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Objective Truth versus Human Understanding in Mathematics and in Chess

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Abstract

This paper begins with a review of the collection *18 Unconventional Essays on the Nature of Mathematics* edited by Hersh (2006). Inspired especially by the contribution by Thurston to that collection, I then go on to discuss, by means of a couple of thought experiments involving computer "oracles", the nature of mathematics as a human activity, hopefully providing some balance to to the simplified view (sometimes held by research mathematicians such as myself) of the discipline as purely a quest for objective truth.

Keywords: Philosophy of mathematics, Platonism, computer-assisted proof, thought experiment, chess

1 A stimulating collection of essays

To anyone who has experienced the inescapable force and logical necessity of a mathematical proof, the Platonic existence of numbers and their properties – independently of us humans who think and argue about them – is obvious. The first time I encountered this argument,[2] I felt immediately excited: Yes! That is exactly how it is. Surely this settles the old question of whether new mathematical results are discovered or invented!

My excitement didn't last very long, however, because on second thought I realized that the argument has the same structure as the following proof of God's existence, which I think is flawed: *To anyone who has met God, His existence can no longer be in doubt.* Clearly, if we are to take the question about the ontological status of mathematical concepts seriously, we need to do better than merely to refer to an intuitive feeling shared by many (probably most) mathematicians, and there is a lot to say about the subject.

Of course, a mathematician is not obliged to consider this issue, but if he or she chooses to do so, then a nice entrance gate to the literature on this and other topics in the philosophy of mathematics is the recent collection *18 Unconventional Essays on the Nature of Mathematics*

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[2]This was in Hacking (1999), although of course the argument does not originate from him, and in fact I even extrapolated a bit from what he writes, because all he says is that those who have *not* had this experience do not realize why a mathematician would be inclined towards Platonism.

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edited by Reuben Hersh (2006). The present paper begins as a review of this collection. Then, inspired by what I have read, I will move on in Sections 2 and 3 to some thoughts of my own concerning objective truth versus human understanding in mathematics – and in chess.

All of the texts in Hersh's collection have previously appeared elsewhere. Most of them are fairly recent, but a few classics are also included, such as Alfréd Rényi's "A Socratic Dialogue on Mathematics", which is a very fitting opening of the collection, as it beautifully states (and even begins to answer) some of the basic questions that the other essays are concerned with, such as: What are mathematical objects? Do they exist? If they do not exist in the same ordinary sense as physical objects, how can it be that they are useful to the real world? And the aforementioned issue concerning new mathematical results: discoveries or inventions?

Although the matematicians, philosophers, and social scientists that contribute to this volume cannot be said to represent a common "school" of philosophy of mathematics, a kind of joint theme nevertheless emerges, namely the emphasis on distinctly human aspects of mathematics. Few of the authors seem to share my Platonic Hunch.[3]

Does it really matter whether Platonism is true or not, other than for the somewhat esoteric purpose of predicting whether or not, once we encounter an extraterrestrial species advanced enough to have come up with (say) radio astronomy[4] and interplanetary travel, they will be familiar with things like prime numbers, the Central Limit Theorem, and the Newton–Raphson method? Timothy Gowers, in his thoughtful contribution "Does Mathematics Need a Philosophy?" that contains many illuminating concrete examples, suggests at first that the answer is no:

> Suppose a paper were published tomorrow that gave a new and very compelling argument for some position in the philosophy of mathematics, and that, most unusually, the argument caused many philosophers to abandon their old beliefs and embrace a whole new -ism. What would be the effect on mathematics? I contend that there would be almost none, that the development would go virtually unnoticed. And basically, the reason is that the questions considered fundamental by philosophers are the strange, external ones that seem to make no difference to the real, internal business of doing mathematics.

Then, however, he goes on to balance this statement by pointing out, in the following pragmatic argument in favor of formalism, that the choice of philosophy may not be devoid of pedagogical consequences:

> If you are too much of a Platonist or logicist, you may well be tempted by the idea that an ordered pair is *really* a funny kind of set [of the form $\{\{x\},\{x,y\}\}$]. And

[3]In speaking of the Platonic Hunch, I am inspired by philosopher Daniel Dennett's talk of the Zombic Hunch (see, e.g., Dennett, 2005, or Blackmore, 2005). When philosophers speak of zombies, they mean imaginary creatures that are exact copies of ourselves, speaking and acting like us, the only exception being that zombies lack subjective experience. The Zombic Hunch is the intuitive inclination, shared by most of us, to think that zombies are possible, at least in principle. Dennett says they are not, granting only that the purely materialistic approach to explaining conciousness (identifying consciousness with certain physical processes in the brain) which he advocates also needs to explain the Zombic Hunch. Likewise, in my mind, any philosophy of mathematics that denies Platonism faces the challenge of explaining the Platonic Hunch.

