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Abstract
We present a novel decomposition method to decompose an eighth-degree polynomial equation, into its two constituent fourth-degree polynomials, as factors, leading to its solution. The salient feature of the octic equation solved here is that, the sum of its four roots being equal to the sum of the remaining four roots. We derive the condition to be satisfied by coefficients so that the given octic is solvable by the proposed method.

Key words: Octic equation; polynomials; factors; roots; coefficients; solvable polynomial equation; polynomial decomposition; solvable octic.

2000 Mathematics Subject Classification: 12D05: Polynomials (Factorization)

1. Introduction

It is well known from the works of Ruffini, Abel and Galois that the general polynomial equations of degree higher than the fourth cannot be solved in radicals [1 – 4]. This does not mean that there is no algebraic solution to these equations. The algebraic solutions to the general quintic have been obtained with symbolic coefficients. Hermite solved the Bring-Jerrard quintic using elliptic functions, and Klein gave a solution to the principal quintic using hypergeometric functions [5, 6]. The general sextic can be solved in terms of Kampe de Feriet functions, and a restricted class of sextics can be solved in terms generalized hypergeometric functions in one variable, using Klein's approach to solving the quintic equation [7]. There is not much literature on the solution to polynomial equations beyond sixth-degree, except in the form of numerical methods. We hope this paper will fill this gap to some extent.

In this paper we present a novel technique of decomposing a solvable octic equation into constituent fourth-degree polynomial factors, thereby facilitating the determination of all of its roots. The salient feature of the octic solved here is that, the sum of its four roots is equal to the sum of its remaining four roots. The condition required to be satisfied by the coefficients, in

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order that the given octic is solvable in radicals with the proposed technique, is derived. At the end of the paper we solve some numerical examples illustrating the applicability of this method.

2. Formulation of equations for solving the octic equation:

Let the octic equation for which the solution is sought be:

$$x^8 + a_7x^7 + a_6x^6 + a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 = 0 \quad (1)$$

where a_0, a_1, a_2, a_3, a_4, a_5, a_6, and a_7 are the real coefficients. In the method proposed here, we attempt to represent the octic (1), in the form as shown below.

$$[(x^4 + b_3x^3 + b_2x^2 + b_1x + b_0)^2 - p^2(x^4 + c_3x^3 + c_2x^2 + c_1x + c_0)^2]/(1 - p^2) = 0 \quad (2)$$

where b_0, b_1, b_2, b_3, and c_0, c_1, c_2, c_3 are the unknown coefficients of the respective fourth-degree polynomials in the above equation. The parameter, p, is also an unknown to be determined. The merit of representing the octic (1) in the form of (2) is obvious: notice that (2) can be easily factorized as:

$$\{[(x^4 + b_3x^3 + b_2x^2 + b_1x + b_0) - p(x^4 + c_3x^3 + c_2x^2 + c_1x + c_0)]/(1 - p)\}$$

$$\{[(x^4 + b_3x^3 + b_2x^2 + b_1x + b_0) + p(x^4 + c_3x^3 + c_2x^2 + c_1x + c_0)]/(1 + p)\} = 0 \quad (3)$$

When each factorial term is equated to zero, we obtain the following two fourth-degree polynomial equations.

$$[(x^4 + b_3x^3 + b_2x^2 + b_1x + b_0) - p(x^4 + c_3x^3 + c_2x^2 + c_1x + c_0)]/(1 - p) = 0 \quad (4)$$

$$[(x^4 + b_3x^3 + b_2x^2 + b_1x + b_0) + p(x^4 + c_3x^3 + c_2x^2 + c_1x + c_0)]/(1 + p) = 0 \quad (5)$$

The solution to the above quartic equations can be obtained easily by the Ferrari's method. The roots of the quartic equations, (4) and (5), will be the roots of the given octic (1), when it gets represented in the form of octic (2). To achieve this the coefficients of octic (1) should be same as that of octic (2). However notice that the coefficients of octic (2) are not explicitly written. Therefore we expand and rearrange (2) in descending powers of x as shown below in order to expose the coefficients for comparison.

$$x^8 + [2(b_3 - c_3p^2)/(1 - p^2)]x^7 + \{[(b_3^2 + 2b_2) - (c_3^2 + 2c_2)p^2]/(1 - p^2)\}x^6$$
$$+ \{2[(b_1 + b_2b_3) - (c_1 + c_2c_3)p^2]/(1 - p^2)\}x^5$$
$$+ \{[(b_2^2 + 2b_0 + 2b_1b_3) - (c_2^2 + 2c_0 + 2c_1c_3)p^2]/(1 - p^2)\}x^4$$
$$+ \{2[(b_0b_3 + b_1b_2) - (c_0c_3 + c_1c_2)p^2]/(1 - p^2)\}x^3 + \{[(b_1^2 + 2b_0b_2) - (c_1^2 + 2c_0c_2)p^2]/(1 - p^2)\}x^2$$
$$+ [2(b_0b_1 - c_0c_1p^2)/(1 - p^2)]x + [(b_0^2 - c_0^2p^2)/(1 - p^2)] = 0 \qquad (6)$$

