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Abstract: *This study is the outcome of a research that investigated how students who were assigned varying levels of geometric thinking attempted problems requiring some amount of algebraic thinking in geometry. The study reports that students' thinking in geometry also requires facility with algebra and as such there is a need for a framework that provides a more inclusive view of what constitutes geometric thinking in school mathematics.*

Key words: geometric thinking, van Hiele framework, school geometry

1. Introduction

The problem solving movement of the 1980s witnessed a burgeoning interest in the processes of learning that was enhanced further by the rise of constructivism within the field of mathematics education. The emphasis was not only on general mathematical thinking but also on thinking in specific content domains. Subsequently, several studies focused on geometric thinking (Burger & Shaughnessy, 1986; Devilliers & Njisane, 1987; Senk, 1989), on probabilistic thinking (Jones, Langrall, Thornton, & Mogill, 1999), on algebraic thinking (Herbert & Brown, 1999) and on statistical thinking (Mooney, 2002; Groth, 2003). In many such studies, the idea was to identify a framework that could aid in assessing a student's level of thinking in that particular content domain so as to facilitate instruction. One of the earliest known frameworks for thinking in a content domain within mathematics education is in geometry, proposed by the van Hieles in the 1950s (husband and wife) in the Netherlands. The primacy of the van Hiele framework attests to the very special status of geometry in mathematics as an essential component of school mathematics curricula all over the world.

2. Van Hiele Theory

The van Hiele theory has since been extensively used in studies to conceptualize students' thinking in geometry at various levels. The van Hieles proposed five hierarchical levels that describe growth in student thinking in geometry. These levels are not necessarily age-bound as in

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the Piagetian cognitive developmental theory. The van Hieles initially used Level 0 to be the lowest level and Level 4 to be the highest of the five levels (Crowley, 1987), however some authors use a numbering from 1 to 5 (Pegg & Davey, 1998). The levels are consecutively: Recognition, Analysis, Informal Deduction, Formal Deduction, and Rigour. Van Hiele (1986) mentioned that tracing the levels of thinking in geometry is not a simple affair, for the levels are not situated in the subject matter but in the thinking of man. Although van Hiele claimed that the roots of his theory are found in the theories of Piaget, progression from one level to the next is not the result of maturation or natural development. All depends on the quality of the experience that one is exposed to.

One of the earliest studies using the van Hiele framework in the United States was carried out by Usiskin (1982) at the University of Chicago. Since, a significant amount of research in school geometry has focused on the van Hiele levels of thinking (e.g., Burger, & Shaughnessy, 1986; Senk, 1989; Guitiérrez, Jaime, & Fortuny, 1991). An increased focus on the van Hiele theory has led researchers to capitalize on the strengths of the theory and also to highlight some shortcomings. The van Hieles developed their theory in the late 1950s, at a time when school geometry was primarily Euclidean in nature. However, the nature of school geometry has undergone major changes since the time when the van Hiele framework was developed.

3. The Nature of School Geometry

The reform of the 1960s in mathematics education brought major changes in the school geometry content. New approaches to geometry such as coordinate, transformational, and vector approaches were emphasized in the school curriculum. Although, the reform movement met with several obstacles, it was nevertheless significant in establishing a prominent place for algebraic approaches to the teaching and learning of geometry in school mathematics. There is a greater emphasis now in the geometry curriculum on writing algebraic expressions, substitution into an expression, setting up and solving equations; all of which require an understanding of the notion of variable and unknown. Whereas geometry has a separate subject status in the high school curriculum in several countries, such as the United States, it is integrated in an inclusive mathematics curriculum in many other countries. High school geometry builds on elementary school geometry which traditionally has emphasized measurement and the informal development of the basic concepts required in geometry at the high school level. The topics on measurements of perimeter, of area, and of volume which are revisited in the high school curriculum provide excellent opportunities for further applications of algebraic concepts in geometry.

