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Bharath Sriraman, Editor

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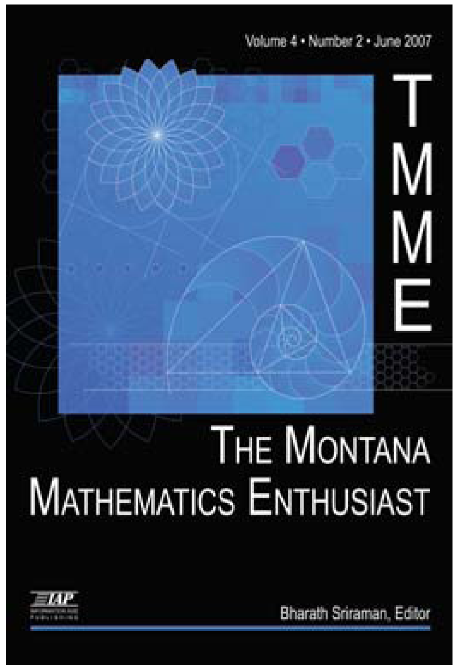

The Montana Mathematics Enthusiast is an eclectic internationally circulated peer reviewed journal which focuses on mathematics content, mathematics education research, innovation, interdisciplinary issues and pedagogy. The journal is published by Information Age Publishing and the electronic version is hosted jointly by IAP and the Department of Mathematical Sciences- The University of Montana, on behalf of MCTM. Articles appearing in the journal address issues related to mathematical thinking, teaching and learning at all levels. The focus includes specific mathematics content and advances in that area accessible to readers, as well as political, social and cultural issues related to mathematics education. Journal articles cover a wide spectrum of topics such as mathematics content (including advanced mathematics), educational studies related to mathematics, and reports of innovative pedagogical practices with the hope of stimulating dialogue between pre-service and practicing teachers, university educators and mathematicians. The journal is interested in research based articles as well as historical, philosophical, political, cross-cultural and systems perspectives on mathematics content, its teaching and learning.

The journal also includes a monograph series on special topics of interest to the community of readers The journal is accessed from 102+ countries and its readers include students of mathematics, future and practicing teachers, mathematicians, cognitive psychologists, critical theorists, mathematics/science educators, historians and philosophers of mathematics and science as well as those who pursue mathematics recreationally. The 40 member editorial board reflects this diversity. The journal exists to create a forum for argumentative and critical positions on mathematics education, and especially welcomes articles which challenge commonly held assumptions about the nature and purpose of mathematics and mathematics education. Reactions or commentaries on previously published articles are welcomed. Manuscripts are to be submitted in electronic format to the editor in APA style. The typical time period from submission to publication is 8-11 months. Please visit the journal website at http://www.montanamath.org/TMME

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MATHEMATICS FOR THE CLASSROOM

EDITORIAL: Expanding spheres of influence- the zenith, the nadir and everything in-between

Bharath Sriraman
The University of Montana

The Montana Mathematics Enthusiast has now entered its fifth year in existence. The changes the journal has been through since its inception has been healthy, especially it's ever increasing sphere of influence in the intellectual community. Being a meticulous keeper of journal records, I have watched with both awe and enthusiasm (no-pun intended) on the far reaches from the world the journal has been accessed. In terms of the number of countries the journal has been accessed from, we have reached a zenith at 102 give or take a few. Last week for the very first time the journal was accessed from Niger, Senegal, Chad and Algeria. While this was pleasing from a statistical (rarity) point of view, there was nevertheless a pang of regret that the journal has so far under-represented three regions of the world, namely Africa, South America and Southwest Asia – and this is our statistical nadir.

Several changes are evident. The editorial board has been expanded to include mathematics education researchers outside the Anglo-American domain of influence. The journal now exists in a print form, published by Information Age Publishing, in addition to the online version remaining free to the community. Presently efforts are being channeled at soliciting manuscripts from researchers in Southwest Asia, Africa and South America. I have received e-mails of interest from colleagues in Turkey and Iran interested in publishing their work in mathematics education in English. In addition, a focus issue on statistics education around the world has materialized as a result of the International Conference on Teaching Statistics (ICOTS-7) in Brazil. Vol.6 of the journal will include several papers from researchers in South America and Central Europe who participated in ICOTS-7. Another focus issue in the works is Non-European mathematics, which will include submissions from colleagues in the African continent.

At a recent conference in Germany, I received some very flattering compliments about the journal. I was asked if there was any particular issue that was representative of the true aims and scope of the journal. This issue [vol.5, no.1] represents the true spirit of the journal both in terms of its content and the geographic reach. The description of the journal states that it "exists to create a forum for argumentative and critical positions on mathematics education, and especially welcomes articles which challenge commonly held assumptions about the nature and purpose of mathematics and mathematics education." To this end, in this journal issue, I am proud to present to the readers an entire forum on the topic of **Ethics and Values in Mathematics Teaching and Learning**. The forum grew out of a provocative submission from Ted Eisenberg, which resulted in a critique from Renuka Vithal and insightful commentaries from Wolff-Michael Roth and Brian Greer. The process followed to handle the "sensitive" nature of Eisenberg's manuscript is commented on by Ted himself in his paper. Essentially an open peer review process was structured where the author was told the identities of the reviewers and vice versa.

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The product of this strategy is the stimulating forum presented in this issue. I would like to personally thank Ted Eisenberg, Wolff-Michael Roth, Brian Greer and Renuka Vithal for being willing participants in this project. In addition Alan Bishop and Kurt Stemhagen have contributed papers pertaining to the issue of ethics and values in mathematics education.

This journal issue represents all continents except (regrettably) South America. Murad Jurdak (Lebanon) contributed a paper entitled "The Action Map as a Tool for Assessing Situated Mathematical Problem Solving Performance" which is rooted in activity theory. The other feature articles include a paper from M.K. Akinsola (Botswana) on a study conducted with pre-service teachers on the psychology of problem solving. Both these papers are quantitative in nature and adequately portray the place of such methodologies in mathematics education. At the other end of the spectrum the issue has three theoretically based reflective papers. Kristin Umland reflects on the current state of research in the area of mathematical cognition. Yuichi Handa's article reflects on teaching a poorly conceived lesson in relation to the literature on comparative lesson study. The featured Montana article by David Davison and Johanna Mitchell analyzes philosophies of mathematics emerging from the ongoing "math" wars and reform efforts in the U.S.A. They analyze "How is Mathematics Education Philosophy Reflected in the Math Wars?"

Another special paper in this issue is a practical application of the thought experiment of Imre Lakatos to mathematics education classrooms. The paper from South Korea by Jaehoon Yim, Sanghun Song and Jiwon Kim on mathematically gifted elementary students' revisiting of Euler's polyhedron theorem explores how the constructions of mathematically gifted fifth and sixth grade students using Euler's polyhedron theorem compare to those of mathematicians as discussed by Lakatos in *Proofs and Refutations*. In their study, eleven mathematically gifted elementary school students were asked to justify the theorem, find counterexamples, and resolve conflicts between the theorem and counterexamples. This journal issue also includes two articles aimed at practitioners in the classroom on the geometric nature of proof by Sue Waring and Steve Humble (a.k.a Dr. Maths in the U.K).

I hope that the 166 journal pages that comprise this issue do not represent a zenith but indicate to the community that interest in mathematics education is present in the far reaches of the globe- and that the journal's philosophy of open access and a spirit of community has been instrumental in fostering interest in under-represented regions of the world in publishing their research. The journal will continue to work on its sense of agency in making the world of publishing a more equitable enterprise for under represented voices and issues in the ongoing mathematics education debates.

Flaws and Idiosyncrasies in Mathematicians: Food for the Classroom?

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Abstract
This paper raises an ethical question: should aspects of a mathematician's personality, political beliefs, physical handicaps, and the ironies surrounding their life be mentioned parenthetically or otherwise in our lessons? What about the political and social norms of the times in the countries in which they lived? There are no hard and fast guidelines on this other than to use good taste; but what is in good taste to one is often in bad taste to another. At the very least this paper presents tidbits of information and innuendo about mathematicians the reader might not know. But hopefully this paper will help the reader develop a personal stance on this issue.

0. Setting the Stage: Ethical Dilemmas

Ethical dilemmas are those gut-wrenching situations that are inescapable in life. They come in different degrees of magnitude and severity–but what they have in common is that they push us to the core of our personal moral beliefs. Each of us can easily think of such situations and decisions we have made in this realm. Some decisions we have made with the authority of certainty; others that to this day we don't know if we acted correctly; and still others that we feel uncomfortable in discussing. But making ethical decisions is a part of life–and sometimes they are not easy to make. Within the university world the arena of problems and situations for which ethical decisions have to be made seems to be unlimited in scope and number. Animal research, stem cell research, genetic engineering, affirmative action admission policies are of one magnitude; accepting grants from individuals and foundations with tainted histories, grants with strings attached, researching sensitive topics such as terrorist profiling, ethnic profiling, etc., are of another magnitude. Universities often have ethics committees to oversee such dilemmas. Ethical dilemmas exist on large, small, and personal scales–even in the mathematics classroom.

1. Introduction

Following are vignettes that reveal flaws in character and idiosyncratic behavior within some of the best-known individuals in the annals of mathematics. They focus on the mathematician's foibles, but the stories also give us a glimpse into the political atmospheres of the times in which they lived. On the surface, mentioning them in the classroom adds a bit of spice to our lessons,

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but in adding that spice are we not tacitly endorsing gossip and stereotypes, and taking on the role of being a bully by smirking at those with paranoia and differences, be them real or imagined? Should such peripheral material about the lives of mathematicians be included in our lessons? This paper discusses this question on both an individual and larger scale.

2. A Sampling of Vignettes

2.1 Girolamo Cardano (1501-1576) is famous for the formulas that bear his name; formulas that enable us to solve cubic polynomial equations (of the form $ax^3 + bx^2 + cx + d = 0$ where a, b, c, and d are integers) in terms of their coefficients. (Just as it is possible to construct formulas to solve quadratic equations in terms of their coefficients, so it is possible to construct formulas for cubic equations. Actually, the coefficients need not be restricted to the integers; the formulas Cardano built also work if the coefficients are complex numbers.) Cardano is also famous for fundamental work in probability theory, and he is considered to be one of the first to have systematically studied games of chance. But Cardano and his associates stole the formulas for solving the cubic equations from a man called Tartaglia (the stutterer), by duping him into revealing them after making a solemn pledge to him that they would be shown to no others. In 1545 Cardano published the formulas in his book the Ars magna, and as you might have guessed, there is no mention of Tartaglia's name. This seems to be one of the first documented cases of intellectual thievery in mathematics. Cardano has been called one of the most wicked and eccentric men in the history of mathematics, for it is said that once in a fit rage he cut off the ears of his younger son; it is also said that he died by his own hand to fulfill an earlier self-calculated prediction of his death date, least it be said that he made mistakes in his calculations! (Ball (1960), Eves (1964)).

The above revelations usually generate a few smiles from students, but in fairness it should also be mentioned that we don't know if any of the above is true. Orestin Ore, like Ball and Eves, also an accomplished mathematician in his time, claims that Cardano died peacefully in his sleep, and that he is unfairly portrayed as a wicked man rather than as one with idiosyncratic behavior. Although Ore does not deny the story of Cardano stealing the formulas for solving cubic equations, he paints Cardano as an eccentric genius who was more like Dennis the Menace, than Ivan the Terrible (Ore (1953)). But with respect to numbers, more mathematical historians seem to line up with Ball and Eves than with Ore (see for example Burton (1991), Cajori (1980), Katz (1992), and Stillwell (1989))[2].

2.2 Isaac Newton (1656-1742) is a name that is known in most households throughout the educated world. Newton is famous for the development of the calculus, and for many of the laws and notions in school and undergraduate-level physics. Recently however, it has been hypothesized that Newton suffered from Asperger's disease, which is a form of autism (Mirsky,

[2] Here is how Ore (op. cit.) described the conflicting impressions: "Cardano's character was an enigma to many of his contemporaries and it must be admitted that it has remained so to most of his biographers through the centuries which have passed. He is a man who has been praised and vilified; by some he has been called a genius, by others a poseur, some have presented him as a benefactor to mankind, others frankly believe him to be an evil spirit, indeed, a monster. One should expect that the analysis of his works would eventually bring a satisfactory clarification, but unfortunately his books can give some support to almost any view."

(2005)). Newton was emotionally frigid, actively discouraged human contact, was known to laugh only once in his life (when a colleague asked what use Euclid could be), and died bragging that he was a virgin and thus uncontaminated (Green (2005)). Newton's name is attached to the saying: If I have seen a little further than others it is because I have stood on the shoulders of giants. But that saying has been found in written form nearly 500 years before Newton was born–he was certainly not its originator, although he is probably the most famous person to have ever said it. The phrase can be seen, for example, etched into the widows of the Chartres Cathedral outside of Paris, that was erected in the year 1195 (Pappus, (1999)).

Newton was secretive and his behavior irascible; he had difficulty handling criticism and he carried grudges to the extreme. It seems that he was a very difficult person to be around. Some historians attribute Newton's peculiar and exasperating behavior to the fact that he was also an alchemist, and that he often handled mercury which is known to affect behavior in ways similar to the descriptions that we have on him; in other words, his Scrooge-type personality was brought on by himself and perhaps caused the nervous breakdown that he suffered at the age of 37 (Johnson and Wolbarsht (1979)). Undoubtedly Newton was a genius; but he seems to have been a genius with serious social problems[3].

2.3. Albert Einstein (1879-1955) is also a household name; but there seems to be some question as to the role his first wife Mileva Maric played in his landmark papers on the theory of general relativity. They met in their student days and they married in 1903. In letters released in 1986 by Einstein's grandson there are statements that clearly show that Einstein and Maric corresponded on scientific topics during their student days, and also after they were married in that they lived apart for a few years. In the letters are statements referring to our work, our theories, and our investigations. Moreover, an editorial assistant claims that the original landmark papers of 1905 carried the names of two authors on them: Einstein and Maric (Pais, (1992)). But the original papers have long been lost, and in Einstein's divorce settlement from Maric it states that if he was ever awarded a Nobel Prize, the prize money should be given to her; he was awarded the Nobel Prize in 1921, and the money went to Maric (Isaacson, (2007)). There is quite a bit of convincing evidence that Einstein was dyslexic. West (1991) and Whitrow (1967) document quite a few of the common signs and specifically discuss Einstein's propensity toward them; he had poor verbal memory, he was weak with foreign languages, his early childhood shows learning problems in school, etc. Mirsky (2003, 2005) goes even further by strongly suggesting that Einstein, like Newton, suffered from Asperger's disease. In building his case he sites the work of a researcher at Cambridge who claims that the common markers of Asperger were there: obsessive focus on a subject of interest, poor relationships, communication difficulties, etc. As in the case of Cardano, no one knows if any of the above is true, even though hundreds of papers have been written that speculate on his relationship with Maric, and his alleged dyslexia. At the very least, there are gray clouds over Einstein's image, and most students, as well as the general public, are unaware of them[4].

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[3] History has often attributed theorems, proofs, ideas and statements to individuals who had nothing to do with them. See Ezra Brown's article "Whodunit?" in Math Horizons (April 2007, p. 24.) and on a broader landscape: The Dictionary of Mis-information by T. Burnam, (Crowell Publishing Co., NYC, 1975) and Serendipity: Accidental Discoveries in Science by R. Roberts (Wiley Sons, NY, 1989).

[4] There are so many entertaining stories about Einstein that one could tell them all day. But one or them that I like is when Einstein, who had the popularity of a rock-star, advocated civil disobedience as a

2.4. Kurt Gödel (1906-1978) shook the mathematical world to its very foundations in 1930 by proving that in every sufficiently complex axiomatic system, it is always possible to construct a statement that cannot be proved true in the system, nor can it be proved to be false. In other words, there will always be open questions. It was once thought that Fermat's Last Theorem, that $a^n + b^n = c^n$ has no non-trivial solution for integer values n greater than two, was an example of this; but in 1995 Andrew Wiles showed that Fermat's Last Theorem was indeed true. Many other easily stated problems and questions are now given as examples of being intractable in the spirit of Gödel. E.g., are there infinitely many twin primes (prime numbers that differ by 2); Goldbach's conjecture that every even number can be written as the sum of two odd primes; and the rule of three. (If n is even, then send n to n/2; and if n is odd, then send n to (3n + 1)/2. The conjecture says that the above rules will eventually send every positive integer to one. E.g.,5 → 16 → 8 → 4 → 2 → 1, but so far, no one has proved it.) Much has been written about Gödel's paranoia, but one of them did him in; he thought someone was trying to poison him and so his way of handling this was to stop eating; he did, and about two weeks later he died from voluntary starvation (Krantz (1990); Goldstein (2005)).

2.5. Andre Bloch's (1893-1948) name is encountered in many different fields of mathematics. He did fundamental work in the areas of function theory, number theory, geometry, algebraic equations, and kinematics. But he made his discoveries working in a world far removed from normality. In 1917 he was having a quiet dinner with his brother, aunt and uncle in their family apartment; for some incomprehensible reason, he rose during the meal and murdered each of them! He then calmly went into the street, stopped the first police officer he saw, and confessed what he had done. Bloch spent the next 31 years of his life in a psychiatric hospital, pushing back the frontiers of mathematics (Cartan & Ferrand (1988)).

2.6. Ludwig Bieberbach (1886- 1982) and Oswald Teichmüller (1913-1943). The Bieberbach Conjecture was concerned with certain transformations of the unit disc into other planar regions. Such transformations are called univalent transformations or univalent functions; they distort shapes but they preserve angles between curves. Univalence means that two different points are never transformed into the same point. A point on the unit disc can be represented by a complex number z, and a univalent function f transforms z into f(z).This function has a Taylor polynomial expansion $f(z) = z + a_2z^2 + a_3z^3 + \cdot\cdot\cdot$, where the coefficients a_2, a_3, a_4, etc. are fixed complex numbers.

Bieberbach conjectured that for all such functions f, the Taylor polynomial is such that for each coefficient a_n, $|a_n|$ is not greater than n. He posed this conjecture in 1916 and it remained an open question until 1984 when Louis de Branges proved it to be true. It is widely acknowledged that Bierberbach played a major role in the development of univalent function theory. He also

legitimate form of protest. The Chicago Daily Tribune wrote: It is always astonishing to find that a man of great intellectual power in some directions is a simpleton or even a jackass in others (Isaacson (p 528), op. cit.). In this spirit, Einstein loved sailing; he had a small sailboat and he often went out alone. There are many stories of him getting lost while sailing, on getting caught in rough waters at the bay of a storm, etc.– and help had to be sent to rescue him. But although Einstein was an avid sailor, he had never learned to swim–and he never wore a life-jacket–even when he was sailing alone. Is this rational behavior from one of the world's smartest men?

played a major role in spreading hatred toward Jews and in helping German Universities take gigantic leaps into the world of bigotry and prejudice. His philosophy simply put went like this: individuals of different races should not mix; Jewish professors should not teach German students, and German professors should not teach Jewish students. Even after the war Bieberbach denied that the Holocaust had existed, and that Germany had committed atrocities against humanity during it. The scientific world went crazy over Bieberbach's notions, but history shows that he succeeded in getting German academics to adopt the notion of "Aryan"[5] mathematics, a society that was void of Jews. There is a quote attributed to Einstein showing how deeply his mistrust and suspicion ran with respect to the Germany people at that time. If relativity is proved right, the Germans will call me a German, the Swiss will call me a Swiss citizen, and the French will call me a great scientist. If relativity is proved wrong, the French will call me a Swiss, the Swiss will call me a German, and the Germans will call me a Jew (Schwartz and McGuinness (1979); a deep analysis of Einstein's feelings in this realm can be found in Isaacson (2007)).

Teichmüller, on the other hand, is remembered for original contributions to the theory of Riemann surfaces, and there is a theory dealing with the moduli of Riemann surfaces that goes by his name (Boos-Bavnbek, (1995)). But Teichmüller too was unashamedly an anti-Semite. When Teichmüller was 20 years old he headed a mob of brown-shirts that refused to let Landau[6] enter an auditorium at Göttingen to teach a calculus course; he told Landau that the students did not want to take instruction from a Jew. There are pictures showing Teichmüller lecturing his own students dressed in full Nazi regalia (Shields (1988), Mehrtens,(1987, 1989)). Teichmüller was instrumental in not only destroying the great mathematician Landau, but he also played a major role in destroying the great mathematical center at Göttingen (Chowdhury, (1995)). However in recent years, a movement seems to be cropping up to whitewash his image, and those of his kind. And how is this done? Simply by ignoring the Nazi aspects of their lives, by pushing their mathematical accomplishments, and by staying mum about their beliefs of Aryan/Germanic superiority and of their attitudes and behaviors towards those of other origins (Boss-Bavnbek, (1995)). This is all part of a dark chapter in the history of mathematics for it wasn't just one person going crazy, much of Europe was going crazy at the same time. Worse, this anti-Semitism seems not to have been confined only to Europe. Evidence is surfacing that it ran deep in the United States too, but in more subtle forms. It appears that leading mathematicians in the States were blackballing Jewish immigrants fleeing Nazi Germany from obtaining employment in major universities. Specifically, George David Birkhoff at Harvard, one of most influential

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[5] Editorial Note: The word "Aryan" as appropriated and abused by the Nazi's to distinguish/label the Germanic race as the "master" race and perpetuate horrific atrocities on the Jewish people has a benign existence in the Eastern world for over 3500 years. In the domain of philology as well as contemporary linguistics, Indo-Aryan is a branch of the Indo Iranian languages. In Sanskrit and Avestan (old Persian), the word *Arya* which has been in existence for over 3500 years <u>is not</u> a racial designation but a term of respect, meaning "honorable" or "noble".

[6] Edmund Landau was a child prodigy who completed his doctorate in number theory (under the supervision of Georg Frobenius at the University of Berlin) at the age of 22; two years later he completed his "habilitation" in the area of analytic number theory; he was mostly interested in the distribution of prime numbers. He succeeded Hermann Minkowski at Gottingen and he was known as being both an outstanding teacher and an outstanding researcher. But after his confrontation with Teichmüller, he never again lectured in Germany.

mathematicians in the States at that time, led a campaign to block Jewish mathematicians from major universities. Names, charges, and counter-charges can be seen in MacLane (1994)), but the story seems not to have ended in the 1940's. This type of blackballing seems to have continued into the late 1980's and 90's against the Jewish-Russian mathematicians fleeing the Soviet Union, with similar charges and countercharges being thrown by those on each side of the issue (Sdravksovska, (1989), Birman, (1992), Axler (1992), or simply type "anti-Semitism and mathematicians" into Google, or some other search engine).

2.7. Alan Turing (1912- 1954). Without a doubt Alan Turing helped England and its allies win World War-II. Turing was the head of a team that cracked the Enigma code that led to Hitler's defeat, and Turning machines are now studied as part of the mathematics curriculum in most universities throughout the world. Books and plays have been written about his genius, but there is also a dark side to his story. Turing was a homosexual and one night in 1952, he picked up a young man on the street and took him home to bed. Not long afterwards Turing's house was burgled, and he suspected the young man. Turing went to the police with his suspicions and in telling the story he revealed to them that he was a homosexual. But homosexuality was against the law in England in those days and the police arrested him on the spot. He was sent to trial and he was convicted of England's indecency act; he was forced to undergo hormone treatment that made him obese and impotent. He became severely depressed and on July 7, 1954 he went to his bedroom carrying an apple and a jar of cyanide solution. He was found dead the next day (Davis (1987), Singh,(1999), Whitemore (1991)). Some say that the icon of Apple Computers is a tribute to Turing and his genius. Many honors carrying his name have been recently established. The Turing Prize is often considered the Nobel Prize of computing, and many universities around the world have buildings and rooms named after him.

3. Enlarging the Lens

The above list of vignettes could easily be expanded, but I believe that the point is clearly enough stated: do such stories belong in the classroom? Should students know that Euler lost the sight in his right eye at the age of 30; that he lost the sight in his other eye at the age of 63, and yet completely blind, he continued to produce an average of one mathematical paper per week (Dunham, (1999))? What about that Wronski ended up insane (Agnew (1960))[7]; that the famous John Horton Conway often lectures barefooted; that Einstein often wore shoes without first putting on socks, that Ron Graham (former president of the AMS) often does a handstand in the middle of a lecture–or starts juggling oranges and other objects at will during his lectures? What about the fact that Erdös had no home, and that he simply roamed the world looking for individuals with whom to do mathematics that were willing to take him in? What about the controversy between Erdös and Atle Selberg (a permanent member in the Institute of Advanced Study at Princeton) over the ownership of a theorem surrounding the Riemann Hypothesis; two

[7] Josef Höené Wronski (1778-1853) was named Josef Höené at birth, but he took the name of Wronsk after is marriage in 1810, and from that point on when writing papers, he used the name of Höené Wronski without a first name. He is mostly known for his work in the philosophy of mathematics, although he also did some fundamental work in differential equations. The Wronskian of n functions u_1, $u_2, \cdots, u_n$ is the determinate of order n which has these functions as the elements of the first row, and their k^{th} derivative as the elements of the $(k + 1)^{st}$ row $(k = 1, 2, \cdots, n - 1)$. The functions are linearly dependent on an interval, if and only if, the Wronskian is zero on that interval.

men in the mathematical community who are known for their modesty, who didn't have a vain bone in their bodies, and who were academically generous to a fault–how could a controversy crop up between them; but it did (du Sautoy,(2003)). What about John Nash (a mathematician who won a Nobel Prize in 1994 for work he did in game theory) being schizophrenic, and that one of his sons, who also has a doctorate in mathematics, is schizophrenic too? Should students hear such things from us in our lessons?

Every discipline has such tales and tidbits of gossip and intrigue. Should teachers bring up the physical infirmities of Stephen Hawking (of black hole fame in physics), or the irony that Beverly Sills, one of the foremost divas of the Metropolitan (NYC) operatic stage, has children who are deaf and who have never heard a single note their mother has sung; What about the fact that Beethoven was deaf; that it is said that Paul McCartney (the former Beatle who recently wrote an opera) cannot read music, or that Mozart seems to have been a musical genius through whom some say God spoke, but who was a scoundrel in real life. Every field, yes every field, has such stories.
Here is how E.T. Bell addressed this topic in his classic book: Men of Mathematics; Another characteristic calls for mention here...several have asked that I address the sex lives of great mathematicians. In particular these inquirers wish to know how many of the great mathematicians have been perverts–a somewhat indelicate question, possibly, but legitimate enough to merit a serious answer in these times of preoccupation with such topics. His answer was: None. (Bell continues on saying that the majority of mathematicians were happily married and that they brought up their children in civilized and intelligent ways (Bell, (1965)). Bell's answer seems to be flippant for the point of expediency; he simply didn't want to address such questions, taking the stance that mathematicians are on average, no different than anyone else–except of course when it comes to mathematics. But if Bell is correct, the above vignettes show there are more than a few anomalies around. The question is, should such aspects of their lives be mentioned in the classroom? Knowing that there were laws in England in the 1950's that forced the police to arrest Turing is, I believe, important for it shows how English society at that time looked at homosexuality. Admittedly Turing brazenly flaunted his homosexuality, but still, it was English law that drove him over the brink; a man who assuredly helped England win the war and on whom the English government had showered much praise and appreciation. Does knowing about Turning's homosexuality detract from our appreciation of his mathematics, or does it add a subtle dimension to it?

Anglin (1992) claims that there are many ways to present mathematics and its history. He approaches this topic by asking a series of questions. Several of them are: Should a history of mathematics revolve around individuals and their private lives? Or should a history of mathematics be organized in terms of nations or races? Or should a history of mathematics be told in terms of chronological periods? Whichever way is chosen for presenting the history really isn't of much interest to us, because we are simply asking if such things as Newton's alleged virginity should be mentioned in the classroom, or Bloch's murdering his family, or Turing's homosexuality, or the fact that many mathematicians seem to have spent time in mental institutions? Should we only mention the positive? E.g., that Euler did wonderful mathematics even when he was completely blind; that Solomon Lefschetz (chairman of mathematics at Princeton) lost both of his hands in a chemistry experiment in his youth; that ended his hopes of becoming a chemist–so he became a brilliant mathematician instead; or that Norbert Wiener was terribly insecure in most areas of life, but that he was gigantically successful as a mathematician? I have posed these questions to colleagues and I have received responses covering the entire range

from an emphatic and emotionally delivered no, we should only address their mathematics and not the stories around them, to an emphatic yes, the stories make the mathematicians all the more human.

The general consensus of opinion is that teachers should: i) know the above and other similar stories but ii) only present to students those with which they themselves feel comfortable in discussing. But it seems that even here–with this practical guide of doing that which one feels comfortable with–there are problems. Why? Because by ignoring the distasteful, history is going to be distorted, and that doesn't seem to be right. Let me explain.

4. Discussion: A Personal Bottom Line

I have argued that each of us should be aware of the above vignettes and of the many other similar stories that are easily accessible to us in the literature– and that we should use our discretion in presenting them to students. Mentioning that Einstein often went without socks, and when he did wear them he sometimes put them over his shoes, that he liked to study barefooted with his feet in cold water because he thought it helped him concentrate, that he was often forgetful to the point of being comical, etc., is fine with me. It is also fine with me to mention his alleged dyslexia, for I believe that his accomplishments become all the more astonishing, and that it drives home to students the fact that dyslexia and intelligence are two distinct and independent phenomena; as are physical infirmities and intelligence, as well as sexual orientation, and political beliefs and intelligence, etc. But I admit that although there are many things I do not feel comfortable in discussing, they cannot be left unsaid. Let me start with the Nazi business of Bieberbach and Teichmüller by relating a story about the music of Richard Wagner in Israel.

In Israel, my country, the music of Richard Wager is not played in public; it is not played on the radio and it is not played in public concerts. As far as the older Israeli public is concerned, Wagner did not exist–or at least they wish he hadn't. Why? Because Wagner was a rabid anti-Semite; Hitler claimed that Wagner's music inspired him, and Jews were marched to their death in the concentration camps during WW-II listening to Wagner's music being blasted over loudspeakers. That was more than 60 years ago, and still his music is boycotted in Israel, at least in public. There have been many conductors who have argued that it is time to bury the past–and they have scheduled Wagner into their programs –but fisticuffs have often broken out within the audience between those in favor and those opposed to listening to Wagner, and fisticuffs have even broken out between the members of the orchestra during rehearsals! Even when it is well advertised that Wagner will be played and that some patrons might want to skip that particular concert, well organized demonstrations meet the concert-goers outside the concert hall, and perpetrators are often planted in the concert halls who are bent to do their utmost to stop the concert before the first note of Wagner can be heard. But within the academic musical world in Israel, Wagner most certainly does exist; his music is studied, and so is his goal of trying to unite drama, art, and music into an art form larger than its constituent components. Wagner's political beliefs and the inspiration his music gave Hitler are not generally discussed in the music academies, although most pupils in this country are well aware of them. But should the same turning of a blind eye be done when speaking about Bieberbach and Teichmüller and their mathematics? On a general level, I don't know the answer to this, and I feel very uncomfortable in discussing this nasty business. On one hand, I want to take the easy way out and simply ignore it all; but I know that these men hurt many individuals, and the evil they did to them should not be whitewashed. Smart people in one domain sometimes do stupid things in other domains.

Bieberbach denied that the Holocaust existed. In today's Germany he would be brought to trial for speaking such beliefs (Haaretz, 2006). Birkhoff has been accused of anti-Semitism and so has Shafarevich. I think it is wrong to ignore their activities in this domain, and to let history portray them, through omission, as being more humane and understanding then they were. So in the classroom I have taken the stance that one's mathematics should not be divorced from other aspects of their life, or from the political and social atmospheres of the times in which they lived. I don't dwell on it, but if a person was a murderer, or a scoundrel, or an anti-Semite, or if he overcame some mental or physical malady, I believe that it should all be mentioned to our students. Knowing such stories will not only enrich our lessons, but they will hopefully influence our students to emulate the good and to despise the bad. Life means interacting with others, and this applies to mathematicians too; knowing the flaws in character and the strengths and weaknesses of the individuals whose mathematics we teach, can only help our students to think and reflect, and that is what our profession is all about–moreover, speaking about such things seems to be the right thing to do. And hard as it is to accept, there seems to be common denominator between the individuals mentioned above. Each of them was passionate and fiercely independent about what they believed in; each stubborn to a fault; each was a work-a-holic; and each made an impact on the lives of others in their generation, and generations to come. Do such elaborations belong in the classroom? You bet they do.

Bibliography

Anglin, W.S. Mathematics and History, *The Mathematical Intelligencer*, 14(4), 1992, pp. 6-12.

Agnew, Ralph. *Differential Equations*, McGraw-Hill, 1960, p.169.

Axler, Sheldon. Sheldon Axler Replies (to statements made by Shafarevich in Zdravkovska cited below), *The Mathematical Intelligencer*, 14(2), 1992, pp. 3-4.

Ball, W. W. Rouse. *A Short Account of the History of Mathematics*, Dover, 1960, pp. 217-21.
Bell. E.T. *Men of Mathematics*, Simon and Schuster, 1965, p. 9.

Birman, Joan. On Shafarevich's Essay Russophobia in Zdravkovska cited below, *The Mathematical Intelligencer*, 14(2), 1992, p. 4.

Booss-Bavnbek, B. Remembering Teichmüller, *The Mathematical Intelligencer*, 17(2), 1995, pp. 15-20.

Burton, David. *Burton's History of Mathematics: An Introduction*, Wm. C. Brown Publishers, 1991.

Cartan, Henri & Ferrand, Jacqueline. The Case of Andre Bloch, *The Mathematical Intelligencer*, 10(1), 1988, pp. 23-26.

Cajori, Florian. *A History of Mathematics*, Chelsea Publishing Company, 1980.

Chowdhury, M.R. Landau and Teichmüller, *The Mathematical Intelligencer*, 17(2), 1995, pp.12-20.

Davis. Martin. Mathematical Logic and the Origin of Modern Computers, in *Studies in the History of Mathematics* by Ester Philips, ed. pp. 137-165.

Dunham, William. *Euler: The Master of Us All*. MAA Publication, 1999, pp. xxiii-xxvi.

Du Sauty, Marcus. *The Music of the Primes*, Perennial, 2003.

Eves, Howard. *Introduction to the History of Mathematics*, Holt, Rinehart and Winston, Inc., 1964, pp. 220-224.

Goldstein, Rebecca. *Incompleteness: The Proof and Paradox of Kurt Gödel*, W.W. Norton, 2005.

Green, David. Quoted in Mirsky, (cited below.)

Haaretz, *The International Tribune*, 87(26590), Nov. 15, 2006, p. 4.

Isaacson, Walter. *Einstein: His Life and Universe*, Simon and Schuster, 2007.

Johnson, L.W. & Wolbarsht, M. L. Mercury Poisoning: A Probably Cause of Isaac Newton's Physical and Mental Ills, *Notes and Records of the Royal Society of London*, 34(1), July,1979.

Katz, Victor J. *A History of Mathematics: An Introduction*. Harper Collins College Publishers, 1992.

Krantz, Steven G. Mathematical Anecdotes, *The Mathematical Intelligencer*, 12(4), 1990.

MacLane, Saunders. Jobs in the 1930's and the Views of George D. Birkhoff, *The Mathematical Intelligencer*, 16(3), 1994, pp. 9-10.

Mehrtens, Herbert. Ludwig Bieberbach and Deutsche Mathematik in *Studies in the History of Mathematics*, (Vol 26; MAA publication, Esther R. Phillips, editor), pp. 195-241.

__________________. The Gleichschaltung of Mathematical Societies in Nazi German, *The Mathematical Intelligencer*, 1(3), 1989, pp. 48-6.

Mirsky, Steve. Antigravity, *Scientific American*, (August 2003, June 2005, and October 2005), pp. 94, 86, & 82 (respectively)).

Ore, Oystein. *Cardano: the Gambling Scholar*, Dover, 1953, (chapters 1,2, & 3).

Pais, Abraham. *Einstein Lived Here*, Clarendon Press, Oxford, 1994, pp.15-16.

Pappas, Theoni. *Newton Was No Sweet Cookie, Mathematical Scandals*, Wide World Publishing/TETRA, 1999, pp. 86-95.

Schwartz, Joseph & McGuinness, Michael. *Einstein for Beginners*, Pantheon Books, 1979, p.3.

Shields, Allen. Klein and Bieberbach Mathematics, Race and Biology, *The Mathematical Intelligencer*, 11(3), 1988, pp. 7-11.

Singh, Simon. *Cracking the Enigma in The Code Book: The Secret History of Codes and Code Breaking,* Fourth Estate Publishing, 1999, pp. 143-189.

Stillwell, John. *Mathematics and Its History*, Springer-Verlag, 1989.

West, Thomas. *In the Mind's Eye*, Prometheus Books, 1991, pp. 118-129.

Whitemore, Hugh. Writing about Alan Turing, *The Mathematical Intelligencer*, 13(4), 1991, pp. 26-30.

Whitrow, G.J. *Einstein:The Man And His Achievement*, Dover Publications, 1967.

Zdravksovska, Smilka. Listening to Igor Rostislavovich Safarevich, *The Mathematical Intelligencer*, 11(2), 1989, pp. 16-28.

Historical Tidbits, the Shoah, and the Teaching of Mathematics

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Abstract
In this extended essay, I use cultural-historical activity theory to look at the questions Theodore Eisenberg raises about the inclusion of historical facts, both historical tidbits and ethically questionable tendencies and horrific actions (the Shoah), in the teaching of mathematics. I conclude by suggesting that the ultimate answer has to be one that involves a decision, which means that an answer cannot be provided a priori or be *determined* by any antecedent. *Deciding* to include this or that in a mathematical curriculum is an *ethical act*.

> Pardonner le pardonnable, le véniel, l'excusable, ce qu'on peut toujours pardonner, ce n'est pas pardonner. [To forgive the forgivable, the venial, the excusable, that which one can always forgive, is not forgiving at all.] (Derrida, 2005, p. 32)

In his article "Flaws and Idiosyncrasies in Mathematicians: Food for the Classroom," Theodore Eisenberg raises an interesting issue: Should mathematics teaching merely focus on mathematical concepts or should mathematics students (at school and university levels) also know about the lives of the mathematicians who first articulated a theorem or solution, the cultural context within which some mathematicians have worked (Nazism, Russian dictatorship), etc.? Some of the examples he features are those of Einstein wearing shoes without socks—I never wear socks, and always sandals rather than shoes, even during visits to central Canadian cities in the winter—and Alan Turing, often considered to be the father of computer science, being homosexual.

Eisenberg raises other issues that are more serious, concerning, for example, the appropriation and appreciation of the products of labor by anti-Semitic scholars and artists. He has not addressed another situation, that of anti-Semitic philosophers or philosophers who did not declare opposition to the Nazi regime, such as Martin Heidegger. For me, therefore, there are two levels of questions. First, should we use and enjoy the productions of these people—Heidegger's philosophy, Wagner's music, the findings and productions by Nazi scientists and mathematics? Some individuals do not appear to mind, as we can see from the fact that the novel *Seven Years in Tibet*, written by the Heinrich Harrer, a member of Hitler's elite SS, recently was turned into a film for a second time. Here, producers, participants in the making of the film, and audiences willingly contribute to the perpetuation of a part of Harrer's autobiography. More so, the author's subsequent autobiography *Beyond Seven Years in Tibet, My Life Before, During and After* has been released in 2006.

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What should we do about the findings of psychological studies that clearly would no longer pass any human research ethics board? One such study was conducted by Stanley Milgram. In this study, ordinary people began to "punish" other ordinary people with electrical shocks of increasing intensity—and despite increasing expressions of pain—obeyed the experimentalist to punish their non-compliant victims even harder. Many of the participating subjects left the experiment traumatized because they had found themselves committing horrendous violence—they did not know that their "victims" were actually faked—similar to the once committed by Nazi torturers. Nowadays, Milgram's study probably would not pass the human research ethics requirements on "minimal risk," such as those that the Canadian National Council on Ethics in Human Research, representing the Tricouncil (which units the three councils funding research in (a) social sciences and humanities [SSHRC], (b) science and engineering [NSERC], and (c) health research [CIHR]), adheres to. And excuses such as "I was simply following orders" no longer will cut muster.