Equating the coefficients of (1) and (6), we obtain the following eight equations as given below.

$$[2(b_3 - c_3p^2)/(1 - p^2)] = a_7 \qquad (7)$$

$$\{[(b_3^2 + 2b_2) - (c_3^2 + 2c_2)p^2]/(1 - p^2)\} = a_6 \qquad (8)$$

$$\{2[(b_1 + b_2b_3) - (c_1 + c_2c_3)p^2]/(1 - p^2)\} = a_5 \qquad (9)$$

$$\{[(b_2^2 + 2b_0 + 2b_1b_3) - (c_2^2 + 2c_0 + 2c_1c_3)p^2]/(1 - p^2)\} = a_4 \qquad (10)$$

$$\{2[(b_0b_3 + b_1b_2) - (c_0c_3 + c_1c_2)p^2]/(1 - p^2)\} = a_3 \qquad (11)$$

$$\{[(b_1^2 + 2b_0b_2) - (c_1^2 + 2c_0c_2)p^2]/(1 - p^2)\} = a_2 \qquad (12)$$

$$[2(b_0b_1 - c_0c_1p^2)/(1 - p^2)] = a_1 \qquad (13)$$

$$[(b_0^2 - c_0^2p^2)/(1 - p^2)] = a_0 \qquad (14)$$

There are nine unknowns (b_0, b_1, b_2, b_3, c_0, c_1, c_2, c_3, and p), but only eight equations [(7) to (14)] to solve. Therefore one more equation has to be introduced so that all the nine unknowns can be determined. However introducing an equation (involving the unknowns) imposes certain condition on the roots (and hence the coefficients) of the octic equation (1). The equation chosen will dictate the type of solvable octic. The equation, we are introducing here is as follows:

$$b_3 = a_7/2 \qquad (15)$$

As we notice from our further analysis given in this paper, the above equation leads to a solvable octic in which the sum of its four roots is equal to the sum of its remaining four roots. Observe that the equation as given by (15) is not the only equation (or condition) to be introduced to solve the octic by this method. There are several ways of choosing the equation, like $b_2 = c_2$, or $b_1 = c_1$, or, $b_0 = c_0$, etc. However the equation introduced will decide the type of solvable octic. Substituting the value of b_3 [using (15)] in equation (7), c_3 is evaluated as:

$$c_3 = a_7/2 \qquad (16)$$

The values of b_3 and c_3 are substituted in equations, (8) to (11), resulting in expressions, (8A) to (11A), respectively, as given below.

$$b_2 = c_2p^2 + F_2(1 - p^2) \quad (8A)$$

where F_2 is given by:

$$F_2 = (4a_6 - a_7^2)/8$$

$$b_1 = c_1p^2 + (a_5/2)(1 - p^2) - (a_7/2)(b_2 - c_2p^2) \quad (9A)$$

$$b_0 = c_0p^2 + (a_4/2)(1 - p^2) - [(b_2^2 - c_2^2p^2)/2] - (a_7/2)(b_1 - c_1p^2) \quad (10A)$$

$$b_0 = c_0p^2 + (a_3/a_7)(1 - p^2) - (2/a_7)(b_1b_2 - c_1c_2p^2) \quad (11A)$$

Now we have seven equations, (8A), (9A), (10A), (11A), (12), (13), and (14), with seven unknowns, b_0, b_1, b_2, c_0, c_1, c_2 and p, to be determined from them. We attempt to determine these unknowns through the process of elimination. Using the expression (8A), we eliminate b_2 from the equations, (9A), (10A), (11A), and (12), resulting in the following expressions, (9B), (10B), (11B), and (12A), respectively.

$$b_1 = c_1p^2 + F_1(1 - p^2) \quad (9B)$$

where F_1 is:

$$F_1 = (a_5 - a_7F_2)/2$$

$$b_0 = c_0p^2 - (a_7/2)(b_1 - c_1p^2) + [a_4 - F_2^2 + (c_2 - F_2)^2p^2][(1 - p^2)/2] \quad (10B)$$

$$b_0 = c_0p^2 + (a_3/a_7)(1 - p^2) - (2/a_7)\{[c_2p^2 + F_2(1 - p^2)]b_1 - c_1c_2p^2\} \quad (11B)$$

$$b_1^2 + 2b_0[c_2p^2 + F_2(1 - p^2)] - (c_1^2 + 2c_0c_2)p^2 = a_2(1 - p^2) \quad (12A)$$

