as to the subsequent one. Each utterance has to be interpreted according to the contexts in which it is expressed (Linell, 1998).

The students' utterances are presented in the left column, while the right column shows important heuristic strategies used in the students' solution process and other aspects from the analysis. Four categories of strategies are identified, given the following abbreviations in the right column: VS (Visualising strategy), MS (Monitoring strategy),
QS (Questioning strategy), and LS (Logical strategy). The name in bold shows the student that initiates a strategy. This is exemplified in the following utterance from the introductory segment of the first episode analysed below:

223	**Gry**	…(6 sec.)… Should we write what we've done so far?... we have drawn...	MS, QS (1. Monitoring question categorised as looking back. 2. Recapitulation of the solution process).

Gry's initiative consists of two monitoring strategies (MS), a m*onitoring question* that elicits and triggers the mathematical activity of the group, and a *recapitulation* of the ongoing solution process. The monitoring questions are also included in another category QS.

4. Approaching and making sense of problem 1Ba

The students took about 70 minutes to come up with a solution on problem 1A, in addition, to spending 20 minutes discussing alternative ways of doing the problem. So, the students were concerned with problem 1A for the whole of the first meeting.

The aim of presenting the following episode is to identify crucial heuristic strategies used in the students' solution process of problem 1Ba. More specifically, the analysis of the students' discussion focuses on how the students attribute meaning to the concept of distance from a point to a line. The episode is taken from the second meeting and all five group members are present. The analysis of the episode is organised in four thematic segments.

The dialogue preceding this episode has shown that the students have read problem 1Ba, discussed the formulation and drawn a figure of an equilateral triangle. They are now ready to do the measurements in a). Gry, who was absent from the first meeting, has taken the responsibility of being the process writer. That means that she is concerned with writing down ideas and strategies that emerge in the problem-solving process.

4.1 Recapitulation and the role of the process-writer

223	**Gry**	…(6 sec.)… Should we write what we've done so far?... we have drawn...	MS, QS (1. Monitoring question categorised as looking back. 2. Recapitulation of the solution process).
224	Unn	We measure the distance from the line to the point...	Attuned response. The concept of distance is introduced.
225	Roy	Well each... each of us has placed the point *P* in different places... in order to get… well… shouldn't we make a table?... or?...	Linked to (223) and (224). Recapitulation – confirmative question focusing on the next activity: to make a table.
226	Unn	Yes…	Agreement.
227	Roy	Yes (low voice)	Agreement.
228	Unn	Then we place all the measurements… (in a table)	Following up Roy (225).
229	Roy	Yes… d_b is… d_a is… d_c is…(8 sec.)…	Roy starts making a table - Silence
230	**Gry**	Then we have… we have placed those points in different ways… and then we have drawn… no… how should we say it… those lines?…	MS, QS (1. Monitoring question categorised as looking back, asking for an explanation, 2. Recapitulation, returning to the concept of distance).
231	Roy	Well (simult.)	Attunement.
232	Liv	We have measured… (simult.)	Attunement.
233	**Roy**	Then we have d_a d_b d_c... those are the distances from *P* to the sides of the triangle… we have to say (write) that it's the shortest distance to the sides... we've chosen...	MS (Recapitulation, summing up for process writing, giving an explanation, making explicit the distances).

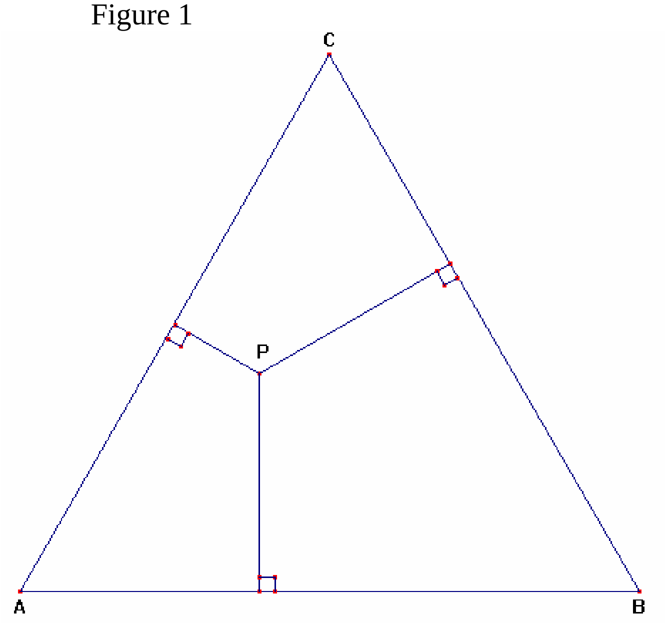
The dialogue illustrates two important monitoring strategies: Monitoring questions (also questioning strategy) and recapitulation of the solution process. It also shows the important role of Gry as the process writer since she elicits and triggers the mathematical activity in the group by initiating two crucial questions. Her *monitoring question*, categorised as *a looking-back question* (223), stimulates a *recapitulation* of the solution process, promoting the establishment of a common ground for the further discussion. The concept of distance from a point to a line has been introduced in the discourse. Gry's second *looking-back question* (230) challenges her colleagues to give an explanation how they have drawn the lines from *P* to the sides of the triangle. Both monitoring questions have been initiated after some silence in the group (223), (229). The students are attuned to Gry's initiatives. In his explanation, Roy makes it clear that they are concerned with the distances d_a , d_b , d_c, emphasising that they have chosen d_a , d_b , d_c to be 'the shortest distances' (233) from *P* to the sides of the triangle.