And what should we do about the studies Nazi doctors did on hypothermia using concentration camp interns from Auschwitz, Birkenau, and Dachau leading to the death of many "research participants" (really, subjects subjected to atrocities)?

And how does the idea of forgiveness play in here?

Eisenberg's paper raises many questions and, fortunately, the author is not subject to the hubris of offering simple answers to these difficult questions. In science education, there is an ongoing debate about the usefulness of teaching not just science content but the nature of science, which means, providing students with opportunities to learn about how science is practiced—including its contingent nature that the science studies literature reported over the past three decades. Surely, what we do in everyday life generally, and how we understand ourselves specifically, mediates what we do professionally. My own activities of intensely gardening (supplying year-round all vegetables we need), cooking (I do the cooking at home), building (I finish the basement, lay tiles and hardwood floors, etc.) have given me an appreciation of the role of the body in knowing; and I have exploited this understanding in the theories of knowing, learning, and meaning with respect to mathematics in the lives of professional scientists. Thus, for example, over 50 percent of research biologists could not interpret a graph that appeared in a first-year university textbook of their own field. Yet some did provide successful interpretations, and these drew on their everyday experiences—for example, going hiking in the local mountains or fertilizing plants and vegetables in their gardens—as resources in their interpretations.

To get a better handle on these issues and questions, I use cultural-historical activity theory, because it makes me look at the systems within which such things as mathematical theorems, technological artifacts (atomic bomb, rockets), scientific knowledge, philosophical masterpieces, musical oeuvres, or paintings and sculptures are produced and reproduced. In the following, I outline the theory and then use it to look at the issues that Eisenberg raises in his article.

A Cultural-historical Activity Theoretic Perspective

The Historical Roots of Cultural-Historical Activity Theory

Cultural-historical activity theory was founded by Russian psychologists (e.g., Leont'ev, 1978) discontent with the way in which most Western psychologist reduced human activity to the

intentions and actions of individuals, on the one hand (as apparent in the famous Vygotsky–Piaget debate), or to the determination of human agency by environmental factors, on the other (behaviorism). They proposed, instead, to use entire activities as the unit of analyzing human productions; here, an activity is denoted by a verb such as farming, manufacturing tools, tailoring, hunting/fishing, doing university-based research, and so forth. Different activity systems together allow societies to survive, as the needs of individual human beings are satisfied through the exchange of resources to meet fundamental needs, such as food, clothing, and shelter. Thus, it would be unthinkable today to have a mathematician living like Diogenes in a barrel without doing something in exchange for which he or she would receive food, clothing to live in Canadian climates, and a heated home.

Activity Theory in Its Present-Day Form

Activity theory later was taken up in the West, where, in one of its two main versions (Engeström, 1987), the structural aspects are highlighted in a mediational triangle (Figure 1). Before explaining the figure in its details, I must highlight three important points. First, the triangle has to be thought as consisting of two mutually constitutive layers, one describing the material world, the other describing how the material world is reflected in human consciousness. Thus, as Alexei N. Leont'ev frequently is quoted to have said/written, the object exists twice—once materially, once in the consciousness that reflects the material world. Second, the triangle only represents the *structural* aspects of human activities only, pushing the agency required to mobilize structure into the background. Thus, while looking at Figure 1, readers need to keep in mind that it represents the structure of activity, but that it really requires *agency* to mobilize the resources available in this structure. Third, the triangle constitutes a static representation pushing the historical aspect of the theory into the background. Thus, as its name suggests, cultural-historical activity theory emphasizes the historically and culturally contingent aspects of human consciousness. Therefore, what is possible today in terms of mathematical proofs particularly and mathematical praxis more generally would not have been thinkable 50 or 100 years ago, or, to sharpen this issue, it would not have been possible yesterday. To understand activity systems, such as the one producing new mathematical knowledge, we therefore always need to study mathematical culture in its historical dimensions. The question Eisenberg raises about teaching some of the contingent elements in mathematicians' lives can be answered in the affirmative, for anything that happens in an activity system leaves its mark on the activity system, including, for example, its outcomes (mathematical knowledge) and its subjects (mathematicians as persons).

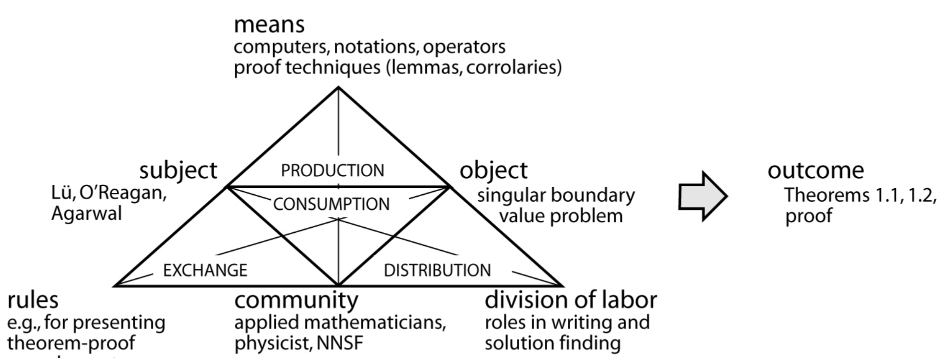

Figure 1. The *structure* of cultural-historical activity theory contains 6 main *moments* that cannot be reduced to each other. Activity as a whole, therefore, is the unit of analysis.

Cultural-historical activity theorists take activity as the minimal unit of analysis. Thus, the triangle in Figure 1 as a whole needs to be considered when we want to know how, for example, new mathematical theorems are produced. Because activity is the minimal unit, none of the terms in the figure denotes an "element" (as some researchers falsely do, even those who self-declare to be practicing cultural-historical activity theory). Rather, these terms denote *moments*, that is, parts that can be articulated on heuristic grounds but cannot be thought independent from other isolable parts because all of those aspects *mutually constitute each other* (Roth & Lee, 2007). Philosophically inclined readers may think of the term *singular plural*, where the whole constitutes the parts and the parts constitute the whole; mathematically inclined individuals know analogous phenomena in systems of coupled differential equations for dynamical systems that cannot be separated in which the current value of certain variables appear as parameters in the evolution of other variables. This then makes it immediately clear that from the chosen theoretical perspective, we cannot think of mathematical theorem production in terms of a mathematician's mental structure and content.

In activity, three levels of events need to be distinguished yet at the same time understood in their mutually constitutive nature: activity, action, and operations (Leont'ev, 1978). An activity—consistent with its origin in the German concept *Tätigkeit* and the Russian concept deyatel'nost'—refers to a form of event at the societal level that contributes to sustaining the life form. Thus, farming, teaching, producing tools, fishing and the likes are activities—doing a mathematical problem in high school is a task. Activities are interconnected, exchange people, products, and money and in so doing, contribute to meet human needs. Activities therefore are oriented toward object-constituted motives. More so, activities contribute to the *sense* of actions (Figure 2), which concretely realize activities. Actions are oriented toward the goals individual and collective subjects set themselves to transform the relevant object into an outcome (product). These last three sentences point us to the dialectical relationship between activities and actions (Roth, 2007a). Actions realize activities, but activities provide the sense for an action: the same action is associated with a different sense in a different activity (showing the middle finger to a teacher who requests silence is different to showing the middle finger when a team mate requests receiving the ball). Actions and the goals they pursue are realized by operations, which are not conscious but determined by the context—we walk to the fridge to get some ice, but the walking itself is realized by steps that we do not think about. But operations are produced only in the service of realizing goal-driven actions. There therefore is another dialectical relationship between conscious, goal-directed actions and contextually determined operations, each presupposing the other. Here, goal-directed actions serve as a *referent* in the unconscious "selection" of operations. Together, the two dialectical relationships between activity and actions, on the one hand, and actions and operations, on the other hand, denote a process that I term *meaning*. As actions may become routinized, they turn into operations; operations also may be "copied" unconsciously while someone participates with others in research or daily activities (by means of a process that has come to be termed *mimesis*). In this way, operations really constitute *crystallized* forms of cultural practices (i.e., patterned actions).

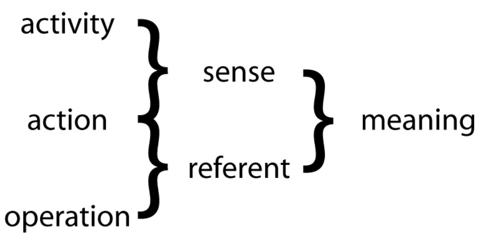

Figure 2. An activity, although it constitutes the *unit of analysis* should be analyzed in terms of three levels that stand in dialectical (mutually constitutive) relations.

Emotions, as I recently showed in an analysis of mathematics in the workplace (Roth, 2007b), are central to events at the conscious and unconscious levels. At the selection of goals, human beings will select those that have a higher valence, that is, that promise some sort of pay-off associated with satisfaction (higher salary, well being). Mathematicians do research and write papers because of the positive emotional valence that comes with innovation and achievement, because publication leads to pay raises, or because of some other reason associated with some pay-off. At the unconscious level, our current emotional states (feeling down, elated) are part of the contexts that shape the production of operations. We know that there are days that we do not feel like doing research or where we do not feel like writing, and no external force driving us will improve the results.

In the following, I use a recently paper published in a journal of applied mathematics (Lü, O'Regan, & Agarwal, 2007) as an exemplary case to explain Figure 1, though not having followed them around with my camera, I am not in a position to write about the emotional aspects in the way I have done it for fish culturists (Roth, 2007b).

An Exemplary Case of an Activity System in Applied Mathematics

The three authors of the paper "Existence to singular boundary value problems with sign changing nonlinearities using an approximation method approach" set out to produce two theorems concerning singular boundary value problems, theorems that—in the words of the authors—constitute the original contribution of the work. In terms of the theory, the three authors constitute the *subject*, the singular boundary value problems the *object*, and the theorems the intended *outcome* of the activity. What they do is mediated by the tools they have available, which may have been some form of electronic *means* to communicate between their institutions located in China, Ireland, and Australia, respectively. That we cannot reduce the different moments also is immediately evident, as the object of activity (boundary value problems) and the outcomes (theorems) define the nature of the subject, applied mathematicians, but the nature of the subject as mathematicians defines the object. To return to the analogy with the coupled differential equations, the temporal evolution of the object and the temporal evolution of the subject cannot be thought (modeled) independently because the state of one at a point in time enters the evolution equation of the other. More so, in a world where difference is required for thinking, the object defines the very nature of the subject. Thus, we would not find everyday folk doing singular boundary value problems: Solving such problems *makes sense* within the community of mathematicians and within activity systems of mathematics; it does not make (immediate) sense in other communities, where this might be considered something outlandish (think about what Einstein's coworkers in the patent office might have thought about him if they knew he was working on what came to be known as relativity theory). Also, we cannot

understand what has been produced without the means of production, which mediate between subject and object. Thus, mathematical activity has a mediated nature.

There are further mediations at work to understand the actions of the mathematicians. For example, the *division of labor* that the three authors have chosen mediates the relation between the subject and object—the "flavor" of the solution proposed may depend on who does what and who takes the lead. The community of applied mathematicians also mediates the relationship between subject and object, as it will be the recipient and "consumer" of the outcomes of this activity. Therefore, what constitutes a legitimate object of mathematical activity and who constitutes a legitimate mathematician depends on (is mediated by) the *community* of mathematicians. This also is immediately evident when we think of the first people to read a manuscript: editors and reviewers. The manuscript has to address the concerns of these recipients ("consumers" [Figure 1]) to make it into a scientific journal in the first place. Thus, the three mathematicians do not just develop theorems and proofs, but they do so in a way that they presuppose others to recognize as legitimately mathematical. More so, much of what mathematicians do does not require conscious reflection: Few scholars I have met know, in terms of formal rules, how to write a good paper: they know to write a paper in the same way they know how to walk or in the way children speak grammatically correct without knowing formal grammar. That is, much of what mathematicians do happens at the level of operations, which may have been the result of explicit actions that have crystallized or that they may have appropriated by unconsciously emulating others within the culture. The Chinese funding agency NNSF, acknowledged in the first footnote, also mediated the object, as its grant enabled the pursuit of the solution and the production of the theorems. Finally, there are *rules* that mediate between the mathematicians and their object. Thus, for example, to solve the singular boundary value problem requires a particular procedure, the proposal of the theorem and its proof, including the production of lemmas and corollaries that are required to achieve the *outcome* in the concrete way that it present itself to readers ("consumers") of the article.

Consequences of Activity Theory for Thinking about Tidbits

Two main points need to be made here. First, if human activity is mediated then all moments of activity make their mark on the outcome, including the means of production, the particulars of the (individual, collective) subject, and the community. For example, the arrival of computers on the scene in the 1960s allowed new forms of doing mathematics to emerge, even though mathematical purists do not accept the use of computers as legitimate. More so, what is acceptable mathematics is a function of the current state of the mathematical culture, which is a characteristic of the mathematical community of the day. But so was the theory of the delta function that the physicist Paul Dirac introduced, but which formal mathematicians did not initially accept as a legitimate object of inquiry until a rigorous definition of distributions as functionals was produced a few years later (Balakrishnan, 2003). In a strong sense, therefore, particulars of the individual and collective subject make their mark on the outcomes. Simple, mundane, and everyday experiences may therefore mediate the solution to scientific problems. For example, one story about the discovery of the chemical structure of benzene suggests that Friedrich August Kekulé had a daydream of a snake biting its tail. Other versions of the discovery say that he might have seen a dance with multiple couples joining up in a ring. (The 2005 Nobel Prize in Chemistry was given "for the development of the metathesis method in organic synthesis," a process explained in terms of a "ring dance with partner exchange" between alkene and catalyst pairs.) Quite innocuous events, images, and observations may provide solutions to

important scientific and mathematical findings, for which *individuals* are credited, though they *received* rather then intended the insight provoked by their being part of everyday collective and material life.

Now if we were flies on the wall watching mathematicians at work, then to understand what is happening, mathematicians' actions, we would need to look at the activity system as a whole. (I am aware of at least two studies that looked at mathematical activity in real time: Livingston [1987] videotaped the reproduction of Gödel's theorem by two mathematicians, and Mertz and Knorr-Cetina [1997] studied theoretical physicists working out some aspect of string theory, that is, the BRST cohomology of the W-algebra.) We cannot just be concerned with presupposed contents of the mathematicians' minds, but we have to take into account the means they use, the community that they intend the products of their labor for, the (tacit/implicit and explicit) rules they adhere to, the division of labor they enact, and so forth.

Implications of an Activity-Theoretic Perspective

Cultural-historical activity theory allows us to better appreciate the relationship between individual and collective. The individual but realizes a possibility that exists at the collective level. The simultaneous emergence of the verb "to google" in the Anglo-Saxon world is but an example of this fact. Another example is that of language emergence: At the very instance that a (first) human being articulated a first word or phrase, he or she had to presuppose that the listener already understood, and therefore, the first speaker was not the first linguistically competent individual after all given that the recipient of the message (listener) had to be equally competent.

From cultural-historical activity theory we can learn two main things pertinent to the issues that Eisenberg raises. First, the outcomes of activity bear the marks of every single moment that one can identify in the system as a whole. Second, and arising from the first, there are strongly viewed no individual contributions, because individual achievements are the outcomes of historical reconstructions where the system as a whole has been abstracted and made to disappear. Thus, the shoemaker or factory worker producing Einstein's shoes, the tailor who cut and sewed his suit, the farmer producing the wheat for his bread, the architect and construction worker making his home all have been abstracted, though Einstein could not have lived his life without them. Third, cultural-historical activity theory teaches us that we produce and reproduce society at a point in time that is culturally and historically contingent. Had Einstein lived 50 years before, he likely would not have been in the position and would not have had the resources to produce general or specific relativity theory (for which he has become most well known), his paper on the photoelectric effect (for which he received the Nobel Prize), or any of the other contributions that he now is celebrated for. More so, 50 years later, he would not have been in the position to invent these theories, as someone else would have likely invented them because the time was ripe and the resources available for framing and solving these problems. This is so because at the collective cultural level, there are action possibilities; at the time of Einstein, a reformulation of a number of issues in physics could be undertaken. Sooner or later someone else would have realized these possibilities.

Now, we cannot know whether wearing or not wearing socks has contributed in any way to the production of relativity theory or any other of the contribution. But it might have been the case that not wearing socks—like taking walks in ice-cold creek water that the Bavarian priest and

hydro-therapist Sebastian Kneipp recommended (Einstein went to school in Munich, the capital of Bavaria)—contributed to a sufficiently healthy condition that allowed him to do the work he did. In this case, if he had been a sickly person, the association between relativity theory or photoelectric effect and Albert Einstein might not have come about. In phenomenology, it is accepted that our bodies constitute what we can know (e.g., Merleau-Ponty, 1945; Henry, 2003). Knowing means knowing to act, not in a reflective way, but in the same way that we know how to walk upright without thinking, in the same way that we talk to our neighbors on the street without having to think about what to say, in the same way we teach mathematics and statistics lectures without having to stop and search for words. This form of knowing leads to the production, in real time, of behaviors that are marked by contingencies: we stumble or stutter during a lecture, we produce incongruencies and malapropisms, we bend the nail rather then getting it into the wood or wall, and so forth. And from such contingencies derive images that produce solutions to the hard problems that exist in the science and mathematics. Take the following examples.

Einstein used the image of an elevator to consider issues concerning relativity. Now this required his knowledge of elevators, and perhaps he had ridden elevators over and over again, such as I had done when I was a child in the hotels near the campground where my family staid during its summer vacations. From a phenomenological perspective, this is entirely intelligible: his experience has changed his way of understanding, and this understanding, intuitive and inarticulate as it may have been, became a resource in his thinking about relativity theory as Kekulé's image of the snake biting its tail, an age-old image existing at the cultural level for a long time, mediated his solution to the benzene structure. Saying that Einstein was fond of riding elevators might be considered a tidbit, but without this experience and the tidbit it gave rise to (if this were to be the case), he would not have been able to think through these issues at all. Does such a tidbit warrant inclusion in the teaching of physics or mathematics of general relativity: yes and no. On a historical level, Einstein may not have been able to produce the principles of general relativity, but someone else might have produced it. On an epistemological level, it would help us understand that experience *is* required for anything like conceptual knowledge—a main point in praxis theories and phenomenological theories of knowing. We do not need to know about his habit—if this were in fact the case—of riding elevators, because someone else would have stated the principles of relativity because they constituted a general possibility. And if someone else had produced them slightly before or after, its statements would be connected to different personal experiences.

Some contingencies and quirks easily can be abstracted from the scientific and mathematical productions; or viewed differently, the marks these contingencies and quirks on the outcomes of scientific and mathematical activity can be considered minor or invisible so that we may disattend to them. Einstein's quirky habit of wearing shoes without socks may be among those. But in the case of Sir Isaac Newton, we have some outcomes of his activities that became contributions to mathematics and physics, leading to celebrations of his outstanding qualities and "genius." But other productions were so much marked by his twisted, tortured, and mystical nature that they did not make it into the annals and history of standard science (White, 1999). This biographer also notes Newton's homosexual tendencies, his ability to hold grudges for decades, and his egomaniac and very petty nature. Thus, Newton's contributions to alchemy and his productions concerning Old Testament prophecies—he thought that the design of Solomon's temple was a code for the entirety of recorded human history—did not become acceptable contributions to any official science and therefore do not feature in today's science (together with his laws) or

mathematics textbooks (together with his calculus). That is, to understand Newton's production, we need to understand all these tidbits. To understand Newton as a person, we do need to know about his phantasms, his alchemy, and so forth. These tidbits allow us to understand that Newton was just another person, with all its idiosyncrasies. But to understand the law of gravity, we do not need to know these tidbits. And further, it is not Newton alone who is responsible for the law of gravity or the law relating force and acceleration now bearing his name: $F = \mathrm{m}a$ or the calculus in the form he proposed. The scientific community has taken care that the quirks are irrelevant and only those productions come to be recognized as contributions to science that are without the contingencies and particularities that characterized Newton's other productions.

Anti-Semitism

The same will be the case concerning the other main issue that Eisenberg brings forth: Should we accept the productions of anti-Semitic scholars and artists? The answer is not easy and my inclination is to say that the answer and solution must be inherently contradictory to allow us making the choice. If the answer were inherently possible and straightforward, it would not require a choosing and taking a stand, and therefore could be delivered in a mechanical and mechanistic way. It would not take a human being to implement, but could be programmed into a computer, which would produce the pre-determined and pre-programmed solution.

Should we accept the productions made within a society that has anti-Semitic tendencies or made by individuals who also make anti-Semitic statements? That is, should we reject the mathematical and scientific advances made during the Nazi regime, including scientists and mathematicians with declared or undeclared Nazi tendencies or sympathies? History shows that—for pragmatic reasons—such tendencies and sympathies often are neglected and even forgotten. (See also my introductory example of the Heinrich Harrer book and film.) Rockets were developed during the Third Reich, and so was the knowledge and the technology for the atomic bomb, both subsequently further developed in the USA and the USSR, including the collaboration of emigrated and captured German scientists. Is a rocket or an atomic bomb anti-Semitic? It probably is not. Is an atomic bomb anti-Japanese or anti-Nazi? Well, it has been used by the Americans who, like Canadians, interned their citizens of Japanese origins despite their allegiance to the new home country. And it has been used to kill "innocent" Japanese in Hiroshima and Nagasaki, who, as in Germany, may not have adhered to the public ideology but have remained silent for fear of being interned and killed in concentration camps. Are scientists responsible? Most scientists will respond "no," conferring the responsibility for the bomb to politicians. Others will not be so sure and will want to make scientists ethically responsible for their production.

Should we not read the work of the German philosopher Martin Heidegger because of his allegiances with the Nazi regime? Some readers may not want to read him for this reason. Others may claim that his work, such as *Sein und Zeit* (Being and Time) does not bear evident marks of these tendencies and therefore, like the atomic bomb, can be considered as a philosophical achievement acceptable to be discussed in scholarly circles. Do we reject Jean-Paul Sartre because he showed sympathies for the repressive regime of the USSR? Do we reject the productions of those U.S. scholars and artists that were devout Marxists and Soviet friendly (and for that persecuted by McCarthy)? Or should we reject those who assisted McCarthy in the persecution of his fellow citizens? Should we reject the productions by present day Israeli scholars because they live in, and perhaps support, a political system that causes havoc for Palestinian families who

have nothing to do with the attacks of militants and suicide bombers? Should we reject the scientific and mathematical findings of U.S. citizens because they live in a country that has the death penalty, that, in the eyes of many people around the world engages in unethical and inhuman interrogation, internment, and repression practices (Abu Graib; Guantanamo Bay; 100,000 civilian "collateral damage" in Iraq as a by-product of "fighting global terrorism")? The US is, after all, one of the countries that Amnesty International cites for human right violations of the kind that individuals from other nations are tried for in the world court at The Hague. This list of questions shows that there are no easy solutions; in fact, any solution may be the possible impossible itself. Personally, it is somewhere along these lines that I would like to place myself for pragmatic purposes. It would force me to make a decision in each and every case, in each and every course I teach, always requiring me to think about the unsolvable mystery of (collective) human consciousness that leads us to these aporetic situations.

Coda: Should We Teach the Tidbits of History?

In the manner of Jacques Derrida, one of my most favorite philosopher, who avoids giving simple answers to complex problems, I make another turn: Though announcing the end (Coda!), I make another beginning. It is a truly Nietzschean (eternal) beginning and renewal. Therefore I make another return concerning the question whether we should be teaching about Einstein's socks: In another area of my research, gesture studies, it is well known that some hand-arm movements are coincidental, that is, without function in the conversation; these are referred to as "grooming" movements, such as scratching one's arm during a conversation. Other hand-arm movements do have a function because they contribute to understanding on the part of the speaker or listener: for example, when the listener scratches her head, the speaker may take this to be an indication that the listener does not understand or has difficulties understanding. How are human beings capable to separate scratching one's from signaling lack of understanding? Pragmatically, we do separate the two forms of hand-arm movements; and if there were a misinterpretation to occur, subsequent speaker- or listener-initiated transactional turns would seek to rectify misalignment. How do we separate the wheat from the chaff, and is the chaff of relevance?

In mathematics (science) education, does it matter for a student to know whether Einstein wore socks or not? On the one hand, it does not matter teaching about it: wearing socks and the outcome of Einstein's thinking processes, e.g., general relativity theory, appear to be unrelated. On the other hand, it does matter: we are less prone to deify, as this often happens, a human being who, after all, is subject to birth, death, and (eating, drinking, defecation, clothing) needs as all other human beings. The emperor has no clothes; and Einstein had no socks when he slipped into his shoes without them. Einstein was special, as we all are; and he was not so special, as we all are. He realized cultural possibilities; as we all do. And he realized some in a way that he became celebrated for; as some of us are when we receive awards for work attributed to us (I have a few of those). But these rewards are from communities that have enabled and accepted the very innovations that we produce—in giving me an award a society actually rewards itself, for I would not have published if the community had not been ready for it. Einstein built on the knowledge produced by others before him, including Albert Michelson and Edward Morley's experiments on the constancy of the speed of light; he knew of the Fitzgerald-Lorentz contraction, and he knew of the transformations that convert the observation of measurements in different systems of reference, which were named by the French mathematician Henri Poincaré after the Dutch physicist and mathematician Hendrik Lorentz (the Lorentz

transformations). Knowing about Einstein's habit to wear shoes without socks is but one piece of evidence to recognize that the emperor does not wear clothes.

My answer to the question whether we should teach historical and biographical tidbits has to remain contradictory: I personally like to live in a world where we the emperor has no clothes if this is the case. In my work with graduate students and colleagues, I always make this point clear whenever someone asks me about my scholarly productivity or produces some other laudatory comment, for example, about the number of prizes and awards I have received. I always comment that all these accomplishments would have been impossible without the community that was the very source of the possibilities that I concretely realized, but which someone else could have realized as well. So sometimes I point out that I, too, do not wear clothes, experience pain, suffer, am elated, and so forth.

The question whether we should teach the tidbits of mathematics and science history depends on how we see ourselves. The answer therefore has to remain aporetic, forcing us to *make* choices rather than accepting present conditions that dictate to us whether to include tidbits and the Shoah in mathematics (science) teaching. Are we like dog *trainers*, getting the best to perform whatever we teach? Or are we *educators* interested in more than the mechanical transmission of knowledge and skills? Should high school students know about the context within which mathematical knowledge was produced? Definitely so! Does this mean knowing about the presence or absence of socks on Einstein's feet? Perhaps. Should university mathematics students know about Einstein's socks? Perhaps, especially if they do not continue to pursue graduate studies in mathematics and become professional mathematicians. *Education* means that we know how the world works; training means that we acquire some routine skills without worrying about their epistemological and ontological nature. (As a graduate student in physics, I complained to my professors that they were teaching us *mere* skills, and therefore that university was little different from vocational school. I said that physics had so many epistemological and ontological consequences that we should be discussing. But they responded that training us in certain skills *was* the purpose of university education.) It therefore also means that we live in a world without gods. Einstein's mannerisms concerning his socks is a good way to push a god off the pedestal and to recognize him as but another human being who has done his part to reproduce and produce everyday, mundane, immortal society. Einstein wears no socks in the same way that the emperor does not wear clothes.

Aporia

At the end of his article, Theodore Eisenberg asks the really hard question about what to do with "the Nazi business of Bieberbach and Teichmüller" and other issues surrounding Nazism, anti-Semitism, and the Shoah. Eisenberg states that he "feel[s] uncomfortable in discussing this nasty business." It is not my place to lecture him or anyone else how to deal with this problem, which really is an aporia, a problem without solution, or rather, a problem with contradictory solutions. (This, especially and because of my German origins, and especially and because my parents were only children at the time. These contingencies cannot be excuses, which is a very biblical theme, as we know from the concept of "original sin.") The solution has to be as aporetic as the problem. Let me explain.

In making a decision about whether to include historical facts in the teaching of mathematics, as well as in decisions about whether to include the work of anti-Semitic (pro-Nazi, pro-Serbian

nationalist, anti-American) scholars within the community of mathematicians (scientists, artists, culture generally), we must not forget the concept of *forgiveness*. Here I do not mean the simple concept of forgiveness—can I forgive this person, can I not forgive that person—but rather the advanced concept of forgiveness in all its complexity (Derrida, 2005). Derrida points out that we can only and truly forgive the unforgivable, because if we forgive the forgivable, we have not really done anything particular. A computer can forgive the forgivable using an algorithm. And, as the title *Pardonner: L'Impardonnable et L'Inprescriptible* (To Pardon: On the Unpardonable and Imprescriptable) suggests, pardon generally and pardoning the unpardonable specifically *cannot be prescribed* (which is why it is not my place to lecture anyone on how to deal with the issue). If we can forgive Bieberbach, Teichmüller and the likes, forgiveness becomes mechanical or a matter of exchange. If we do not forgive the unforgivable, then we do not make a decision and simply submit to the condition. Forgiving the unforgivable, however, is the most difficult task we face. To make his point, Derrida discusses the case of the Russian-born philosopher Vladimir Jankélévitch (his family emigrated because of the pogroms against Jews), who, in a little book entitled *Le Pardon* (The Pardon), had suggested that pardoning a sin is the greatest challenge to judicial logic. Jankélévitch took a hard-line stance and suggested, in *L'Imprescriptible*, that the Shoah (Holocaust) attained such inexpiable singularity that renders impossible any form of pardon. Derrida also analyzes poem "Todtnauberg," written by the German- and French-speaking poet Paul Celan (born into a Jewish family in Romania) after his visit of Heidegger at his home in Todtnauberg, a poem in which he points to (in his usual oblique style) what he had hoped to hear so much:

. . .	. . .
die in dies Buch	the in this book
geschriebe Zeile von	written line of
einer Hoffnung, heute,	a hope, today,
auf eines Denkenden	for a thinker's
kommendes	coming
Wort	word
im Herzen,	in the hear
. . .	. . .

But his host (Heidegger) did not pronounce it: the request to be pardoned for his allegiance to the Nazi regime. Derrida takes up the complete opposition Jankélévitch showed with respect to any forgiveness of the Nazi crimes and shows that a solution to this problem dignified to be named such has to remain aporetic and contradictory. Derrida suggests that the pardon *has to be asked for*, to be just, for the fact to be just, and because the one asking is just, and, because to be just one has to be unjust (i.e., asking for forgiveness of the unforgivable). But Oswald Teichmüller and Ludwig Bieberbach are no more; they cannot in any way ask for forgiveness. Yet we must be in a position to forgive the unforgivable that they enacted. The upshot is that we may pardon the unpardonable, forgive the unforgivable; and this, too, can become part of our teaching (mathematics, science, philosophy, music, art).

Epilogue

I am glad Theodore Eisenberg took up the challenge to address not only the small problems like Einstein's socks but also the real hard and unsolvable problem of the Shoah (and similar atrocities, the genocides in Rwanda and Serbia, etc.). If there were a simple solution, it would not

be a real problem. I like the internal contradiction the author leaves at the end—feeling uncomfortable with the "nasty business," but at the same time, as a step toward forgiving what remains unforgivable, "teaching the strengths and weaknesses of the individuals whose mathematics we teach." I see it as a move toward a better world, hopefully one without atrocities, one in which people of all races and believes resort to mechanisms other than violence to resolve their unavoidable differences—whether they are Catholics and Protestants in Ireland, East and West Germans, North and South Koreans, or the within-Semite differences between Israelis and Palestinians, now living divided on the two sides of an emerging concrete wall.

References

Balakrishnan, V. (2003). All about the Dirac delta function (?). *Resonance, 8*, 48–58.

Derrida, J. (2005). *Pardonner: L'impardonnable et l'imprescriptible* [To pardon: On the unpardonable and imprescriptable]. Paris: L'Herne.

Eisenberg, T. (2008). Flaws and idiosyncrasies in mathematicians: Food for the classroom? *The Montana Mathematics Enthusiast*, 5(1), 3-14.

Engeström, Y. (1987). *Learning by expanding: An activity-theoretical approach to developmental research.* Helsinki: Orienta-Konsultit.

Henry, M. (2003). *Philosophie et phénoménologie du corps: Essai sur l'ontologie biranienne 5ième ed* [Philosophy and phenomenology of the body: An essay on Biranien ontology, 5h ed.]. Paris: Presses Universitaires de France.

Leont'ev, A. N. (1978). *Activity, consciousness and personality*. Englewood Cliffs, NJ: Prentice Hall.

Livingston, E. (1986). *The ethnomethodological foundations of mathematics*. London: Routledge and Kegan Paul.

Lü, H., O'Regan, D, & Agarwal, R. P. (2007). Existence to singular boundary value problems with sign changing nonlinerarities using an approximation method approach. *Applications of Mathematics, 52,* 117–135.

Merleau-Ponty, M. (1945). *Phénoménologie de la perception* [Phenomenology of perception]. Paris: Gallimard.

Merz, M., & Knorr-Cetina, K. (1997). Deconstruction in a "thinking" science: Theoretical physicists at work. *Social Studies of Science, 27*, 73–111.

Milgram, S. (1963). Behavioral study of obedience. *Journal of Abnormal and Social Psychology, 67*, 371–378.

Roth, W.-M. (2007a). Introduction: Heading the unit of analysis. *Mind, Culture, and Activity, 14*, •••–•••.

Roth, W.-M. (2007b). Motive, emotion, and identity at work: A contribution to third-generation cultural historical activity theory. *Mind, Culture and Activity, 14*, 40–63.

Roth, W.-M., & Lee, Y. J. (2007). "Vygotsky's neglected legacy": Cultural-historical activity theory. *Review of Educational Research, 77,* 186–232.

White, M. (1999). *Isaac Newton: The last sorcerer*. Toronto: HarperCollins.

Comments provoked by "Flaws and idiosyncrasies in mathematicians: Food for the classroom?" by Theodore Eisenberg

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Abstract
In mathematics classes, it is appropriate for many reasons to discuss mathematicians as people with lives, personal problems, both influenced by and influencing cultural movements and societal issues. Mathematics is a human activity, and mathematicians are human. Eisenberg's paper raises important and fascinating issues, such as the extent to which intellectual achievements can be kept separate from the personality and actions of their creator (such as Wagner). However, in my reactions, I suggest ways in which I believe the discussion needs to be broadened and refocused.

Nature of the sample of vignettes
The set of vignettes presented in Section 2 could serve to reinforce inaccurate and harmful beliefs about the nature of mathematics:

• Academic mathematics is solely a European intellectual achievement
Until comparatively recently, histories of mathematics were extremely Eurocentric. Such bias has been substantially corrected by scholars such as Swetz (1994), Joseph (1992), and Powell and Frankenstein (1997). There is no lack of examples of eminent non-European mathematicians – Ramanujan (Rao, 1998) immediately comes to mind (and in personal and social terms, his relationship with Hardy is of particular interest (Kanigel, 1991)). The representation of mathematics as the creation of solely dead, white males can be considered as a form of symbolic violence against non-European and female students.

On a specific point, the claim that "Isaac Newton … is a name that is known in most households throughout the educated world" is doubly disturbing. In any Western society, such as the United States, I find it hard to believe that the claim is accurate, unless there is an implicit qualification by social class. The claim also suggests that people living in, say, China or Peru are not educated.

• The history of mathematics is a history of great individual achievements rather than of collective and cumulative intellectual effort (see below). An example that also has a bearing on the previous point is the argument by Almeida and Joseph (2007) that elements of calculus had been developed in Kerala at least 200 years before Newton and Leibniz and that lines of communication between Kerala and Europe mean that it was likely that this work was known in Europe.

• Mathematics was all done in the past

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The paper contains few references to recent and contemporary mathematicians, for which excellent sources, which also portray the individuals as fully-rounded human beings are the sets of interviews by Albers and Alexanderson (1985), and Albers, Alexanderson, and Reid (1990).

• Women do not do significant mathematics
Anglin (1992), begins a section headed "How should the historian tackle the scarcity of women mathematicians?" with the forthright declaration that:

> Men and women are equal intellectually. Apparent differences between male and female mathematical ability are due to social factors such as cultural systems in which men take all the educational opportunities for themselves (p. 8).

This statement could be nuanced by reference to particular factors that relate to mathematics as opposed to, say, literature. Moreover, as Martin Gardner points out, in his introduction to Albers et al. (1990): "Although social forces inhibiting the entrance of women into mathematics may be abating, they are still very much with us". In terms of teaching, some suggestions may be made. First, there is no excuse these days for referring to the generic mathematician as "he". Second, reference should be made to female mathematicians who have made significant contributions (which avoiding the temptation, as Anglin (1992, p. 8) recommends, to exaggerate the role of women in mathematics). Most importantly, the social and cultural conditions should be carefully considered (several instructive examples can be found among the delightful collection of anecdotes assembled by Wells (1997)). It should also be considered whether there are aspects of academic mathematics as human practice that are differentially alienating to females, such as its (perceived or real) tendency to emotional detachment and its long and inglorious involvement in the making of war.

What kind of history of mathematics?
The mere fact that Eisenberg feels it necessary to inform students about the human weaknesses of mathematicians is itself a comment on the inadequacy of history teaching in general, whereby history is presented to the young as nationalistic propaganda in terms of flawless heroes – military leaders, political leaders, intellectuals, industrialists, and so on.

What form should a history of mathematics take? Several possible organizing frameworks listed by Anglin (1992) are cited by Eisenberg: individuals and their personal lives, nations or races, chronological periods. However, Anglin (p. 7) also comments that "There is no reason why one could not write a history of mathematics entirely from a communitarian point of view" and, later (p. 8) that "it may be more illuminating to relate a piece of mathematics to its social environment than to a fictitious anecdote about the private life of the author of that piece of mathematics". I like the suggestion made by Davis (1985) of a balanced approach that he calls "Jamesian" by reference to a quotation by Henry James (1917): "The community stagnates without the impulse of the individual; the impulse dies away without the sympathy of the community". Such an approach rightly rejects the "great men" (literally, in the case of Bell (1965)) style of history. Moreover, Davis also raises a number of deep questions about the role of mathematics in society, such as: "Why do we, today, allow our military strategies to be so mathematized and computerized when the difference of one bit in a program may send all down the road to oblivion?".

In the examples discussed by Eisenberg, it may be useful to distinguish three levels of increasing relevance and importance:

1. Details of the personal life and character of the individual.
There is a certain justification for introducing such details, in the service of enlivening a lesson, and stressing the humanity of the individuals whose work is being discussed. Anglin (1992, p. 11) also suggests making reference to blunders by mathematicians, which entertainly underlines their humanity and undermines the view of mathematics as immaculately conceived. I would suggest minimizing the use of personal references in a disconnected way and with an emphasis on "spice". Rather, it is appropriate to choose anecdotes in order to make important points. For example, consider the letter that Charles Babbage wrote to Tennyson about one of his poems, as cited in Wells (1997, p. 51). In this letter, Babbage suggested replacing the lines "Every moment dies a man/ Every moment one is born" with "Every moment dies a man/ Every moment $1^1/_6$ is born", with the comment that, while this is not strictly correct, it is sufficiently accurate for poetry. This anecdote, while amusing, also could be the starting point for a discussion of alternative ways of seeing the world.

I find it strange that Eisenberg at one point states that "we are simply asking if such things as Newton's alleged virginity should be mentioned in the classroom", since elsewhere he does go beyond such a narrow focus. It would be of great interest to know whether the incidences of various forms of behavior are higher among top-flight mathematicians than among comparable groups (e.g. great scientists or artists – or, indeed, people in general), and to investigate whether some causal relationship could be established. A subtle methodology is necessary for such comparisons, otherwise many well-known subjective mechanisms come into play...

2. Aspects of the mathematician's life in relation to the social and political milieux of his/her time.
A good example, to which Eisenberg refers, is the persecution of Turing in England after the Second World War on account of his homosexuality, despite his crucial contribution to that war in leading to effort to break the German codes (the play about Turing has the double-meaning title "Breaking the Code").