Using (9B) we now eliminate b_1 from equations (10B), (11B), (12A), and (13), resulting in expressions (10C), (11C), (12B), and (13A) respectively as shown below.

$$b_0 = c_0p^2 + F_0(1 - p^2) + [(c_2 - F_2)^2 (1 - p^2) p^2/2] \quad (10C)$$

where F_0 is given by:

$F_0 = (a_4 - a_7F_1 - F_2^2)/2$, and

$$b_0 = c_0p^2 + [(a_3 - 2F_1F_2)/a_7](1 - p^2) + (2/a_7)(1 - p^2)(c_1 - F_1)(c_2 - F_2)p^2 \quad (11C)$$

$$2b_0(c_2 - F_2)p^2 + 2F_2b_0 - 2c_0c_2p^2 = [a_2 - F_1^2 + (c_1 - F_1)^2p^2](1 - p^2) \quad (12B)$$

$$b_0(c_1 - F_1)p^2 + F_1b_0 - c_0c_1p^2 = (a_1/2)(1 - p^2) \quad (13A)$$

We are left with five equations, (10C), (11C), (12B), (13A), and (14), involving five unknowns namely b_0, c_0, c_1, c_2, and p. Using (10C) we eliminate b_0 from equations (11C), (12B), (13A), and (14), resulting in the expressions, (11D), (12C), (13B), and (14A) as shown below.

$$\{p^2[(a_7/4)(c_2 - F_2)^2 - (c_2 - F_2)(c_1 - F_1)] - F_3\}(1 - p^2) = 0 \quad (11D)$$

where F_3 is given by:

$F_3 = (a_3 - a_7F_0 - 2F_1F_2)/2.$

$$\{p^2[p^2(c_2 - F_2)^3 + F_2(c_2 - F_2)^2 - 2(c_2 - F_2)(c_0 - F_0) - (c_1 - F_1)^2] - F_4\}(1 - p^2) = 0 \quad (12C)$$

where F_4 is given by:

$F_4 = (a_2 - F_1^2 - 2F_0F_2)$

$$\{(c_1 - F_1)[p^4(c_2 - F_2)^2 - 2p^2(c_0 - F_0)] + p^2F_1(c_2 - F_2)^2 - 2F_5\}(1 - p^2) = 0 \quad (13B)$$

Where F_5 is defined as:

$F_5 = (a_1 - 2F_0F_1)/2$

$$[p^2(c_0 - F_0)^2 - p^4(c_0 - F_0)(c_2 - F_2)^2 - p^2F_0(c_2 - F_2)^2 - (p^4/4)(1 - p^2)(c_2 - F_2)^4 + F_6](1 - p^2) = 0$$

(14A)

where F_6 is given by:

$F_6 = a_0 - F_0^2$

We observe that the term $(1 - p^2)$ emerges as a factor in the above equations [(11D), (12C), (13B), and (14A)]. However we cannot equate $(1 - p^2)$ to zero, since this term appears in the denominator in equation (2), and subsequently in many equations also. Therefore factoring out this term from the equations, (11D), (12C), (13B), and (14A), we obtain following expressions respectively.

$$p^2[(a_7/4)(c_2 - F_2)^2 - (c_2 - F_2)(c_1 - F_1)] - F_3 = 0 \quad (15)$$

$$p^2[p^2(c_2 - F_2)^3 + F_2(c_2 - F_2)^2 - 2(c_2 - F_2)(c_0 - F_0) - (c_1 - F_1)^2] - F_4 = 0 \quad (16)$$

$$(c_1 - F_1)[p^4(c_2 - F_2)^2 - 2p^2(c_0 - F_0)] + p^2F_1(c_2 - F_2)^2 - 2F_5 = 0 \quad (17)$$

$$p^2(c_0 - F_0)^2 - p^4(c_0 - F_0)(c_2 - F_2)^2 - p^2F_0(c_2 - F_2)^2 - (p^4/4)(1 - p^2)(c_2 - F_2)^4 + F_6 = 0 \quad (18)$$

At this stage we have four equations [(15), (16), (17), and (18)] involving four unknowns, c_0, c_1, c_2, and p. Continuing the process of elimination, we now eliminate $(c_1 - F_1)$ from equations, (16) and (17), using equation (15). The equations, (16) and (17), get transformed into equations, (16A) and (17A), respectively as shown below.

