Investigations of a chip-firing game

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INVESTIGATIONS OF A CHIP-FIRING GAME

by

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We investigate a variation on a chip-firing game in which a node $v$ may fire only if it is possible to transfer a chip to each neighbor of $v$ and discard a chip from $v$. The discarded chip provides the variation, so we call this kind of chip-firing game a burn-off game. Chip-firing games have been used by physicists to study a property of complex systems called self-organized criticality. The main goal of this dissertation is to study burn-off games using strictly mathematical methods.

The chip distributions that occur during a sequence of burn-off games are called legal configurations. We characterize legal configurations, and then show that each legal configuration is equally likely, in the long run, to arise during a sequence of burn-off games played on any connected graph. In the case of complete graphs, this allows us to find a closed formula for the probability of experiencing a burn-off game of any desired length. For connected graphs, we describe a method for calculating the same probabilities.
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Chapter 1

Introduction

1.1 Models

Over the past twenty years, physicists have modeled complex physical systems with simple computer models that display some of the important characteristics of their real-life counterparts. One example of such a complex system is the earth’s crust. Sudden transfers of energy within the earth’s crust create seismic events, such as earthquakes, that are felt on the surface. Most of these events are brief and minor, while a few cause great damage.

On August 14, 2003, a power blackout affected over 50 million people in Canada and the northeastern United States. In [17], Sara Robinson relates that engineers who are studying the history of power blackouts consider the electrical grid to be a complex system in which blackouts are events just like earthquakes are
events in the earth's crust. Other complex systems that have been modeled include
snowfields, traffic, and the human brain (see Section 1.4). All of these examples
qualify as systems in the sense that they are composed of many small interacting
parts. The earth's crust may be viewed as a collection of small platelets of earth that
press against each other and occasionally shift. The electrical grid is a web of
interacting wires and power stations. A snowfield, at its finest level, is a collection
of snowflakes that put weight on each other until a group of them slide downhill.

These systems are complex in that their behaviors, while predictable in
one sense, are entirely unpredictable in another. If we consider these systems from a
broad perspective, we can be fairly secure in our predictions about them. For
example, small earthquakes occur with great frequency, while large earthquakes are
uncommon. While geologists are able to make relatively accurate predictions about
when and where the next large earthquake will strike, they rely on macro-level
observations, like fault lines and historical data, to make their claims as opposed to
closely observing the tiny interacting chunks of earth. The system they study is far
too complex for such a micro-level analysis to be helpful; the same is true for power
blackouts and avalanches.

The models for these complex systems have one feature in common: they
simulate a network of individuals that, when stimulated, may interact with their
neighbors by distributing some sort of stress. In fact, all of the models mentioned
thus far are special cases of "chip-firing games", studied in a recent paper by Björner et al. [6]. We turn now to a discussion of these games.

1.2 Chip-firing games

A chip-firing game takes place on a connected graph; all graphs in this thesis are simple. We assume throughout this dissertation that the reader is familiar with both basic graph theory, as introduced, e.g., in [19], and basic probability theory, as introduced, e.g., in [10]. (A glossary of notation and terminology appears in Appendix 1.) The nodes of the graph take the role of the individuals in the network. Integer-valued stresses upon each individual are represented by chips contained on the nodes. When a node fires, chips are moved to its neighbors in accordance with a firing rule. A node cannot fire until the number of chips it contains meets or exceeds a threshold, called the node’s critical number.

A generalized chip-firing game that includes as special cases the models discussed in Section 1.1 has three features:

(a) the game takes place on a connected graph;

(b) each node $v$ has a critical number $k_v$;

(c) if the number of chips on $v$ is at least $k_v$, then $v$ can fire, sending a chip to $k_v$ of its neighbors, according to some rule.

If no node is able to fire, we say that the game is relaxed. A node $v$ containing exactly $k_v$ chips is critical. If $v$ contains more than $k_v$ chips, we say $v$ is supercritical.
In Figure 1.1, the node numbers enumerate the chips resting on their respective nodes. The game operates as follows: each node $v$ has critical number $k_v = \deg(v)$; when a node fires, it sends one chip to each neighbor. The figure shows a typical sequence of configurations under these stipulations. In each configuration, the shaded node is the one about to fire, which leads to the next configuration.

Note that in the first configuration, all but the lower-left node may fire. Bjorner et al. [6] showed that when more than one node may fire, the order in which the nodes are fired does not affect either the number of firings until the game is relaxed or the final configuration of the chips. Note also that the final configuration would lead to a configuration identical to the initial one if the shaded node fired. This suggests that this particular chip-firing game will be of infinite length. Bjorner et al. proved that if the number of chips on a graph exceeds a certain value (that depends on the numbers of nodes and edges in the graph), the chip-firing game will indeed never terminate.
The chip-firing game studied in this dissertation is similar to that of [6], with one small difference that will force all games to be of finite length. Before we turn our attention to this new game, we formally define “chip-firing game” and a few related terms.

1.3 Definitions involving chip-firing games and burn-off games

Let $G = (V, E)$ be a connected graph. For each node $v$, we place a nonnegative number $C(v)$ of chips on $v$. A particular distribution $C : V \rightarrow \mathbb{N}$ of chips on $V$ is called a configuration. Any node $v$ that contains at least as many chips as its critical number $k_v$—that is, $C(v) \geq k_v$—may be fired; we call such a node $v$ fire-able. A node that fires sends a chip to each of its neighbors.

With these definitions in mind, we consider the following chip-firing game: Begin with any configuration on $G$. If there exists a node that may be fired, we fire it. This constitutes a turn of the game. As the turns of a game progress, we say that the configuration is relaxing. If no node may be fired, the game ends; otherwise, a new turn begins in which any fire-able node is fired. The length of the game is the number of turns taken from the initial configuration until the game ends. If at the start of the game no node may be fired, the game has length zero. If no node is able to fire, we say that the configuration is relaxed. To play a new game, a chip is placed on a randomly selected node. This node is called the seed. We also use seed as a verb to refer to the process of making this selection.
In most models, including the Bjorner et al. model [6], we take $k_v$ to be the degree $\text{deg}(v)$ of $v$. A theorem in [6] states that in such a chip-firing game, the nodes may be fired in any allowable order, and

(a) the length of the game will not be affected;

(b) the final configuration will not be affected.

This theorem implies that if the graph contains more than one fire-able node, the choice of which node to fire has no bearing on the length of the game or the final configuration of the chips. The same paper [6] points out that some chip-firing games may be of infinite length. For example, if the number of chips on the graph exceeds twice the number of edges, then, by the pigeonhole principle, we may always find at least one node that contains enough chips to fire. The paper also establishes that if the number of chips on the graph is less than the number of edges, then every chip-firing game is of finite length.

In our consideration of models for real-world phenomena, it is helpful to study a slightly different chip-firing game in which all games are of finite length. Our goal is to enumerate games of each possible length, and infinite games obstruct this analysis. Further, since many models are linked to phenomena such as earthquakes and avalanches that involve friction and kinetics, it is reasonable to build into those models some recognition that the energy in the system may be lost through escaping heat. Indeed, Kauffmann [12, pp. 71-92] argues that complex
systems spontaneously exhibit both order and complexity at the cost of thermodynamic energy loss.

Therefore, we consider the following variation of the chip-firing game above: a node \( v \) may only be fired when \( C(v) \geq \deg(v) + 1 \), and when \( v \) is fired, one of its chips is lost from the system. The chip that escapes at each firing models the loss of energy from the system. We call such a game a burn-off game. Such a system is referred to as "dissipative" by physicists, in contrast with "conservative" systems like classical chip-firing games, that do not lose energy as they are processed. We also note that critical takes on a new meaning in the context of burn-off games: a critical node is one that will fire when a chip is added to it.

The study of modified chip-firing games is not uncommon. In [9], Eriksson summarizes and extends the results of [6] by considering games on directed graphs and games where the edges are weighted. In the latter case, the values placed on the nodes are not limited to integers but allow any real numbers. A node fires when its value is negative, and this value is added to the values on its neighbors. Both modified games exhibit properties of the original described in [6], although in some ways they differ drastically.

Eriksson continues to explore modified chip-firing games in [8], where he considers graphs that mutate between games. After a node fires, the edges incident with the node may be erased and then new edges may be added. The mutations
occur in a predetermined order; this order may or may not have a finite period. As in the other papers mentioned so far, close attention is given to discovering the conditions under which the games are finite or infinite.

The variant studied by Biggs in [4] includes a vertex that can always fire, but will only do so if no other node in the graph can fire. Biggs confirms the results in [6] for his variant and demonstrates that the set of “stable”, “recurrent” configurations of a graph has the structure of an abelian group.

A final example of recent interest in this topic is [5]. Bitar and Goles consider the periodic nature of chip-firing games in which all nodes that can fire are fired simultaneously. These “parallel” chip-firing games are shown to simulate logical functions, like OR and NOT, making chip-firing games of nontrivial interest to computer scientists.

1.4 Self-organized criticality

Every firing in a burn-off game removes a chip from the system; clearly, every burn-off game is of finite length. Later (see Lemma 4.8) we will see that the length of a burn-off game is bounded from above by the number of nodes. For example, burn-off games played on the graph in Figure 1.1 will be of lengths zero through four. Figure 1.2 shows the results of 10,000 simulated burn-off games played on the graph in Figure 1.1; the code, written in Smallbasic, appears in
Appendix 2. The simulation proceeds as discussed in Section 1.3 (p. 5). Note that longer games occur with lower frequency.

![Frequency of Game Length](image)

**Figure 1.2. Results of 10,000 burn-off games**

Empirical results like these are common in papers that investigate simple models of complex systems. In [7], for example, Dhar models a pile of sand by assigning numerical values to cells in a lattice. Each value represents the steepness of the sandpile in that region. Stresses are added at random until a cell’s steepness exceeds some threshold—at which point the sand slips. In the model, the simulated stress is distributed to the cell’s neighbors, and again the cells are checked for instability. The simulated avalanche continues until all cells of the grid have steepnesses within the threshold, at which point the number of cells that slipped during the avalanche is recorded. This process continues through many computer-driven iterations, and when the data is compiled, it compares favorably with measurements taken from real sandpiles.
Complex systems like this sandpile model potentially possess what is called *self-organized criticality* (hereafter abbreviated SOC). The important features of SOC are listed in [1], a nontechnical introduction to the subject that appeared in *Scientific American*. The primary characteristic of SOC is this: the system, although complex, displays predictable behavior when viewed at a macroscopic level. Such systems are simulated using computer software that collects data relating the size and frequency of events. In a sandpile model, the events are avalanches (as measured by the number of sand particles that slipped during the event), and they can occur every time a bit of sand is added to the system. In a burn-off game, the events are games (as measured by the number of cells that fire), and they can occur every time a chip is added to a cell.

In some systems, the relationship between the size and frequency of events in the system follows a power law; that is, if $S$ is the size of an event, and $F_S$ is the frequency of events of size $S$, then $F_S \approx Cy^S$ for some real constants $C$ and $\gamma$. A power law relationship is a hallmark of a system that is in a self-organized critical state; on page 1 of [11], for example, the author states, "Although the dynamical response of the systems is complex, the simplifying aspect is that the statistical properties are described by simple power laws." The results in Figure 1.2 suggest this relationship for some $\gamma < 1$. We estimate $\gamma$ using regression after considering an earthquake model due to Bak and Tang [3].
Their simple model works on a 50 × 50 lattice. The neighbors of each cell are defined to be those horizontally and vertically adjacent. A cell fires if its value is at least four, increasing the value of each neighbor by one while itself decreasing in value by four. Cells on the edge of the lattice operate as if an invisible border of cells surrounds the lattice: if a cell on the edge fires, one stress (or two stresses, in the case of corner cells) is (are) lost from the system. Figure 1.3 shows the results of 10,000 such games simulated with a Smallbasic program (the code appears in Appendix 2).

![Figure 1.3. Results of 10,000 Bak games](image)

Before the first game, every cell in the lattice receives between one and four stresses with uniform probability. In Section 2.3, we discuss an alternative method of initializing a game. Note that results for games of length zero and of
length greater than 35 are omitted, and that the results are compressed into groups of five; both of these decisions were made simply to neaten the presentation. The lefthand bar in each pair shows the results from the simulation, while the righthand bar depicts the standard (least squares) exponential regression. For this data, the regression gives $C = 7.834$ and $\gamma = 0.924$, so $F_s \approx 7.834(0.924)^s$. The results in [3] exhibit the same behavior. The agreement between the simulated data and the regression is at best marginally satisfying. Applying the same analysis to the data in Figure 1.2, we find that $C = 39$ and $\gamma = 0.622$, so $F_s \approx 39(0.622)^s$ (see Figure 1.4). The results in this dissertation provide a way to predict the simulated data mathematically, and Figure 4.6 (p. 96) demonstrates the close agreement we can find using our methods.

![Figure 1.4. Results of 10,000 burn-off games, revisited](image)

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Since SOC was originally identified [3], its possible links with real-world phenomena have been widely explored. While earthquakes [3] and sandpile avalanches [7] are the events most commonly connected with SOC, investigation has also proceeded along less obvious lines. Nagel and Paczuski [15] created a simple traffic model and studied its self-organized critical behavior, using traffic jams as the “events” in their model. The model considers a single lane of traffic in which all cars move at or below the maximum allowable speed, responding to the car ahead according to preset rules of acceleration and deceleration. The model then introduces a small random element that interferes with the deterministic motion of the cars. Resultant traffic jams are noted, and the system is allowed to relax into its original deterministic state. The authors show that the frequency of the jams has a power law relationship with the size of the jams.

Stassinopolous and Bak [18] turned their attention to the human brain, modeling the brain as a graph in which neurons are nodes and their connections are edges. Each neuron possesses a firing threshold: when its neighbors fire, they add a charge to the neuron; if the charge exceeds the threshold, the neuron fires. An “event” in this model is a thought—if the proper neurons fire, the thought is a successful one, and the neurons that were involved in the thought are “rewarded” by having their thresholds decreased.
Liu et al. [14] investigate the fractal nature of the sandpile model studied in [3] when the model is allowed to relax from uniform initial conditions—in other words, when every cell in the sandpile begins at the same slope. The authors experiment with different boundary shapes (e.g., square, triangular) and different initial conditions. Despite the uniformity of the initial configurations, the sand exhibits behavior typical of systems in a SOC state.

As an example of how the theory of SOC has reached a non-mathematical audience, we note a recent article (2002) in Atlantic Monthly [16] that investigates a wide variety of systems demonstrating elements of SOC behavior, from persons of different ethnic backgrounds tending to live in ethnic neighborhoods, to rival religious groups with tendencies toward inter-group violence, to a possible explanation for the disappearance of the Anasazi culture of the southwestern United States in 1300 A.D.

1.5 SOC and burn-off games

As discussed in Section 1.4, a system that exhibits SOC should demonstrate a power law relationship between the frequency and size of the events in the system [1]. These systems can be modeled by chip-firing games, and the particular kind of chip-firing game in question in this work has the burn-off feature that not only more realistically models real-world phenomena, but also allows us to investigate mathematically whether such chip-firing games exhibit SOC. Later, we
discover the closed formula (5.15) relating burn-off game length to frequency on complete graphs; restricting the model to complete graphs, however, produces analytic results that do not indicate the presence of SOC. Nevertheless, the model exhibits interesting properties of its own, which are the focus of this dissertation.

Jensen [11] devotes a chapter to a discussion of computer models of SOC. On p. 29, his list of features that characterize the models he studies are all characteristics of burn-off games. These features are: “the dynamical variable or field is updated in every time step according to some algorithm”; “the choice of the updating algorithm is, to some degree, arbitrary”; and “the criteria for choosing the relevant definitions are, for the most part, simplicity and intuition”. A burn-off game exemplifies these features. Of attempts to formalize the study of SOC, such as [7], Jensen writes, “Despite their undeniable beauty, the exact solutions have one drawback: the specific mathematics tends to be tailored to the details of the solved model.” In this dissertation, we find analytical results that apply to burn-off games played on any connected graph; although these results are not closed formulas, they are certainly not “tailored to the details” of the model. Our strictly mathematical approach adds a level of rigor to a field typically centering more on empirical results.
1.6 Outline

In Chapter 2, we show that burn-off games possess an important feature of other chip-firing games and discover an algorithm that helps determine all configurations of chips that can occur during a sequence of burn-off games on a connected graph. These results are crucial in subsequent chapters.

In Chapter 3, we investigate complete graphs and find a relationship between chip configurations that arise during burn-off games and spanning trees of related graphs. As an aside, we demonstrate how this result produces a new proof of Cayley’s Theorem for enumerating the spanning trees of a complete graph.

Before we use the results of Chapter 3 to find a closed formula relating burn-off game lengths to frequency, we extend the results on complete graphs to connected graphs in Chapter 4. The chapter concludes with a method for determining the probability that a burn-off game on a connected graph will be of any given length. This method is used to generate analytically results like those in Figure 1.1, which were generated by computer simulation.

Finally, Chapter 5 contains the remaining results for burn-off games on complete graphs. The methods of Chapter 4 are shown to confirm what we discover about complete graphs in Chapter 3. We conclude with ideas for further research.
Chapter 2

Preliminary results

2.1 Introduction

This chapter establishes results that are fundamental to subsequent chapters. We first show that the order in which nodes are fired in a burn-off game is irrelevant to the final configuration of chips when the game ends. After discussing how we may characterize those relaxed configurations that can occur at the beginning of a sequence of burn-off games, we present an algorithm that recognizes all relaxed chip configurations that can occur during the sequence of games.
2.2 Burn-off games exhibit strong convergence

Burn-off games enjoy the property that all are of finite length. This is clear: after every turn, the total number of chips decreases by one, so eventually no node will be able to fire. We now show that burn-off games possess an additional important property called strong convergence, as defined in [9], namely, that nodes may be fired in any order without affecting the length or final configuration of a game.

Proposition 2.1 In a burn-off game on a connected graph $G = (V,E)$, the nodes may be fired in any order without affecting the length or final configuration of the game.

In our proof below, we follow the argument in [6], establishing the same conclusion for the chip-firing game considered there. The authors show that the "language" of such games possesses three properties that together imply their version of the assertion. In order to discuss the essential ingredients of their argument, we need to introduce some terminology and notation.

Label the nodes of $G$. It may be the case that a particular configuration on $G$ can be played in more than one way. For each firing sequence, write down the labels that correspond to the nodes in the order that they are fired. The resulting sequence of labels is a word, and the set of all possible words, over all possible initial configurations, is a language $L$. The empty word $\lambda$ is a member of every language,
corresponding to a game of length zero. A word need not bring a configuration to its eventual relaxed state.

Let \( n = |V| \) be the number of vertices. If \( \alpha \) is a word, we define its score vector \( [\alpha] \in \mathbb{N}^n \) as follows:

\[ [\alpha]_i = k \text{ if the node } i \text{ occurs } k \text{ times in } \alpha. \]

For example, suppose that a four-node graph has labels \( \{1, 2, 3, 4\} \). If \( 23424 \in \mathcal{L} \) (indicating that node 2 fires first, followed by node 3, and so on), then \( [23424] = (0, 2, 1, 2) \), because node 1 fires 0 times, node 2 fires 2 times, and so on. The issue here is whether a firing sequence different from 23424 could result in the same final configuration. Now we are ready to state the three aforementioned properties sufficient for Proposition 2.1.

**Definition 2.2**

1. \( \mathcal{L} \) is **left-hereditary** if, whenever a word belongs to \( \mathcal{L} \), every initial segment of the word also belongs to \( \mathcal{L} \). For example, if \( 23424 \in \mathcal{L} \), then so must \( \lambda, 2, 23, 234, \) and 2342 be elements of \( \mathcal{L} \).

2. \( \mathcal{L} \) is **locally-free** if, for any \( \alpha \in \mathcal{L} \) and any two different nodes \( x \) and \( y \) with \( \alpha x \in \mathcal{L} \) and \( \alpha y \in \mathcal{L} \), we also have \( \alpha xy \in \mathcal{L} \). For example, if \( 2342 \in \mathcal{L} \), and both \( 23421 \in \mathcal{L} \) and \( 23424 \in \mathcal{L} \), then so also are \( 234214 \) and \( 234241 \) elements of \( \mathcal{L} \).
(3) \( \mathcal{L} \) is permutable if whenever \( \alpha, \beta \in \mathcal{L} \), with score vectors \([\alpha] = [\beta]\), and \( \alpha x \in \mathcal{L} \) for some node \( x \), then \( \beta x \in \mathcal{L} \). In other words, if one word is just a permutation of another, then both may be extended by the same symbol to obtain a new word of \( \mathcal{L} \). For example, if \( 23424 \in \mathcal{L} \) and \( 22344 \in \mathcal{L} \), and \( 23424 \) may be extended to \( 234241 \in \mathcal{L} \), then \( 223441 \) is also an element of \( \mathcal{L} \).

Proof of Proposition 2.1. We follow the strategy of [6], where the authors show that any two words (in a language possessing the properties of Definition 2.2) that have the same score will describe two different firing sequences, yet will result in the same final configuration. Thus, we verify that if \( \mathcal{L} \) is the collection of all chip-firing words in a burn-off game, then \( \mathcal{L} \) possesses the properties of Definition 2.2. Recall that in this collection we include all configurations that have not yet relaxed, but are merely on their way to a relaxed configuration.

That \( \mathcal{L} \) is left-hereditary is clear: for a game to have progressed from one configuration to another necessarily means that all intermediate configurations must have also belonged to \( \mathcal{L} \).

To see that \( \mathcal{L} \) is locally free, consider a configuration that allows for two different nodes, \( x \) and \( y \), to fire. Firing one of them (say, \( x \)) sends one chip to each neighbor of \( x \) (and burns one chip from the system). Thus, a node \( y \) that could have
fired before \( x \) fired can certainly still do so, since the number of chips on \( y \) has either remained unchanged or increased by one.

Finally, to see that \( L \) is permutable, we argue that any two partial games corresponding to words \( \alpha \) and \( \beta \) with the same score must lead to the same configuration. There are only two ways that the number of chips on a node \( x \) can change: if \( x \) fires, the number of chips on \( x \) decreases by \( \text{deg}(x) + 1 \); if a neighbor of \( x \) fires, the number of chips on \( x \) increases by one. If two partial games \( \alpha \) and \( \beta \) have the same score, then \( x \) fires the same number of times in each game, as do the neighbors of \( x \). Thus, the configurations at the end of either partial game are identical. \( \square \)

The property that nodes may be fired in any order without affecting the length or final configuration of a burn-off game will be essential to almost every argument presented in this dissertation.