3. Cases wherein the mathematical practices were directly influenced by, or influenced political/ social events.
In my opinion, university curricula for mathematicians and future mathematics teachers ought to include at least one course on the social history of mathematics. This should deal with in-depth analyses of such issues as the anti-Semitic activities within their academic practices of mathematicians in Germany (Segal, 2003) and elsewhere. It should also address more general sociocultural topics such as the interplay between probability and statistics, social science, and world-views (Hacking, 1975, 1990). Moreover, forms of mathematical practice other than academic mathematics (in other word, ethnomathematics (D'Ambrosio, 2006)) should be considered in such a course.

A final point regarding the writing of history is the question of accuracy, the problem of knowing whether a story is true (a problem which, ironically, has become magnified in the "Information Age"). Accordingly, care should be taken in characterizing the provenance of anecdotes (see comment by Anglin (1992) cited above). As with all history, that of mathematics is subject to the

proliferation of myth through failure to consult primary sources. Moreover, as pointed out by Anglin (1992) historians of mathematics are particularly susceptible to subjective biases related to their own perspectives about the nature of mathematics. The whole question of the reliability of documentary evidence, and of information gleaned from the Internet, is appropriate for discussion with students.

Ethical responsibilities of mathematicians and mathematics educators

Details of the personal ethics of mathematicians (on which Eisenberg tends to concentrate) are much less important, in my view, than their larger ethical responsibilities to society. This view has been most eloquently expressed by Ubi D'Amrosio (2003):

> It is clear that Mathematics is well integrated into the technological, industrial, military, economic and political systems and that Mathematics has been relying on these systems for the material bases of its continuing progress. It is important to look into the role of mathematicians and mathematics educators in the evolution of mankind. … It is appropriate to ask what the *most universal mode of thought* – Mathematics – has to do with the *most universal problem* – survival with dignity.
>
> I believe that to find the relation between these two universals is an inescapable result of the claim of the universality of Mathematics. Consequently, as mathematicians and mathematics educators, we have to reflect upon our personal role in reversing the situation. (Emphasis in original).

In this respect, discussion by Eisenberg of anti-Semitism in the academy is very appropriate. I would have liked to have seen a least a sketch of other possible topics. In particular, the history of the involvement of mathematicians in the development of nuclear weapons comes to mind. Accounts of the Manhattan Project (e.g. Rhodes, 1986) paint a picture of a group of mainly men motivated by patriotism, camaraderie, competition, and intellectual challenge, with little thought given to the deeper consequences of their work – at least until after the Hiroshima and Nagasaki attacks, and with honorable exceptions. The fact that this analysis is not just a matter of past history is made clear by an article under the title "Rival US labs in arms race to build safer nuclear bomb" (Vartabedian, 2006):

> "I have had people working nights and weekends," said the head of the Los Alamos design team. "This is a chance to *exercise skills that we have not had a chance to use for 20 years*." At Livermore Labs, a similar picture: The lab is running supercomputer simulations around the clock, and teams of scientific experts working on all phases of the project "*are extremely excited*." (Emphasis added).

Final comment

The paper is appropriately provocative (in the best sense of the word) and correct in its central point that mathematics education should reflect the nature of mathematics as human activity. The comments above reflect the various directions in which I think that central message should be extended and strengthened and the ways in which the paper provoked me personally.

References

Albers, D. J. & Alexanderson, G. L. (Eds.). (1985). *Mathematical people: Profiles and interviews*. Boston: Birkhaüser.

Albers, D. J., Alexanderson, G. L., & Reid, C. (Eds.). (1990) *More mathematical people*. Boston: Harcourt Brace Jovanovich.

Almeida, D. F & Joseph, G. G. (2007). Kerala mathematics and its possible transmission to Europe. *Philosophy of Mathematics Education Journal*, *20* (June). Retrieved 7/9/07 from: http://www.people.ex.ac.uk/PErnest/pome20/index.htm

Anglin, W. S. (1992). Mathematics and history. *The Mathematical Intelligencer*, *14*(4), 6-12.

Bell, E. T. (1965). *Men of mathematics*. New York: Simon and Schuster.

D'Ambrosio, U. (2003). The role of mathematics in building up a democratic society. In Madison, B. L., & Steen, L. A. (Eds.), *Quantitative literacy: Why numeracy matters for schools and colleges. Proceedings of National Forum on Quantitative Literacy, National Academy of Sciences, Washington, DC, December, 2001*. Princeton, NJ: National Council on Education and the Disciplines. (Available at: http://www.maa.org/ql/qltoc.html)

D'Ambrosio, U. (2006) *Ethnomathematics: Link between traditions and modernity*. Rotterdam, The Netherlands: Sense Publishers.

Davis, P. J. (1985). Introduction: Reflections on writing the history of mathematics. In D. J. Albers D. J., & G. L. Alexanderson (Eds.) *Mathematical people: Profiles and interviews* (pp. xi-xvi). Boston: Birkhaüser.

Gardner, M. (1990). Introduction. In D. J. Albers, G. L. Alexanderson, & C. Reid (Eds.) *More mathematical people* (pp xi-xv). Boston: Harcourt Brace Jovanovich.

Hacking, I. (1975). *The emergence of probability*. Cambridge: Cambridge University Press.

Hacking, I. (1990). *The taming of chance*. Cambridge: Cambridge University Press.

James, W. (1917). *Selected papers on philosophy*. London: Dent and Sons.

Joseph, G. G. (1992). *The crest of the peacock: Non-European roots of mathematics*. London: Penguin.

Kanigel, R. (1991). *The man who knew infinity: A life of the genius Ramanujan*. New York: Scribners.

Powell, A. B. & Frankenstein, M. (Eds.) (1997). *Ethnomathematics: Challenging Eurocentrism in mathematics education*. Albany, NY: SUNY Press.

Rao, K. S. (1998). *Srinivasa Ramanujan: A mathematical genius*. Chennai: EastWest Books.

Rhodes, R. (1986). *The making of the atomic bomb*. New York: Simon and Schuster.

Segal, S. L. (2003). *Mathematics under the Nazis*. Princeton, NJ: Princeton University Press.

Swetz, F. (1994). *From five fingers to infinity*. Chicago: Open Court.

Vartabedian, R. (2006). Rival US labs in arms race to build safer nuclear bomb. *Los Angeles Times*, June 13. Retrieved 7/4/07 from: http://fairuse.100webcustomers.com/fuj/latimes52.htm

Wells, D. (1997). *The Penguin dictionary of curious and interesting mathematics*. Penguin: London.

Critique on Eisenberg's article Flaws and idiosyncrasies in mathematicians...

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I enjoyed reading the article and learned some interesting (and disturbing) information about well known mathematicians that is not so well known. However in its current form it is more appropriate for the popular media rather than as an academic or scholarly article.

Having said that it does raise some serious questions of ethics and values that all mathematics educators should be engaging. The increasingly popular view that mathematics teaching be socially contextualised means that this kind of historical information may be communicated in lecture rooms and classroom with little understanding or awareness of the "hidden curriculum" being enacted.

The questions are posed but left "in the air" as it were. It is largely a descriptive account. The article could be theoretically strengthened and give a more grounded set of perspectives from which to consider the problem if for example a section were to be included on how such questions may be addressed from different ethical theoretical points. The discussion section could also draw on some of the debates and literature that advocates (and is against) greater use of history to teach and learn mathematics. The ethics of teaching any history of mathematics which includes biographies of mathematicians is the specific and rather novel issue being raised. But one is left disappointed that a deeper engagement is absent. A serious mathematics curriculum issue is put on the table but not explored from the vantage of different curriculum standpoints. For instance how would ethnomathematics, critical, feminist or socio-contructivists respond to this challenge? There is also some repetition in the sections and appears as more of the same. I hope these comments will be helpful to the author in improving the article

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Reaction to the Reactors

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1. A comment. I discussed the contents of my paper with colleagues on many different occasions. By their reactions they seem to divide themselves into two camps. One camp takes the stance of: Are you crazy to mention such things in the classroom? We are charged to teach mathematics, and that's it. Who cares if Einstein never learned to swim? The other camp however takes an opposite stance: Such items are really important for students to know because they help students to think about the lives and times of the men who created the material they are studying; our lessons are richer because of such stories, and are our students are richer too. Obviously I have taken the latter viewpoint, but I am well aware that by simply presenting the topics and tidbits that I do, I am publicly displaying areas and concerns in my own personal belief system. I like to think that I am helping students decide their own stances on such issues, but I am well aware that some tidbits and stories are chosen at the expense of others. When I first heard that Beverly Sills had deaf children who never heard a note their mother had sung, I said to myself, that is an example of tragic irony–and I have used this as my exemplar of tragic irony ever since. And when I first heard that Einstein (and subsequently others) called George David Birkhoff "the biggest anti-Semite in America" I was speechless; but it is a fact that G.D. Birkhoff did his utmost to keep Jews from major universities in America. Saunders Mac Lane, another giant of 20th century mathematics (and a contemporary of Garrett Birkhoff, G.D.'s son who, like his father was also the chairman of the mathematics department at Harvard, but who most definitely held opposite views on this issue than his father) came to G.D.'s defense, but his defense was limp. MacLane said: Look, those were the times; George David simply wanted to give American jobs to Americans. It's as simple as that. Such a defense is easy to understand, but hard to swallow. Jobs, particularly in the academic world, should go to the most qualified candidate–handing out academic positions on other criteria is a recipe for mediocrity– such is the case today, and such was the case in the 1940's. There is more to teaching mathematics than the mathematics itself–I am well aware that it is not politically correct these days to point out flaws, foibles, infirmities and ironies in the lives of others–but on the other hand, in some cases it seems wrong to ignore them.

2. The Editor's Modus Operandi. Before submitting this paper to Bharath Sriraman, editor of this Journal, I sent him a note stating that I had written what I consider to be a controversial paper, and asked him if I had the right to request that it not sent to certain individuals on his Editorial Board for review. I went on to say that I personally know several members on the Board–and some of them sit in the UK; the University and College Lecturers Union in the UK had recently voted to boycott Israeli academics (because of Israel's treatment of the Palestinians), and this boycott was turning into an international incident (a situation Bharath completely knew about). So as to not put my UK colleagues into an awkward situation, I requested that the paper

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not be sent to them. And not wanting to put my Israeli colleagues on the board into the awkward situation of judging a fellow countryman, I also requested that he not to send it to them. Bharath immediately agreed to both requests and so I sent him the paper, which he read within a day or so. He then surprised me by giving ME a list of potential referees and asking ME if I had any objections to them! This had never before happened to me, and I must admit that I felt good about this modus operandi. As it turned out, I received three reviews of the paper. As you can see from the above, two wrote very extensive and penetrating position papers; while the third Prof. Renuka Vithal, Dean of Education at the University of KwaZulu-Natal in South Africa, made specific recommendations for improving the article. So, let me begin with some of her suggestions.

3. Professor Vithal felt that the style of writing was not appropriate for a scholarly journal. She specifically stated that the paper lacks a theoretical framework to present the issues that have been raised. She wrote in her review: "The questions are posed but left "in the air" as it were. It is largely a descriptive account. The article could be theoretically strengthened and give a more grounded set of perspectives from which to consider the problem if for example a section were to be included on how such questions may be addressed from different ethical theoretical points." She went on to say: "A serious mathematics curriculum issue is put on the table but not explored from the vantage of different curriculum standpoints. For instance how would ethnomathematics, critical, feminist or socio-constructivsts respond to this challenge?" Upon reading her comments I said to myself: I am out of my depth. Theoretical frameworks to discuss why some brilliant men (Bieberbach and Teichmüller and their ilk) bullied their Jewish colleagues to the point of driving them to the brink of sanity and beyond? (Beyond? Yes, beyond. Felix Hausdorff, e.g., (of topologlical space fame; a name known to just about everyone in the mathematics community) and his wife couldn't get out of Germany in the mid-1940's; jointly they committed suicide rather than submit to the end Hitler had planned or them.) What kind of theoretical framework is there for presenting a curriculum to describe the world as it was? The world had run amuck; are there theoretical frameworks to explain genocide, for turning human beings into animals, for understanding man's inhumanity to man?

Much to my surprise, there are such theoretical frameworks with theological, philosophical and humanitarian viewpoints–but I am hardly qualified to discuss them in anything but a superficial way. One would think that the Golden Rule: Do unto others as....says it all, but it doesn't. Let me just mention one experiment that stands as one of the central pillars in the theoretical foundation of social psychology, and that is very close recent events of our generation. The experiment is mentioned by columnist Michael Shermer in his article in Scientific American (August, 2007, p. 22-23), which tries to understand the Abu Ghraib prison scandal. Suspected terrorists were kept by USA forces at the Abu Ghraib prison in Iraq, and pictures and video-tapes were leaked to various presses around the world showing USA Military Police guards committing psychological and physical atrocities on the prisoners. The world was shocked, and statements of shame and disbelief filled the news; how could such a thing happen?

Shermer, mentions the research of 40 years ago by Prof. Philip Zimardo, a social psychologist at Stanford University. Zimardo randomly assigned students to being guards or prisoners in a mock prison environment. Psychological tests given prior to the experiment showed the students to be "normal" on personality and morality scales, but by the sixth day into the experiment, the guards had changed into cruel sadists, and the prisoners had turned into emotionally shattered tragedies. Out of fear of the direction the experiment was heading, Zimardo stopped it before its

completion date. Then the analyses and different explanations as what had happened began, but the bottom line was crystal clear: the capacity to do good and evil is in every one of us–and each can be brought to the surface without much effort. (See Zimbardo's recent text: The Lucifer Effect: Understanding How Good People Turn Evil, (2007, Random House), or a Robert Levine's reivew of it in American Scientist, 95(3),Sept/Oct. 2007, pp. 440-442.)

This is all getting rather far from the theme of my paper, but apparently theories do exist to explain both good and bad, rational and irrational, moral and immoral behavior. It boils down to understanding the environment–the environment shapes the moment, and the moment shapes the man. I think that Professor Vithal is absolutely correct–if we are going to enter this arena of talking about the lives of the individuals whose mathematics we study, we must also speak about the times in which they lived. I will expand on this notion when I address several of the concerns of Professors Roth and Greer. But putting many of the stories into larger landscapes to understand some of the absurdities in behavior as mentioned in the vignettes, seems to denigrate a fundamental goal of most educational systems in the world, and that is the goal of trying to teach one to think for themselves. Perhaps we can justify the behavior of some of the individuals mentioned above, but is it wrong to expect one to go against the tide when the tide is going against one's ethical beliefs? The students in Zimardo's experiment were normal before the experiment started–and yet they easily slipped into sadistic and unconscionable roles. Yes, the moment makes the man–and many in the annals of our discipline seems to have failed to rise to the moment.

4. Professor Wolff-Michael Roth raised several extremely important issues. The first is that there are tidbits, and then again, there are tidbits–and they are of completely different orders of magnitude. How right he is, but often they differentiate themselves as they are presented–Einstein not wearing socks is a tidbit of knowledge; but the chapter in the annals of "Deutsche Mathematik" with all of the inhumanity that it brought with it, and which still lingers today, is much more than a tidbit of history. I think that Professor Roth's reaction paper is brilliant, well-reasoned, well-written, pertinent to the issues raised, and more importantly, it gives the paper a theoretical framework that I could never have constructed by myself. For that, for demonstrating a deep understanding of the issues, and for looking at them from a different perspective, I thank him. His "cultural-historical activity theoretic perspective" with its accompanying examples and elaborations make a lot of sense to me–and I agree with his comment that "...if human activity is mediated then all moments of activity make their mark on the outcome...." To me, this justifies presenting many of the stories and comments in the manner that I have in the paper. I understand how he takes this further concluding e.g., that if Einstein had not discovered relativity, "..sooner or later someone else would have..." but this gets us into the polemic of: Is science created by man or is science uncovered by man?–and I do not want to enter that arena because I haven't the academic skills to defend my opinions which, by the way, support both stances on the issue. (I am a bit like Prof. Roth when he asked if it is important that students know about Einstein's socks–and then he stated: "on the one hand, it does not matter..... and on the other hand, it does matter...."). Convincing arguments for each side of the polemic can be presented –and there seems to be no correct answer.) But let's move to the "social-historical problem" presented in the paper which Professor Roth addresses. In the paper it is argued that if we ignore the anti-Semitic behavior of Bieberbach, Teichmüller and others, then by omission, we are whitewashing history–and that seems to be ethically wrong. The magnitude of the Shoah is hard to comprehend–six million Jews were sent to their death; Israel was born because of this genocide, but that is an embarrassing reason for the world to sanction the birth of a country.

Singling out Bieberbach and Teichmüller is not a condemnation of all Germans, although the world certainly has proof enough as to the harm that often results from extreme nationalism. I do not believe in collective punishment; one should not be held responsible for the actions of their fathers or forefathers. One should be judged only by their own actions. But of course, that hardly ever happens, particularly in Israel. I can give many examples of wonderful young Germans who come to Israel to voluntarily work with retarded adults, to work on kibbuzim, and in health and social agencies. I don't know what motivates them to come, but they come–and I am very proud and appreciative that they are here–yet others say: they are here because of guilt. I don't know if this is true, but they are voluntarily here doing important work–and I appreciate them. And maybe someday the music of Richard Wagner will be heard in Israeli concert halls, but I doubt if that will happen in the near future. Admittedly there are hundreds of similar situations from around the world that have also produced victims and villains–and I think it is morally correct to tailor-make our lessons to discuss them, particularly if our students have in some way been affected by them. The world will not improve unless we make peace with these situations–and that means understanding them. There is much to the notion of ethnomathematics, and I think that Professor Roth has given us a wonderful theoretical justification for building lessons within this framework. But one thing we should remember, and that is that every issue has more than one side to it–and that as teachers, our job is try to accurately present all sides of the issue, and to give to our students the tools and the knowledge to judge things for themselves.

5. Professor Brian Greer. I have known Brian for many years and he has always impressed me as being sensitive to and caring about the feelings of others; but my paper certainly touched some nerves with him in unintentional ways–and I am sincerely sorry if I have offended him or other readers. I had no idea that when I wrote that Newton's name is known in most households throughout the educated world, that it would be interpreted in ways other than I had intended, that his name is known to the hoi-polloi. Anyway, Brian has a point; I did not intend to insult or slur anyone–and if I have, please accept my apologies. OK, now let me try to address some of Brian's concerns.

Brian claims that the choice of the vignettes "... reinforce inaccurate and harmful beliefs about the nature of mathematics as being solely a European intellectual achievement....and that mathematics is a discipline that has been pushed forward by white male individual achievements." Well, I believe in mentioning to students where and how notions originate– but sometimes we simply don't know their origins: E.g., "It has long been believed that India first introduced the number 0. Now, however, it's known that the Maya of southern Mexico and Guatemala (ca 300 B.C.–A.D. 900) discovered and used zero independently of, and possibly before the mathematicians of India" (Smith (1996), Agnesi to Zeno (Key Curriculum Press), p. 47). OK, so this is a claim for India and the Mayans. But then in another text we read: "About A.D. 150, the Alexandrian astronomer Ptolemy began using the omicron (o, the first letter of the Greek o____ "nothing"), in the manner of our zero, not only in a medial but also in a terminal position" (Burton, D., op.cit. p. 23). So who should be credited with introducing the notion of a zero? (See End note 2_ above.)

Many individuals have raised another of Brian's concerns, that the development of mathematics is presented as having been done exclusively by white, male, Europeans. Although I can understand this concern, let's face it, the mathematics that is studied and known in the Western world emanates from the Greeks and it is based on Aristotelean logic. Historians will admit that various non-European and non-Western groups and individuals had independently discovered

various notions and theorems–but the notions and theorems that were discovered, e.g., the Pythagorean Theorem, Frieze Designs (which are so prevalent in the art and embroidery work of ancient peoples), arithmetical patterns, etc. were not developed into a body of work as they were in Ancient Greece (and in the European/Western world)-nor were they abstracted (as they were in the E/W world) during the centuries that followed their discovery. Yes, the mathematics of many non-Western civilizations was very advanced–but mathematics as a body of knowledge has taken on a particular characterization. Unlike in the field of medicine, there aren't two different and competing mathematical worlds. In medicine we can compare methods from the West against alternative methods of treatment–with the bottom line being: has the patient's health improved or not? In mathematics this privilege doesn't exist. So I think what Brian is really saying is that we should let students know that non-white/ non-Europeans have also made major contributions to mathematics–and of course, he is correct. But Brian raised the name of Ramanujan as an example, and on this one I think that Brian is wrong.

Srinivasa Iyengar Ramanujan was a self-schooled genius who died at the age of 33; he is included in Ioan James' highly acclaimed text Remarkable Mathematicians (MAA publication,2002), and he seems to be the only non-Westerner included in James' text. He is considered to be the greatest mathematician in India's history, but he was already in the E/W mold of doing mathematics when Hardy and Littlewood took him under their tutelage, and this brings us back to the point made above. The rules and standards for judging the worth of a piece of mathematics are set– and I sincerely believe that they are applied without prejudice to one and all; and as I see it, the E/W rules are not going to change. Although Brian and others have voiced concern about "other" cultures not being represented even in the history of mathematics classrooms, I am not convinced of the validity of this concern. I don't think that Brian means that we should talk about individual men and women who are of non-E/W backgrounds, but rather of the mathematics developed in non-E/W cultures. It is true, that non-E/W cultures were mathematically advanced, but to recognize for example that other cultures used Frieze diagrams is not the same as them having had an understanding as to why there are only a finite number of them. Brian's concern then comes down to talking about individuals from all walks of life as having developed mathematics–and of course, this can and should be done.

Along this line, Brian raises the lack of women in the history of mathematics in general, and in the vignettes in particular. As mentioned earlier, one is free to choose the vignettes they wish to mention–for they are chosen with an agenda in mind; and if one wants to emphasize women in mathematics, great, so be it. There are some wonderful stories about women in the annals of mathematics but as Brian is well aware, there aren't many of them (see: NCTM's Women in Mathematics and Science, 1996). Worse, women are not well represented on the lists of the "great ones" in mathematics. Type "great names in mathematics" into Google (or some other search engine) and various surveys will appear; Archimedes, Newton and Gauss, in some order, are almost always listed in the top 5 names– and Einstein's name often appears in the top 15. Interestingly, and appropriate to Brian's comment, there are lists with specialized concerns; e.g., there are lists of great mathematicians of African origins, lists of great women in mathematics, and also lists of mathematicians from specific countries. The material is available, one simply has to use it.

Brian takes issue with some of my comments on Turing and I admit to feeling very uncomfortable in class when I speak about his life and his death. But I try to convince students that there seems to be a common denominator connecting many of the personalities discussed in

the vignettes, and that includes Turing too. The common denominator is that the individuals were driven to success; they thought about their mathematical problems day and night; they had strong personalities and a strong code of ethics; they thought for themselves; they were tenacious and obstinate, confident, and competitive. (Many educators have questioned how we can instill these characteristics in students–for these elements seem to be the key to success in mathematics and in most other fields too. Einstein said that creativity is fostered in democratic societies; and R.L. Moore (of U. of Texas fame) proved that mathematicians are developed in competitive atmospheres (type R.L. Moore into Google). Turning shared these characteristics too–he flaunted his homosexuality to the presiding judge at his trial, knowing full well that he was backing the judge into a corner. The judge actually gave Turing a choice, hormone treatment or a year in jail; Turing chose the former. One has to respect Turing because he stood up for what he believed in–and unfortunately the world lost one of the greatest geniuses of our time because of it. I didn't mean to single out Turing's homosexuality, but rather his tenacity and his genius. Brian makes a call that "university curricula for mathematicians and future mathematics teachers ought to include at least one course on the social history of mathematics." And he goes on to say that this course should include some of the topics I have raised in the paper. I fully agree with him.

My paper turned out to be provocative with the referees–and if it gets a few other individuals to think about the issues raised, I will be thrilled. The concerns raised by Brian and Wolf-Michael have helped place the paper into a larger landscape than in it was originally set. I thank them for their responses– they have given each of us even more to think about–and more food for the classroom.

6. A final comment. The problem raised in the paper is an old one. Michael Fried, a colleague of mine at BGU and who has published in this journal, recently sent me a quote alerting me to a discussion that took place more than two and a quarter centuries ago.

Dear Ted, I was reading James Boswell's Life of Johnson this morning–don't ask me why!–and came across a passage that I thought would make a good opening quote, albeit a long one, for your paper on telling the truth in history. It really gives both sides of the coin, which does, indeed, have two sides:

Talking of biography, I said, in writing a life, a man's peculiarities should be mentioned, because they mark his character. JOHNSON. 'Sir, there is no doubt as to peculiarities: the question is, whether a man's vices should be mentioned; for instance, whether it should be mentioned that Addison and Parnell drank too freely: for people will probably more easily indulge in drinking from knowing this; so that more ill may be done by the example, than good by telling the whole truth.' Here was an instance of his varying from himself in talk; for when Lord Hailes and he sat one morning calmly conversing in my house at Edinburgh, I well remember that Dr. Johnson maintained, that 'If a man is to write A Panegyrick, he may keep vices out of sight; but if he professes to write A Life, he must represent it really as it was:' and when I objected to the danger of telling that Parnell drank to excess, he said, that 'it would produce an instructive caution to avoid drinking, when it was seen, that even the learning and genius of Parnell could be debased by it.' And in the Hebrides he maintained, as appears from my Journal, that a man's intimate friend should mention his faults, if he writes his life. (My edition is an abridged version in the Portable Johnson and Boswell (Louis Kronenberger, ed.). There, the quotation is on pp.254-5.)
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I don't think a collective decision on this polemic is possible; so again I ask, where do you personally stand on it? Will you mention some of the tidbits concerning the lives of some of the individuals listed above the next time you speak about them in class? It certainly seems to be something to think about.

Final comments on Eisenberg's paper

Brian Greer
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I must begin by expressing appreciation for the spirit in which Ted accepted my comments as intended to be intellectually and not personally provocative. Let me also assure him that I was not at all offended by the paper – it would be more accurate to say that I was stimulated by it to voice some strong disagreements. Beyond the particular aspects on which we have different perspectives, I applaud his decision to raise ethical issues, too often ignored in writing about mathematics education.

I have little to add to my original response.

In relation to the accreditation of mathematical discoveries, there are two points. One is that, even if the historical record is unclear, it is still possible to acknowledge what is partially known, or even speculation. For example, Hacking (1975, p. 8) commented that "it is reasonable to guess … that a good deal of Indian probability lore is at present unknown to us". The second is to ask the question – to what extent is the information unavailable, or not widely known, because it has been suppressed? I referred, in my initial response, to a paper by Almeida and Joseph on the development of calculus in Kerala. In this paper they argue that:

> This inclination for ignoring advances in and priority of discovery by non-European mathematicians persisted until even very recent times. … A possible reason for such puzzling standards in scholarship may have been the rising Eurocentrism that accompanied European colonization.

In relation to Ramanujan, I find odd the statement that he was "already in the [European/Western] mold of doing mathematics when Hardy and Littlewood took him under their tutelage" in the light, for example, of Hardy's statement that his gifts were "so unlike those of a European mathematician trained in the orthodox school".

The paucity of women achieving eminence in mathematics is not disputed; rather, the question of interest is why that is so.

Finally, I would like to re-emphasize that history has not stopped. As Ted comments at the beginning of his paper, people in universities are increasingly steeped in ethical issues that need to be faced as humankind strives to achieve, as Ubi D'Ambrosio puts it, "survival with dignity". I was fascinated to learn recently that in the draft of his famous (and prescient) speech on the dangers of the military-industrial complex, Eisenhower originally included "academic" (Giroux, 2007).

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Primo Levi (1989, p. 175) wrote:

> It would please me (and it seems to me neither impossible nor absurd) if in all scientific departments one point were insisted on uncompromisingly: what you will do when you exercise your profession can be useful, neutral, or harmful to mankind.
> … Within the limits that you will be granted, try to know the end to which your work is directed. We know the world is not black and white and your decision may be probabilistic and difficult: but you will agree to study a new medicament, you will refuse to formulate a nerve gas.

References

Giroux, H. (2007). *The university in chains: Confronting the military-industrial-academic complex*. Boulder, CO: Paradigm Publishers.

Hacking, I. (1975). *The emergence of probability*. Cambridge: Cambridge University Press.

Levi, P. (1989). *The mirror maker*. New York: Schocken Books.

Values in Mathematics and Science Education: similarities and differences.

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Introduction

In the modern knowledge economy, societies are demanding greater mathematical and scientific literacy and expertise from their citizens than ever before. At the heart of such demands is the need for greater engagement by students with school mathematics and science. As the OECD/PISA definition of numeracy puts it:

> "Mathematical literacy is an individual's capacity to identify and understand the role that mathematics plays in the world, to make well-founded judgements and to use and engage with mathematics in ways that meet the needs of that individual's life as a constructive, concerned and reflective citizen"(OECD, 2003)

Values are an inherent part of the educational process at all levels, from the systemic, institutional macro-level, through the meso-level of curriculum development and management, to the micro-level of classroom interactions (Le Métais, 1997) where they play a major role in establishing a sense of personal and social identity for the student. However the notion of studying values in mathematics education is a relatively recent phenomenon (Bishop, 1999). According to Chin, Leu, and Lin (2001), the values portrayed by teachers in mathematics classrooms are linked to their pedagogical identities. Seah and Bishop (2001) describe the values held by teachers as representing their 'cognisation' of affective variables such as beliefs and attitudes, and the subsequent internalisation of these values into their respective affective-cognitive personal system.

Even in science education the study of values in classrooms is not a major focus of research. Nevertheless, in mathematics and science education values are crucial components of classrooms' affective environments, and thus have a crucial influence on the ways students choose to engage (or not engage) with mathematics and science. Clearly the extent and direction of this influence will depend on the teachers' awareness of, respectively, values ascribed to the particular discipline, the values carried by their selection from the available pedagogical repertoire, and their consciousness or otherwise of imposing their own personal values (Pritchard & Buckland, 1986).

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Data from a previous research project, Values and Mathematics Project (VAMP) has shown that teachers of mathematics are rarely aware of the values associated with teaching mathematics (FitzSimons, Seah, Bishop & Clarkson, 2000). Furthermore, any values 'teaching' which does occur during mathematics classes happens implicitly rather than explicitly (Bishop, 2002).
(Various relevant papers from that study, and from other authors, are available from this website: http://www.education.monash.edu.au/centres/scienceMTE/vamppublications.html)

This paper will report on ideas developed from a more recent research project concerned with values in both mathematics and science education.

Theoretical framework

Comparing values teaching in different subject areas is a relatively novel research approach and some parallel research on teachers of mathematics and history by Bills and Husbands (2004), which builds on the ideas of Gudminsdottir (1991) from English and history teachers, also shows what can be learnt from this approach.

It was decided that for this study, in order to have some basis for the mathematics and science comparisons it would be necessary to use an established theoretical framework for the values studied. We used the six values cluster model developed by the author (Bishop, 1988), based on his analysis of the writings concerning the activities of mathematicians throughout Western history and culture. It is important to stress that the emphasis in that analysis was not primarily on which values might be, are, or should be, emphasised in mathematics education, but rather on the development of mathematics as a subject throughout Western history.

In this model, six value clusters are structured as three complementary pairs, related to the three dimensions of ideological values, sentimental values, and sociological values. These three dimensions are based on the original work of White (1959), a renowned culturologist, who proposed four components to explain cultural growth. White nominated these as technological, ideological, sentimental (or attitudinal), and sociological, with the first being the driver of the others. Bishop (1988) argued that mathematics could be considered as a symbolic technology, representing White's technological component of culture, with the other three being considered as the values dimensions driven by, and also in their turn driving, that technology. The six sets of value clusters are structured as shown in *Figure 1*.

1. Epistemology of the Knowledge (Ideological values)
1a Rationalism
Reason Explanations Hypothetical Reasoning Abstractions Logical thinking Theories
1b Objectivism
Atomism Objectivising Materialism Concretising Determinism Symbolising Analogical thinking

2. How individuals relate to the knowledge (Sentimental or attitudinal values)
2a Control
Prediction Mastery over environment Knowing Rules Security Power
2b Progress
Growth Questioning Alternativism Cumulative development of knowledge Generalisation

3. Knowledge and Society (Sociological values)

3a Openness
Facts Universality Articulation Individual liberty Demonstration Sharing Verification
3b *Mystery*
Abstractness Wonder Unclear origins Mystique Dehumanised knowledge

Figure 1. Values of Western Mathematical Knowledge (Bishop, 1988)

The six value clusters that Bishop (1988) originally identified are described as follows:

> The particular societal developments which have given rise to Mathematics have also ensured that it is a product of various values: values which have been recognised to be of significance in those societies. Mathematics, as a cultural phenomenon, only makes sense if those values are also made explicit. I have described them as complementary pairs, where *rationalism* and *objectism* are the twin ideologies of Mathematics, those of *control* and *progress* are the attitudinal values which drive Mathematical development, and, sociologically, the values of *openness* and *mystery* are those related to potential ownershiop of, or distance from Mathematical knowledge and the relationship between the people who generate that knowledge and others. (Bishop, 1988, p.82)

Values in Mathematics and Science

Regarding their similarities, both mathematics and science are taken as ways of understanding that are embedded in rational logic - focusing on universal knowledge statements. Both are seen by society in general as essential components of schooling, rivalled only by literacy. Hence, teachers of each face substantial political and social pressures from outside the school (e.g., system-wide assessments of student performance, purposes for teaching seen as being directly related to technological development, etc.). In their teaching, both involve following routines, although not exclusively. Both involve modelling, albeit with different emphases. Similarly each is incorporated into the other's applications but in an asymmetrical relationship.

On the other hand, science curricula/texts commonly contain a section on "The Nature of Science" while mathematics rarely contains the equivalent. While the values embedded in

mathematics teaching are almost always implicit, in science teaching some are quite explicit. For example, curriculum movements such as *Science-Technology-Society* make some values explicit and central to the intended learning outcomes; laboratory work seeks to make explicit such values as 'open mindedness,' 'objectivity,' etc.; and content described as *The Nature of Science*, for example, also makes some values explicit (see also UNESCO, 1991).

Among the general public, although the concept of 'a science industry' or 'scientific industries' is widely recognised, this is not the case for mathematics. In the popular media (e.g., magazines, newspapers, books, radio, television), science receives much more attention than mathematics, despite there having been a few recent movies featuring mathematical prodigies. Even when it is present, mathematics is generally subsumed under science. In the case of the popular pursuit of gambling, where mathematical thinking might be considered to play an important role, this is generally not the case as 'luck' seems to be considered a critical factor for many people.

Yet mathematics plays a much more prominent role as a gatekeeper in society than does science. For example, it is often used as a selection device for entry to higher education or employment, even when the skills being tested are unrelated to the ultimate purpose. In broad terms (e.g. modelling or simulations which reduce costs and/or danger), mathematics is considered to be publicly important; at the very same time as it is considered to be personally irrelevant (Niss, 1994), apart from the obvious examples of cooking, shopping and home maintenance. Politically, mathematics has been ascribed a *formatting role* in society (Skovsmose, 1994).

Differences in values between Mathematics and Science, as perceived by the educators in the project.

The project involved two mathematics educators and two science educators, and in the first part of the project there was considerable discussion and analysis of this initial framework, particularly in relation to whether the same structure could hold for science (see Corrigan, Gunstone, Bishop & Clarke, 2004). As a result of this analysis, a comparison of values between the mathematics and science educators was achieved, as shown in *Figure 2*.

Mathematics	Science
Rationalism Reason Explanations Hypothetical reasoning Abstractions Logical thinking Theories	***Rationalism*** Reason Explanations Hypothetical reasoning Abstractions Logical thinking Theories
Empiricism Atomism Objectivising Materialism Concretising Determinism Symbolising Analogical thinking	**Empiricism** Atomism Objective Materialisation Symbolising Analogical thinking Precise Measurable Accuracy Coherence Fruitfulness Parsimony Identifying problems
Control Prediction Mastery over environment Knowing Rules Security Power	**Control** Prediction Mastery over problems Knowing Rules Paradigms Circumstance of activity
Progress Growth Questioning Cumulative Development of knowledge Generalisation Alternativism	***Progress*** Growth Cumulative development of knowledge Generalisation Deepened understanding Plausible alternatives
Openness Facts Universality Articulation Individual liberty Demonstration Sharing Verification	***Openness*** Articulation Sharing Credibility Individual liberty Human construction
Mystery Abstractness Wonder Unclear origins Mystique Dehumanised knowledge Intuition	**Mystery** Intuition Guesses Daydreams Curiosity Fascination

Figure 2: Comparison between values associated with mathematics and science.

As can be seen there is a considerable amount of agreement, but there are some important differences. As far as the Ideological dimension is concerned there are both similarities and differences. In the cluster of Rationalism there is much agreement, as both subjects require the use of all the logic skills available and thus emphasise the range of values associated with those skills. With the value cluster of Objectism, which became recast as 'Empiricism' in order to accommodate the scientists' approach, there is also some agreement, but the highly empirical nature of science means that it has many more value aspects there than does mathematics. The experimental and observational activities of science bring other values into play than we can find in doing mathematics.

For the Sociological dimension, with the complementary pairing of Control and Progress, there was once again some agreement between the mathematics and science educators about the Control value cluster, with its emphasis on prediction, mastery, and procedural rules. However the circumstances of the activity and different paradigms are significant in science but have little meaning in mathematics. Likewise with Progress, the idea of the cumulative development of knowledge is clearly similar, but the role of science in continuing to deepen understanding of a phenomenon again has no parallel in mathematics development.

Some other differences appear with the Sentimental dimension, that is the way individuals relate to the knowledge of the subject. In relation to the Openness value cluster, the emphasis of science on credibility and human construction are significant, compared with the idea of 'facts' in mathematics and the value of verification, sometimes via proof. With Mystery, which itself is a rather mysterious category, the dehumanised nature of mathematics with its abstractness and unclear notions of the origins of ideas contrasts strongly with the intuition, daydreaming, and empirically-based guesses of the scientists.

When considering these contrasts it is important to remember that this framework involves pairs of clustered values along the three dimensions. So the two clusters should not be considered as dichotomous, but rather as complements of each other. For example, Openness is the complement of Mystery, and therefore both clusters are present to some extent in that value dimension. Furthermore, what the model suggests is not that science and mathematics are markedly different but that there are strong similarities in their values, as befits their common heritage. There are however some interesting and, in terms of education, revealing different values represented also.

Mathematics and science teachers' values and practices

We now turn to some of the data collected from the primary and secondary teachers by means of specially constructed questionnaires. They were based on the three complementary pairs, Rationalism and Empiricism, Control and Progress, Openness and Mystery, discussed above. The same structure was used for the mathematics and the science questionnaires and for the primary and secondary teachers, although there were some minor adjustments in the descriptions of teaching situations. 13 primary teachers of years 5/6 and 17 secondary teachers of years 7/8 volunteered to answer these questionnaires. Primary teachers in the state system in Australia teach both subjects to their classes, and we also chose secondary teachers who taught both subjects to the same classes.
Questions 1 and 2 of the questionnaires ask for the extent to which particular activities are emphasised in practice in the teacher's mathematics (and science) classes. The items in these questions are designed to explore, in sequence, aspects of Rationalism and Empiricism, Control and Progress, Openness and Mystery. So, the first three statements in Qu.1 all relate to the value of Rationalism, and so on through the 18 items in Question 1.

Question 2 uses the same structure (a group of 3 items relating to each of the 6 values in the three pairs) to ask about the frequency of use of specific classroom activities. For all the statements in Questions 1 and 2, we scored the responses as 4 (for "Always"), 3 ("Often"), 2 ("Sometimes"), 1 ("Rarely"), and also calculated means. We recognise that in doing this we have taken an ordinal scale and treated it as if it was a ratio scale.

To facilitate comprehension of the results, we have combined the data for Questions 1 and 2, and in the data reported below, for example, a teacher's view of the frequency of emphasis on Rationalism in his/her class' activities is represented by the mean score for the six items relating to that value cluster in the two questions.

