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Abstract

This paper presents a new construct: the conceptual tool 'Flexibility'. The construct was a result of an attempt to extract experiences of teaching and learning with the use of laptops. It was further developed and refined on the basis of four small-scale teaching experiments. The teaching experiments, being part of a development project in upper secondary school mathematics, investigated the use of laptops for teaching differential equations from a modeling point of view. The research was double-aimed: one objective was to conclude the project with some recommendations for the design of teaching, in the form of guidelines suitable for a wider dissemination amongst upper secondary mathematics teachers. The other main aim was to draw on the project's experiences for theory development within a framework based on Realistic Mathematics Education (RME) and related ideas. The construction of 'Flexibility' served both these aims.

Keywords: computer environments; flexibility; instrumental genesis; technology; teaching and learning; laptops; differential equations; Realistic mathematics education (RME)

1. Background

In Denmark, a recent reform of the structure and the curriculum in upper secondary school encompassed introducing the use of CAS in mathematics, chemistry and physics. Teaching with CAS was not required to follow authorized plans or materials: the design of teaching sequences, planning and preparation of extra teaching materials etc. were still the individual teacher's responsibilities. Thereby, the reform made new demands on the teachers' professional development: it was far from obvious how the use of CAS should be integrated in the individual teacher's repertoire of teaching instruments[2]. A metropolitan area six-year school development project, titled 'World Class Math and Science'[3], partly served as a precursor for the reform,

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[2] For a discussion of the generation of CAS use as the teacher's teaching instrument, see (Andresen 2006 p 265-275)
[3] www.matnatverdensklasse.dk

although the government did not sponsor the project. Part A of the development project[4] encompassed experiments with laptop classes in upper secondary school mathematics.

I was employed at the Danish University of Education to do the research in part A of the development project for my Ph.D. in mathematics education. Following the standpoint described in Wittmann (1998) and Lesh & Sriraman (2005), that mathematics education is a design science, I believe that research and development has to be linked to teaching and learning practice at crucial points. Improvement of teaching practice must be merged with the progress of the whole field of educational research and reversely, research progress linked to the development of teaching practice. So, the research for the Ph.D. project became double-aimed: one objective was to conclude the World Class project with some recommendations for the design of teaching, in the form of guidelines suitable for a wider dissemination amongst upper secondary mathematics teachers. The other main aim was to draw on the project's experiences for theory development within the theoretical framework based on Realistic Mathematics Education (RME) and related ideas. My construction of 'Flexibility' served both these aims.

1.1 Objectives of the construction

With this background, the initial goal of the research was to extract knowledge from the project's teaching experiments in a way that made this knowledge useful. More precisely, the aim was to identify, articulate and conceptualise the participants' shared experiences of improved learning and, subsequently, to turn these experiences into a form that would:

- Allow teachers to take them into account for improvement of their own teaching
- Serve as a contribution to math education theory

In yearly evaluation interviews, participants from the World Class A project repeatedly had expressed a shared experience of improved learning: in the interviews, a number of students and teachers described their feelings of getting "a better overview" and "a deeper understanding", apparently as a result of the use of laptops. These remarks were developed further with explanations like "because you can easily get series of graphs", "you do not get stuck in technical details", "it is easy to see examples" or "in stead of remembering a lot of techniques you are allowed to concentrate on the ideas". Such answers are in accordance with some expectations to CAS (and computer) use, widespread amongst math teachers. Teachers, like many researchers[5], believe that besides to favour visualisations, the CAS routines can be incorporated into the design of teaching as shortcuts, to facilitate students' focusing on ideas, structures and conceptions.

Hence, in the research project the challenge was to critically sort out examples of fruitful support of students' learning from the general feeling of 'flow in the classroom'. So the objective turned out to be pointing out important elements of learning activities, for teachers to aim at in their future design and preparation of teaching sequences and single lessons. These elements of learning activities are important, according to two criteria:

1. The activity should promote the student's actual work with mathematics in an observable way

[4] See (Andresen 2006 chapter 3, p 21-39) for a detailed description

[5] See for example (Drijvers, 2003, p 92). Hypothetically, the work with CAS may offer the students a shortcut to reification or a shortcut to working with "objects" as if the processes were reified.

2. Arguments based on the theoretical framework should support the claim that the activity could promote the student's learning

The mental actions 'change of perspective' and 'change of representation' turned out to be a common denominator for the important elements in focus of my interest. They gave inspiration to my choice of the term 'flexibility' to denote the new construct in the following definition.

1.2 Definition: Flexibility of mathematical conceptions

Definition

> *The flexibility of a mathematical conception constructed by a person is the designation of all the changes of perspective and all the changes between different representations the person can manage within this conception.* (Andresen, 2006, p. 136):

In this definition of the conceptual tool flexibility, the term 'change of perspective' means change between different facets of the mathematical conception in question, regarded as the student's construct[6]. My selection of a number of complementary pairs of perspectives intended to make flexibility an operational tool. Two main considerations determined my choice of pairs, in accordance with (1.) and (2.) above:

- Changes within the pairs of perspective should be recognisable for the teacher (or any observer)
- The changes should be pivots for the students' learning process.

Within the notion of flexibility, the term 'representation' is used in the sense of the media of expression or communication. The objective of the construction was not to categorise all mental changes. The overall aim was to offer teachers a few building blocks in the form of simple design heuristics. My design of these building blocks should be appropriately based on the research to ensure that the use of them was likely to support the students' learning.

