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A reflection on mathematical cognition: how far have we come and where are we going?

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Abstract: Any theory of mathematical cognition and learning must ultimately articulate with our current understanding of how the mind works and with current theories of how knowledge is acquired from both an individual and social perspective. Certainly the human mind is very complex; understanding how it works in general and identifying the components that contribute to "doing mathematics" in particular is no easy task. To understand how the mind does mathematics, we must identify what mathematicians actually do in the context of everyday cognition. Here we discuss some of what is known and about this and point out directions for future work.

The mind and mathematics

Human minds did not evolve to do abstract mathematics per se, and yet we can use them to do mathematics none the less; there must be an explanation of how we got from there to here. Sperber (1996) says, "Materialism… does not commit scientists who espouse it to describing the objects of their discipline and the causal process in which these objects enter into the vocabulary of physics. What it does commit them to is describing objects and processes in a manner such that identifying the physical properties involved is ultimately a tractable *problem*, not an unfathomable *mystery* (to use Noam Chomsky's famous distinction…)." (p. 10) Cognitive scientists seek material explanations for how the mind works in general; the missing link we must pursue is how mathematical cognition and learning can be explained in terms of the "usual" processes of the mind. Despite the delicate and difficult work needed for such a program, we must insist on our goal being a theory of how the mind does mathematics without begging any questions. It is argued here that mathematical structures are the cognitive imprint of the structural relationships we naturally perceive in the world, refined into idealized "objects" that can be studied with an idealized reasoning structure.

Susan Haack talks about scientific reasoning as a refinement and extension of our everyday capacities for empirical inquiry. She quotes Thomas Huxley: "The man of science simply uses with scrupulous exactness the methods which we all, habitually and at every minute, use carelessly" and then Albert Einstein: "[T]he whole of science is nothing more than a refinement of everyday thinking."(p. 95) At some appropriate level of generality, mathematical thinking must also arise as a refinement of our everyday cognitive capacities to reason about the world. A proper accounting of how this takes place is our best hope for explaining the "unreasonable

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effectiveness of mathematics" to help us make predictions about the world (to borrow Eugene Wigner's famous phrase). Given the usual perception that abstract mathematics is about anything but the everyday world, we must pick our path carefully. We have taken the first step by noting that how we think about mathematics must be explainable in terms of our standard-order cognitive systems. The next step is to ask, what are these systems and how are they honed to learn and do mathematics?

First, let's illustrate what we might mean by "systems." The prevailing wisdom is no longer that the mind is some general purpose thinking machine, but rather the fusion of different specialized "cognitive modules." Gallistel (1999) says:

> Despite long-standing and deeply entrenched views to the contrary, the brain no longer can be viewed as an amorphous plastic tissue that acquires its distinctive competencies from the environment acting on general purpose cellular-level learning mechanisms. Cognitive neuroscientists, as they trace out the functional circuitry of the brain, should be prepared to identify adaptive specializations as the most likely functional units they will find. At the circuit level, special-purpose circuitry is to be expected everywhere in the brain, just as it currently is expected routinely in the analysis of sensory and motor function. (p. 1190)

If we are to ask which cognitive modules and learning mechanisms are harnessed for mathematics, we should first look for specific cognitive modules that are adapted for "proto-mathematics." This has been the approach of scientists such as Stanislas Dehaene who study our understanding of numbers and basic arithmetic. Dehaene (1997) says:

> Newborns readily distinguish two objects from three and perhaps even three from four, while their ears notice the difference between two sounds and three. Hence, the brain apparently comes equipped with numerical detectors that are probably laid down before birth. The plan required to wire up these detectors probably belongs to our genetic endowment…. [Most] likely, a brain module specialized for identifying numbers is laid down through spontaneous maturation of cerebral neuronal networks, under direct genetic control and with minimal guidance from the environment. Since the human genetic code is inherited from millions of years of evolution, we probably share this innate protonumerical system with many other animal species. (p. 61-62)

Typically, work in cognitive science has focused on the acquisition and representation of specific mathematical topics such as numbers (as in Dehaene's work), algebra word problems, geometric proofs etc. Dehaene claims that "As humans, we are born with multiple intuitions concerning numbers, sets, continuous quantities, iteration, logic, and the geometry of space."(p. 246) While there has been work done in general in these other areas, it has not developed to the same depth as it has with our understanding of whole numbers and operations. Take, for example, the Handbook of Mathematical Cognition; of the 27 chapters dedicated to mathematics and the mind, most treat numbers and operations, with the exception of Lakoff and Nunez's[2] article on conceptual metaphor and a few chapters devoted to mathematical disability.