Questions 3 and 4 are the questions which concern the teachers' preferences for the six value clusters described above. The structure of these questions is that each question contains 6 statements to be ranked by the teachers. Each statement relates to one of the values clusters, for example, the statement "It develops creativity, basing alternative and new ideas on established ones" relates to the value of Progress. The other statements follow closely the other value descriptors although their order is different in the two questions. Note also that although the teachers knew we were studying values, they were not made aware of the value structure underlying the questions and the various statements.

Tables 1-4 show the results from the two groups of teachers in terms of their rankings of the six values clusters. In brackets are the means of (a) the frequencies in Questions 1 and 2, and (b) the rank orders in Questions 3 and 4.

Table 1 Teachers' preferred values and their preferred teaching practices: rank orders: Primary Mathematics

	Rationalism	Empiricism	Control	Progress	Openness	Mystery
Qus. 1/2	4 (2.64)	2 (2.80)	1 (2.95)	5 (2.44)	3 (2.65)	6 (2.25)
Qu. 3	2 (2.30)	1 (1.46)	6 (5.23)	4 (3.15)	3 (3.53)	5 (3.61)
Qu. 4	3 (3.66)	1 (1.33)	5 (3.75)	2 (3.00)	3 (3.66)	6 (3.83)

We can see that from Table 1 that there is a close similarity between the primary teachers' views on questions 3 and 4, and some close correlation between them and questions 1/2 particularly regarding Empiricism, Openness and Mystery. However, the ranks for Control stand out as being markedly different.

Table 2 Teachers' preferred values and their preferred teaching practices: rank orders: Primary Science

	Rationalism	Empiricism	Control	Progress	Openness	Mystery
Qus. 1/2	2 (3.05)	3 (2.90)	1 (3.07)	4 (2.57)	5 (2.47)	6 (1.91)
Qu. 3	2 (2.75)	1 (1.41)	6 (4.91)	4 (3.41)	5 (3.66)	3 (3.00)
Qu. 4	4 (3.41)	1 (1.41)	6 (4.75)	3 (3.33)	5 (3.83)	2 (2.58)

For Science the primary teachers again express similar views for Questions 3 and 4, and once again the ranks for Control are markedly different from that in Questions 1/2. Mystery is also ranked differently in practice from the teachers' preferred views.

Table 3 Teachers' preferred values and their preferred teaching practices: rank orders: Secondary Mathematics

	Rationalism	Empiricism	Control	Progress	Openness	Mystery
Qus. 1/2	2 (2.15)	3 (2.05)	1 (2.75)	5 (1.93)	4 (1.99)	6 (1.79)
Qu. 3	1 (1.94)	2 (2.05)	6 (4.52)	4 (3.88)	3 (3.35)	5 (4.29)
Qu. 4	1 (1.70)	2 (1.82)	3 (3.44)	4 (4.00)	4 (4.00)	6 (4.47)

The secondary teachers rank Rationalism highest for mathematics in terms of their preferred values (Questions 3 and 4) but, like their Primary colleagues, they place Control in the highest rank in practice.

Table 4 Teachers' preferred values and their preferred teaching practices: rank orders: Secondary Science

	Rationalism	Empiricism	Control	Progress	Openness	Mystery
Qus.1/2	1 (2.86)	3 (2.61)	2 (2.84)	5 (2.30)	4 (2.33)	6 (2.03)
Qu. 3	4 (3.18)	1 (1.25)	6 (5.87)	4 (3.18)	3 (3.06)	2 (2.81)
Qu. 4	3 (3.12)	1 (1.25)	6 (4.12)	2 (3.00)	5 (4.06)	4 (3.33)

For the secondary teachers and science, Questions 3 and 4 show us that the teachers' main value preference is for Empiricism, but in practice they favour Rationalism with Control coming a close second. Once again we can see differences with respect to Control, but this time also with Mystery.

The comparisons between the values in mathematics and science for the two groups of teachers show interesting differences, reflecting their concerns with the curriculum and teaching at their respective levels. For the primary teachers, concerning Ideology, they prefer Empiricism over Rationalism for both science and mathematics, though both are important, rankings which are also reflected in the findings for their preferred practices. At the primary level of course much mathematical work is empirical in nature. For the Sentimental dimension, Control is much less favoured than Progress also for both, but the practices are very different. Another main difference between the subjects appears in the Sociological dimension where Openness and Mystery reverse their positions with the two subjects, the first being more favoured than the second in mathematics and the reverse in science. This difference does not translate to the practices however, with the science practices being ranked much more like the mathematics practices.

For the secondary teachers, concerning the Ideological dimension, they favour Rationalism for mathematics and Empiricism for science, disagreeing with the primary teachers. For the Sentimental dimension, the secondary teachers largely agree with their primary colleagues and for the Sociological dimension, they again agree with their primary colleagues favouring Openness for mathematics compared with Mystery, and reversing these for science. Indeed Mystery for

science is ranked 2 and 4 by the secondary teachers and ranked 2 and 3 by the primary teachers, showing how significant they consider that aspect to be.

Conclusions and implications

The comparison of the values between the science and mathematics educators in the project has revealed perceptions of some important differences between the two subjects. It has also helped to clarify the values structure underlying the current project. In particular, regarding the Ideological dimension, there was evidence that mathematics educators favour the cluster of Rationalism while science educators emphasises Empiricism.

With the Sociological dimension, while both subjects favour Control, the values of Progress differ, with science seeking to deepen understanding of relationships rather than constructing new knowledge as in mathematics. Concerning the Sentimental dimension, there are important differences in both the Openness and Mystery clusters with science seeming to relate more to the humanising aspects of knowledge compared with mathematics.

The comparisons between the values in mathematics and science for the teachers also show interesting differences, reflecting their concerns with the curriculum and teaching at their respective levels. At the primary level the teachers favour Empiricism over Rationalism for both science and mathematics, though both are important, and this contrasts with the findings above. At the primary level of course much mathematical work is empirical in nature. For the Sociological dimension, Control is much less favoured than Progress also for both. The main difference between the subjects appears in the Sentimental dimension where Openness and Mystery reverse their positions with the two subjects, the first being more favoured than the second in mathematics and the reverse in science. This difference shown by the primary teachers reflects the educational implications of the educators' views above.

For the secondary teachers, the Ideological dimension reflects the educators' views, with mathematics favouring Rationalism and science favouring Empiricism, disagreeing with the primary teachers. For the Sociological dimension, the secondary teachers largely agree with their primary colleagues and for the Sentimental dimension, they again agree with their primary colleagues favouring Openness for mathematics compared with Mystery, and reversing these for science. Indeed mystery for science is ranked 2 and 4 by the secondary teachers and ranked 2 and 3 by the primary teachers, showing how significant they consider that aspect to be.

In general, the conceptualisation put forward for this project has begun to show interesting and interpretable results. Discussions with the teachers have revealed an interest in the issues of values teaching in all subjects, but also a lack of vocabulary, and conceptual tools to enable them to develop explicitly the values underlying mathematics education. One of the goals of this project was by contrasting mathematics and science, to help teachers develop those conceptual tools further. As we have seen, and as has been shown above, the contrasts between these two closely related forms of knowledge are provocative, and already reveal worthwhile challenges particularly for mathematics teaching to pursue.

For example, the difference between the emphasis on Empiricism at primary level and on Rationalism at secondary level implies some important challenges for explicit values development

in the teaching of mathematics at those two levels. How should that values development be smoothed across the primary/secondary divide?

The differences in the views on Progress are also revealing, with the development of understanding in science contrasting with the construction of new knowledge in mathematics. How can we reconstruct our views of the mathematics curriculum so that progress through that curriculum is not just a matter of acquiring new knowledge but of ensuring that it also deepens learners' understanding of what has been taught before?

Finally could the dehumanised, highly abstract and mystique-laden value of Mystery of mathematics which appears to be such an obstacle to mathematics learners be made more explicit so that it could be challenged by the more humanised and personal intuitive nature of that value which science appears to enjoy?

However, before jumping to too many conclusions, we must remember that the data are from questionnaires and consist of teachers' reported views of their preferences and their practices. We do not know the extent to which their rankings of these practice statements reflect their actual practices. However, the data for science at the secondary level, where teachers emphasises other values than mathematics, indicates the usefulness of comparing subjects and their values emphases.

Finally one can see that, if the data reported here are valid, the differences show that teachers' values in the classroom are shaped to some extent by the values embedded in each subject, as perceived by them. This implies that changing teachers' perceptions and understandings of the subject being taught may well change the values they can emphasise in class. Further if teachers wish to emphasise values other than those they currently emphasise, it is possible to learn strategies from their teaching of other subjects.

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REFERENCES

Bills, L. and Husbands, C. 2004) Analysing of Embedded Values in History and Mathematics Classrooms. Paper presented at the British Educational Research Association Conference in Manchester, 2004 (September 2004).

Bishop, A. J. (1988). Mathematical enculturation: A cultural perspective on mathematics education Dordrecht, the Netherlands: Kluwer Academic Publishers.

Bishop A.J. (1999). Mathematics teaching and values education - an intersection in need of research. Zentralblatt fur Didaktik der Mathematik, 31(1), 1-4

Bishop, A.J. (2002, April) Research policy and practice: the case of values. Paper presented to the Third conference of the Mathematics Education and Society Group, Helsingor, Denmark.

Chin, C., Leu, Y.-C., & Lin, F.-L. (2001). Pedagogical values, mathematics teaching, and teacher education: Case studies of two experienced teachers. In F.-L. Lin & T. J. Cooney (Eds.), *Making sense of mathematics teacher education* (pp. 247-269).Dordrecht, The Netherlands: Kluwer Academic Publishers.

Corrigan, D.J., Gunstone, R.F., Bishop, A.J. & Clarke, B. (2004) Values in science and mathematics education: mapping the relationships between pedagogical practices and student outcomes. Paper presented at Summer School of the European Science Educational Research Association, Mulheim, Germany, August, 2004.

FitzSimons, G.E., Seah, W.T., Bishop, A.J. & Clarkson, P.C. (2000). What might be learned from researching values in mathematics eduication? In T.Nakahara & M. Koyama (eds.), *Proceedings of the 24th conference of the International Group for the Psychology of Mathematics Education* (Vol. 1), (P.153) Hiroshima: Hiroshima University.

Gudmunsdottir, S. (1991) Values in pedagogical content knowledge. *Journal of Teacher Education*, 41(3) pp 44-52

Halstead, M. (1996). Values and values education in schools. In J. M. Halstead, & M. J. Taylor (Eds.), *Values in education and education in values* (pp. 3-14). London: Falmer

Kluckholm, C. (1962). *Culture and Behavior* New York: Macmillan.

Kohlberg, L. (1981). *The philosophy of moral development : Moral stages and the idea of justice* (1st Edition) San Francisco: Harper & Row.

Krathwohl, D. R., Bloom, B. S., & Masia, B. B. (1964). *Taxonomy of educational objectives: The classification of educational goals (Handbook II: Affective domain).* New York: David McKay.

Le Métais, J. (1997). *Values and aims underlying curriculum and assessment.* (International Review of Curriculum and Assessment Frameworks Paper 1). London: School Curriculum and Assessment Authority.

Niss, M. (1994). Mathematics in society. In R. Biehler, R. W. Scholz, R. Strässer, & B. Winkelmann (eds.), *Didactics of mathematics as a scientific discipline* (pp. 367-378). Dordrecht: Kluwer Academic Publishers.

Nixon, J. (1995). Teaching as a profession of values. In J. Smyth (Ed.), *Critical discourses on teacher development* (pp. 215-224). London: Cassell.

OECD (2003). *Assessment Framework – mathematics, reading, science and problem solving: knowledge and skills.* Paris: OECD.

Peters, R. (1970) *Ethics and education.* London: Allen and Unwin

Pritchard, A. J., & Buckland, D. J. (1986). *Leisure, values and biology teaching.* (Science and Technology Education, Document Series No. 22) Paris: UNESCO.

Raths, L. E., Harmin, M., & Simon, S. B. (1987). Selections from 'values and teaching'. In J. P. F. Carbone (Ed.), *Value theory and education* (pp. 198-214). Malabar, FL: Robert E. Krieger.

Rokeach, M. (1973). *The nature of human values.* New York: The Free Press.

Seah, W.T. (1999). The portrayal and relative emphasis of mathematics and mathematics educational values in Victoria and Singapore lower secondary school textbooks: a preliminary study. Unpublished Master of Education thesis, Monash University, Melbourne, Australia.

Seah, W. T., & Bishop, A. (2001). Teaching more than numeracy: The socialisation experience of a migrant teacher. In J. Bobis, B. Perry, & M. Mitchelmore (Eds.), *Numeracy and beyond.* (pp. 442-450). Turramurra, NSW: Mathematics Education Research Group of Australasia Incorporated

Skovsmose, O. (1994). *Towards a philosophy of critical mathematics education.* Dordrecht: Kluwer Academic Publishers.

UNESCO (1991). *Values and ethics and the science and technology curriculum.* Bangkok: Asia and the Pacific Programme of Educational Innovation for Development, UNESCO-729.

White, L.A. (1959). *The evolution of culture.* New York: McGraw-Hill.

Doin' the Math: On Meaningful Mathematics-Ethics Connections

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Abstract: In this essay mathematics is conceived of as intentional human activity. Since intention implies choice, there are ethical dimensions to making mathematical choices. Embracing these dimensions requires acknowledging the contextual nature of mathematics. First John Dewey's philosophy of mathematics and a reconsideration of mathematical empiricism are posited as ways to foster a context sensitive understanding of mathematics. Next, I address the ways in which existent conceptions of mathematics—even those which support reform in mathematics education—are insufficient with regard to their ability to recognize its human dimensions. The essay concludes with a distinction between mathematics education that ethically applies existing versions of mathematics and mathematics education that seeks to recast mathematics as a necessarily and undeniably ethical enterprise.

> All of that time where did it go?
> What did you do and what have you got to show for it?
> Doin' the math is kind of a bummer
> You best avoid crunchin' that number
>
> Where are they now and what are they doin'?
> Everyone's ancient at your high school reunion
> Doin' the math don't bring satisfaction
> There's no more addition now it's all subtraction
>
> A monkey, a dog, a horse, a giraffe
> *They're all gonna die but they don't do the math*
> Doin' the math is kind of a bummer
> You best avoid crunchin' that number
>
> --Singer-songwriter Loudon Wainwright III, from *Doin' the Math*

A central argument of this paper is that mathematics is an intentional human activity and that—since intention implies choice—there are ethical dimensions to making mathematical choices. Embracing these connections requires moving away from how we typically conceive of mathematics. Accenting the intentional aspects of engaging in mathematical activity is one effective way to counter the predominant ways of thinking about mathematics and mathematical knowledge, namely that it is different *in kind* than most other forms of knowledge. Blurring the

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sharp distinctions between mathematics and other activity/knowledge makes possible new ways to think about mathematics in the context of its teaching and learning.

As I sat down to begin the task of writing this paper, the first strains of *Doin' the Math* emanated from my office computer speakers. The song is a wry but somewhat bleak account of the inevitability of growing old. When Wainwright refers to "doin' the math" what he presumably means is something akin to thinking about aging, dwelling on the inevitable, or something along those lines. I doubt that the songwriter was trying to make a profound philosophical statement about the nature of mathematics and yet, to my ear—toward the song's end—that's exactly what he did. I was half-listening to the music when Wainwright crooned, "*A monkey, a dog, a horse, a giraffe…they're all gonna die but they don't do the math*."

Wainwright's point, as I take it, is that the years are increasing for all of us, animal friends included, but that this increase in years is understood as *adding* only by humans. This is significant, as it forms the basis of a powerful plain language philosophical counter-argument to the ubiquitous commonsense understanding of mathematics as beyond the human pale. This extra-human, often Plato-inspired conceptualization of mathematics can best be summed up by the response I often get when I mention that my work considers what happens when we choose to think of mathematics as sets of tools humans have constructed to help solve our problems. A common response is that this cannot be so and typically some version of the "if everyone on the planet died tomorrow 2+2 would still equal 4" argument is employed. The power of Wainwright's claim is that it suggests that if "everyone died tomorrow" mathematical activity would *cease*. Certainly, giraffes and other animals would still be aging, but there would be no *addition* of years, as addition requires intentional activity.

My sense is that getting people—particularly those responsible for mathematics teaching and curriculum design—to fundamentally change their way of conceiving of mathematics will require more than just Wainwright's lyrics. Thus, in this paper I argue for a recognition, even an embracing, of the human and hence, of the ethical dimensions of mathematics. Andrew Ward (2007) argues similarly for recognition of the science-ethics connection. In order for the two to be thought of as coexistent, he claims that science must be thought of differently; namely, its contextuality must be put in the foreground. Here, I apply the same strategy with the mathematics-ethics connection, but with mathematics it is a tougher argument to make, as many mainstream versions of mathematics do not acknowledge that it has *any* context, let alone that we can choose to focus on contextual factors. This paper is a call for such a reconceptualization.

To support this call, I first provide a summary of John Dewey's philosophy of mathematics—positing it as a way to think about the nature of mathematics that requires acknowledgement of its contexts. Next, I argue that in order for the context of mathematical activity to be appreciated, mathematical empiricism needs to be given consideration. Next, I address the ways in which existent conceptions of mathematics—even those which support reform in mathematics education—are insufficient with regard to their ability to foster awareness of its context and hence its human dimensions. In the final section of the paper, I make a distinction between mathematics education that ethically applies existing versions of mathematics and mathematics education that seeks to recast mathematics as a necessarily and undeniably ethical enterprise.

What are generally taken to be sterile, extra-human, ethics-neutral mathematical knowledge and techniques have, to paraphrase William James, the trail of the human serpent all over them.

Simply saying so is insufficient and the case needs to be made that rethinking the nature of the mathematical enterprise can help us make meaningful mathematics-ethics connections and, subsequently, to pave the way for an engaging brand of school mathematics that draws sustenance from these connections.

Dewey's *Psychology of Number*: "Doin' the Math" *is* the Math

I probably found Wainwright's song so compelling because it serves as the musical complement to one of Dewey's central points about the nature of mathematics. In *The Psychology of Number and its Applications to Methods of Teaching Arithmetic* (1895), James McLellan and Dewey[2] posit that mathematics exists when existential circumstances give way to a need for consideration of quantity. Dewey surmises that mathematics originated when human questions turned from the crude question of "how much?" to the more refined query of "how many?" Thus, from the start, Dewey frames mathematics in terms of its activity. One of Dewey's most potent (and humorous) quotes on this topic even employs an animal metaphor quite similar to Wainwright's. In making the point that mathematics is intentional human activity, Dewey claims, "There are hundreds of leaves on the tree in which the bird builds its nest, but it does not follow that the bird can count" (p. 23).

Dewey's unorthodox operationalization of the term "psychology" is crucial to understanding his philosophy of mathematics. Contrary to most philosophers and philosophers of mathematics of his day, rather than viewing an individual's psychology as an impediment to or distorting factor of clear apprehension of truth, Dewey saw it as a critical component of coming to know. This is one reason why *Psychology of Number*, a philosophical look at how children come to grasp the concept of number, is such a clear expression of Dewey's philosophy of mathematics. That is, how children come to know mathematical concepts centers on the mental activities (i.e., psychology) of children as they encounter various empirical situations. Dewey described this simple sense of quantity as coming about in light of the human need to measure in order to solve practical problems and to improve lives (p. 42).

Dewey saw the commonly understood distinction between counting and measuring as getting in the way of understanding how children organically come to know number. Counting relates to determining how *many* of something there are and measuring involves determination of how *much* of something there is. In other words, the counting-measuring distinction relates to whether something is a series of parts of one whole, or a related group made up of individual units. Dewey's pragmatic answer held that—depending on context— they may be either. The reasons the individual engaged in the mathematical activity in the first place must be taken into account when answering the question.

Deweyan mathematics is defined and understood by its use. The concept of a particular number (say three) does not reside within a group of three apples, beanbags or any other objects any more than it does in the symbol "3." Three, as a construct, emerges from activities requiring quantification as a means to an end. Dewey's accompanying pedagogy accordingly focuses on measurement, as all counting is measuring and all measuring is counting. Making measurement the vehicle for mathematical explorations ensures, according to Dewey, that number symbols will

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[2] Although the *Psychology of Number* was co-written with McLellan, for the remainder of this paper I will refer only to Dewey. See Stemhagen (2003) for a justification of this decision as well as for a fuller description of Dewey's philosophy of mathematics.

always be linked to concrete units and encourages active, empirically-oriented, and contextualized conceptions of mathematical enterprises. Finally, Dewey understands measurement as taking place in contexts wider than simply the act of measuring. He uses the measurement of a field as an example. In a genuine mathematical inquiry, simply finding the area of the field will not meaningfully measure it. To do so, wider contexts must be considered—that is, what is the field capable of producing? How does it relate to our lives? To answer these questions counting and/or measuring must be employed (e.g., amount of produce, the price it will bring at market, the costs of growing the produce, etc.). Dewey sums up the ways in which mathematical inquiries are inseparable from our broader aims: "All numerical concepts and processes arise in the process of fitting together a number of minor acts in such a way as to constitute a complete and more comprehensive act" (p. 57).

Any number of calculations could be done to measure the fields, but the ones that relate to how we actually live our lives are the calculations that will help us successfully conclude our inquiry. In other words, mathematics is more than just crudely counting or measuring; it requires thoughtful consideration about a multitude of contextual factors. Thus, Dewey's version of mathematics emphasizes the interplay between empirical objects and our actions; it acknowledges the importance of the role of human intent in the construction of mathematical knowledge. To Dewey, the development of mathematics is driven by the ways in which we use it. In fact, it is not too strong of a claim to sum up Dewey's philosophy of mathematics as mathematics is its use. To borrow from (and add to) Wainwright lyrics, "doin' the math" *is* the math.

(Re)Opening the Door to Mathematical Empiricism

While Dewey was certainly no simple mathematical empiricist, his attention to context, particularly physical contexts, possibly leaves him susceptible to critique from those skeptical of the place of empiricism in philosophy of mathematics. By simple empiricism, I am referring to the idea that mathematics exists "out there" in the physical world. That is, the reason why 2+2=4 is because that is what is true in the physical world. To the simple empiricist, the idea of number resides in the environment and one's development of mathematical knowledge takes place as one observes the environment. Although it is beyond the scope of this paper to fully make the case for a reconsideration of the merits of mathematical empiricism,[3] it is interesting that Gottlob Frege's attack on J.S. Mill's mathematical empiricism had much to do with the subsequent marginalization of empiricism as a viable philosophy of mathematics (Kitcher, 1980).

This event is noteworthy because the part of Mill's position that Frege so savagely attacked was, by my read, actually the part whereby Mill went beyond that of a simple empiricist and treaded lightly into the territory of the mathematics-as-human-activity camp. Mill claimed that "… Two pebbles and one pebble are equal to three pebbles…affirms that if we put one pebble to two pebbles, those very pebbles are three" (Mill, p. 168). Here Mill suggests that there is a "we" required to "put" together the pebbles to make three. I see this as a nascent affirmation of the human hand in the creation of mathematics. Kitcher (1980) agrees, stating: "Thus the root notion in Mill's ontology is that of a collecting, an activity of ours, rather that that of a collection, an abstract object (p. 224). To Mill's notion that number comes about from arranging objects, Frege responded: "if Mill is right, we are very lucky that not all objects in the world are nailed down, for otherwise it would be false that 2+1=3" (1997, p. 94).

[3] It should be noted that I am uncertain about the worth of making such a case.

If the premise that the contexts of mathematical inquiry matters is on the mark, then Frege's critique is errant. At first blush, it appears that Frege is attacking the notion that the truth of mathematical statements resides in physical contexts. I think this misses the point, as "nailing down" objects prevents their arrangement and not their physical existence. Clearly there is still a physical context, it is just that Frege's idea of "nailed down" suggests a lack of mobility that, if it was actually the case, may very well have affected the direction of the development of mathematics. If our physical reality was so different that our genuine inquiries had no need or place for the grouping of objects, it is difficult to imagine how (and maybe more importantly, why) mathematics as a discipline would have developed as it did.[4]

All of this suggests Mill is not thinking like a simple empiricist—in focusing on the activity of arranging he is nodding toward the ways in which our choices (in this case choosing to engage in mathematical grouping) create mathematics. Frege mischaracterized Mill's point when he stated that without the moving of objects that addition would be *false*. Dewey, Mill, and other non-simple empiricists might argue that rather than false, without the need and ability to rearrange physical objects it might be that 2+1=3 would be *irrelevant*.

With the tasks of brief explication of Dewey's philosophy of mathematics and a quick plug for the merits of reconsidering empiricism in mathematics complete, let us consider how it is that such a way of thinking about mathematics and mathematics education can help to shed light on the intersections between mathematics and ethics. Dewey's description of the origins and nature of mathematics as emerging from willful human interaction with the environment is one way to make the case that the mathematics and ethics are inseparable. If mathematics comes about as we engage in inquiries in order to live better in the world, it follows that ethics is never far from mathematics.

What's Wrong with the Ways We Think about School Mathematics?

The "math wars" is a label given to the dispute between two mathematics education factions. Traditionalists or back-to-basics proponents argue that the aim of mathematics education should be mastery of a set body of mathematical knowledge and skills. The philosophical complement to this version of the teaching and learning of mathematics is mathematical absolutism. Reform-oriented mathematics educators, on the other hand, tend to see understanding as a primary aim of school mathematics. Constructivism is often the philosophical foundation for those espousing this version of mathematics education.[5]

Given this paper's focus on the task of establishing the importance of human contexts to mathematics it should not be hard to see that the traditionalist's point of view, to the degree that it conceives of mathematics class as a place for the transmission of preexistent, extra-human mathematical truths and skills, is not going to be of much value. The more interesting claim is that reformers, to the degree that they rely on constructivism as an undergirding philosophical

[4] For those who are questioning how it is that I can make the leap from children's rudimentary mathematical understandings to the endeavors of contemporary professional mathematicians, I suggest reading Kitcher's *The Nature of Mathematical Knowledge* (1983). In it, he works to link the simple origins of mathematics to today's complex discipline.

[5] I do not wish to claim that philosophies and pedagogies correspond perfectly to one another. Weber's notion of selective affinity (1996) is useful here as while there are no hard fast rules, there seems to be a tendency for particular ways of thinking about mathematics to have some relation to certain pedagogies.

support also do not have much to offer with regard to the contextualization of mathematics. Reform mathematics has had to work against very firmly entrenched and stubborn traditional mainstream perceptions of mathematics. As a result, some of its reliance on philosophical constructivism has fostered a preoccupation with the ways in which individual children make sense of new mathematical ideas in light of their existing understandings. While I applaud the reform movement's efforts to make school mathematics more learner-centered, this focus can lead to a mathematics education that is overly individual and cognitive. The constructivist focus primarily on the individual's construction of mathematical knowledge, can lead to neglect of other contextual factors, such as social and environmental factors.

Toward a Strong Form of Contextual Recognition

In an effort to answer a very important question—one that serves as the title of their essay—*What is Mathematics Education For?*, Greer and Mukhopadhyay (2003) refer to a contemporary shift whereby mathematics is increasingly being thought of as a human activity with a requisite increase in "recognition of the historical, cultural, and social contexts of both mathematics and mathematics education" (p. 2). While I embrace their vision of a mathematics education that recognizes these connections, I believe that for mathematics to play a meaningful role in making our world a more just place, we need to embrace a strong form of recognition of context and to move away from how we typically conceive of mathematics. Furthermore, rethinking the purpose of school mathematics as a means to arm students with tools for social justice, while certainly an improvement over school mathematics as drill-and-kill or even as sets of isolated individual constructions, might miss the point. It might miss the point because it is a *post hoc* application of mathematics to our social problems. In other words, a strong form of recognition as to the historical, social, and cultural contexts of mathematics and mathematics education requires means that—in addition to using pre-decided upon mathematical knowledge and skills to improve our social circumstances—we need to acknowledge that mathematics itself is fundamentally historically, socially, and culturally situated.[6] In other words, rather than simply finding ethical uses for mathematics, we need to teach mathematics in a way that recognizes that it is not different in kind than other enterprises (particularly ethical ones).

Distinguishing between Ethics as Application and Inseparable, Meaningful Mathematics-Ethics Connections

I have argued that Dewey's philosophy of mathematics with a small dose of appreciation for the role of empiricism in mathematics is one way to pave the way for meaningful mathematics-ethic bonds. In making this claim I have implied a distinction that I wish, at this point, to make explicit. While still certainly not in the mainstream, there is much work being done to study the socio-ethical contexts and possibilities of school mathematics. Moses and Cobb's *Radical Equations* (2001) is a good example of the moral/ethical questions related to choosing to use school mathematics achievement as a sorting mechanism, whereby those who can make it through Algebra I get to go to college and those who don't do not. There are also ethical dimensions related to the performance of marginalized groups in math class. While this certainly relates to Moses' look at mathematics and poor Southern African-American students, scrutiny of the posited gender gap in mathematics performance is also a prime example of this mathematics-

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[6] Jean Lave's anthropological explorations of the situated nature of cognitive activity, particularly of mathematical thinking, provide a good point of departure for such acknowledgement. See, for example, Lave, Murtaugh, and de la Rocha (1984).

ethics connection. Finally, there are clearly ethical dimensions related to choosing to apply mathematics in order to illuminate social justice issues, with Gutstein's *Reading and Writing the World with Mathematics* (2006) serving as a recent example of this line of scholarship. While I believe deeply in the worth of the social justice-oriented projects above, I see such efforts as consisting primarily of using mathematics as it exists in an ethical matter. Although I would certainly never argue against the merits of using mathematics in an ethical matter, I believe that right-minded application is not the extent of the mathematics-ethics connection and that there are deeper, more fundamental connections that relate to the very origin and nature of mathematics.

According to Dewey, the ethically fertile questions of how we can best arrange ourselves provide the origins of mathematics. Thus, the distinction I am making is between ethics as application of existing mathematics for good intent versus the notion of a reconceptualization of mathematics as a fundamentally ethics-laden enterprise.[7] There is a similar question hotly debated in philosophy of technology, where some argue that the commonsense understanding of technology as a neutral set of tools that can be used for good or ill ignores the ethics that are built into technological artifacts from the start (Winner, 1986). Winner gives an example of such non-neutrality by noting how the telephone, rather than being a neutral communication tool to be implemented by users as they see fit, possesses powerful tendencies toward certain social arrangements and has greatly contributed to fundamentally different social arrangements. Winner sums up this technological non-neutrality by asking: "As we make things work, what kind of world are we making?" (p. 17). Part of his solution to this problem is to call for a reconciliation between the making and use of technology. Winner sees our lack of understanding about technology as emerging from a sharp division of labor between those who make technological artifacts and those who consume them.

Davis and Hersh (1981) similarly warn: "The social and physical worlds are being mathematized at an increasing rate. The moral is: We'd better watch it, because too much of it may not be good for us" (p. xv). One way to counter unchecked mathematization and the making-use distinction in mathematics education might be to help students experience some of what goes on in the world of those who are involved in the making of mathematics and in the mathematization of our experience. Hersh (1997) explains mathematics as divided into two areas, front and back. The front is the highly polished finished product of mathematicians and the back is the area where mathematicians are busy engaging in the messy but often practically fruitful activities of mathematicians. He uses the analogy of a restaurant. In the dining room (front) everything is to appear orderly and under control. Those in the front (students) are not privy to all that goes on behind the scenes (in the back) in order to create the seamless experience of dining in the front.

The idea is that if students can get involved in the messy but engaging practices of mathematical creation that it will go a long way toward ensuring that mathematics class is a place where

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[7] Efforts that seek to both reconceptualize mathematics *and* to apply the reconceptualization for social betterment deserve the highest praise. An example of this vein of scholarship is Jo Boaler's work to broaden what we mean by mathematical rationality and to apply it to making school mathematics more inclusive. See, for example, Boaler (1997) Boaler and Greeno (2000). Gutstein also engages in thoughtful philosophical reconstruction in his project and accordingly he also deserves recognition. Since his philosophizing operates instrumentally as a means to support his pedagogy, I position his overall project on the application side of the divide.

children's experiences are grounded in genuine inquiry. That is, our live, human problems could serve as a starting point for the teaching and learning of mathematics in class. The value of what gets taught and learned there could be measured against how well the products of mathematics class address the initial problem. Recalling this essay's opening claim that mathematics should be thought of a form of intentional human activity, I hope that this exploration has helped to render a vision of school mathematics in which students are encouraged to engage in intentional ethical activity to identify problems that lend themselves to mathematical inquiry and that they meaningfully engage in "doin' the math."

References

Boaler, J. (1997). E*xperiencing school mathematics: Teaching styles, sex, and setting.* Buckingham, UK: Open University Press.

Boaler, J. & Greeno, J. (2000). Identity, agency, and knowing in mathematics worlds. In J. Boaler (Ed.), *Multiple perspectives on mathematics teaching and learning.* Westport, CT: Ablex Publishing.

Davis, P. & Hersh, R. (1981). T*he mathematical experience*. Brighton, England: Harvester.

Frege, G. (1974) Foundations of arithmetic. In M. Beaney (Ed.), *The Frege Reader.* Malden, MA, Blackwell.

Greer, B. and Mukhopadhyay, S. (2003). What is mathematics education for? *The Mathematics Educator*, 13(2), 2-6.

Gutstein, E. (2006). *Reading and writing the world with mathematics: Toward a pedagogy for social justice.* New York: Routledge.

Hersh. R. (1997). *What is mathematics, really?* New York: Oxford University Press.

Kitcher, P. (1980). Arithmetic for the Millian. *Philosophical Studies*, 37, 215-236.

Kitcher, P. (1983). *The nature of mathematical knowledge*. New York, Oxford University Press.

Lave, J., Murtaugh, M., & de la Rocha, O. (1984). The dialectic of arithmetic in grocery shopping. In B. Rogoff & J. Lave (Eds.), *Everyday cognition: Its development in social context* (pp. 67-94). Cambridge, Massachusetts: Harvard University Press.

McLellan, J. & Dewey, J. (1895). *The Psychology of Number and its Applications to Methods of Teaching Arithmetic*. New York: D. Appleton and Company.

Mill, J. S (1967). *A system of logic*, 8th ed. London: Longman's, Green and Co.

Moses, R., & Cobb, C. (2001). *Radical equations: Math literacy and civil rights*. Boston: Beacon Press.

Stemhagen, K. (2003). Toward a pragmatic/contextual philosophy of mathematics: Recovering Dewey's 'Psychology of Number'. In 2003 Philosophy of Education Yearbook. Urbana, IL: Philosophy of Education Society.

Stemhagen, K. (2007). Empiricism, contingency and evolutionary metaphors: Getting beyond the "math wars". *International Electronic Journal of Mathematics Education*, 2(2), 91-105.

Ward, A. (2007). Ethics and observation: Dewey, Thoreau, and Harman. *Metaphilosophy*, 38(5), 591-611.

Weber, M. (1996). *The Protestant ethic and the spirit of capitalism*. Los Angeles: Roxbury Publishing Company.

Winner, L. (1986). *The whale and the reactor: A search for limits in an age of high technology*. Chicago: The University of Chicago Press.

The Action Map as a Tool for Assessing Situated Mathematical Problem Solving Performance

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ABSTRACT. The aim of this paper is to investigate the appropriateness, concurrent and construct validity of action map as a tool for assessing situated problem solving performance. Action map is rooted in activity theory whose stipulations are compatible with situated problem solving. Thirty-one last year secondary students were given three tasks with real –world context. Based on the analysis of the written solutions and interviews, evidence is presented on the appropriateness and validity of action map as an instrument to assess situated problem solving performance.

Despite many calls for including applications as a major goal of teaching mathematics citing a variety of social, psychological, pedagogical reasons and justifications, assessment lagged behind in developing appropriate tools to assess situated problem solving (de Lange, 1996). Existing assessment taxonomies, rubrics, and models are lacking in that they are not embedded in a theory that adequately explain the complexity of interaction with reality in situated problem solving. We believe that the action map is an appropriate assessment tool for situated problem solving and at the same time is embedded in activity theory (Leont'ev, 1981) that stipulates that human behavior and thinking are inseparable and occur within meaningful contexts as people conduct purposeful goal-directed activities. The aim of this paper is to describe the action map as an instrument for assessing situated problem solving and to present evidence in support of its construct and concurrent validity. The action map is based on activity theory whose conceptual framework is compatible with situated problem solving. Concurrent validity will studied in relation to an assessment rubric for problem solving.

Activity Theory

Activity theory was developed by Leont'ev (1981). He defined activity as: "…the unit of life that is mediated by mental reflection. The real function of this unit is to orient the subjects in the world of objects. In other words, activity is not a reaction or aggregate of reactions, but a system with its own structure, its own internal transformations, and its own development." (p.46). A central assertion of activity theory is that our knowledge of the world is mediated by our interaction with it, and thus, human behavior and thinking occur within meaningful contexts as people conduct purposeful goal-directed activities. This theory strongly advocates socially organized human activity as the major unit of analysis in psychological studies rather than mind or behavior. Leont'ev (1981) identified several interrelated levels or abstractions in theory of activity. Each level is associated with a special type of unit. The first most general level is associated with the unit of *activity* that deals with specific real activities such as work, play, and learning. The second level of analysis focuses on the unit of a *goal-directed action* that is the process

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subordinated to a conscious goal. The third level of analysis is associated with the unit of *operation* or the conditions under which the action is carried out. Operations help actualize the general goal to make it more concrete.

Human activity can be realized in two forms: “mental” activity or internal activity and practical objective or external activity (Leont’ev, 1981). The fundamental and primary form of human activity is external and practical. This form of activity brings humans into practical contact with objects thus redirecting, changing and enriching this activity. The internal plane of activity is formed as a result of internalizing external processes. “Internalization is the transition in which external processes with external, material objects are transformed into processes that take place at the mental level, the level of consciousness” (Zinchencho & Gordon, 1981, p.74).

Three types of actions in mental activities had been identified: perceptual, mnemonic, and cognitive (Zinchencho & Gordon, 1981). Perceptual actions are those by which the human being maintains contact with the environment. They are initiated by stimuli from the environment and enriched on the basis of prior experience. Mnemonic actions refer to actions, which involve recognition, reconstruction, or recall (Piaget & Inhelder as cited in Zinchencho & Gordon, 1981). Cognitive actions involve thinking in terms of images of real objective processes (Gal’perin cited in Zinchencho & Gordon, 1981).

Activity theory was selected as a conceptual model in this study because it advocates socially organized human activity as the major unit of analysis in psychological studies rather than mind or behavior and because it makes the assumption that thinking and doing are inseparable

Action Map

The action map is a schematic representation of organization and sequence of the actions of the objective content of an activity (see Figure 1) using the method of structural-analysis (Zinchencho & Gordon, 1981). This method was used because it puts forward an operational analytic method derived from activity theory itself. It provides a way for representing the structure of activity as a system of interconnected units with potential relationships among them and among types of connections. In the systematic-structural approach, it is assumed that the structure of actions and operations, the internal transitions from one action to another, and their sequential organization depend on the objective content of activity. Thus the identification of the organization and sequence of the actions of the objective content of an activity provides a characterization of its level, form, and type (Zinchencho & Gordon, 1981).

A number of studies used activity theory to investigate work activities (Millroy, 1992 ; Masingila, 1996 ; Pozzi et al, 1998). Jurdak and Shahin (2001) used activity theory to compare work and learning activities. It is in the last study that structural analysis was used systematically and action map was used as a tool without actually using the name ‘action map’.

Methodology

Sample

The sample consisted of 31 grade 12 students selected from four private schools in Beirut, Lebanon. Their teachers nominated the students as being from the highest achievers in mathematics in their classes. All the students were in the last grade of secondary school and were

in the general science stream, which prepares students for university studies in mathematics, sciences, and engineering.

Problem Tasks

The problem tasks were constructed to meet the three criteria set for situated problem solving. First, the problem situation has to be *real* to the population of the students concerned. By that we mean that the situation is within the current experiential space of students. Second, the problem has to be formulated in a context in the sense that the problem solver may have to put boundary conditions or introduce assumptions and data and to engage in a process of mathematization to formulate the problem in mathematical terms. Third, the problem task should lend itself to multiple approaches and different levels of treatments.