$$2p^4(c_0 - F_0)(c_2 - F_2)^3 = p^6(c_2 - F_2)^5 + p^4F_7 (c_2 - F_2)^4 + p^2F_8(c_2 - F_2)^2 - F_3^2 \quad (16A)$$

where F_7 and F_8 are given by:

$F_7 = (16F_2 - a_7^2)/16,$

$F_8 = (a_7F_3 - 2F_4)/2.$

$$[8F_3 - 2a_7p^2(c_2 - F_2)^2](c_0 - F_0)$$
$$= 8F_5(c_2 - F_2) + 4p^2F_3(c_2 - F_2)^2 - 4p^2F_1(c_2 - F_2)^3 - p^4a_7(c_2 - F_2)^4 \quad (17A)$$

Now there are three equations [(16A), (17A), and (18)] containing three unknowns, c_0, c_2, and p. Using equation (17A), we eliminate $(c_0 - F_0)$ from equations, (16A) and (18), to obtain the equations, (16B) and (18A), respectively as given below.

$$p^6(c_2 - F_2)^6 + F_9p^4(c_2 - F_2)^4 + F_{10}p^2(c_2 - F_2)^2 + F_{11} = 0 \quad (16B)$$

Where F_9, F_{10}, and F_{11} are given by:

$F_9 = (a_7F_8 + 8F_5 - 4F_3F_7)/(a_7F_7 - 4F_1)$

$F_{10} = -[(a_7F_3^2 + 4F_3F_8)/(a_7F_7 - 4F_1)]$

$F_{11} = 4F_3^3/(a_7F_7 - 4F_1)$

$$p^8(c_2 - F_2)^8 + F_{12}\, p^6(c_2 - F_2)^6 + F_{13}\, p^4(c_2 - F_2)^4 + F_{14}\, p^2(c_2 - F_2)^2 + F_{15} = 0 \quad (18A)$$

where F_{12}, F_{13}, F_{14}, and F_{15} are given by:

$F_{12} = (4a_7^2F_0 - 16F_1^2 - 8a_7F_3)/a_7^2$

$F_{13} = (16F_3^2 + 64F_1F_5 - 4a_7^2F_6 - 32a_7F_0F_3)/a_7^2$

$F_{14} = (32a_7F_3F_6 + 64F_0F_3^2 - 64F_5^2)/a_7^2$

$F_{15} = - 64F_3^2F_6/a_7^2$

We are left with two equations, (16B) and (18A), involving two unknowns, c_2 and p. However since these unknowns (c_2 and p) occur only as a inseparable product term $[p(c_2 - F_2)]$ in both the equations [(16B) and (18A)], it will not be possible to determine them separately. This situation can be illustrated more clearly by observing the following simple example.

Let us attempt to determine the two variables, u and v, from the following two equations.

$uv + k = 0,$
$u^2v^2 + muv + n = 0$

where k, m, and n are coefficients in the above equations. Notice that the variables, u and v, occur as an inseparable product, uv, in these equations. By substituting the value of uv from the first equation in the second equation, we obtain following expression.

$k^2 - mk + n = 0$

Thus instead of getting values for u and v from the above two equations, what resulted is a relation among the coefficients. In a later section of the paper we use this technique to derive the condition to be satisfied by the coefficients of the given octic equation (1), so that the octic is solvable through the proposed method.

The above example illustrates that, we cannot determine c_2 and p from equations, (16B) and (18A), instead we can get an expression relating the coefficients (of the given octic) from these equations. Since we are left with no further equation, determining c_2 and p (separately) appears to be an impossible task. In the next section, we describe a technique by which the unknowns, c_2 and p, are successfully evaluated.

At present let us observe the equations (16B) and (18A) more closely. The equation (16B) is a sixth-degree polynomial equation in $p(c_2 - F_2)$, while equation (18A) is a eighth-degree polynomial equation in $p(c_2 - F_2)$. However since both of these equations contain only even powers of $p(c_2 - F_2)$, the degrees of these equations can be reduced to half by the following variable transformation.

$$g = p^2(c_2 - F_2)^2 \qquad (19)$$

Thus the equations (16B) and (18A) are transformed into cubic and quartic equations respectively as shown below.

$$g^3 + F_9g^2 + F_{10}g + F_{11} = 0 \qquad (20)$$

$$g^4 + F_{12}g^3 + F_{13}g^2 + F_{14}g + F_{15} = 0 \qquad (21)$$

The cubic equation (20) yields three roots of g, while the quartic equation (21) provides four roots of g. The root, which is common to both the equations, (20) and (21), is the desired value of g, we are looking for. In the next paragraph, we describe a method to extract this common root.