[4]Or perhaps I should say astronomy based on electromagnetic radiation at wavelengths unavailable directly to their own sensory organs.

> if you teach that to undergraduates, you will confuse them unnecessarily. The same goes for many artificial definitions. What matters about them is the basic properties enjoyed by the objects being defined, and learning to use these fluently and easily means learning appropriate replacement rules rather than grasping the essence of the concept.

Inevitably in a multiple-contributor collection such as Hersh's, the quality of the contributions is somewhat uneven.[5] For instance, Gian-Carlo Rota's "The Pernicious Influence of Mathematics upon Philosophy" contains repeated condemnations of philosophers guilty of "mathematizing" their subject, without exemplifying that practice by even a single name or a single reference. That is poor scholarship. And Rafael Núñez' essay "Do *Real* Numbers Really Move? Language, Thought and Gesture: The Embodied Cognitive Foundations of Mathematics" suffers from the author's limited familiarity with the mathematics he discusses. One example is when he muses on the use of temporally loaded words like *approaches* and *tends* in statements like "the sum $s_n = \sum_{i=0}^{n} a_i$ approaches $s = \sum_{i=0}^{\infty} a_i$ as n tends to infinity" where no time dynamics is involved, without considering the obvious explanation that we tend to think about s as arising by *starting* with a_0, *then* adding a_1, *then* adding a_2, and so on.

When I set out to read *18 Unconventional Essays*, I was particulary curious about what the two contributors – Donald MacKenzie and Andrew Pickering – who, in view of some of their earlier writings,[6] belong firmly to what I perceive as the "other side" (i.e. the side of social-constructivists and postmodernists) in the infamous Science Wars[7], had to offer on the subject of mathematics. It turned out that they both give fairly reasonable and reader-friendly summaries of factual case histories: in MacKenzie's case mainly about formal verification in computer science, and in Pickering's about Sir William Rowan Hamilton's fascinating search for higher-dimensional extensions of complex numbers culminating in 1843 in the discovery of quaternions. MacKenzie reveals his position most clearly when he writes, concerning the existence of several different kinds of logic employed by contemporary computer scientists, that

> [i]t cannot be guaranteed that [this plurality] would remain intact in a situation where there are major financial or political interests in the validity or invalidity of a particular chain of formal reasoning,

thus suppressing the possibility that the inner "logic" (if I may use that word in an informal sense) of the science itself eventually reveals that only one of the logics (and here I revert to the strict meaning of the word) is in the end intellectually – as opposed to commercially or politically – rewarding. As to Pickering, he gives his position away at an early stage of his essay where he explains his theoretical approach. His goal is to understand why, in a subject such as mathematics that deals purely with abstract concepts that the researchers are

[5]The collection would also have gained from a more active editing and a more careful proofreading. Annoying examples abound, such as the reference on p. 57 to the non-existent Figure 1, or the confusing duplication of the figure on p. 261. Jody Azzouni's choice in "How and Why Mathematics is Unique as a Social Practice" to italicize every 20th or so word is likewise annoying, and should have been moderated by an editor.

[6]See, e.g., MacKenzie (1981) or Pickering (1984).

[7]See, e.g., the collection edited by Ashman and Baringer (2001) for a variety of views on these, or Weinberg (2001) for one that I find myself in basic agreement with.

free to define in any way they please, a researcher can ever run into problems. Why indeed. We mathematicians are so used to the fact that not all mathematical objects and definitions are equally fruitful and interesting, and that once the definitions are set we cannot do as we please with the resulting objects, that the question sounds utterly silly – whereas it is easy to imagine that to a sociologist of science working in the postmodern "anything goes" tradition it sounds quite reasonable.