The monitoring strategies of posing looking back questions and recapitulation the ongoing solution process are related to the students' focus on the process writing. These strategies are also indicators of mutuality since the students, in this way, are establishing a common ground for the problem-solving activity.

4.2 Questioning generating a discussion about the concept of distance

The second segment is almost a continuation of the first one. Liv has been working on her figure, and she has constructed the perpendiculars from *P* to their intersections with the sides of the triangle (see figure 1 below). The dialogue below shows how some questions generate a mathematical discussion about the concept of distance among the students.

Figure 1

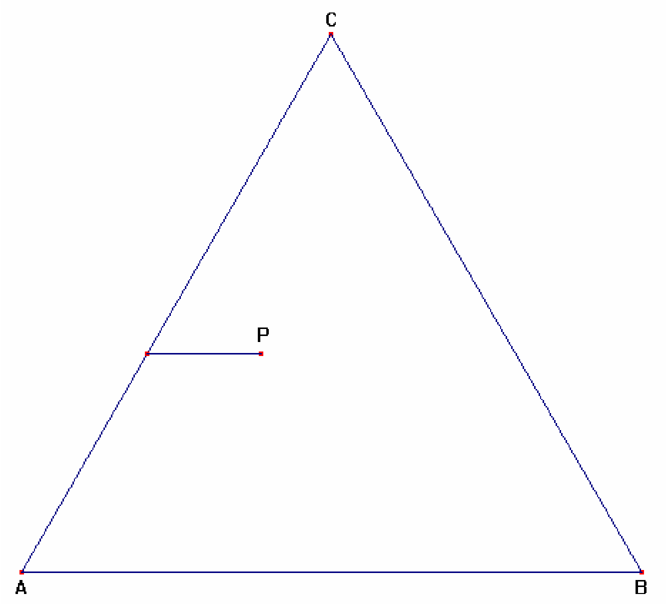


237	**Unn**	*90* degrees?... is that relevant?...	MS, QS, VS (Monitoring questions related to figure 1, asking for an explanation n).
238	Roy	Well... in order to get the shortest… (inaudible, simult.)	Attuned response.
239	Liv	*90* degrees is the shortest distance...(simult.)	Attuned response.
240	**Unn**	Yes but why couldn't we draw it just right out there?…	QS, VS (Why-question related to figure 2, introducing an alternative way of doing the measurements).
241	**Roy**	Hmm?…	QS (Brief following up question, asking for clarification).
242	**Unn**	Why do you draw it up there?...	QS (Why-question related to figure 1, challenging the '90 degrees' perspective).
243	Liv	*90* degrees is the shortest distance... (simult.)…	Repetition of the same explanation.
244	Roy	*90* degrees on that line there... (simult.)…	Explanation is linked to the figure.
245	**Mia**	Should it be *90* degrees?...	QS (Following up question, asking for further explanations).
246	Liv	Yes... *90* degrees is the shortest distance...	Repetition of the same explanation.

The dialogue shows three open questions initiated by the same student, Unn. These questions stimulate a discussion about the concept of distance among the students. Unn has probably seen that Liv has been working on her figure. This seems to stimulate the *monitoring question* (237) that challenges the students to consider why they should use angles of *90* degrees in order to do the measurements (see figure 1). The *why question* (240) comes up with an alternative way of doing these measurements in which the distance should be measured along the line through *P* parallel to the base of the triangle (see figure 2 below). The third question (242) challenges the students to justify why 'the shortest distance' is the length of the perpendicular from *P* to its intersection with the side of the triangle. These questions are all related to the students' attempt at drawing a figure, the strategy of visualising. Mia enters the dialogue (245) with a *following up question* that challenges Roy and Liv to reconsider their argumentation. However, the response of the *90*-degree angle has been constantly repeated (246).

The dialogue throughout this segment has shown that the three strategies *monitoring*, *questioning*, *visualising* (translating the problem into a visual representation) stimulate the students to become aware of two alternative ways of interpreting distance from a point to a line.

Figure 2



4.3 Elaboration on the two different perspectives

As a continuation of the dialogue, these two different perspectives on the idea of distance are discussed.