Neither did the construction aim to establish any one-to-one correspondence between every mathematical conception and its exposure within each perspective. Supposedly, it is clear that any mathematical conception can be expressed in each perspective in more than one way, and it may be exposed in more than one way in each representation. The perspectives and representations, referred to in the definition, are presented in a later paragraph in this paper. Before that, the next paragraph tells about the theoretical basis for the concept of flexibility.

[6] According to L. P. Steffe and P. W. Thompson (Steffe & Thompson 2000 p 268-269), the experience of students' learning allows the researcher to inquire the students' mathematical realities. These realities are called *students' mathematics* and by (partly) knowing them, it is possible for the researchers to construct a model of students' mathematics, called *mathematics of the students*. Students' mathematics, which the students have constructed as a result of their interactions in their physical and sociocultural milieu, is indicated by what the students say and do when they engage in mathematical activity. In contrast, mathematics of the students is part of the shared knowledge in the classroom, compatible with the educational goals.

2. Basic ideas beyond the concept of 'flexibility'

Literature studies played an important role for the development and refinement of my earliest idea of 'flexibility'. This paper only presents three main ideas from the framework that forms the basis for my definition of 'flexibility'[7]: i) a specific dynamical approach to concept formation that combines main heuristics of Seymour Papert and Jean Piaget, introduced by Edith Ackermann, ii) vertical and horizontal mathematising in the RME sense realised in Koeno Gravemeijer's four level model and iii) the French theory of Instrumental Genesis.

2.1. A dynamical approach to concept formation

To facilitate cognitive growth, Edith Ackermann presents the idea of a bi-directional interplay between "diving in" and "stepping back" in (Ackermann 1990 p 6). Ackermann refers to both Jean Piaget and Seymour Papert[8] as constructivists who see children as the builders of their own cognitive tools, as well as builders of their external realities. Both Piaget and Papert consider knowledge as a personal experience to be constructed. Further, they both acknowledge adaptation as

> *.. the ability to maintain a balance between stability and change, closure and openness, continuity and diversity, or, in Piaget's words, between assimilation and accommodation. (...) The main difference is that Piaget's interest was mainly in the construction of internal stability whereas Papert is more interested in the dynamics of change. (Ackermann, 1990, p.4)*

The main point in Ackermann's description of Papert's view is: *diving into* situations rather than looking at them from a distance, *connectedness* rather than separation are powerful means of gaining understanding: becoming one with the phenomenon under study is a key to learning.
Ackermann's description of Piaget's view can be summarised like this: the way children progressively become *detached* from the world of concrete objects and local contingencies is closely related to their gradually becoming able to mentally manipulate symbolic objects within the realm of hypothetical worlds, so that rules and invariants are means of interpreting and organizing the world, and *abstract and formal thinking* are the most powerful way to handle complex environments.

Ackermann states that her own perspective is

> *..an integration of the above views. Along with Piaget I view separateness through progressive decentration as a necessary step toward reaching a deeper understanding. I see constructing invariants as the flipside of generating variation. (..) I share Papert's idea that diving into unknown situations, at the cost of experiencing a momentary sense of loss, is a crucial part of learning. (..) My claim is, that both "diving in" and "stepping back" are equally important in getting such a cognitive dance going. (Ackermann 1990 p 6)*

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[7] The complete theoretical framework is presented and discussed in (Andresen 2006, chapter 5)

[8] In a footnote, Ackermann says: "*Describing the difference between Piaget and Papert has been useful for me, and might be of general interest for the reasons mentioned in the text. It is through working directly with both thinkers (first, at the Piaget Institute, and currently at MIT) that I became progressively convinced of the need for integrating structural and differential approaches in describing human development.*" (Ackermann 1990 p 27)

Ackermann's idea of a 'cognitive dance' gave me the inspiration to construct 'flexibility' as a means to capture the dynamics of concept formation rather than static descriptions of cognitive structures. My method was to identify and group changes of perspective and changes between representations.

2.2. Vertical and horizontal mathematising in RME

Within the paradigm of RME, guided reinvention (or progressive mathematising), didactical phenomenology and emergent models are the three key heuristics for the design of teaching. In contrast to the cognitive theories of concept formation, 'modelling' was acknowledged to be an issue of interest in the World Class project's teaching culture. Moreover, a 'guided reinvention' design is to a great extend in accordance with the prevailing norms for good teaching. So it seemed reasonable to include RME into the framework in the following way. The four-level-model of activity intends to capture the way students' thinking evolves (Gravemeijer & Stephan 2002 pp 159-160):

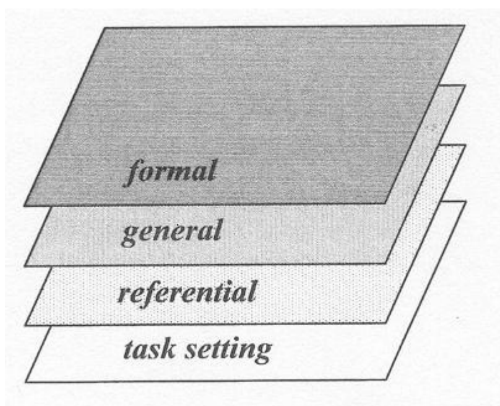


Fig 1. Levels of activity (Gravemeijer and Stephan, 2002, p. 159)

In the model, the activities at the level of task setting take place in a situation that is experientially real for the students. This enables the students to reason with a model of the problem but still think about acting in the situation. For experienced students an experientially real situation may also be mathematical. At the next, referential, level the students act with a model that is meaningful because it signifies an experientially real activity for them. At the general level the type of activity has changed since the students' attention has shifted from the contextual meaning to the mathematical relations involved. The students' activities are no longer dependent on situation-specific imagery. Finally, the students will no longer need the support of a model for their more formal mathematical reasoning, which constitutes the fourth level of activity.