Rényi, as mentioned above, touches upon the question of the "unreasonable effectiveness of mathematics in the natural sciences" that was so elegantly phrased in the classical paper of Eugene Wigner (1960), and a couple of other contributors – Eduard Glas, and Hersh himself – also find reasons to discuss Wigner. I feel, however, that both Glas and Hersh misunderstand the essence of Wigner's question.[8] Glas, in his contribution "Mathematics as Objective Knowledge and as Human Practice", writes that

> [t]he effectiveness of pure mathematics in natural science is miraculous only to a positivist, who cannot imagine how formulas arrived at entirely independently of empirical data can be adequate for the formulation of theories supposedly inferred from empirical data. But once it is recognized that the basic concepts and operations of arithmetic and geometry have been designed originally for the practical purpose of counting and measuring, it is almost trivial that all mathematics based on them remains applicable exactly to the extent that natural phenomena resemble operations in geometry and arithmetic sufficiently to be conceptualized in (man-made) terms of countable and measurable things, and thus to be represented in mathematical language.

Well, yes. But the true miracle – the one that Wigner is concerned with – is that nature is sufficiently ordered in such a way as to "resemble operations in geometry and arithmetic" and admit laws that can be "represented in mathematical language". Once that is the case, it is hardly miraculous that Darwinian evolution followed by human cultural developments equips us with the capacity to grasp these things.

Hersh, in a similar vein, portrays Wigner's miracle as being that of how often pure mathematics turns out to have profound applications in physics when no such application was originally intended. But again, the miracle of nature's amenability to mathematical description wouldn't go away even if it were the case that the mathematics were developed in direct contact with the physical application.

The best essay in Hersh's collection is, in my mind, William Thurston's "On Proof and Progress in Mathematics", which originally appeared as part of a discussion in the *Bulletin of the American Mathematical Society* in 1993–94 concerning the role of rigor and proof versus conjecture and speculation in mathematics, and the relation between mathematics and theoretical physics.[9] Thurston's essay lacks the philosophical ambition of most of the others, and instead provides a series of down-to-earth observations concerning how mathematics is done

[8] It has become a bit of a bad habit of mine to criticize other writers' understanding of Wigner's question; readers who know some Swedish may turn to Häggström (2004) for an earlier instance. In all these cases, it seems that I reveal my Platonic Hunch quite clearly.

[9] This discussion is very much worth rereading in its entirety; see Jaffe and Quinn (1993), Thurston (1994), Atiyah et el. (1994) and Jaffe and Quinn (1994).

and how mathematicians interact, drawing generously on the author's personal experience. What I read as his main point is that the view that "progress made by mathematicians consists of proving theorems" is a harmful simplification, resulting in a malign one-sidedness towards considering *theorem-credits* in the way we evaluate each other's qualifications. Progress in mathematics, according to Thurston, consists of improving our *understanding* of mathematics, and this is a much broader goal than mere theorem-proving. As one illustration to this point, he discusses the famous computer-assisted proof by Kenneth Appel and Wolfgang Haken of the four-color theorem,[10] noting that

> when Appel and Haken completed [their proof] using a massive automated computation, it evoked much controversy. I interpret the controversy as having little to do with doubt people had as to the veracity of the theorem or the correctness of the proof. Rather, it reflected a continuing desire for *human understanding* of a proof, in addition to knowledge that the theorem is true. [emphasis in original]

I wholeheartedly agree with Thurston's views on these matters as he expresses them in his essay. Expanding the research frontiers of mathematics is a collective enterprise, where progress achieved by one mathematician builds upon that of others. What sort of progress, then, can serve to be built upon in this manner? The mere production of true mathematical statements, perhaps accompanied by formal proofs, does not suffice: what a mathematician needs in order to be able to make further progress is to find out *why* these statements are true, in a way that she can truly internalize. We may think of such understanding as a (more or less rudimentary) map of the mathematical terrain, while the true statements and their formal proofs merely provide isolated snapshots, of little use for further exploration.

The remainder of this paper will be devoted to a couple of thought experiments that I hope will serve to illustrate and illuminate these issues. In particular, I hope that even readers who share my deeply felt Platonic Hunch will be convinced that there is a side to what we set out to accomplish in mathematics which is distinctly human. First, in Section 2, I will describe my thought experiment in the setting of the game of chess, where a dichotomy between objective truth and human understanding exists which is similar to that in mathematics, the advantage of the chess setting being that the situation there is more clear-cut than in mathematics and not at all controversial. Then, in Section 3, I will adapt the thought experiment to the setting of mathematics.