256	**Unn**	But I don't agree... I think that this is very illogical... because if I should have measured this...	LS (If-then structure, building up a logical cause effect argument).
257	Roy	It's nearly like this... (Roy refers to the perpendicular on his figure)	Focus on his figure.
258	Unn	Then I would only have laid the ruler there and said how far it is right out…	Continuation of (256), repetition of own perspective.
259	Liv	But that's not the shortest distance...	Attunement, challenging Unn (258).
260	**Unn**	I would only have done it like this... if it's *90* or *60* degrees... actually that doesn't make any difference to me...	LS (Begin the if-then structure, sticking to her own perspective).
261	**Roy**	But then... the distance differs then...	LS (Complete the if-then structure, giving the argument).

The dialogue shows that the students are now ready to defend their own arguments and challenge the other students' points of view. By employing the logical strategy (LS), using an *if-then structure*, Unn repeats the perspective of measuring the distance along the line through *P* parallel to the base of the triangle (256), (258). From a mathematical point of view, this difficulty seems to be quite stable since it is natural or logical to lay the ruler parallel to a horizontal base. The argument against this idea has been repeated, claiming that this is not 'the shortest distance' from *P* to the side of the triangle (259). Prior to this discussion, Liv has constantly repeated that they have to focus on the *90*-degree angle.

The dialogue shows the great attunement among the students, particularly when Unn and Roy employ the logical strategy together. Unn introduces the *if-then structure* (260), while Roy is following up and giving the argument (261). By giving examples of *90*-degree angles or *60*-degree angles, indicating that it makes no difference how you measure, the students are introduced to the fact that the two different perspectives lead to different distances. The elaboration of the two different perspectives has brought a new element into the mathematical discussion since some of the students have seen that they do not get comparable results if they measure the lengths along different line segments from *P* to the sides of the triangle.

4.4 Justification leading to agreement

After having elaborated on the two different perspectives on the concept of distance from a point to a line, the dialogue continues with the important *why-question* below:

269	**Liv**	But why did you measure straight out like this?...	QS (Why-question, challenging Unn's perspective).
270	Unn	No... actually I don't know... that's a good question but it can... well...	Uncertainty. Difficulty to give an explanation.
271	**Roy**	Hmm?…	QS (Brief following up question, asking for further explanations).
272	Unn	No it was just accidental that I did it like this... but eeh... we have to find... shouldn't we decide to do this in the same way?...	Questions (269), (271), leading to agreement. Unn's question triggers a confirmation.
273	**Roy**	At any rate, we have to decide that the three angles are equal… if we should find any ratio between those... we can't have one angle of *30* degrees like this and then one of *60* and one of *90*... that's... the three angles have to be equal if we're to find a pattern...	MS (Recapitulation, conclusion. Summary of the ongoing discussion). LS (If-then structure, building up a logical cause effect argument).

The *why-question* (269) challenges Unn to give an argument for her way of measuring the distances from *P* to the sides of the triangle. Unn has problems in giving reasons for her interpretation of distance, and she realises that Liv's question is good (270). In fact she is not that eager to defend her proposition. When Unn is invited to repeat her way of doing the measurements by Roy's *following up question* (271), she admits that her idea was just accidental (272). It seems as if Unn gradually realises that they have to decide to do the measurements in the same way in order to be able to compare their results.

Roy repeats the argumentation for doing the measurements in the same way in order to be able to find a pattern (273). He sums up the most important elements of the ongoing discussion and draws a conclusion. The students have established agreement about the fact that they have to choose a particular and unique distance in order to find a pattern. The participants gradually come round to a single way of interpreting the concept of distance from a point to a line.

The students in this particular group do not have the experience and background in mathematics to go straight to the measurements of d_a, d_b and d_c respectively. However, by using the strategies of *monitoring* (recapitulation, monitoring questions), *questioning* (posing open questions, categorised as *monitoring*, *why* and *following up*) and the logical strategy (*if-then structure*), they make progress in their attempts at attributing meaning of the concept of distance from a point to a line.

4.5 Reconstruction leading to the conjecture $d_a + d_b + d_c$ = constant

The students spent about 32 minutes on problem 1Ba. The analysis of the episode introduced above has been chosen from the first 10 minutes of this discussion. The reconstruction of the solution process for the final 22 minutes of the solution process is briefly summarised below, leading to the conjecture: $d_a + d_b + d_c = constant$.

Roy helps Mia to draw the perpendiculars on her figure (the strategy of visualising the problem, VS). She is still uncertain regarding the prior discussion about choosing the distance from *P* to a side of a triangle to be the length of the perpendicular segment from *P* to its intersection with that side for their measurements. Roy repeats the argument for measuring in the same way in order to be able to find a pattern. In his explanation, he also draws an arbitrary line from *P* to the side of the triangle in order to show Mia that this line segment is longer than the length of the perpendicular from *P* to its intersection with the side of the triangle. The strategy visualising (VS) is then used as a tool for his explanation.