Mathematics and the mind

While keeping one eye on the work of cognitive scientists, we should also ask what the mathematician's perspective has to offer the study of mathematical cognition. Consider the following statement by G. H. Hardy:

> I believe that mathematical reality lies outside us, that our function is to discover or observe it, and that the theorems which we prove, and which we describe grandiloquently as our 'creations,' are simply our notes of our observations.

This sentiment reflects a common one among mathematicians. Technically, Platonism is the philosophy of Plato, but given that there are different interpretations of Plato's philosophy, I will use the term Mathematical Platonism, or simply Platonism, to be a belief that the objects of mathematical study have a reality separate from the human conceptualization of them. Mathematical Platonism differs from realist perspectives in science, where the objects of study tend to be about physical objects of which we have some empirical evidence. This contrasts with mathematical structures, such as groups or hyperspheres, which no one claims exist in the physical world as physical objects.

In a paper titled "Reification as the Birth of Metaphor," Anna Sfard reports on the interviews she has conducted with three renowned mathematicians: a logician, a set theorist, and a specialist in ergodic theory. In these interviews, the three mathematicians talk about the mathematical concepts that they study as if they were concrete in some way. They talk of perceiving images and structures. Her ergodic theorist says, "In those regions where I feel an expert… the concepts, the mathematical objects turned tangible for me." The term that Sfard uses for this cognitive phenomenon is reification (although I am using it more generally than she does): To **reify** is to regard or treat an abstraction as if it had concrete or material existence.

In several pieces of writing, Bertrand Russell reflects on both the reasons for and disillusionment of this experience. In this quote from "Portraits from Memory," he explains how such a sentiment might emerge:

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[2] Lakoff is well known for his application of his theory of conceptual metaphors to many different domains. In a critique of Lakoff's work in Moral Politics, Jesse Walker says, "The problem is that [Lakoff] has… a model that may have some explanatory power but which he has stretched far beyond its limits." This is exactly the fault one can ascribe to the work of Lakoff and Nunez. Dennett's analysis of Skinner comes to mind:

> Skinner was a greedy reductionist, trying to explain all the design (and design power) in a single stroke. The proper response to him should have been: "Nice try—but it turns out to be much more complicated than you think!" And one should have said it without sarcasm, for Skinner's *was* a nice try. (p. 395)

And so the criticism of Lakoff and Nunez's work should go, *mutatis mutandis*.

> Mathematics is, I believe, the chief source of the belief in eternal and exact truth, as well as in a super-sensible intelligible world. Geometry deals with exact circles, but no sensible object is exactly circular; however carefully we use our compasses, there will be some imperfections and irregularities. This suggests the view that all exact reasoning applies to ideal as opposed to sensible objects; it is natural to go further, and to argue that thought is nobler than sense, and the objects of thought are more real than those of sense-perception. Mystical doctrines as to the relation of time to eternity are also reinforced by pure mathematics, for mathematical objects, such as number, if real at all, are eternal and not in time. Such eternal objects can be conceived as God's thoughts. Hence Plato's doctrine that God is a geometer, and Sir James Jeans' belief that He is addicted to arithmetic.

Given that there is no way to empirically verify the existence of timeless and tenseless mathematical objects, believing in them is a matter of faith. Yet the experience of doing mathematics is very compellingly like studying "real" objects. Reuben Hersh says that most mathematicians are Platonists in their day-to-day work but only Formalists on Sundays. So they behave as Platonists behind closed doors, but when asked for a public accounting of their activities, retreat behind Formalism.