As far as physical plausibility is concerned, the thought experiments will go from bad to worse. I don't think that is a problem. Writes Richard Dawkins, in the opening chapter of his book *The Extended Phenotype* (1982): "Thought experiments are not supposed to be realistic. They are supposed to clarify our thinking about reality."

2 Truth versus understanding in chess

Chess is played on a board with a certain finite but extremely large number of possible positions. Two players, White and Black, take turns making moves according to certain

[10]See, e.g., Appel and Haken (1977).

rules – moves that alter the position on the board. The rules include in particular two – the three-fold repetition rule and the 50-move rule – that prevent the game from going on indefinitely.[11] As a result, the number of moves of a game is bounded.[12] In certain positions, the rules stipulate that Black is check-mate, and when such a position is reached the game ends and White is declared winner. In others, White is check-mate and Black is declared winner, while in yet others the game is declared drawn. In all other positions, the game goes on.[13]

When a chess player sits down to think about what move to play, her thinking can roughly be described as having two aspects, which we may call *combinatorial* and *abstract*. By the combinatorial aspect, I mean looking ahead into concrete possible sequences, such as:

> If I play my knight to f3, then my opponent can either play his knight to c6 or his pawn to d6. In the former case, I have the choice of playing my bishop to b5 (which will probably be answered by pawn to a6) or maybe to c4, while in the latter case I should probably instead play my pawn to d4.

And so on. This has to be combined with the abstract aspect, which means looking at a position to evaluate it using various general principles. The most basic such principle involves counting the pieces on the board weighted by their values, where one rule-of-thumb says that a knight or a bishop is worth about three pawns, a rook five pawns and a queen nine. This is then supplemented by various corrections, such as penalizing doubled or isolated pawns or rewarding rooks on open files; in general, the stronger a chess player is, the more such corrective factors is she able to account for in her abstract evaluation. The evaluation typically results in a more-or-less nuanced statement such as "due to Black's isolated c-pawn, White has a small advantage which can be expected to endure into the endgame" or "in compensation for the sacraficed pawn, White has a lead in development and pressure against f7, resulting in fairly equal chances for the two players".

Idealizing slightly, the art of playing good chess may be described as consisting of looking combinatorially into the various positions that can result, evaluating each of them abstractly, and choosing the moves that result in the best prospects under worst-case assumptions concerning what the other player does. It is desirable to be able to look many moves ahead, but since the number of possible future positions grows exponentially in how far one looks, some pruning of the tree of variations will then be necessary, and a very important skill is to judge which branches are irrelevant enough to admit pruning without harming the quality of the final decision.

[11] Henceforth, by a "position", I mean to include not only where the chess pieces are located, but also certain other aspects that are physically visible not on the board but only on the obligatory score sheets kept by the two players. The most obviously important such aspect is "who's turn is it to move?", but there are also others pertaining to three-fold repetition and the 50-move rule.

[12] Knowing the rules of the game, it is not hard to check that 5100 moves is an upper bound, and that this bound is close to sharp. That a game goes on for n moves means (following standard chess terminology) that White moves n times and Black moves n or $n-1$ times. In practice, games longer longer than 100 moves occur rarely, and games longer than 200 hardly ever.

[13] I am ignoring the fact that in all these other positions, the game can also end, if one of the players resigns in view of hopeless prospects, or if the two players agree on calling it a draw. This additional complication has little effect on the following discussion.

Chess-playing computer programs use the same two aspects as humans, but with a different balance: computer programs are vastly superior as regards the combinatorial aspect (they can consider many more variations) whereas humans are comparatively better at the abstract aspect. On balance, the strongest computer programs today are at least as strong as the best human chess players.

Sometimes, it is possible to make a definite assesment of a position that leaves no room for doubt or further refinement; such will be the case, for instance, if White works out a procedure by which she can check-mate Black within the next three moves no matter what Black does. In this case it makes sense to say that the position is *objectively* a win for White.

I now claim that *every* position has an objective value, and that there are only three possibilities: either the position is a win for White, a draw, or a win for Black; let us denote these by the numerical values 1, $\frac{1}{2}$ and 0 (thus taking White's point of view). The claim follows by considering the so-called game tree for chess; this is well known, but for the reader's convenience I explain in the Appendix how the argument goes.