So, progressive mathematising implies the students' raising up through the four levels. The level-raising does not happen as a total, one-directional movement: the student may in one context of content be at one level and in another context at another level. Besides, the raising from one level to another happens over time, where the student switches between two levels in both directions several times. The levels, further, have different character. It follows that in terms of RME, concept formation by modelling can be expressed as changes between levels: horisontal mathematising happens as changes from *situational, task setting* level to *referential* level when models emerge, and vertical mathematising happens as changes from the level of *model of* (referential) – to the level of *model for* (general) and to further up to the *formal* level. In the description below, the first three levels in the four-level-model are considered as perspectives on the mathematical conceptions in question. Level-raising is considered as the result of bidirectional changes between these perspectives, which are included in 'flexibility'.

2.3. The French theory of Instrumental Genesis

The French theory of instrumental genesis is based on the idea that an artefact, for example a CAS calculator, does not in itself serve as a tool. It becomes useful, and then denoted an 'instrument', only after the user's formation of (one or more) mental utilisation scheme(s). Such utilisation schemes connect the artefact with conceptual knowledge and understanding of the way it may be used to solve a given task. The utilisation schemes contribute to the formation of instrumented action schemes. So, an instrument consists of the tool, for example a laptop with the CAS software Derive, the student's mental utilisation schemes and the task or problem to be solved. (Drijvers, 2003, pp. 96-97).

The process in which the artefact becomes an instrument is called 'instrumental genesis'. The process proceeds through activities in

> *The two-sided relationship between tool and learner as a process in which the tool in a manner of speaking shapes the thinking of the learner, but also is shaped by his thinking.* (Drijvers & Gravemeijer, 2005, p. 190).

The two directions of the process can be linked to the construction of epistemic and pragmatic knowledge, respectively. The distinction between construction of epistemic and pragmatic knowledge is reflected in the definition of flexibility, which discerns between pairs of perspectives with relation to the construction of epistemic knowledge and, partly corresponding, pairs with relation to pragmatic knowledge.

This outline of the theory of instrumental genesis reveals the underlying French framework: the scheme concept, encompassing utilisation schemes and instrumented action schemes, was introduced by Vergnaud (Trouche, 2005, p.149). Since the mental utilisation schemes are not directly accessible for study and analysis, the concept of 'instrumented techniques' is of special interest: instrumented techniques are the external, visible and manifest parts of the instrumented action scheme. Still, an instrumented technique involves conceptual elements, since the technique reflects the schemes. This leads me to the following two crucial conclusions:

- The study of a student's development and use of instrumented techniques is useful to enlighten the student's development of those instrumented action schemes, to which the techniques relate
- Development of mathematical conceptions cannot be studied if use of technology is considered separate from the student's other activities

The first point stresses the importance of empirical studies of students' work. The second point opposes my research to the standpoint, that teaching may be performed independently of what tools the students have at their disposal. It is also in contrast to the view, that the influence of for example computer use can be overlooked as if it were just a matter of 'digitising the pencil case'. This is in line with what Jean-Baptiste Lagrange stresses in (Lagrange, 2005, pp. 131-132):

> *The traditional opposition of concepts and skills should be tempered by recognising a technical dimension in mathematical activity, which is not reducible to skills. A cause of misunderstanding is that, at certain moments, a technique can take the form of a skill.*

Besides the very grouping of the pairs of perspectives, the construction of the 'tool – object' pair was another result of the impact of the theory of instrumental genesis. The pair composed by a 'tool' perspective on a mathematical conception and an 'object' perspective on the same conception is realised for example in problem-solving settings. The tool – object pair corresponds to Anna Sfard's process – object duality (Sfard 1991). The tool perspective on mathematical conceptions is opposed to the 'pure skill' viewpoint and it implies the technical dimension that Lagrange refers to in the above quotation. So, within the notion of perspectives, the term tool denotes mathematical processes, carried out by the student with his instrument to serve a concrete purpose. It should be remarked that the purpose of the activity makes the difference between the tool perspective and Sfard's process perspective, not the instrument. Activities that aim at construction of pragmatic knowledge, involves the tool perspective on the mathematical conception in question. If the purpose is construction of epistemic knowledge, the process perspective is involved. This tool perspective on mathematical conceptions is in accordance with Régine Douady's definition of concept as a tool:

> *We say that a concept is a tool when the interest is focused on its use for solving a problem. A tool is involved in a specific context, by somebody, at a given time. A given tool may be adapted to several problems, several tools may be adapted to a given problem.* (Douady, 1991, p. 115)

Mathematical activities, then, must be considered from a tool perspective when they are considered as parts or elements of a technique in the above mentioned sense. To take the corresponding object perspective, genesis of the instrument is requested. Therefore, the instrumental genesis is in a crucial way linked to and maybe a prerequisite for the change to object perspective.

2.4. Perspectives and representations, referred to in the definition of flexibility

The pairs of perspectives form three groups: perspectives intrinsic to mathematics, and perspectives relevant for construction of epistemic- and pragmatic knowledge respectively. As it was mentioned above, the distinction between epistemic and pragmatic knowledge intended to take into account that the conceptual tool flexibility should resonate with the theory of instrumental genesis. All the changes between perspectives and between representations are bidirectional. Each pair of perspectives can be developed reflexively during the bi-directional changes.