Let us put the philosophical issues aside for a moment. What insights do we gain from these introspective ruminations of mathematicians? Looking at this issue from a broader view point will be helpful here. In explaining the difference between cognitive neuroscience theories about consciousness and, say, motor action, Dehaene and Naccache state,

> What is specific to consciousness, however, is that the object of our study is an introspective phenomenon, not an objectively measurable response. Thus, the scientific study of consciousness calls for a specific attitude which departs from the 'objectivist' or 'behaviorist' perspectives often adopted in behavioral and neural experimentation. In order to cross-correlate subjective reports of consciousness with neuronal or information-processing states, the first crucial step is to take seriously introspective phenomenological reports. Subjective reports are the key phenomena that a cognitive neuroscience of consciousness purport to study….
>
> The idea that introspective reports must be considered as serious data in search of a model does not imply that introspection is a privileged mode of access to the inner workings of the mind. Introspection can be wrong…. We need to find a scientific explanation for subjective reports, but we must not assume that they always constitute accurate descriptions of reality. " (p. 3)

Dehaene's claims about subjective reports being relevant to understanding consciousness have a parallel in mathematical cognition. Mathematicians' tendency to use introspection to think about how mathematics gets accomplished provides important clues for determining how we actually do mathematics. But more than taking the introspective reports at face value, we should consider how mathematicians describe their work as data that should be explained by any complete theory of mathematical cognition.

Thus, it is more important to understand and explain the cognitive reality of mathematical objects than to argue about in what sense they exist. So far, most researchers in human cognition who have addressed the issue of Mathematical Platonism have used our understanding of cognition to dismantle Platonism as an epistemology of mathematics. But it is precisely this subjective experience that we should use to give us clues about how mathematicians do what they do. Dehaene comes close when he says,

> Presumably, one can become a mathematical genius only if one has an outstanding capacity for forming vivid mental representations of abstract mathematical concepts—mental images that soon turn into an illusion, eclipsing the human origins of mathematical objects and endowing them with the semblance of an independent existence. (p. 242-243)

What Dehaene fails to recognize is that we need to understand and explain *how* we form "vivid mental representations of" abstractions and *why* they appear to have an independent existence. Furthermore, it is not just mathematical geniuses who function this way, but anyone who uses mathematics effectively to solve complex problems. Think of the ubiquitous use of the term "mathematical tools," often used as if it stands in opposition to the mathematician's view of mathematical objects. Yet a function is still a conceptual object if it is used to describe the height of a rocket over time. It is just that some people develop or collect tools, and others prefer to use them for some secondary purpose.[3] It is the selective attention of (especially pure) mathematicians on the mathematical objects themselves vs. how they are used that makes them reflect more on the subjective experience of having these mental representations.

Anna Sfard gives a very plausible explanation for the Platonic experience. She says,

> It becomes clear that this 'practical' Platonism is not a matter of deliberate choice, of insufficient sophistication, or a lack of mathematical (or philosophical) maturity. It is because of the very nature of our imagination, because of our embodied way of thinking about even the most abstract ideas, that we spontaneously behave and feel like Platonists. (51)

Thus, to understand how people do mathematics, we must understand the cognitive origin of this experience, that is, we must work to understand the cognitive mechanisms that allow for the reification of abstract ideas. Most writing about this phenomenon has been of a philosophical nature, but what is needed now is a cognitive account of it.

Natural category representation

To develop a theory of how people might represent mathematical objects in the mind, we should look to theories of how they represent more familiar objects such as dogs or tomatoes. The

[3] What of this "mathematics as tools?" Certainly one need not understand the wiring or motor design for a compound miter saw to use it to build a window frame. A fortiori one need not understand the physics behind its electrically powered motor. With physical tools, one can go to the hardware store and purchase them ready-made. In contrast, mathematical tools are constructed in the mind, made up of neuronal connections that are shaped by culturally mediated physical, social, and mental experiences. Because the tools themselves need to be assembled by the learner, their inner workings do need to be made plain to the assembler. Furthermore, the very way that mathematical tools are used requires that their structure be understood so that it can properly be fitted to the problem at hand. The metaphor of mathematics as a tool is dangerous if we expect too much similarity with saws and hammers.

categories that people make to organize what they know about the objects and ideas that they encounter in everyday life are referred to as natural categories.

The classical view of category representation was that all categories were determined by definitions which described necessary and sufficient conditions for category membership. So in a sense, a category was thought to be a well-defined set, and objects were either in the set or not. But considerable evidence suggests that there are many natural categories that we do not represent this way.