Thus, somewhat disappointingly for chess addicts, although the better a chess player becomes the more refined and nuanced evaluations of positions will she be able to make, in the limit as she learns chess perfectly these refinements collapse into just the three values 1, $\frac{1}{2}$ and 0. This feature of the game, although of some interest from a philosophical point of view, is of little or no practical interest to chess players, and is therefore seldom discussed in the chess literature; see, however, Rowson (2005) for an exception.[14]

Today, grandmasters and other advanced level chess players tend to spend large amounts of time and effort on opening preparation before their games and tournaments. The opening is the early part of the game where one can reasonably expect the game to follow *exactly* paths foreseen before the game. There is a huge literature on openings, as well as databases of published games and analyses. An important part of opening preparation, especially on grandmaster level, is the search for improvements, i.e., for better moves than those previously known or published. During the past decade, chess-playing computer programs have become an increasingly widely used tool in this kind of search. Such use of computer programs is perfectly allowed, whereas of course using computers *during* games is not allowed.[15]

Imagine now – and this is the thought experiment I have in mind in the present section – that a computer program producing the objective value of any given position is developed and becomes available to these chess players.[16] It is easy to see that in a position with a given value, at least one of the possible moves in the position leads to a position that achieves this value, while none of the available moves improve on it. Let us assume that in addition to the objective evaluation of the position at hand, the program also informs about exactly which moves in the position are good enough to retain that value; this is in principle no harder than

[14]Rowson (2005) and Nunn (2001, 2002) are the only references on the topic of chess that I cite here. My main source on chess, however, consists of my 26 years (starting at age 13) of experience as a competitive chess player.

[15]The last remark refers to ordinary over-the-board games. For correspondence chess, the situation remains unsettled: some tournaments and organizations allow the use of computers during games, while others rule them out (despite the difficulty of enforcing the rule).

[16]I am assuming that the program is based on the brute-force approach outlined in the Appendix. As explained there, such a program is possible in principle, although a highly unlikely prospect in practice.

the original evaluation.

What will happen? Probably the first thing is that we will find out the answer to a question that pops up every now and then in conversations among chess players: what is the objective value of the initial position? Rowson (2005) agrees confidently with the general consensus among grandmasters and other experts that the value is $\frac{1}{2}$, although strictly speaking the answer is not known.[17] Let us assume, for the sake of the discussion, that this general consensus is correct, so that in other words chess played perfectly is a draw. (For the other two possibilities, my discussion below carries through *mutatis mutandis.*)

Then what? Will chess disappear, as a result of everone realizing that they can no longer win chess games? Not at all.[18] Even in objectively drawn positions, prospects of winning may be good due to the possibility for the opponent to make a mistake and slip into a lost position. After all, we are just humans, and chess is such an extraordinarily complicated game that it seems highly unlikely that even the best of us will be able to learn to play it perfectly.

How about the use of this new computer program – let us call it Orakel – in opening preparation? I predict that after a brief feeling of excitement, chess players would begin to realize that Orakel does not provide them with all the information they are looking for. In each position, Orakel provides a move or a number of moves being objectively the best. But in case of more than one objectively best move, which of these moves should the chess player take to heart in order to play when the position arises in a game? Her goal is to win games against fellow human beings, and from this point of view the different moves that tie for the title "objectively best" are typically far from equally good. She should prefer moves that give as many reasonable-looking options as possible for her opponent to make fatal mistakes, at the same time as making it as easy for herself as possible to avoid such mistakes.

We can even imagine a scenario where she maximizes her practical chances of winning the game by playing a move which is *not* among the objectively best ones, thus deliberately decreasing the objective value of her position in return for setting up a puzzle that will most likely turn out to be too difficult for her opponent to handle. Such play is (apart from our extremely rigorous notion of "objectively best")[19] well-known and usually looked down upon as "cheap trap-setting", but I am convinced that examples abound that are sufficiently subtle that even on the highest grandmaster level the best thing to do in practice is something that is objectively sub-optimal.