Roy brings the idea into the discussion that the triangles should have equal sides if they should find a pattern. The students decide to place some more points in their own triangles, and Roy comes up with the following conjecture: $(d_a + d_b + d_c) / 3 = constant$. The students try out this conjecture, and they observe that they do not need to divide by *3*.

The students spent about *25* minutes working on problem 1Bb without coming up with a convincing argument. In the discussion, they focused on cyclic quadrilaterals, the special quadrilateral kite, and the fact that the conjecture was based on equilateral triangles. They were also conjecturing about cyclic quadrilaterals being similar (Bjuland, 2002).

It might be asked why student teachers should work on this difficult problem without getting any help from their teacher? A supervising teacher could have guided the students in their "zone of proximal development'' (Vygotsky, 1978). If there had been a teacher, when should he become involved in the group discussion? As far as problem 1Ba is concerned, a teacher could just have told them how distance is defined from a mathematical point of view.
However, a teacher often dominates the discussion since his voice represents the mathematical community. Based on the analysis of the group discussion from the solution process of problem 1Ba, there are reasons to believe that these students have established a constructive mathematical discussion. This is due to the fact that they got the opportunity to attribute meaning to the

concept of distance. This knowledge could then be affirmed in one of the plenary lessons in the third part of the project.

5. From sense-making to convincing on problem 3b

The aim of presenting the second episode is to show how these students are able to succeed in finding a solution to problem 3b, which is quite a difficult problem for students with limited mathematical background. The analysis focuses on the students' main heuristic strategies in the solution process.

The students have started a 25-minutes discussion on problem 3 during the second meeting. They read the problem and discuss what is meant by the mathematical symbols 'Ω' and '$\neq$'. From this conversation, I have reconstructed how the students draw an auxiliary figure step by step in order to visualise the problem (see Bjuland, 2002). The students come up with two different figures:

Figure 3

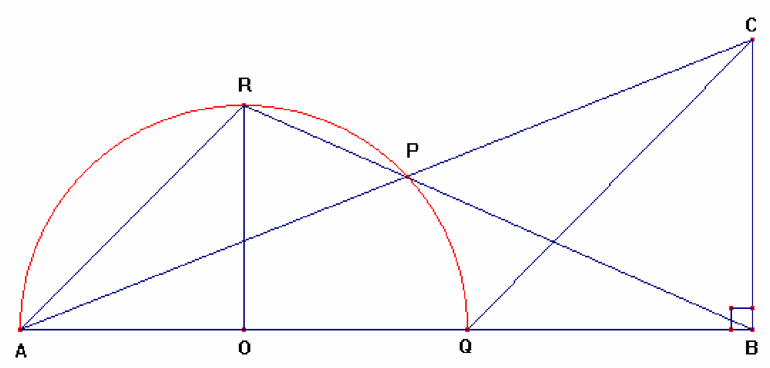
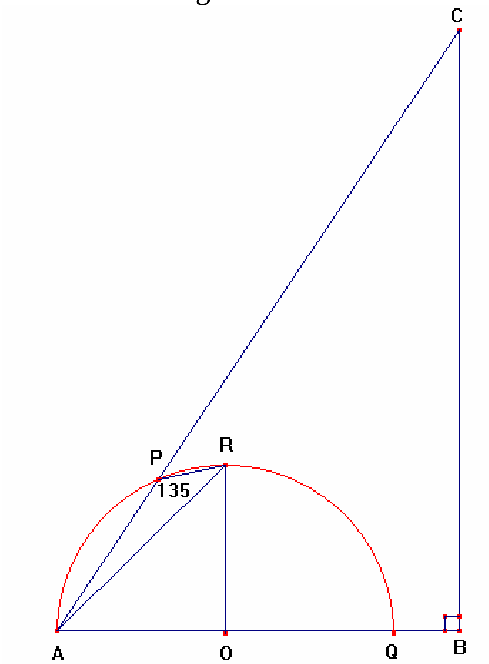


Figure 4



They agree on measuring angle $\angle APR$ by a protractor in order to get an idea of the size of the angle. The measurements from four of the students show that angle $\angle APR$ is *45* degrees. However, one of the students' measurement suggests that angle $\angle APR$ is *135* degrees. The students compare the two figures, but they do not find any mistake. They observe that point *P* is placed to the left of point *R* (figure 4). The student who comes up with the *135*-degree angle of $\angle APR$ draws a new figure in which angle $\angle APR$ is *45* degrees. By doing this, the students avoid the fact that there are two solutions to the problem.

They find a solution for angle $\angle APR$ by introducing an argument that the angle $\angle AOR$ at the centre is double the angle $\angle APR$ at the circumference since they both subtend the same arc *AR* (see figure 3). In a similar way, they find the angle at the circumference $\angle QPA$ by applying Thales' theorem, observing that the angle *AOQ* at the centre is *180* degrees. After some efforts, they come up with a convincing argument that angle $\angle QPC$ is *90* degrees.