For example, Hampton asked subjects to rate whether certain items belonged in certain categories (like whether the kitchen sink was a piece of furniture or a tomato was a vegetable). He found that items fell along a continuum so that almost all subjects agreed that certain items definitely belonged to the category and others definitely did not, but many items fell in a fuzzy range in between where subjects differed on whether they did or did not belong to the category. Interestingly, McCloskey and Glucksberg found that subjects who were asked to make similar judgments repeatedly on the same items were shown to change their minds about category membership much more frequently with these "borderline" cases (tomato/fruit) than on items that were consider to be more "typical" of the category (apple/fruit).

Murphy argues that such fuzziness is a necessary result of our need to understand a world with many fine gradations of things. So in a sense, the categories we form discretize a continuous world. As a result, natural categories have fuzzy boundaries. Yet while their boundaries are fuzzy, their "cores" are very crisp. Even within a given category, certain members are considered more or less typical. For example, sparrows are considered to be very typical birds whereas penguins are not. Rosch and Mervis articulated a theory that typicality is based on family resemblance, which is constituted by the following two conditions:

(1) If they have features common in their category, and
(2) They do not have features in common with other categories.

Murphy states that the typicality phenomenon reveals a prototype structure to our representation of categories. Perhaps it is this prototype structure of our *mental representations* of categories that inspired the original notions of Platonic ideals or Aristotle's essences. Thus, the very notion of essential dogness or roundness reflects some very deep structure of the human mind, and only indirectly reflects the deep structure of the world. Plato's work would then represent the original introspective reports that provide us clues about how the human mind understands the world.

Gelman suggests that essentialist thinking is an early cognitive bias and that young children are natural essentialist thinkers. Clearly it serves us well to see similarities between things we encounter in the world and to look for what hidden structures and relationships might account for these similarities. But must this tendency we have necessarily reflect the exact structure of the world around us? No—this capacity for essentialist thinking can be a useful strategy without representing the world completely faithfully. When we look at a sparrow on the fence, we do not perceive every aspect of the bird such as the heart pumping its blood or the electrical activity of its nervous system, but the visual representation we have of it is enough to help us know something about what it is and what it is likely to do. Dennett says:

> Aristotle had taught, and this was one bit of philosophy that had permeated the thinking of just about everybody… [that] all things—not just living things—had two kinds of properties; essential properties, without which they wouldn't be the *kind* of thing they were, and accidental properties, which were free to vary within the kind…. [Yet the geological record showed that] species were not eternal and immutable; they had evolved over time…. Even today, Darwin's overthrow of essentialism has not been completely assimilated…. The essentialist urge is still with us, and not always for bad reasons. Science does aspire to carve nature at the joints, and it often seems that we need essences, or something like essences, to do the job. (36-39)

If preschoolers are essentialist thinkers, is it really likely that this is the legacy of Aristotle rather than a feature of the human mind? Perhaps it is this push to understand the essences of the things in the world around us that we refine into mathematical thought, for in the world of mathematical objects things really do have essences. We seek to cut nature at the joints because of the structures of our minds. This is useful even if not always an identical reflection of reality. Piaget considered this possibility in his book Structuralism, where he states, "Must we, to make sense of the fact that we are in possession of knowledge of nature, allow for some sort of permanent tie, though not of identity, between "external" structures and the structures of "our" operations? If there is such a connection, we should find it in evidence in "intermediate" regions: biological structures and our own sensory-motor acts should exhibit it in its efficacy." (p. 39)

Pinker and Prince discuss category representation in the context of subclasses of regular and irregular verbs. From their analysis they conclude, "Both family resemblance categories and classical categories can be psychologically real and natural. Classical categories do not have to be the product of rules that are explicitly formulated and deliberately transmitted." (p. 234) Furthermore, Maddox and Ashby present findings from standard cognitive laboratory experiments, neuropsychological patient data[4], and neuroimaging studies in which they argue that there is strong evidence that human category learning is mediated by multiple, qualitatively distinct cognitive and neural systems. Not only do we use different neural circuits to think about different kinds of categories, but there is evidence that different individuals use different neural circuits for the same category or concept. Pinker and Prince conclude,