In the notion of flexibility the term representation means representation system or communication media in a functional sense. So, the representations function partly as the media for externalisations of internal conceptual systems (Lesh and Doerr, 2000, p. 364; Mousoulides, Sriraman & Christou, 2007) and resembling the use of the term in the KOM-report (Niss, 2002). There is no sharp distinction between the four representations. Especially, the technical representation in many cases widely overlaps with the others.

2.4.1. Perspectives intrinsic to mathematics

Local and global position

Local position and global position are intrinsic mathematical perspectives. For example, changes between them occur when a single member of a group of objects is picked out for closer examination or if a single object is incorporated into a collection or family of objects. Using CAS, this change is easily realised for example by the use of facilities as substitution and copy – paste.

General - specific

General status and specific status are intrinsic mathematical perspectives, which concern the *domain of validity*. Inductive reasoning is linked to changes from specific to general perspective, whereas deductive reasoning is linked to changes from general to specific perspective. The use of CAS allows a quick and easy generating of specific objects from general expressions or formulas.

Analytic- constructive

The analytic and constructive perspectives are well-known phases in working with geometry. Examples are *measuring a given thing* and *construction* of a figure with a given measure, respectively. Especially, changes between these two perspectives are of relevance in sequences of modelling at a functional level.

2.4.2. Perspectives linked to construction of epistemic knowledge

The process - object duality.

The process perspective is operational and the object perspective is structural. The distinction and the connection between these two perspectives are encompassed by the reification theory. In contrast with the viewpoint of reification theory, the students' mental activities captured in the notion of flexibility, though, may go in both directions and thereby support the development of both perspectives.

Situated - decontextualised

The situated perspective is used to capture concrete aspects of problems, and of handling actual problems and challenges. The decontextualised perspective is the result of extracting rules, aiming at internal stability, which can occur through abstraction. Changes between situated and decontextualised perspective are linked to reflections, but not necessarily to modelling at the functional level. Therefore, this pair of perspectives relates to formation of epistemic rather than pragmatic knowledge. Changes from situated to decontextualised perspective correspond to the raising from referential to general level in the four-level model described above.

2.4.3. Perspectives linked to construction of pragmatic knowledge

The tool - object duality
The tool perspective[9] focuses on the use of a mathematical conception for solving a problem in a specific context, with some specific aids. The corresponding object perspective means the distant overview-perspective on the tool. Flexibility in this case encompasses the student's distant overview over a collection of mathematical tools, besides his capability to change from the distant over-view perspective into a tool perspective in the actual context.

Model - reality
The changes between reality and model perspective result in level-raising from the first to the second level in Gravemeijer's four-level model. In a number of cases, reality is expressed in natural language, and changes from reality to model then happens in connection with a change of representation from natural- to formal language.

Model of - model for
The changes between model of and model for perspective result in raising from the referential to the general level in Gravemeijer's four-level model. The change is exemplified in the case of calculus by K. Gravemeijer and M. Doorman (Gravemeijer & Doorman, 1999, p.111).

2.5. Representations

The term representation is used in a very broad sense. Consequently, changes between the four representations may occur in different contexts, and happen at a variety of levels. As part of the students' modelling and problem solving activities, the communication media may for example change at a functional level. In contrast, in other cases a single change to natural language may involve cognitive challenges like interpretation of a graph or ascribing meaning to a symbol.
The four representations do not intend to form any classification of media for expression of mathematical conceptions. In line with this, computer language is included as a representation on its own although it overlaps with the other representations, because changes to and from computer language is one part of the instrumental genesis.

Analytic representation (formal language)
Analytic representation includes formal expressions and formulas, algebraic expressions and symbols. Changes to analytic representation are related with symbolising whereas changes from analytic representation are connected with interpretation of symbols and the ascribing of meaning to symbols. The changes may cover more or less complex actions: For example, a shift to formal language may imply a routine translation from the graph of a linear function into the corresponding formal expression. Or it may describe the level raise from situated to formal level or from referential to general level. During modelling processes, the changes to formal language often happens stepwise, passing stages of partly formalised in-between-expressions.

[9] Here, the term tool is used in a broad meaning, not synonymous with artifact in contrast to instrument like in the theory of instrumental genesis

Graphical representation
Graphic representation includes graphs, curves, diagrams and tables, drawings etc. and explanatory gesticulation. Changes to and from graphic representation may be the result of for example the student's construction of a spatial conception of a curve that corresponds to the formal expression of a mathematical conception. Though, the curved line in the calculator's window, created by one press on the button is regarded as a graphic representation of the conception in question as well.

Natural language
The representation natural language encompasses spoken as well as written expressions and includes talk, explanations and negotiations, some texts written by the students or the by teacher, and some textbook texts.

The teacher's assessment has an impact on the communication in the classroom: depending on the social norms in the classroom, the students' can learn standard phrases of explanation by rote. The teacher's assessment of the student's understandings often relies on the student's capability to handle the content 'in his or her own words'. Standard phrases resembling own words, of course, can still be learned by root. Nevertheless, in the notion of flexibility such standard phrases are considered natural language. Natural language can overlap with formal language when technical terms are used, for example in group discussions.
The changes to natural language in many cases cover complex processes of interpretation and construction of meaning.

Technical representation (computer language)
Changes to technical representation encompass the translation from other representations into the version, adapted to the computer or calculator in question. Changes to computer language span from simple routines where students choose a well known instrument, to complex processes of instrumental genesis. Expressions in the other representations and changes between them are mediated in computer language: graphs, formal expressions and natural language can all be expressed in technical language too. In some cases, further, the software allows simultaneously use of formal language and graphic representation.