We can think up various refinements of the three values 1, $\frac{1}{2}$ and 0 to be incorporated in improved versions of Orakel. For instance, when a position is a win for White, it may be considered even better for White the smaller the number of moves is in which White can force check-mate. But neither this refinement, nor any of the variations that come quickly to mind, come anywhere near solving the problem of what is the best shot against a fellow human being. I find it hard to imagine how a definite solution to that problem could possibly be achieved in an automatized manner. For one thing, the best move in terms of maximizing

[17]The other "reasonable" answer is that the value is 1 (White wins), although no proof is known ruling out that the answer is 0 (Black wins). The latter is a weird possibility, implying that the initial position is one of so-called mutual *zugzwang*.

[18]Except possibly for the case of correspondence chess.

[19]Somtimes in the chess literature, one move is pointed out as objectively the best and another as preferable in practice, but with the rigorous terminology employed here both judgements are typically far from objective.

one's chances against a particular opponent will depend on exactly who the opponent is: Is he a beginner, an average club player, or a grandmaster? Was he brought up in the Soviet chess tradition? Does he have an obsession with maintaining a harmonic pawn chain or (at the other end of the spectrum och chess players' tempraments) with launching a direct attack upon the enemy king? All this, and much more, influence what is a good move in practice. In short, human psychology is of crucial importance to a chess player, but Orakel will in itself have little or nothing to offer on this aspect of the game.

I also predict that the specialized literature on chess openings will continue to flourish in the presence of Orakel.[20] The variations exhibited in these books will be supplemented not only by the objective verdicts provided by Orakel, but also by the kind of much more nuanced positional evaluations that today's chess players are used to, which offer valuable advice on how they should play in order to maximize their practical chances. Orakel's objective assessments are, on their own, simply not sufficient for providing human chess players with the understanding that they need.

3 Truth versus understanding in mathematics

The goal for a chess player is to win games against other chess players. The issue of what the goal for a mathematician is is far less clear-cut, but as an aid in thinking more clearly about that, I invite the reader to join me in a variation of the thought experiment of the previous section.[21]

This thought experiment involves Orakel II, a machine that is even more preposterous than Orakel. Orakel II is to number theory what Orakel is to chess. What it does is to answer, simply by brute force search, all questions we may have about the natural numbers. More precisely, for any property of natural numbers, or finite tuplets thereof, expressible in standard arithmetic, Orakel II works its way through all candidate instances and then reports back whether or not an instance was found with the desired property. Some examples of questions we may feed into Orakel II are the following.

(1) Do there exist positive integers x, y, z and $n \geq 3$ such that $x^n + y^n = z^n$?

(2) Does there exist an even number $n \geq 4$ which is not the sum of two primes?

(3) Does there exist a positive integer n such that if we write out the number 2^n in decimal form and read it backwards, then we get an integer power of 5?[22]

[20] A somewhat analogous situation has in fact already occured. If we restrict to chess positions with only at most six peices (including pawns and the two enemy kings) on the board, then databases have been set up which solve this part of the game in the same sense that Orakel solves the entire game. Still, there is a need for books that combine information from the database with pedagogical and humanly understandable explanations. For instance, Nunn (2001, 2002) does exactly this. An anonymous reviewer at Amazon.com asks about Nunn (2001) whether "with the advent of endgame databases, is this book worth buying anymore?", to which I answer, most emphatically, yes!

[21] I am talking about pure mathematics here, where real-world applications – helping engineers construct bridges or assisting geneticists in their search for genes that influence susceptibility to breast cancer – are so remote that they play little or no role in the mathematician's daily work.

[22] For instance, $n = 32$ will not do, because $2^{32} = 65,536$, and $63,556$ is not equal to 5^m for any integer m.

We know since about 1994 that the answer to (1) is no.[23] As to (2), the famous Goldbach conjecture from 1742 says that the answer is no, and this is what everyone still believes to be the case although the search for proof has so far been in vain. Finally, concerning (3), Freeman Dyson speculates in his contribution to Brockman (2006) that the answer is no, but also that this fact is not formally provable within the standard axiomatizations of arithmetic,[24] thus exemplifying Gödel's incompleteness theorem.