2.3 Definition of reverse-firing game

In the literature (e.g., [3], [14]), computer models of complex systems are initialized into a state that the authors assume will exhibit SOC as soon as the model operates on the system. For example, in Bak's paper [3], which studied SOC on a checkerboard-grid, the system was initialized by randomly assigning an integer larger than four (and smaller than some pre-determined upper bound) to each cell
of the grid. Recall that in Bak's model a cell fires if its value exceeds four. Thus, this initialization makes it possible for all cells to fire. This grid is then relaxed until no cells can fire. Bak assumes that the resulting graph will immediately begin to exhibit SOC as the values in the cells begin to be randomly perturbed.

Define a configuration to be supercritical if every node is supercritical. In our analysis, we shall play burn-off games only on those configurations that can result from relaxing a supercritical configuration. In fact, our first task in the following analysis is to establish, for any given connected graph, how to determine if a given configuration of chips can indeed be such a result.

To determine those configurations that are the result of relaxed supercritical configurations, it is instructive to consider what happens when a chip-firing game is played in reverse. A reverse-firing game is defined so as to undo the firing rule of the chip-firing game under consideration. For example, recall the firing rule for the burn-off model: a node $v$ may be fired only when $C(v) \geq \text{deg}(v) + 1$, and when a node is fired, one chip from $v$ is lost from the system. In a reverse-firing game, chips are added to the graph at each turn. The reverse-firing rule for the burn-off model is: select any node $v$ such that all neighbors of $v$ contain at least one chip; from each neighbor, move one chip onto $v$, then add a chip to $v$ from outside the system.
Figure 2.1. An instance of a reverse-firing game. The ellipsis indicates that the fifth configuration (lower left) continues to reverse-fire until the final supercritical configuration (lower right) is reached.

Example 2.3 Figure 2.1 shows this in action on a graph $G$. Suppose we begin a reverse-firing game on $G$ by reverse-firing $v_3$. Nodes $v_2$ and $v_4$, the neighbors of $v_3$, both donate one chip to $v_3$, and another is added from outside the system, for a total of 4 chips on $v_3$. Of course, in the second configuration, $v_3$ is able to fire; if it did, the first configuration would be the result. The figure also displays the results if the nodes are reverse-fired in the order $v_3$, $v_4$, $v_2$, and $v_1$. In the fifth configuration, the nodes $v_1$, $v_2$, and $v_4$ are all able to fire. However, $v_3$ does not have enough chips to fire in a burn-off game. If we continued the reverse-firing game by reverse-firing the nodes again in the same order, the final configuration would be as shown in the
figure. In this configuration, all nodes are supercritical. If this configuration were used to initialize $G$, then the first configuration in the figure would result from playing the burn-off game to relaxation.

While it is certainly possible that this reverse-firing game can continue indefinitely, we will only care to play such games until every node is able to fire. Any configuration, like the first one in Figure 2.1, that can be reverse-fired to a supercritical state shall be called legal. A relaxed legal configuration is a legal configuration in which no nodes may fire. Because the rest of this chapter is concerned with enumerating relaxed legal configurations, we let $L(G)$ denote the number of relaxed legal configurations on a graph $G$. Note that as we define $L$ we drop the stipulation that $G$ be connected. The flexibility allowed by dropping this stipulation will be useful in Theorem 4.9, which considers relaxed legal configurations on disconnected graphs. None of the subsequent results in this chapter require that $G$ be connected.

2.4 Characterizing relaxed legal configurations

Not all relaxed configurations are legal; for example, a graph containing no chips is clearly relaxed, but just as clearly cannot be reverse-fired into a supercritical state. We are now ready to characterize the relaxed legal configurations on any given graph $G = (V, E)$. Our characterization uses the notation $1_A : V \rightarrow \{0, 1\}$ to denote the indicator function of a subset $A \subseteq V$. 

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Proposition 2.4  A relaxed configuration \( C : V \rightarrow \mathbb{N} \) is legal if and only if there exists an ordered partition \((I_k)\) of independent sets \(I_k\) of \(V\) so that, for each \(v \in V\), if \(v \in I_j\), then

\[
C(v) \geq \sum_{x \rightarrow y} 1_{(x \in I_r \text{ and } y \in I_j)}.
\]

For convenience later, we call the property of \((C, (I_k))\) that every node \(v\) contains at least as many chips as it has neighbors that are members of earlier independent sets Property \(P\).

![Figure 2.2. A legal configuration](image)

Example 2.5  Let \(C\) be the configuration shown in Figure 2.2. We may assign the nodes to independent sets as follows: \(I_1 = \{v_1, v_3\}\); \(I_2 = \{v_4\}\); \(I_3 = \{v_2\}\). Nodes \(v_1\) and \(v_3\) have no neighbors in earlier independent sets because they are members of \(I_1\), the earliest independent set of all. Node \(v_4\) has one neighbor, \(v_3\), in an earlier independent set; since \(v_4\) contains at least one chip, Property \(P\) is not violated at \(v_4\). Node \(v_2\) has three neighbors in earlier independent sets, and it contains three chips. Thus, Property \(P\) holds for this choice of \((C, (I_k))\).
Proposition 2.4 implies that since this configuration is relaxed, it is legal; that is, it may be reverse-fired to a supercritical configuration. As we saw in Example 2.3, this configuration is indeed legal. Now we verify (in general) that this legality is equivalent to the existence of a partition of \( V \) with Property \( P \).

**Proof of Proposition 2.4.** Suppose that we have a relaxed legal configuration \( C \). By definition, a legal configuration is one that can be reverse-fired into a configuration where all nodes are supercritical. Further, \( C \) is relaxed, so none of the nodes in \( G \) are supercritical. Thus, in any reverse-firing game that reveals a relaxed configuration to be legal, all nodes must reverse-fire (because reverse-firing is the only way for a node to gain chips during a reverse-firing game).

Consider any reverse-firing sequence that shows \( C \) to be legal. List only the first time each node reverse-fires during the game; suppose that the nodes are reverse-fired in the order \( v_1, v_2, \ldots, v_n \). Put each node \( v_j \) into its own set \( I_j \). Since each \( I_j \) contains only one node, each member of the ordered family \( (I_j) \) is independent. We now need to show that \( (C, (I_j)) \) possesses Property \( P \).

Any given node \( v_j \) will reverse-fire only after each node \( v_1, v_2, \ldots, v_{j-1} \) reverse-fires at least once. Each of these nodes that is a neighbor of \( v_j \) takes a chip from \( v_j \) when it reverse-fires. Since each earlier node is a member of an earlier independent set, \( v_j \) must contain at least as many chips in \( C \) as it has neighbors that
are members of earlier independent sets.

To establish the converse, suppose that the nodes can be partitioned into \( r \) independent sets so that \( (C,(I_k)) \) possesses Property \( P \). Reverse fire the nodes of \( I_k \), for \( k = 1,2,...,r \), in that order (although within sets, the nodes may be fired in any order). We claim that this reverse-firing results in each node increasing its number of chips by one.

Consider a node \( v \in I_j \), and let \( C(v) \) be the number of chips on \( v \) before the reverse-firing process starts. Because of Property \( P \), the node \( v \) contains at least as many chips as it has neighbors in all \( I_k \) with \( k < j \). Suppose that there are \( s \) such neighbors. Each neighbor in these sets \( I_k \) with \( k < j \) reverse-fires before \( v \) does, and each reverse-firing will pull one chip from \( v \). This leaves \( C(v) - s \geq 0 \) chips on \( v \). If any nodes that are in \( I_j \) reverse-fire before \( v \), the number of chips on \( v \) is unaffected, since \( I_j \) is independent. When \( v \) reverse-fires, it pulls a chip from each of its \( \deg(v) \) neighbors and receives one extra chip for the reverse burn-off. Finally, all nodes in sets \( I_m \) with \( m > j \) reverse-fire, and each neighbor of \( v \) in these sets (say there are \( l \) of these) pulls a chip from \( v \). Therefore, after each node has been reverse-fired once, the number of chips on \( v \) is decreased by \( s + l = \deg(v) \) and increased by \( \deg(v) + 1 \) (while never becoming negative), for a net increase of one, as claimed.

Notice that reverse-firing each node once, as described in the preceding paragraph, preserves Property \( P \). Thus, this process may be repeated until all nodes
become supercritical. That is, if \( t = \max_v \{\deg(v) - C(v)\} \), then repeating the process \( t + 1 \) times will result in every node \( v \) containing at least \( \deg(v) + 1 \) chips. Therefore, the original configuration was legal.

2.5 Checking legality of any configuration on any graph

Given a configuration \( C \) (not necessarily relaxed) on a graph \( G \), we may be interested in knowing if \( C \) is legal. Though we will not use it immediately, we take a short detour here to present an algorithm to answer this question. The algorithm will often be useful in Chapter 4 as we consider burn-off games on a connected graph. Our proof of the algorithm's efficacy leans on Proposition 2.4.

Algorithm 2.6

**INPUT:** a graph \( G = (V,E) \) and a chip configuration \( C : V \rightarrow \mathbb{N} \) on \( G \);

**OUTPUT:** answer to question “Is \( C \) legal?”

1. Find \( v \in V \) such that \( C(v) \geq \deg(v) \). If this cannot be done, then stop: \( C \) is not legal. Otherwise, let \( G' = G \).

2. Delete \( v \) from \( G' \). If all nodes are now deleted, then stop: \( C \) is legal. Otherwise, let \( G' = (V', E') \) be the new graph.

3. Find \( v \in V' \) such that \( C\big|_{V'}(v) \geq \deg_{G'}(v) \). If this cannot be done, then stop: \( C \) is not legal. Otherwise, go to step (2).
Example 2.7  Figure 2.3 shows two ways in which Algorithm 2.6 might operate on the configuration depicted. In both passes from left to right, appropriate nodes are deleted until none remain. In the first step, the algorithm may delete either $v_2$ or $v_4$, as they both contain at least as many chips as their degree. At the end of either execution sequence, all nodes are deleted, so, by Proposition 2.7 below, the starting configuration is legal.

![Algorithm execution sequences](image)

Figure 2.3. Two execution sequences of Algorithm 2.6 on the same initial configuration

In the proof of Proposition 2.9, we need the following result.

Lemma 2.8  Let $C : V(G) \to \mathbb{N}$ be a configuration and $G'$ a subgraph of $G$. If $C|_{V(G')} \neq \text{legal on } G'$, then $C$ is not legal on $G$. 

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Proof. We will argue the contrapositive: if $C$ is legal on $G$, then so must $C_{|V(G')}$ be legal on $G'$. By Proposition 2.4, the legality of $C$ on $G$ implies that it is possible to partition $V(G)$ into independent sets $(I_k)$ enjoying Property $P$ (see p. 25).

For each $k$, consider the partition $J_k = I_k \cap V(G')$ of $V(G')$. Collect all nonempty sets $J_k$ into the ordered family $(J_r)$ of independent sets, and focus on a node $v \in V(G')$. Since $(C,(I_k))$ has Property $P$, the node $v$ (considered now in $G$) contains at least as many chips as it has neighbors (in $G$) that are members of earlier independent sets. In $G'$, the node $v$ may have fewer such neighbors, but it obviously cannot have more. Thus, $(C_{|V(G'),(J_k)})$ has Property $P$, so $C_{|V(G')} \text{ is legal on } G'$.  

Let $L$ be the set of legal configurations on $G$. Now we are ready to establish the correctness of Algorithm 2.6.

**Proposition 2.9** Given a graph $G = (V,E)$ and a configuration $C : V \rightarrow \mathbb{N}$, Algorithm 2.6 correctly determines whether $C \in L$.

**Proof.** First, we show that if at any point during the operation of Algorithm 2.6 (say, when we have arrived at a subgraph $G'$) every node $v$ contains fewer than $\deg_{G'}(v)$ chips, then the original configuration is not legal. Suppose, by way of
contradiction, that an original configuration $C^*$ leading to this situation on $G^*$ is legal. By Lemma 2.8, $C^*|_{V(G^*)}$ is legal; then the nodes of $G^*$ may be partitioned into independent sets $(J_r)$ so that $(C^*|_{V(G^*)}, (J_r))$ has Property $P$. Consider a node $v$ that is a member of the last set $J_s$. Since all of its neighbors are members of earlier independent sets, $v$ contains at least $\deg_{G^*}(v)$ chips, contradicting our assumption in the first sentence. Thus, if every $v \in V(G^*)$ contains fewer than $\deg_{G^*}(v)$ chips, then $C^*$ is not legal.

Second, we show that if the algorithm proceeds until all nodes are deleted, then $C$ is legal. Suppose that the algorithm's deletion order is $v_n, v_{n-1}, \ldots, v_1$. Place each node $v_j$ into its own set $I_j$. For $v_j$ to be deleted from $G^*$ by the algorithm, it must contain at least as many chips as it has neighbors in $G^*$. Since every one of these neighbors is in an earlier independent set, our choice of $(C, (I_k))$ has Property $P$, and so $C$ is legal. ■

2.6 The poset of configurations

Now that we have characterized legal configurations, we turn our attention to a result that allows us (in Chapter 4) to enumerate the relaxed legal configurations on any given connected graph. We assume the reader is familiar with the topic of posets as discussed, e.g., in [13].

Chip-firing games proceed in cycles beginning with the random placement of a stress (one chip) that may trigger a game of nontrivial length. We...
therefore consider configurations that arise from adding a single chip to a legal configuration. To this end, it is useful to consider the set \( \mathcal{B} \) of all configurations on a fixed graph \( G \) as a poset \( (\mathcal{B}, \leq) \) whose ordering relates to the numbers of chips on the nodes of \( G \) as follows: for \( P, Q \in \mathcal{B} \) and \( \leq \) the usual (total) ordering on \( \mathbb{N} \), let

\[
P \preceq Q \text{ iff each } v \in V(G) \text{ satisfies } P(v) \leq Q(v).
\]

Recall that a legal configuration is one that can be reverse-fired to a supercritical configuration (see Section 2.3).

**Proposition 2.10** If \( P \) is a legal configuration, then any \( Q \) with \( P \preceq Q \) is also legal.

**Proof.** We clearly need only consider those configurations \( Q \) with \( P < Q \). Such a \( Q \) has at least as many chips on any given node \( v \) as does \( P \), and since \( P < Q \), there exists a node \( x \) with \( P(x) < Q(x) \). Starting from the configuration \( P \), add one chip to \( x \) to create a new configuration \( P' \).

Since \( P \) is legal, there exists a reverse-firing sequence that results in a supercritical configuration. "Freeze" the new chip on \( x \), and carry out the same reverse-firing sequence starting with \( P' \). The frozen chip will not affect the reverse-firing game (since it is frozen), and once \( P \) is reverse-fired to a supercritical configuration, the chip may be "thawed". The resulting supercritical configuration shows that \( P' \) is legal.
If \( P' = Q \), the assertion is proved; if not, the argument above can be repeated with \( P' \) in the role of \( P \). ■

Now suppose that a relaxed legal configuration \( Q \) has a chip added to a randomly selected node \( v \), creating the configuration \( Q^* \). By Proposition 2.10, we know that \( Q^* \) is legal. Suppose that \( v \) may fire in \( Q^* \), and let \( R \) be the relaxed configuration that results. Since \( R \) can be reverse-fired back to the legal configuration \( Q^* \), we know that \( R \) is also legal. Thus, if we initialize a sequence of burn-off games by relaxing an arbitrary supercritical configuration, then all relaxed configurations that occur during the sequence of burn-off games will be legal.

We have demonstrated in this chapter how we may begin playing a series of burn-off games on a connected graph and how to recognize when a relaxed configuration is legal. This knowledge will be important in Chapter 4, where we analyze burn-off games on any given connected graph. First, we consider the special case of complete graphs in Chapter 3.
Chapter 3
Investigations on complete graphs

Before we investigate how the material in Chapter 2 leads to analytical results for general connected graphs, we pause to study complete graphs in this chapter. We find a pair of algorithms that give a one-to-one correspondence between relaxed legal configurations on $K_n$ and spanning trees of $K_{n+1}$. These algorithms not only provide us with a proof of Cayley’s Formula (see [19]), but also prepare us to discuss a similar, but more difficult, pair of algorithms in Chapter 4.

3.1 Statements equivalent to Property $P$

Proposition 2.4 established that a configuration $C$ on a graph is legal if and only if the nodes can be partitioned into independent sets $(I_r)$ so that $(C,(I_r))$
possesses Property $P$. Here we show that Property $P$, for complete graphs, may be reformulated in two ways that will be helpful later. We continue to let $C$ denote a chip configuration, here on $V(K_n)$, and now define $q = (q_1, q_2, \ldots, q_n)$ by $q_i = C(v_i)$.

**Proposition 3.1** Let $C : V \to \mathbb{N}$ be a configuration on $K_n$. The following statements are equivalent:

1. For some partition of $V$ into independent sets $(I_s)$, the pair $(C, (I_s))$ satisfies Property $P$;
2. For each $j = 1, 2, \ldots, n$, at most $j$ of the $q_i$ are at most $j - 1$;
3. For each $k = 1, 2, \ldots, n$, at least $k$ of the $q_i$ are at least $n - k$.

**Proof of (1) $\Rightarrow$ (2).** Suppose that $(C, (I_s))$ has Property $P$. Then for $s = 1, \ldots, r$, every node in $I_s$ contains at least as many chips as it has neighbors that are members of earlier independent sets. In a complete graph, the only way to partition the nodes into independent sets is to assign each node to its own set. Thus, each independent set contains just one node; label that node $v_j$ for each $j = 1, 2, \ldots, n$.

By Property $P$, each $v_{j+1}$ contains at least $j$ chips. The only nodes that might contain $j - 1$ chips or fewer are $v_1, v_2, \ldots, v_j$; this implies that at most $j$ nodes contain at most $j - 1$ chips, which establishes statement (2).
Proof of (2) ⇒ (3). If at most \( j \) of the \( q_i \) are \( j - 1 \) or less, then at least \( n - j = k \) of the \( q_i \) are \( j = n - k \) or more.

Proof of (3) ⇒ (1). It will be convenient to relabel the nodes of \( K_n \) so that the index runs from 0 to \( n - 1 \) and \( C(v_i) \geq C(v_{i+1}) \) for \( i = 0, 1, \ldots, n - 2 \). Define \( p = (p_0, p_1, \ldots, p_{n-1}) \) by \( p_i = C(v_i) \); observe that \( p \) is a permutation of \( q \). By statement (3), at least \( k \) of the \( p_i \) are at least \( n - k \), for \( k = 1, 2, \ldots, n \). Because \( (C(v_i)) \) is nonincreasing, we have \( C(v_j) = p_j \geq n - j - 1 \) for \( j = 0, 1, \ldots, n - 1 \).

Let \( I_j = \{v_{n-j-1}\} \) for \( j = 0, 1, \ldots, n - 1 \); equivalently, \( I_{n-j-1} = \{v_j\} \) for \( j = 0, 1, \ldots, n - 1 \). Since each \( I_j \) contains a single node, it is independent. Since each \( v_j \in I_{n-j-1} \) and the graph is complete, we see that \( v_j \) has \( n - j - 1 \) neighbors in earlier independent sets. We have \( C(v_j) \geq n - j - 1 \) from the preceding paragraph, so \( (C, (I_j)) \) has Property \( P \).

Example 3.2 Consider \( K_6 \), and let \( q = (5, 2, 1, 5, 0, 2) \). Statement (2) shows this configuration to be not legal: let \( j = 3 \), and observe that the statement "at most 3 entries of \( q \) are at most 2" is false. Statement (3), with \( k = 3 \), is "at least 3 entries of \( q \) are at least 3," which is also false.
3.2 Enumerating relaxed legal configurations on a complete graph

Here we establish a formula for $L(K_n)$, the number of relaxed legal configurations $C$ on a complete labeled graph. Our determination includes a parameter to bound the maximum value of $C(v)$ for $v \in V = V(K_n)$. For $n \geq 1$ and $m \geq n - 1$, let $L_{n,m}$ be the number of legal configurations satisfying $C(v) \leq m$ for each $v \in V$. For convenience, we also define $L_{0,m} = 1$ for all $m \geq 0$. In the proof of the next result, we shall find it convenient to use characterizations (2) and (3) in Proposition 3.1.

**Theorem 3.3**  
For all $n \geq 1$ and $m \geq n - 1$, we have $L_{n,m} = (m - n + 2)(m + 2)^{n-1}$.  

**Example 3.4**  
Suppose that we wish to find the number of legal configurations on $K_4$, where no node contains more than 5 chips, as in Figure 3.1. Note that the nodes $v_1$ and $v_3$ have enough chips to fire, so this is not a relaxed legal configuration. To count all relaxed legal configurations, we would fix $m$ equal to the degree of each node in $K_n$, which is $n - 1$. Theorem 3.3 asserts that the number of legal configurations like the one in Figure 3.1 is $L_{4,5} = (5 - 4 + 2)(5 + 2)^{4-1} = 1029.$
Proof of Theorem 3.3. We use induction. Since $L_{0,m} := 1$ for all $m \geq 0$, this satisfies (3.1). For each $m \geq 0$, we extend our base case to include $L_{1,m}$, which counts the number of legal configurations on a single node. By condition (3) in Proposition 3.1, at least one node must contain at least zero chips. Thus, the number of chips occupying our single node lies in the set $\{0, 1, \ldots, m\}$; so $L_{1,m} = m + 1$, which satisfies (3.1). Finally, we observe that for $n \geq 2$, $L_{n,n-2}$ enumerates the legal configurations in which all $n$ nodes contain at most $n - 2$ chips; since condition (2) in Proposition 3.1 requires that at most $n - 1$ nodes may contain at most $n - 2$ chips in a legal configuration, we have $L_{n,n-2} = 0$ for all $n \geq 2$. This result satisfies (3.1), so we include it in our base case.