3. Methods and modes of the research

The initial construction of flexibility was based on teaching experiences, a preparatory classroom study and literature studies[10]. The teaching experiences were partly my personal ones; partly from the World Class A project referred to in the evaluation interviews, and partly second-hand ones, collected over years in informal talk and discussions with teacher-friends and colleagues.
Subsequently, I carried out an empirical study which aimed to

- Identify signs which could indicate flexibility in the students' mathematical conceptions
- Inquire how the teaching, the task, the teacher's questions etc. provoked the students to demonstrate flexibility
- Interpret the role of flexibility for the students' further working with mathematics
-

[10] The research design, the empirical studies and inquiries and their relations and roles in the design are presented and discussed in (Andresen, 2006, chapter 6)

The outcome of the study appeared at different levels. For example, one of the overall conclusions was that flexibility of the students' conceptions in general was prerequisite for acting with competence in the KOM sense (Niss, 2002). In several concrete cases I concluded that the use of certain software commands enhanced the flexibility of the students' mathematical conceptions. Finally, at the level of analysis I concluded that the conceptual tool flexibility was useful to throw light on the students' process of getting used to computer use.

Four small-scale teaching experiments served as cases for the empirical study. The participating teachers designed teaching sequences on differential equations for the experiments based on shared written materials in the form of a booklet, prepared by a group of teachers from the development project (Hjersing, N., Hammershøj, P. and Jørgensen, B. 2004). For the teachers, the aim of the teaching experiments was to develop good practices of teaching differential equations from a dynamical systems point of view. The booklet focused on modeling with a problem solving approach.

Data from the teaching experiments had the form of field notes and film recordings from classroom observations, students written reports, group interviews with teachers and students, and teaching materials. I observed fifty lessons spread over the four classes with 22, 12, 10 and 6 lessons during the winter 2003-2004. The lessons were chosen, restricted by practical circumstances. The four teachers who had voluntarily agreed were very helpful and obliging supportive. Thanks to them, the students were also helpful and open-minded.

I observed almost all lessons on the subject differential equations, taught by one of the booklet's teacher-authors. The goals were: i) to study the authors' overall intensions with the modelling approach to the subject, ii) to qualify my reactions on the booklet to the authors and iii) to inquire the use of laptops in class room teaching. In one other class, my observations focused on the modelling aspect. In the last two classes, I observed students' group work on project tasks. It was my hope that the dialogue and negotiations between the students in the groups would reveal signs of their learning process at a closer and more personal level, compared to the classroom observations.

Data were studied during interpretative analysis, taking social and psychological perspectives into account. The analysis followed the approach, designed by Paul Cobb et al. to meet the following three criteria (Cobb, Stephan, McClain & Gravemeijer 2001 p 116):

- Enable documentation of the collective mathematical development of the classroom community over the extended periods of time covered by instructional sequences
- Enable documentation of the developing mathematical reasoning of individual students as they participate in the practices of the classroom community
- Result in analyses that feed back to inform the improvement of the instructional designs.

4. Episode from a case study

The rest of this paper presents an excerpt from one of the Ph.D. project's nine cases. The objective of this part of the research was to inquire into the hypothesis: the construct 'flexibility' may capture important elements of learning activities, like they were described in the first parts of this paper. The excerpt is the larger part of one (the second) of the case's three episodes. It intends to illustrate the analyses of data following the three main aims, listed above. My analysis of the episode mostly concentrates on the changes between model of perspective and model for perspective and on the inquiry of the model for perspective as it appears in two groups of students' work. Obviously, other perspectives and representations are involved and might have been chosen as the object of analysis as well. In the thesis, other cases concentrate on other perspectives and representations[11].

4.1. Case 8, chemical reactions

The theme of case 8 was changes between model of perspective and model for perspective. This case encompassed episodes 8.1 to 8.3. The main aim of the analyses of the case was to inquire the model for perspective as it appeared in some of the students work. Two groups of students (group 9 and group 11) in two different schools worked with the same project task. The project task concerned with exploration of differential equations' models of the rate of chemical reactions of order zero, one and two. The episodes in case 8 were based on the project task, the written reports from the two groups of students, a transcription of film recordings of one of the group's (group 9) work with the project task during one lesson, and my field notes from the same lesson. The project task took the model of chemical reactions as its starting point[12]. After a few introductory statements concerning the technical handling of the amounts and concentrations of the compounds, for example in the case of precipitation of a compound, the model of the reaction rate was introduced in the task:

> Theoretically, the rate of chemical reactions in general is expressed as:
>
> <u>The rate of combination of two or more chemical compounds is proportional to the product of their concentrations</u>
>
> The relation is expressed like this:
>
> $$Rate = \frac{dc}{dt} = \mp k[A]^x[B]^y[C]^z \quad (1.1)$$
>
> That is, the rate of production/consumption of C is proportional to the concentration of the reactants ([A], [B], [C]) raised to the power of x, y and z respectively. The degree of the exponent denotes the order of the reaction.

Fig 2. Excerpt from page 51 in (Hjersing et al. 2004), (author's translation)

The task's design intended to encourage the student's explorations of this model, aiming to see special cases and recognise them as models of certain reactions of order zero, one and two. The students in the two groups were taught these topics in chemistry in advance. In their study program, they all combined high-level chemistry and high-level mathematics. In the textbook the special cases of the model were deduced from the general expression and then treated mathematically. In that sense, the project task dealt with changes between model of perspective with reference to the chemical setting and model for perspective with reference to the more general model in the mathematical setting. In the following, only episode 8.2 is considered.