> The referents for many words, such as bird and grandmother, appear to have properties of both classical and family resemblance categories. How are these two systems to be reconciled?... [A likely] reconciliation is that people have parallel mental systems, one that records the correlational structure among sets of similar objects, and another that sets up systems of idealized laws. Often a category within one system will be linked to a counterpart in another system. (p. 254)

Understanding concepts is not the only instance where we integrate the information provided by distinct cognitive systems. We use our different senses to try to understand the objects and phenomena that we encounter in our everyday lives, and we integrate the information from the different systems so that we may have the most complete understanding of the things we encounter. For example, as we stand in the buffet line surveying the choices ahead of us, our

[4] Neuropsychological patient studies compare the performance on certain tasks between normal subjects and subjects with neurological deficits (e.g. amnesiacs).

impression of what is there is drastically different if it looks beautiful but smells rotten or looks dreary but smells heavenly. In this way, we expect that the different data we collect should coalesce into an understanding of the things we experience. We do not think that if we have both visual and olfactory information coming in that there are necessarily two stimuli, and if we eat with our eyes closed we may still know from the smell, taste and feel of what we put in our mouths that it is chocolate pudding or caviar or rotten oranges. Thus, we must have a mental representation of the food we eat that can be derived from the integration of different sensory inputs in different combinations. By analogy, we should not be surprised if we find that we use different mental representations to the same mathematical end. Fayol and Seron suggest that in the case of whole numbers and operations, this is exactly the case:

> Any model of number processing must account for the fact that educated adults are able to recognize and produce numbers in the Arabic code and the verbal code. It is therefore necessary to postulate that adults possess mental representations which are able to guide these recognition and production operations. However, researchers disagree as to the role and the format of these representations and their interrelations. It now appears to be well established that the symbolic representations are functionally independent and that they may undergo isolated impairment or be degraded in accordance with specific patterns in brain-damaged patients…. Evidence for the existence of some of these dissociations is also provided by cerebral imaging data which suggests that the verbal and Arabic codes are not processed in the same regions. (p. 8)

Here we see a connection to the mathematics education literature, and the ubiquitous references to teaching mathematical concepts using "multiple representations." We need to make the important distinction between external or public representations, such as graphs or equations, and internal or mental representations. In the mathematics education literature, multiple representations usually refer to the first of these—the external representations. But there is a curious silence about *what* is being represented. If anything, it will be said that they are representations of a concept. But what kind of concept is implicitly assumed? Typically, it is a family resemblance category and not a classical one. For example, consider the following statement taken from the NCTM standards: "In grades 3-5 all students should… develop understanding of fractions as parts of unit whole, as parts of a collection, as locations on number lines, and as division of whole numbers." (p. 148) The part-whole "interpretation" of a fraction requires one definition and the division of whole numbers "interpretation" requires another, and seeing the connection between these requires some real mathematical work (see, e.g. Beckmann). It is the justification of the connection between these that allows for the family resemblance structure for the category "interpretations of fractions," but each of them constitutes its own classical category in its own right. Furthermore, this category has fuzzy boundaries. Would we consider a ratio as an "interpretation" of a fraction? Looking at the range of approaches to the definition of a ratio, we see there would be disagreement about this (see Milgram, p. 219-254).

From the mathematician's perspective, mathematical categories are, in fact, described by definitions giving necessary and sufficient conditions for category membership. But they may arise from experiences with natural categories of objects in the world that are not classical. In mathematical categories, we strive to articulate definitions so that the boundaries are not fuzzy—in fact, if there is confusion about whether something belongs in a particular mathematical category, then we say that the category is not "well-defined" and we strive to clarify the definition. The purpose of devising definitions in mathematics is exactly so that we may communicate precisely about which objects we mean. It is not so much that there are "natural" mathematical categories with a priori definitions; definitions develop to isolate out the objects that interest us in some way, usually along some important structural lines. (For a better understanding of how this works, see the discussion of what a polyhedron is in Lakatos' Proofs and Refutations.) Of course, we inherit definitions from the mathematical culture in which we are embedded, and in this way mathematical definitions have an existence that is independent of us as individuals. Yet the definitions of mathematical objects and structures originate in our experiences with the world as we find it, including the natural world as well as the world of ideas that has developed before we enter it.