On the other hand, Clements and Battista (1992) have claimed that school geometry refers almost universally to Euclidean geometry, even though there are numerous approaches to the study of a particular topic. While this may be true at lower levels, there are very strong connections between algebra and geometry at higher levels. Also, algebra and geometry have strong historical links. The use of literal symbols in the form of variables, constants, parameters and so on abounds in algebra. Symbols abound in school geometry as well. Students work with variables and unknowns when generalizing results or solving problems such as finding unknown sides or angle measures. The idea of a variable is also used in geometry using a variable point as in problems involving loci. Other simple uses of algebra in geometry as far as symbols are concerned involve labelling points or vertices, sides, and angles of figures. Some other connections between algebra and geometry in the high school curriculum arise in problem solving and modelling, and in the various modes of representations – graphical, algebraic, and numeric. The symbolic representations pose problems for the students. Duval (2002) has claimed that there is no direct access to mathematical objects other than through their representations, and thus we can only work on and from semiotic representations, because they provide a means of processing. In geometry, this implies working in different registers (natural language, symbolic, and figurative) and moving in between registers. Algebra offers geometry a powerful form of symbolic representation.

Many of the concepts in geometry have their counterpart in algebra. For example, a point in geometry corresponds to an ordered pair (x, y) of numbers in algebra, a line corresponds to a set of ordered pairs satisfying an equation of the form $ax + by = c$ $(a, b, c \in R)$, the intersection of two lines to the set of ordered pairs that satisfy the corresponding equations, and a transformation corresponds to a function in algebra (National Council of Teachers of Mathematics [NCTM], 1989). Algebraic results can be achieved geometrically and geometrical results can be demonstrated using algebra. For example, Pythagorean theorem for a right triangle having sides of lengths a, b, and c, can be represented algebraically using the formula $a^2 + b^2 = c^2$.

4. Geometric Thinking

As any form of mathematical thinking, geometric thinking is quite difficult to conceptualize. It is definitely a form of mathematical thinking within a specific content domain.

It would be simpler to consider what students are expected to be able do in geometry and accordingly model and understand their thinking. For example, the Standards (NCTM, 2000) highlighted the following aspects of school geometry for grades 9 – 12: analyze characteristics and properties of two- and three-dimensional geometric shapes and develop mathematical arguments about geometric relationships; specify locations and describe spatial relationships using coordinate geometry and other representational systems; apply transformations and use symmetry to analyze mathematical situations; and use visualization, spatial reasoning, and geometric modeling to solve problems (p. 41). The NCTM Standards clearly emphasize the link between algebra and geometry when it mentions describing spatial relationships using coordinate geometry.

In addition, geometric thinking is inherent in the types of skills we want to nurture in students. Hoffer (1981) has proposed a set of five categories of basic skills relevant to school students: (1) visual skills - recognition, observation of properties, interpreting maps, imaging, recognition from different angles; (2) verbal skills - correct use of terminology and accurate communication in describing spatial concepts and relationships; (3) drawing skills- communicating through drawing, ability to represent geometric shapes in 2-D and 3-D, to make scale diagrams, sketch isometric figures; (4) logical skills - classification, recognition of essential properties as criteria, discerning patterns, formulating and testing hypothesis, making inferences, using counter-examples; and (5) applied skills - real-life applications using geometric results learnt and real uses of geometry e.g. for designing packages. Although Hoffer seems to focus on Euclidean geometry, it is difficult to imagine how accurate communication in describing spatial concepts and relationships can be done without some form of algebraic support. Most high schools in the United States follow the Algebra I – Geometry – Algebra II sequence, in the first three years of high school. There is a clear emphasis on the importance of algebra for the study of geometry. So, how do students think in school geometry when solving problems? Furthermore, another question to ask is: Does the van Hiele theory still hold in a context where school geometry has changed considerably to accommodate various algebraic approaches?

This paper is the outcome of a study on students' use of algebraic thinking in geometry at high school level. The research did not specifically focus on levels of geometric thinking but focused on how students used the following three forms of algebraic thinking in high school geometry: symbols and algebraic manipulation, different forms of representation, and generalization. The emphasis is on issues about what constitutes geometric thinking and the need to conceptualize geometric thinking within a broader framework that is inclusive of algebraic thinking in school geometry.

5. Methodology

This qualitative study took place over a three-month period during the first semester of the academic year in two large Midwestern rural high schools in the United States. One geometry class (post-Algebra I) was selected from each high school (school X and school Y). Class A from school X had 21 students and class B from school Y had 18 students. Two tests were administered to the students from these two classes: an algebra test (constructed by the researcher) and a van Hiele test (developed by Usiskin at the University of Chicago, 1982). The algebra test was finalized based on comments from the two classroom teachers and three other experts in the field. The van Hiele test from the Usiskin study has met with criticism (see Crowley, 1990; Wilson, 1990). However, it was deemed relevant for the particular purpose of selecting the focus students in this study and so was not modified. Based on their performances on the two tests, three students were selected from each of the two classes: Anton, Beth, and Mary from class A in school X and Kelly, Phil and Ashley from class B in school Y. It is to be noted that Anton was the only student in the sample with the highest assigned van Hiele level whereas Kelly was the student with the highest algebra test score (27 out of 30). Mary had the lowest algebra test score (5 out of 30) whereas none of the students were assigned a van Hiele level 0.