[11] See the overview over Cases and episodes p. 202-204 in (Andresen 2006)

[12] For an introduction to the rates of chemical reactions and equilibrium see H. F. Holtzclaw, W.R. Robinson and W.H. Nebergall (1984): *General chemistry*, D.C. Heath and Company, USA pp 407-449 or P.W. Atkins (1990): *Physical Chemistry.* Oxford University Press pp775-810

4.2. Episode 8.2: group 9 answering question 3

When the episode took place, two students from group 9 worked with the chemical reactions project whereas the third member of the group was absent. At this time the students had passed the first one and a half page of the task's text, where the reaction rate was modelled for reactions of the different orders. The two students worked with page 53 in the booklet:

Irreversible second order reactions

Consider the irreversible reaction $A + B \rightarrow X + Y$.
One molecule A and one molecule B combine to one molecule of each of the compounds X and Y. The rate of consumption of A and B equals the production rate of X and Y.

We have inquired second order reactions in the simple cases, where the initial concentrations of the reactants were equal. But what happens, if [A] does not equal [B] from the beginning, or if some X or Y is already produced?

This time, we will model the production of X. The differential equation, mentioned earlier, now changes into:

$$\frac{dx}{dt} = kab \quad (1.6)$$

We now want an expression on the right side, which only depends on the immediate [X], and on some initial values of [A] and [B]. So the goal is to express a and b as functions of x.
The immediate concentration of a equals: The initial concentration of a (a_o) minus the 'new' x. The 'new' x equals the actual concentration of x minus the initial concentration of x (x_o). Similar to [B], so all in all gives (1.6) the equation:

$$\frac{dx}{dt} = k(a_0 - (x - x_0))(b_0 - (x - x_0)) \quad (1.7)$$

$$\frac{dx}{dt} = k(a_0 + x_0 - x)(b_0 + x_0 - x) \quad (1.8)$$

Example

2 mol silver nitrate $AgNO_3$ is mixed with 3 mol hydrochloric acid HCl. White silver chloride precipitates and the reaction runs completely. In this case there is ½ mol silver chloride when the reaction starts. The rate constant k is 1.

$$AgNO_3 + HCl \rightarrow AgCl + HNO_3$$

Based on the text we state the following:
k = 1
$a_o = 2$
$b_o = 3$
$x_o = ½$
Then, the differential equation is:

$$\frac{dx}{dt} = 1(2 + \frac{1}{2} - x)(3 + \frac{1}{2} - x) \quad (1.9)$$

Task3

Find the equilibrium points, where the rate equals zero for the differential equation in (1.8) and explain, what this means in practice

Fig3. Page 53 from the booklet (Hjersing et al. 2004), author's translation

In terms of flexibility, (1.1) gives a model for perspective since the level of the equation is general, whereas the perspective of (1.9) is model of because the level is referential – it refers to the actual experiment in the task.
Therefore, (1.8) can give both perspectives on the equation, depending on which of the two others it relates to:
<u>(1.8) in model of perspective with regard to (1.1)</u>
The model (1.8) is a general application of (1.1) to the case of a second order irreversible reaction with two reactants.

In (1.8), $[X]=x$ equals the concentration of the product, $[A]=a=(a_0+x_0-x)$ equals the concentration of one of the reactants, and $[B]=b=(b_0+x_0-x)$ equals the concentration of the other reactant. a_0 is the initial concentration of the compound a, similar for b and x.

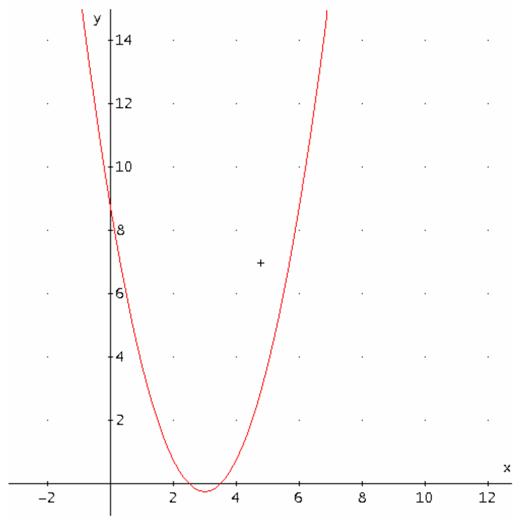
Fig 4. Explanation of (1.8)

So, with regard to (1.1) (1.8) serves as a model at referential level, referring to second order reactions. The perspective of (1.8), then, is model of whereas the perspective of (1.1) is model for.
<u>(1.8) in model for perspective with regard to (1.9)</u>
The model (1.9) is an application of (1.8) to the case of the reaction between Ag^+ and Cl^- where *a* stands for the concentration of Ag^+ (or $AgNO_3$) and *b* stands for the concentration of Cl^- (or HCl). *x* stands for the concentration of AgCl and dx/dt is the formation rate of AgCl. Since (1.9) is concerning with the actual formation of silver chloride, (1.9) is even more concrete than (1.8). So, (1.8) is a more general and decontextualised equation than (1.9). Consequently, in this case (1.8) serves to give the model for perspective and (1.9) serves to give the model of perspective on the equation.