Now fix $n \geq 2$ and $m \geq n - 1$ and assume that (3.1) is valid for each $L_{n-k,m-1}$ for $k = 0, 1, \ldots, n$. To determine $L_{n,m}$, we let $k \in [0, n]$ count the number of nodes containing exactly $m$ chips; there are $\binom{n}{k}$ ways to choose these $k$ nodes. Consider the configuration of chips on the remaining $n - k$ nodes of $K_n$. Focusing on these
nodes, we see from condition (3) of Proposition 3.1, that for \( l = 1, \ldots, n - k \), at least \( l \) of them contain at least \((n - k) - l\) chips. Thus, the configuration on the remaining \( n - k \) nodes is a legal configuration on \( K_{n-k} \) with at most \( m - 1 \) chips on each node. Since the number of such configurations is \( L_{n-k,m-1} \), we have

\[
L_{n,m} = \sum_{k=0}^{n} \binom{n}{k} L_{n-k,m-1}.
\]

We may now apply our inductive hypothesis to simplify the sum:

\[
L_{n,m} = \sum_{k=0}^{n} \binom{n}{k} L_{n-k,m-1} \\
= \sum_{k=0}^{n} \binom{n}{k} ((m - 1) - (n - k) + 2)((m - 1) + 2)^{(n-k)-1} \\
= \sum_{k=0}^{n} \binom{n}{k} \left( \frac{k}{n} \left[ (m + 1) - (m - n + 1) + (m - n + 1) \right] (m + 1)^{n-k-1} \right) \\
= \sum_{k=0}^{n} \binom{n}{k} (m + 1) + \sum_{k=0}^{n-1} \binom{n-1}{k} (m - n + 1)(m + 1)^{n-k-1} \\
= (m - n + 2) \sum_{k=0}^{n-1} \binom{n-1}{k} (m + 1)^{(n-1)-k} \\
= (m - n + 2)((m + 1) + 1)^{n-1}.
\]

As suggested in Example 3.4, \( L_{n,n-1} = (n + 1)^{n-1} \) gives the number of relaxed legal configurations for a burn-off chip-firing game on \( K_n \). We consider this familiar expression again in the following section.
3.3 A connection between $L_{n,n-1}$ and Cayley's Formula

In the preceding section, we determined that $L(K_n) = (n + 1)^{n-1}$. Of course, this is Cayley's Formula (see, e.g., [19]) for the number of spanning trees of $K_{n+1}$. Our next result provides a new proof of this formula.

**Theorem 3.5** The number of relaxed legal configurations on $K_n$ equals the number of spanning trees of $K_{n+1}$.

**Proof.** We establish injections between the set $\mathcal{R}$ of relaxed legal configurations on $K_n$ and the set $\mathcal{S}$ of spanning trees of $K_{n+1}$ by giving algorithms that, given a member of one set, generate a unique member of the other set. Define $A : \mathcal{R} \to \mathcal{S}$ via Algorithm 3.6 below and $B : \mathcal{S} \to \mathcal{R}$ via Algorithm 3.7. In both algorithms, score refers simply to a numeric label assigned to a node. Define $F : V(K_n) \to \mathbb{N}$ as the score function that makes this assignment.

Our first algorithm injectively maps $\mathcal{R}$ to $\mathcal{S}$. 


Algorithm 3.6 \hspace{1cm} \textbf{INPUT:} a complete graph $K_n$ and relaxed legal configuration $q = (q_1, q_2, \ldots, q_n)$; \\
\textbf{OUTPUT:} a spanning tree of $K_{n+1}$. \\

1. Fix a node $v_0$ in $K_{n+1}$. \\
2. Label the remaining nodes $v_1, v_2, \ldots, v_n$. \\
3. Let $M_0 = (v_0)$ and $M_0 = \{v_0\}$. \\
4. Let $F(v_0) = n - 1$. \\
5. Let $Q_0 = 0$ and $i = 0$. \\
6. Until all $V_j$ have been included in some sequence $M_t$, do the following: 
   a. $i + 1 = i$. 
   b. Let $Q_i = Q_{i-1} + |M_{i-1}|$. 
   c. Let $M_t = (v_{i_1}, v_{i_2}, \ldots, v_{i_t})$, for some $t \geq 1$, be the sequence (in increasing subscript order) of all nodes $v_k$ for which $q_k = F(u)$ for some $u \in M_{i-1}$. Let $\overline{M}_t = \{x : x \text{ is an entry of } M_t\}$. 
   d. Add an edge from each $v_k$ in $M_t$ to the node $w \in \overline{M}_{i-1}$ for which $q_k = F(w)$. 
   e. For each $j = 1, 2, \ldots, |M_t|$, let $F(v_{i_j}) = n - Q_i - j$. 

Note: The purpose of the variables $Q_i$ is to record the number of nodes that have been included in an earlier $M_t$. In fact, $Q_i = \bigcup_{j=0}^{i-1} |M_j|$. Thus, in step (6e),
each score is given to exactly one node. Therefore, "the node" selected in step (6d) is indeed unique. For an example trace of this algorithm, please see Example 3.8 starting on p. 47.

Proof that \( A \) is well-defined. Algorithm 3.6 will fail if, at any step, \( \mathcal{M}_i \) is empty, because no further edges can then be added in step (6d). We demonstrate below that no \( \mathcal{M}_i \) is ever empty, but assume for now that this is true. Step (6d) adds an edge from each \( v_k \in \mathcal{M}_i \) to a node that is already part of a single growing component of the subgraph of \( K_{n+1} \) being constructed. The algorithm continues until all nodes of \( K_{n+1} \) are members of some \( \mathcal{M}_i \), so all nodes of \( K_{n+1} \) are eventually connected to the growing component. Note also that exactly \( n \) edges are created by step (6d), one for each node except \( v_0 \). A spanning connected subgraph (of an \((n + 1)\)-node graph \( G \)) with \( n \) edges is necessarily a spanning tree of \( G \); thus, the algorithm indeed constructs a spanning tree of \( K_{n+1} \).

It remains to prove that no \( \mathcal{M}_i \) is empty. We proceed by induction. It is clear in step (3) that \( \mathcal{M}_0 \) is not empty; suppose that \( \mathcal{M}_{i-1} \) is not empty for some fixed \( i > 0 \). (By definition of \( \mathcal{M}_{i-1} \), it is clear that \( \mathcal{M}_{i-1} \) is also not empty.) As Algorithm 3.6 proceeds, the scores assigned to the nodes in step (6e) descend from \( n - 1 \). We must show that at least one entry of \( q \) is large enough to equal the score of one of the nodes in \( \mathcal{M}_{i-1} \). This will guarantee that \( \mathcal{M}_i \) contains at least one element.
When, in step 6(c), the nodes are checked to see if they will be members of $M_i$, the algorithm inspects the scores assigned to the nodes in $M_{i-1}$ (our induction hypothesis ensures that there are nodes in $M_{i-1}$ to inspect). These scores were assigned (during the preceding iteration) in the following order: $n - Q_{i-1} - 1$, $n - Q_{i-1} - 2, \ldots, n - Q_{i-1} - |M_{i-1}| = n - Q_i$. Thus, the lowest score assigned to a node of $M_{i-1}$ is $n - Q_i$.

Since $q$ is a legal configuration, at least $Q_i$ entries of $q$ are at least $n - Q_i$ (see Proposition 3.1). Since $Q_i - 1$ nodes (corresponding to entries of $q$) have been assigned scores (we subtract 1 because $v_0$ does not correspond to an entry of $q$), there is at least one unassigned node $v_k$ containing at least $n - Q_i$ chips (i.e., $q_k \geq n - Q_i$); this number is at least as big as the lowest score found in $M_{i-1}$. We also know that $q_k$ equals one of the scores assigned to a node in $M_{i-1}$, for if $q_k$ exceeded all of those scores, then $v_k$ would have already been assigned to an earlier sequence. It follows by induction that no $M_i$ is empty and, by our earlier remarks, that $A$ is indeed well-defined.

Proof that $A$ is an injection. Let $q = (q_1, q_2, \ldots, q_n)$ and $q^* = (q_1^*, q_2^*, \ldots, q_n^*)$ be two distinct relaxed legal configurations on $K_n$. Let $V(K_{n+1}) = \{v_0, v_1, \ldots, v_n\}$. Let $A(q) = T$ and $A(q^*) = T^*$. We will prove that $A$ is an injection by showing that $T$ and $T^*$ must be distinct.
As \( q \neq q^* \), Algorithm 3.6 must encounter \( q_i \neq q_i^* \) for some \( i \in \{1, \ldots, n\} \). Since the scores are assigned in order of decreasing value, in the first such encounter either \( q_i \) or \( q_i^* \) will be the largest entry of its corresponding sequence. Without loss of generality, suppose that \( q_i^* > q_i \); specifically, find 
\[
\max\{\max_{i \in \{1, \ldots, n\}} \{q_i, q_i^* : q_i \neq q_i^*\}\},
\]
and (if necessary) interchange the labels \( q \) and \( q^* \) so that \( q_i^* \) has this value. We will show that \( v_i \) has different neighbors in \( T \) and \( T^* \), so that \( T \neq T^* \). Suppose that as Algorithm 3.6 operates on \( q \), the sequences constructed in step (6c) are \( M_1, \ldots, M_j \), and suppose that as it operates on \( q^* \), the sequences constructed in step (6c) are \( M_1^*, \ldots, M_j^* \). Now suppose that \( v_i \in M_s \) (for some \( s \in \{1, \ldots, j\} \)) and \( v_i \in M_t^* \) (for some \( t \in \{1, \ldots, k\} \)). We consider two cases.

**Case 1.** \( s = t \)

Since \( q_i, q_i^* \) are the earliest unequal entries encountered, we have \( M_{s-1} = M_{t-1}^* \). Since \( q_i \neq q_i^* \), the node \( v_i \) must be assigned different neighbors, say \( x, y \), from among the nodes of \( M_{s-1} \) and \( M_{t-1}^* \), respectively. Then the edge \( \{v_i, x\} \in E(T) \setminus E(T^*) \), implying that \( T \neq T^* \).

**Case 2.** \( s \neq t \)

Algorithm 3.6 adds an edge between \( v_i \) and some node \( w^* \) in \( M_{t-1}^* \). Since \( q_i, q_i^* \) are the earliest unequal entries encountered, we know that \( M_{t-1} = M_{t-1}^* \). But we have assumed that \( q_i^* > q_i \), so \( t < s \); this implies that \( M_{s-1} \neq M_{t-1}^* \). Thus, the edge \( \{v_i, w^*\} \in E(T^*) \setminus E(T) \), again implying that \( T \neq T^* \).
Our second algorithm injectively maps \( S \) to \( \mathcal{R} \). Recall that \( F : V(K_n) \to \mathbb{N} \) assigns a numeric score to each node of the complete graph on which the chip-firing games are taking place.

**Algorithm 3.7**

**INPUT:** a spanning tree of \( K_{n+1} \);

**OUTPUT:** a relaxed legal configuration \( q = (q_1, q_2, \ldots, q_n) \) on \( K_n \).

1. Fix a node \( v_0 \) in \( K_{n+1} \).
2. Label the remaining nodes \( v_1, v_2, \ldots, v_n \).
3. Let \( N_0 = \{v_0\} \), \( \overline{N}_0 = \{v_0\} \), and \( i = 0 \).

Repeat:

4. \( i + 1 \mapsto i \)
5. Define \( N_i = (v_{i+1}, v_{i+2}, \ldots, v_n) \), for \( t \geq 1 \), as the sequence (in increasing subscript order) of all unassigned nodes that are neighbors in \( T \) of a node in \( N_{i-1} \). Let \( \overline{N}_i = \{x : x \text{ is an entry of } N_j\} \).

Until: all nodes have been assigned.

6. Define \( N = (v_j)^{n+1} \) as the concatenation of all the \( N_j \)'s, in their natural order.

7. For \( k = 1, 2, \ldots, n \), set \( F(u_k) = n - k \) (note that the \( (n + 1)^{st} \) entry is not assigned a score).

8. Let \( q_i \) represent the number of chips on node \( v_i \) of \( K_n \). Determine the vector \( q = (q_1, q_2, \ldots, q_n) \) as follows:

   a. For each \( i = 1, 2, \ldots, n \), \( v_i \) is an entry of some \( N_j \), and is thus the neighbor in \( T \) of some \( v_j \in \overline{N}_{j-1} \).

   b. Let \( q_i = F(v_i) \).
Note: In step (5) we are performing a breadth-first search (see [19]) from \(v_0\) to determine the \(N_i\)'s. For an example trace of this algorithm, see Example 3.9 starting on p. 50.

**Proof that \(B\) is well-defined.** Step (7) makes it clear that \(q\) is relaxed, since the highest score assigned to a node is \(n - 1\). In order to show that \(q\) is legal, we demonstrate that, for \(l = 0, 1, \ldots, n - 1\), at most \(l + 1\) entries of \(q\) could be assigned a value at most \(l\) (refer to Proposition 3.1). In step (7), a node \(v_k\) receives the score \(l\) after \((n - 1) - l\) other nodes have been labeled. As \(v_k\) has \(n\) neighbors in \(K_{n+1}\), it will have \(n - ((n - 1) - l) = l + 1\) unlabeled neighbors in \(T\). These are the only nodes that can be assigned \(q\)-values at most \(F(v_k) = l\) in step (8b). Thus, for \(l = 0, 1, \ldots, n - 1\), at most \(l + 1\) entries of \(q\) are assigned a value at most \(l\). Proposition 3.1 implies that \(q\) is indeed legal. ■

**Proof that \(B\) is an injection.** Let \(T\) and \(T^*\) be two distinct spanning trees of \(K_{n+1}\). Let \(V(K_{n+1}) = \{v_0, v_1, \ldots, v_n\}\). Let \(B(T) = q\) and \(B(T^*) = q^*\). We will prove that \(B\) is an injection by showing that the configurations \(q\) and \(q^*\) must be distinct.

For \(v \in V(T)\), let \(\Gamma_T(v)\) denote the set of neighbors of \(v\) that are assigned the value \(F(v)\) in step (8b); in other words, \(\Gamma_T(v)\) is the set of nodes adjacent to, and one edge further from, \(v_0\) in \(T\). Define \(\Gamma_{T^*}(v)\) analogously.
Since \( T \neq T^* \), we know that \( \Gamma_T(v) \neq \Gamma_T(v) \) for some \( v \in V(K_{n+1}) \). Let \( r = \min\{i : \exists v \in \mathcal{N}_i \cap \mathcal{N}_i^* \text{ with } \Gamma_T(v) \neq \Gamma_T(v)\} \). Choose any \( v \in \mathcal{N}_r \cap \mathcal{N}_r^* \) with \( \Gamma_T(v) \neq \Gamma_T(v) \). Step (8b) assigns the value \( F(v) \) to all \( x \in \Gamma_T(v) \) and the value \( F^*(v) \) to all \( y \in \Gamma_T(v) \). But since \( r \) is a minimum, we have \( F(v) = F^*(v) \); by step (7), we know that this score is assigned exclusively to \( v \). To receive this score in step (8b), a node must be a member of either \( \Gamma_T(v) \) or \( \Gamma_T(v) \). Since \( \Gamma_T(v) \neq \Gamma_T(v) \), we have \( q \neq q^* \); thus, \( B \) is injective.

Since we have demonstrated injections between the set \( \mathcal{R} \) of relaxed legal configurations on \( K_n \) and the set \( S \) of spanning trees of \( K_{n+1} \), we have \( |\mathcal{R}| = |S| \), which finally completes the proof of Theorem 3.5.

Notice that Theorems 3.3 and 3.5 together yield a new proof of Cayley's Formula. Indeed, the first of these implies that \( |\mathcal{R}| = (n+1)^{n-1} \) (see the remark at the beginning of this section), while the second yields \( |\mathcal{R}| = |S| \). Thus, \( |S| = (n+1)^{n-1} \), or, as the result is more typically presented, the number of spanning trees of \( K_n \) is \( n^{n-2} \).

**Example 3.8** We illustrate Algorithm 3.6. Consider the configuration on \( K_5 \) shown in Figure 3.2. We know it is legal by Algorithm 2.6 (the verification algorithm could,
e.g., successfully delete the nodes in the order \( x_4, x_5, x_2, x_1, x_3 \). For this configuration, \( q = (1,3,1,4,3) \), and \( n = 5 \).

![Figure 3.2. A legal configuration on \( K_5 \)](image)

Now we trace the execution of Algorithm 3.6 for this configuration as the input. Steps (1) through (5) result in the node labels \( v_0, \ldots, v_5 \) (with \( v_0 \) fixed), \( M_0 = \{v_0\} \), \( F(v_0) = n - 1 = 4 \), \( i = 0 \), and \( Q_0 = 0 \). Now we perform the steps in (6) until every \( v_k \) is a member of some \( M_i \) (see Figure 3.3). In the first iteration of step (6), we let \( i = 1 \), \( Q_1 = Q_0 + |M_1| = 0 + 1 = 1 \), and \( M_1 = \{v_4\} \) (because \( q_4 = 4 \), the score assigned to \( v_0 \in M_0 \)). We therefore connect \( v_4 \) to \( v_0 \) with an edge, and \( v_4 \) is assigned the score \( n - Q_1 - 1 = 5 - 1 - 1 = 3 \). (See the leftmost graph in Figure 3.3.) In the second iteration of step (6), we let \( i = 2 \), \( Q_2 = Q_1 + |M_2| = 1 + 1 = 2 \), and \( M_2 = \{v_2, v_3\} \), because \( q_2 = q_3 = 3 \), the score assigned to \( v_4 \in M_1 \). We therefore connect both \( v_2 \) and \( v_3 \) to \( v_4 \), and \( v_2 \) is assigned the score \( n - Q_2 - 1 = 5 - 2 - 1 = 2 \), whereas \( v_3 \) is assigned the score \( n - Q_2 - 2 = 5 - 2 - 2 = 1 \). (See the rightmost graph in Figure 3.3.)

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In the final iteration, we let $i = 3$, $Q_3 = Q_2 + |\overline{M}_{3-1}| = 2 + 2 = 4$, and $\overline{M}_3 = \{v_1, v_3\}$, because $q_1 = q_3 = 1$, the score assigned to $v_5 \in \overline{M}_2$. We therefore connect both $v_1$ and $v_3$ to $v_5$, completing the spanning tree. We may assign scores to $v_1$ and $v_3$, but all nodes are now members of some $M_i$, so the algorithm terminates.

The lower graph in Figure 3.3 shows the spanning tree $T$ of $K_6$ associated with the legal configuration $q = (1, 3, 1, 4, 3)$ on $K_5$. The interested reader will find that if Algorithm 3.7 is given $T$, the output will be $q$, but we illustrate Algorithm 3.7 with a new example.
Example 3.9  Consider the spanning tree of $K_7$ shown in Figure 3.4. We trace the execution of Algorithm 3.7 when this tree is the input.

Three iterations of steps (4) and (5) give us $N_0 = (v_0)$, $N_1 = (v_1, v_5)$, $N_2 = (v_3, v_4, v_6)$, and $N_3 = (v_2)$. Thus, $N = (v_0, v_1, v_5, v_3, v_4, v_6, v_2)$. Step (7) assigns scores to the nodes as shown in Figure 3.5.
Step (8) assigns the value 5 to $q_1$, since $v_1 \in \mathcal{N}_1$, and its neighbor in $\mathcal{N}_0$ is $v_0$. Next, step (8) assigns the value 1 to $q_2$, since $v_2 \in \mathcal{N}_3$, and its neighbor in $\mathcal{N}_2$ is $v_4$. Continuing in this fashion, we find that the relaxed legal configuration on $K_6$ associated with the spanning tree of $K_7$ in Figure 3.4 is $q = (5, 1, 4, 3, 5, 4)$.

In Chapter 5, we will use Proposition 3.1, Theorem 3.3, and Theorem 3.5 to find the probabilities associated with each possible burn-off game length on a complete graph. First, however, we extend the results of this chapter to the more general case of a connected graph.
Chapter 4

Results for connected graphs

We will return to the preceding chapter's investigation of complete graphs in Chapter 5. In this chapter, we generalize the results of Chapter 3 to connected graphs. We begin by generalizing Theorem 3.5, and then discuss a method for enumerating the pairs of relaxed legal configurations and seeds that result in burn-off games of any desired length.

In Section 2.6, we saw that the relaxed configurations that occur in a sequence of burn-off games are legal. Theorem 4.13, which concludes the present chapter, determines the (stationary) probability distribution of these relaxed legal
configurations. Thus, we consider a burn-off game as an experiment starting with a configuration \( C \) on a connected graph \( G = (V,E) \) in which each node has uniform probability \( 1/|V| \) of being the seed; the resulting relaxed legal configuration is the outcome.

As we will see in Theorem 4.1, the set \( R \) of relaxed legal configurations on \( G \) is finite; therefore, we may consider the elements of \( R \) as states in a Markov chain (see, e.g., [10]). We employ this idea in Section 4.4, where we finally answer the question of whether there is a connection between burn-off games and SOC introduced in Section 1.4.

### 4.1 Enumerating relaxed legal configurations on any connected graph

A burn-off game played on a connected graph \( G = (V,E) \) with a relaxed legal configuration consists of two phases: first, we choose a seed at random and place a chip on it; second, we allow the game to play until we obtain a relaxed legal configuration. We continue by choosing a seed in this configuration, and so on. In this section, we enumerate such configurations.

We create the graph \( G^* = (V^*,E^*) \) by adding a new node \( x \) adjacent to every node in \( G \). Specifically, \( V^* = V \cup \{x\} \) and \( E^* = E \cup \bigcup_{w \in V} \{x, v\} \). Perhaps surprisingly, relaxed legal configurations on \( G \) are related to spanning trees of \( G^* \) just as in Theorem 3.5 for complete graphs.
Theorem 4.1  The number of relaxed legal configurations on $G$ is the number of spanning trees of $G^*$. 

We may use the Matrix Tree Theorem (as discussed, e.g., in [19]) to count the spanning trees of $G^*$. 

Proof. We establish algorithmically injections back-and-forth between the set of relaxed legal configurations on $G$ and the set of spanning trees of $G^*$. Let $\mathcal{R}$ be the set of relaxed legal configurations on $G$ and $S$ the set of spanning trees of $G^*$. Define $A : \mathcal{R} \rightarrow S$ via Algorithm 4.2 below, and $B : S \rightarrow \mathcal{R}$ via Algorithm 4.3 below.
Algorithm 4.2

**INPUT:** a connected graph \( G = (V,E) \) with \( V = \{v_1, v_2, \ldots, v_n\} \), and a relaxed legal configuration \( C : V \to \mathbb{N} \);

**OUTPUT:** \( A(C) \), a spanning tree \( T^* \) of \( G^* \).

1. Let \( i = 1 \).
2. Let \( M_i \) be the sequence (in increasing subscript order) of nodes \( v_k \) such that \( C(v_k) = \text{deg}_G(v_k) \); let \( M_i = \{x : x \text{ is an entry of } M_i\} \).
3. For each \( v_k \in M_i \), add \( v_k \) to \( V(T^*) \) and \( \{x, v_k\} \) to \( E(T^*) \).
   (If \( V(T^*) = V \), then stop.)
4. \( i + 1 \rightarrow i \).
5. Let \( M_{i+1} \) be the sequence (in increasing subscript order) of the nodes not yet included in \( V(T^*) \) that are neighbors of nodes in \( M_i \). Let \( M_i = \{x : x \text{ is an entry in } M_i\} \).
   For each \( u \in M_i \), execute steps (6) through (9):
   6. For \( r = 1, 2, \ldots, i - 1 \), let \( N_r = (v_{r1}, v_{r2}, \ldots, v_{rk_r}) \) be the sequence (in increasing subscript order) of the \( k_r \) \( G \)-neighbors of \( u \) that appear in \( M_r \). Let \( N_i = \{x : x \text{ is an entry in } N_i\} \).
   7. Let \( s = \bigcup_{r=1}^{i-1} N_r \) and \( N = (v_k, v_{k2}, \ldots, v_{k_s}) \) be the sequence determined by concatenating the sequences \( N_1, N_2, \ldots, N_{i-1} \).
   8. If \( C(u) < \text{deg}_G(u) - s \), then delete \( u \) from \( M_i \) and \( M_{i+1} \).
   9. Otherwise, \( C(u) = \text{deg}_G(u) - j \) for some \( j \) with \( 1 \leq j \leq s \). Add \( u \) to \( V(T^*) \) and \( \{u, v_j\} \) to \( E(T^*) \).
10. If \( V(T^*) = V \), then stop. Otherwise, go to step (4).
Note the similarity between Algorithm 4.2 and Algorithm 3.6; for a complete graph $K_n$, adding a special node $x$ adjacent to every other node simply creates $K_{n+1}$, and the output of Algorithm 3.6 is a spanning tree of $K_{n+1}$. However, the two algorithms will not necessarily produce the same outputs. In step 6(d) of Algorithm 3.6, our choice of edge depends on the sequence formed in step 6(c); this sequence is created by arranging nodes simply in increasing subscript order. Compare this approach to that of steps (6) and (7) of Algorithm 4.2, which impose greater restrictions on this sequence.