Table 1
Focus Students

School	Students	Algebra Score (out of 30)	Van Hiele Level (scale 0 – 4)
	Anton	23	4
School X	Mary	5	1
	Beth	13	1
	Kelly	27	3
School Y	Ashley	17	1
	Phil	24	1

The six focus students were interviewed four times for about 40 minutes each time. During these interviews the focus students were asked to solve sets of problems which involved the use of algebra in geometry. The problems were finalized based on the schools' mathematics programs with the help of three experts in the field. These problems included the use of variables and unknowns, the writing and solution of simple linear equations, the writing and solution of linear simultaneous equations in two unknowns, the substitution of values in expressions, and the recall and use of formulae within geometry. Besides, the two classrooms were observed for about three months and 12 lessons from each class were videotaped. Artifacts, such as tests, quizzes, and homework of the focus students were also collected. The two teachers from these two classrooms were interviewed twice for about 30 minutes each time.

6. Discussion

It is not possible to describe in detail how each of the focus students attempted the set of tasks in geometry, as the research had a different focus and was not specially looking at levels of thinking in geometry. So, in what follows, a general description is given about how these students with varying levels of performance on the algebra and van Hiele tests solved the problems involving algebraic thinking in geometry.

Three aspects of algebraic thinking were investigated: the use of symbols and algebraic relations, the use of representations, and the use of generalizations within geometrical contexts. Four of the focus students were assigned van Hiele Level 1 (scale 0-4): Mary, Beth, Phil, and Ashley. In comparison to each other, these students showed different dispositions towards the use of algebraic thinking in geometry. Mary and Beth had difficulties in the use of each of the three aspects of algebraic thinking enumerated above. They encountered difficulties when working with variables and unknowns, and writing and solving equations. They also found the use of different representations difficult. They could work with some linear geometric patterns with some help but not with non-linear ones. However, they could remember some of the formulae, like the one for finding distance between two points in the coordinate plane. Mary, who had the lowest score on the algebra test (5 out of 30), struggled in most of the problems that she was asked to solve. She had a dislike for mathematics because she did not like the many formulae she had to remember. In general, Mary and Beth had a more instrumental approach to the use of algebraic thinking in geometry.

On the other hand, Phil and Ashley (with algebra test scores of 24 and 17 out of 30 respectively) were better than Mary and Beth at using the different forms of algebraic thinking. They could confidently work with variables and unknowns. It was interesting to note that Ashley did quite well in most of the interview problems that she tackled. Phil and Ashley had a more positive approach to problem solving and a quest for finding the solution, compared to Mary and Beth. However, both Phil and Ashley were not good at remembering formulae. In this episode with the interviewer (R), Ashley is solving the problem: *If an isosceles triangle has two sides of lengths 10 cm and 4 cm, what will be its perimeter?*

> Ashley: So...it will be 10, ... [*she mumbles and draws a triangle showing the sides as 10, 10, 4*]

R: So what would be the perimeter in that case?
Ashley: Just 24.
R: Ok. What could it be otherwise?
Ashley: 18
R: Can you show it on a diagram?
Ashley: Hmm...[*she draws some diagrams*]
R: So now, are both of these possible, only one of them is possible, what do you think?
Ashley: Hmm...this one couldn't really be possible [*referring to the 4, 4, 10 triangle*] because it is 10. If you add the two sides it does not give 10.
R: In other words the answer will be ..what?
Ashley: 24.
R: Alright…and suppose I give you a triangle having sides *a*, *b*, and *c*...[*I draw a triangle ABC with sides a, b, c*] what condition must be satisfied by *a*, *b*, and *c*?
……
R: So if I have the sides of lengths *a*, *b*, and *c* what condition should be satisfied then?
Ashley: So $a + b \geq c$.
R: Can you write it down?
A: [*she* writes $a + b \geq c$] …and the others…[$b + c \geq a$, $c + a \geq b$, *she writes these statements without any prompts*]
R: When you have an 'equal to' sign what will happen?
Ashley: It will be equilateral?...I don't know..
R: Hmm...suppose I had like 10, 5, and 5 what would happen in that case? [*I use a line segment of length 10 and illustrate the condition]*
Ashley: Oh.. that wouldn't be a triangle.. it would be just a line.
R: Ok…

Although, Ashley had a fairly low algebra score, she was quick to generalize to a triangle with sides of lengths *a*, *b*, and *c*. In comparison, Anton who had the highest assigned van Hiele level and a fairly high algebra score had major difficulties with this problem.