The equations, (20) and (21), are rewritten as follows for our convenience.

$$g^3 = -(F_9g^2 + F_{10}g + F_{11}) \qquad (20A)$$

$$(g + F_{12})g^3 = -(F_{13}g^2 + F_{14}g + F_{15}) \qquad (21A)$$

Using (20A), we substitute the value of g^3 in (21A) to obtain following expression.

$$F_9g^3 + (F_{10} + F_9F_{12} - F_{13})g^2 + (F_{11} + F_{10}F_{12} - F_{14})g + F_{11}F_{12} - F_{15} = 0 \qquad (21B)$$

Again using (20A), g^3 is eliminated from (21B), resulting in the following quadratic equation.

$$(F_{10} + F_9F_{12} - F_{13} - F_9^2)g^2 + (F_{11} + F_{10}F_{12} - F_{14} - F_9F_{10})g + F_{11}F_{12} - F_{15} - F_9F_{11} = 0 \qquad (21C)$$

For the sake of convenience, the above quadratic is rearranged as follows.

$$g^2 + h_1g + h_0 = 0 \qquad (21D)$$

where h_0 and h_1 are given by:

$h_0 = (F_{11}F_{12} - F_{15} - F_9F_{11})/(F_{10} + F_9F_{12} - F_{13} - F_9^2)$

$h_1 = (F_{11} + F_{10}F_{12} - F_{14} - F_9F_{10})/(F_{10} + F_9F_{12} - F_{13} - F_9^2)$

Using (21D), the value of g^2 is substituted in the cubic equation (20A), to obtain the following quadratic equation.

$$h_1g^2 + (h_0 + F_9h_1 - F_{10})g + F_9h_0 - F_{11} = 0 \qquad (20B)$$

Again using (21D), we eliminate g^2 from (20B) to obtain a linear equation in g as follows.
$(h_0 + F_9h_1 - F_{10} - h_1^2)g + F_9h_0 - F_{11} - h_0h_1 = 0$

From the above linear equation the common root of g is found out as:
$g = h_2/h_3$ (20C)

where h_2 and h_3 are given by:
$h_2 = h_0h_1 + F_{11} - F_9h_0$
$h_3 = h_0 + F_9h_1 - F_{10} - h_1^2$

Once g is determined, the product, $p(c_2 - F_2)$, can be evaluated from (19). However to determine c_2 and p separately in the absence of any further equation requires some novel technique, which will be presented in the next section.

3. A discussion on the value of p^2

From equations (11D), (12C), (13B), and (14A), we notice that the term $(1 - p^2)$ emerges as a factor in these equations. However we were constrained not to equate p^2 to unity, as it amounts to division by zero in equation (2). Instead let us examine the consequences, when p^2 approaches unity, but will not attain unity value. In other words we are applying the limiting process and evaluating the expressions (that contain p), when p^2 tends to unity. Thus as a first step, let us evaluate c_2 in the limit as p^2 tends to unity, by rearranging the expression (19) and applying the limit as shown below.

$$c_2 = \lim_{p^2 \to 1} [F_2 \pm (g/p^2)^{1/2}] \quad (22)$$

Simplifying (22) results in two values of c_2 as:

$c_{21} = F_2 + (g)^{1/2}$

$c_{22} = F_2 - (g)^{1/2}$ (23)

Consider the expression (15). After rearranging (15) and substituting for $(c_2 - F_2)$ by utilizing (19), we apply the limit as p^2 tends to unity to determine c_1, as shown below:

$$c_1 = \lim_{p^2 \to 1} \{F_1 + (a_7/4p)[\pm (g)^{1/2}] - (F_3/p)[\pm 1/(g)^{1/2}]\}$$

After simplifying, the above expression yields two values of c_1 (corresponding to two values of c_2 respectively) as:

$c_{11} = F_1 + \{(a_7/4)(g)^{1/2} - [F_3/(g)^{1/2}]\}$

$$c_{12} = F_1 - \{(a_7/4)(g)^{1/2} - [F_3/(g)^{1/2}]\} \quad (24)$$

In the same manner, the expression (17A) is rearranged and the term, $(c_2 - F_2)$, is eliminated using (19), and then limit as p^2 approaches unity, is applied as shown below, to facilitate evaluation of c_0.

$$c_0 = \lim_{p^2 \to 1} \{F_0 + [(4F_3g - a_7g^2)/(8F_3 - 2a_7g)] + [(\pm)(8F_5 - 4F_1g)(g)^{1/2}]/[p(8F_3 - 2a_7g)]\}$$