The process of searching, constructing, and screening resulted in three tasks (Appendix A) that were judged by the researcher to be meaningful and satisfy the three criteria. The *Car Loan* Task presents a situation where two options for payment in installments for a car. The student is to decide which option is better and why. The *Cell Phone Task* presents a situation where two actual offers for a cell phone from two companies are presented with all the specifications as advertised. The student is to decide which offer is better and to rationalize the decision. In the *BMI Task,* the formula for the body mass index together with a table of norms and BMI chart. The student is asked to rationalize how the chart was produced from the table of norms.

Assessment Rubric

A rubric adopted from the QUASAR project (Lane, 1993) was used to assess the solutions of the problems. The rubric assesses mathematical knowledge, strategic knowledge, and communication. *Mathematical knowledge* is defined as the degree to which the student shows understanding of the task's mathematical concepts and principles; uses appropriate mathematical terminology and notations; and executes algorithms completely and correctly. *Problem solving* is defined as the degree to which the student may use relevant outside information of a formal or informal nature; identifies all the important elements of the task and shows understanding of the relationships between them; reflects an appropriate and systematic strategy for solving the task; and gives clear evidence of a solution process, and solution process is complete and systematic. *Communication* is defined as the degree to which the student gives a complete response with a clear, unambiguous explanation and/or description which may include an appropriate and complete diagram; communicates effectively to the identified audience; presents supporting arguments which are logically sound and complete; may include examples and counter-examples. The five scale points were defined as follows:
0 (*No Answer*), 1 (*Inadequate*), 2 (*Minimal*), 3 (*Competent*), 4 (*Exemplary*).

Procedure

In each of the four schools, the selected students were asked to come to a designated room in the school. The investigator explained the purpose of the study to them and their queries were addressed. Each student was asked to read the three tasks and choose one of them. While solving the task, each student was asked by the investigator about how their approach of the solution of the task. The problem solving session lasted for 60 to 90 minutes. Students were allowed to use calculators and computers and to ask questions about the task during the session. All interviews were audio-taped. It should be mentioned that the tapes were not used for the purpose of this study and were intended for another aspect of the study which focuses on studying situated problem solving as an activity. All what the students wrote during the problem solving session

was collected and properly identified. The written solutions constituted the basic documents that were subjected to documentary analysis in two distinct ways. First, using the assessment rubric, two raters assessed each solution and a comparison of a sample of the two ratings showed a high degree of agreement between the two raters (at least 90% in the three categories).

Second, the written solutions were subjected to structural analysis (Zinchencho & Gordon, 1981). This was an iterative process in which a researcher reviewed the written solution of each student to identify the actions and putting a short description of each action. The sequence of these actions, as they unfolded based on the written solution, were identified. The descriptions of the actions were put in boxes and connected with arrows to indicate the sequence of actions. The actions were then classified into one of three categories (perceptual, mnemonic, and cognitive). The constructed action map was then validated against the written solution and modified accordingly. This iterative process continued until the action map was judged as accounting for almost all the actions, their sequence, and their type. Another researcher did a second validity check by comparing the written solution with the constructed action map. The final product was a figure similar to those in Figure 1, which represents two rather contrasting action maps for two students who the cell- phoned task.

Data Analysis

Five variables were identified from the action map: Relative frequency of mnemonic actions (R/MN), relative frequency of cognitive actions (R/COG), relative frequency of perceptual actions (R/PER), number of actions (ACTIONS), and number of loops (LOOPS) (a loop was defined as a triangle formed by the arrows that indicate the sequence of actions). Four variables were identified from the assessment rubric as follows: Math knowledge, problem solving, communication, and total (the sum of the three variables).

Two statistical analyses were done. A stepwise multiple regression was performed to identify the variables in the action map that predict problem solving performance as measured by the assessment rubric. Second a factor analysis with a Quartimax rotation was done to examine the construct validity of the action map by identifying the structure of the action map and the factors therein. To illustrate these variables we calculated their values for the two examples in Figure 1 (Table 1)

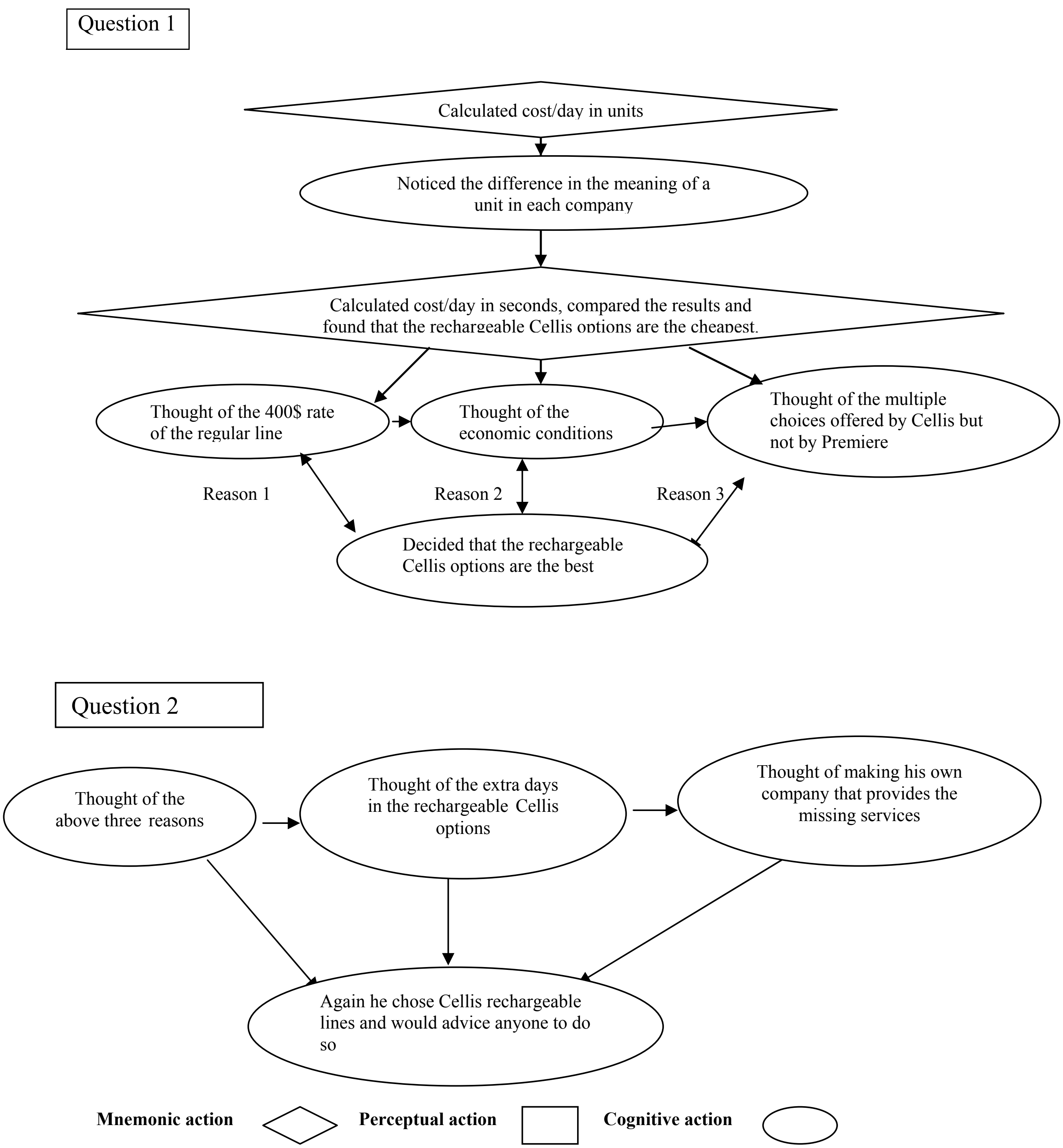
Table 1: Values of the variables for examples 1 &2

	Example 1	Example 2
R/MN[1]	2/11	14/16
R/COG[2]	9/11	2/16
R/PER[3]	0/11	0/16
ACTIONS[4]	11	16
LOOPS[5]	6	2
Math Knowledge	4	2
Problem Solving	4	2
Communication	4	2

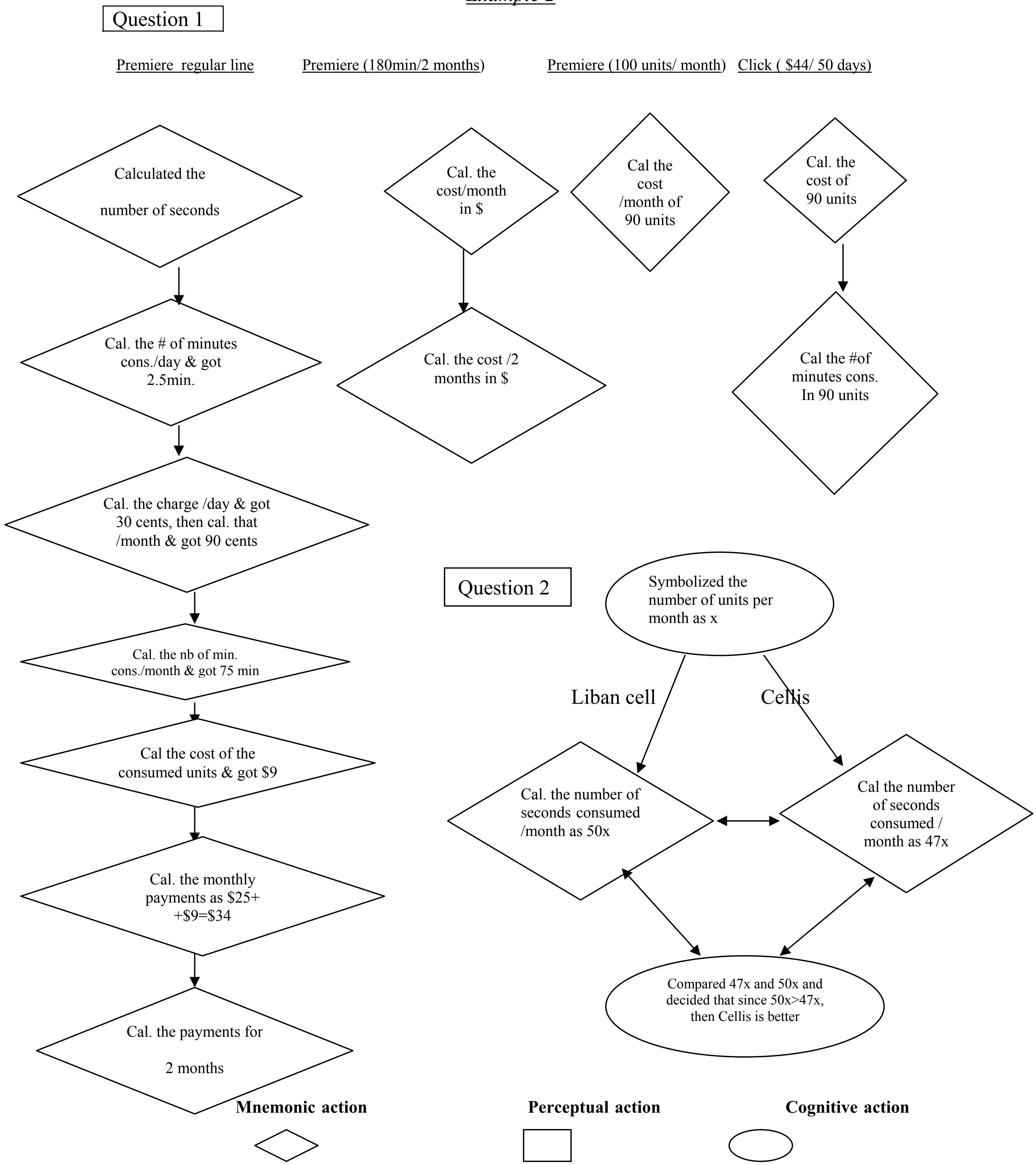
[1]Relative frequency of mnemonic actions
[2]Relative frequency of cognitive actions
[3]Relative frequency of perceptual actions
[4]Number of actions
[5]Number of loops

Figure1. An example of an action map (Cell Phone Task)

<ins>*Example 1*</ins>



Example 2


Question 1
Premiere regular line
Premiere (180min/2 months)
Premiere (100 units/ month)
Click ($44/ 50 days)
Calculated the number of seconds
Cal. the # of minutes cons./day & got 2.5min.
Cal. the charge /day & got 30 cents, then cal. that /month & got 90 cents
Cal. the nb of min. cons./month & got 75 min
Cal the cost of the consumed units & got $9
Cal. the monthly payments as $25+ +$9=$34
Cal. the payments for 2 months
Cal. the cost/month in $
Cal. the cost /2 months in $
Cal the cost /month of 90 units
Cal. the cost of 90 units
Cal the #of minutes cons. In 90 units
Question 2
Symbolized the number of units per month as x
Liban cell
Cellis
Cal. the number of seconds consumed /month as 50x
Cal the number of seconds consumed / month as 47x
Compared 47x and 50x and decided that since 50x>47x, then Cellis is better
Mnemonic action
Perceptual action
Cognitive action

The action map seems to be an adequate tool for assessing situated problem solving in at least two ways. First, it provides a representation of not only the product but also the process of problem solving in the sense that it maps the actions and their sequence as they unfold in the problem solving process thus providing a visual representation of the internal structure of the activity. Second, it captures the interaction between the problem solver and reality because it describes the sequence of actions as they occur simultaneously in the internal plane (thinking) as well as the external plane (doing).

Reliability

Cronbach α for the rubric across its five levels and for the action map across its variables (relative frequency of mnemonic actions, relative frequency of cognitive actions, relative frequency of perceptual actions, number of actions, and number of loops) are reported in Table 2. In spite of the small sample in this study, Cronbach α was moderate high, indicating a reasonable internal consistency for the action map.

Table 2.Cronbach α for the Rubric and Action Map

Task	*Rubric*	*Action Map*
Car	.63	.73
Cell phone	.67	.69
BMI	.88	.52

Concurrent Validity

The concurrent validity of the action map relative to the assessment rubric seems to be quite high. The results of the stepwise multiple regression (Table 3) indicate that, as measured by the action map, the predictors of mathematical knowledge, problem solving, communication, and overall performance, as measured by the rubric, fall into two categories. The first category consists of the relative frequency of the type of

Table 3. Results of Regression Analysis

Variable	Math knowledge		Problem solving		communication		Total	
	R	R^2	R	R^2	R	R^2	R	R^2
R/MN[1]	-	-	.54	.30	.45	.21	.59	.35
R/COG[2]	.57	.33	-	-	-	-	-	-
R/PER[3]	-	-	-	-	-	-	-	-
ACTIONS[4]	.70	.49	.77	.59	-	-	.76	.57
LOOPS[5]	-	-	.80	.64	-	-	-	-

[1]Relative frequency of mnemonic actions
[2]Relative frequency of cognitive actions
[3]Relative frequency of perceptual actions
[4]Number of actions
[5]Number of loops

actions (relative frequency of mnemonic actions, relative frequency of cognitive actions, relative frequency of perceptual actions) and the second of the structure of the action map (number of actions and number of loops). For mathematical knowledge, the relative frequency of cognitive actions, and number of actions, account for 49% of the variance. For problem solving, relative frequency of mnemonic actions, number of actions, and number of loops account for 64% of the variance. For communication, relative frequency of mnemonic actions account for 21% of the variance. For the overall performance (total score), relative frequency of mnemonic actions and the number of actions account for 57% of the variance. In general, performance in problem solving increases with the increase in the relative frequency of cognitive actions (or the decrease in frequency of mnemonic actions since this is negatively correlated with the relative frequency of cognitive actions as indicated in Table 4) and the increase in number of actions and number of loops. In other words the quality of problem solving is dependent on the frequency of cognitive actions and the complexity of the structure of the action map.

It is quite remarkable that these two categories of variables in the action map (a tool embedded in activity theory) account for high percentage of problem solving as measured by the assessment rubric, which has different assumptions.

Table 4.Correlation Matrix

	R/MN[1]	R/COG[2]	R/PER[3]	ACTIONS[4]	LOOPS[5]
R/MN[1]	1	-.85	-.53	.10	.01
R/COG[2]	-.85	1	.23	.03	.14
R/PER[3]	-.53	.23	1	-.25	-.26
ACTIONS[4]	.10	.03	-.25	1	.62
LOOPS[5]	.01	.14	-.26	.62	1

[1]Relative frequency of mnemonic actions
[2]Relative frequency of cognitive actions
[31]Relative frequency of perceptual actions
[4]Number of actions
[5]Number of loops

Construct Validity

We examined further the structure of the of the action map by performing a factor analysis with Quartimax rotation on the variables derived from the action map. The analysis provided support to the two- factor structure (Table 5): Factor 1 with high loadings on the type of action (relative frequency of mnemonic actions, relative frequency of cognitive actions, relative frequency of perceptual actions) and Factor 2 with high loadings on the structure of the action map which reflect the complexity of the activity(number of actions and number of loops)

Table 5: Factor Structure of the action map

Variable	Factor 1	Factor 2
-Ratio of cognitive actions to total number of actions(R/COG)	.90	.18
-Ratio of perceptual actions to total number of actions (R/PER)	.59	-.45
-Number of actions (ACTIONS)	-.05	.86
-Number of loops (LOOPS)	.05	.89
-Ratio of mnemonic actions to total number of actions (R/MN)	-.97	.04
%of variance	42.17	35.46

The usability of the action map calls for addressing practical questions as to what and how the it may be used as an assessment tool. This study has demonstrated that the action map may be used as a theory-embedded alternative tool to the rubric in assessing performance on mathematical problem solving by trained researchers in a research context. One would conjecture that the action map may be used by teachers to assess problem solving performance of situated problem tasks outside the classroom, assuming that teachers are trained in using action map. It remains an open question whether the action map can be constructed from an audio tape of problem solving through the thinking –aloud technique. The decision to use the action map as an alternative to the rubric is to be mediated by curricular goals of mathematics as well the comparative costs and benefits of the two tools. It should be mentioned that we are not making any claim that action map may be used to assess traditional procedural knowledge or conceptual understanding.

This study has also demonstrated that the action map may be constructed from the written solutions of students only. Our experience shows that the action map requires less time to construct than a rubric, however, the construction of an action map for an individual student will require much more than to administer than an already available rubric.

In conclusion, the action seems to be a promising tool for assessing situated problem solving. It is a tool which is embedded in a theory compatible with the assumptions of situated problem solving and the same time is usable in assessment of problem solving in mathematics classes as a viable alternative to rubrics.

Endnote:

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References

Lane, S. (1993).The conceptual framework for the development of a mathematics performance assessment instrument for QUASAR. *Educational Measurement: Issues and Practice*, *12*(2), 16-23.

de Lange, J. (1996).Using and applying mathematics in education. In Bishop et al. (Eds.), *International Handbook of Mathematics Education* (pp. 49-97). Netherlands: Kluwer Academic Publishers.

Jurdak, M. & Shahin, I. (2001). Problem solving activity in the workplace and the school: the case of constructing solids. *Educational Studies in Mathematics, 47*, 297-315

Leont'ev, A.N. (1981). The problem of activity in psychology. In Wersch, J.V. (Ed.), *the concept of activity in Soviet Psychology*. New York: M.E.Sharp

Masingila, J. O. (1996). Mathematics practice in carpet laying: A closer look at problem solving in context, ERIC Document Reproduction Service, ED398 068.

Millroy, W.L. (1992). An ethnographic study of the mathematical ideas of a group of Carpenters. *Journal for Research on Math Education Monographs*: *5*, 0883-9530

Pozzi, S.; Noss, R.; & Hoyles, C. (1998). Tools in practice, mathematics in use. *Educational Studies in Mathematics Education* 36:105-122.

Zinchenko, V. P & Gordon, V. M. (1981). Methodological problems in analyzing activity, In Wertsch (Ed.), *the concept of activity in Soviet psychology* (pp.72-133). NY: M.E.Sharpe.

Appendix A
Context Problem Tasks
A.1 *Car Loan Task*

Rasamny Youniss Company is making a special offer on Nissan-Almera cars, model 1999, and automatic/full option for $13950 cash. Now, you have two options for payment in installments, either through the bank or through the company itself. Through the bank, and with a down payment of $5,000, you can pay with a 12% annual interest on the balance, $305 at the end of each month. However, the second option, and with a down payment of $5,000 you can repay, in equal monthly installments for 36 months at an annual interest rate of 7.5% on the total.

1) Suppose you wanted to pay the whole remaining amount after 6 months. In each option, how much do you have to pay to close your account?
2) Which is the most convenient option for paying for the car?

A.2 *Cell Phone Task*

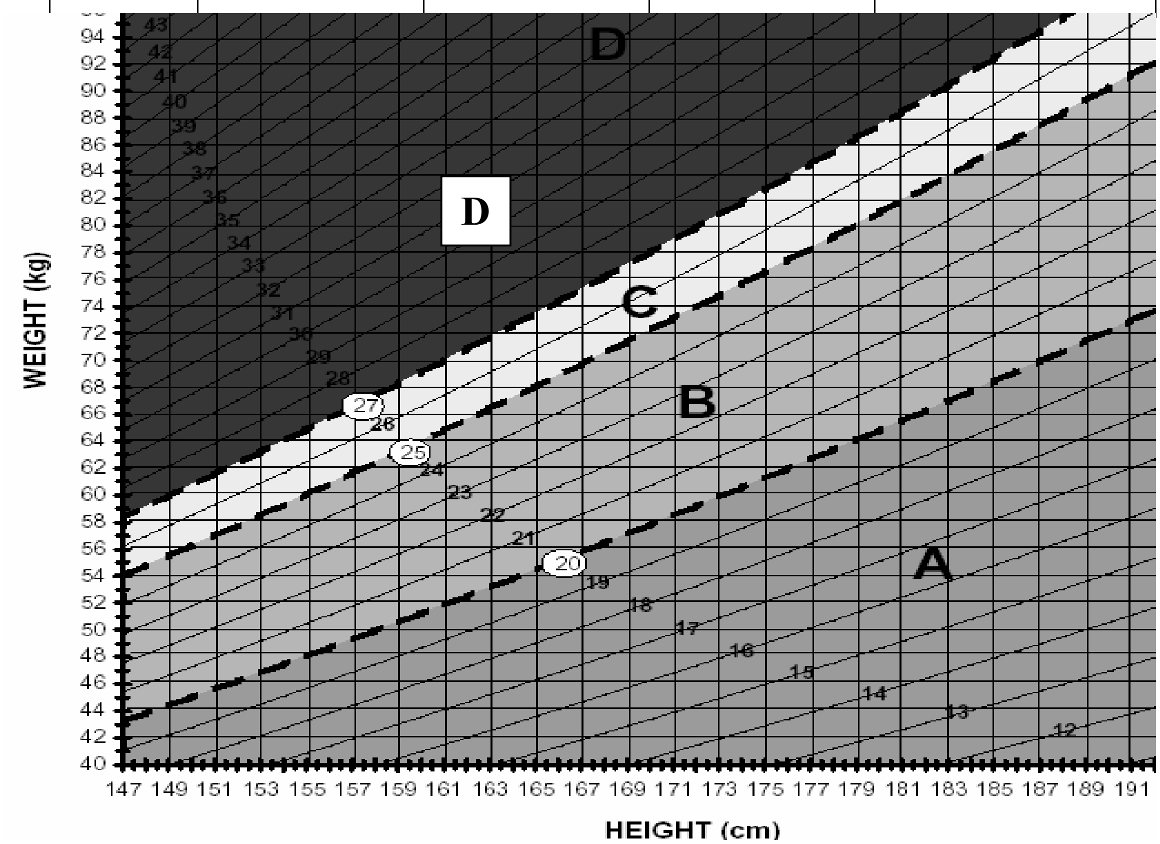
If you want to get a mobile phone, Libancell and Cellis offer multiple services. Both can give you a regular line for $400 with $25 fixed monthly payment and the call will be charged 12 cents/ min. An alternative plan is providing monthly rechargeable cards with a certain number of units. While Libancell offers a Premiere line, Cellis provides a Click line. To get a Premiere line you have to pay $75 a fixed amount for the line and you can recharge it every two months for 103000 L.L. (180 units with duration of 50 second/unit) or 68000L.L (180 units with duration of 50 seconds/unit). To get a Click line you have to pay $75 a fixed amount for the line and you can recharge it through buying separate cards with prices varying according to the time it serves. A $22 (90 units) rechargeable cards serves for 15 days with 5 extra days for receiving calls only, $33 (135 units) cards serves for 25 days with 10 extra days for receiving calls only, and $44 (180 units) card serves for 40 days with 10 extra days for receiving calls only. With Cellis click line, the unit duration is 47 seconds.

1) Suppose that you consume 3 units per day on the average. Which of the options is the cheapest? Explain.
2) Given the number of the units consumed daily, which of the three options is the cheapest? Explain.

A.3 *BMI Task*

Finding out your body mass index (BMI) is a quick way to figure out if your weight is healthy for your height. Nutritionists have developed refined ways to interpret BMI values, for instance, different BMI values can mean you are underweight, ideal weight, slightly overweight or obese. BMI can be calculated as W/h^2, W=weight (kg), h=height (m).Given the following norms, find a mathematical way that may be used to transform these norms into the chart below:

Symbol	A	B	C	D
BMI	< 20	20 ---25	25---27	27>
Condition	Underweight	Correct	Overweight	Obese



Relationship of some psychological variables in predicting problem solving ability of in-service mathematics teachers

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Abstract
This paper examines some psychological variables in predicting problem solving ability of in-service mathematics teachers. The sample consists of 122 in-service teachers enrolled in degree programme. Five standardized instruments were used to collect the data on teachers' mathematics anxiety, mathematics teaching efficacy belief, locus of control, study habits and problem solving ability. Multiple regression, Chi-square analysis,and Pearson moment correlation coefficient were used to analyze the data. The results show that mathematics anxiety, mathematics teaching efficacy belief, locus of control and study habits all have significant relationships with problem solving ability with mathematics anxiety having the highest and study habits the lowest as stated above. Implications for mathematics teacher education were discussed.

Key Words: Mathematics anxiety; mathematics teaching efficacy belief; locus of control; study habits; problem solving ability; in-service teachers

Introduction

Teachers' beliefs about mathematics have a powerful impact on the practice of teaching (Uusimaki, & Nason, 2004; Charalambos, Philippou & Kyriakides, 2002; Ernest, 2000). It has been suggested that teachers with negative beliefs about mathematics influence a learned helplessness response from students, whereas the students of teachers with positive beliefs about mathematics enjoy successful mathematical experiences that result in them seeing mathematics as a discourse worthwhile of study (Karp, 1991). Thus, what goes on in the mathematics classroom may be directly related to the beliefs teachers hold about mathematics. Hence, it has been argued that teacher beliefs play a major role in their students' achievement and in their formation of beliefs and attitudes towards mathematics (Emenaker, 1996). Addressing the causes of negative beliefs held by pre-service primary teacher education students about mathematics therefore is crucial for improving their teaching skills and the mathematical learning of their students (Uusimaki & Nason, 2004). Reboli, & Holodick (2002) reported that the National Council of Teachers of Mathematics in its 1991 publication Professional Standards for Teaching Mathematics (NCTM, 1991) and the current Mathematics Program Standards for the National Council for Accreditation of Teacher Education (NCATE, 1998) stress the importance of the

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disposition of the classroom teacher towards mathematics. They maintain that if students are to develop a disposition to do mathematics, it is essential that the teacher communicate a positive attitude towards mathematics. Additionally, teachers need to establish a supportive classroom learning environment that fosters the confidence of students to learn mathematics. Unfortunately, research has reported that many pre-service elementary teachers have negative attitudes toward mathematics, are not confident in their own mathematics ability, and claim to have a high level of anxiety towards mathematics (Harper & Daane, 1998; Tooke & Lindstrom, 1998). So it is important for mathematics teacher educator to continually search for more personal factors that could hinder elementary school teachers from adequate delivery of instructions to their pupils.

The issue of mathematics teachers' self-efficacy, study habits, locus of control, anxiety towards the teaching and learning of mathematics as well as their problem solving ability is the concerns of this study.

Problem solving

Problem solving has a special importance in the study of mathematics. A primary goal of mathematics teaching and learning is to develop the ability to solve a wide variety of complex mathematics problems (James W. Wilson, Maria L. Fernandez, and Nelda Hadaway (1993). Stanic and Kilpatrick (1988) traced the role of problem solving in school mathematics and illustrated a rich history of the topic. To many mathematically literate people, mathematics is synonymous with solving problems -- doing word problems, creating patterns, interpreting figures, developing geometric constructions, proving theorems, etc. On the other hand, persons not enthralled with mathematics may describe *any* mathematics activity as problem solving (Wilson, Fernandez, and Hadaway, 1993). Problem solving is an integral part of all mathematics learning. In everyday life and in the workplace, being able to solve problems can lead to great advantages. However, solving problems is not only a goal of learning mathematics but also a major means of doing so. Problem solving means engaging in a task for which the solution is not known in advance. Good problem solvers have a "mathematical disposition"--they analyze situations carefully in mathematical terms and naturally come to pose problems based on situations they see.

Good problems give students the chance to solidify and extend their knowledge and to stimulate new learning. Most mathematical concepts can be introduced through problems based on familiar experiences coming from students' lives or from mathematical contexts. As students try different ideas in solving problems, the teacher can help them to converge their ideas towards the solution of the problem, thus providing a meaningful introduction to a difficult concept.
Students need to develop a range of strategies for solving problems, such as using diagrams, looking for patterns, or trying special values or cases. These strategies need instructional attention if students are to learn them. However, exposure to problem-solving strategies should be embedded across the curriculum. Students also need to learn to monitor and adjust the strategies they are using as they solve a problem.

Teachers play an important role in developing students' problem-solving dispositions. They must choose problems that engage students. They need to create an environment that encourages students to explore, take risks, share failures and successes, and question one another. In such supportive environments, students develop the confidence they need to explore problems and the ability to make adjustments in their problem-solving strategies. (NCTM, 2000).

"The first rule of teaching is to know what you are supposed to teach. The second rule of teaching is to know a little more than what you are supposed to teach. . . . Yet it should not be forgotten that a teacher of mathematics should know some mathematics, and that a teacher wishing to impart the right attitude of mind toward problems to his students should have acquired that attitude himself" Polya (p. 173). It then follows that teacher of mathematics should by themselves be comfortably with problem solving otherwise they might not be able to effectively inculcate the attitude of problem solving to their students.

Many teachers do recognize that non traditional strategies are necessary to meet the learning needs of their increasingly diverse students. Embracing change can be unsettling, but these teachers venture into new territory, opening a world of discovery for themselves and their students. For they know that a young mind carefully nurtured may be the next big thinker to solve another of the world's mysteries (Jarrett, 2000).

The importance of students' (and teachers') beliefs about mathematics problem solving lies in the assumption of some connection between beliefs and behavior. Thus, it is argued, the beliefs of mathematics students, mathematics teachers, parents, policy makers, and the general public about the roles of problem solving in mathematics become prerequisite or co-requisite to developing problem solving (Wilson, Fernandez, & Hadaway,1993). The question then is: Are the teachers who are suppose to the lay the good foundation for the student's problem solving capacity themselves good problem solvers?

Teacher Efficacy and Academic Achievement

Teacher efficacy has proved to be powerfully related to many meaningful educational outcomes such as teachers' persistence, enthusiasm, commitment and instructional behavior, as well as self-efficacy beliefs (Tschannen-Moran, & Hoy, 2001). A teacher's efficacy belief is a judgment of his or her capabilities to bring about desired outcome of student engagement and learning even among those students who may be difficult or unmotivated (Armor, Corroy-Oseguera, Cox, King, McDonnell, Pascal, Panly & Zellar, 1976) and this judgment may have a powerful effect on students learning. According to Bandura (1977) self-efficacy is mediated by a person's beliefs or expectations about his/her capacity to accomplish certain tasks successfully or demonstrate certain behavior (Hackett, & Betz, 1981). This expectation determines whether or not a certain behavior or performance will be attempted, the amount of effort the individual will contribute to the behavior, and how long the behavior will be sustained when obstacles are encountered. (Brown, 1999). Some researchers belief that greater efficacy enable teachers to be less critical of students when they make errors (Ashton & Webb, 1986), to work longer with a student who is struggling (Gibson,& Dembo, 1984) and be less inclined to refer a difficult student to special education (Soodak & Podell, 1993). Researches have also shown that teachers with a high sense of efficacy exhibit greater enthusiasms for teaching (Allinder, 1994) have greater commitment to teaching (Coladarci, 1992) and are more likely to stay in teaching (Burley, Hall, Villeme, & Brockmeier, 1991). Teacher's sense of efficacy has also been related to student outcome such as achievement (Armor et al, 1976) motivation (Midgley, Feldlanfer, & Ecceles, 1988). In addition teachers' efficacy beliefs also related to their behavior in the classroom. The effort they invested in teaching, the goals they set, and their level of aspirations are products of their efficacy beliefs. Teachers with a strong sense of efficacy tend to exhibit greater level of planning and organization (Allinder, 1994) are more open to new ideas (Guskey, 1988) and are more willing to experiment with new methods to better meet the needs of their students (Stein & Wang,1988).

When individual have low self-efficacy expectations regarding their behavior, they limit the extent to which they participate in the endeavor and are more apt to give up at the first sign of difficulty (Brown, 1999). In order words low efficacy beliefs may serve as barrier to teachers teaching effectiveness and efficacy. When teachers have a low self-efficacy, their teaching may tends to be characterized by authoritative, teacher-centred roles with a less clear understanding of the various development levels of their students. To Rubeck and Enochs (1991) teachers who were week in content knowledge background tended to have significantly lower personal efficacy than did teachers with strong content background. Teachers with a high self efficacy may tend to teach in ways characterized by the use of inquiry approaches more students centred ,beliefs that they can help any students overcome learning and succeed, and are more knowledgeable of their students development levels.

The role of self-efficacy helps to examine why people's performance attainment might differ even when they have similar knowledge and skills (Pajares & Miller, 1995). From the fore going review, it is clear that the way teachers view themselves and their roles in the teaching context is at least partially derived from their self-efficacy beliefs. The issue of pre-service teachers' mathematics efficacy beliefs is therefore very important for them to be able to carry out their primary function of teaching diligently and effectively. In the present study the extent of the relevance of the construct to problem solving ability of in-service mathematics teachers is part of the major concern.

Mathematics Anxiety and Achievement in Mathematics

According to Tooke (1998) mathematics anxiety has been the topic of more research than any other area in the affective domain and has become very popular research topics for both mathematics educators and educational psychologists. Mathematics anxiety has serious consequences in both daily life and in work, and has its roots in teaching and teachers (Williams, 1988) and has been tied to poor academic performance of students, as well as to the effectiveness of elementary teachers (Bush, 1989; Hembree, 1990). Mathematicians and mathematics educators have great concern that teachers' attitudes toward mathematics may affect more than their students' values and attitudes toward mathematics; these attitudes may affect the effectiveness of the teaching itself (Teague & Austin-Martin,1981). Mathematics anxiety is more than a dislike toward mathematics. Smith (1997) characterized mathematics anxiety in a number of ways, including: (a) uneasiness when asked to perform mathematically, (b) avoidance of math classes until the last possible moment, (c) feelings of physical illness, faintness, dread, or panic, (d) inability to perform on a test, and, (e) utilization of tutoring sessions that provide very little success. Mathematics anxiety has been defined as a state of discomfort which occurs in response to situations involving mathematical tasks which are perceived as threatening to self esteem (Cemen, 1987). In turn, these feelings of anxiety can lead to panic, tension, helplessness, fear, distress, shame, inability to cope, sweaty palms, nervous stomach, difficulty breathing, and loss of ability to concentrate (Cemen, 1987; Posamentier & Stepelman, 1990). Although only a small proportion of persons suffer from a propensity to experience this condition, it is important to recognize how it can lead to a very debilitating state of mind. Those persons with severe cases of mathematics anxiety are limited in college majors and career choices. There is a particular concern in the case of elementary teachers, because it is has been reported that a disproportionately large percentage experience significant levels of mathematics anxiety (Buhlman & Young, 1982; Levine, 1996). This leads to doubts as to their potential effectiveness in teaching mathematics to young children (Trice & Ogden, 1986). NCTM (1989) recognizes math anxiety as a problem and has specifically included in its assessment practices, since a

teacher's job is to assess his/her students' mathematical dispositions (Furner, J.M., & Breman, B.T (2004). NCTM (1989) has included the following in its Standards document for teaching mathematics:

"As mathematics teachers it is our job to assess students' mathematical disposition regarding:
-confidence in using math to solve problems, communicate ideas, and reason;
-flexibility in exploring mathematical ideas and trying a variety of methods when solving problems;
-willingness to persevere in mathematical tasks;
-interests, curiosity, and inventiveness in doing math;
-student ability to reflect and monitor their own thinking and performance while doing math;
-focus on value of and appreciation for math in relation to its real-life application, connections to other disciplines, existence in other cultures, use as a tool for learning, and characteristics as a language" (p. 233).

According to Hadfield and McNeil (1994) the causes of mathematics anxiety can be divided into three areas: environmental, intellectual, and personality factors. Environmental factors include negative experiences in the classroom, parental pressure, insensitive teachers, mathematics presented as rigid sets of rules, and non participatory classrooms (Dossel, 1993; Tobias, 1990). Intellectual factors include being taught with mismatched learning styles, student attitude and lack of persistence, self-doubt, lack of confidence in mathematical ability, and lack of perceived usefulness of mathematics (Cemen, 1987; Miller & Mitchell, 1994). Personality factors include reluctance to ask questions due to shyness, low self esteem, and viewing mathematics as a male domain (Cemen, 1987; Gutbezahl, 1995; Levine, 1995; Miller, & Mitchell, 1994).

Many researchers attempt to trace the evolution of mathematics anxiety among high school and college students back to their elementary school classroom experiences. When early school experiences get the blame for mathematics anxiety, the elementary teacher is usually labeled as the responsible party. Mathematically anxious teachers are said to pass their anxieties on to their students (Buhlman & Young, 1982). They are also often doubted as to their effectiveness as teachers of mathematics (Hadfield & McNeil, 1994; Kelly & Tomhave, 1985). According to Brush (1981), mathematically anxious teachers tend to use more traditional teaching methods, such as lecture, and concentrate on teaching basic skills rather than concepts. This is contrary to the current movement toward teaching mathematical concepts and problem solving through cooperative learning and projects (National Council of Teachers of Mathematics, 1989). It is certainly agreed upon by most educators that elementary school teachers are at a disadvantage if they possess mathematics anxiety, and to admit their fears and attempt to overcome them would not only be in their best interest, but also be in the best interest of their students. Amelioration any perceived mathematics anxiety noticed in pre-service teachers during their training may go a long way in reducing these cankerworms and thereby making them a more effective mathematics teacher.

The changes in levels of mathematics anxiety among future teachers in two different mathematics materials and methods classes were investigated by (Vison, Haynes, Sloan, & Gresham, 1997). The changes were a function of using: (a) Bruner's framework of developing conceptual knowledge before procedural knowledge, and (b) manipulative to make mathematics concepts more concrete. The sample included 87 pre-service teachers enrolled in mathematics methods courses. Two strategies were used to gather data both at the beginning and ending of each quarter. First, future teachers completed 98-item, Likert-type questionnaires. Second, some of the factors that influence the levels of mathematics anxiety were determined through the use of

questionnaire-guided narrative interviews. Multivariate analysis of variance was employed as the quantitative measure for comparing mathematics anxiety both at the beginning and ending of the quarter. Data revealed a statistically significant reduction of mathematics anxiety levels. Turkey's HSD was used to determine that a significant difference in mathematics anxiety levels occurred between the classes in the fall and winter quarters. Results of the study have implications for teacher education programs concerning the measurement of mathematics anxiety levels among future teachers and the determination of specific contexts in which that anxiety can be interpreted and reduced (Vinson, 2001).