The students start working on problem 3b about 30 minutes into the third meeting. Figure 3 is the starting point for the mathematical discussion. The dialogue preceding this episode has shown that figures are drawn based on the information given in the problem, representing geometrical visualisations of the problem. Relevant subconfigurations from figure 3 have been found in order to make sense of the problem. The students have focused particularly on quadrilateral *QBCP* on a separate figure. Two main questions and one idea have emerged in the discussion: Is *QBCP* a cyclic quadrilateral? Is triangle ΔQPC similar to triangle ΔQBC? The idea of reflecting triangle ΔQPC around the axis of reflection *QC* has also been discussed (Bjuland, 2002).

The episode is organised in three thematic segments. The analysis of the first segment illustrates that the monitoring strategy of looking back on the solution process, bringing formerly acquired ideas into the discussion, is crucial for the students' attempt at coming up with a reasonable solution.

5.1 Looking back on ideas generated in the solution process

927	**Liv**	We have *APR* (angle) which is *45* (degrees) there... then we have…	MS (Recapitulation of solution on problem $3a_1$). Continuation in (929).
928	Roy	That's why we could find…	Attunement. Continuation in (930).
929	Liv	Then there's *90* degrees there... then there's *180* degrees there... isn't there?...	Focus on the figure, probable $\angle$QPA (90°), $\angle$APC (180°), related to (930).
930	**Roy**	That's why we could find *PQC*... *QPC*... yes but eeh... we have that quadrilateral and we have some vertical angles... reflection... cyclic quadrilateral...	MS (Recapitulation of solution on problem $3a_2$ and different ideas previously discussed).
931	Gry	Yes but it doesn't meet… (the circle does not meet the vertices of the quadrilateral)	Following up Roy (930). The idea of a circle, circumscribing the quadrilateral QBCP.
932	**Liv**	Similarity then?...	QS (Brief following up question, is triangle ΔQPC similar to triangle ΔQBC?).

The monitoring strategy of *recapitulation the ongoing solution process* is brought into the discussion by Liv's initiative (927). The use of the personal pronoun *we* (927), suggests that the students have developed a shared understanding of the solution for problem 3a. The students recapitulate previously acquired ideas and solutions in order to come up with a direction for the reasoning process (927) – (932). In one respect these utterances are elaborations on Liv's monitoring initiative (927). However, I have chosen to emphasise Roy's recapitulation (930) as a monitoring strategy since he focuses on specific ideas previous discussed in the solution process. Roy's initiative also stimulates Gry (931) to elaborate on one of these ideas since she focuses on the circle that circumscribes quadrilateral QBCP. The monitoring strategy of *recapitulation* helps the students to establish common ground, giving all of them the opportunity to participate in the mathematical discussion.

5.2 Focusing on the particular idea of cyclic quadrilaterals

933	**Unn**	Is it a cyclic quadrilateral?...	MS, QS (Monitoring question, asking for confirmation).
934	Roy	Yes… it's possible to find the centre... it's possible to find a centre...	Response. Related (932), a circle, circumscribing QBCP.
935	**Mia**	She got the circle around the whole... (Mia speaks about Liv's figure)…	MS (Monitoring other students' work, bringing the idea into the discussion).
936	**Gry**	Did she?...	
937	Mia	Yes…	QS (following up question attuned to Mia).
938	Gry	I don't…	
939	**Liv**	It's just... well it's just to bisect... then you get the centre... but it's not…	Response. Related to (931). VS (modify her figure by constructing the centre by bisecting the sides. Suggestion of rejecting this idea).

Based on all the ideas that have emerged in the discussion, *a monitoring question* encourages the students to focus on the particular idea whether quadrilateral *QBCP* is a cyclic quadrilateral or not (933). The dialogue illustrates that the students elaborate on this idea (934) – (939). Liv has, through all her drawings of quadrilateral *QBCP* (Bjuland, 2002), also modified her figure by drawing a circle that circumscribes the quadrilateral. However, she does not explain her work to the other students. Instead, it is Mia who brings this important element into the discussion (935), informing her colleagues about Liv's modified figure. This monitoring strategy provokes more attention (936) – (938), stimulating Liv to focus on her circle (939). She suggests how they could construct the centre of this particular circle in order to obtain a more accurate figure, indicating that she uses the strategy of modifying her visual representation. However, she also suggests that the construction of the circle is not the best idea in order to come up with a solution. Prior to this discussion, Liv has focused on the idea of similarity as a possible direction for the solution process.

In the continuation of the dialogue, Liv is still concerned with the idea of similarity. However, one of the students recapitulates the characteristics of a cyclic quadrilateral by searching for help from her textbook (953). This seems to help the students to conclude that *QBCP* is a cyclic quadrilateral (954) – (956).