The example and task 3 (in fig 3.) both aim to make the students change in both directions between the general mathematical model (1.8.) and the model of the actual reaction (1.9). Therefore, in terms of flexibility, they encourage to changes between model for and model of perspective. These changes happened stepwise in the actual case.
In the written report, the answer from group 9 to task 3 (fig 3.) was:

- Where does the rate for the differential equation equal 0? Where the rate equals zero are the equilibrium points.
To make the differential equation equal zero, one of the 3 factors in the differential equation has to equal zero.

$$\#35: \quad f(x) := 1\cdot\left(2 + \frac{1}{2} - x\right)\cdot\left(3 + \frac{1}{2} - x\right)$$



$$\#36: \quad \text{SOLVE}(f(x), x, \text{Real})$$

$$\#37: \quad x = 3.5 \vee x = 2.5$$

It is seen from the graph, that the equilibrium points are 2,5 and 3,5.
This means that the concentrations are unchanged in these points. From a mathematical point of view, both these points are useful, but in practice only 2,5 is of relevance because the reaction will stop here.

Fig 5. The students' answer

Apparently, the students linked the notion of equilibrium points with the roots of the polynomial. In their text, they interpreted the term equilibrium point in natural language in three (slightly) different ways, namely as

1) Points of the reaction where the reaction rate equals zero
2) Points of the reaction where the concentration is unchanged
3) A point where the reaction stops

In all three interpretations the equilibrium points were seen in a model of perspective. The students made a change to formal language when they turned to the equation #35 (fig 5.) and simultaneously to graphic representation, which was used in parallel (mixed in between to analytic expressions) with the formal language.

The students explicitly interpreted the actual values of the roots as concentrations. This interpretation revealed that they could handle the changes between equilibrium points in the model perspective (referential level) to the concentrations in the reality perspective. Later on, the students changed to a model for perspective on the equilibrium points, where they interpreted the equilibriums as the roots of the polynomial. They referred to the distinction between referential level (that is, model of perspective) and general level (that is, model for perspective) with the remark "*From a mathematical point of view*" in contrast to practice, which was understood to be at the referential level, that is, in a model of perspective.

By the students interpretation of the equilibrium point as a point where the reaction stops and where the rate equals zero, they linked the conception of equilibrium closely to the conception of rate. From the excerpt of text from the report (fig 5.), it is uncertain to say whether the students' chemical conception of equilibrium was a dynamic or a static one. The third point, (3), above and the following excerpts from the recordings of the students' work with task 3 may suggest that the students perceived the conception of chemical equilibrium in a static way even if the conception of rate was well described in their report's introduction (mentioned in the previous episode[13]). The students tended to confuse the rate and the rate constant:

S2: We look for the equilibrium point. Rate and rate constant is not the same, isn't it?
S1: The rate…
S1: Let us just…
S2: The rate, is it k or is that simply the rate constant
S1: It is the rate constant
S2: Then I guess we have to isolate k

In this way, then, the students were tempted to completely reduce the complexity of the problem. The dialogue revealed that they might doubt whether the equations (1.1) (fig 2.) , (1.8) and (1.9) (both fig 3.) modelled the reaction rate mathematically (general level) as well as chemically (referential level). According to my interpretation, hence, what confused them was the combination of mathematical and chemical models. Combining them is the very issue of changing between model for perspective and model of perspective in this case.
The dialog continued:

S1: No, because the rate is zero
S2: Then we simply have to write it…
S1: I do not understand what we are supposed to do. (reads) "The equilibrium points where the rate is zero"
S2: Where equal amounts are being produced. Where the amount left is the same as what is produced, isn't it?

Here, S2 referred to the chemical conception of dynamical equilibrium. In the next remark, S1 interpreted this involvement of a chemical conception as a change from (mathematical) model perspective to (chemical) reality perspective, but S2 maintained the model perspective by referring to *theory*:

[13] Episode 8.1 page 251-252 in (Andresen 2006)

S1: This is explaining what it means in real world, isn't it?
S2: But didn't we learn theory about this in chemistry?
S1: I do not understand what they... when the rate is zero, is it simply the rate constant?

The students called the teacher who referred to page 16-17 in the booklet, where the equilibrium solutions were considered:

$$\frac{dP}{dt} = k \cdot P \cdot \left(1 - \frac{P}{N}\right)$$

(...)
The right side of the equation is a polynomial of second degree in P with the roots
$P(t) = 0$, $P(t) = N$
It is positive between the roots and negative outside.

Fig 6. Excerpt from (Hjersing et al. 2004, p.16), author's translation

S1: (reads) the right hand side of the equation is a polynomial of second degree, find the roots,..
T: So what is this?
S1: I am reading... Equilibrium solutions – that is what we are looking for
T: Equilibrium solutions, what is it then?
S2: It was negative and positive second degree

The problem had now been transformed to the simpler one of finding the roots of a polynomial of second degree. This problem was expressed in graphic representation, natural and formal language simultaneously. The problem was seen in a model for perspective[14] as far as it had to be solved in a purely technical context with no connection with the chemical model.
When the dialogue continued, the teacher tried to establish the connection between the simpler problem and the question of equilibrium solutions to the differential equation. In terms of flexibility, the teacher insisted on keeping the changes between model for perspective and model of perspective in focus of interest. Shortly after, the episode was concluded with the teacher having accomplished the guidance of the students. The students went to the computer room for doing some graphs.