For an example trace of Algorithm 4.2, see Example 4.5 starting on p. 67.

Proof that $A$ is well-defined. Not only must we be sure that Algorithm 4.2 outputs a spanning tree $T^*$, but also we must check that it does not halt before doing so. To establish both of these results, we look at each step in turn.

Step (2). We have already seen in Algorithm 2.6 that in a legal configuration, at least one node contains at least as many chips as its degree. Thus $\mathcal{M}_1$ is nonempty.

Step (3). It is clear that $T^*$ is thus far a tree; in fact, it is a star.

Step (5). We must establish that $\overline{M}_i$ is nonempty so that the “for each $u \in \overline{M}_i$" step is
not quantifying over an empty set. We proceed by induction. In the discussion of Step (2) above, we observed that $\overline{M}_1$ is nonempty. By construction, all nodes in $\overline{M}_1$ are critical. Because $C$ is a legal configuration, we may apply Algorithm 2.6 to $G$ and delete all of the nodes (in any order) in $\overline{M}_1$.

With these statements as our base case, our induction hypothesis is in two parts: for fixed $i > 1$, suppose that (a) $\overline{M}_1,\overline{M}_2,...,\overline{M}_{i-1}$ are nonempty; and (b) we may apply Algorithm 2.6 to $G$ and delete the nodes in $\overline{M}_1,\overline{M}_2,...,\overline{M}_{i-1}$ without halting.

Let $M = \bigcup_{j=1}^{i-1} \overline{M}_j$. Lemma 2.8 states that the configuration on any subgraph of a graph (on which we have a legal configuration) must itself be legal. So, if our application of Algorithm 2.6 has deleted exactly the nodes of $M$, then at least one of the remaining nodes $u$ of $G - M$ must be critical in $G - M$. Suppose that $u$ is not a neighbor of any node in $M$. Because $u$ is critical in $G - M$, and none of its neighbors have been deleted in our application of Algorithm 2.6, we see that $u$ is not a neighbor of any node in $M$. But this places $u$ in $\overline{M}_1$, which contradicts the choice of $u$ in $G - M$.

Thus, we know that $u$ is a neighbor of some node in $M$. Now if $u$ is not a neighbor of a node in $\overline{M}_{i-1}$, it must be adjacent to, say, $s \geq 1$ nodes in $\overline{M}_1,\overline{M}_2,...,\overline{M}_{i-2}$. Thus, $u$ has been considered previously by step (8) and has been deleted each time. Therefore, $C(u) < \deg_G(u) - s$. This shows—back in our application of Algorithm 2.6—that if we have deleted all of the nodes in $\overline{M}_1,\ldots,\overline{M}_{i-1}$,
including the \( s \) neighbors of \( u \), then \( u \) will not be critical in \( G - M \). This contradicts the fact that \( u \) is critical in \( G - M \), so \( u \) must be a neighbor of a node in \( \overline{M}_{i-1} \).

Because \( u \) is critical in \( G - M \), step (8) will not delete \( u \) from \( \overline{M}_i \). Thus, \( \overline{M}_i \) is nonempty; this fulfills part (a) of the induction hypothesis. We claim that any node \( w \) placed in \( \overline{M}_i \) by step (5) will survive past step (8) only if it, too, is critical in \( G - M \). For \( w \) to survive step (8), we require \( C(w) \geq \deg_G(w) - s \), where \( s \) is the number of \( G \)-neighbors of \( w \) that appear in \( M \). Since \( \deg_G(w) - s \) simply equals \( \deg_{G - M}(w) \), we know \( w \) is critical in \( G - M \). Thus, all nodes in \( \overline{M}_i \) may be deleted as we apply Algorithm 2.6. This fulfills part (b) of the induction hypothesis.

**Step (6).** Step (5) assures us that these neighbors exist.

**Step (8).** The argument given above for step (5) assures us that \( \overline{M}_i \) remains nonempty after all nodes of \( \overline{M}_i \) have been processed in step (8).

**Step (9).** It is impossible to create a cycle in this step because step (5) only considers those nodes that are not yet part of \( T^* \).

**Step (10).** This step assures us that \( T^* \) will be a spanning tree of \( G^* \).
Observe that step (9) adds at least one edge to $T^*$ since $M_i$ remains nonempty. Once $n - 1$ edges have been added to $T^*$, step (10) will halt the algorithm. Since $A$ does not halt until it outputs a spanning tree $T^*$, $A$ is well-defined.

Proof that $A$ is an injection. Let $C \in \mathcal{R}$ and $C' \in \mathcal{R}$ be two distinct relaxed legal configurations on $G$. We prove that $A$ is an injection by showing that the spanning trees $A(C)$ and $A(C')$ must be distinct. As Algorithm 4.2 operates on $C$ and $C'$, it must encounter a node $v$ for which $C(v) \neq C'(v)$. Let us call such a $v$ special. Step (8) might remove $v$ from consideration; if this occurs for both inputs $C$ and $C'$, then we consider a future pass of the algorithm. Because Algorithm 4.2 includes every node in the output before it halts, we know that eventually we will find a special node that is not removed by step (8) concurrently for both inputs $C$ and $C'$.

Now if $v$ is removed by step (8) for one input but not the other, then step (9) will connect $v$ to a different neighbor for the two inputs. On the other hand, suppose $v$ is not removed by step (8) for either input; because $C(v) \neq C'(v)$, step (9) will connect $v$ to a different neighbor for the two inputs. In either case, $A(C)$ and $A(C')$ must be distinct, and $A$ is an injection.

Before we turn to Algorithm 4.3, we recall one definition. Let $G = (V,E)$
be a connected graph, and \( u, v \in V \). As in [19], we define the distance from \( u \) to \( v \) (denoted \( d_G(u, v) \)) as the least length of a \( u, v \)-path.

**Algorithm 4.3**

**INPUT:** a spanning tree \( T^* \) of \( G^* \) and an ordering \( V = \{v_1, v_2, \ldots, v_n\} \) of the nodes of \( G \);

**OUTPUT:** \( B(T^*) \), a relaxed legal configuration \( C : V \rightarrow \mathbb{N} \) on \( G \).

1. Let \( M_0 = \{x\} \).
2. Let \( m = \max_{v \in V} \{d_G(x, v)\} \). For \( j = 1, 2, \ldots, m \), let \( M_j \) be the sequence (in breadth-first order, breaking ties lexicographically by subscript) of nodes \( v \) for which \( d_G(x, v) = j \). Let \( \bar{M}_j = \{x : x \text{ is an entry of } M_j\} \).
3. For each \( u \in \bar{M}_i \), let \( C(u) = \deg_e(u) \).

For \( i = 2, 3, \ldots, m \),

for each \( u \in \bar{M}_i \), following the ordering in \( M_i \),

execute steps (4) through (7):

4. For \( r = 1, 2, \ldots, i - 1 \), let \( N_r = \{v_{r,1}, v_{r,2}, \ldots, v_{r,k_r}\} \) be the sequence (in their \( M_r \)-ordering) of the \( k_r \geq 0 \) \( G \)-neighbors of \( u \) that appear in \( M_r \). Let \( \bar{N}_r = \{x : x \text{ is an entry of } N_r\} \).
5. Let \( s = \left| \bigcup_{r=1}^{i-1} \bar{N}_r \right| \).
6. Let \( N = \{v_{i,1}, v_{i,2}, \ldots, v_{i,s}\} \) be the sequence determined by concatenating the sequences \( N_i, N_{i-1}, \ldots, N_1 \).
7. For some \( t \in \{1, 2, \ldots, s\} \), we have \((v_{i,t}, u) \in E(T^*)\). Let \( C(u) = \deg_e(u) - t \).
Note the similarity between Algorithm 4.3 and Algorithm 3.7, where a spanning tree of $K_n$ is the input and a configuration on $K_n$ is the output. However, the breadth-first order imposed by step (2) of Algorithm 4.3 is more restrictive than the search performed by steps (4) and (5) of Algorithm 3.7. Thus, given the same input, the two algorithms do not necessarily produce the same outputs.

For an example trace of Algorithm 4.3, see Example 4.6 starting on p. 69.

Proof that $B$ is well-defined. In step (2), we partition $V$ into sequences $M_1, M_2, \ldots, M_m$. Step (3) assigns chips to the nodes in $\overline{M}_1$, while step (7) assigns chips to the nodes in $\overline{M}_2, \ldots, \overline{M}_m$. Therefore, Algorithm 4.3 at least produces a function $C : V \rightarrow \mathbb{N}$.

Now we use Algorithm 2.6 to establish that $C$ is legal. Since $T^*$ is a spanning tree of $G^*$, we know that $\overline{M}_1$ is nonempty (see step (3)); hence, there is at least one node $u$ such that $C(u) = \deg_G(u)$. Thus Algorithm 2.6, given $C$ as input, may delete the nodes in $\overline{M}_1$. This fact is the base case in an induction argument that proves that in Algorithm 2.6, the nodes in $\overline{M}_1, \overline{M}_2, \ldots, \overline{M}_m$ can be deleted in the order given by this list. Suppose that this is true for $\overline{M}_1, \overline{M}_2, \ldots, \overline{M}_{k-1}$, where $2 \leq k < m$. For any $u \in \overline{M}_k$, step (7) assigns $C(u) = \deg_G(u) - t \geq \deg_G(u) - s$. Recall that $s$ counts the neighbors in $G$ of $u$ that are in $\bigcup_{r=1}^{k-1} \overline{M}_r$; in our induction hypothesis, we have assumed that these neighbors have been deleted from $G$, resulting, say, in a subgraph $G'$. If other nodes in $\overline{M}_k$ have been deleted before we consider $u$, then
\( \deg_{G'}(u) \) does not increase. Thus, we have \( \deg_{G'}(u) \geq \deg_{G'}(u) \), so \( u \) can be deleted in step (2) of Algorithm 2.6.

Proof that \( B \) is an injection. Suppose that \( T_1, T_2 \in S \) satisfy

\[
C_1 := B(T_1) = B(T_2) =: C_2 := C;
\]

we'll show that then \( T_1 = T_2 \).

Let \( V = V(G), \ n = |V| \), and write the breadth-first orderings of \( V \) determined during the computation of \( B(T_1) \) and \( B(T_2) \) as \( (u_i)_{i=1}^n \) and \( (w_i)_{i=1}^n \), respectively. To complete the proof, we shall find it useful to establish the following lemma.

**Lemma 4.4** Under the hypothesis that \( C_1 = C_2 \), if there exists an integer \( j \geq 1 \) such that \( u_i = w_i \) for all \( i \in \{1, \ldots, j\} \), then the subtree \( H_1^* \) of \( T_1^* \) induced on \( \{x, u_1, \ldots, u_j\} \) is identical to the subtree \( H_2^* \) of \( T_2^* \) induced on \( \{x, w_1, \ldots, w_j\} \).

**Proof.** We induct on \( j \). First note that \( H_1^*, H_2^* \) are indeed subtrees of \( T_1^*, T_2^* \), respectively, since the sequences \( (u_i), (w_i) \) are defined by breadth-first search on these trees. It is also clear from the definitions of \( (u_i), (w_i) \) that \( u_1, w_1 \) are adjacent to \( x \) in \( H_1^*, H_2^* \), respectively. In the case when \( j = 1 \), these subtrees both consist of 2-vertex trees containing the edge \( \{x, u_1\} = \{x, w_1\} \) and are therefore identical.
Now fix $j > 1$, assume that the lemma holds for smaller instances of $j$, and suppose that $u_i = w_i$ for all $i \in \{1, \ldots, j\}$. Let $G_0^*$ denote the subgraph of $G^*$ induced on the common vertex set $U := \{x, u_1, \ldots, u_j\}$ of $H_1^*, H_2^*$, and let $G_0 = G_0^* - x$. We consider four executions of Algorithm 4.3; in each case, the input vertex ordering is inherited from $G$.

The first pair of executions computes $D_1 := B(H_1^*)$ and $D_2 := B(H_2^*)$, two configurations on $G_0$. Since $(u_i)_{i=1}^j$, $(w_i)_{i=1}^j$ are initial segments of $(u_i)$, $(w_i)$, it is evident from Algorithm 4.3 that $D_1, D_2$ are obtained from $C_1, C_2$ by replacing $\deg_G$ in steps (3), (7) by $\deg_{G_0}$ and restricting the resulting functions to $U$. Since $C_1 = C_2$, we have $D_1 = D_2$. For $k = 1, 2$ and for each node $u \in V(G_0)$, let $t_k(u)$ denote the value of $t$ in step (7) as Algorithm 4.3 determines $D_k(u)$; if $D_k(u)$ is determined in step (3), we take $t_k(u) := 0$. Then

$$D_k(u) = \deg_{G_0}(u) - t_k(u) \quad \text{for } k = 1, 2 \text{ and each } u \in V(G_0). \quad (4.1)$$

The second pair of executions computes $D'_1 := B(H_1^* - u_j)$ and $D'_2 := B(H_2^* - w_j)$, two configurations on $G_0^* := G_0 - u_j = G_0 - w_j$. For $k = 1, 2$ and for each node $u \in V(G_0^*)$, define $t'_k(u)$ analogously with $t_k(u)$; now we have

$$D'_k(u) = \deg_{G_0^*}(u) - t'_k(u) \quad \text{for } k = 1, 2 \text{ and each } u \in V(G_0^*). \quad (4.2)$$

Since $(u_i)_{i=1}^j$, $(w_i)_{i=1}^j$ are respectively breadth-first orderings of $V(H_1^*)$, $V(H_2^*)$, the sequences $(u_i)_{i=1}^{j-1}$, $(w_i)_{i=1}^{j-1}$ are such orderings of $V(H_1^* - u_j)$, $V(H_2^* - w_j)$.
Thus, during the second pair of executions of Algorithm 4.3 described above, every sequence $M_i$ (in the statement of the algorithm) is the same as during the first pair of respective executions, except, in passing from the first pair to the second, the final node of $M_m$ (resp. $u_j, w_j$) has been deleted. Therefore

$$t'_k(u) = t_k(u) \quad \text{for } k = 1, 2 \text{ and each } u \in V(G'_0). \quad (4.3)$$

Since $D_1 = D_2$, the equations in (4.1) imply that

$$t_1(u) = t_2(u) \quad \text{for each } u \in V(G_0). \quad (4.4)$$

Comparing (4.4) with (4.3), we see that

$$t'_1(u) = t'_2(u) \quad \text{for each } u \in V(G'_0). \quad (4.5)$$

It follows from (4.2), (4.5) that $D'_1 = D'_2$. Since these are configurations on $G'_0$, whose vertex set is $U \setminus \{u_j\} = U \setminus \{w_j\}$, the induction hypothesis implies that $H'_1 - u_j = H'_2 - w_j$. Finally, from (4.4), we have $t_1(u_j) = t_2(u_j)$, and in Algorithm 4.3, this means that the node $u_j = w_j$ has the same neighbor in $H'_1 - u_j$ as in $H'_2 - w_j$. Therefore $H'_1 = H'_2$.

It follows from Lemma 4.4, with $j = n$, that if $(u_i)$ and $(w_i)$ agree entirely, then $T'_1 = T'_2$. Thus, it remains only to address the case when $u_i \neq w_i$ for some $i \in \{1, \ldots, n\}$, and here we'll reach a contradiction.

First, notice that according to Algorithm 4.3, for any $u \in V$, we have
$C(u) = \deg_G(u)$ if and only if $u$ is adjacent to $x$ in both of $T_1^*, T_2^*$. Therefore, $T_1^*, T_2^*$ do not differ in their adjacencies to $x$, and the sequences $(u_i)$, $(w_i)$ agree in their initial entries, corresponding to the (necessarily nonempty) neighbor sets of $x$ in $T_1^*, T_2^*$. If there are $L$ such neighbors, then $u_i = w_i$ for $i \in \{1, \ldots, L\}$, and we're assuming that $L < n$.

Let $i_0$ denote the least $i$ such that $u_i \neq w_i$. Since $L < i_0 \leq n$, it is easy to see that Algorithm 4.3 reaches step (7) in defining $C_1(u_{i_0})$ and $C_2(w_{i_0})$. Let $j = i_0 - 1$, and define $H_1^*, H_2^*$ as in the statement of Lemma 4.4. Since

$$u_i = w_i \quad \text{for } i \in \{1, \ldots, j\},$$

(4.6)

Lemma 4.4 shows that $H_1^* = H_2^*$. From (4.6), we also see that $w_{i_0}$ does not appear in the subsequence $(u_i)_{i=1}^j$, and $u_{i_0}$ does not appear in the subsequence $(w_i)_{i=1}^j$. Thus, in computing $B(T_1^*)$, Algorithm 4.3 processes $u_{i_0}$ before $w_{i_0}$, while in computing $B(T_2^*)$, it processes $u_{i_0}$ after $w_{i_0}$.

Now consider the instants during the two executions of Algorithm 4.3 when step (7) defines $C_1(u_{i_0})$ and $C_2(u_{i_0})$. In particular, for $k = 1, 2$, define $t_k$ as in the proof of Lemma 4.4, so that

$$C_k(u_{i_0}) = \deg_G(u_{i_0}) - t_k(u_{i_0}) \quad \text{for } k = 1, 2.$$

Since $C_1 = C_2$ by hypothesis, we have

$$t_1(u_{i_0}) = t_2(u_{i_0})$$

(4.7)
As Algorithm 4.3 executes on $T_1^*$ and is processing $u = u_{i_0}$, denote the sequence $N$ in step (6) by $N_1$. Likewise, during execution on $T_2^*$ and while processing the same node, denote the corresponding sequence by $N_2$. The entries of $N_1$ are the $G$-neighbors of $u_{i_0}$ lying (strictly) closer to $x$ in $T_1^*$ than $u_{i_0}$. Similarly, the entries of $N_2$ are the $G$-neighbors of $u_{i_0}$ lying closer to $x$ in $T_2^*$ than $u_{i_0}$. Since $H_1^* = H_2^*$, the sequence $N_1$ forms an initial segment of the sequence $N_2$. It follows from this and (4.7) that the $T_1^*$-neighbor of $u_{i_0}$ closer to $x$ (than $u_{i_0}$) in $T_1^*$ and the $T_2^*$-neighbor of $u_{i_0}$ closer to $x$ in $T_2^*$ are the same. A similar argument shows that the $T_1^*$- and $T_2^*$-neighbors of $w_{i_0}$ closer to $x$ (than $w_{i_0}$) in these trees are identical. Under these conditions, Algorithm 4.3 necessarily processes $u_{i_0}$ and $w_{i_0}$ in the same order during the computations of $B(T_1^*)$, $B(T_2^*)$. But we concluded two paragraphs earlier that this is not the case. This contradiction shows that the case when $u_i \neq w_i$ for some $i \in \{1, \ldots, n\}$ is impossible and therefore completes the proof. \[\blacksquare\]
Example 4.5  Consider the legal configuration shown in Figure 4.1. We show how Algorithm 4.2 associates this configuration with a spanning tree $T^*$ of $G^*$. 

![Figure 4.1. A legal configuration used to illustrate Algorithm 4.2](image)

The two critical nodes are $v_5$ and $v_7$, so $M_1 = (v_5, v_7)$; the edges $\{x, v_5\}$ and $\{x, v_7\}$ are added to $T^*$. The nodes adjacent to those in $M_1$ comprise $M_2 = (v_1, v_2, v_4)$. Inspecting $v_1$ we find that $N = (v_5)$, so $s = 1$. Since $C(v_1) = 2 = 3 - 1 = \text{deg}(v_1) - s$, step (9) adds the edge $\{v_1, v_5\}$ to $T^*$. Inspecting $v_2$ we find that $N = (v_5, v_7)$, so $s = 2$. Now $C(v_2) = 1 = \text{deg}(v_2) - s$, so $C(v_2) = \text{deg}(v_2) - 2$; step (9) adds the edge $\{v_2, v_7\}$ to $T^*$. (Note that if $C(v_2)$ were 2 rather than 1, step (9) instead would have added the edge $\{v_2, v_5\}$ to $T^*$.) Inspecting $v_4$ we find that $N = (v_7)$, so $s = 1$. However, $C(v_4) = 1$ and $\text{deg}(v_4) - s = 3 - 1 = 2$; since $C(v_4) < 2$, step (8) deletes $v_4$ from $M_2$.

The (unused) nodes adjacent to those in $M_2 = (v_1, v_2)$ are $M_3 = (v_3, v_6)$. Inspecting $v_3$ we find that $N = (v_1)$, so $s = 1$. Since $C(v_3) = 1 = \text{deg}(v_3) - s$, step (9) adds the edge $\{v_3, v_1\}$ to $T^*$. Inspecting $v_6$ we find that $N = (v_1, v_2)$, so $s = 2$. However, $C(v_6) = 0$ and $\text{deg}(v_6) - s = 3 - 2 = 1$; since $C(v_6) < 1$, step (8) deletes $v_6$. 

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from $M_3$. The (unused) node adjacent to the one in $M_3 = (v_3)$ is $M_4 = (v_4)$.

Inspecting $v_4$ we find that $N = (v_7, v_3)$. Note the order in which the two elements of $N$ appear; this occurs because step (7) specifies that when $i = 4$, the order of the neighbors as they appear in $M_1, M_2, M_3$ must be preserved. Since each feature of the algorithm has now been illustrated in this example, we conclude simply by mentioning that the edges $\{v_3, v_4\}$ and $\{v_4, v_6\}$ are added to $T^*$. Figure 4.2 shows the output of Algorithm 4.2.