953. Unn: A cyclic quadrilateral is a quadrilateral which can be circumscribed by a circle… in cyclic quadrilaterals opposite angles are supplementary angles… together they subtend the whole circumference… (Unn reads from her textbook)…
954. Liv: Mmm…
955. Mia: Yes but that shows that this is a cyclic quadrilateral…
956. Roy: Yes…

The textbook is here used as an important strategy in order to come up with a shared understanding about this particular concept. However, there is still a discussion about whether the opposite angles should be *90* degrees each. The dialogue in the next segment shows how this discussion develops among the participants.

5.3 Breakthrough: coming up with a convincing argument

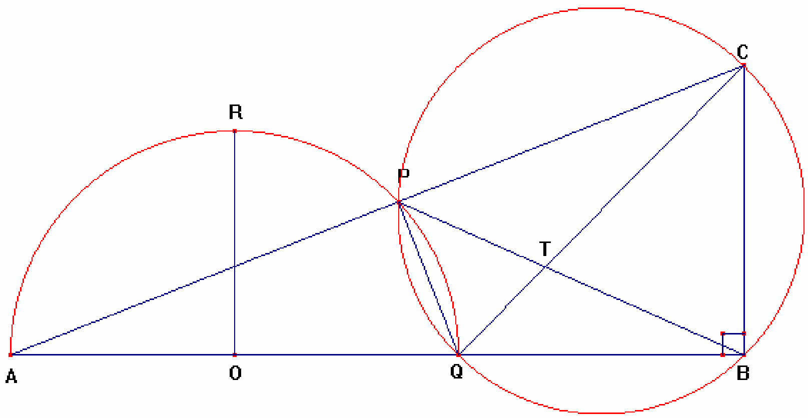
962	Roy	Yes two of those are *90* and they're *180* together... but those others don't need to be... they are *180* together... but one of those can be *30* and one can be…	Characteristics of a cyclic quadrilateral.
963	Mia	One is... together... one is... yes one of those can be *60* and the other... can't it?... it just says *180* together...	Following up (962).
964	Unn	But... yes... but that we already knew...	
965	**Mia**	Yes... but she has at any rate... she has roughly done it to get the centre... but it	Confirmation of the characteristics. MS (Monitoring Liv's figure, informing about the circle, circumscribing QBCP).
966	**Liv**	Well... we can do it exactly then... then we get the centre... here is the centre... like this... do you think it has something to do with angles at the circumference?… Roy?…	VS (Attuned to Mia (965), responding by making a proper construction of the circle. Introducing the idea of angles at the circumference).
967	**Roy**	Hmm?…	QS (Brief following up question, asking for a further explanation).
968	Liv	If we can find angles at the circumference here...	Repetition of the idea of angles at the circumference.
969	Mia	In a circle…	Difficult to interpret, maybe reading from her textbook.
970	Liv	Yeeees!... look at this... there we are!... (she laughs and laughs)...	Breakthrough, find a solution.

In his explanation, Roy (962) makes it clear that the particular quadrilateral *QBCP* consists of two opposite angles which are *90* degrees each. However, he goes on to emphasise that the other opposite angles do not need to be equal. By introducing examples in which one of the angles could be *30* degrees (962) or *60* degrees (963), both Roy and Mia stress that the sum of two opposite angles should be *180* degrees.

As shown in the analysis of the previous segment, Mia informed the other students about Liv's modified figure in which Liv has drawn a circle that circumscribed the cyclic quadrilateral *QBCP* (935). Sequentially linking it to the discussion above about the cyclic quadrilateral, Mia recapitulates this by reminding the students about this modified figure (965). The initiative of repeating this important step seems to trigger the breakthrough in the solution process (965) – (970). The strategy of monitoring another students' work provokes Liv to do the construction of the circle exactly (966). By focusing on the circle, Liv looks back on the idea of some angles at the circumference and brings this into the discussion by her request for agreement directed to Roy. His brief *following up question* (967) triggers a repetition of this idea. By focusing on some angles at the circumference (968), Liv comes up with a solution to the problem (970). Her strong affective response suggests that she has observed that angle $\angle BQC$ and angle $\angle BPC$ are both angles at the circumference, subtending the same segment of the circle.

The students' breakthrough in the solution process has been brought about by four crucial steps, leading to the following figure:

Figure 5



The first step has been to recapitulate previously acquired ideas and solutions, conjecturing about cyclic quadrilaterals in particular. The second step has been to conclude that *QBCP* is a cyclic quadrilateral. This conclusion has been triggered by *a monitoring question* and the students' initiative of recapitulating the characteristics of a cyclic quadrilateral by searching for help from a theoretical source. The third step in the solution process is to construct the circle that circumscribes quadrilateral *QBCP* (see figure 5 above). This has been stimulated by the monitoring strategy from one of the participants in which the students are reminded of the modified figure where this circle has been approximately drawn around *QBCP*. Based on this, the construction of the circle has been done exactly. It has then been observed that angle $\angle BQC$ and angle $\angle BPC$ are both angles at the circumference, subtending the same arc. By Thales' theorem, these angles are then equal. This is the fourth step in the solution process, and the students have come up with a proper solution.