Simplifying the above expression, we obtain two values for c_0 (corresponding to two values of c_2 respectively) as:

$$c_{01} = F_0 + (g/2) + [(8F_5 - 4F_1g)/(8F_3 - 2a_7g)](g)^{1/2}$$

$$c_{02} = F_0 + (g/2) - [(8F_5 - 4F_1g)/(8F_3 - 2a_7g)](g)^{1/2} \quad (25)$$

In the same fashion, we determine b_0, b_1, and b_2, by applying the limit to the expressions (10C), (9B), and (8A) respectively. The values of b_0, b_1, and b_2 are obtained as shown below.

$$b_0 = c_0$$
$$b_1 = c_1$$
$$b_2 = c_2 \quad (26)$$

We have determined all the unknowns in the octic (2), which means we are able to successfully represent the given octic (1) in the form of (2). Using the results of (26), along with the earlier determined values of b_3 and c_3, the octic (2) gets converted into an interesting octic as shown below.

$$[x^4 + (a_7/2)x^3 + c_2x^2 + c_1x + c_0]^2 = 0 \quad (27)$$

Looking at the above equation (27) emerged after evaluation of all unknowns, one may feel little disappointed thinking that, what resulted after all the exhaustive mathematics, is a tame octic equation with repeated roots. However it is not the complete story, as the next section reveals that, the octics with distinct roots also can be solved using this approach.

4. Decomposition of the octic equation:

Since c_0, c_1, and c_2 have two values each, equation (27) yields two distinct quartic polynomials as factors of octic equation (2) as shown below.

$$[x^4 + (a_7/2)x^3 + c_{21}x^2 + c_{11}x + c_{01}][x^4 + (a_7/2)x^3 + c_{22}x^2 + c_{12}x + c_{02}] = 0 \quad (28)$$

Equation (28) proves that, we have arrived at the solution to the given octic equation (1), by decomposing it into a pair of quartic polynomials as its factors. When each of the quartic factorial term in the above equation is equated to zero, we obtain two quartic equations as shown below.

$$x^4 + (a_7/2)x^3 + c_{21}x^2 + c_{11}x + c_{01} = 0$$

$$x^4 + (a_7/2)x^3 + c_{22}x^2 + c_{12}x + c_{02} = 0 \quad (29)$$

These quartics can be solved by the well-known Ferrari's method to obtain four roots each, and eight roots in total, which are the required roots of the given octic equation (1). In the coming sections we shall study the behavior of the roots, and the condition to be satisfied by the coefficients.

5. Behavior of the roots

Let x_1, x_2, x_3, and x_4 be the roots of first quartic equation, and x_5, x_6, x_7, and x_8 be roots of second quartic equation in the equation set (29). Notice that the coefficients of x^3 in these quartics are equal, which means that the sum of the roots of first quartic is equal to that of second quartic, as shown below.

$$x_1 + x_2 + x_3 + x_4 = x_5 + x_6 + x_7 + x_8 \quad (30)$$

Individually each sum (of four roots) is given by:

$$x_1 + x_2 + x_3 + x_4 = -(a_7/2)$$

$$x_5 + x_6 + x_7 + x_8 = -(a_7/2) \quad (31)$$

From (30) we note that, the given octic equation is solvable by this method if the sum of one group of its four roots is equal to the sum of its remaining four roots. From (31) we observe that the sum of the four roots in each group (and hence in each quartic equation) is a real number. Since equation (30) relates the roots of the given octic equation (1), we note that one of the roots can be expressed in terms of remaining seven roots, and hence this root is not independent. If we denote x_8 as a dependent root, then it is expressed in terms of other independent roots as:

$x_8 = x_1 + x_2 + x_3 + x_4 - (x_5 + x_6 + x_7)$

Since the roots of the given octic are related, the coefficients also have to be related, and in the next section we shall derive the condition to be satisfied by the coefficients.

6. Condition for the coefficients

From the expression (20C) we note that the parameter, g, is evaluated in terms of parameters, F_9, F_{10}, F_{11}, h_0, and h_1. Again note that h_0 is a function of F_9, F_{10}, F_{11}, F_{12}, F_{13}, and F_{15}, whereas h_1 is a function of F_9, F_{10}, F_{11}, F_{12}, F_{13}, and F_{14}. Thus eventually g will be a function of F_9, F_{10}, F_{11}, F_{12}, F_{13}, F_{14}, and F_{15}, which are functions of the coefficients of octic (1). Hence g is ultimately a function of coefficients, a_0, a_1, a_2, a_3, a_4, a_5, a_6, and a_7. Using equation (20C) we substitute the value of g in the quadratic equation (21D) to obtain the following expression.

$$(F_9h_0 - F_{11} - h_0h_1)^2 + h_1(F_9h_0 - F_{11} - h_0h_1)(h_1^2 - F_9h_1 - h_0 + F_{10}) + h_0(h_1^2 - F_9h_1 - h_0 + F_{10})^2 = 0 \quad (32)$$

The expression (32) provides a relation among the coefficients, a_0, a_1, a_2, a_3, a_4, a_5, a_6, and a_7. These coefficients have to satisfy the condition (32) in order that the given octic is solvable in this fashion. In the following numerical examples we find out the roots of the solvable octic, whose coefficients satisfy the condition (32).