4.3. Conclusion of case 8

The changes between model of and model for perspective were in this case realised as changes between a mathematically formalised chemical model and a more general mathematical model. It was understood in the project task's text that the students knew the formalised chemical models in advance. So, the students were supposed in advance to manage changes between the chemical reality, in the form of proceedings of the actual reactions, and the formalised chemical models. In the episodes, the changes between the two perspectives went in both directions: the report from group 11 for the major part took the model for perspective (illustrated in episode 8.1), but the

[14] This situation illustrates a tool perspective on the solving of quadratic equation

task's questions provoked changes to and fro a model of perspective and further interchanges between model and reality perspective (it was demonstrated in episode 8.3). The report from group 9 divided into parts that took the model of perspective and other parts that took a model for perspective. The links between the two perspectives, though, were not always clearly revealed (it was shown in episode 8.3).

Episode 8.2 demonstrated how the teacher guided the students to change between model of and model for perspective. The episode pointed to the important role of the guidance from the teacher, as well as from the task's questions.

During the complete case, changes between reality and model perspectives and between model of and model for perspectives were realised on the relation, modelled by the differential equations (1.1), (1.8) and (1.9). The same relation was expressed in all the four different representations. According to the definition of flexibility, the individual student's flexibility of the conception of this mathematical relation (e.g. the differential equations model), in this context, was developed to the extent that he or she was able to manage these changes. Working with the tasks in the case seemed to support this flexibility since the individual student was more likely to manage the changes after the teaching sequence.

5. Conclusion

To conclude this paper three questions are discussed: what contribution does the construction of flexibility add to the field, are the promises, criteria and requests from the first part of the paper kept, and what questions does this paper not answer?

5.1. What is new?

The introduction of a new construct like flexibility with its definition and technical notion to the field of math education, which may seem almost overloaded with a diversity of notions and terms already, has to be justified. One important argument is that the introduction of flexibility summarises, connects and simplifies key elements of well established and acknowledged theory. The issue of visualisation, for example, is included in flexibility in terms of change to graphic representation. The relation between the levels in Gravemeijer's model is another example. The novel idea beyond the construction of flexibility is to focus on the dynamics as a common denominator, which leads to consider all these key elements in a new light and in a new combination.

The case illustrates how flexibility intends to serve as a tool for clarification: The case's task presented a rich and complex structure of models at different levels, and links between them. The task was suitable for the students' exploration of the relation, modelled by the differential equations, exactly because it was sufficiently complex to offer possibilities of open ended inquiries. During the case, one of the students' main difficulties concerned the changes between the levels of the model, represented in the three equations. Another main difficulty was the transformation of the problem to the simpler one of finding the roots of a second degree polynomial, followed by interpretation of the result. These main difficulties are closely related to the main learning potentials of the task, so attempts to avoid them by omitting parts of them for simplification would be of little use. The analysis of the case demonstrates how these main issues of the task can be interpreted in terms of flexibility. My claim is that the teacher could strengthen

the guidance of the students and make it more explicit without giving the answers, if he or she had such an interpretation in mind.
The case also illustrates that the apparent conflict of having the equation (1.8) at two different levels is cancelled when focus, in terms of flexibility, is on the dynamics of changing between levels rather than on placing conceptions at the levels.

5.2. Does flexibility keep the promises?

According to the initial goal of the research mentioned in this paper, flexibility should serve as a tool for teachers to take the project's experiences into account for improvement of their own teaching. The case gives an example of flexibility's potentials for improved guidance, according to my interpretation. To meet the goal, though, more focused teaching experiments, based on an elaborated description of the single elements of flexibility would be the next step.
The second request, which concerns the contribution to math education theory, was already discussed in a previous paragraph. The objectives of the construction set two demands on flexibility: signs of flexibility should be observable, and claims of its relevance should be theoretically founded. The construction of flexibility intended to capture important learning activities which promote the student's actual work with mathematics in an observable way. The case demonstrates that the changes of perspective and changes between different representations are observable. The presentation in this paper of three basic ideas beyond the concept of flexibility serves to justify[15] the claim, that the changes promote the student's learning.

5.3. What is left?

It follows from the research design that the effect was not tested on teaching designs, which aim at development of flexibility of the mathematical conceptions in question. A number of guidelines are presented to conclude the Ph.D. thesis. The guidelines are meant for teachers who want to aim at flexibility in the design of their teaching. A large scale inquiry of the effect could include teaching materials and teaching designs based on these guidelines, pre- and after tests and qualitative evaluation. The guidelines are (Andresen, 2006, pp. 294-295):

Teaching that aims to support flexibility in the mathematical conceptions with and without the use of CAS should be based on the following principles:

- The design of tasks and problems ensures that changes of perspective go in both directions in all the pairs of perspectives that form "flexibility":
    1. Local - global
    2. General – specific
    3. Analytic- constructive
    4. Process - object
    5. Situated – decontextualised
    6. Tool – object
    7. Model - reality
    8. Model of - model for
- The design of tasks and problems ensures that changes between representations go in both directions between *graphic representation, analytic representation (or formal language), natural language* and *technical representation (or computer language)*

[15] Further justification builds on the discussions all through chapters 5 and 7 in (Andresen, 2006).

- Expressive work and explorative work with mathematical models are both important. The teaching is designed to ensure that the students over time are encouraged to model a number of key conceptions including all the four levels situated, referential, general and formal.
- A diversity of strategies is not only accepted, but appreciated in the classroom. The students are encouraged to try out ideas and techniques. Results, ideas and strategies are discussed and negotiated with open minds in the classroom.

The relations between flexibility and the shortcuts mentioned in the first part of this paper, and the role of flexibility in the instrumental genesis, apparently, are issues in focus of interest for the continued work with development and refinement of the construct flexibility. It should be remarked that although the project took the use of laptops as its starting point, flexibility is not restricted to mathematical conceptions within a computer environment.

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