![Figure 4.2. The spanning tree $T^*$ associated (by Algorithm 4.2) with the legal configuration in Figure 4.1](image-url)

Figure 4.2. The spanning tree $T^*$ associated (by Algorithm 4.2) with the legal configuration in Figure 4.1
Example 4.6 Consider the spanning tree $T^*$ of $G^*$ shown in Figure 4.3. We show how Algorithm 4.3 associates this spanning tree with a legal configuration $C$ on $G$.

Step (2) gives $M_1 = (v_3,v_5)$, $M_2 = (v_1,v_7,v_6)$, and $M_3 = (v_2,v_4)$. Note how the requirements in step (2) affect the subscript order in $M_2$. Step (3) assigns $C(v_3) = \text{deg}_G(v_3) = 3$ and $C(v_5) = \text{deg}_G(v_5) = 3$. Inspecting $v_1$, we find $N = (v_3,v_5)$. Since the edge $\{v_1,v_3\}$ is in $T^*$, we have $t = 1$, so step (7) assigns $C(v_1) = \text{deg}_G(v_1) - t = 4 - 1 = 3$. (Note that if the edge $\{v_1,v_5\}$ had been in $T^*$ instead, step (7) would have assigned $C(v_1) = 2$.) For $v_7$ and $v_6$, step (7) assigns $C(v_7) = 1$ and $C(v_6) = 1$. At this point $i$ becomes 3. Inspecting $v_2$, we find $N = (v_3,v_5,v_1)$. Note the order in which the elements of $N$ appear; this occurs because step (6) specifies that the order of the neighbors as they appear in $M_1,M_2$ must be preserved. Since the edge $\{v_2,v_1\}$ is in $T^*$, we have $t = 3$, so step (7) assigns $C(v_2) = \text{deg}_G(v_2) - t = 3 - 3 = 0$. Step (7) also assigns $C(v_4) = 1$, since $\{v_4,v_6\} \in E(T^*)$. Figure 4.4 shows the output of Algorithm 4.3.
4.2 An extension of Theorem 4.1

By deleting certain edges from $G^*$ (defined in Section 4.1), we may enumerate the relaxed legal configurations that correspond to games of length zero on $G$. This result will be useful in Chapter 5, but we mention it here because it requires Algorithm 4.2. For $v \in V$, let $T_v$ denote the number of spanning trees of $G^* - xv$. Let $(C, v) \in \mathcal{R} \times V$ denote a choice of a relaxed legal configuration $C$ on $G$ and a seed $v \in V$.

**Proposition 4.7** The number of pairs $(C, v)$ that result in a game of length zero is $\sum_{v \in V} T_v$.

**Proof.** As shown in the discussion of Algorithm 4.2, an edge $(x, v)$ in $T^*$ forces $v$ to be critical in the corresponding relaxed legal configuration, whereas $v$ will
specifically not be critical when that edge is missing in $T^*$. So by removing the edge \( \{x,v\} \) from $G^*$ and enumerating the spanning trees, we will count the number of relaxed legal configurations in which $v$ is not critical. Now if $v$ is the seed, it will not fire, and the game length will be zero. We sum over all $v \in V$ to count all the pairs $(C, v)$ that result in a game of length zero.

We will require Proposition 4.7 in Chapter 5 due to the unusual nature of games of length zero; in all other games, at least one node must fire, so the seed must be critical. A game of length zero requires a seed that is not critical, making games of length zero a special case.

4.3 Enumerating games of any length on any connected graph

Before we consider a theorem that allows us to enumerate the burn-off games of any length on any connected graph $G$, we prove that there are a finite number of game lengths to investigate.

Lemma 4.8 During a burn-off game that starts with a relaxed legal configuration on $G = (V,E)$, no node may fire more than once.

Proof. Let $v$ be the seed. When $v$ fires, it loses all of its chips. For $v$ to fire a second
time, it must gain at least deg(v) + 1 chips from its neighbors. By the pigeonhole
principle, this will happen only if at least one neighbor of v fires at least twice. Thus,
v cannot be the first node to fire more than once.

Suppose that w \in V, with w \neq v, is the first node to fire a second time. Say there are
\begin{equation}
    c \leq \deg(w)
\end{equation}
chips on w when the seed is chosen, and k neighbors of w have fired once before w
fires for the first time. Just before w fires, it will contain c + k > \deg(w) chips. After w
fires, it will contain c + k - (\deg(w) + 1) chips. For w to fire again it must contain at
least \deg(w) + 1 chips, so it must gain at least \( t := \deg(w) + 1 - (c + k - (\deg(w) + 1)) \)
chips from its neighbors. But
\[
    t = (\deg(w) - k) + (\deg(w) - c) + 2 > \deg(w) - k,
\]
by (4.8). Now \deg(w) - k is precisely the number of neighbors of w that did not fire
before w fired. So for w to gain \( t > \deg(w) - k \) chips, either one of these neighbors
must fire at least twice before w fires a second time, or one of the k neighbors that
fired before w fired must fire a second time. Either case contradicts our assumption
that w was the first node to fire a second time. \( \blacksquare \)

So given any connected graph \( G = (V,E) \), we know that the length \( g \) of
any burn-off game must satisfy \( 0 \leq g \leq |V| \). While the relatively simple methods of
Section 4.2 do not extend to enumerating games of any length, the methods introduced in this chapter nevertheless give formulas for the number of such games.

First we need some new notation. For \( v \in V \), let \( T_{v,g} \) denote the set of subtrees of \( G \) with \( g \) nodes and including \( v \). For a graph \( X \) with subgraph \( X_1 \) and a node \( u \in V(X) \), let \( \Gamma_{X_1}(u) \) be the set of neighbors of \( u \) that lie in \( V(X_1) \). For a connected graph \( G = (V,E) \), let \( (C,v) \in \mathcal{R} \times V \) denote a choice of relaxed legal configuration \( C \) and seed \( v \). Finally, we note that this proof considers legal configurations on disconnected graphs and remind the reader that the function \( L \) enumerating relaxed legal configurations may operate on such graphs (see p. 24).

**Theorem 4.9** The number of pairs \((C,v)\) resulting in a game of length \( g > 0 \) is

\[
\sum_{v \in V} \sum_{T \in T_{v,g}} L(G - T).
\]

**Proof.** For \( v \in V \), let \( \mathcal{R}_{v,g} \) denote the set of relaxed legal configurations on \( G \) such that if \( v \) is seeded, then the resulting burn-off game will be of length \( g \). For \( R_1, R_2 \in \mathcal{R}_{v,g} \), define the relation \( \simeq \) as follows: suppose that when \( v \) is seeded in \( R_1 \) and \( R_2 \), the nodes that fire in each game induce the same subgraph \( H \) of \( G \); suppose also that \( R_1|_H = R_2|_H \). If both of these conditions hold, we say \( R_1 \simeq R_2 \). It is clear that \( \simeq \) is an equivalence relation on \( \mathcal{R}_{v,g} \). Let \( \mathcal{Q}_{v,g} \) be the set of equivalence classes of
To prove Theorem 4.9, it will be helpful to establish injections $A : T_{v^g} \rightarrow Q_{v^g}$ and $B : Q_{v^g} \rightarrow T_{v^g}$.

Define $A : T_{v^g} \rightarrow Q_{v^g}$ as follows. Let $T \in T_{v^g}$, and let $H$ be the subgraph of $G$ induced on $V(T)$. Create $H'$ as follows: to each $u \in V(T)$, append $\deg_G(u) - \deg_H(u)$ leaves to $u$. Let $J$ be this set of leaves. Now let $T'$ be the spanning tree of $H'$ consisting of $T$ and $J$. Create $T^*$ by appending the node $x$ and the edge $(x, v)$ to $T'$.

Use $T^*$ (with $H'$ as the underlying graph) as the input in Algorithm 4.3; let $C^*$ be the output configuration. Let $Q$ be a configuration on $G$ defined by $Q(v) = C^*(v)$ and $Q(u) = C^*(u) + 1$ for each $u \in V(H) - v$. Let $Z$ be any relaxed legal configuration on $G - H$. Define $Q(w) = Z(w)$ for each $w \in V(G - H)$. Now $Q$ is a configuration on $G$. We demonstrate below that $Q \in R_{v^g}$; thus, we may let $\overline{Q}$ denote the equivalence class of $Q$. Finally, let $A(T) = \overline{Q}$.

Claim 1. $A$ is well-defined.

Proof of claim. To show that $Q \in R_{v^g}$, we will demonstrate that (a) $Q$ is a relaxed legal configuration on $G$; and (b) seeding $v$ in $Q$ results in a burn-off game of length $g$.

(a) $Q$ is a relaxed legal configuration on $G$.

Because $v$ is the only neighbor of $x$ in $T^*$, only $v$ is critical in $C^*$ (see step
(7) of Algorithm 4.3). As we define $Q$, then, adding a chip to each $u \in V(T) - v$ does not make any of these nodes supercritical. We choose $Z$ to be any relaxed legal configuration on $G - T$, so none of the nodes in $V(G - T)$ are supercritical. Therefore, $Q$ is relaxed.

We appeal to Algorithm 2.6 to demonstrate the legality of $Q$. We defined $C^*$ using Algorithm 4.3, so $C^*$ is a legal configuration on $H'$. Thus, if Algorithm 2.6 operates on $C^*$, it will provide a deletion sequence $S$ of $V(H')$. Since every $w \in J$ is a leaf, each $\deg_H(w) = 1$. Since only $v$ is critical in $C^*$, we must have $C^*(w) = 0$. Without loss of generality, then, we may permute $S$ so that $V(H)$ is processed before $J$ and see that this new deletion sequence $S'$ also satisfies the requirements of Algorithm 2.6. In passing from $C^*$ to $Q$, we let $Q(v) = C^*(v)$ and $Q(u) = C^*(u) + 1$ for each $u \in V(H) - v$. Because $\deg_{H'}(x) = \deg_G(x)$ for every $x \in V(H)$, Algorithm 2.6 may begin to process $Q$ on $G$ in the same order found in the initial subsequence of $S'$ containing the nodes of $V(H)$. Since we extended $Q$ to $V(G - T)$ by choosing any legal configuration $Z$ on the subgraph $G - T$, Algorithm 2.6 may finish processing $Q$, thereby confirming the legality of $Q$.

(b) Seeding $v$ in $Q$ results in a game of length $g$.

We first show that each node in $T$ fires, and then show that none of the nodes in $G - T$ fire. Since $T$ has $g$ nodes, and by Lemma 4.8 no node may fire twice,
the resulting game will be of length $g$.

Clearly, $v$ can fire. For $u \in V(T) - v$, let

$$S_u = \{w \in \Gamma_H(u) : d_{H'}(w, x) < d_{H'}(u, x)\}$$

and $s_u = |S_u|$. By step (7) of Algorithm 4.3, we have

$$C^*(u) \geq \deg_{H'}(u) - s_u = \deg_G(u) - s_u.$$

We defined $Q(u) = C^*(u) + 1$, so once the nodes in $S_u$ fire, the number of chips on $u$ will be at least $\deg_G(u) + 1$, allowing $u$ to fire as well.

For $w \in V(G - T)$, let $s_w = |\Gamma_T(w)|$. In each relaxed legal configuration $Z$ on $G - T$, we must have $Z(w) \leq \deg_{G-T}(w) = \deg_G(w) - s_w$. Because the nodes in $T$ contribute a total of $s_w$ chips to $w$ once they have all fired, the number of chips on $w$ will never exceed $\deg_G(w)$. Since we define $Q(w) = Z(w)$, we know $w$ will not fire when $v$ is the seed.

We have shown that $Q$ is a relaxed legal configuration on $G$ such that if $v$ is seeded, the resulting game will have length $g$; thus, we know that $Q \in R_{v,g}$. Hence, $A$ is well-defined.

Claim 2. $A$ is injective.

Proof of claim. We will show that for distinct trees $T_1, T_2 \in T_{v,g}$, we have $A(T_1) \neq A(T_2)$. For this argument, we let $Q_{T_1}, Q_{T_2}$ denote one of the relaxed legal configurations on $G$ that result as we find $A(T_1), A(T_2)$ respectively. (Note that $A(T_1)$
does not equal $Q_{T_1}$ but rather $\overline{Q}_{T_1}$; similarly, $A(T_2) = \overline{Q}_{T_2}$.)

First suppose that $T_1$ and $T_2$ contain the same $g$ nodes. Because $T_1$ and $T_2$ share the same vertex set, we know $H_1$ and $H_2$ will be identical. The creation of $H'_1$ (and $H'_2$) does not involve the structure of $T_1$ (and $T_2$), so $H'_1$ and $H'_2$ will be identical as well. Consequently, we know $J_1 = J_2$, which implies that what makes $T_1^*$ and $T_2^*$ distinct is the distinct structures of $T_1$ and $T_2$. When we use $T_1^*$ and $T_2^*$ as inputs to Algorithm 4.3, the injective nature of the algorithm implies that $C_1^*$ and $C_2^*$ will be distinct; thus, $Q_{T_1} |_{T_1}$ and $Q_{T_2} |_{T_2}$ will be distinct. Because $V(T_1) = V(T_2)$, we have $\overline{Q}_{T_1} \neq \overline{Q}_{T_2}$. Thus, $A(T_1) \neq A(T_2)$.

Now suppose that $T_1$ and $T_2$ do not contain the same $g$ nodes, and that $A(T_1) = A(T_2) = Q$ for some $Q \in Q_{\mathbb{V}}$. When we showed above that $A$ is well-defined, we saw that seeding $v$ in $Q$ results in a game in which precisely the nodes in the underlying tree fire. But the original trees $T_1, T_2$ considered in this case are distinct. The deterministic nature of burn-off games, discussed in Proposition 2.1, prohibits this result; the same set of nodes must fire in any burn-off game played on a given configuration with seed $v$. Thus, $A(T_1) \neq A(T_2)$. ■

Having established that $A : T_{\mathbb{V}g} \to Q_{\mathbb{V}}$ is a well-defined injection, we turn our attention to showing the same is true of $B : Q_{\mathbb{V}g} \to T_{\mathbb{V}g}$, defined as follows. Let $\overline{Q} \in Q_{\mathbb{V}g}$, so that $Q \in \overline{Q}$. Let $H$ denote the subgraph induced on the nodes that fire if
\( v \) is seeded in \( Q \).

Because \( Q \in \overline{Q} \), seeding \( v \) in \( Q \) results in a burn-off game in which the nodes of \( H \) fire. With \( h = |V(H) - v| \), let \( F = (v, u_1, u_2, \ldots, u_h) \) be such a firing sequence of \( V(H) \). For \( m = 1, \ldots, h \), let \( l_m \) denote the number of \( H \)-neighbors of \( u_m \) that precede \( u_m \) in \( F \). At the time \( u_m \) fires, it must contain at least \( \deg_{G}(u_m) + 1 \) chips, so

\[
Q(u_m) \geq \deg_{G}(u_m) + 1 - l_m.
\]

This inequality is clearly equivalent to

\[
Q(u_m) \geq |\Gamma_{G-H}(u_m)| + \deg_{H}(u_m) + 1 - l_m,
\]

and since \( \deg_{H}(u_m) \geq l_m \), we may subtract \( |\Gamma_{G-H}(u_m)| \) from the right-hand side without it becoming negative. On the left-hand side, subtracting \( |\Gamma_{G-H}(u_m)| \) amounts to removing that many chips from \( u_m \). Let \( Q^*_H \) denote the configuration on \( H \) that results if, for each \( m = 1, \ldots, h \), we remove \( |\Gamma_{G-H}(u_m)| \) chips from \( u_m \). Thus, we have

\[
Q^*_H(u_m) \geq \deg_{H}(u_m) + 1 - l_m, \quad \text{for all } m = 1, \ldots, h.
\]

Since \( \deg_{H}(u_m) \geq l_m \), we may remove one additional chip from each \( w \in V(H) - v \).

Let \( Q_H \) denote the resulting configuration on \( H \), so that

\[
Q_H(u_m) \geq \deg_{H}(u_m) - l_m, \quad \text{for all } m = 1, \ldots, h.
\]

Note that \( v \) is the only node in \( V(H) \) that is critical in \( Q_H \).

Our intention is to input the graph \( H \) and the configuration \( Q_H \) into Algorithm 4.2. The algorithm requires that \( H \) is connected and \( Q_H \) is a relaxed legal
configuration. Since $H$ is a subgraph of $G$ induced on the nodes that fire during a burn-off game, $H$ is connected. Our choice of $Q$ comes from an equivalence class of the relation $\simeq$ on $R_{v,g}$, so $Q$ is a relaxed configuration on $G$. For each $u \in V(H)$, we remove $|\Gamma_{G-H}(u)|$ chips from $u$, so $Q_H^*$ is a relaxed configuration on $H$. In creating $Q_H$ from $Q_H^*$, we remove a chip from each $w \in V(H) - v$, so $Q_H$ is a relaxed configuration on $H$.

Finally, we appeal to Proposition 2.4 to show that $Q_H$ is a legal configuration on $H$. Partition $V(H)$ into independent sets as follows: for $t = 0, \ldots, h-1$, let $I_t = \{u_{h-t}\}$, and let $I_h = \{v\}$. For all $t = 0, \ldots, h-1$, those $H$-neighbors of $u_{h-t}$ that follow $u_{h-t}$ in $F$ appear in $\cup_{r=1}^{t} I_r$. Therefore,

$$Q_H(u_{h-t}) \geq \deg_H(u_{h-t}) - l_{h-t} = \sum_{w \sim u_{h-t}} 1_{\{w \in I_r \text{ and } r > t\}}$$

for all $t = 0, \ldots, h-1$.

It is easy to check that the analogous inequality holds for $v$, and this proves that $Q_H$ is legal.

We apply Algorithm 4.2 with the connected graph $H$ and the relaxed legal configuration $Q_H$ on $H$. The algorithm outputs a spanning tree $T^*$ of $H^*$. Because $v$ is the only node in $V(H)$ that is critical in $Q_H$, the only node adjacent to the special node $x$ in $H^*$ is $v$. Let $T = T^* - x$. Finally, define $B(\overline{Q}) = T$. This tree is clearly a member of $\mathcal{T}_{v,g}$, so $B$ is well-defined.
Claim 3. \( B \) is injective.

Proof of claim. We will show that for distinct \( \overline{Q}, \overline{Q}' \in Q_{v,g} \), we have \( B(\overline{Q}) \neq B(\overline{Q}') \).

Let \( Q, Q' \) be representatives of \( \overline{Q}, \overline{Q}' \) respectively. Let \( H, H' \) denote the subgraphs induced on \( G \) by the \( g \) nodes that fire when \( Q, Q' \) respectively are seeded at \( v \).

First, we consider the case where \( H = H' = H_0 \). Because \( \overline{Q} \) and \( \overline{Q}' \) are distinct, we know \( Q|_{H_0} \neq Q'|_{H_0} \). Therefore, \( Q_{H_0} \) and \( Q'_{H_0} \) will be distinct relaxed legal configurations on \( H_0 \). The injective nature of Algorithm 4.2 ensures that \( B(\overline{Q}) \neq B(\overline{Q}') \).

Second, we consider the case where \( H \neq H' \). When each of these subgraphs is used as the underlying graph in an iteration of Algorithm 4.2, the output is a spanning tree of that subgraph (with the edge \( \{v, x\} \), which we subsequently delete). Since \( H \neq H' \), these two trees must be distinct, so \( B(\overline{Q}) \neq B(\overline{Q}') \).

Assisted by the following claim, finally, we'll be able to turn our attention to the inner sum that appears in the statement of Theorem 4.9. Given \( T \in T_{v,g} \), let us denote \( A(T) \) by \( \overline{Q}_T \).

Claim 4. For each \( T \in T_{v,g} \), we have \( |\overline{Q}_T| = L(G - T) \).

Proof of claim. Because \( \overline{Q}_T \) is an equivalence class of the relation \( \approx \) on \( R_{v,g} \), it collects
all relaxed legal configurations that agree on $V(H)$. Thus, two elements of $\bar{Q}_T$ can differ only on $V(G - T)$. By Lemma 2.8, the legality of $Q \in \bar{Q}_T$ on $G$ implies the legality of $Q|_{G - T}$ on $G - T$. Hence, $|\bar{Q}_T| \leq L(G - T)$. Let $L$ represent any relaxed legal configuration counted by $L(G - T)$, and use $L$ for $Z$ in the definition of $A(T)$ (p. 74). This has the effect of extending $L$ to the rest of $G$ using $Q|_T$, which is common to all $Q \in \bar{Q}_T$. Because we used $A : T_{xg} \rightarrow Q_{xg}$ in bringing about this extension, the resulting configuration is legal on $G$. Since this extension is clearly injective, we have $|\bar{Q}_T| \geq L(G - T)$.

To complete the proof of Theorem 4.9, it suffices to show that for each $v \in V$, the number $N$ of relaxed legal configurations $C$ that result in a game of length $g$ when seeded at $v$ is $S := \sum_{T \in T_{xg}} L(G - T)$. From the definition of $R_{xg}$, we have $N = |R_{xg}|$, so it remains to show that $|R_{xg}| = S$.

Since both of $A, B$ are injections, and both of $T_{xg}, Q_{xg}$ are finite sets, it follows that $A$ is in fact a bijection. (The same is true of $B$, but we won't need this fact.) Thus, as $T$ runs through $T_{xg}$, its image $A(T) = \bar{Q}_T$ runs through $Q_{xg}$, and it follows that

$$|R_{xg}| = \sum_{\bar{Q} \in Q_{xg}} |\bar{Q}| = \sum_{T \in T_{xg}} |\bar{Q}_T| = \sum_{T \in T_{xg}} L(G - T) = S,$$

where Claim 4 justifies the third equation. ■
Example 4.10  We illustrate Theorem 4.9 on the graph $G$ in Figure 4.5. Here we consider a game of length three that starts with $v$ as the seed. One of the subtrees $T \in T_{v,3}$ is shown with bold lines. To apply Theorem 4.9 in evaluating $L(G - T)$, we add the node $x$ to $G - T$, resulting in the graph on the right.

![Figure 4.5. Determination of one term in the sum in Theorem 4.9](image)

Theorem 4.1 depends on Algorithm 2.6, which may take disconnected graphs as input (see p. 24). By Theorem 4.1, then, we have $\tau(G - T + x) = L(G - T)$. Using the Matrix Tree Theorem (or by inspection), we find that $L(G - T) = 8$. Thus, this particular choice of subtree through $v$ adds 8 legal configurations to the sum in Theorem 4.9. This is just one of the subtrees contributing to the inner sum; there are four more such subtrees.