7. Numerical examples

Following numerical examples enhance our understanding of the proposed method. Consider the octic equation shown below.

$$x^8 + 2x^7 - 25.1475x^6 - 62.86875x^5 + 51.94875x^4 + 95.47125x^3 - 78.72625x^2 + 17.3725x - 1.05 = 0$$

The coefficients of above octic have been obtained from the roots, which satisfy the relation (30). Using the expressions (15) and (16), the values of b_3 and c_3 obtained are: $b_3 = 1$, $c_3 = 1$. The parameters, F_0 to F_{15}, are determined as shown below, from the corresponding expressions derived in section 2.

$F_0 = -41.12646255474101$, $F_1 = -18.36062622070313$, $F_2 = -13.07374954223633$,

$F_3 = -151.1801398726457$, $F_4 = -1491.192986908885$, $F_5 = -746.4213570143718$,

$F_6 = -1692.435922218831$, $F_7 = -13.32374954223633$, $F_8 = 1340.012847036239$,

$F_9 = -242.5149916348265$, $F_{10} = 16339.83020529136$, $F_{11} = -295355.2632059779$,

$F_{12} = -908.2356715938659$, $F_{13} = 217987.6293012944$, $F_{14} = -19859925.14863872$,

$F_{15} = 618901739.0483127$.

Next, the parameters, h_0, h_1, h_2, and h_3, are determined from the corresponding expressions given in section 2 as shown below.

$h_0 = 10504.27396520806$, $h_1 = -216.0867489139377$

$h_2 = -17745.7612467463$, $h_3 = -124.7631910180035$

The parameter, g, is found out from (20C) as:

$g = 142.2355512226805$

From this value of g the coefficients of constituent quartic equations given in (29) are determined as shown below.

$c_{21} = -1.147495$, and, $c_{22} = -25$,

$c_{11} = 0.2787476$, and, $c_{12} = -37$,

$c_{01} = -0.01749611$, and, $c_{02} = 60$.

The quartic equations formed with these coefficients are:

$$x^4 + x^3 - 1.147495x^2 + 0.2787476x - 0.01749611 = 0$$

$$x^4 + x^3 - 25x^2 - 37x + 60 = 0$$

Solving the above quartic equations by Ferrari's method, all the eight roots of the octic equation are found out as: 0.1, 0.25, 0.4, – 1.75 (for the first quartic), and, 1, – 3, – 4, 5 (for the second quartic).

Let us solve another octic equation shown below.

$$x^8 + 2x^7 – 11.875x^6 – 12.5625x^5 + 45.66016x^4 + 14.58203x^3 – 53.34473x^2 + 1.620117x + 12.91992 = 0$$

As in the previous example, the coefficients of above octic have been obtained from the roots, which satisfy the relation (30). Using the expressions (15) and (16), the values of b_3 and c_3 obtained are: $b_3 = 1$, $c_3 = 1$. The parameters, F_0 to F_{15}, are determined as shown below, from the corresponding expressions derived in section 2.

$F_0 = 1.953125$, $F_1 = 0.15625$, $F_2 = – 6.4375$, $F_3 = 6.34375$, $F_4 = – 28.22265625$,

$F_5 = 0.5048828125$, $F_6 = 9.105224609375$, $F_7 = – 6.6875$, $F_8 = 34.56640625$,

$F_9 = – 17.34765625$, $F_{10} = 68.400634765625$, $F_{11} = – 72.94073486328125$,

$F_{12} = – 17.66015625$, $F_{13} = – 72.42822265625$, $F_{14} = 2177.700668334961$,

$F_{15} = – 5862.76876449585$.

Next, the parameters, h_0, h_1, h_2, and h_3, are determined from the corresponding expressions as shown below.

$h_0 = 40.2431640625$, $h_1 = – 15.53515625$, $h_2 = 0$, and $h_3 = 0$.

Notice from (20C) that g becomes indeterminate since $h_2 = 0$, and $h_3 = 0$. This indicates there are more (than one) common roots between the cubic equation (20) and the quartic equation (21). Therefore we evaluate g from the quadratic equation (21D). The roots (g_1, g_2) of quadratic equation, (21D), are the common roots between (20) and (21), and are given by:

$g_1 = 3.28515618609$, $g_2 = 12.25$.