Anton (Level 4) and Kelly (Level 3) were the two focus students with the highest van Hiele levels and fairly high algebra scores (23 and 27 out of 30 respectively). In general they were much better in using algebraic thinking in geometry than the other focus students, except possibly Ashley. They could work with variables, unknowns, and equations with greater confidence. But they did have a few difficulties as well. It is worth noting that Anton and Kelly were not very good at remembering formulae. They seemed to be generally more relational or conceptual in their use of algebraic thinking. In this episode with the interviewer (R), Kelly is asked to solve the problem: *The supplement of an angle is four times its complement. What is the angle?*

R: So if you don't know the angle how do you start the problem?
Kelly: With *x*?
R: Ok…let's try so if *x* is the angle. First, what would be its complement?

Kelly: …is it 90-*x*?...I don't know…
R: Hmm… like for 40 it was 50 isn't it? For *x* it will be…[*she writes* 90-*x*]. Ok. And what will be the supplement?
Kelly: 180-*x*?
R: Ok. So one of them is four times the other which one is four times the other?
Kelly: Four times its complement is the supplement.
R: Ok, alright. Can you write an equation from there?
[*she writes* 180 - *x* = 4(90-*x*)] Can you solve it? [*she eventually solves the problem but has some difficulties with the algebra*.]

Kelly initially had a difficulty understanding the geometrical concepts supplement and complement of an angle. She had no difficulty in setting up the equation correctly. However, she did seem to have a few difficulties in reaching the solution. On the other hand, Anton did not do well in this problem. He proceeded to work on the problem as he described it below:

Anton: Let us see, supplement of an angle. Let's just say *S* for the supplement and minus *A* for angle equals four times *C*, the complement. [*he writes* *S*-*A* = 4*C*].. hmmm so *S* = 180… no wait we are talking about the complement. I keep thinking that the total…hmmm but *A* + *S* = 180 and *A* + *C* = 90 so 4*C* would equal *S*-*A*. So 4*C* + *A* = supplement [*he writes* 4*C*+ *A* =*S*]. So if you take this …

Anton was completely lost with his use of variables and did not recover from his initial slip to set up the right equation to get a solution. He knew what the complement and the supplement of angle stood for. Thus Anton, who had the highest van Hiele score and understood the geometrical concepts, had difficulties with the algebra in this question.

The results of this study show that algebraic thinking has strong connections to thinking in geometry. There is a significant amount of algebra in the geometry curriculum at high school level. Hence, students studying geometry need to be well prepared in algebra. The use of tests such as the van Hiele test targets students' thinking in an exclusively Euclidean context and tend to give a limited view of students' thinking in school geometry, which incorporates a significant amount of algebra. It is important to have tests for assessing students' thinking in geometry that would include the use of algebraic thinking as well. Phil and Ashley were both assigned van Hiele level 1, but they were both quite versatile in their use of algebra in problems requiring algebraic thinking in geometry. On the other hand, Anton was assigned the highest van Hiele level of four, and he had a few difficulties working on the selected problems. Hence, the van Hiele levels of thinking in geometry should be interpreted with greater care.

Most researchers agree that there is a certain hierarchic development of cognition as far as geometry is concerned and the van Hiele levels provide a valuable framework for studying geometric thinking. Even studies in non-Western contexts have provided support for the van Hiele theory. For example, De Villiers and Njisane (1987), working with high school students in South Africa, investigated how eight different Geometric Thinking Categories (GTCs) corresponded with the van Hiele model. The categories were: recognition and representation of figure types; visual recognition of properties; use and understanding of terminology; verbal description of a figure (or its recognition for a verbal description); one step deduction; longer deduction; hierarchical classification; and reading and interpreting given definitions. They found that *hierarchical classification* was the most difficult GTC for pupils. Roughly they concluded that the first two GTCs corresponded to the first van Hiele level, the next two to the second van Hiele vele and the next two to the third van Hiele level.