4.4 All relaxed legal configurations are equally likely

Theorem 4.9 does not by itself accomplish our goal of mathematically predicting whether a sequence of burn-off games will exhibit the size versus
frequency relationship exhibited by a system in a state of SOC. The results in this section give us the last tool we need in our pursuit of this goal. Recall that for a connected graph $G = (V,E)$ we denote the set of relaxed legal configurations on $G$ by $\mathcal{R}$.

**Proposition 4.11** For $C \in \mathcal{R}$ and $v \in V$, there exists at least one $C^* \in \mathcal{R}$ such that if $C^*$ is seeded at $v$, then the resulting relaxed configuration is $C$.

**Proof.** If removing a chip from $v$ in $C$ results in a legal configuration $C^*$, then $C^*$ has the desired property, for if $C^*$ is seeded at $v$, then the resulting configuration is immediately $C$ after a game of length zero.

So we may suppose either that $C(v) = 0$ or that removing a chip from $v$ in $C$ creates an illegal configuration in $G$. In either case, we define $C^*$ by first defining a subgraph $Y$ of $G$. If $C(v) = 0$, then let $Y$ be the subgraph induced on $v$ alone. Now suppose that $C(v) > 0$. Because $C$ is a legal configuration, we may use Algorithm 2.6 to delete all of the nodes in some sequence according to the conditions specified in this algorithm. Since removing a chip from $v$ creates an illegal configuration on $G$ (see the preceding paragraph), there exists a set of nodes $X = \{v,y_1,y_2,\ldots,y_k\}$, with $k \geq 1$, that induces a connected subgraph of $G$ where only $v$ is critical in $C|_X$. Let $Y$ be the subgraph induced on the largest such set; the arguments below do not require
this choice to be unique.

Let \( C^*(v) = \text{deg}_G(v) \). For each \( u \in V - v \), let

\[
C^*(u) = (C(u) - |\Gamma_G(u) \cap Y|) \mod \text{deg}_G(u) + 1. \tag{4.9}
\]

Note that

\[
C(u) \geq |\Gamma_G(u) \cap Y| \text{ for } w \in V(Y) \tag{4.10}
\]

because otherwise we must include \( w \) in \( V(Y) \). We claim that \( C^* \) has the desired property. To establish this claim, we will show that (a) \( C^* \) is a legal configuration; and (b) if \( C^* \) is seeded at \( v \), the resulting relaxed configuration is \( C \).

(a) \( C^* \) is a legal configuration.

Since \( C \) is legal, using Algorithm 2.6 we may delete the nodes in \( V \) in some sequence \( \hat{S} = (u_i)_{i=1}^{N} \). For \( r = 1, \ldots, |V| \), let \( G_r \) be the subgraph of \( G \) that exists just prior to the deletion of \( u_r \) by Algorithm 2.6. Select an edge \( e = \{u_j, u_i\} \) such that (without loss of generality) \( u_j \) precedes \( u_i \) in \( \hat{S} \). Because Algorithm 2.6 can delete \( u_j \) from \( G_r \), we know \( C_{G_r}(u_j) \geq \text{deg}_{G_r}(u_j) \geq 1 \). Suppose that we delete \( e \) from \( G \) to create the graph \( G' \) (so that \( \text{deg}_{G'}(u_j) = \text{deg}_{G}(u_j) - 1 \)), and simultaneously create configuration \( C' \) on \( G' \) by removing one chip from \( u_j \) (so that \( C'_{G'}(u_j) = C_{G}(u_j) - 1 \)) and one chip from \( u_i \), if possible (we discuss this case later in the paragraph). We claim that Algorithm 2.6 may delete the nodes in \( V(G') \) in the same sequence \( \hat{S} \). The only nodes affected by the deletion of \( e \) are \( u_j \) and \( u_i \), so we consider each in turn.
Define $G'_r$ analogously to $G_r$. If we delete the nodes $u_1, \ldots, u_{j-1}$ of $G'$ in that order, we have $\deg_{G'_j}(u_j) = \deg_{G_j}(u_j) - 1$. To thereupon delete $u_j$ we require $C'_{G'_j}(u_j) \geq \deg_{G'_j}(u_j)$, which follows from

\[
C'_{G'_j}(u_j) = C'_{G_j}(u_j)
= C_G(u_j) - 1
= C_{G_j}(u_j) - 1
\geq \deg_{G_j}(u_j) - 1
= \deg_{G'_j}(u_j).
\]

We now turn to $u_i$. Because Algorithm 2.6 can delete $u_i$ from $G_i$, then $C_{G_j}(u_i) \geq \deg_{G_j}(u_i)$. If $C_{G_j}(u_i) = 0$, then all of the $G$-neighbors of $u_i$ must precede $u_i$ in $S$. In this case, $\deg_{G_j}(u_i) = 0$, so $C'_{G_j}(u_i) \geq \deg_{G_j}(u_i)$ even if we were not able to remove a chip from $u_i$ as described in the paragraph above. If $C_{G_j}(u_i) > 0$, then we were able to remove a chip from $u_i$ as described above. Here the same argument given for $u_j$ applies to $u_i$. Thus, Algorithm 2.6 may delete the nodes in $V(G')$ in the order given in $S$; this assures us that the new configuration on $G'$ is legal.

Let $S$ be the subsequence of $\hat{S}$ that contains the nodes in $Y$. With the help of the ideas in the paragraphs above, we now establish that Algorithm 2.6, acting on $G$, can also delete the nodes in $\hat{S} \setminus S$ before it deletes the nodes in $S$. Create the subgraph $\hat{G}$ of $G$ by deleting every edge in $E(G)$ that is incident with a node in $V(Y)$. At each deletion, remove one chip from the nodes incident with these edges, if possible. Let $\hat{C}$ be the configuration that results on $\hat{G}$. We know from (4.10) that
\( C_G(w) \geq |\Gamma_Y(w)| \) for \( w \in V(Y) \), so \( C_{\hat{G}}(w) \geq 0 \); that is, we will be able to remove a chip from \( w \) for each edge incident with \( w \) that we delete. By the argument given above, we know \( \hat{C} \) is legal. Because we have deleted every edge in \( E(G) \) that is incident with a node in \( V(Y) \), we know that \( \deg_G(y) = 0 \) for all \( y \in Y \). Therefore, if we use Algorithm 2.6 to investigate the legality of \( \hat{C} \), we may delete the nodes in \( V(Y) \) at any time we choose; in particular, we choose to delete them after we delete the nodes in \( V(G - Y) \). Let \( \hat{Q} \) be the sequence in which we delete the nodes in \( \hat{G} \), specifically, let \( \hat{Q} = (w_1, \ldots, w_\alpha, w_1, \ldots, w_\beta) \), where \( \alpha + \beta = |V| \), \( w_\alpha \in V(G - Y) \), and \( w_\beta \in Y \). Define \( \hat{G}_r \) analogously to \( G_r \). For \( w_\alpha \in V(G - Y) \), we know \( C_{\hat{G}_r}(w_\alpha) > \deg_{\hat{G}_r}(w_\alpha) \).

We claim that the nodes in \( V(G - Y) \) may be deleted from \( G \) by Algorithm 2.6 in the same order they appear in \( \hat{Q} \). Restore the edges that were deleted during the creation of \( \hat{G}_r \), along with the chips that were removed from the nodes. After we restore both the deleted edges and the removed chips, we have

\[
C_{\hat{G}_r}(w_\alpha) = C_{\hat{G}_r}(w_\alpha) + |\Gamma_Y(w_\alpha)| \geq \deg_{\hat{G}_r}(w_\alpha) + |\Gamma_Y(w_\alpha)| = \deg_{\hat{G}_r}(w_\alpha).
\]

Thus, we may let \( w_\alpha \) be the \( a \)-th node deleted as Algorithm 2.6 investigates the legality of \( G \). Let \( W \) denote this sequence \( (w_\alpha)_{\alpha=1}^n \). Since \( C_{|Y} \) is legal by Lemma 2.8, we may use Algorithm 2.6 to delete the nodes in \( V \) in some order. Let \( W' \) be this sequence, which we note is a permutation of \( S \).

So, if we concatenate the sequences \( W \) and \( W' \) to form the sequence \( T \), we know that Algorithm 2.6, acting on \( G \), can delete the nodes in \( V \) in the sequence \( T \).
Now concatenate the sequences $W'$ and $W$ to form the sequence $T^*$. We argue now that when we check $C^*$ for legality using Algorithm 2.6, we can delete the nodes in the order found in $T^*$.

First, we show that $v$ must be the initial entry in $W'$. If $v$ is the only entry in $W'$, this is clear; further, since $v$ is critical in $C^*$, Algorithm 2.6 may delete $v$ as it works on $C^*$. In this case, we may pick up the argument as it resumes after (4.14). If $v$ is not the only entry in $W'$, then suppose $y \in Y$, with $y \neq v$, is the first entry in $W'$. By definition of $Y$, we have $C(y) < \text{deg}\_7(y)$; thus, at least one neighbor of $y$ in $Y$ must be deleted before Algorithm 2.6 may delete $y$. This immediately gives our contradiction. Suppose then that $W' = (v, y_1, y_2, \ldots, y_k)$. For $i = 1, 2, \ldots, k$, let $t_i$ denote the number of neighbors of $y_i$ that precede $y_i$ in $W'$. For each $i$ we have

$$C(y_i) < \text{deg}\_7(y_i). \quad (4.11)$$

But since $y_i$ can be deleted (from $Y$) in the order specified by $W'$, we have

$$C(y_i) \geq \text{deg}\_7(y_i) - t_i. \quad (4.12)$$

In (4.9) we set

$$C^*(y_i) = (C(y_i) - \text{deg}\_7(y_i))(\text{mod} \deg_G(y_i) + 1).$$

But (4.11) implies that

$$C^*(y_i) = \text{deg}_G(y_i) + 1 + C(y_i) - \text{deg}\_7(y_i). \quad (4.13)$$

Now (4.12) and (4.13) together imply that
By (4.14) we conclude that in \( C^* \) the nodes in \( W' \) can be deleted in the order given by this sequence. It remains to show that the nodes in \( W \) may be deleted sequentially following the deletion of every node in \( W' \). By (4.9), the only nodes in \( W \) for which \( C(w_i) \neq C^*(w_i) \) are those with neighbors in \( Y \). Specifically, for \( i = 1, 2, \ldots, a \), we have \( C^*(w_i) = C(w_i) - |\Gamma_Y(w_i)| \) by (4.9) and (4.10). However, we have just seen that all of the nodes in \( V(Y) \) may be deleted first as Algorithm 2.6 checks the legality of \( C^* \), so the \( |\Gamma_Y(w_i)| \) neighbors of \( w_i \) in \( Y \) have been deleted when the algorithm begins to delete the nodes in \( W \). Let \( t'_i \) equal the number of neighbors of \( w_i \) that precede \( w_i \) in \( W \). Because \( C \) is a legal configuration on \( G \), we know that \( C(w_i) \geq \deg_G(w_i) - t'_i \). Thus,

\[
C^*(w_i) = C(w_i) - |\Gamma_Y(w_i)| \\
\geq \deg_G(w_i) - t'_i - |\Gamma_Y(w_i)|,
\]

which implies that Algorithm 2.6 may delete \( w_i \) in the order it appears in \( W \). We have now established that we may delete all of the nodes in the configuration \( C^* \) in the manner detailed in Algorithm 2.6; thus, by Proposition 2.9 we know that \( C^* \) is a legal configuration. 

\[\Box\]
(b) If $v$ is seeded in $C^*$, then the resulting relaxed configuration is $C$.

In our argument for part (a), we saw that an application of Algorithm 2.6 may delete the nodes of $G$ when they are in the configuration $C^*$ by deleting first the nodes in $W'$ and then those in $W$. We now argue that when $C^*$ is seeded at $v$, precisely the nodes in $W$ fire during the game.

It is clear that $v$ will fire when it receives one chip because $C^*(v) = \deg_G(v)$. For each $y_i$ in $W' = (v, y_1, y_2, \ldots, y_k)$, we have $C^*(y_i) > \deg_G(y_i) - t_i$ by (4.14). If the nodes in $(v, y_1, y_2, \ldots, y_{i-1})$ fire, then the number of chips on $y_i$ increases to $C^*(y_i) + t_i > \deg_G(y_i)$. Thus, $y_i$ may now fire as well; so, all nodes in $W'$ fire during the game. When we create $C^*$ from $C$ we remove $|\Gamma_Y(w)|$ chips from each node $w \in V(Y)$ (see (4.9) and (4.10)). But if we fire all of the nodes in $V(Y)$, then $w$ gains exactly that many chips. Since $w$ is not supercritical in $C$, it is also not supercritical at any point in the game that starts on $C^*$. Thus none of the nodes in $W$ fire during the game. Further, it is evident that when $C^*$ has relaxed, the number of chips on $w$ is $C(w)$.

We now establish that when $C^*$ has relaxed, the number of chips on each $y_i \in V(Y) - v$ is $C(y_i)$. Since $y_i$ fires during the game, it loses $\deg_G(y_i) + 1$ chips. But all $\deg_G(y_i)$ of its neighbors also fire during the game, so it gains this many chips. Thus, from (4.13) we see that when $C^*$ has relaxed, the number of chips on each $y_i \in V(Y) - v$ is $C(y_i)$. 

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Finally, we recall that $C^*(v) = \deg_G(v)$ by definition. When $v$ is seeded and it fires, it loses all of its chips, but then it gains one chip from each of its $\deg_Y(v)$ neighbors in $Y$. When the game finishes, $v$ will thus contain $\deg_Y(v)$ chips, which, as we saw when we defined $Y$, is $C(v)$. We have therefore established that $C^*$ relaxes to $C$ when $v$ is the seed. ■

We intend to prove that all relaxed legal configurations are equally likely. As a step in this direction, it is helpful for us to show that choosing two seeds in a given configuration results in two distinct relaxed configurations; before we do, we pause to prove the following useful result.

**Lemma 4.12.** Let $C$ be a relaxed legal configuration on a connected graph $G$. Consider a burn-off game of length at least two that results in the configuration $C'$. Let $H$ be the subgraph induced on the nodes that fire during the game. After relaxation, only the seed $v$ will be critical in $H$.

**Proof.** The seed $v$ must fire, since the game length is at least two. When $v$ fires it loses all its chips. Each of its neighbors in $H$ fires, sending a chip back to $v$. By Lemma 4.8, none of these nodes may fire a second time. Thus, at the end of the game, $C'(v) = \deg_H(v)$; that is, $v$ is critical in $H$.  

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Consider \( u \in V(H) - v \). We know that \( C(u) \leq \deg_G(u) \) as the game begins. By Lemma 4.8, each node in \( H \) fires exactly once. Thus, each of the \( \deg_H(u) \) neighbors of \( u \) in \( H \) fires and adds one chip to \( u \). By Proposition 2.1, the firing order has no effect on \( C' \). Thus, \( C'(u) = C(u) + \deg_H(u) - (\deg_G(u) + 1) < \deg_H(u) \). □

Now we proceed to the main result that, along with Proposition 4.11, allows us to conclude that all relaxed legal configurations in a burn-off game on a given connected graph are equally likely.

**Proposition 4.13** Given \( C \in \mathcal{R} \) on a connected graph \( G = (V,E) \), it is not possible to choose two distinct seeds and relax the resulting configurations to the same legal configuration.

**Proof.** Define \( R_v : V \to \mathbb{N} \) as the relaxed configuration that results from seeding \( C \) at \( v \). We wish to show that for \( u, v \in V \), with \( u \neq v \), we have \( R_u \neq R_v \). Suppose, for a contradiction, that \( R_u = R_v \). If \( u \) is the seed and does not fire, then \( R_u(u) = C(u) + 1 \). If \( v \) is the seed and does not fire, then \( R_v(u) = C(u) \). Thus, \( R_u \neq R_v \), which contradicts our assumption. Thus, at least one of \( u, v \) must fire when chosen as the seed.

Suppose first (without loss of generality) that if \( G \) is seeded at \( u \), then \( u \) fires, while if \( G \) is seeded at \( v \), then \( v \) does not fire. Since \( v \) is not critical in \( G \) and we
have assumed $R_u = R_v$, then $C(u) = R_v(u) = R_u(u)$. If $u$ is chosen as the seed, it fires and loses all of its chips. Because $R_v(u) = R_v(u)$, $u$ must regain $\deg(u)$ chips. This happens only if every neighbor of $u$ fires after $u$ itself fires. Now for every neighbor of $u$ to fire, at least one of those neighbors must itself be critical in $G$. Suppose that $w$ is such a neighbor. In the game where $v$ is the seed and does not fire, we thus know that $w$ is critical in $G$. Because $R_w(w) = R_v(w)$, it must be true that $w$ again becomes critical, after it fires, in the game where $u$ is the seed. But this is impossible by Lemma 4.12.

Now we may suppose that both $u$ and $v$ fire if chosen as the seed. First, we show that if $v$ is the seed (without loss of generality), then $u$ must also fire. Suppose by way of contradiction that $u$ does not fire, so $R_u(u) = \deg(u)$. If $u$ and $v$ are neighbors, then clearly $u$ must fire if $v$ is the seed; we therefore assume for the remainder of the proof that $u$ and $v$ are not neighbors. Then in the game where $u$ is the seed, every neighbor of $u$ must also fire so that $R_u(u) = \deg(u)$. Therefore, at least one neighbor $w$ of $u$ must be critical in $G$. By Lemma 4.12, $R_u(w) < C(w)$. But $R_u = R_v$ implies that $w$ must also fire in the game where $v$ is the seed. Since $u$ is critical in $G$, it too will fire when $w$ fires. This contradicts our assumption. Thus, if either $u$ or $v$ is the seed, the other node must fire during the game.

So in the game where $v$ is the seed, at least one neighbor of $v$ must also fire so that $u$ will fire. Suppose that $k$ neighbors of $v$ fire. After $v$ fires, it contains zero
chips, so \( R_v(v) = k \). Because \( R_u = R_v \), we must have \( R_u(v) = k \) as well. In the game where \( u \) is the seed, at least one neighbor of \( v \) must fire in order to allow \( v \) to fire. Say that \( w \) is such a neighbor. Now after \( v \) fires, it contains zero chips, so \( k \) of its neighbors must now fire to achieve \( R_u(v) = k \). Since \( w \) has already fired in this game, it may not be one of these \( k \) neighbors (by Lemma 4.8). So at least \( k + 1 \) neighbors of \( v \) fire in the game that starts with \( u \) as the seed.

Thus, we have \( k \) neighbors of \( v \) firing in the game where \( v \) is the seed and at least \( k + 1 \) neighbors of \( v \) firing in the game where \( u \) is the seed. There is therefore a node \( z \) that is adjacent to \( v \) and that fires when \( u \) is the seed but not when \( v \) is the seed. The number of chips on \( z \) at the end of a game in which it does not fire must be at least \( C(z) \). But the number of chips on \( z \) at the end of a game in which it does fire must be at most \( C(z) + \deg(z) - (\deg(z) + 1) < C(z) \). Thus, \( R_u(z) < R_v(z) \), contradicting our assumption that \( R_u = R_v \). \[ \blacksquare \]

We may now proceed with the main result of this section. Recall that we are considering a sequence of burn-off games as a Markov chain where the states are the relaxed legal configurations and the transitions are determined by the configurations that result when a node is chosen as the seed and the configuration is relaxed. Let \( D \) be the digraph that represents this Markov chain, and let \( P \) be the
transition probability matrix. Let $\mathcal{R}$ be the set of relaxed legal configurations on $G$ and let $r = |\mathcal{R}|$.

**Theorem 4.14** All relaxed legal configurations in a sequence of burn-off games on a connected graph $G = (V,E)$ are equally likely.

**Proof.** Let $C^* \in \mathcal{R}$ be a relaxed legal configuration on $G$. When we initiate a burn-off game on $G$ by seeding one of its nodes, $C^*$ relaxes to a legal configuration (by Proposition 2.1). As a result, we know that

$$\text{outdeg}_D(C^*) \leq |V| \quad \text{for all } C^* \in \mathcal{R}. \quad (4.15)$$

Summing over all relaxed legal configurations, we have

$$\sum_{C^* \in \mathcal{R}} \text{outdeg}_D(C^*) \leq r|V|. \quad (4.16)$$

In Proposition 4.11 we saw that for any legal configuration $C$ and node $v$ there is at least one legal configuration $C^*$ that relaxes to $C$ when $v$ is seeded. Then in Proposition 4.13 we found that such $C^*$ must be distinct for each choice of $v$. Together, these results imply that

$$\text{indeg}_D(C) \geq |V| \quad \text{for all } C \in \mathcal{R}. \quad (4.17)$$

Summing over all relaxed legal configurations, we have

$$\sum_{C \in \mathcal{R}} \text{indeg}_D(C) \geq r|V|. \quad (4.18)$$
Inequalities (4.16) and (4.18) together give
\[ r|\mathcal{V}| \geq \sum_{C \in \mathcal{R}} \text{outdeg}_D(C) = \sum_{C \in \mathcal{R}} \text{indeg}_D(C) \geq r|\mathcal{V}|, \]

which establishes the identity
\[ \sum_{C \in \mathcal{R}} \text{outdeg}_D(C) = \sum_{C \in \mathcal{R}} \text{indeg}_D(C) = r|\mathcal{V}|. \] (4.19)

Taking (4.19) along with (4.15) and (4.17) respectively, we conclude that
\[ \text{outdeg}_D(C) = \text{indeg}_D(C) = |\mathcal{V}| \quad \text{for all } C \in \mathcal{R}. \] (4.20)

In a burn-off game, the seed is chosen at random, so every nonzero entry in \( P \) is 1/|\mathcal{V}|. Since \( \text{indeg}_D(C) = |\mathcal{V}| \) for all \( C \in \mathcal{R} \), it follows that \( P \) is doubly stochastic. Therefore (see, e.g., [10]), the entries of the stationary distribution are all equal. ■

Theorem 4.9 and Theorem 4.14 together give us a method by which we may mathematically determine, without resorting to computer simulations, the probabilities associated with the lengths of burn-off games on any given connected graph. They do not, however, provide a closed formula for finding these probabilities. In Chapter 5, we return to the special case of complete graphs and, in this more restrictive setting, find such a closed formula. We conclude the present chapter with an example to show how our mathematical results can be compared to our earlier empirical ones.
Example 4.15 Consider the graph $G$ shown in Figure 1.1 on p. 4. Applying Theorem 4.1 to $G$, we find that there are 40 relaxed legal configurations. Because $G$ has four nodes, there are 160 pairs $(C,v)$ of relaxed legal configurations and seed choices. Of these, 82 pairs result in a game of length zero (Proposition 4.7). We know that burn-off games on $G$ may not have length greater than four (Lemma 4.8).