More common roots indicate that there are as many ways to group the eight roots, which satisfy the condition (30). Let us choose one of the two common roots, $g = 12.25$, and then proceed to determine the coefficients, c_{01}, c_{11}, c_{21}, and c_{02}, c_{12}, c_{22}, of quartic equations given in (29). These coefficients are found out as:

$c_{21} = – 2.9375$, and, $c_{22} = – 9.9375$.

$c_{11} = 0.09375$, and, $c_{12} = 0.21875$,

$c_{01} = 0.84375$, and, $c_{02} = 15.3125$.

The two quartic equations formed with these coefficients are as shown below.

$x^4 + x^3 - 2.9375x^2 + 0.09375x + 0.84375 = 0$

$x^4 + x^3 - 9.9375x^2 + 0.21875x + 15.3125 = 0$

Solving the above quartics by Ferrari's method, we obtain their roots as: 0.75, 1, – 0.5, – 2.25 for the first quartic; and 1.75, 2, – 1.25, – 3.5 for the second quartic.

If we choose the other common root, $g = 3.28515618609$, then the two quartic equations formed are as shown below.

$x^4 + x^3 - 8.25x^2 + 2.75x + 3.5 = 0$

$x^4 + x^3 - 4.625x^2 - 2.4375x + 3.691405 = 0$

The roots of the first quartic are: 1, 2, – 0.5, and, – 3.5, and the roots of second quartic are: 0.75, 1.75, – 1.25, and, – 2.25. Note that these roots are same as that obtained earlier (with $g = 12.25$).

Let us solve one more octic equation, whose roots are complex. Consider the following octic:

$x^8 - 10x^7 + 53x^6 - 166x^5 + 389x^4 - 790x^3 + 1787x^2 - 2314x + 1690 = 0$

Again the coefficients in the above equation are determined from the roots, which satisfy the condition (30). Using the expressions (15) and (16), b_3 and c_3 are obtained as: $b_3 = -5$, $c_3 = -5$. The parameters, F_0 to F_{15}, are determined as shown below, from the corresponding expressions derived in section 2.

$F_0 = 31.5$, $F_1 = -13$, $F_2 = 14$, $F_3 = -55.5$, $F_4 = 736$, $F_5 = -747.5$, $F_6 = 697.75$, $F_7 = 7.75$,

$F_8 = -458.5$, $F_9 = -12.76470588235294$, $F_{10} = 2783.705882352941$,

$F_{11} = 26816.29411764706$, $F_{12} = 54.56$, $F_{13} = -1673.36$, $F_{14} = -171585.76$,

$F_{15} = -1375516.44$.

Next, the parameters, h_0, h_1, h_2, and h_3, are determined from the corresponding expressions as shown below.

$h_0 = 884.1560157086755$, $h_1 = 107.2395573009639$

$h_2 = 132918.7853218638$, $h_3 = -14768.75392465153$.

Using (20C) the value of g is found out as: $g = -9$. Since the value of g is negative, the coefficients of the quartic equations shown in (29) are complex; as can be seen from the expressions (23), (24), and (25). Thus the coefficients, $[c_{21}, c_{22}]$, $[c_{11}, c_{12}]$, and, $[c_{01}, c_{02}]$, are evaluated from (23), (24), and, (25), as given below.

$c_{21} = 14 + 3i$, and $c_{22} = 14 - 3i$,

$c_{11} = -13 - 26i$, and, $c_{12} = -13 + 26i$,

$c_{01} = 27 + 31i$, and $c_{02} = 27 - 31i$.

The quartic equations formed with the above coefficients are:

$$x^4 - 5x^3 + (14 + 3i)x^2 - (13 + 26i)x + 27 + 31i = 0$$

$$x^4 - 5x^3 + (14 - 3i)x^2 - (13 - 26i)x + 27 - 31i = 0$$

Above quartic equations are solved to obtain the eight roots of the given octic equation. The roots are found out as:

$(1 - 1i)$, $(2 - 3i)$, $(3 + 2i)$, $(-1 + 2i)$, for the first quartic; $(1 + 1i)$, $(2 + 3i)$, $(3 - 2i)$, $(-1 - 2i)$, for the second quartic.

The numerical calculations in these examples are performed using BASICA software in double precision mode.

8. Conclusions

A novel polynomial decomposition technique is presented, to solve certain solvable octic equations. The criteria for the roots and the coefficients to satisfy, in order that the octic is solvable by the method given, are derived. Some numerical examples are solved using the proposed method.

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