Trujillo (1999) through administration of the Revised Mathematics Anxiety Rating Scale (R-MARS) to 50 pre-service elementary teachers identified the five most mathematically anxious teachers. Each of the five identified participants was interviewed with regard to her mathematics experiences in elementary school, high school, college, and family setting. Their perceptions as to the causes of their specific anxieties about mathematics were expressed. Their future plans to deal with their anxieties about teaching mathematics when they join the teaching profession were also voiced. Negative school experiences, lack of family support, and general test anxiety were trends found within the backgrounds of the participants. Despite their current apprehensions regarding the study and teaching of mathematics Trujillo(1999) found out that most of the subjects were very confident and optimistic as to the possibility of setting aside their fears in order to develop into effective teachers of mathematics themselves.

All of the prospective elementary teachers in this study had environmental, cognitive, and personality factors that contributed to their levels of mathematics anxiety. They all had negative classroom experiences and minimal family support, they all suffered from mathematics test anxiety, and they all had fears in regard to teaching mathematics themselves. He also found out that they all are aware of their negative feelings toward mathematics, and they are all determined to prevent the passage of their negative feelings on to their own students.

Haper and Daane (1998) analyzed math-anxiety levels among elementary pre-service teachers before and after a mathematics methods course, noting factors that promoted math anxiety. Interviews and surveys indicated significant reductions in math anxiety at the end of the course. Anxiety stemmed from rigid and structured classroom instructional practices. The main factors causing math anxiety were word problems and problem solving. Poole (2001) says many prospective and current elementary teachers admit, although reluctantly, that their weakest subject area is math. This weakness is compounded by their lack of confidence and poor attitudes toward the subject. He also says many of these teachers attribute unhappy, negative experiences in the "early grades" as sources of their weakness. Whatever the reasons, educators should not foster 'negative feelings' about math. They should implement programs that enhance learning about, and improve attitudes toward, the subject of mathematics, particularly for prospective elementary teachers.

Teacher attitudes have been a major focus of many research studies involving mathematics anxiety. Teague and Austin-Martin (1981) investigated teachers' mathematics anxiety and its relationship on teaching performance. The results indicated a correlation between the two variables. In addition, mathematics methods courses were found to reduce anxiety towards mathematics, but not significantly change attitudes towards mathematics. Similarly, Olson and Gillingham (1980) concluded from their study that attitude toward mathematics and mathematics anxieties were not significantly related. On the other hand, Arem (1993) structured a popular

self-help book, on the premise that a positive attitude toward self and mathematics serves as a solid foundation for overcoming math anxiety.

Teacher variables have been studied to determine effects upon student achievement and mathematics anxiety. Van de Walle (1973) investigated third- and sixth-grade teachers' formal (mathematical emphasis on rote memory) and informal (probing and trial-and-error) perceptions of mathematics. Findings indicated a positive effect on students' mathematical comprehension when teachers exhibited informal perceptions and evidence of positive attitudes, such as low mathematics anxiety. Furoto and Lang (1982) studied teaching strategies designed to foster students' positive self-concepts and their subsequent effects on attitudes, anxieties, and achievement in mathematics. The study revealed a positive relationship between students' achievement and teacher attitudes, as well as, a reduction in mathematics anxiety levels as a result of positive self-concepts.

From an academic standpoint, Post (1992) warned that negative attitudes toward mathematics can produce negative results in mathematics due to the reduction of effort expended toward the math activity, the limited persistence one exerts when presented with an unsolved problem, the low independence levels one is willing to endure, and whether or not a certain kind of activity will even be attempted. Cruikshank and Sheffield (1992) wrote that they were unconvinced that elementary school children suffer from mathematics anxiety. Instead, they argued that teachers, who fail to implement seven important measures, cause their students to learn math-anxious behaviors. These measures include teachers who: (a) show that they like mathematics; (b) make mathematics enjoyable; (c) show the use of mathematics in careers and everyday life; (d) adapt instruction to students' interests; (e) establish short-term, attainable goals; (f) provide successful activities; and (g) use meaningful methods of teaching so that math makes sense. Martinez (1987) has noted that, "Math-anxious teachers can result in math-anxious students" p.117. Sovchik (1996) offered the relationship between mathematics anxiety and future students as one that is passed from teachers to students. Teachers, Sovchik warned, must first examine the symptoms of math anxiety to see if they themselves exhibit any. In addition to that, teachers were encouraged to incorporate strategies in the classroom to alleviate mathematics anxiety altogether. In a study conducted by Scholfield (1981), teacher attitudes were directly linked to student performance in and student attitudes toward mathematics. Results indicated that high-achieving teachers produced high-achieving students with least-favorable attitudes toward mathematics. Those teachers who were classified middle- or low-achieving in their abilities to teach mathematics had students whose attitudes were the most-favorable, yet maintained the lowest achievement scores. Akinsola (2002) study mathematics anxiety and its relationship to in-service teacher's attitude to the studying and teaching of mathematics and found significant relationships between teachers' mathematics anxiety and their attitudes towards the studying and teaching of mathematics. Teachers with high mathematics anxiety tend to avoid studying and teaching of mathematics. This study will like to see what relationship exist between mathematics anxiety and problem solving ability of the subject.

Locus of Control and Academic Achievement.

It has often been said that obtaining a good education is the key to being successful in the world. But what determine been successful while in school? While many factors may contribute to school achievement, one variable that is oven overlooked is locus of control (Grantz, 2006).

Locus of control refers to an individual's generalized expectations concerning where control over subsequent events resides (WikEd, 2005). In the context of education, locus of control refers to the types of attributions we make for our success and for/or failures in school tasks (Grantz, 2006). Locus of control is grounded in expectancy-value theory, which describes human behavior as determined by the perceived likelihood of an event or outcome occurring contingent upon the behavior in question, and the value placed on that event or outcome. More specifically, expectancy-value theory states that if (a) someone values a particular outcome and (b0 that persons believes that taking a particular outcome action will produce that outcome, then (c) they are more likely to take that particular action (WikEd, 2006). Locus of control is the perceived source of control over our behavior. It influence the way we view ourselves and our opportunities (Gershaw, 1989). Rotter (1966) classified locus of control into a bipolar dimension from internal to external. Internal control is the term used to describe the belief that control of future outcomes resides primarily in oneself. In other words, people with internal locus of control believe they control their own destiny (Gershaw, 1989). External control refers to the expectancy that control is outside oneself, either is in the hand of other powerful people or due to fate/chance or luck.

Research has shown that having an internal locus of control is related to higher academic achievement (Findley & Cooper. 1983) , students with internal locus of control earn better grades and work harder(Grantz,1999) and include spending more time on home work as well as studying longer for test. These make sense because if you believe working hard pay off then you are likely to do so (Grantz, 1999).

External locus of control may be caused by continued failure in spite of continued attempts at school tasks (Bender,1995) and a high external locus of control, in turn, leads to a lack of motivation for study and school in general (Grantz,1999). If one has an external locus of control, he may feel that working hard is futile because their efforts have only brought disappointment. Ultimately, they may perceive failure as being their destiny (Grantz, 1999). In other words, students with an external locus of control are more likely to respond to failure by giving up hope and not trying harder (Anderman & Midgley, 1997). Out of the 36 studies reviewed by Bar-Tal and Bar-Zohar (1997) on locus of control and academic achievement 31 of the studies indicated a significant relationship with internals having higher achievement than external.

Becker (1987) comparing student teachers' with internal locus of control and external locus of control during the student teaching experience found that student teachers with internal locus of control expressed more confidence in themselves than student teachers with external locus of control. Also student teachers with internal control attempted to check for their students' understanding of concept more frequently than student teachers with external control. The result of this study underlying the importance of the locus of control construct as factor that could affect the pattern of instructional delivery by teachers.

Weiner asserts that people attribute their successes and failures to internal or external reinforces. An "internal person" attributes successes and failures to her ability or to her effort. An "internal person" attributes her performance to causes for which she assumes personal responsibility. An "external person" attributes her performance to factors for which she has no responsibility and over which she has no control. If she fails, the "external person" assumes that the task was too difficult or that she was unlucky (or both). If the "external person" succeeds, she attributes her success to the easiness of the task or to luck. (Weiner, 1986)

Bandura's theory of observational learning concerns learning from models (Bandura, 1969). He asserts that much behavior is acquired through observing and imitating other people. He contends that new patterns of behavior are learned through observing behavior without the observer overtly responding or receiving any reinforcements in the exposure setting. He writes, "Modeling influences . . . can create generative and innovative behavior." (Bandura, 1977: 40-41)
He argues that observers watch models performing responses, which embody a certain principle. Later the observers behave in a way stylistically similar to the model's behavior, even though the observer is not mimicking the model's specific responses, because the observer has applied what she has learned from the model to a new, but related, situation. (Bandura, 1977).

Bandura and Walters assert that teachers as role models may have three types of effects on students (Howard, 1996). The first is the "modeling effect," which involves the student's direct imitation of the model's behavior. The second is the "disinhibitory effect," which involves the student's observing the consequences of the model's actions and consequently choosing behavior in opposition, if the model's observed consequences were undesirable. For example, when female faculty members are regarded with low esteem by school administrators and are not treated as equals, the effect may be to inhibit female students' aspirations toward the teaching profession. The third modeling effect is the "eliciting effect," which involves the increased susceptibility in a student to the influence of the role model. For example, a female teacher who holds high expectations for female students' achievement may have an increased probability of influencing the female students' performance through cues which elicit a positive response in the students. (Bandura & Walters, 1969). Teachers are visibly in a position to be imitated by their students and having an internal locus of control or external locus of control can affect the directions of student learning.

This study will like to determine if there is a relationship between locus of control and problem solving ability of in-service teachers.

Study habits and Academic Achievement

There are many factors responsible for underachievement like, motivation, study habits, attitude towards teacher, attitude towards education, school and home background, concentration, mental conflicts, level of aspiration, self-confidence, examination fear, etc.(Sirohi,2004). Poor habits of study not only retard school progress but develop frustration, destroy initiative and confidence and make prominent the feeling of worthlessness towards himself and the subject of study whereas effective methods ensure success, happiness and sense of accomplishment (Smith & Littlefield, 1948). All too often, students perform poorly in school simply because they lack good study habits. In many cases, students don't know where to begin, don't fully understand the material, are not motivated by it, or feel that there was too much work given to them with too little time to complete or study it. If their studying skills do not improve, these students will continue to test poorly and not perform to their fullest potential

In a study of underachievement in relation to study habits and attitudes by Sirohi (2004) the most significant factor contributing to underachievement is poor study habit which has been indicated by 100% underachievers in their study.

Good work habits and skills are not acquired theoretically or in vacuum, it is proper habit of work and insistence on them in every detail and over a long period of time that create right attitudes and values (Secondary Education Commission, 1952).

Since learning is not a team sport but an activity that involves solely the student and the knowledge, it behooves on individual students to set a good work or study habits rather than been vagarious. Since certain skills need to be acquired at an early age—particularly mathematics and reading, writing, and thinking in one's native language—it is important that the idea of self-teaching be inculcated in the earlier years so that learning these essential skills will automatically lead to the development of good study habits.

There is a general need of teaching students the use of general study habits and each subject teacher, as he teaches specific subject skills, should call attention to this general habits. The question is: Are the elementary school teachers themselves have good study habits for them to be able to impact it to their students? What is the relationship of teachers' study habits to their problem solving ability?

Research questions

(1) How much did mathematics teaching efficacy belief, locus of control, study habits and mathematics anxiety (when taken together) contribute to the prediction of problem solving ability of in-service teachers?

(2) What is the relative contribution of each of the variables to the prediction of problem solving ability among the subjects?

(3) Is there is a significant relationship between:

(a) in-service teachers with high and low mathematics teaching efficacy belief and problem solving ability.

(b) in-service teachers with high and low mathematics anxiety and problem solving ability

(c) in-service teachers with internal and external locus of control and problem solving ability.

Method

Design

The design employed in this study was an ex-post facto type. In such a research, the investigation does not have a direct control of independent variables because their manifestations have already occurred or because they are inherently not manipulable. What the researcher did in present study was to examine the four psychological variables(independent variables-mathematics teaching efficacy belief, locus of control, study habits and mathematics anxiety) and problem solving ability(dependent variable) as it occurred rather than creating these manifestations.

Participants

Data for this study were collected from a total of 122 in-service mathematics teachers enrolled in the B.Ed primary education programme in the Department of Primary Education, University of Botswana. The sample included 92 females and 30 males. The mean age of the participants was 37years. Their ages range from 29 to 50 years while their working experience ranged from 6 to 25 years.

Instrumentations

(1) **Mathematics Teaching Efficacy Belief Instrument (MATEBI**)

The MATEBI consists of 25 items in a four-point Likert type scales ranging from strongly agree, agree, disagree and strongly disagree. The MATEBI was adopted from Enochs and Riggs (1990).Though the Enochs and Riggs scale is a five-point Likert scale the present study contains four because the investigator want all participant to have an opinion on all items. So the

undecided portion was removed. The internal consistency of the MATEBI score was measure by Cronback's coefficient of alpha. The coefficient alpha is the function of the extent to which items in a test have commonality and is the lower limit of the reliability of a set of test scores (Cortinal, 1993). The reliability of scale scores will naturally be influenced not only by the instrument used but also by the sample composition and variability (Davis, 1987). It is therefore important to report reliabilities coefficient for the actual data collected (Vacha-Haase, Kogan, & Thompson, 2000), Isaac, Friedman, & Efret, 2002). The MATEBI was subjected to Cronback alpha reliability coefficient. It was found to be 0.91.

(2) **Locus of Control Scale**
The locus of control behavior scale based on Rotter (1966) was used as measuring instrument. It consists of 13 paired items. The instrument has a coefficient alpha of 0.82.

(3) **Mathematics Anxiety Rating Scale**
The mathematics anxiety rating scale by Richardson & Suinn, 1972 modified by Akinsola (2002) was used to measure the teachers' mathematics anxiety. It consisted of 30 items on a five-point Likert scale. The instrument yielded a reliability index of 0.79.

(4) **Problem Solving Ability Inventory**.
This was assessed with the aid of Rodman, Dean and Rosati (1986) instrument which was modified by Yokomoto, Buchanan, & Ware (1995) to reflect a shift in emphasis toward problem solving. The inventory is divided into two parts, with the first set (items 1-11) assessing student beliefs and attitudes towards problem solving in learning and testing process whilst the second set (items 12-16) assessed student appreciation for mathematics , algebra word problems, and puzzles, and it also assessed student self-perception of their competencies as problem solvers. Students could select "strongly agree," agree," "disagree," or "strongly disagree" as their response on each item. To ensure the suitability of the instrument for the current study however, it was subjected to test-retest reliability analysis. The obtained reliability coefficient was 0.77

(5) **Study Habits Scale**
The study habits questionnaire was a 35 items (3-point scale) adapted from Nneji (2002) study habits questionnaire. It is a 3 point Likert Scale featuring Mostly, Occasionally and Only. A test-retest reliability coefficient of 0.79 was obtained when given to fifty in-service secondary school teachers to score.

Procedure

Data Analysis Procedure
The stepwise multiple regression procedure(backward solution) was used to examine the joint and separate contribution of mathematics self-efficacy, locus of control, study and mathematics anxiety to the prediction of problem solving ability while chi-square analysis and Pearson moment correlation coefficient were used to determine significant relationship between the various aspects of the independent variables on the dependent variable.

Results
The research question was interested in knowing the joint contribution of the independent variables (mathematics teaching efficacy belief, locus of control, study habits and mathematics

anxiety) and dependent variable (problem solving ability). The results of multiple regression analysis are presented in Table 1 below:

Table 1: Multiple regression analysis on problem solving data

Multiple R	= 0.79251
R-Square (R^2)	= 0.62807
Adjusted R-Square	= 0.62431
Standard Error (SE)	= 5.38421

Analysis of Variance

Source	Df	SS	MS	F-Ratio	P
Regression	4	10,025.36120	2506.34030	83.60652	<0.05
Residual	117	3507.40326	29.97781		

The above table shows that the predictor variables contributed 62.81% of the variable in problem solving ability. The table further reveals that the analysis of variance of the multiple regression data yields an F-ratio of 83.60652 which is significant at 0.05.

The results presented in table 2 below show the contribution of each of the variable to the prediction of problem solving ability. The table contains the standardized regression weight for each of the variables which from 3.1963 to 7.32625 and standard error of estimate which ranged from 0.09043 to 0.41162. The t-observed for each variable ranged from 4.11965 to 13.61185 which are all significant at 0.05 levels.

Table 2: Testing the significance of regression weight.

Variable	B	SEB	Beta	T obs.	Signif. T
Math Anxiety	-5.60291	0.41162	0.732635	-13.61185	.000
Math Self-Efficacy	4.26312	0.33617	0.42007	12.68144	.000
Locus of Control	-0.58233	0.09224	0.36241	-6.31320	.001
Study Habits	0.37254	0.09043	0.31963	4.11965	.001
Constant	68.42371	4.27311			.000

The data were also analyzed using chi-square test. The chi-square result shows that there is no significant relationship between internal locus of control and problem solving ability while significant relationship was found between external locus of control and problem solving ability. In-service teachers with internal locus of control had a higher problem solving ability and those with external locus of control had a lower problem solving ability. Also with chi-square test, no significant relationship was found between in-service teachers with low mathematics anxiety and problem solving ability whilst significant relationship was found between in-service teachers with

high mathematics anxiety and problem solving ability. However, the result shows that in-service teachers with low mathematics anxiety had a higher problem solving ability whilst in-service teachers with high mathematics had a low problem solving ability. Similarly, in-service teachers with high mathematics teaching efficacy had high problem ability while those with low mathematics teaching efficacy had low problem solving ability.

Discussion of finding

The result of this study as evident from Table 1 has shown that the four construct of mathematics anxiety, mathematics teaching efficacy belief, locus of control and study habits contributed 62.81% of the variance of problem solving ability in that order. The multiple R value of 0.79251 signified a high correlation between the predictor and the predicted variables. The result indicated that the predictor variables are potent contributors to the problem solving ability of the in-service mathematics teachers. This was further corroborated by the F-value of 83.60652 which was significant at 0.05 levels. The result thus shows that these variables without exceptions have high predictive value in relation to problem solving ability.

The result revealed that mathematics anxiety contributed mostly to problem solving ability in mathematics thus imply that the more the mathematics anxiety of in-service teachers the weaker their problem solving ability. The image and fear of mathematics is molded and shaped by past experiences and that it is very difficult to teach something you don't possess yourself. These tensions and pressures in the teacher towards problem solving may inhibit sustainable confidence in the delivery of mathematics instruction thereby making them slothful and less effective. Mathematics anxiety is a very real fear for millions of people but the problems becomes acute when the person most afraid of problem solving is standing in front of the classroom trying to teach the subject. The National Council of Teachers of Mathematics in its 1991 publication, Professional Standards for Teachers of Mathematics (NCTM, 1991) and the Mathematics Programme Standards for Accreditation of Teacher Education (NCATE, 1998) stress the importance of the disposition of the classroom teacher towards mathematics. It was their view that if students are to develop a disposition to do mathematics, it is essential that their teacher communicates positive attitude towards mathematics, so the prospective teacher of mathematics has to be kind in words and deeds to them. To break the cycle of poor attitudes generating poor attitudes and provide students with positive experience in learning mathematics taking fears into account can help the teacher approach the subject with attitude that students can learn these subjects, and be sensitive to students who fail due to a lack of confidence. Teachers may also want to take extra care to teach these subjects well and to encourage questions. These presuppose that the teacher himself/herself is not fearful of the subject.

Mathematics teaching efficacy beliefs represents a person's evaluation of his or her ability or competency to reach or overcome a mathematics tasks or obstacles. Low self –efficacy has been linked to increase cheating, lack of concentration, low motivation, lack of persistence, and depression (Finn & Frone, 2004). Conversely, high self-efficacy has been associated with pursuit and achievement goals, problem solving and persistence (Vrugt, Langeries, & Hoogstrate, (1997).Consequently these factors are related to the problem solving effort of student. In other words mathematics self- efficacy may influence how successful students are ready to engage in problem solving activities. Students with high mathematics self-efficacy may be ready to confront and solve any problem that comes there way whereas students with low mathematics self-efficacy may distance themselves from engaging in solving problems they might have perceived as difficult thereby leading to poor achievement in mathematics. Students with high mathematics

self-efficacy are likely to persist in solving any kind of mathematics problems while those with low mathematics self-efficacy easily abandoning solving mathematics problems they considered too tasking. High mathematics self-efficacy students are likely to increase personal accomplishment, more absorbed in their mathematics and set and maintain more challenging mathematics goals.

Thus the way a teacher judge his/her capability to organize and execute the course of action required to attain designated types of performance in mathematics like problem solving may likely affect the way they approach the task. So, attempting problem solving tasks in mathematics by in-service teachers depends on their level of mathematics self-efficacy as revealed from this study. In other words a teacher with a high level of mathematics self-efficacy will be willing to expend energy, effort and time in solving problem and encouraging his student in the act of problem solving. On the other hand, a teacher with a low mathematics self-efficacy may not be willing in exerting energy, effort and time on mathematics problem solving. Such a teacher will hardly encourage his/her students to persist on solving mathematical problems they might have considered too tough to handle. It may be concluded that teachers with high mathematics self-efficacy are likely to be more apt in fostering and encouraging their students to tackle mathematics problems of all colour-ation whilst teacher with low mathematics self-efficacy may not be enthusiastic and committed in encouraging their students to embark on problem solving in mathematics since they themselves are not kin at problem solving as teachers model the behaviour they wish their students to exhibit.

In this study locus of control was find to correlate with problem solving ability of the in-service teachers. Further investigation by chi-square revealed significant difference between in-service teachers with external locus of control and mathematics problem solving ability. Locus of control which is the tendency to ascribe achievements and failures to either internal factors that they can control(effort, ability, motivation) or external factors that are beyond control (chance, luck, others' actions) is an important factor that could affect the ways a teacher performs his teaching role. It could be ascertain that teachers who believe that effort and ability are essential in the learning of mathematics are likely to motivate and encourage their students to tackle and solve problems in mathematics whereas those who believe that luck, fate, chance or powerful others might not be favourably disposed towards encouraging their students in engaging in strenuous mathematics problem solving because they themselves attached their successes to luck, chance or fate.

Study habits have been shown to contribute to students' failure in mathematics (Mangaliman, 2007). Study habits correlated least with problem solving ability in this study. Nonetheless, there is a significant relationship between study habits and problem solving ability of in-service teachers. For teachers to encourage good study habits in their students they themselves have to be an epitome of good study habits.

Conclusion

Attitude cannot be easily separated from learning because they are acquired through the process of learning which involves interactions of several variables. As illustrated by the definition of learning by Farrant (1994) that "learning is a process of acquiring and retaining attitudes, knowledge, understanding, skills and capabilities" p 107. According to Farrant's definition, learners are not born with attitudes but instead they acquire them when they get in contact with the new world. This position is supported by Olaitan (1994) that "attitude can be learned and

teachers should strive hard to develop the right attitudes in their pupils particularly towards acquiring manipulation skills" p27. Attitudes differ according to how learners perceive what they are taught and whoever is teaching them. This position is supported by Jonassen (1996) who defined attitudes as "how people perceived the situation in which they find themselves" p485. He then asserted that if learners are not assisted or encouraged to perceive positively most of the things they learn, their performance in class will be affected. Thus the crucial roles of teachers as facilitator of positive attitudes of students.

Most mathematics teachers are obtuse to student problems in mathematics thereby failing to educe the best from them. Mathematics teachers with lack of understanding and acceptance often provide a psychological climate which may precipitate negative attitude and avoidance to mathematics by students. This should no be so. As mathematics teachers we should always seek for avenue by which we will be making our students elated at the end of our interaction with them in the classroom. Methods which are perdurable should be employed always to sustain student continuous interest in learning mathematics. This is the only way by which we may be able to gear and stir them up and change their negative perception towards the learning of mathematics. This presupposes that mathematics teachers themselves are positively oriented towards the learning and teaching of the subject.

Our teacher training programme must be evaluated on their ability to prepare mathematics teachers for students that have or may have develop discomfort for mathematics and who may end up teaching the elementary/junior schools where these feelings have been found to begin. By studying the pre-service and in-service teachers, we cannot only ascertain what future and present teachers are feeling but we can work with them towards alleviating their own discomfort with mathematics as well as prepare them for students they may encounter with similar feelings. The power of process resides in the key pathways through which mathematics teachers learn, grow, and improve their practices. A high mathematics teaching efficacy, a low mathematics anxiety, and internal locus of control and a good study habits are essential factors for would be mathematics teacher to be able to perform his teaching tasks creditably and optimally.

A teacher's competence, a teacher's identity, a teacher's 'self' is woven into the fabric of everyday events in a way which means that they have little choice but to be committed to outcomes of events that involve, at one and the same time, both the pupils' and the teachers' careers in the school (Denscombe, 1995). It is therefore necessary for mathematics teachers to be percipient of students' mood and by so doing they may be able to reduce student often nasty experiences in mathematics classroom.

Reference.

Akinsola, M.K. (2002). In-service elementary teachers' mathematics anxiety and its relationship to teachers' attitude towards the studying and teaching of mathematics. Nigerian Journal of Applied Psychology, 7(1), 188-202.

Allinder, R.M. (1994). The relationship between efficacy and the instructional practices of special education teachers and consultants. Teacher Education and Special Education, 17, 86-95.

Anderman, E.M & Midgley, C. (1997).Changes in personal achievement goals and the perceived classroom goal structure across the transition to middle level schools. Contemporary Educational Psychology, 22,269-298

Arem, C. (1993). Conquering Math Anxiety: A Self-Help Workbook. Pacific Grove, CA: Brooks/Cole Publishing Company.

Armor, D., Conroy-Oseguera, P., Cox, M., King ,N., Mcdonnell, L., Pascal, A., Panly, E., & Zellar, G (1976). Analysis of the school preferred reading programmes in selected Los Angels miniority schools. REPORT NO 2007-LAUSD, Santa Monica. C.A: Rand Corporation (ERIC Document Production Service. No 130243

Bandura, A. (1969). Principle of Behaviour Modification. Holt, Rinehart and Winston. New York.

Bandura, A. (1977). Self-efficacy: Towards a unifying theory of behavioral change. Psychological Review, 84,191-215.

Bar-Tal, D & Bar-Zohar, Y (1977). The relationship between perception of locus of control and academic achievement. Contemporary Educational Psychology 2, 181-199.

Becker, N.L. (1987). A comparison of student teachers with internal locus of control and student teachers with external locus of control during the student teaching experience. Ph.D dissertation, University of Nebraska-Lincoln.

Bender, W. N. (1995). Learning Disabilities: Characteristics, Indentification, and Teaching Strategies. (2nd Ed.). Needham Heights, Mass: Allyn & Bacon

Brown, B.L (1999). Self-efficacy beliefs and career development. RRIC Identifier ED 429187.
Brush, L.R. (1981). Some thoughts for teachers on mathematics anxiety. Arithmetic Teacher, 29(4), 37-39.

Buhlman, B.J., & Young, D.M. (1982). On the transmission of mathematics anxiety. Arithmetic Teacher, 30(3), 55-56.

Burley,W.W., Hall,B.W., Villeme,M.G., & Brockmeier, L.L. (1991). A path analysis of the mediating role of efficacy in first-year teachers' experiences, reactions, and plans. Paper presented at the annual meeting of the American Educational Research Association, Chicago.

Bush, W. S. (1989). Mathematics anxiety in upper elementary school teachers. School Science and Mathematics, 89, 499-504.

Cemen, P.B. (1987). The nature of mathematics anxiety. (Report No. SE 048 689). Stillwater, OK: Oklahoma State University. (ERIC Document Reproduction Service No. ED 287 729)

Charalambos, C., Philippou, G., & Kyriakides, L. (2002). Towards understanding teachers' philosophical beliefs about mathematics. Paper presented at the International Group for the Psychology of Mathematics Education (PME), Norwich UK.

Coldarci, T (1992). Teachers' sense of efficacy and commitment to teaching. Journal of Experimental Education, 60. 323-337.

Cornell, C. (1999). I hate math! I couldn't learn it, and I can't teach it! In *Childhood education.* Washington, USA: Association for Childhood Education.

Cortina, J.M. (1993). What is coefficient alpha? An examination of theory and application. Journal of Applied Psychology, 78, 98-104.

Cruikshank, D.E., & Sheffield, L.J. (1992). Teaching and Learning Elementary and Middle School Mathematics. New York: Merrill, 24.

Davis, R.V. (19870 Scale construction. Journal of Counselling psychology, 34, 481-489.

Denscombe, M. (1995). Teachers as an Audience for research: the acceptability of ethnographic approaches to classroom research. Teacher and Teaching: theory to practice, 1(2), 173-191

Dossel, S. (1993). Maths anxiety. Australian Mathematics Teacher, 49(1), 4-8.

Dutton, W.H. & Dutton, A. (1991). Mathematics Children Use and Understand. Mountain View, CA: Mayfield Publishing Company.

Emenaker, C. (1996). A problem-solving based mathematics course and elementary teachers' beliefs. School Science and Mathematics', 96(2), 75-85.

Ernest, P. (2000). Teaching and learning mathematics. In V.Koshy, P. Ernest, & R. Casey (Ed.), Mathematics for primary teachers. London, UK: Routledge.

Farrant, J.S (1994). Principle and Practice of Education. Singapore. Longman.

Findley, M. J., & Cooper, H. M. (1983). Locus of control and academic achievement: A literature review. Journal of Personality and Social Psychology, 44, 419-427.

Finn, K.V., & Frone, M.R (2004).Academic Performance and Cheating. Moderating of school identification and self-efficacy. Journal of Educational Research, 97(3), 115-122.

Friedman,I.A., & Kass, E. (2002). Teacher self-efficacy: a classroom organization conceptualization. Teaching and Teacher Education, 18, 675- 686.

Furner, J.M., & Breman, B.T. (2004). Confidence in their ability to do mathematics: The need to eradicate math anxiety so our future students can successfully compete in a high-tech globally competitive world. Philosophy of Mathematics Education Journal 18.

Furoto, D.M., & Lang, M. (1982). Relationship of self concept enhancement to anxiety_and achievement in college mathematics. (ERIC Document Reproduction Service No. ED 218 088, 26 pages).

Gershaw, D.A (1989). Line on life: locus of control. Retrieved 11/11/2006 from http://virgil.azwestern.edu/~daq/lol/control locus.html

Gibson, S., & Dembo, M. (1984). Teacher efficacy; A construct validation. Journal of Educational Psychology, 76. 569-582.

Grantz, M. (2006). Do you have the power to succeed? Locus of control and its impact on education. Retrieve 30/5/2006 from http://www.units.muohio.edu/psybersite/control/education.shtml

Guskey, T.R. (1988).Teacher efficacy, self-concept, and attitudes towards the implementation of instructional innovation. Teaching and teacher education, 4, 63-69.

Gutbezahl, J. (1995). How negative expectancies and attitudes undermine females' math confidence and performance: A review of the literature. Amherst, MA: University of Massachusetts. (ERIC Document Reproduction Service No. ED 380 279)

Hackett, G., & Betz, N (1981). A self- efficacy approach to the career development of women. Journal of Vocational Behaviour.18, 3, 326-339.

Hadfield, O.D, & McNeil, K. (1994). The relationship between Myers-Briggs personality type and mathematics anxiety among preservice elementary teachers. Journal of Instructional Psychology, 21(4), 375-384.

Harper, W.H. & Daane, C.J. (1998). Causes and reduction of math anxiety in preservice elementary teachers. Action in Teacher Education. Vol. 19, No. 4, pp. 29-38.

Hembree, R. (1990). The nature, effects, and relief of mathematics anxiety. Journal for Research in Mathematics Education, 21, 33-46.

Howard, D.E. (1996). The relationship of internal locus of control and female role models in female college students. Ph.D dissertation abstract, The University of Texas at Austin.

Ineeta, V & Irohi, S (2004). A study of underachievement in relation to study habits and attitudes. Journal of Indian Education.

Jarrett, D (2000). Problem solving: Getting to the heart of mathematics. Northwest Teacher, 1(1).
Jonassen, D. (1996). Education, communication and Technology. London. Macmillan.

Karp, K. S. (1991). Elementary school teachers' attitudes toward mathematics: The impact on students' autonomous learning skills. School science and mathematics, 91(6), 265-270.

Kelly, W.P., & Tomhave, W.K. (1985). A study of math anxiety / math avoidance in preservice elementary teachers. Arithmetic Teacher, 32(5), 51-53.

Levine, G. (1995). Closing the gender gap: Focus on mathematics anxiety. Contemporary Education, 67(1), 42-45.

Levine, G. (1996). Variability in anxiety for teaching mathematics among pre-service elementary school teachers enrolled in a mathematics course. Paper presented at the annual meeting of the American Educational Research Association, April 12, 1996, New York, NY. (ERIC Document Reproduction Service No. ED 398 067).

Mangaliman, R.A.(2007). Factors affecting students' failures in mathematics. Unpublished Ph.d thesis, Saint Louis University, Baguio City, Ched-CA. Retrieved 06/02/2007 from, http://dspace.slu.edu.ph/bitstream/123456789/125/1/ROGELIO+A.+MANGALIMAN.pdf.

Martinez, J.G.R. (1987). Preventing math anxiety: A prescription. Academic Therapy, 23, 117-125.

McLeod.D.B. (1992). Research on affect in mathematics education: A reconceptualization. In Grouws, D. A. (Ed.). Handbook of research on mathematics teaching and learning. New York: MacMillan.575-596.

Midgley, C., Feldlanfer, H., & Ecceles, J (1989). Change in teacher efficacy and student self-and task-related beliefs in mathematics during the transition to junior high school, Journal of Educational Psychology, 81, 247-258.

Miller, L.D., & Mitchell, C.E. (1994). Mathematics anxiety and alternative methods of evaluation. Journal of Instructional Psychology, 21(4), 353-358.

National Council for Accreditation of Teacher Education.(1998). Program for initial preparation of K-4 teachers with an emphasis in mathematics, 5-8 mathematics teachers, 7-12 mathematics teachers. [Online].Available: http://www.ncate.org

National Council of Teachers of Mathematics. (1989). Curriculum and evaluation standards for school mathematics. Reston, VA: Author.

National Council of Teachers of Mathematics (1991). Professional standards of teaching mathematics. Reston, VA.

Olaitan, S.O. (1994). Agricultural Education in the Tropics. Tokyo, Macmillan.

Olson, A.T., & Gillingham, D.(1980). Systematic desensitization of mathematics anxiety among pre-service elementary teachers. Alberta Journal of Educational Research, 26, 120-127.

Pajares, F., & Miller, M. D.(1995). Mathematics self-efficacy and mathematics performances: The need for specificity of assessment. Journal of Counseling Psychology, 42, 190-198

Polya, G. (1973). How to solve it. Princeton, NJ: Princeton University Press. (Originally copyrighted in 1945).

Poole, G (2001). ETSU treating math anxiety at is root with budding teachers. Retrieved 23/12/2006: http://www.etsu.edu/etsu/news/20010265.htm

Posamentier, A.S., & Stepelman, J.S. (1990). Teaching secondary school mathematics (3rd ed.). New York: Merrill.

Post, T.R. (ed.). (1992). Teaching Mathematics in Grades K-8. Boston: Allyn & Bacon.

Reboli, D. & Holodick, N. (2002). Using Technology to Reduce Math Anxiety in Preservice Elementary Teachers. In C. Crawford et al. (Eds.), Proceedings of Society for Information Technology and Teacher Education International Conference 2002 (pp. 1099-1100). Chesapeake, VA: AACE.

Rebeck, M, & Enochs, L. (1991).A path analytic model of variables that influence science and chemistry teaching self-efficacy and outcome expectancy in middle science teachers. Paper presented at the annual meeting of the national association of research in science teaching, Lake Geneva, WI.

Report of the Secondary Education Commission (Oct. 1952–June 1953). Government of India.Richardson,F & Suinn, R.(1992). The Mathematics Anxiety Rating Scale: Psychometric Data. Journal of Counselling Psychology, 19(6), 551-554.

Roadman, S.M., Dean, R.K., & Rosati, P.A.(1986). Self-perception of engineering students' preferred learning style related to MBTI type. Proceeding of the ASEE annual Conference, pp 1303-1313.

Rotter, J.B (1966) Generalized expectancies for internal versus external control of reinforcement, Psychological Monographs, 80, (1, Whole No. 609).

Scholfield, H.L. (1981). Teacher effects on cognitive and affective pupil outcomes in elementary school mathematics. Journal of Educational Psychology, 73, 462-471.

Sirohi , V (2004). A study of underachievement in relation to study habits and attitudes. Journal of Indian Education (May), 14-19.

Smith, S. & Littlefield (1948). An Outline of Best Methods of Study. New York: Barnes and Noble Inc.

Smith, S.S. (1997). Early Childhood Mathematics. Boston: Allyn & Bacon.

Soodak., & Podell, D(1993). Teacher efficacy and student problem as factors in special education referral. Journal of Special Education, 27, 86-95

Sovchik, R.J. (1996). Teaching Mathematics to Children. New York: HarperCollins.

Stanic, G., & Kilpatrick, J. (1988). Historical Perspectives on Problem Solving in the Mathematics Curriculum. In R. I. Charles & E. A. Silver (Eds.), The teaching and assessing of mathematical problem solving (pp. 1-22). Reston, VA: National Council of Teachers of Mathematics.

Stein, M.K., & Wang, M.C (1988). Teacher development and school improvement. The process of teacher change. Teaching and Teacher Education, 4, 171-189

Teague, P. T., & Austin-Martin, G. (1981). Effects of a mathematics methods course on prospective elementary school teachers' math attitudes, math anxiety, and teaching performance. Paper presented at the annual meeting of the Southwest Educational Research Association. Dallas, TX.

Tooke, D. & Lindstrom, L.C. (1998). Effectiveness of a mathematics course in reducing math anxiety of preservice elementary teachers. School Science and Mathematics, 98, 136-139.

Trice, A.D., & Ogden, E.D. (1986). Correlates of mathematics anxiety in first-year elementary school teachers. Educational Research Quarterly, 11(3), 2-4.

Trujillo, K. M (1999). Tracing the Roots of mathematics anxiety through in-depth interviews with pre-service elementary teacher**.** College Student Journal

Tschannen-Moran, M, & Hoy, A.W (2001). Teacher efficacy: Capturing an elusive construct. Teacher and Teaching 17, 783-805.

Uusimaki, L & Nason, R (2004). Causes underlying pre-service teachers' negative beliefs and anxieties about mathematics. Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education, 4, 369–376

Vacha-Haase, T., Kogan, L.R., & Thompson, B (2000). Sample composition and variabilities in published studies versus those in test manuals: validity of score reliability inductions. Educational and Psychological Measurement, 60, 509-522

Vinson,B.M, Haynes, J., Sloan, T., & Gresham, R.(1997) A comparison of pre-service teachers' mathematics anxiety before and after a method class emphasizing manipulates. Paper presented November 12-14 at the annual meeting of the MidSouth Educational Research Association in Nashville, TN

Vinson, B.M. (2001). A comparison of pre-service teachers' mathematics anxiety before and after a method class emphasizing manipulates, Early Childhood Education Journal, 29(2), 89-94

Vrugt, A.T, Langeres, M.P, & Hoogstraten, J (1997). Academic self-efficacy and malleability of relevant capabilities as predictors of exam performance. Journal of Experimental Education, 66(1), 61-73.

Williams, W. V. (1988). Answers to questions about math anxiety. School Science and Mathematics, 88, 95-104.

Wilson, J.W, Fernandez, M.L and Hadaway, N (1993). Mathematical problem solving. In Wilson, P. S. (Ed.). Research Ideas for the Classroom: High School Mathematics. New York: MacMillan.

Yokomoto, C.F., Buchanan, W.W., & Ware, R. (1995). Problem solving: An assessment of student attitudes, expectations, and beliefs. Proceedings of the ASEE/IEEE FIE 95 Conference.