![Figure 4.6. Comparison of analytic results to previously simulated results in Figure 1.2, p. 9](image)

Four applications of Theorem 4.9 show that the number of pairs resulting in games of length one, two, three, and four are 35, 16, 15, and 12 respectively. By Theorem 4.14, each relaxed legal configuration is equally likely to appear during a sequence of burn-off games on $G$. Thus, we may calculate the probability distribution of the game length. These results are shown in Figure 4.6: the lefthand bars record the results of 10,000 computer simulations, previously appearing in...
Figure 1.2; the righthand bars are those found in this example using the methods of the present chapter. The close visual agreement between the analytical and simulated data was confirmed by a $\chi^2$ goodness-of-fit test. Even with the level of significance $\alpha$ as high as 0.1, this test did not reject the hypothesis that our analytical results correctly model the simulated data.
Chapter 5

Distribution of game length on a complete graph

Our main goal in this chapter is to find a closed formula for the probability distribution of the burn-off game length on a complete graph. By finding this distribution, we shall draw conclusions about whether such games exhibit SOC behavior.

5.1 Overview

Refer to the introduction of Chapter 4 for a description of the probability model we associate with burn-off games. Let us define the random variable $X$ to be
the length $g \geq 0$ of a burn-off game on $K_n$, and, as in Chapter 3, let $\mathcal{R}$ be the set of relaxed legal configurations on $K_n$. To determine the distribution of $X$, we condition on the configuration $q \in \mathcal{R}$ that is seeded:

$$\Pr\{X = g\} = \sum_q \Pr\{X = g \mid q \text{ is seeded}\} \Pr\{\text{seeded configuration is } q\}. \quad (5.1)$$

In Section 5.2, we find that $\Pr\{\text{seeded configuration is } q\}$ does not depend on $q$, but only on $n$. In Section 5.3, we find an expression for

$$P_{g,q} := \Pr\{X = g \mid q \text{ is seeded}\}$$

in terms of just $n$ and $g$. These results yield a formula for finding the probability of any game length we choose.

5.2 All relaxed legal configurations on $K_n$ are equally likely

Recall from the introduction to Chapter 4 that we view burn-off games as a sequence of experiments in which a pair $(q, v) \in \mathcal{R} \times \mathcal{V}$ results in an outcome $q^* \in \mathcal{R}$. In this section, we investigate the long-run probability that, during the relaxed phase of a burn-off chip-firing game on $K_n$, the chips are distributed on the nodes in a particular configuration $q$. We know from Theorem 4.14 that this long-run probability does not depend on $q$; we include this alternate proof because it results in an explicit formula in the case of complete graphs.
Theorem 5.1  In a sequence of burn-off games played on $K_n$, the stationary probability of a burn-off game relaxing to configuration $q \in \mathcal{R}$ is $\frac{1}{(n + 1)^{n-1}}$.

Recalling Theorem 3.3, we see that the denominator here is the number of relaxed legal configurations for a burn-off chip-firing game on $K_n$. Theorem 5.1 simply states that every relaxed legal configuration is equally likely to appear during the relaxed phase of a sequence of burn-off games. Since the proof occupies a number of pages, we first provide an outline.

Let $q$ be a legal configuration on $K_n$, and $v \in V(K_n)$. We will find a legal configuration $q^*$ such that, if $v$ is the seed, the resulting relaxed configuration is $q$ (cf. Proposition 4.11, where our proof is less transparent). We will also show that if the chips are in a configuration $q^*$, it is impossible to choose two different nodes as the seed and result in the same configuration $q$. Thus, we will show that there are at least $n$ configurations on which a particular choice of seed will result in the configuration $q$.

Thanks to the deterministic nature of chip-firing games (see Proposition 2.1), it is easy to see that for any given configuration $q^*$, there are at most $n$ configurations that can result from adding a chip to a node. Every chip-firing game must relax to just one configuration, and since there are $n$ nodes to choose as the seed, there are at most $n$ possible outcomes.
These bounds together imply that for each configuration $q$ there are
exactly $n$ configurations $q^*$ that relax to $q$. Since each node is equally likely to be
chosen as a seed, we will be able to use a key property of Markov chains with
doubly-stochastic transition matrices to complete our argument.

Proof of Theorem 5.1. Let $q$ be a legal configuration on $K_n$ and $v \in K_n$. We seek a
legal configuration $q^*$ such that, if $q^*$ is seeded at $v$, the resulting relaxed
configuration is $q$. We consider two cases. Before jumping in, let us note that if
$q(v) = 0$, then Proposition 3.1 implies that we cannot have $q(u) = 0$. Thus, any
configuration with $q(v) = 0$ will be addressed in Case 2.

Case 1. There exists $u \neq v$ such that $q(u) = q(v) \geq 1$.

Suppose that $q(u) = q(v) = j$. In this case, it suffices to take $q^*$ as the
configuration agreeing with $q$ except for one fewer chip on $v$. Since $q$ is legal,
Proposition 3.1 shows that at most $j + 1$ nodes contain $j$ or fewer chips. But
$q(u) = q(v) = j$ implies that at most $j - 1$ nodes contain $j - 1$ or fewer chips. When a
chip is removed from $v$ to obtain $q^*$, at most $j$ nodes will contain $j - 1$ or fewer chips.
Thus, $q^*$ is a legal configuration; clearly, $q^*$ relaxes to $q$ (after zero steps).

Case 2. No other node contains the same number of chips as does $v$.

Denote the number of chips on the nodes of $K_n$ with a vector as follows:
where \( x_{n-1} \geq x_{n-2} \geq \ldots \geq x_1 \geq x_0 \) (we shall refer to this as the canonical order on \( q \)), and \( x_i \) is the number of chips on \( v \). Later, it will be useful to rewrite these inequalities as: \( x_{i-k} \geq x_{i-(k+1)} \) for \( 0 \leq k < i \), and \( x_{n-j} \geq x_{n-(j+1)} \) for \( 1 \leq j < n-i \).

Since \( q \) is legal, at most \( i \) nodes contain \( i-1 \) or fewer chips. In addition, \( q \) is written in canonical order, so \( x_i \geq i \). If \( x_i > i \), then we may reduce the number of chips on \( v \) by one to create \( q^* \), a configuration in which at most \( i \) nodes contain \( i-1 \) or fewer chips. Therefore, it remains only to consider those cases where \( x_i = i \).

Since \( q \) is legal, we know that \( x_{i-k} \geq i-k \), for \( 0 \leq k < i \), and that \( x_{n-j} \geq n-j \), for \( 1 \leq j < n-i \). We therefore know that \( x_{i-1} = i-1 \), because \( i = x_i \geq x_{i-1} \geq i-1 \), and \( v \) is the only node containing \( x_i \) chips. Since all legal configurations have at least one critical node, and \( q \) is written in canonical order, we also have \( x_{n-1} = n-1 \), the common degree in \( K_n \). For later reference, we restate and label these findings.

\[
\begin{align*}
x_i &= i. \\
x_{i-1} &= i - 1. \\
x_{i-k} &\geq i - k, \text{ for } 0 \leq k < i. \\
x_{n-1} &= n - 1. \\
x_{n-j} &\geq n - j, \text{ for } 1 \leq j < n - i. \\
x_{i-k} &\geq x_{i-(k+1)} \text{ for } 0 \leq k < i. \\
x_{n-j} &\geq x_{n-(j+1)} \text{ for } 1 \leq j < n - i. 
\end{align*}
\]
The configuration $q^*$ that we claim relaxes to $q$ when seeded at $v$ is

$$q^* = (x^*_1, x^*_{n-1}, \ldots, x^*_2, x^*_1, x^*_0, x^*_{n-1}, x^*_{n-2}, \ldots, x^*_{n-(n+1)}),$$

where

$$x^*_i = n - 1,$$
$$x^*_{i-k} = x_{i-k} - x_i + n \quad \text{for } 1 \leq k \leq i,$$
$$x^*_{n-j} = x_{n-j} - (x_i + 1) \quad \text{for } 1 \leq j \leq n - (i + 1).$$

Example 5.2 Consider $K_6$, and $q = (5,4,4,2,1,0)$. Suppose we pick the node containing 2 chips as our special node $v$. Note that $x_i = 2 = i$, and that $v$ is the only node containing this many chips—these two qualities fit the requirements of Case 2.

According to the definition of $q^*$, we have $q^* = (5,5,4,2,1,1)$. Since $x^*_i = 5$, if we add a chip to $v$, then we obtain the following firing sequence:

$$(6,5,4,2,1,1)$$
$$(0,6,5,3,2,2)$$
$$(1,0,6,4,3,3)$$
$$(2,1,0,5,4,4).$$

If re-written in canonical order, the final configuration is $q$.

Following our outline, to complete the proof of Theorem 5.1, we will first establish four facts about $q^*$:
(a) \( q^* \) is in canonical order;
(b) \( q^* \) is a legal configuration;
(c) if \( v \) is seeded in \( q^* \), then \( q^* \) will relax to \( q \);
(d) starting from \( q^* \), it is not possible to seed a node other than \( v \) and relax to \( q \).

(a) \( q^* \) is in canonical order.

The configuration \( q^* \) is in canonical order if its entries are nonincreasing.

To demonstrate this, we observe that the entries \( x_i^*, x_{i-1}^*, \ldots, x_2^*, x_1^*, x_0^* \) are nonincreasing. First note that

\[
\begin{align*}
x_{i-1}^* &= x_{i-1} - x_i + n \\
&= (i - 1) - i + n \\
&= n - 1,
\end{align*}
\]

so that \( x_i^* = n - 1 = x_{i-1}^* \). Now for \( 0 < k < i \),

\[
\begin{align*}
x_{i-k}^* &= x_{i-k} - x_i + n \\
&\geq x_{i-(k+1)} - x_i + n \\
&= x_{i-(k+1)}^*.
\end{align*}
\]

where the inequality holds by (5.7). Observe that \( x_0^* \geq x_{m-1}^* \) because
Finally, we show that the entries $x_{n-1}^*, x_{n-2}^*, \ldots, x_{n-(i+1)}^*$ are nonincreasing.

For $1 \leq j \leq n - (i + 1)$,

\[
x_{n-j}^* = x_{n-j} - (x_i + 1) \\
\geq x_{n-(i+1)} - (x_i + 1) \\
= x_{n-(i+1)}^*
\]

where the inequality holds by (5.8). Thus, $q^*$ is in canonical order. ■

(b) $q^*$ is a legal configuration.

If a configuration $r = (r_{n-1}, r_{n-2}, \ldots, r_j, r_{j-1}, r_{j-2}, \ldots, r_2, r_1, r_0)$ is in canonical order, and for all $0 \leq j \leq n - 1$, only the entries $r_{j-1}, r_{j-2}, \ldots, r_2, r_1, r_0$ are $j - 1$ or less, then at most $j$ entries of $r$ are $j - 1$ or less, so $r$ is legal (by Proposition 3.1). In other words, if $r$ is in canonical order and $r_j \geq j$ for all $0 \leq j \leq n - 1$, then $r$ is legal. We have established in part (a) that $q^*$ is in canonical order, so here we show that $q_j^* \geq j$ for all such $j$.

We begin by noting that $x_i^* = n - 1$, so that $q_{n-1}^* \geq n - 1$. We turn our
attention to the terms that follow \( x_i^* \). First, for \( 0 < k \leq i \), note that \( q_{n-(i+1)}^* = x_{i-k}^* \) and that
\[
\begin{align*}
x_{i-k}^* &= x_{i-k} - x_i + n \\
&\geq i - k - i + n \\
&= n - k \\
&> n - (k + 1),
\end{align*}
\]
where the first inequality holds by (5.2) and (5.4). Second, for \( 1 \leq j \leq n - (i + 1) \), note that \( q_{j-1}^* = x_{n-(i+j)}^* \) and that
\[
\begin{align*}
x_{n-(i+j)}^* &= x_{n-(i+j)} - (x_i + 1) \\
&= x_{i+j} - x_i - 1 \\
&\geq i + j - i - 1 \\
&= j - 1,
\end{align*}
\]
where the inequality holds by (5.2) and (5.6). Since \( q^* \) is in canonical order and each of its entries is sufficiently large, \( q^* \) is legal.

(c) If \( q^* \) is seeded at \( v \), then the resulting relaxed configuration is \( q \).

Recall that \( q_{-1}^* = x_i^* \), the number of chips on \( v \) in the configuration \( q^* \). Since \( x_i^* = n - 1 \) (by (5.9)), then \( v \) will fire when \( q^* \) is seeded at \( v \). We proceed to show that this causes all of the nodes represented in the next \( i \) entries of \( q^* \) also to fire but none of the remaining \( n - (i + 1) \) nodes.

For convenience, we will refer to the nodes of \( K_n \) by their corresponding positions in \( q^* \), so, e.g., the node starting with \( x_i^* \) chips will be called node \( (n - 1) \)
and the node starting with $x_{n-(i+1)}^*$ chips will be called node 0. Note that for $k = 0, 1, \ldots, i$, there are $x_{n-k}^*$ chips on the node $n-(k+1)$.

We demonstrate now that the node $n-k$, for $2 \leq k \leq i+1$, will fire if all nodes $n-a$, for $a = 1, 2, \ldots, k-1$, have fired. The node $n-k$ begins with $x_{n-(k-1)}^*$ chips, and, using the estimates from part (b) above, we have $x_{n-(k-1)}^* > n-k$. Thus, after the nodes $n-a$, for $a = 1, \ldots, k-1$, have fired and thus added $k-1$ chips to the node $n-k$, the number of chips on this node will exceed $(n-k) + (k-1) = n-1$. Therefore, the node $n-k$ indeed fires.

We show now that none of the nodes $n-a$, for $a = i+2, i+3, \ldots, n$, will fire. We need only check the status of the node $n-(i+2)$, because $q^*$ is in canonical order, implying that the node $n-(i+2)$ contains at least as many chips as any of the nodes $n-a$ for which $a = i+3, \ldots, n$. The number of chips on the node $n-(i+2)$ is $x_{n-1}^* = x_{n-1} - (x_i + 1) = (n-1) - (i+1) = n-i-2$. By the argument above, all nodes earlier than the node $n-(i+2)$ in the canonical ordering have fired. There are $i+1$ such nodes, so $i+1$ chips are added to the node containing $x_{n-1}^* = n-i-2$ chips in $q^*$. The new number of chips on the node $n-(i+2)$, then, is $(n-i-2) + (i+1) = n-1$, so that node does not fire.

Finally, we argue that $q^*$ relaxes to $q$. First, consider the nodes that do fire, namely, the nodes containing $x_{n-k}^*$ chips (where $k = 0, 1, \ldots, i$). Recall from (5.9) that $x_{n-k}^* = x_{n-k} - x_i + n$, and note that $i+1$ nodes fire during the course of the game,
including the node in question. When a node fires, it loses $n$ chips, so the new number of chips on a node that begins with $x_{i-k}^*$ chips is $x_{i-k}^* + i - n = x_{i-k} - x_i + n + i - n = x_{i-k}$, since $x_i = i$ (by (5.2)).

Now consider the nodes that do not fire, namely, the nodes with $x_{n-j}^*$ chips on them (where $j = 1, 2, \ldots, n - (i + 1)$). Since $x_{n-j}^* = x_{n-j} - (x_i + 1)$, and $i + 1$ nodes fire (adding one chip each to every other node), the new number of chips on a node that begins with $x_{n-j}^*$ chips is $x_{n-j}^* + (i + 1) = x_{n-j} - (x_i + 1) + (i + 1) = x_{n-j}$, again using (5.2). Thus, we see that for both sets of nodes, the corresponding entries of $q^*$ are transformed to those of $q$ by the firing sequence.

(d) Starting from $q^*$, it is not possible to seed at a node $u \neq v$ and relax to $q$.

We show that if we start with the configuration $q^*$ and seed a firing sequence at a node $u \neq v$, the resulting configuration cannot be identical to $q$.

One of two outcomes can occur: (1) neither $u$ nor $v$ fires if a chip is added, or (2) one or both fire if a chip is added. In the first case, the conclusion clearly follows: adding a chip to $u$ increases the number of chips on $u$ by one, but does not change the number of chips on $v$, while adding a chip to $v$ has the opposite effect.

As we turn to the second case, we recall Lemma 4.8 which implies that a burn-off game started in a relaxed configuration may take at most $n$ turns. (To establish that such a game may take exactly $n$ turns, consider $K_n$ with every node
containing $n - 1$ chips.) Note also that if a node $x$ fires once during the course of a game of length $g$, then the number of chips on $x$ increases by $g - 1$ (or $g$, if $x$ is the seed) and decreases by $n$ (one chip moves to each neighbor, and one burns off). On the other hand, if $x$ does not fire during the course of such a game, then the number of chips on $x$ increases by $g$. Thus, if $u$ is the seed, and it fires once, the number of chips on $u$ increases by $g$ and decreases by $n$.

Now we consider case (2): let $u$ be the node that fires if a chip is added to it and $v$ be the node that may or may not fire. We examine these two possibilities in turn. Suppose that $v$ contains fewer than $n - 1$ chips, so that if it were chosen as the seed, it would not fire—in this case, the number of chips on $u$ would remain constant. If $u$ is the seed, it will fire and lose all of its chips. For $u$ to regain its $n - 1$ chips, $g$ would have to be equal $n$ (by Lemma 4.8, $g$ cannot exceed $n$). But if $g = n$, then the number of chips on $v$ increases by $g - 1 = n - 1$ and decreases by $n$, resulting in the loss of one chip from $v$.

Suppose instead that $v$ contains $n - 1$ chips, so that if it were chosen as the seed, it would fire, and the number of chips on it would increase by $g$ and decrease by $n$. In this case, were $u$ to be the seed, $v$ would fire as well, which would increase the number of chips on $v$ by $g - 1$ and decrease it by $n$. In these cases, the resulting numbers of chips on $v$ differ, so the final configurations cannot be the same. ■
Combining the results (a) through (d) with the remarks in the outline of this proof (p. 100), we have now established that for every relaxed configuration \( q \), there are exactly \( n \) relaxed configurations that, when seeded, will relax to \( q \).

Consider the burn-off chip-firing game as a Markov chain, where the states are the relaxed legal configurations, and the transitions are determined by the configurations that result when a node is seeded. Let \( D \) be the digraph representing this Markov chain. If \( \mathcal{R} \) is the set of relaxed legal configurations, then

\[
 n |\mathcal{R}| \leq \sum_{C \in \mathcal{R}} \text{indeg}_D(C) = \sum_{C \in \mathcal{R}} \text{outdeg}_D(C) \leq n |\mathcal{R}|. \tag{5.10}
\]

The first inequality follows from the conclusions just reached, and the second is justified in the outline of the proof (p. 100). In a burn-off game, the seed is chosen at random, so the transition probabilities are all \( \frac{1}{n} \). Since by (5.10) we have \( \text{indeg}_D(C) = \text{outdeg}_D(C) = n \) for all \( C \in \mathcal{R} \), the transition matrix for this Markov chain is doubly stochastic. It follows (see, e.g., [10]) that the entries of the stationary distribution are all equal. In other words, we have shown that all relaxed legal configurations on \( K_n \) are equally likely. We saw earlier that the number of legal configurations on \( K_n \) is \( (n+1)^{-1} \). Therefore, the stationary probability of a burn-off game relaxing to configuration \( q \) is \( \frac{1}{(n+1)^{-1}} \). \( \blacksquare \)
5.3 The probabilities associated with varying game lengths

In this section, we consider the conditional probability that, during a burn-off chip-firing game on $K_n$, a firing sequence will be of a particular length $g \geq 0$, given that the game is in a particular legal configuration $q$. In other words, we investigate

$$P_{g,q} = \Pr\{X = g \mid q \text{ is seeded}\},$$

for a nonnegative integer $g$ and $q \in \mathcal{R}$. Recall this is the conditional probability appearing in Section 5.1.

Recall that a node in a relaxed configuration is critical if it contains as many chips as its degree, that is, if the node will fire with the addition of a single chip. Call a node unstable if it will fire only when a critical node is chosen as the seed. Finally, call a node a dud if it cannot fire during any game that is played on its configuration. Either a node will fire when seeded (so is critical) or will not (so is unstable or a dud). But an unstable node will fire when a critical node is seeded, whereas a dud will not. Since each node must be one of these three types, the nodes are partitioned into critical nodes, unstable nodes, and duds.

Example 5.3 Consider the configuration $q = (6, 6, 5, 4, 1, 1, 0)$ on $K_7$. This configuration has two critical nodes, each containing 6 chips. Either of these nodes will fire if chosen as the seed. The configuration has two unstable nodes that contain...
5 and 4 chips. If either of these is seeded, it will not fire; however, if one of the
critical nodes is seeded, both of the critical nodes will fire, and both will contribute
one chip to the node containing 5 chips, allowing that node to fire. Now, with the
addition of the three chips provided by these three firings, the node containing 4
chips will fire as well. Since none of the remaining three nodes will collect enough
chips to fire, these nodes are duds.

Given a configuration $q$, let $c$ denote the number of critical nodes and $u$
the number of unstable nodes. Notice that there are only two possible game lengths
that can result when a random node is seeded: either the node will not fire,
resulting in a game of length zero; or the node will fire, resulting in a game of length
$c + u$, by Lemma 4.8. For this reason, we will profit from counting the configurations
that have $c$ critical nodes and $u$ unstable nodes, for fixed nonnegative integers $c$
and $u$.

We handle the case $c = 1$ separately after considering the case $c \geq 2$.

**Proposition 5.4** For all integers $2 \leq c \leq n$, $0 \leq u \leq n - c$, and $d = n - (c + u)$, the
number of configurations on $K_n$ with $c$ critical nodes, $u$ unstable nodes, and $d$ duds is

$$\binom{n}{d} L_{d,d-1} \binom{n-d}{u} L_{u,n-d-3}.$$ (5.11)
Proof. Let $Q$ be the set of configurations that satisfy the conditions of our proposition. We shall work toward determining the size of $Q$ by first examining all possible chip placements on the duds.

If a critical node of $q \in Q$ is chosen as the seed, then all $c + u$ critical and unstable nodes will fire, contributing $c + u$ chips to each dud. Therefore, each dud contains at most $n - (c + u) - 1 = d - 1$ chips, for otherwise the added chips will cause it to fire. Since there are $\binom{n}{d}$ ways to choose which of the $n$ nodes will be duds, and $L_{d,d-1}$ ways to place chips on these duds, the first two factors in (5.11) account for the chip placement on the duds.