A case study: what are circles?

We come equipped with the ability to see patterns in both images and sound. Very young children easily recognize and name circles. Thus, even children recognize the category of round things, but the category of round objects has blurry edges:

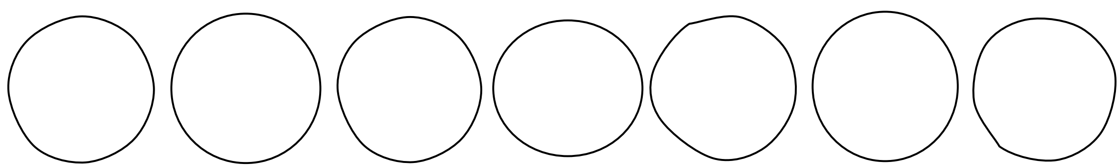

Figure 1: Which of these figures are round? Are any of them circles?

Given what we know of category representations, this category must also be equipped with a category prototype. This is no different than the category prototypes for dogs or tomatoes. Then why do we mathematicize circle shapes and not dog shapes? Note that roundness is a secondary property of an object and objects from many different categories can be round. We still recognize an oval plate as a plate. But would we recognize a snake-shaped dog? From the cognitive perspective, circles are the collection of things in the world that also happen to be round.

But for this proto-mathematical category to mature in to an actual mathematical category, we analyze what it means to be "round." We see there are subtleties involved—do we mean round like a plate or round like a ball?[5] We develop mutually exclusive categories for these different kinds of round things—circles and spheres—and then come up with necessary and sufficient conditions for an object to be in one of our new mathematical categories of round things. And in doing so, we create an object that exactly embodies each of these kinds of roundness; nothing more and nothing less. Once the structural definition is in place (the set of points equidistant from some chosen center in either the plane or 3-space, respectively) we can imagine a circle or sphere without any other properties—not a ripple in a pond or a soccer ball, but a new object that embodies the ideals of the respective categories of round things. So a mathematical circle is a round object with no properties other than those which follow logically from "roundness" defined in some appropriate way.

[5] Note that many books for pre-school children do not distinguish between these, and in fact show pictures of balls under the heading of "circle."

Recalling Russell's comment about sensible vs. ideal objects, we realize that there really is no perfect circle in this world as we have defined it. But through the power of thought experiments and careful reasoning, there it is, represented in the mind.

In everyday reasoning we usually do not need (or even want) to separate out roundness from the other properties of round objects we experience in the world. The kinds of inferences we make that connect past experiences with current observation serve us well: "The last time I saw a tree with round red things on it I could eat them." In mathematical reasoning, however, we do carefully separate out distinct characteristics. This is a hallmark of mathematical thinking, and we could not do it without our ability to learn and think about categories with a classical structure. In general, we can think of mathematics as the *science of essentialism*.

Yes, Virginia, there are mathematical objects

Many artificial categories that can be defined in this classical way that are used in psychology experiments are basically arbitrary, but for mathematical definitions that are inspired by phenomena in the world, they can be designed to capture in some essential way the salient features[6] of the category prototype. But even when they have concrete origins, they define a new abstract set of objects that have their own attributes and that are discovered through deduction. In Dubinsky's words, "Abstraction, in general, is the determination in a given situation, which may be a mathematical object, a procedure, or a combination of the two, of *what is essential* in a component of the situation. In mathematical abstraction, one generally expresses this essence in some systematic manner." (Emphasis added.) Mathematicians represent these essential features by defining an object or structure that embodies them, and it is these objects that are represented in the mind.

It is from this perspective that we might think of mathematical objects as the cognitive imprint of structural reality in the same way that visual images are the cognitive imprint of physical reality. Just as our mental representations of the objects we see are not equivalent to photographs of those objects, so our mental representations of structures and relationships in the world are not "snapshots" of those phenomena, but rather idealizations of them that are meaningful in our current web of knowledge.

Once mathematical concepts take on their own cognitive reality, we can look for attributes and structure in the world of these abstract objects and abstract again. The most basic example is numbers and operations. How else can we talk of the "properties of numbers" and "number systems?" After number systems reify, we can think of groups, which are defined to capture the particular operational relationships we see in number systems, (or the set of symmetries of a square, etc.).