6. Discussion and conclusion

In order to respond to the research question, I focused my attention on the identification of heuristic strategies expressed in the mathematical discussion of a group of adult students working on two geometrical problems without teacher intervention. Related to this question, I also focused on the critical function of these strategies in order for the students to make mathematical progress in the solution process. Table 2 below summarises these findings.

Problem 1Ba	Problem 3b
Focus: The concept of distance from a point to a line	Focus: The concept of cyclic quadrilateral
Heuristic strategies with critical mathematical function: 1. Visualising strategy (VS). Transforming a written mathematical text into a visual representation: a) Draw a figure of an equilateral triangle. b) Place different positions of P as interior points of the triangle. c) Draw the distances d_a, d_b, d_c from P to the sides of the triangle. d) Use of figures as a support for their mathematical explanations. 2. Monitoring strategy (MS). Establishing common ground for the problem-solving activity: a) Recapitulation - Looking back on the solution process, returning to the concept of distance from a point to a line, - Conclusion, summing up the discussion for process writing. b) Monitoring questions - Looking back on the solution process, related to process writing 3. Questioning strategy (QS). Posing open questions, stimulating the mathematical discussion. a) Monitoring questions (see 2b). b) Why questions. c) Following up questions (asking for further clarifications and explanations) 4. Logical strategy (LS). Building up a logical cause effect argument a) If-then structure	Heuristic strategies with critical mathematical function: 1. Visualising strategy (VS). Transforming a written mathematical text into a visual representation: a) Draw an auxiliary figure step by step in order to visualise the problem. b) Identify a subconfiguration from the initial figure, the quadrilateral *QBCP*. c) Draw *QBCP* on a separate figure. d) Modify this figure by drawing a circle, circumscribing the quadrilateral. 2. Monitoring strategy (MS). Establishing common ground for the problem-solving activity: a) Recapitulation - Looking back on suggested ideas from the solution process, - Looking back on previously acquired solutions. b) Monitoring questions - Bringing specific ideas into the discussion. c) Monitoring other students' work - Bringing these ideas into the discussion 3. Questioning strategy (QS). Posing open questions, stimulating the mathematical discussion. a) Monitoring questions (see 2b). b) Following up questions (asking for further clarifications and explanations). 4. Using the textbook as a tool in order to discuss a particular mathematical concept.

Table 2: Critical heuristic strategies used in the solution process

The diversity of strategies indicates central elements of mathematical reasoning, corresponding to the second component of Schoenfeld's (1985, 1992) framework. I have particularly identified how the following categories of strategies have been constructive for the students' solution process: *visualising*, *monitoring*, *questioning*, and *logical strategy*. The analysis has also revealed that the students, almost simultaneously, use the strategies of modifying their figures, posing open questions, and monitoring their solution process. It seems as if these strategies are all related to the students' metacognitive activity. Lester (1994) confirms that metacognitive awareness plays an import role in problem solving, but such an activity during problem solving is not common among novice problem solvers. The study carried out by Goos et al. (2002) also reported that unsuccessful problem solving was characterised by students' poor_metacognitive decisions throughout the problem-solving process. It might be asked why the students in this particular group are able to monitor their solution process. What stimulates this monitoring activity?

The elements of metacognitive training in combination with cooperative learning were critical components in my design of the instructional context introduced in the methodology part. These aspects are important in order to benefit from collaborative problem solving. This is also emphasised in the study carried out by Kramarski and Mevarech (2003). As far as my particular group is concerned, the conjecturing atmosphere established in the discussion has showed that the communication is mutually supportive. Graumann (1995) emphasises the importance of establishing mutual relationships between participants in dialogues. The common ground among the students is one crucial condition for learning to take place.

The students are attuned to each other's perspectives. They show willingness to pose open questions, and they seek to explain and justify by collaborating with each other in order to come up with important ideas and possible solutions on the mathematical problems. In this respect the analysis of the dialogue is also an indicator of the students' equal status in the group even though one of the students seems to have more power based on the students' mathematical backgrounds and their experiences. In Stacey's study (1992) a lot of ideas were generated in the dialogues among the junior secondary students. However, these students did not seem to monitor or to carefully consider all the group initiatives. Motivation for learning and maturity among student teachers are of course aspects to take into consideration why these students have established a community that shares a commitment to caring, collaboration, and a dialogic mode of making meaning (Wells, 1999).