A reflection on mathematical cognition: how far have we come and where are we going?

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Abstract: Any theory of mathematical cognition and learning must ultimately articulate with our current understanding of how the mind works and with current theories of how knowledge is acquired from both an individual and social perspective. Certainly the human mind is very complex; understanding how it works in general and identifying the components that contribute to "doing mathematics" in particular is no easy task. To understand how the mind does mathematics, we must identify what mathematicians actually do in the context of everyday cognition. Here we discuss some of what is known and about this and point out directions for future work.

The mind and mathematics

Human minds did not evolve to do abstract mathematics per se, and yet we can use them to do mathematics none the less; there must be an explanation of how we got from there to here. Sperber (1996) says, "Materialism… does not commit scientists who espouse it to describing the objects of their discipline and the causal process in which these objects enter into the vocabulary of physics. What it does commit them to is describing objects and processes in a manner such that identifying the physical properties involved is ultimately a tractable *problem*, not an unfathomable *mystery* (to use Noam Chomsky's famous distinction…)." (p. 10) Cognitive scientists seek material explanations for how the mind works in general; the missing link we must pursue is how mathematical cognition and learning can be explained in terms of the "usual" processes of the mind. Despite the delicate and difficult work needed for such a program, we must insist on our goal being a theory of how the mind does mathematics without begging any questions. It is argued here that mathematical structures are the cognitive imprint of the structural relationships we naturally perceive in the world, refined into idealized "objects" that can be studied with an idealized reasoning structure.

Susan Haack talks about scientific reasoning as a refinement and extension of our everyday capacities for empirical inquiry. She quotes Thomas Huxley: "The man of science simply uses with scrupulous exactness the methods which we all, habitually and at every minute, use carelessly" and then Albert Einstein: "[T]he whole of science is nothing more than a refinement of everyday thinking."(p. 95) At some appropriate level of generality, mathematical thinking must also arise as a refinement of our everyday cognitive capacities to reason about the world. A proper accounting of how this takes place is our best hope for explaining the "unreasonable

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effectiveness of mathematics" to help us make predictions about the world (to borrow Eugene Wigner's famous phrase). Given the usual perception that abstract mathematics is about anything but the everyday world, we must pick our path carefully. We have taken the first step by noting that how we think about mathematics must be explainable in terms of our standard-order cognitive systems. The next step is to ask, what are these systems and how are they honed to learn and do mathematics?

First, let's illustrate what we might mean by "systems." The prevailing wisdom is no longer that the mind is some general purpose thinking machine, but rather the fusion of different specialized "cognitive modules." Gallistel (1999) says:

> Despite long-standing and deeply entrenched views to the contrary, the brain no longer can be viewed as an amorphous plastic tissue that acquires its distinctive competencies from the environment acting on general purpose cellular-level learning mechanisms. Cognitive neuroscientists, as they trace out the functional circuitry of the brain, should be prepared to identify adaptive specializations as the most likely functional units they will find. At the circuit level, special-purpose circuitry is to be expected everywhere in the brain, just as it currently is expected routinely in the analysis of sensory and motor function. (p. 1190)

If we are to ask which cognitive modules and learning mechanisms are harnessed for mathematics, we should first look for specific cognitive modules that are adapted for "proto-mathematics." This has been the approach of scientists such as Stanislas Dehaene who study our understanding of numbers and basic arithmetic. Dehaene (1997) says:

> Newborns readily distinguish two objects from three and perhaps even three from four, while their ears notice the difference between two sounds and three. Hence, the brain apparently comes equipped with numerical detectors that are probably laid down before birth. The plan required to wire up these detectors probably belongs to our genetic endowment…. [Most] likely, a brain module specialized for identifying numbers is laid down through spontaneous maturation of cerebral neuronal networks, under direct genetic control and with minimal guidance from the environment. Since the human genetic code is inherited from millions of years of evolution, we probably share this innate protonumerical system with many other animal species. (p. 61-62)

Typically, work in cognitive science has focused on the acquisition and representation of specific mathematical topics such as numbers (as in Dehaene's work), algebra word problems, geometric proofs etc. Dehaene claims that "As humans, we are born with multiple intuitions concerning numbers, sets, continuous quantities, iteration, logic, and the geometry of space."(p. 246) While there has been work done in general in these other areas, it has not developed to the same depth as it has with our understanding of whole numbers and operations. Take, for example, the Handbook of Mathematical Cognition; of the 27 chapters dedicated to mathematics and the

mind, most treat numbers and operations, with the exception of Lakoff and Nunez's[2] article on conceptual metaphor and a few chapters devoted to mathematical disability.

Mathematics and the mind

While keeping one eye on the work of cognitive scientists, we should also ask what the mathematician's perspective has to offer the study of mathematical cognition. Consider the following statement by G. H. Hardy:

> I believe that mathematical reality lies outside us, that our function is to discover or observe it, and that the theorems which we prove, and which we describe grandiloquently as our 'creations,' are simply our notes of our observations.

This sentiment reflects a common one among mathematicians. Technically, Platonism is the philosophy of Plato, but given that there are different interpretations of Plato's philosophy, I will use the term Mathematical Platonism, or simply Platonism, to be a belief that the objects of mathematical study have a reality separate from the human conceptualization of them. Mathematical Platonism differs from realist perspectives in science, where the objects of study tend to be about physical objects of which we have some empirical evidence. This contrasts with mathematical structures, such as groups or hyperspheres, which no one claims exist in the physical world as physical objects.

In a paper titled "Reification as the Birth of Metaphor," Anna Sfard reports on the interviews she has conducted with three renowned mathematicians: a logician, a set theorist, and a specialist in ergodic theory. In these interviews, the three mathematicians talk about the mathematical concepts that they study as if they were concrete in some way. They talk of perceiving images and structures. Her ergodic theorist says, "In those regions where I feel an expert… the concepts, the mathematical objects turned tangible for me." The term that Sfard uses for this cognitive phenomenon is reification (although I am using it more generally than she does): To **reify** is to regard or treat an abstraction as if it had concrete or material existence.

In several pieces of writing, Bertrand Russell reflects on both the reasons for and disillusionment of this experience. In this quote from "Portraits from Memory," he explains how such a sentiment might emerge:

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[2] Lakoff is well known for his application of his theory of conceptual metaphors to many different domains. In a critique of Lakoff's work in Moral Politics, Jesse Walker says, "The problem is that [Lakoff] has… a model that may have some explanatory power but which he has stretched far beyond its limits." This is exactly the fault one can ascribe to the work of Lakoff and Nunez. Dennett's analysis of Skinner comes to mind:

> Skinner was a greedy reductionist, trying to explain all the design (and design power) in a single stroke. The proper response to him should have been: "Nice try—but it turns out to be much more complicated than you think!" And one should have said it without sarcasm, for Skinner's *was* a nice try. (p. 395)

And so the criticism of Lakoff and Nunez's work should go, *mutatis mutandis*.

> Mathematics is, I believe, the chief source of the belief in eternal and exact truth, as well as in a super-sensible intelligible world. Geometry deals with exact circles, but no sensible object is exactly circular; however carefully we use our compasses, there will be some imperfections and irregularities. This suggests the view that all exact reasoning applies to ideal as opposed to sensible objects; it is natural to go further, and to argue that thought is nobler than sense, and the objects of thought are more real than those of sense-perception. Mystical doctrines as to the relation of time to eternity are also reinforced by pure mathematics, for mathematical objects, such as number, if real at all, are eternal and not in time. Such eternal objects can be conceived as God's thoughts. Hence Plato's doctrine that God is a geometer, and Sir James Jeans' belief that He is addicted to arithmetic.

Given that there is no way to empirically verify the existence of timeless and tenseless mathematical objects, believing in them is a matter of faith. Yet the experience of doing mathematics is very compellingly like studying "real" objects. Reuben Hersh says that most mathematicians are Platonists in their day-to-day work but only Formalists on Sundays. So they behave as Platonists behind closed doors, but when asked for a public accounting of their activities, retreat behind Formalism.

Let us put the philosophical issues aside for a moment. What insights do we gain from these introspective ruminations of mathematicians? Looking at this issue from a broader view point will be helpful here. In explaining the difference between cognitive neuroscience theories about consciousness and, say, motor action, Dehaene and Naccache state,

> What is specific to consciousness, however, is that the object of our study is an introspective phenomenon, not an objectively measurable response. Thus, the scientific study of consciousness calls for a specific attitude which departs from the 'objectivist' or 'behaviorist' perspectives often adopted in behavioral and neural experimentation. In order to cross-correlate subjective reports of consciousness with neuronal or information-processing states, the first crucial step is to take seriously introspective phenomenological reports. Subjective reports are the key phenomena that a cognitive neuroscience of consciousness purport to study….
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> The idea that introspective reports must be considered as serious data in search of a model does not imply that introspection is a privileged mode of access to the inner workings of the mind. Introspection can be wrong…. We need to find a scientific explanation for subjective reports, but we must not assume that they always constitute accurate descriptions of reality. " (p. 3)

Dehaene's claims about subjective reports being relevant to understanding consciousness have a parallel in mathematical cognition. Mathematicians' tendency to use introspection to think about how mathematics gets accomplished provides important clues for determining how we actually do mathematics. But more than taking the introspective reports at face value, we should consider how mathematicians describe their work as data that should be explained by any complete theory of mathematical cognition.

Thus, it is more important to understand and explain the cognitive reality of mathematical objects than to argue about in what sense they exist. So far, most researchers in human cognition who

have addressed the issue of Mathematical Platonism have used our understanding of cognition to dismantle Platonism as an epistemology of mathematics. But it is precisely this subjective experience that we should use to give us clues about how mathematicians do what they do. Dehaene comes close when he says,

> Presumably, one can become a mathematical genius only if one has an outstanding capacity for forming vivid mental representations of abstract mathematical concepts—mental images that soon turn into an illusion, eclipsing the human origins of mathematical objects and endowing them with the semblance of an independent existence. (p. 242-243)

What Dehaene fails to recognize is that we need to understand and explain *how* we form "vivid mental representations of" abstractions and *why* they appear to have an independent existence. Furthermore, it is not just mathematical geniuses who function this way, but anyone who uses mathematics effectively to solve complex problems. Think of the ubiquitous use of the term "mathematical tools," often used as if it stands in opposition to the mathematician's view of mathematical objects. Yet a function is still a conceptual object if it is used to describe the height of a rocket over time. It is just that some people develop or collect tools, and others prefer to use them for some secondary purpose.[3] It is the selective attention of (especially pure) mathematicians on the mathematical objects themselves vs. how they are used that makes them reflect more on the subjective experience of having these mental representations.

Anna Sfard gives a very plausible explanation for the Platonic experience. She says,

> It becomes clear that this 'practical' Platonism is not a matter of deliberate choice, of insufficient sophistication, or a lack of mathematical (or philosophical) maturity. It is because of the very nature of our imagination, because of our embodied way of thinking about even the most abstract ideas, that we spontaneously behave and feel like Platonists. (51)

Thus, to understand how people do mathematics, we must understand the cognitive origin of this experience, that is, we must work to understand the cognitive mechanisms that allow for the reification of abstract ideas. Most writing about this phenomenon has been of a philosophical nature, but what is needed now is a cognitive account of it.

Natural category representation

To develop a theory of how people might represent mathematical objects in the mind, we should look to theories of how they represent more familiar objects such as dogs or tomatoes. The

[3] What of this "mathematics as tools?" Certainly one need not understand the wiring or motor design for a compound miter saw to use it to build a window frame. A fortiori one need not understand the physics behind its electrically powered motor. With physical tools, one can go to the hardware store and purchase them ready-made. In contrast, mathematical tools are constructed in the mind, made up of neuronal connections that are shaped by culturally mediated physical, social, and mental experiences. Because the tools themselves need to be assembled by the learner, their inner workings do need to be made plain to the assembler. Furthermore, the very way that mathematical tools are used requires that their structure be understood so that it can properly be fitted to the problem at hand. The metaphor of mathematics as a tool is dangerous if we expect too much similarity with saws and hammers.

categories that people make to organize what they know about the objects and ideas that they encounter in everyday life are referred to as natural categories.

The classical view of category representation was that all categories were determined by definitions which described necessary and sufficient conditions for category membership. So in a sense, a category was thought to be a well-defined set, and objects were either in the set or not. But considerable evidence suggests that there are many natural categories that we do not represent this way.

For example, Hampton asked subjects to rate whether certain items belonged in certain categories (like whether the kitchen sink was a piece of furniture or a tomato was a vegetable). He found that items fell along a continuum so that almost all subjects agreed that certain items definitely belonged to the category and others definitely did not, but many items fell in a fuzzy range in between where subjects differed on whether they did or did not belong to the category. Interestingly, McCloskey and Glucksberg found that subjects who were asked to make similar judgments repeatedly on the same items were shown to change their minds about category membership much more frequently with these "borderline" cases (tomato/fruit) than on items that were consider to be more "typical" of the category (apple/fruit).

Murphy argues that such fuzziness is a necessary result of our need to understand a world with many fine gradations of things. So in a sense, the categories we form discretize a continuous world. As a result, natural categories have fuzzy boundaries. Yet while their boundaries are fuzzy, their "cores" are very crisp. Even within a given category, certain members are considered more or less typical. For example, sparrows are considered to be very typical birds whereas penguins are not. Rosch and Mervis articulated a theory that typicality is based on family resemblance, which is constituted by the following two conditions:

(1) If they have features common in their category, and
(2) They do not have features in common with other categories.

Murphy states that the typicality phenomenon reveals a prototype structure to our representation of categories. Perhaps it is this prototype structure of our *mental representations* of categories that inspired the original notions of Platonic ideals or Aristotle's essences. Thus, the very notion of essential dogness or roundness reflects some very deep structure of the human mind, and only indirectly reflects the deep structure of the world. Plato's work would then represent the original introspective reports that provide us clues about how the human mind understands the world.

Gelman suggests that essentialist thinking is an early cognitive bias and that young children are natural essentialist thinkers. Clearly it serves us well to see similarities between things we encounter in the world and to look for what hidden structures and relationships might account for these similarities. But must this tendency we have necessarily reflect the exact structure of the world around us? No—this capacity for essentialist thinking can be a useful strategy without representing the world completely faithfully. When we look at a sparrow on the fence, we do not perceive every aspect of the bird such as the heart pumping its blood or the electrical activity of its nervous system, but the visual representation we have of it is enough to help us know something about what it is and what it is likely to do. Dennett says:

> Aristotle had taught, and this was one bit of philosophy that had permeated the thinking of just about everybody… [that] all things—not just living things—had two kinds of properties; essential properties, without which they wouldn't be the *kind* of thing they were, and accidental properties, which were free to vary within the kind…. [Yet the geological record showed that] species were not eternal and immutable; they had evolved over time…. Even today, Darwin's overthrow of essentialism has not been completely assimilated…. The essentialist urge is still with us, and not always for bad reasons. Science does aspire to carve nature at the joints, and it often seems that we need essences, or something like essences, to do the job. (36-39)

If preschoolers are essentialist thinkers, is it really likely that this is the legacy of Aristotle rather than a feature of the human mind? Perhaps it is this push to understand the essences of the things in the world around us that we refine into mathematical thought, for in the world of mathematical objects things really do have essences. We seek to cut nature at the joints because of the structures of our minds. This is useful even if not always an identical reflection of reality. Piaget considered this possibility in his book Structuralism, where he states, "Must we, to make sense of the fact that we are in possession of knowledge of nature, allow for some sort of permanent tie, though not of identity, between "external" structures and the structures of "our" operations? If there is such a connection, we should find it in evidence in "intermediate" regions: biological structures and our own sensory-motor acts should exhibit it in its efficacy." (p. 39)

Pinker and Prince discuss category representation in the context of subclasses of regular and irregular verbs. From their analysis they conclude, "Both family resemblance categories and classical categories can be psychologically real and natural. Classical categories do not have to be the product of rules that are explicitly formulated and deliberately transmitted." (p. 234) Furthermore, Maddox and Ashby present findings from standard cognitive laboratory experiments, neuropsychological patient data[4], and neuroimaging studies in which they argue that there is strong evidence that human category learning is mediated by multiple, qualitatively distinct cognitive and neural systems. Not only do we use different neural circuits to think about different kinds of categories, but there is evidence that different individuals use different neural circuits for the same category or concept. Pinker and Prince conclude,

> The referents for many words, such as bird and grandmother, appear to have properties of both classical and family resemblance categories. How are these two systems to be reconciled?... [A likely] reconciliation is that people have parallel mental systems, one that records the correlational structure among sets of similar objects, and another that sets up systems of idealized laws. Often a category within one system will be linked to a counterpart in another system. (p. 254)

Understanding concepts is not the only instance where we integrate the information provided by distinct cognitive systems. We use our different senses to try to understand the objects and phenomena that we encounter in our everyday lives, and we integrate the information from the different systems so that we may have the most complete understanding of the things we encounter. For example, as we stand in the buffet line surveying the choices ahead of us, our

[4] Neuropsychological patient studies compare the performance on certain tasks between normal subjects and subjects with neurological deficits (e.g. amnesiacs).

impression of what is there is drastically different if it looks beautiful but smells rotten or looks dreary but smells heavenly. In this way, we expect that the different data we collect should coalesce into an understanding of the things we experience. We do not think that if we have both visual and olfactory information coming in that there are necessarily two stimuli, and if we eat with our eyes closed we may still know from the smell, taste and feel of what we put in our mouths that it is chocolate pudding or caviar or rotten oranges. Thus, we must have a mental representation of the food we eat that can be derived from the integration of different sensory inputs in different combinations. By analogy, we should not be surprised if we find that we use different mental representations to the same mathematical end. Fayol and Seron suggest that in the case of whole numbers and operations, this is exactly the case:

> Any model of number processing must account for the fact that educated adults are able to recognize and produce numbers in the Arabic code and the verbal code. It is therefore necessary to postulate that adults possess mental representations which are able to guide these recognition and production operations. However, researchers disagree as to the role and the format of these representations and their interrelations. It now appears to be well established that the symbolic representations are functionally independent and that they may undergo isolated impairment or be degraded in accordance with specific patterns in brain-damaged patients…. Evidence for the existence of some of these dissociations is also provided by cerebral imaging data which suggests that the verbal and Arabic codes are not processed in the same regions. (p. 8)

Here we see a connection to the mathematics education literature, and the ubiquitous references to teaching mathematical concepts using "multiple representations." We need to make the important distinction between external or public representations, such as graphs or equations, and internal or mental representations. In the mathematics education literature, multiple representations usually refer to the first of these—the external representations. But there is a curious silence about *what* is being represented. If anything, it will be said that they are representations of a concept. But what kind of concept is implicitly assumed? Typically, it is a family resemblance category and not a classical one. For example, consider the following statement taken from the NCTM standards: "In grades 3-5 all students should… develop understanding of fractions as parts of unit whole, as parts of a collection, as locations on number lines, and as division of whole numbers." (p. 148) The part-whole "interpretation" of a fraction requires one definition and the division of whole numbers "interpretation" requires another, and seeing the connection between these requires some real mathematical work (see, e.g. Beckmann). It is the justification of the connection between these that allows for the family resemblance structure for the category "interpretations of fractions," but each of them constitutes its own classical category in its own right. Furthermore, this category has fuzzy boundaries. Would we consider a ratio as an "interpretation" of a fraction? Looking at the range of approaches to the definition of a ratio, we see there would be disagreement about this (see Milgram, p. 219-254).

From the mathematician's perspective, mathematical categories are, in fact, described by definitions giving necessary and sufficient conditions for category membership. But they may arise from experiences with natural categories of objects in the world that are not classical. In mathematical categories, we strive to articulate definitions so that the boundaries are not fuzzy—in fact, if there is confusion about whether something belongs in a particular mathematical category, then we say that the category is not "well-defined" and we strive to clarify the definition. The purpose of devising definitions in mathematics is exactly so that we may

communicate precisely about which objects we mean. It is not so much that there are "natural" mathematical categories with a priori definitions; definitions develop to isolate out the objects that interest us in some way, usually along some important structural lines. (For a better understanding of how this works, see the discussion of what a polyhedron is in Lakatos' Proofs and Refutations.) Of course, we inherit definitions from the mathematical culture in which we are embedded, and in this way mathematical definitions have an existence that is independent of us as individuals. Yet the definitions of mathematical objects and structures originate in our experiences with the world as we find it, including the natural world as well as the world of ideas that has developed before we enter it.

A case study: what are circles?

We come equipped with the ability to see patterns in both images and sound. Very young children easily recognize and name circles. Thus, even children recognize the category of round things, but the category of round objects has blurry edges:

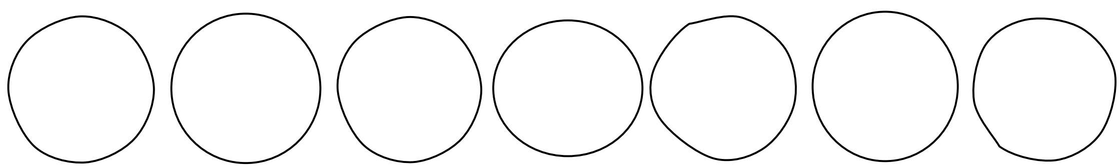

Figure 1: Which of these figures are round? Are any of them circles?

Given what we know of category representations, this category must also be equipped with a category prototype. This is no different than the category prototypes for dogs or tomatoes. Then why do we mathematicize circle shapes and not dog shapes? Note that roundness is a secondary property of an object and objects from many different categories can be round. We still recognize an oval plate as a plate. But would we recognize a snake-shaped dog? From the cognitive perspective, circles are the collection of things in the world that also happen to be round.

But for this proto-mathematical category to mature in to an actual mathematical category, we analyze what it means to be "round." We see there are subtleties involved—do we mean round like a plate or round like a ball?[5] We develop mutually exclusive categories for these different kinds of round things—circles and spheres—and then come up with necessary and sufficient conditions for an object to be in one of our new mathematical categories of round things. And in doing so, we create an object that exactly embodies each of these kinds of roundness; nothing more and nothing less. Once the structural definition is in place (the set of points equidistant from some chosen center in either the plane or 3-space, respectively) we can imagine a circle or sphere without any other properties—not a ripple in a pond or a soccer ball, but a new object that embodies the ideals of the respective categories of round things. So a mathematical circle is a round object with no properties other than those which follow logically from "roundness" defined in some appropriate way.

[5] Note that many books for pre-school children do not distinguish between these, and in fact show pictures of balls under the heading of "circle."

Recalling Russell's comment about sensible vs. ideal objects, we realize that there really is no perfect circle in this world as we have defined it. But through the power of thought experiments and careful reasoning, there it is, represented in the mind.

In everyday reasoning we usually do not need (or even want) to separate out roundness from the other properties of round objects we experience in the world. The kinds of inferences we make that connect past experiences with current observation serve us well: "The last time I saw a tree with round red things on it I could eat them." In mathematical reasoning, however, we do carefully separate out distinct characteristics. This is a hallmark of mathematical thinking, and we could not do it without our ability to learn and think about categories with a classical structure. In general, we can think of mathematics as the *science of essentialism*.

Yes, Virginia, there are mathematical objects

Many artificial categories that can be defined in this classical way that are used in psychology experiments are basically arbitrary, but for mathematical definitions that are inspired by phenomena in the world, they can be designed to capture in some essential way the salient features[6] of the category prototype. But even when they have concrete origins, they define a new abstract set of objects that have their own attributes and that are discovered through deduction. In Dubinsky's words, "Abstraction, in general, is the determination in a given situation, which may be a mathematical object, a procedure, or a combination of the two, of *what is essential* in a component of the situation. In mathematical abstraction, one generally expresses this essence in some systematic manner." (Emphasis added.) Mathematicians represent these essential features by defining an object or structure that embodies them, and it is these objects that are represented in the mind.

It is from this perspective that we might think of mathematical objects as the cognitive imprint of structural reality in the same way that visual images are the cognitive imprint of physical reality. Just as our mental representations of the objects we see are not equivalent to photographs of those objects, so our mental representations of structures and relationships in the world are not "snapshots" of those phenomena, but rather idealizations of them that are meaningful in our current web of knowledge.

Once mathematical concepts take on their own cognitive reality, we can look for attributes and structure in the world of these abstract objects and abstract again. The most basic example is numbers and operations. How else can we talk of the "properties of numbers" and "number systems?" After number systems reify, we can think of groups, which are defined to capture the particular operational relationships we see in number systems, (or the set of symmetries of a square, etc.).

[6] Even though the case we have described in detail is a category organized by a physical feature, the same argument can work with the appropriate kinds of functionally defined categories. For instance, if you have three gumball machines where the first gives two gumballs for a quarter, the second gives three and the third gives five, then we can see that these gumball machines all have in common a multiplicative rule and # gumballs = *a* * # quarters would be a mathematization of the category prototype.

To understand mathematical cognition, we need to understand how external representations (such as definitions, equations, and graphs) mediate the corresponding mental representations of mathematical categories (such as functions). Given the earlier discussion of the interaction of different cognitive systems that support the conceptualization of a single thing, we should expect these verbal, symbolic, and pictorial representations to coalesce into the perception of a single thing—a mathematical object.

Idealized objects require idealized reasoning

Situations that we encounter in everyday life and the relationships between objects we find there are complex. We rarely know all of the facts that are relevant to these situations, so we must make decisions with incomplete information. Because no knowledge of the real world is absolutely certain and is never complete, such decisions are probabilistic in nature. But the ideal objects that we study in mathematics are very different—because we can choose their properties and the situations that we wish to consider them in, we can and do reason about them in an idealized way.

LeFevre et al state that "a substantial portion of the research on mathematical cognition is focused on the processing of arithmetic problems, especially single-digit arithmetic." (p. 365) The heavy emphasis on numbers and arithmetic is reflected in the conceptualization of mathematics held by the cognitive science community. For instance, Gallistel and Gelman state, "Mathematics is a system for representing and reasoning about quantities, with arithmetic as its foundation." Of course, mathematics is about much more than quantity, and Gallistel and Gelman get closer when they say, "From a formalist perspective, arithmetic is a symbolic game, like tic-tac-toe. Its rules are more complicated, but not a great deal more complicated. Mathematics is the study of the properties of this game and of the systems that may be constructed on the foundation that it provides." While a complete and universally satisfactory definition of mathematics is difficult to formulate, we can say at least that it is a complex system of *reasoning* about both quantity and space and the abstract structures derived from them (whether one takes a formalist perspective or not). Clearly, our reasoning modules developed well before the advent of modern mathematics, so we should look to see what types of cognitive modules are being proposed that might reasonably be conscripted for mathematical purposes.

According to Markovits and Barrouillet, early theories on the study of the development of reasoning were eventually shown to be in contradiction with substantial empirical evidence. As a very simplistic summary, early theories of human reasoning essentially assumed that people reason logically (Piaget famously held this view), and ample evidence has shown that this is not true in many situations. Murphy states that the role of concepts is to help us organize our knowledge about the things we encounter in the world, and that the inferences (though not strictly "logical") that we make based on these conceptual structures are quite important.

Pinker and Prince state: "Bobick (1987), Shepard (1987) and Anderson (1990) have attempted to reverse-engineer human conceptual categories in terms of their function in people's dealings with the world. They have independently proposed that categories are useful because they allow us to infer objects' unobserved properties from their observed properties (see also Rosch 1978; Quine 1969). (p. 247)" On the other hand, logical reasoning requires us to separate out what we know from everyday experience and to look in a purely structural way at an argument. Our reasoning

about many everyday situations is certainly not purely logical, yet we use our reasoning structures we come equipped with to do purely logical reasoning.

Given the likely modularity of the human mind, it is quite possible that there is more than one reasoning system rather than one single all-purpose system. For example, Sperber and Girotto discuss Cosmides' theory that people have "a 'social contract algorithm' specialized in reasoning about social contracts [that allow us] to detect parties that were not abiding by the terms of the contract." Sperber and Girotto, in turn, argue that, "People tend to be guided not by any form of reasoning but by context-sensitive intuitions of relevance (see also Evans, 1989). Intuitions of relevance are activated by the pragmatic mechanism involved in comprehending the task (just as they are by any comprehension process)." They go on to say that their approach "is in no way hostile to evolutionary psychology. In fact, the relevance-guided comprehension mechanism involved in the selection task is viewed as an evolved module specialized for the comprehension of communicative intentions, and more specifically as a sub-module of a Theory-of-Mind mechanism." Regardless of the details of future directions such theories may take, this discussion reminds us that we must be mindful to look for theories that provide a proper balance between general and domain-specific reasoning abilities as they are employed to do mathematics.

An example of the detailed kinds of theories we would need to understand mathematical reasoning is given in Gopnick et al (2004), where they have developed a theory of causal reasoning based on Bayes nets. Such work makes it clear that purely logical reasoning would actually not be sufficient for making basic inferences about certain kinds of everyday situations. If we were able to know with certainty all of the relevant facts about what causes an illness, for instance, then pure logic would be the most accurate way of reasoning. But in practice, what we know at any given time is tentative and fragmentary. Thus, everyday reasoning is likely to devise solutions to situations that optimize the likely outcomes given less-than-optimal information. Gopnik et al connect this abstract analysis to the particulars of likely information types:

> The epistemological difficulties involved in recovering causal information are just as grave as those involved in recovering spatial information. Hume (1739/1978) posed the most famous of these problems, that we only directly perceive correlations between events, not their causal relationship. How can we make reliably correct inferences about whether one event caused the other? Causation is not just correlation, or contiguity in space, or priority in time, or all three, but often enough, that is our [only] evidence.

Just as our sensory systems have evolved to help us represent the objects we encounter faithfully enough to be useful to us to navigate our surroundings, find food, avoid danger etc., our reasoning[7] abilities must have evolved to help us reliably see structures and relationships in the world—that is, to reason (relatively) reliably about what we perceive in the world. As Gopnik et. al. said when discussing cognitive representations of the causal relations among events, "Given the adaptive importance of causal knowledge, one might expect that a wide range of organisms would have a wide range of devices for recovering causal structure." Perhaps the mechanisms that allow for the mental representations of causal relationships in the world are analogous to our

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[7] Here I use the term **reasoning** in the broad sense it is meant in the cognitive psychology literature, which I take to be any process by which someone draws conclusions based on their current understanding of a given situation, and not in the narrower sense of "logical reasoning."

ability to see structures and relationships in the world that we can then study deliberately through more formalized mathematics.

How might this work? Gopnick's Bayes net models for representations of causal knowledge include both causal chains and associated probabilities. Logical arguments can be seen as a particular subset of such causal structures where the associated probabilities are always assumed to be zero or one. In this way, we can think of mathematical reasoning as an idealization of a certain kind of everyday reasoning.

In summary

While one must keep them in perspective, there is a benefit to be gained from trying to explain the introspective reports of mathematicians. In the words of Jacques Hadamard:

> Will it ever happen that mathematicians will know enough about the physiology of the brain, and neurophysiologists enough of mathematical discovery, for efficient cooperation to be possible?

We may find in the end that some modified version of Platonism holds; not that mathematical structures exist in some timeless or tenseless place, but as structures themselves they reflect or approximate to greater and lesser extent the structure of the natural world and natural systems. The cognitive reality of mathematical structures may, in fact, result from a refinement of natural concepts that are designed to give us a better understanding of the world, and in that way, have a "reality" that is objective—as much as any mental representation can be. Mathematics is the study of all possible ideal worlds, and in so far as any of these bear a resemblance to the actual world, we find the power of mathematics to describe what we see.

I would like to expand on the opening lines of G.H. Hardy's "A Mathematician's Apology":

> A Mathematician, like a painter or poet, is a maker of patterns.

Let us generalize this to encompass a broader definition of what mathematics is and who qualifies as a mathematician.

Painters and musicians create images and sounds that can either be representational or abstract—each utilizing critically important cognitive/sensory systems. Writers use language to create both fiction and non-fiction. The medium for mathematicians' creative works is structure and reasoning, and applied mathematicians' work is representational while pure mathematicians' work may not be. But just as we may find both intrinsic beauty as well as links between the abstract works of artists, musicians, and fiction writers to the world of human experience, so at times we find our pleasure in pure mathematics from both its intrinsic beauty and the serendipitous insights we gain from it into the world around us. It is the combination of the need to accurately represent the world with our sensory and reasoning systems and our capacity for creativity that gives rise to the paradox of the "unreasonable effectiveness of mathematics" to describe the world around us.

References

Ashby*,* G. Maddox*,* T. (2005) Human Category Learning. *Annual Review of Psychology*, *56*, 149-178.

Beckmann, S. (2005). *Mathematics for Elementary Teachers with Activities*. Addison Wesley Higher Education

Campbell, J. (ed.) (2005) *Handbook of mathematical cognition*. New York, NY: Psychology Press

Cosmides, L. (1989). The logic of social exchange: Has natural selection shaped how humans reason? Studies with the Wason selection task. *Cognition, 31*, 187-276.

Dehaene, S. (1997) *The number sense: how the mind creates mathematics*. New York: Oxford University Press

Dehaene, S. and Naccache L. (2001). Towards a cognitive neuroscience of consciousness: basic evidence and a workspace framework. In Dehaene (ed.) *The cognitive neuroscience of consciousness.* Cambridge, Mass.: MIT Press

Dennett, D. (1995). *Darwin's dangerous idea : evolution and the meanings of life*. New York: Simon & Schuster

Dubinksy, E. (2000). Mathematical Literacy and Abstraction in the 21st Century, *School Science and Mathematics, 100,* 289-297

Fayol, M and Seron, X. (2005). About numerical representations: Insights from neuropsychological, experimental, and developmental studies., In Campbell, J (ed.) *Handbook of mathematical cognition*. New York: Psychology Press

Gallistel, C. R. (1999). The replacement of general-purpose learning models with adaptively specialized learning modules. In M.S. Gazzaniga, (ed.). *The Cognitive Neurosciences. 2d ed.* (1179-1191) Cambridge, MA. MIT Press

Gallistel C. R., Gelman, R. (2005) Mathematical Cognition. In K Holyoak & R. Morrison (Eds) *The Cambridge handbook of thinking and reasoning.* (559-588) Cambridge University Press

Gelman, S. (2003) *The essential child: origins of essentialism in everyday thought*. New York: Oxford University Press

Gopnick, A., Glymour, C., Sobel, D., Shulz, L., Kushnir, T, & Danks, D. (2004). A theory of causal learning in children: Causal maps and Bayes nets. *Psychological Review, 111*, 1- 31.

Haack, S. (2003) *Defending Science-Within Reason: Between Scientism and Cynicism.* Amherst, New York: Prometheus Books

Hadamard, J. (1945) *An Essay on the Psychology of Invention in the Mathematical Field.* Princeton, NJ: Princeton University Press

Hampton, J.A. (1979) Polymorphous concepts in semantic memory. *Journal of Verbal Learning & Verbal Behavior*, *18*, 441-461.

Hardy, G. H.. (1940). *A Mathematician's Apology*. Cambridge: Cambridge University Press

Hersh, R. (1997) *What is mathematics, really?* New York: Oxford University Press

Lakatos, I. (1976) *Proofs and refutations: the logic of mathematical discovery* edited by John Worrall and Elie Zahar. New York: Cambridge University Press

Lakoff, G. and Núñez, R. (2000) *Where mathematics comes from: how the embodied mind brings mathematics into being.* New York: Basic Books

LeFevre, J., DeStefano, D. and Coleman, B. (2005). Mathematical cognition and working memory. In Campbell, J. (ed.) Handbook of mathematical cognition. New York: Psychology Press

Markovits H. and Barrouillet P. (2004) Introduction: Why is understanding the development of reasoning important? *Thinking and Reasoning, 10,* 113–121

McCloskey, M. and Glucksberg, S. (1979) Decision processes in verifying category membership statements: Implications for models of semantic memory. *Cognitive Psychology, 11,*1-37

Milgram, R.J. (2005). *The Mathematics Pre-Service Teachers Need to Know.* math.stanford.edu/ftp/milgram/FIE-book.pdf

Murphy, G. (2002) *The big book of concepts.* Cambridge, Massachusetts: MIT Press

National Council of Teachers of Mathematics (NCTM). (2000). *Principles and standards for school mathematics.* http://www.nctm.org/standards/overview.htm

Piaget, J (1970) *Structuralism.* Translated and edited by Chaninah Maschler. New York: Harper & Row

Pinker, S. and Prince, A. (2002). The Nature of Human Concepts: Evidence from and Unusual Source. in Jackendoff, R., Bloom, P., Wynn, K. (eds.) *Language, Logic, and Concepts: essays in memory of John Macnamara* Cambridge, Mass.: MIT Press

Rosch, E. Mervis, C. (1975). Family resemblances: Studies in the internal structure of categories. *Cognitive Psychology, 7*, 573-605

Russell, B. (1956). *Portraits from memory, and other essays.* New York: Simon and Schuster

Sfard, A. (1994).Reification as the Birth of Metaphor. *For the Learning of Mathematics, 14,* 44-55

Sperber, D. (1996) *Explaining culture : a naturalistic approach.* Oxford, UK: Blackwell

Sperber, D. (2004) Modularity and relevance: How can a massively modular mind be flexible and context-sensitive? In Carruthers Peter , Laurence Stephen and Stich Stephen, (eds.) *The Innate Mind: Structure and Content*.

Sperber, D., Girotto, V. (2003) Does the Selection Task Detect Cheater-Detection? In: Fitness, J. & Sterelny, K. (eds.), *New directions in evolutionary psychology*, Macquarie Monographs in Cognitive Science, Psychology Press

Wigner, E. (1960) The Unreasonable Effectiveness of Mathematics in the Natural Sciences. *Communications in Pure and Applied Mathematics, 13*, 1-14

Reflections upon Teaching a Poorly-Conceived Lesson

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Abstract
Using a "failed" mathematics lesson as a mini-case study, I explore some of the challenges inherent in teaching an inquiry-style mathematics lesson. The ensuing discussion centers around two issues: the preparation and subsequent scaffolding involved in guiding students to desired understandings, along with the tension inherent in "telling" and "not telling" in an inquiry-style pedagogy. I resort to personal reflection, as well as to Wenger's (1998) theoretical construct of *participation* and *reification* in revealing the underlying link between the manner of lesson preparation and consequent engagement of students during the enactment of the lesson.

Reflections upon Teaching a Poorly-Conceived Lesson

One of the difficulties in attempting to transform one's teaching practices from a traditional lecture format to a more student-centered mode of instruction is that one oftentimes relinquishes a certain degree of control and predictability in the classroom. As opposed to simply planning out a lecture with the expectation that students will ask questions when they do not follow *the course of reasoning as set out by the instructor*, the new practice requires one to more carefully map out—that is, to anticipate and be prepared for— various contingencies: in particular, such issues as, in what manner might the ideas be best embedded within activities (as opposed to directly telling them), what degree of scaffolding might be necessary and optimal for the different levels of students in the class, in what manner might one want to go down divergent learning paths depending upon student input, and so on. As such, in opening up the classroom discourse, by its very nature, student-centered teaching practices introduce new and necessary skills for the teacher to acquire both in preparation and in interaction with the students (see Lampert, 2001).

When the lesson plan has been carefully thought out in these terms and more, the teaching itself can be a real thrill. It affords a richness of dialogue, and even learning opportunities for the teacher that lecture-based teaching does not often enable. On the other hand, when done poorly, this other way of teaching can leave both teacher and students alike in pedagogical predicaments that the traditional approach safely steers clear of for the most part. I am particularly interested in this latter case, of when things are not aright, of when such *attempts* can go messily wrong. I am interested in this, for I believe that it is in exploring *when* and *how* a particular pedagogical approach fails that one sometimes comes to a deeper understanding of how it might come to succeed.

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In this short paper, I would like to relate my own teaching experiences using an "activity-based" mathematics lesson that in my opinion came up short. The lesson is chosen not for its uniqueness, but for how it epitomizes a collection of "failed" lessons from a semester's teaching. In describing my interactions with the lesson, I hope to begin characterization of a few choice principles in lesson planning that may perhaps carry over into curriculum design.