[6] Even though the case we have described in detail is a category organized by a physical feature, the same argument can work with the appropriate kinds of functionally defined categories. For instance, if you have three gumball machines where the first gives two gumballs for a quarter, the second gives three and the third gives five, then we can see that these gumball machines all have in common a multiplicative rule and # gumballs = *a* * # quarters would be a mathematization of the category prototype.

To understand mathematical cognition, we need to understand how external representations (such as definitions, equations, and graphs) mediate the corresponding mental representations of mathematical categories (such as functions). Given the earlier discussion of the interaction of different cognitive systems that support the conceptualization of a single thing, we should expect these verbal, symbolic, and pictorial representations to coalesce into the perception of a single thing—a mathematical object.

Idealized objects require idealized reasoning

Situations that we encounter in everyday life and the relationships between objects we find there are complex. We rarely know all of the facts that are relevant to these situations, so we must make decisions with incomplete information. Because no knowledge of the real world is absolutely certain and is never complete, such decisions are probabilistic in nature. But the ideal objects that we study in mathematics are very different—because we can choose their properties and the situations that we wish to consider them in, we can and do reason about them in an idealized way.

LeFevre et al state that "a substantial portion of the research on mathematical cognition is focused on the processing of arithmetic problems, especially single-digit arithmetic." (p. 365) The heavy emphasis on numbers and arithmetic is reflected in the conceptualization of mathematics held by the cognitive science community. For instance, Gallistel and Gelman state, "Mathematics is a system for representing and reasoning about quantities, with arithmetic as its foundation." Of course, mathematics is about much more than quantity, and Gallistel and Gelman get closer when they say, "From a formalist perspective, arithmetic is a symbolic game, like tic-tac-toe. Its rules are more complicated, but not a great deal more complicated. Mathematics is the study of the properties of this game and of the systems that may be constructed on the foundation that it provides." While a complete and universally satisfactory definition of mathematics is difficult to formulate, we can say at least that it is a complex system of *reasoning* about both quantity and space and the abstract structures derived from them (whether one takes a formalist perspective or not). Clearly, our reasoning modules developed well before the advent of modern mathematics, so we should look to see what types of cognitive modules are being proposed that might reasonably be conscripted for mathematical purposes.

According to Markovits and Barrouillet, early theories on the study of the development of reasoning were eventually shown to be in contradiction with substantial empirical evidence. As a very simplistic summary, early theories of human reasoning essentially assumed that people reason logically (Piaget famously held this view), and ample evidence has shown that this is not true in many situations. Murphy states that the role of concepts is to help us organize our knowledge about the things we encounter in the world, and that the inferences (though not strictly "logical") that we make based on these conceptual structures are quite important.

Pinker and Prince state: "Bobick (1987), Shepard (1987) and Anderson (1990) have attempted to reverse-engineer human conceptual categories in terms of their function in people's dealings with the world. They have independently proposed that categories are useful because they allow us to infer objects' unobserved properties from their observed properties (see also Rosch 1978; Quine 1969). (p. 247)" On the other hand, logical reasoning requires us to separate out what we know from everyday experience and to look in a purely structural way at an argument. Our reasoning about many everyday situations is certainly not purely logical, yet we use our reasoning structures we come equipped with to do purely logical reasoning.

Given the likely modularity of the human mind, it is quite possible that there is more than one reasoning system rather than one single all-purpose system. For example, Sperber and Girotto discuss Cosmides' theory that people have "a 'social contract algorithm' specialized in reasoning about social contracts [that allow us] to detect parties that were not abiding by the terms of the contract." Sperber and Girotto, in turn, argue that, "People tend to be guided not by any form of reasoning but by context-sensitive intuitions of relevance (see also Evans, 1989). Intuitions of relevance are activated by the pragmatic mechanism involved in comprehending the task (just as they are by any comprehension process)." They go on to say that their approach "is in no way hostile to evolutionary psychology. In fact, the relevance-guided comprehension mechanism involved in the selection task is viewed as an evolved module specialized for the comprehension of communicative intentions, and more specifically as a sub-module of a Theory-of-Mind mechanism." Regardless of the details of future directions such theories may take, this discussion reminds us that we must be mindful to look for theories that provide a proper balance between general and domain-specific reasoning abilities as they are employed to do mathematics.