Now we shall account for the number of ways we may place chips on the $u$ unstable nodes of $q$. Since nodes containing $n - 1$ chips are critical, unstable nodes contain at most $n - 2$ chips. The greatest number of chips that may be added during a game to an unstable node is $c + (u - 1)$, which implies that the smallest number of chips that an unstable node may contain at the outset is $n - (c + (u - 1)) = d + 1$. Thus, unstable nodes contain between $d + 1$ and $n - 2$ chips.

Within the set of $u$ unstable nodes, there is one that contains the fewest chips, and it may contain anywhere from $d + 1$ to $n - 2$ chips. If two unstable nodes $v$ and $w$ both contain $d + 1$ chips, then there would be at most $c + (u - 2)$ other nodes that could fire and add chips to $v$ and $w$. These additional chips would increase the number of chips on $v$ and $w$ to at most
\[(d + 1) + c + (u - 2) = c + d + u - 1 = n - 1,\]

which is not enough for either of them to fire. Thus, both \(v\) and \(w\) would be duds, a contradiction. Therefore, if the unstable node containing the fewest chips contains \(d + 1\) chips, then the unstable node containing the next-to-fewest chips must contain between \(d + 2\) and \(n - 2\) chips.

By a similar argument, we see that the node containing the \(k^{th}\) fewest chips (for \(k = 1, 2, \ldots, u\)) among the unstable nodes must contain at least \(d + k\) and at most \(n - 2\) chips. (Note that \(d + u \leq n - 2\) because we are considering the case where \(c \geq 2\).)

Consider the vector \((x_1, x_2, \ldots, x_u)\), where \(x_k\) is the number of chips on the unstable node containing the \(k^{th}\) fewest chips. We have just seen that \(d + k \leq x_k \leq n - 2\). Subtracting \(d + 1\) yields \(k - 1 \leq x_k - (d + 1) \leq n - d - 3\), for \(k = 1, 2, \ldots, u\). Since \(x_k \geq d + 1\), we see that counting the number of ways to distribute chips onto the unstable nodes is equivalent to counting the number of legal configurations on \(u\) nodes where each node can have at most \(n - d - 3\) chips (cf. characterization (3) of Proposition 3.1). By the definition of \(L_{nm}\) (preceding Theorem 3.3), we have \(L_{u,n-d-3}\) ways to distribute the chips. Since there are \(\binom{n-d}{u}\) ways to choose which of the \(n - d\) non-duds will be unstable and \(L_{u,n-d-3}\) ways to place chips on these unstable nodes, the second two factors in (5.11) account for the chip placement on the unstable nodes.
Once we have placed chips on all duds and unstable nodes, it remains only to place chips on the critical nodes. Since all critical nodes must contain \( n - 1 \) chips, and we have no choice as to which nodes will be critical (with the unstable nodes and duds already decided), the placement of these remaining chips is uniquely determined. Thus, for \( c \geq 2 \), the total number of legal configurations on \( K_n \) with \( c \) critical nodes, \( u \) unstable nodes, and \( d \) duds is

\[
\binom{n}{d} L_{d,d-1} \binom{n-d}{u} L_{u,n-d-3}.
\]

We shall find it convenient to simplify the expression (5.11). The next result is an easy consequence of Theorem 3.3.

**Corollary 5.5** With the parameters as in Proposition 5.4, the number of relaxed legal configurations on \( K_n \) is

\[
\binom{n}{d} (d+1) \binom{n-d}{u} (c-1)(n-d-1)^{u-1}.
\]

Since the expression in Corollary 5.5 equals zero when \( c = 1 \), it does not apply in this case. We therefore turn to the case \( c = 1 \) now.

**Proposition 5.6** The number of relaxed legal configurations on \( K_n \) with one critical node is \( n^{n-1} \).
Proof. A relaxed legal configuration with just one critical node cannot have any unstable nodes since, when the critical node fires, it gives just one chip to each other node, which is not sufficient to fire any of the non-critical nodes. Thus, in the proof of Proposition 5.4, once we count the number of ways to distribute the chips on the duds, the remaining choice for the critical node is determined. We find that the number of relaxed legal configurations on $K_n$ with $c = 1$ critical node and $d = n - 1$ duds is

$$\left(\begin{array}{c} n \\ d \end{array}\right)L_{d,d-1} = \left(\begin{array}{c} n \\ n-1 \end{array}\right)L_{n-1,n-2} = n^{n-1}.$$  

In our investigation of $P_{g,q} = \Pr\{X = g \mid \text{legal configuration is } q\}$, we may now group together all configurations $q$ that have $c \geq 2$ critical nodes, $u$ unstable nodes, and $d$ duds. Each of these has probability $\frac{c}{n}$ of having a game of length $g \geq c$, since one of the $c$ critical nodes must be chosen in order for this to occur. Thus,

$$\sum_{q:c\geq 2} P_{g,q} = \sum_{c=2}^{g} \left(\begin{array}{c} n \\ d \end{array}\right)(d + 1)^{d-1} \left(\begin{array}{c} n-d \\ u \end{array}\right)(c-1)(n-d-1)^{n-1} \frac{c}{n}.$$  \hspace{1cm} (5.12)

The sum stops at $g$ because we cannot have more than $g$ critical nodes without having a game length longer than $g$, should a critical node be chosen as the seed.

Note that if a critical node is chosen as the seed, then $c + u = g$ (by Lemma 4.8). Now (5.12) may be simplified to a form involving only $n$, $g$, and $c$: 

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\[ \sum_{q \geq 2} P_{g,q} = \sum_{n-g}^g \binom{n}{n-g}(n-g+1)^{q-1} \binom{g}{g-c}(c-1)(n-(n-g)-1)^{q-c-1} \left( \frac{c}{n} \right) \]

\[ = \frac{1}{n} \binom{n}{g}(n-g+1)^{q-1} \sum_{c=2}^g \binom{g}{c} c(c-1)(g-1)^{q-c-1}. \quad (5.13) \]

The summation in this last expression may be simplified further.

**Lemma 5.7** For \( g \geq 2 \), we have \( \sum_{c=2}^g \binom{g}{c} c(c-1)(g-1)^{q-c-1} = g^{q-1} \).

**Proof.** The lefthand side is

\[ \sum_{c=2}^g \binom{g}{c} c(c-1)(g-1)^{q-c-1} = \sum_{c=2}^g \frac{g!}{(c-2)!(g-c)!} (g-1)^{q-c-1} \]

\[ = g \sum_{c=2}^g \frac{(g-2)!}{(c-2)!(g-2)-(c-2)!} (g-1)^{q-2-(c-2)} \]

\[ = g \sum_{c=0}^{g-2} \binom{g-2}{c} (g-1)^{q-2-c} \]

\[ = g(1 + (g-1))^{q-2}. \]

Applying Lemma 5.7 to (5.13) yields

\[ \sum_{q \geq 2} P_{g,q} = \frac{1}{n} \binom{n}{g}(n-g+1)^{q-1} g^{q-1} = \binom{n-1}{g-1}(n-g+1)^{q-1} g^{q-2}. \quad (5.14) \]

Note that this expression does not depend on \( q \).

A necessary condition for having a game length \( g \) at least two is that
\[ \Pr\{X = g\} = \sum_{k \geq 2} P_{g,k} \cdot \frac{1}{(n+1)^{n-1}} = \binom{n-1}{g-1} \frac{(n-g+1)^{n-g-1} g^{-2}}{(n+1)^{n-1}}. \] (5.15)

Proposition 5.6 and Theorem 5.1 together imply that
\[ \Pr\{X = 1\} = \frac{n^{n-1}}{n(n+1)^{n-1}} = \frac{n^{n-2}}{(n+1)^{n-1}}. \] (5.16)

Conveniently, (5.16) agrees with (5.15) with \( g = 1 \); the remark following Corollary 5.5 necessitated the separate handling of this case here. We delay discussion of the remaining case \( g = 0 \) until Section 5.4, where we will employ the methods of Chapter 3.

At this point, we may use our analytical results to see if a sequence of burn-off games on a complete graph exhibit one of the main properties of a system in a state of SOC: namely, that the frequency of games and their lengths have a power law relationship (as discussed on p. 10). We show now that the relationship is in fact vertically symmetric and thus does not follow any power law.

**Proposition 5.8** For \( 1 \leq g \leq n \), we have \( \Pr\{X = g\} = \Pr\{X = n - g + 1\} \).

**Proof.** Using (5.15), we have
\[
\Pr(X = n - g + 1) = \left( \frac{n - 1}{n - g} \right) \frac{(n - (n - g + 1) + 1)^{n-(n-g+1)-1} (n - g + 1)^{n-g+1-2}}{(n + 1)^{n-1}} \\
= \left( \frac{n - 1}{g - 1} \right) \frac{g^{g-2}(n - g + 1)^{n-g-1}}{(n + 1)^{n-1}} \\
= \Pr(X = g).
\]

We can thus conclude that a burn-off chip-firing game on a complete graph will not possess one of the key features of a system that exhibits SOC. That is, we do not find a power law relationship (as discussed in Section 1.4) between the size (i.e. game length) and frequency (i.e. probability) of events in the system. A complete graph features total communication among its nodes during a chip-firing game. In other words, any event triggered by one node will affect every other node. Systems that feature SOC generally do not communicate so thoroughly. It is typical (see, e.g., [3] and [7]) for SOC models to represent the individuals in a system as nodes with low degree on a large graph.

5.4 Establishing these results using the methods of Chapter 3

In this section, we show how our results from Chapter 3 verify our results for complete graphs established in the present chapter.

First we recover the formula \( L_{n,n-1} = (n + 1)^{n-1} \) (see p. 40) using Theorem 4.1 for enumerating the legal configurations on a general connected graph. For \( K_n \), the special node \( x \) in Theorem 4.1 makes \( G^* \equiv K_{n+1} \). Thus, Theorem 3.5 is
really a corollary of Theorem 4.1, and we have \( L_{n,n-1} = \tau(K_{n+1}) = (n + 1)^{n-1} \) by Cayley’s Formula (see, e.g., [19]).

We now enumerate the games of length zero on \( K_n \) using Proposition 4.7. Recall that for each node \( v \), we remove the edge \( \{x, v\} \) from \( G^* \) and enumerate the spanning trees of the resulting graph. We use the Matrix Tree Theorem (see, e.g., [19]) to count the spanning trees. We reduce the Laplacian matrix of \( G^* \) by eliminating the row and column associated with \( v \), resulting in the \( n \times n \) matrix

\[
A = \begin{bmatrix}
  n & -1 & -1 & -1 \\
  -1 & n & -1 & \cdots & -1 & -1 \\
  -1 & -1 & n & -1 & \cdots & -1 \\
  \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
  -1 & -1 & -1 & n & \cdots & -1 \\
  -1 & -1 & -1 & \cdots & n & 1
\end{bmatrix}
\]

To enumerate the spanning trees of \( G^* \), we find \( \det(A) \). Denoting the rows of \( A \) by \( (r_j)_{j=1}^n \) and performing the elementary row operations \( \sum_{j=1}^n r_j \rightarrow r'_j \), followed by \( r'_1 + r_j \rightarrow r'_j \) (for \( j = 2, \ldots, n \)), we find the determinant to be \( (n-1)(n+1)^{n-2} \). There are \( n \) nodes, so the number of pairs \( (q, v) \) with \( q \in \mathcal{R} \) and \( v \in V \) that result in a game of length zero when \( v \) is the seed is \( n(n-1)(n+1)^{n-2} \). Given \( q \), a node \( v \) is selected as the seed with probability \( 1/n \). By Theorem 5.1, each legal configuration on \( K_n \) occurs with probability \( 1/(n+1)^{n-1} \). Thus, conditioning on the choice of \( (q, v) \) with \( q \in \mathcal{R} \) and seed \( v \) we find that
\[
\Pr\{X = 0\} = \sum_{(q,v)} \Pr\{X = 0 \mid (q,v)\} \Pr\{v \mid q\} \Pr\{q\}
\]
\[
= n(n - 1)(n + 1)^{n-2}(1/n)(1/(n + 1)^{n-1})
\]
\[
= \frac{n - 1}{n + 1}. \quad (5.17)
\]

As a check on our work, we pause to show that the probabilities in (5.15), (5.16), and (5.17) sum to one.

**Proposition 5.9** For each positive integer \(n\), we have
\[
\frac{n-1}{n+1} + \frac{n^{n-2}}{(n+1)^{n-1}} + \sum_{g=2}^{n} \frac{\binom{n}{g}(n-g+1)^{n-g-1}g^{g-1}}{n(n+1)^{n-1}} = 1. \quad (5.18)
\]

**Proof.** We first manipulate (5.18) into a form that is amenable to a combinatorial proof. Multiplying both sides of the equation by \(n(n+1)^{n-1}\), we see that it will be equivalent to establish
\[
(n-1)n(n+1)^{n-2} + n^{n-1} + \sum_{g=2}^{n} \binom{n}{g}(n-g+1)^{n-g-1}g^{g-1} = n(n+1)^{n-1}.
\]

As noted following equation (5.16), the second term may be absorbed into the sum. Collecting together the remaining terms external to the sum, we see that (5.18) is equivalent to
\[
\sum_{g=1}^{n} \binom{n}{g}(n-g+1)^{n-g-1}g^{g-1} = 2n(n+1)^{n-2}. \quad (5.19)
\]

Finally, using the identity \(\binom{n}{g} = \binom{n+1}{g} \frac{n-g+1}{n+1}\) we reduce (5.19) to
To see that (5.20) holds, first observe that the right side enumerates the pairs \((T, \varepsilon)\), where \(T\) is a spanning tree of \(K_{n+1}\) for which one edge \(e\) (of its \(n\) edges) has been distinguished and oriented (in one of the two possible directions). The left side also enumerates these pairs. Given \((T, \varepsilon)\), notice that deleting the oriented edge \(\varepsilon\) from \(T\) leaves behind a spanning forest of \(K_{n+1}\) with two components \(L, R\) (that we may consider ordered from left to right). If \(|V(L)| = g\), for an integer \(g\) with \(1 < g < n\), then \(|V(R)| = n + 1 - g\). Conversely, given such a spanning forest, we can recover \((T, \varepsilon)\) by selecting a node \(x\) of \(L\) and a node \(y\) of \(R\) and letting \(\varepsilon = (x, y)\). On the left side of (5.20), the factor \(\binom{n + 1}{g}\) accounts for the selection of \(V(L)\) (hence for the selection of \(V(R)\)). Since \(L, R\) are, respectively, spanning trees of the induced (complete) subgraphs \(K_{n+1}[V(L)], K_{n+1}[V(R)]\), the factors \(g^{g-2}\) and \((n + 1 - g)^{n-1-g}\) are delivered by Cayley's Formula. Finally, the factors \(g, (n + 1 - g)\) count the number of ways to select the vertices \(x \in V(L)\) and \(y \in V(R)\) determining \(\varepsilon\).

Finally, we recapture the general formula (5.15), namely that for \(g \geq 2\),

\[
\Pr\{X = g\} = \binom{n-1}{g-1} \frac{(n-g+1)^{n-1-s}g^{g-2}}{(n+1)^{n-1}}.
\]

We appeal to Theorem 4.9. For each \(v\) in \(K_n\), we find all subtrees having \(g\) nodes that
include $v$. There are \( \binom{n-1}{g-1} \) ways to choose the $g-1$ nodes $u \neq v$ for these subtrees; each choice results in a complete subgraph $K_g$ of $K_n$, for which there are $g^{g-2}$ spanning trees (by Cayley's Formula). Now we delete each subtree in turn, counting the number of legal configurations on the resulting graph. When $K_g$ is deleted from $K_n$, the graph $K_{n-g}$ remains. By Theorem 4.1 (or Theorem 3.5), the number of legal configurations on $K_{n-g}$ is $(n-g+1)^{n-g-1}$. Thus, Theorem 4.9 yields 

\[
\binom{n-1}{g-1}g^{g-2}(n-g+1)^{n-g-1}
\]

as the number of ways to have a game of length $g \geq 2$ on $K_n$. Since each legal configuration is equally probable by Theorem 5.1, we see that

\[
\Pr(X = g) = \binom{n-1}{g-1} \frac{(n-g+1)^{n-g-1}g^{g-2}}{(n+1)^{n-1}}
\]

and have thus re-derived (5.15).

In Chapter 3, we investigated burn-off games on complete graphs with a direct approach. Our methods of Chapter 4 were more general. In the present chapter, we have re-established several important results from Chapter 3 using the methods of Chapter 4. We revisited the link between Cayley’s Theorem and relaxed legal configurations, enumerated the pairs $(q,v) \in R \times V$ leading to games of length zero, and confirmed the probability distribution of burn-off game lengths. Of final note is the fact that we were able to obtain an independent, combinatorial proof that these probabilities sum to one. We conclude the dissertation with a brief list of suggestions for further research.
5.5 Further research

Early models of SOC allowed stresses to escape the system. In [3], earthquake stresses that reach the edge of the grid are considered to be lost. In [7], avalanches can spill sand out of the system as if it were falling from a table. To represent these models as chip-firing games, at least one node must have the capacity to hold an infinite number of chips. This leads to a natural question: can a chip-firing game with such a node be reconciled with the results in this dissertation concerning burn-off games?

Many modified chip-firing games have been studied, including games with mutating edges [8] and games played in parallel [5]. It would be interesting to determine whether our results extend to these situations as well.

Finally, Theorems 4.9 and 4.13 together provide us with a method for determining the probability distribution of the burn-off game length on a general connected graph. We do not, however, have a closed formula for this distribution as we do in Chapter 5 for complete graphs. Finding such a formula presents a tantalizing open problem.
Appendix 1
Glossaries

Glossary of notation

<table>
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<tr>
<th>Symbol</th>
<th>Definition</th>
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<tr>
<td>~</td>
<td>is a neighbor of</td>
</tr>
<tr>
<td>( \tau(G) )</td>
<td>the number of spanning trees of the graph ( G )</td>
</tr>
<tr>
<td>( \text{indeg}(v) )</td>
<td>in a directed graph, the number of incoming arcs incident with ( v )</td>
</tr>
<tr>
<td>( \text{outdeg}(v) )</td>
<td>in a directed graph, the number of outgoing arcs incident with ( v )</td>
</tr>
</tbody>
</table>
Glossary of terminology

configuration on a graph $G = (V, E)$, a distribution $C : V \to \mathbb{N}$ of chips on $V$

connected describes a graph in which every pair of nodes is joined by a path

critical in a classical chip-firing game (e.g., [6]), describes a node that contains as many chips as its degree; in a burn-off game, describes a node that will fire if a chip is added to it

critical number in a chip-firing game, the least number of chips a node can contain for it to fire

deletion (edge) the removal of a single edge from a graph, not including its ends

deletion (node) the removal of a single node, and all edges incident with it, from a graph

fire the process in a chip-firing game by which the chips on a node are redistributed

independent describes a set of nodes inducing a subgraph with an empty edge set

induced subgraph of a graph $G = (V, E)$, with $V' \subseteq V$, the subgraph $H = (V', E')$ of $G$, where $E' = \{(x, y) \in E : x, y \in V'\}$

language a collection of words

legal describes a configuration that may result from relaxing a supercritical configuration, or result from relaxing another legal configuration that has been seeded

length of a game the number of nodes that fired during a chip-firing game, during one iteration of a seed-to-relaxation sequence

relax the process in a chip-firing game that begins with seeding a node and continues until no nodes can fire

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<table>
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<th>Term</th>
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<tr>
<td>relaxed</td>
<td>describes a configuration in which no node can currently fire</td>
</tr>
<tr>
<td>seed</td>
<td>as a noun, the node to which a chip is added to initiate a chip-firing game; as a verb, the process of adding a chip to such a node</td>
</tr>
<tr>
<td>simple</td>
<td>describes a graph that has no loops or multiple edges</td>
</tr>
<tr>
<td>supercritical</td>
<td>describes a node that contains more than a critical number of chips, or describes a graph in which every node is supercritical</td>
</tr>
<tr>
<td>word</td>
<td>the concatenation of labels that corresponds to the sequence of nodes that fire as a legal configuration relaxes to another legal configuration</td>
</tr>
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Appendix 2

Smallbasic code

The following code, written in Smallbasic, was used to produce the results shown in Figure 1.2.

```
option base 1
dim gameLengths(0 to 4)
adjMatrix = [0,1,0,0;1,0,1,1;0,1,0,1;0,1,1,0]
config = [0;0;0;0]
critNumbers = [1;3;2;2]
input "How many games";numGames
for thisGame = 1 to numGames
   ' Initialize
gameLength = 0 : canFire = true
   ' Choose a node at random as seed
   seed = int(4 * rnd + 1)
   ' Add a chip to the seed
   config(seed,1) = config(seed,1) + 1
```
repeat
    didFire = false
    for thisNode = 1 to 4
        if config(thisNode,1) > critNumbers(thisNode,1) then
            gameLength = gameLength + 1
            tmpVector01 = [0;0;0;0]
            tmpVector01(thisNode,1) = 1
            tmpVector02 = adjMatrix * tmpVector01
            config(thisNode,1) = config(thisNode,1) -
            (critNumbers(thisNode,1) + 1)
            config = config + tmpVector02
            didFire = true
            endif
        next thisNode
    until didFire = false

    gameLengths(gameLength) = gameLengths(gameLength) + 1
next thisGame
print gameLengths
The following code, written in Smallbasic, was used to produce the results shown in Figure 1.3.

dim arrGrid(0 to 51,0 to 51)
dim arrGameLengths(50)

' Initialize arrGrid
for x = 1 to 50
  for y = 1 to 50
    arrGrid(x,y) = int(4 * rnd + 1)
  next y
next x

input "Number of games";conNumberGames

for ctrGameNumber = 1 to conNumberGames
  ' select seed
  x = int(50 * rnd + 1)
  y = int(50 * rnd + 1)
  ' perturb seed
  arrGrid(x,y) = arrGrid(x,y) + 1
  ' see if any node can fire
  conGameLength = 0
  repeat
    flgDidFire = false
    for i = 1 to 50
      for j = 1 to 50
        if arrGrid(i,j) > 3 then
          flgDidFire = true
          conGameLength = conGameLength + 1
          arrGrid(i,j) = arrGrid(i,j) - 4
          arrGrid(i-1,j) = arrGrid(i-1,j) + 1
        end if
      next j
    next i
    if flgDidFire then break
  end repeat
end for
arrGrid(i+1,j) = arrGrid(i+1,j) + 1
arrGrid(i,j-1) = arrGrid(i,j-1) + 1
arrGrid(i,j+1) = arrGrid(i,j+1) + 1
endif

next j
next i
until flgDidFire = false
' record the game length
' first, collect all games of length longer than 50 into one category
if conGameLength > 50 then conGameLength = 50
arrGameLengths(conGameLength) =
arrGameLengths(conGameLength) + 1
next ctrGameNumber

print arrGameLengths
References


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