In all these examples, there are infinitely many cases to check, so what could I possibly mean when I say that Orakel II will do it for us? Well, Orakel II is a bit like an ordinary computer, except that the speed at which it works varies much more dramatically: it carries out its first operation in 1 second, its second operation in $\frac{1}{2}$ second, its third in $\frac{1}{4}$ second, and so on. After $\sum_{i=0}^{\infty} 2^{-i} = 2$ seconds, Orakel II has carried out infinitely many operations, and stops so we can read off the results it has stored in its memory.[25] The memory is infinite in terms of number of bits, but not in terms of physical size, as the first bit occupies 1 Å, the second $\frac{1}{2}$ Å, the third $\frac{1}{4}$ Å, and so on.

Imagine now that, at the time Orakel II is finally manufactured, we still haven't figured out how to prove (or disprove) Goldbach's conjecture. The machine is advertised as telling us everything we could possibly want to know about number theory, so we set it up to solve Goldbach's conjecture, press the "start" button, and after just over two seconds it tells us that indeed, Goldbach was right. Are we happy? Well, not for long. After a brief feeling of excitement, we realize that this is not the answer we wanted. That Goldbach was right we sort of knew all along, but what we really wanted to know was *why* no even number $n \geq 4$ exists which is not the sum of two primes. Orakel II has told us nothing about this.

OK, next try. We program Orakel II to work its way through all syntactically correct formal proofs within standard arithmetic, and check for each of them whether it is a valid proof of Goldbach's conjecture. If we are out of luck, it will turn out that no such proof exists (so we end up with another witness to Gödel's incompleteness theorem, but no useful information as to why Goldbach's conjecture is true). But suppose we are lucky, and Orakel II does produce a proof. This time, our feeling of excitement is *very* brief, because we quickly notice that the proof is $12,804,771$ steps long. Orakel II assures us that this is the shortest formal proof there is. After a couple of years of trying to comprehend the proof and to translate it into high-level mathematical language, we have managed to understand bits and pieces of it, but as to the overall structure of the proof itself, we are completely bamboozled.

So we give up on that particular proof, and turn to Orakel II again, asking for *other* formal proofs of Goldbach's conjecture. It generously provides us with a long list of such proofs, and after ten years M., one of our most brilliant mathematicians, announces that she, together with two of her assistants, has been able to translate a $19,228,630$-step formal proof into high-level mathematical language, and that the proof as a whole actually makes sense. They are initially met by a bit of skepticism from the mathematical community, but after a few months of patiently describing their work in seminars, they have managed to convince a

[23] Wiles (1995), Taylor and Wiles (1995).

[24] Dyson provides good arguments for believing the first part of his speculation, but fails to do so for the second part.

[25] Each bit in the memory can then take *three* values: 0, 1 and $*$, where $*$ means that the value of the bit at time t failed to converge as $t \to 2$.

couple of dozen of the best experts in the field of the soundness of their outline. The riddle of Goldbach's conjecture is finally considered solved, and M. and her two assistants go on to receive medals, prizes, and all sorts of other recognition for having solved it.

Could we have programmed Orakel II to find this, as it turned out in the end, highly elegant solution to the problem for us? I find that hard to imagine, and even if it were possible, it would have had to involve deep insights into human psychology. It seems that some logical structures are much easier for us to grasp than others, for reasons that are hard-wired into our brains.[26] The extraterrestrial beings we encountered in Section 1 may very well be hard-wired in a different manner, giving them a taste for mathematical proof that differs drastically from ours.[27] While they, too, know about the Central Limit Theorem, it may take us some time to figure out that we and they are talking about the same thing, and it may be that their reaction to our favorite proofs of that result is something along the lines of "Well, yes, that is formally correct, but why do it in such a strange and convoluted manner?" followed by the presentation of an alternative proof that looks utterly weird to us.[28]

These thought experiments may, as acknowledged already at the outset, be a bit on the wild side, but I think they nevertheless help us think clearly about objective truth versus human understanding. Like Orakel's $\{0, \frac{1}{2}, 1\}$-valued verdict of a chess position fails to provide us with the understanding we are looking for, Orakel II's $\{\text{true}, \text{false}\}$-valued verdict of number-theoretic statements is equally insufficient. And not even the formal proofs provided by Orakel II suffice to give us the desired understanding. That human psychology is an important aspect of what is to be considered a satisfactory solution to a mathematical problem seems like an unavoidable conclusion even for a hard-headed Platonist.

Appendix: The game tree argument

Here I describe the game tree representation of chess; for more on game trees and how to analyze them I recommend Berlekamp, Conway and Guy (1982).