However, research has also shown that there exist some concerns regarding this theory. For example, Pandiscio and Orton (1998) have claimed that one of the weaknesses of the van Hiele theory is that it appears to lack generality and thus each strategy may need to be revised for different content domains. On the other hand, Senk (1989) claimed that van Hiele did not acknowledge the existence of a "nonlevel"; instead she asserted that all students entered geometry at ground level, which is Level 0 (scale 0 – 4), with the ability to identify common geometric features by sight.

Another issue that some researchers (Guitiérrez, Jaime, & Fortuny, 1991) have brought forth is that a student can possibly develop two consecutive van Hiele levels of reasoning at the same time. They found that, depending on the complexity of the problem, students used several levels of reasoning. However, they claimed that this was not be interpreted as a rejection of the hierarchical structure of the van Hiele theory, but rather that the theory should be adapted to the complexity of human reasoning processes. People do not behave in a simple, linear manner, which the assignment of one single level would lead us to expect.

Regarding proof in geometry, research has shown that high school students' achievement in writing geometric proofs is positively related to the van Hiele levels of geometric thought and to achievement on standard nonproof geometry content (see Senk, 1989). According to the van Hieles, students below Level 2 (scale 0-4) should not be able to do proofs at all other than memorization; students at Level 2 might be able to do short proofs based on empirically derived premises; but only students at Levels 3 or 4 should be expected to write formal proofs consistently (Levels are based on a scale of 0 to 4). Senk (1989) claimed that this is only partially supported by her research. Students at Level 2 or higher substantially outperformed students at lower levels; however when concurrent knowledge of standard nonproof content was controlled, students at Level 3 or 4 did not score consistently higher than those at Level 2.

The van Hieles developed their levels of thinking in geometry while the secondary geometrical content was mostly Euclidean, but the advent of the mathematics reform in the 1960s changed the face of geometry. Different types of geometries were introduced such as coordinate, vector, and transformational. This in turn introduced a significant amount of algebraic manipulation in high school geometry. Do the van Hiele levels of thinking still hold in a geometrical context where algebra plays a significant role?

With a view to give an alternative framework for studying geometric thinking, researchers have proposed combining the van Hiele theory with other popular theories. For example, Olive (1991) analyzed the Logo work of 30 ninth-grade students from three different theoretical perspectives: the van Hiele levels of thinking, the SOLO taxonomy (Structure of the Observed Learning Outcomes from Biggs, & Collis, 1982), and Skemp's (1987) model of mathematical understanding. Pandiscio and Orton (1998), in their theoretical paper, argued for a synthesis of van Hiele's and Piaget's perspectives. Pegg and Davey (1998) have also argued for a synthesis of the van Hiele and SOLO models for research in geometry. Some studies have looked at other cognitive aspects of learning geometry. For example, Chinnapan (1998) examined the nature of prior mathematical knowledge that facilitates the construction of useful problem representations in geometry. Lawson and Chinnapan (2000), for instance, explored the relationship between problem-solving performance and the organization of students' knowledge. They reported findings on the extent to which content and connectedness indicators differentiated between high- and low-achieving groups of students undertaking geometrical tasks.

7. Conclusion

There is agreement among researchers about a hierarchic development of geometrical thinking. The van Hiele theory provides a strong framework. However, research has also shown that this theory has some limitations. Amongst others, there have been concerns about the distinctness of the levels of thinking and the possibility of different levels in different topic areas. The present study adds another dimension to this issue, namely that the levels of thinking in geometry cannot ignore the significant connection between algebra and geometry.

The argument that students who were assigned varying van Hiele levels performed differently on the problems in geometry that required algebraic thinking, rests on the validity of the van Hiele test and a broader definition of school geometry. The instrument from the Usiskin (1982) uses only multiple choice items. This instrument is known to have been criticized by several researchers, such as Crowley (1990) and Wilson (1990). However, this instrument provided base-line data for selecting the focus students. In future studies, a more conclusive test that can combine written tests and interviews, may possibly provide better information.

Besides, the connections between algebra and geometry, levels of thinking in school geometry may also be influenced by technology. Various types of Dynamic Geometry Software (DGS) are now available for students to explore geometrical concepts. In addition, earlier work with LOGO has shown that students with LOGO experience gained more than control students in geometry (Scally, 1987). As such there are various avenues to explore when considering geometric thinking. There is still a strong interest in how students think in geometry. It is essential that researchers come up with models or frameworks that address some of the shortcomings of the van Hiele theory or possibly come up with a totally new framework. This study simply reports on the need for such a framework.

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