An example of the detailed kinds of theories we would need to understand mathematical reasoning is given in Gopnick et al (2004), where they have developed a theory of causal reasoning based on Bayes nets. Such work makes it clear that purely logical reasoning would actually not be sufficient for making basic inferences about certain kinds of everyday situations. If we were able to know with certainty all of the relevant facts about what causes an illness, for instance, then pure logic would be the most accurate way of reasoning. But in practice, what we know at any given time is tentative and fragmentary. Thus, everyday reasoning is likely to devise solutions to situations that optimize the likely outcomes given less-than-optimal information. Gopnik et al connect this abstract analysis to the particulars of likely information types:

> The epistemological difficulties involved in recovering causal information are just as grave as those involved in recovering spatial information. Hume (1739/1978) posed the most famous of these problems, that we only directly perceive correlations between events, not their causal relationship. How can we make reliably correct inferences about whether one event caused the other? Causation is not just correlation, or contiguity in space, or priority in time, or all three, but often enough, that is our [only] evidence.

Just as our sensory systems have evolved to help us represent the objects we encounter faithfully enough to be useful to us to navigate our surroundings, find food, avoid danger etc., our reasoning[7] abilities must have evolved to help us reliably see structures and relationships in the world—that is, to reason (relatively) reliably about what we perceive in the world. As Gopnik et. al. said when discussing cognitive representations of the causal relations among events, "Given the adaptive importance of causal knowledge, one might expect that a wide range of organisms would have a wide range of devices for recovering causal structure." Perhaps the mechanisms that allow for the mental representations of causal relationships in the world are analogous to our

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[7] Here I use the term **reasoning** in the broad sense it is meant in the cognitive psychology literature, which I take to be any process by which someone draws conclusions based on their current understanding of a given situation, and not in the narrower sense of "logical reasoning."

ability to see structures and relationships in the world that we can then study deliberately through more formalized mathematics.

How might this work? Gopnick's Bayes net models for representations of causal knowledge include both causal chains and associated probabilities. Logical arguments can be seen as a particular subset of such causal structures where the associated probabilities are always assumed to be zero or one. In this way, we can think of mathematical reasoning as an idealization of a certain kind of everyday reasoning.

In summary

While one must keep them in perspective, there is a benefit to be gained from trying to explain the introspective reports of mathematicians. In the words of Jacques Hadamard:

> Will it ever happen that mathematicians will know enough about the physiology of the brain, and neurophysiologists enough of mathematical discovery, for efficient cooperation to be possible?

We may find in the end that some modified version of Platonism holds; not that mathematical structures exist in some timeless or tenseless place, but as structures themselves they reflect or approximate to greater and lesser extent the structure of the natural world and natural systems. The cognitive reality of mathematical structures may, in fact, result from a refinement of natural concepts that are designed to give us a better understanding of the world, and in that way, have a "reality" that is objective—as much as any mental representation can be. Mathematics is the study of all possible ideal worlds, and in so far as any of these bear a resemblance to the actual world, we find the power of mathematics to describe what we see.

I would like to expand on the opening lines of G.H. Hardy's "A Mathematician's Apology":

> A Mathematician, like a painter or poet, is a maker of patterns.

Let us generalize this to encompass a broader definition of what mathematics is and who qualifies as a mathematician.

Painters and musicians create images and sounds that can either be representational or abstract—each utilizing critically important cognitive/sensory systems. Writers use language to create both fiction and non-fiction. The medium for mathematicians' creative works is structure and reasoning, and applied mathematicians' work is representational while pure mathematicians' work may not be. But just as we may find both intrinsic beauty as well as links between the abstract works of artists, musicians, and fiction writers to the world of human experience, so at times we find our pleasure in pure mathematics from both its intrinsic beauty and the serendipitous insights we gain from it into the world around us. It is the combination of the need to accurately represent the world with our sensory and reasoning systems and our capacity for creativity that gives rise to the paradox of the "unreasonable effectiveness of mathematics" to describe the world around us.

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