The tree representing all possible chess games consists of nodes and directed links, and is built up as follows. Starting with a node v corresponding to the initial position, we create one new node w for every position that can be reached by White's first move, together with a link from v to w. We then continue building the tree inductively: for each node we create an outgoing link corresponding to each possible move, together with a new node.[29] A game of chess now corresponds to a branch in this tree, starting at the initial node v, following outgoing links until a node is reached that lacks outgoing links. Note, crucially, that since the

[26] Pinker (1995) presents plenty of evidence in this direction.

[27] As additional support for this wild-looking speculation, let me simply note that in order to find diverging opinion about what constitutes satifaction and beauty in a mathematical proof, I need not even visit other planets: going out of my office and crossing the corridor will in fact suffice.

[28] What I'm saying here is that Erdős' God is probably a bit anthropomorphic. For readers who do not know what I am talking about here, I quote from the preface of the marvellous book by Aigner and Ziegler (2004): "Paul Erdős liked to talk about The Book, in which God maintains the perfect proof for mathematical theorems, following the dictum of G.H. Hardy that there is no permanent place for ugly mathematics. Erdős also said that you need not believe in God but, as a mathematician, you should believe in The Book."

[29] Thus, positions that can be reached via more than one move order appear as more than one node in the tree.

number of moves in each position is finite, and the number of moves in a game is bounded, the game tree is finite – notwithstanding the number of nodes will be Vast (Very much larger than ASTronomical), to borrow a term from Dennett (1995).

We can now go on to assign values (0, $\frac{1}{2}$ or 1) to the nodes of the tree, in the following manner. First, all nodes without outgoing links correspond to positions where the rules of the game stipulate that the game ends with a specific result: 0, $\frac{1}{2}$ or 1. We then continue as follows. Whenever at least one node in the tree remains to be assigned a value, there is one whose outgoing edges all lead to nodes that already have a value.[30] Let w be such a node, and denote the nodes that its outgoing edges point at by $w_1, \ldots, w_k$. If w corresponds to a position where White is to move, then White, playing perfectly, should choose a move that leads to as favorable a position as possible, corresponding to a node w_i that makes Value(w_i) as large as possible among the available options. Thus,

$$\text{Value}(w) = \max\{\text{Value}(w_1), \ldots, \text{Value}(w_k)\}\,.$$

Similarly, if w corresponds to a position where Black is to move, then

$$\text{Value}(w) = \min\{\text{Value}(w_1), \ldots, \text{Value}(w_k)\}\,.$$

In this way, we eventually assign values to all nodes of the tree, and the game of chess is, in a sense, solved.

Note that the game tree argument not only gives an existence result for the objective value of any chess position, but also provides an algorithm for finding that value. However, the Vast-ness of the game tree prevents us from carrying this out in practice by implementing and running the algorithm on a computer.[31]

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[30] To see this, start at the initial node v, follow outgoing edges, always choosing to go to a node that has not yet been assigned a value. When this is no longer possible, we stand at a node of the desired kind. (Note that, with this procedure, the initial node v will be the last one to be assigned a value.)

[31] Using a technique known as dynamical programming, the waste of time computing power arising from the fact that each chess position is represented by many nodes in the tree can be avoided, but not even this is enough to make the actual implementation of this "final solution" to chess anywhere near feasible.

The sheer physical implausibility of such implementations makes it tempting to deduce that we will never be able to solve chess in the sense of being able to tell the objective value of any given position. But other approaches leading to such a solution are conceivable. Someone may come up with a procedure for abstract evaluation of any position landing in a $\{0, \frac{1}{2}, 1\}$-valued assesment Eval(w) that agrees with Value(w) at all positions where the rules of the game stipulate that it ends, together with an argument showing that in any position with White (resp. Black) to move, all moves reach positions w_i satisfying $\text{Eval}(w_i) \leq \text{Eval}(w)$ (resp. $\text{Eval}(w_i) \geq \text{Eval}(w)$) with equality for at least one of the possible moves. An induction argument then shows that $\text{Eval}(w) = \text{Value}(w)$ for all positions w. The evaluation procedure might even be simple enough for chess players to learn by heart, thus killing the game. This scenario, although not physically preposterous like the brute force solution, still seems quite unlikely to ever occur in reality. But then again, I might be wrong about this.

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