It is important for students in their teaching training programme to experience collaborative problem solving in small groups. Teaching mathematics through problem solving could stimulate students to develop a thorough understanding of mathematical concepts (Lester and Lambdin, 2004). The students in this particular group have without teacher involvement elaborated their understanding of the particular mathematical concepts: distance from a point to a line and cyclic quadrilaterals. The detailed analyses give us insight into how important the sense-making process is for the students' discussion on geometrical concepts and for the development of a mathematical solution (problem 3). The findings of Schoenfeld (1985, 1992) based on college and high-school students, working on unfamiliar problems, show that the students immediately jump into implementation after an initial quick reading and analysis of the problem. I am aware of the fact that it is difficult to make a comparison between Schoenfeld's novices and my adult students when it comes to their way of approaching and making sense of a given problem, due to different variables such as time and the nature of the problem. However, my analysis makes an important contribution to research since my findings reveal that it is possible for adult students with limited mathematical backgrounds to succeed in finding a solution to a complex geometry problem without teacher involvement.

Even though I take into consideration that problem 1 and problem 3b are different in nature, problem 3b is not an easy problem for these students. First of all they have to focus on a subconfiguration, a quadrilateral, in their figures. Then they have to find out that this is a cyclic quadrilateral. Thirdly, they have to know that it is possible to construct a circle that circumscribes the quadrilateral before using Thales' theorem and arguing for the fact that two angles are equal. The analysis has revealed that these students have identified and argued why the quadrilateral is cyclic. From a mathematical point of view, this is not obvious, and students need quite a lot of geometrical expertise to see this. We could ask why they manage to do this without teacher intervention? The teaching part in the first period of the project plays a crucial role. It is also possible that the monitoring awareness established in the group has helped them to carefully consider the ideas generated in the conversation before rejecting them. The analysis has particularly focused on the important role of the process writer that stimulates the mathematical discussion by generating utterances categorised as *looking-back questions*. She is concerned with recapitulating the solution process or the last idea introduced in the dialogue. This is promoting the establishment of a common ground for the further discussion. It seems as if the process writer really plays a critical role in the process of problem solving. This is an important result that I cannot find adequately addressed in earlier research. It could be tempting to ask what happened in the other 20 groups of students? Is this just a nice story from one group?

In Bjuland (2002), I have carefully analysed one of the other groups of students while working on the same geometry problems. The findings from this group are also promising. By using constructive heuristic strategies, particularly the strategy of posing monitoring questions, these students are also able to find a solution on problem 3b without any involvement from their teacher. In this group all students contribute with important monitoring strategies that stimulate the progress in the solution process. Since only three of the groups were randomly selected for observation, I have only the group reports from the other groups as empirical materials. It is therefore difficult to report on those groups as far as monitoring activity is concerned.

From a social scaffolding perspective (Wells, 1999; Wood, Bruner and Ross, 1976), I have already posed the following question: Could some open questions from a teacher have stimulated the students in their solution process on problem 1? It is also likely that a teacher could have guided the students in their discussions with stimulating questions in order to obtain an effective solution process. If that is the case, why should these students work on two geometry problems without getting any help from a teacher? In the group work, designed in the autumn of a later semester at this particular teacher-training college, other students worked on the same geometrical problems as reported from in my study. Being a teacher throughout these meetings, I had the opportunity to observe the students during their work and identify their difficulties with the problems. When great frustration appeared in a group, I observed that it was constructive for the solution process to stimulate the students' group dialogue by posing an open question linked to their discussion. Maybe such situations are the most suitable for teachers to become involved in the group discussion.

7. Final remarks

This study has focused on how elements of mathematical reasoning (the students' heuristic strategies) are expressed in dialogues. The corpus includes the students' utterances in interaction, socially contextualised in a problem-solving context. The analysis has been focused on student conversation in one collaborative small group of students working on two geometry problems. The findings have revealed that monitoring activity, related to the use of the strategies of monitoring, questioning, and visualising, is crucial for mathematical progress in the solution process and for having a constructive discussion about mathematical concepts. As a pedagogical implication, this finding suggests that teacher education must stimulate metacognitive training in combination with cooperative learning among the students in order to develop problem-solving skills. More specifically, this also means that students in teacher education must be aware of the critical role of the process writer in the process of problem solving. I am fully aware that the corpus of this study is limited and therefore questionable as the groundwork for generalisation. Still, pre-service teacher educators and researchers can potentially use the findings of this study to help design and implement instruction that stimulate students to develop their problem-solving skills in collaborative small groups.

One possible direction for future research would be to focus more closely on observation, analysis and interpretation of the conversation of adult students working in groups on problems from other topics of mathematics. Is there anything about geometry, in general, or these tasks, in particular, that stimulate students to develop their reasoning or their problem-solving skills?

Acknowledgements

I would in particular like to thank the management at NLA College of Teacher Education that provided the grant for doing this research. Thanks also go to my colleagues Maria Luiza Cestari, Simon Goodchild and Hans Erik Borgersen at Agder University College for their constructive comments on a previous version of this article.

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