A Post-Introductory Prelude

Prior to my involvement in a collaborative lesson-planning group at a university-level mathematics education department, I taught for approximately six years at the collegiate level within mathematics departments. The primary distinction between my prior teaching experiences and my currents ones are as follows: whereas I planned and taught alone before, I am currently involved with three to four other instructors in the lesson planning.[1] In addition, our lessons are more of the activity-based lessons, while I used to teach mostly in a traditional lecture or recitation-style format.

I want to relate here a series of incidents from my first semester during this transition that I think relevant to the discussion to follow. First, to set the scene: the teaching concerns a sequence of mathematics content and methods courses for prospective elementary teachers. As mentioned, part of the teaching involves collaborative planning of the lessons. We gather together, on average, once a week for one to two hours to discuss our learning goals and the manner in which we would like to structure our lesson to accommodate those learning goals.

I recall one particular lesson, sometime during mid-semester that I somehow was not able to "bring to life." I found myself ill-prepared for fostering the kind of lively discourse among the students that up until that point, I felt I had successfully encouraged. I remember having two competing thoughts: the first was that student-centered pedagogy was hard, and that perhaps it was my inexperience with this type of teaching that left me *feeling* somewhat a failure as a teacher. The second thought was that maybe the lesson didn't really fit my *pedagogical style,* and it wasn't that I was a failure as I thought.

The real revelation occurred during our next planning meeting. Somehow, I let it be known that my confidence as a teacher was waning. Then one of the other instructors confided that she too felt the same way, and immediately, yet another instructor admitted to similar sentiments. The fourth instructor, who had been involved in the group planning process longer than two out of the three of us, mentioned that *the lesson plan had been problematic for him!*

What struck me about the last colleague's comment, and also the convergence of our so-called failures in teaching, was that until that point, I had been conceiving teaching in terms of individual and highly idiosyncratic teaching abilities, or more specifically, in terms of one's *personal ability* to bring a lesson to life in the classroom. Somehow, in the previous lessons when the lesson seemed to go well, I had secretly thought, *what a great and talented teacher I am that I am able to bring these half-baked lessons to life and to engage the students in lively and meaningful discourse!* And suddenly, I realized that there could be a lot more to it than that. It dawned on me that my "performance" abilities might have had a little less to do with some innate talents, and that instead it was more the lesson plans that seemed either to offer the space within which to bring out my capacities as a teacher or constrain them by their ill-planned nature.

As if that one experience were not enough, during subsequent lessons, whether I felt, again, the same old sense of success or of failure, I noticed that my experiences were regularly matched by the experiences of my colleagues. I found myself fascinated and intrigued by this fact, that even with our wildly divergent teaching styles and teaching backgrounds, we were having remarkably similar teaching experiences over each of the lessons. What this did for my thinking was that whereas I had previously placed much of the credit as well as blame of teaching upon my performing abilities (and in some ways upon my personality), I began to think instead of the lesson plan as the more meaningful point of focus in terms of what worked and did not work in a lesson.[2]

In retrospect, I think that the lessons seemingly failed when we had not carefully enough thought about the relevant mathematical and pedagogical issues involved and had instead glossed over some crucial details in our planning. It is not a particularly deep or earth-shattering insight, but I do think that it is an important one. In fact, it is the primary point that I now hope to elaborate upon more concretely in the context of a so-called failed lesson. I hope also to make a secondary point, again in the context of the lesson, regarding the tension between "telling" and "not telling" in mathematics teaching. I will first present the lesson, then an account of my own experience of teaching it, followed by reflections upon the lesson.

The Lesson[3]

The lesson motivates an alternate algorithm—sometimes referred to as the *common denominator approach*— to the invert-and-multiply procedure for division of fractions. Here is an example:

$$\frac{2}{3} \div \frac{1}{4} = \frac{8}{12} \div \frac{3}{12} = 8 \div 3 = \frac{8}{3}.$$

The idea is that when given a division-of-fractions problem, one first finds the common denominator. Once both fractions are rewritten with a common denominator, only the numerators need be considered in the division—that is, the denominators can essentially be "ignored" in the subsequent calculation.

To begin, the exploration lays out its intent by stating, "The purpose of this exploration is to help you to understand this [alternate] algorithm and why it works" (p. 154). Its aim is for students to arrive at such an understanding through a process of *discovery*, as will be made clear through the ensuing description. The first task begins by offering five story problems, all of which model a division operation. Here is a modified example:

> *Rick has 4 lbs of sugar. Each time he wants to make a cake, he uses 2/3 lbs of sugar. How many cakes can he make with the sugar he has?*

For each of the story problems, the students are asked to represent the problem and its solution using a diagram. One could imagine a student beginning by drawing four rectangles to stand for the four pounds, where each of the rectangles is then partitioned into thirds to accommodate the eventual "taking away" of the 2/3 pounds at a time. After counting that this can be done six times, the student would conclude that the answer (by use of a measurement model for division) is six.

The activity then asks the students to "consider carefully" and "describe" what "[they] did in order to arrive at [their] answer." The students are subsequently asked to "try to connect what

[they] did on paper to how the problem could be solved using only numbers." One hint is offered in "For example, in the first problem, the original number sentence is $4 \div 2/3$, but regardless of the diagram you draw, you will divide each of your four units into thirds, and thus you are now solving $12/3 \div 2/3$" (p. 154)

An ideal description in response to the prompt for the sample problem above would sound like this (I have italicized the key turn in reasoning):

> I drew four boxes to represent the four pounds. I needed to take away 2/3 pounds at a time, and see how many times I could do that. So, I needed to partition each of the boxes into thirds. So, instead of just 4, I now had 12/3. So, the problem really is $12/3 \div 2/3$. At this point, I am asking how many times can I take away 2/3 from 12/3. *But at this point, it doesn't matter that each of the pieces are called thirds. It's really just a matter of taking away two pieces at a time from 12 pieces, and seeing how many times you can do that. Whether you call those pieces 'thirds' or 'ones'—or even 'fourths' or 'fifths,' for that matter—it's still the question of "how many times can you take away 2 from 12?" They would all give you the same answer of 6. So, $12/3 \div 2/3$ is the same as $12 \div 2$."*

Lastly, the activity closes with the following instructions: "After completing the five problems, look for commonalities in all the problems that lead to a generalization (rule) that you could use in all the division problems" (p. 154). The goal, obviously, is for the students to come to know and to understand, through a process of discovery, the 'common-denominator algorithm.'

How the Lesson Actually Proceeded

Here is how the flow of this lesson transpired from my perspective as the instructor. First, I noticed how most all of the students were able to represent, with relative ease, the problem and its solution through a diagram. We had already covered the topic of representing division solution strategies with diagrams in previous class meetings.

But when asked to describe what they did in order to arrive at their answer, suddenly no one in the entire class appeared to know what the problem was "getting at." Many of those who vocalized their confusion were able to describe what they had done in their diagrams, but mostly in procedural ways, such as, "I drew 4, and then I took away 2/3 pieces six times, and so the answer is six." When pressed, some students were able to mention the repartitioning of 4 into 12/3 but little more. Not one student was able to describe the key turn in reasoning that this activity was meaning to raise. We had established a class norm of justifying mathematical claims by this time in the semester, but at the same time, previous discussions had centered on much simpler and more straightforward ideas and conjectures.

I recall thinking to myself that the lesson was not working. It was not *leading* the students to the desired mathematical insight and thinking as we, the instructors, had expected. I tried asking various questions that I thought might lead to the desired key turns in reasoning, but to no avail. Unable to generate the appropriate kind of scaffolding question that might gently *guide* the students' thinking toward the desired end, I ended up *telling* the students what the activity was getting at (the common denominator approach) and showed how the diagram for the first story problem 'proved' that such an approach worked. I was essentially modeling for the students what they were to do with the remaining four problems. During my elaboration, I guessed that less than half the students had grasped the argument I was expressing, though at the time, I was at a loss as to how I could deal with the situation in a better way. Secretly and with some distress,

I was hoping that those students who understood my explanation would later explain it to those who didn't during the subsequent group work.
After allowing for some time for group work on the remaining problems, I asked a few students to explain their explanations in regards to problems two through five. As expected, few if any of the students showed that they had truly grappled with the reasoning process and instead had somehow *proceduralized* the reasoning into something fairly routine and superficial. They now knew *how* the new common-denominator algorithm itself worked and could be applied, but the conceptual reasoning as grounded in diagrammatic referents—more or less, the *why*—somehow got lost in the process. And again, my attempts at having students explain the *why* were met with confused and somewhat hostile glares. Interestingly, all four instructors had similar experiences with the lesson.

Reflecting Upon the Lesson
The central question for me is this: what might be the lessons learned from such a lesson? What general principles might I extract from this experience? In reflecting back, I think of the activity as assuming too much of the students' ability to see what the designers of the activity clearly saw. While we, the instructors, knew *how to look* at the reasoning process involved in the use of the diagram in order to see, or extract out, a justification for the alternate algorithm embedded within it, the students, in fact, had no idea of what to even look for. Nowhere in the lesson, up until that point, was it mentioned that the point was to derive and justify the common-denominator approach to division of fractions. In some ways, it could be asserted that the pre-service teachers were asked to stare at their diagrams and somehow chance upon a half-baked proof for an algorithm whose existence they had no knowledge, nor awareness of—what, in retrospect, I might suggest as a tricky and next-to-impossible task.

A question arises: if indeed I have captured some essential aspect of the lesson—specifically in how such an activity purposefully withholds important information that might otherwise help a student to "see" the mathematical relationship or concept—how could such an activity ever see the light of day, especially in a published textbook? Of course, a group of us chose this activity for our own lesson! So, though I might not capably answer the question in regards to its publication, I might at least speculate as to how a group of us might have overlooked its particular shortcomings. I do so through a particular theoretical framework.

In his book *Communities of Practice (1998)*, Etienne Wenger discusses the interplay between *participation* and *reification*. Participation is defined as the "complex process that combines doing, talking, thinking, feeling, and belonging. It involves our whole person including our bodies, minds, emotions, and social relations" (p. 56). Meanwhile, reification is defined as the "process of giving form to our experience by producing objects that congeal this experience into thingness" (Wenger, 1998, p. 58), and would include mental objects such as concepts and even words. As an example, a book is a reification of someone's thinking, with the act of thinking being a form of participation. Even a person's understanding of something is a kind of reification, as long as it has *solidified* into something that one might call an understanding, whether correct or not. Wenger points out the interplay between the two, that participation often leads to a reification, which in turn affords further participation, and so on. So a lesson plan can be looked upon as one very concrete reification of the act of participating in discussion and/or preparation, while that same lesson plan also gives rise to the kinds of participation (in the form of thinking, discussing, solving, and so on) that are available to students. Wenger also points out

the notion of *premature reification*—that is, the idea of arriving at reifications before sufficient participation has been realized.
Thus, in terms that Wenger has introduced, one way to describe what happened during the lesson is to say that the students had prematurely moved to reifying their understanding before sufficient participation with the ideas had occurred. Telling a student to "carefully consider" what they had done in making a set of diagrams is not a sufficient prompt toward the kind of full participation necessarily for the kind understanding (reification) sought of the reasoning involved. Without sufficient participation, the resulting understanding is oftentimes fragile, as Wenger posits. Or in the case of the lesson, it was not in any way complete nor deep.

Put differently, one could say that *the activity's intent of having the students come to an understanding of why the algorithm works was circumvented due to insufficient engagement with the key ideas.* Of course, expecting students to "discover" a new piece of mathematics without sufficient scaffolding, or support, more often than not will fail to foster the desired engagement.

Yet, one might also go backwards in this back-and-forth chain between participation and reification. That is, *insufficient grappling with the mathematical ideas amongst ourselves (the instructors) during our planning sessions appears the culprit in the deficiency within the lesson (the reified object)—a deficiency which led to our students mirroring our own insufficient engagement with the mathematics.* That is, to say that the lesson plan "failed" to engage the students in *full participation* (leading to a failure to reify a desired understanding) is in some ways to say that the participatory act on the part of the lesson planners was not deep or thorough enough. One mirrors the other.

Another relevant factor, I believe, was the rather unreflective grappling with the role of telling and not telling in mathematics pedagogy. It occurs to me that it would be fairly easy to internalize the message of reform as, "Let students discover the mathematics, rather than telling them." This approach to mathematics teaching, no doubt works in some contexts, but it also fails quite miserably, as can be seen, in other contexts such as this. The challenge is in knowing when telling or guiding would unnecessarily clamp down on what otherwise might be productive thinking and when it would be beneficial, perhaps even necessary, and lead to fruitful learning. It is certainly not a trivial issue (see Chazan & Ball, 1999 for a discussion on the use of "judicious telling").

In Summary
Two critical points emerge. The first and primary point relates to a level of detail in support offered or not offered within an activity—sometimes, referred to as "scaffolding." When viewed from Wenger's framework (reification and participation), implicit in a presence or lack of scaffolding within a lesson is the presence or lack of careful thought (or in the discussion, if group planning) around the planning and construction of the lesson. Another way to think of it would be to say that *the lack of engagement on the students' part during the lessons makes manifest the shortcomings in the lesson plans, which in turn reflects the deficiency in the engagement with the relevant mathematical ideas during the designing of the lesson plans themselves.*

As a teacher, I have in the past thought, *if it takes me, the instructor, a little effort to solve a problem, then chances are half the students won't be able to solve it.* Perhaps others have had similar (and useless) thoughts. In light of the point I am raising, I might offer an amendment to the thought and offer it instead as a workable pedagogical principle: *if it takes a teacher a little effort to solve a problem (solved, likely through tacit understanding of the underlying concepts), that's likely an indication that s/he needs to be very*

clear and explicit on the mathematics at a finer-grained level, appropriate to his/her students' mathematical maturity and understanding. That is, there must further engagement (participation) with the pertinent mathematics. Further, such a teacher also needs then to take his or her finer-grained understanding to reconstruct a scheme for approaching the problem with an eye on how best to elucidate, or bring about understanding of the key points. Through careful thinking of these issues, the teacher might be better prepared with appropriate scaffolding questions; or for the curriculum developer, these key insights might be contained and explicitly surfaced within the activity/lesson, if the implementing teacher is to have a chance at successfully bringing the lesson to life.

The second point relates to the pedagogical awkwardness that results in withholding a piece of information that would otherwise help the students to "see" a particular mathematical concept or relationship, especially under the guise of allowing for "discovery" or "not telling." The essential factor, in this case, might not be in the *form* of the teaching – telling or not telling—but in whether telling, or not telling, is supporting or hampering the engagement level of the students. I would posit that it is the degree of student engagement, and not the *form* of the teaching, that stands as the first principle in both lesson/curriculum design as well as in teaching. It is the question of how best to raise and maintain a high level of cognitive engagement, and in turn reflection upon subject matter that tells us what pedagogical actions a teacher might adopt in a teaching situation.

In closing, I would like to note how I began by describing this and other such lessons as "failed" lessons. A point worth making is that a so-called "failed" lesson, in fact, has the potential to becoming a meaningful *success* toward professional growth if one takes the time to extract out new learnings from them, and perhaps that is one underlying message of this entire tale.

References

Bassarear, T. (2001) *Mathematics for elementary school teachers: Explorations (2nd ed.),* Boston: Houghton Mifflin Company.

Chazan, D., & Ball, D. (1999). Beyond Being Told Not to Tell. *For the Learning of Mathematics, 19*(2), 2 - 10.

Lampert, M. (2001). *Teaching problems and the problems of teaching.* New Haven, CT: Yale University Press.

Stigler, J., & Hiebert, J. (1999). *The Teaching Gap: Best Ideas from the World's Teachers for Improving Education in the Classroom.* New York: Free Press.

Wenger, E. (1998). *Communities of practice: Learning, meaning, and identity.* Cambridge: Cambridge University Press.

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Endnotes:

1. I am not speaking here of "lesson study," as in Japanese lesson study (see Stigler & Hiebert, 1999), but instead, I mean planning each of a semester's lessons together with colleagues, and all of us teaching that same lesson.
2. An interesting benefit that I have noticed regarding such a view of teaching is that one feels less threatened by observers and criticism. Whereas before this experience I might have felt that any criticism of the teaching was directed at me as teacher, I have begun thinking of criticism as being pointed more to the lesson, and less to myself. This in turn has helped ease the transition from working in isolation as a teacher to becoming part of a larger teaching community.
3. The lesson consists essentially of going over a handout ("Exploration 5.12: An alternative algorithm for dividing fractions) from Bassarear (2001). As such, I will refer to it interchangeably as "lesson," "activity," or "exploration."

The mathematically gifted elementary students' revisiting of Euler's polyhedron theorem

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Abstract: This paper explores how the constructions of mathematically gifted fifth and sixth grade students using Euler's polyhedron theorem compare to those of mathematicians as discussed by Lakatos (1976). Eleven mathematically gifted elementary school students were asked to justify the theorem, find counterexamples, and resolve conflicts between the theorem and counterexamples. The students provided two types of justification of the theorem. The solid figures suggested as counterexamples were categorized as 1) solids with curved surfaces, 2) solids made of multiple polyhedra sharing points, lines, or faces, 3) polyhedra with holes, and 4) polyhedra containing polyhedra. In addition to using the monster-barring method, the students suggested two new types of conjectures to resolve the conflicts between counterexamples and the theorem, the exception-baring method and the monster-adjustment method. The students' constructions resembled those presented by mathematicians as discussed by Lakatos.

Key words: counterexample, elementary students, Euler's polyhedron theorem, Lakatos, mathematically gifted

1. Introduction

One perspective on mathematics education states that it is important to analyze and reconstruct the historical development process of mathematical knowledge for improving mathematics teaching and learning. A number of scholars including Clairaut (1741, 1746), Branford (1908), Klein (1948), Toeplitz (1963), Lakatos (1976), Freudenthal (1983, 1991), and Brousseau (1997) share this perspective. This view usually assumes a close relationship between the historical genesis and individual learning process, and supposes that students, with the assistance and guidance of a teacher are capable of constructing knowledge similar to that obtained historically by mathematicians. In particular, Lakatos (1976) demonstrated this view in his book, *Proofs and Refutations*, through an imaginary conversation between a

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teacher and pupils. The teacher and pupils support and criticize one another's claims from the perspective of various historical figures. However, the knowledge construction carried out by the teacher and pupils as presented by Lakatos is, in fact, the construction performed by prominent mathematicians including Euler, Legendre, and Cauchy. Lakatos' (1986) quasi-empirical view seems to ask students to learn mathematics by working like mathematicians (Chazan, 1990) prompting the question, "Is it also possible for elementary students to carry out knowledge constructions based on Euler's polyhedron theorem similar to those produced by mathematicians as discussed by Lakatos'?" In seeking a response to this question, this study focuses on (1) the knowledge constructions of mathematically gifted elementary students in comparison to those of mathematicians as discussed by Lakatos (1976), (2) how mathematically gifted fifth and sixth grade students justify Euler's polyhedron theorem, (3) the figures they suggest as counterexamples to Euler's polyhedron theorem, and (4) how they react when presented with counterexamples.

2. Background

2. 1. Literature Review

Sriraman found (2003) that the problem solving behaviors of mathematically gifted high school students' and those of non mathematically gifted students differed significantly. He reported that gifted students invest a considerable amount of time in trying to understand the problem situation, identifying the assumptions clearly, and devising a plan that was global in nature. Previous studies on the cognitive processes of mathematically gifted students have focused on generalization, abstraction, justification, and problem-solving (Krutetskii, 1976; Lee, 2005; Sriraman, 2003; 2004). Lee (2005) also found that mathematically gifted students have a tendency to advance to higher-level reasoning through reflective thinking.

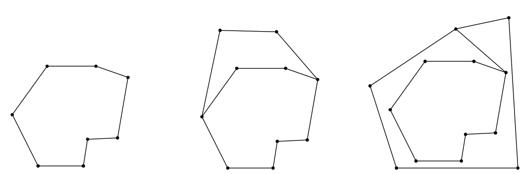
Some researchers have analyzed the knowledge construction of students based on Lakatos' perspective (Athins, 1997; Boats et al., 2003; Borasi, 1992; Cox, 2004; Nunokawa, 1996; Reid, 2002; Sriraman, 2006). For example, Sriraman (2006) reconstructed the quasi-empirical approaches of six above average high school students' attempts to solve a counting problem and present the possibilities for mathematizing during classroom discourse in the spirit of Lakatos. Cox (2004) reported that the ability of high school students to proof improved after introducing them to the process of 'conjecture → proof → critique → accept or reject' in geometry classes. Borasi (1992) described the process where two high school students revised the definition of polygon and concluded that working on polygon "à la Lakatos" provided the context for valuable mathematical thinking and for activities that encourage participants to make use of their mathematical intuition and ability. Reid (2002) analyzed the problem-solving process of fifth-grade students and categorized their process of dealing with counterexamples based on monster-barring and exception-barring into three reasoning patterns. Athins (1997) reported that he observed a case of monster-barring on angles in a fourth grade mathematics class.

2.2. Euler's polyhedron theorem in Lakatos' *Proofs and Refutations*

In Lakatos' (1976) *Proofs and Refutations*, some justifications for Euler's theorem such as Cauchy's proof that appeared in the history of mathematics are shown in the dialogues between the teacher and pupils. For example, Lakatos has pupils Zeta and Sigma say the following explanation (pp.70-72).

Step 1 : For a polygon, $V = E$.

Step 2 : For any polygon $V - E = 0$ (Fig. 1 (a)). If I fit another polygon to it (not necessarily in the same plane), the additional polygon has n_1 edges and n_1 vertices; now by fitting it to the original one along a chain of n'_1 edges and $n'_1 + 1$ vertices we shall increase the number of edges by $n_1 - n'_1$ and the number of vertices by $n_1 - (n'_1 + 1)$; that is, in the new 2-polygonal system there will be an excess in the number of edges over the number of vertices: $E - V = 1$; (Fig. 1 (b)); for an unusual but perfectly proper fitting see Fig. 1 (c). 'Fitting' a new face to the system will always increase this excess by one, or, for an F-polygonal system constructed in this way $E - V = F - 1$.



(a) (b) (c)

Figure 1

Step 3 : I can easily extend my thought-experiment to 'closed' polygonal systems. Such closure can be accomplished by covering an open case-like polygonal system with a polygon-cover: fitting such a covering polygon will increase F by one without changing V or E. Or, for a closed polygonal system – or closed polyhedron – constructed in this way, $V - E + F = 2$.

Following the conjecture and proof, there appear counterexamples that refute the conjecture and proof. Lakatos called a counterexample that refutes lemma or subconjecture a *local* counterexample, and a counterexample that refutes the original conjecture itself a *global* counterexample (pp. 10-11). He suggested six types of counterexamples which appeared in the history of mathematics as described below.

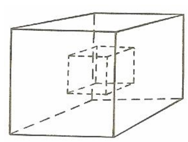


Figure 2. Hollow cube (p.13)

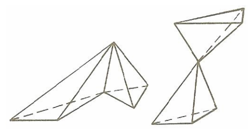


Figure 3. Two tetrahedra with a common edge or vertex (p.15)

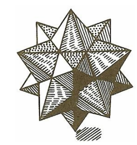


Figure 4 . Star-polyhedron (p.17)

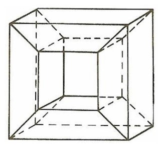


Figure 5. Picture-frame (p.19)

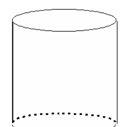


Figure 6. Cylinder (p.22)

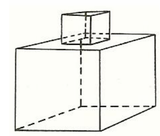


Figure 7. Crested cube (p.34)

When a general counterexample is presented, there are five options. The first option considers the refuted conjecture incorrect and rejects it. The second option is to use the method of monster-barring in which the counterexample is seen as a monster, and the original conjecture is maintained (pp.16-23). This method generates clearer definition, but it is not useful from a heuristic point of view because it does not improve the conjecture. The third option is the method of exception-barring in which the original conjecture is changed into a revised conjecture by adding a conditional clause that mentions an exception (pp.24-27). This method does not guarantee that all exceptions are specified, and leaves the question of what is the range in which the theorem is valid. The fourth option is the method of monster-adjustment where the perspective under which the example was considered as a counterexample is seen as distorted, and the counterexample is interpreted as an example by readjusting the perspective (pp.30-33). The fifth option is the method of lemma-incorporation, where careful analysis of the proof is made to identify the guilty lemma. The lemma can then be incorporated in the conjecture to improve the refuted conjecture (pp.33-42).

3. Methodology

3.1. Participants

Although there are diverse definitions of mathematical giftedness, there is no one universally accepted definition (e.g., Bluton, 1983; Miller, 1990; Gagne, 1991). In this study, Gagne's (1991) definition of mathematically gifted students as "students who are distinguished by experts to have excellent ability and potential for great achievements" was applied. Eleven fifth and sixth-grade male students (aged 10 -12) from different Korean elementary schools in Gyeonggi province participated in the study. Five students were in the fifth-grade and six were in the sixth-grade. The sixth grade students were attending an advanced program for mathematically gifted students; three (A, B, and C) in a Korean

government sponsored university program, and three (D, E, and F) in an office of education program. The fifth-grade students (G, H, I, J, and K), having passed a screening process which included a written test, an in-depth interview, and recommended from their school principal, were scheduled for admission to the university program. All students were motivated and confident of mathematics.

3.2. Tasks

The participants were presented with the following tasks:

Task 1: Explain what you know about the relationship between vertices (V), edges (E) and faces (F) in polyhedra. Explain how the relationship is justified.

Task 2: Is $V - E + F = 2$ true in all polyhedra? If not, when is it not true?

Task 3: If you consider a counterexample a polyhedron, how would you revise the theorem?

If you believe a counterexample is not a polyhedron, how would you revise the definition of a polyhedron?

Task 1 was designed to identify the participants' knowledge of the polyhedral theorem and to determine how they justify the theorem. Task 2 was developed to establish the types of counterexamples the participants identified. Task 3 was designed to observe how the participants resolved the disparity between the theorem and the counterexample.

The participants were familiar with the relationship between vertices, edges and faces, $V - E + F = 2$, before taking part in this study. However, they had not previously examined whether the theorem was true in all polyhedra, nor had they sought counterexamples to the theorem.

3.3. Data Collection and Analysis

This study was designed based on Yin's (2003) multiple case study methodology. The eleven participants were presented with the tasks in a set order and interviewed between November 2005 and January 2007. Each participant was video-taped by one researcher while they worked on the tasks and later while being interviewed by another researcher. The participants completed the tasks in approximately two hours. The video clips, transcriptions, observation reports and participants' worksheets were analyzed.

The analysis was conducted on three types of data collected: (a) the types of justification, (b) types of counterexamples, and (c) the methods for solving the conflict. The types of justification and counterexamples presented by the participants were analyzed using open coding (Strauss and Corbin, 1998). The types of justification were divided into two categories, and the counterexamples were categorized into four types, three of which were subdivided into two to three subtypes. The analysis of the participants' attempts to deal with the disparity between the counterexamples and the conjectures highlighted by the counterexamples was made using selective coding (Strauss and Corbin, 1998) which was based on "the method of monster-barring," "the method of exception-barring," "the method of monster-adjustment" and "the method of lemma-incorporation" suggested by Lakatos (1976). Cross-tabulation

analysis was performed, and the results were examined by peers (Merriam, 1998).

4. Results

4.1. Participants' justification of Euler's polyhedron theorem

The participants' justification of the theorem can be divided in two ways; 1) to classify polyhedra into several categories and justify the theorem for each category of polyhedra, and 2) to attempt general justification without classifying polyhedra. The majority of participants justified the theorem by classifying polyhedra into categories and justifying the theorem. Participant D, in Episode 1 below, demonstrated this by logically explaining that the theorem is justified in prisms, pyramids, and prismoids.

Episode 1:

Participant D: First, in prisms, it seems to be justified in all cases.

Interviewer: Why is that?

Participant D: (Drawing figures) Well, look at an n-angle prism. A rectangular prism, it's called that because the bases are rectangles. So, there are four vertices on the top face and four on the bottom face, so, the number of vertices is $2n$. Also, the number of edges is $3n$ because there are four edges on the top face, four on the bottom face, and four on the lateral sides. And, the number of faces is $n+2$ because there are four faces on the lateral sides plus the top and bottom faces. In the case of a pentagonal prism, also, the number of faces is $n+2$, as there are five lateral faces plus the bases (top and bottom faces). '$V-E+F$' stands for 'number of vertices – number of edges + number of faces,' and in n-angle prisms, it is '$2n-3n+(n+2)$,' so '$V-E+F$' equals 2.

Interviewer: Yes.

Participant D: So, I'm done with prisms... in pyramids, too, it is justified all the time.

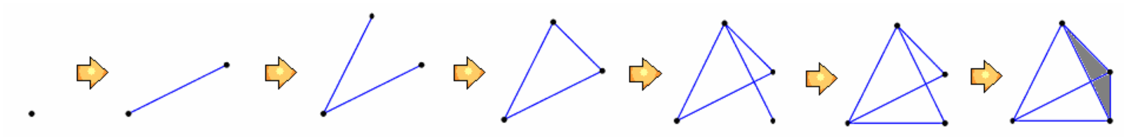
Interviewer: Please explain.

Participant D: ..... an n-angle pyramid. It's justified because the number of its vertices is $n+1$, and it has $2n$ edges and $n+1$ faces. If you add the number of vertices and the number of faces, and then subtract the number of edges, you get 2.

Participant D provided explanations using polyhedra such as rectangular prism in the case of prisms, pyramids, and prismoids. Rectangular prism is a generic example (Mason and Pimm, 1984) which represents general n-angle prism. In the case of regular polyhedra or a polyhedron like the soccer ball, D investigate the theorem application by counting the numbers of points, edges, and faces of specific solids.

Participant B did not categorize solid figures but instead attempted generalized justification. He started with a point and verified $V-E+F$ as the number of points, lines, and faces gradually increased. According to him there is only one V at first, but V and E or E and F increases by 1 respectively as procession is made from (a) to (g) and,

$V-E+F$ is maintained at 1. In the last stage, when one face is covered in (g), he proved that $V-E+F=2$, based on the fact that the number of F increases by 1. This justification is similar to the explanation of pupils Zeta and Sigma in Lakatos (1976, pp. 70-72).



(a) (b) (c) (d) (e) (f) (g)

Figure 8

After justifying Euler's theorem, all the participants expressed the view that there might be a polyhedron with which Euler's theorem was not true. For example, participant D, as indicated in Episode 2, thought that the theorem would not hold in all polyhedra.

Episode 2:

Participant D: Well... first, it is justified in regular polyhedra without exception, because there are only five kinds of regular polyhedra. I think it is justified in all of the five, and then, it is justified, first, in prisms and pyramids. So, I think it is justified in the majority of general polyhedra...

Interviewer: Then, do you think there are some cases in which it doesn't apply?

Participant D : *In some cases... I think it won't apply in all cases.* (Starts drawing figures to find solids with which the polyhedral theorem is not true)

Although participant B justified the theorem using a general method, he tried to find a counterexample, thinking that there still might be one. All the participants express the view that there had to be an example in which the theorem does not apply.

4.2. Solid figures suggested by participants as counterexamples

Participants suggested various types of solid figures as counterexamples to the theorem. The solid figures suggested by the participants were categorized into the four groups below.

4.2.1. Solids with curved surfaces

Six participants (B, C, E, F, H and I) suggested solids with curved surfaces such as a cone (Fig. 9), a cylinder (Fig. 10) and a sphere (Fig. 11) as counterexamples. Each participant had a different reason for suggesting the cone as a counterexample. Participant F drew the net of a cone in order to count the points, lines, and faces. He claimed that the circle in the net was not counted as an edge because it was a curve, but the radius of the sector had to be counted as an edge because it was a straight line ($V=1$, $E=2$, $F=2$, $V-E+F=1$).

Participant E insisted that the radius of the sector in the net could not be counted as an edge because it was not actually seen in the solid, and thus, $V = 1$, $E = 0$, $F = 2$, $V - E + F = 3$. Participant H said that a cone provided a counterexample, "Because you can't say how many edges there are in a circle."

Figure 9

Figure 10

Figure 11

4.2.2. Solids made of multiple polyhedron sharing points, lines, or faces

Nine participants (A, B, C, D, E, F, G, H and I) cited solids made of two polyhedra sharing points, lines, and faces as counterexamples. These solids can be divided into (1) solids that completely share some points, lines, or faces (Fig. 12 through Fig. 15), and (2) solids that only partially share lines or faces (Fig. 16 through Fig. 19).

4.2.2.1. Solids that completely share points, lines, or faces

In solids that share one point as shown in Fig. 12, the theorem holds in each polyhedron and two polyhedra share a point, $V - E + F = 3$. Participants also suggested solids that share an edge (Fig. 13) and those that share a face completely (Fig. 14 or Fig. 15) are counterexamples.

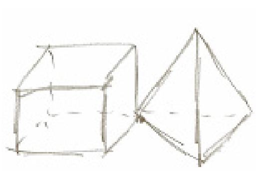


Figure 12

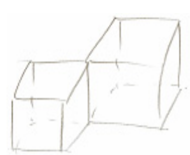


Figure 13

Figure 14

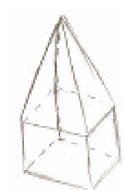


Figure 15

4.2.2.2. Solids that partially share lines or faces

Solids such as in Fig. 14 and Fig. 15 raised the issue with participants of whether it is appropriate to consider shared faces as separated faces. Participants suggested that modified solids that partially share lines or faces were counterexamples.

Figure 16

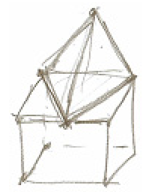


Figure 17

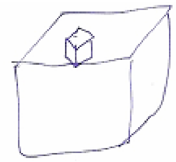


Figure 18

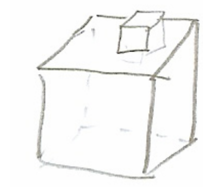


Figure 19

Where edges are partially shared (Fig. 16) and where an edge is divided (Fig. 17) the participants reflected on how to count the number of edges. And a counterexample such as Fig. 19 led the participants to contemplate the question, "Is it appropriate to consider the face created by joining two faces a face?" Lakatos (1976, p.74) called this a "ring-shaped face."

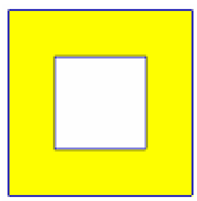


Figure 20. Ring-shape face

4.2.2.3. Polyhedra with holes

The third type of solids that the participants (A, B, C, G, J and K) suggested as counterexamples is solids with holes as shown in Fig. 21 through Fig. 23. These counterexamples also prompted the participants to rethink the definition of face.

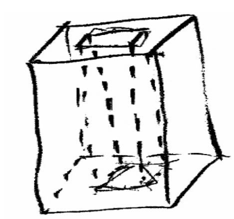


Figure 21

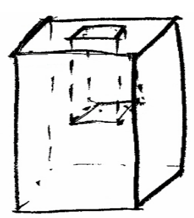


Figure 22

Figure 23

4.2.2.4. Polyhedra containing other polyhedra

Eight participants (A, B, C, D, F, G, J and K) also suggested solids that are polyhedra containing other polyhedra are counterexamples. Counterexamples of this type can be subdivided into three subtypes. The first is the type in which other solids -not sharing any face, point or line- exist in certain solids (Fig. 24). The second is the type in which two solids completely share a face (Fig. 25). The third type is one in which a figure exists inside another

and the two figures share part of a face.

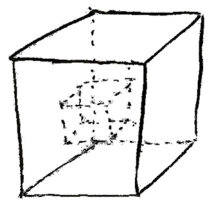


Figure 24

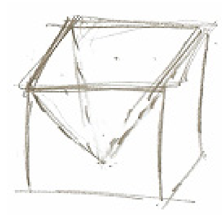


Figure 25

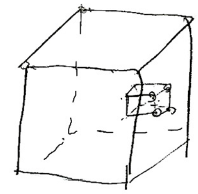


Figure 26

4.2.3. Participants' responses to the disparity created by counterexamples

The participants' responses to the disparity between counterexamples and the theorem are divided into four categories; the method of monster-barring, the method of exception barring; the method of monster adjustments and new conjectures.

4.2.3.1. The method of monster-barring

Participants D and E used the method of monster-barring. In Episode 3, participant E, suggested cones, cylinders, and spheres as counterexamples, and wondered how to determine the numbers of points, lines, and faces in these figures. He then stated, "*A polyhedron is a solid figure made of multiple polygons*", and that the curved surface is not a polygon, and thus, solids with curved surfaces are not polyhedra but monsters.

Episode 3:

Participant E: Cones have curved surfaces, so I think they will not work.

Interviewer: What's wrong with curved surfaces?

Participant E: Because in a curved surface, you can't count the number of edges, and faces... Can you count the number of faces? But the number of vertices is one... I think there is no edge, in the definition that I think of.

Interviewer: Can you say a cone is a polyhedron? Euler's theorem is about polyhedra.

Participant E: When you talk about curved surfaces, a sphere has a curved surface, and a sphere has one face, ... but no distinguishable edge or point, I guess there are none.

Interviewer: What do you think is the definition of a polyhedron?

Participant E: I think it is made of faces that have angles. (Writing down the definition) "Polyhedron = solid figure made of multiple polygons"

Participant D also used the method of monster-barring where polyhedra existed in other polyhedra. He used the method of monster-barring stating that a polyhedron signified "one" solid figure, and that the polyhedron in which there is another polyhedron meant two different solid figures. He modified the definition of polyhedron as "one solid figure

surrounded by multiple polygons."

4.2.3.2. The method of exception-barring

Participants A, D, F, G, and I were observed attempting the method of exception-barring. Participant F defined a polyhedron as a "figure made of faces." So the solid figures with curved surfaces are polyhedra because curved surfaces are faces. To exclude cones, cylinders, and spheres as exceptions, participant F modified the original conjecture to "*In all polyhedra excluding those made of curved faces,* $V - E + F = 2$." Participant I, Episode 4, also used the method of exception-barring by modifying the theorem to "*In polyhedra that do not include a circle,* $V - E + F = 2$."

Episode 4:

Interviewer: (Pointing to the sphere and cylinder.) Then, can we call them polyhedra, too?
Participant I: It has one or more faces... We can call them polyhedra.
Interviewer: Then, don't we need to modify this ($V - E + F = 2$) ?
Participant I: Yeah...
Interviewer: How can we change it?
Participant I: (Thinking hard) So, if a circle is included... I guess only the polyhedra without any circles belong to this category ($V - E + F = 2$), don't they?

Participant I suggested two rectangular solids that share one edge (Fig. 27) are another counterexample. Then he redefined the theorem to "In polyhedra that do not include a circle and are not attached to other polyhedra, $V - E + F = 2$." Participant G found solid figures with holes as counterexamples, and modified the theorem to "In polyhedra which are not completely penetrated by a hole, $V - E + F = 2$."

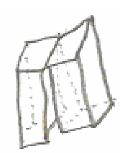
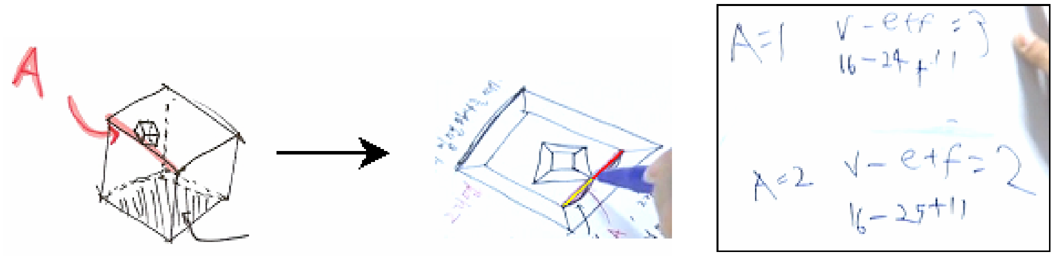


Figure 27

4.2.3.3. The method of monster-adjustment

Participants B, D, E, F, and G tried the method of monster-adjustment to convert a counterexample into an example. Participant B thought, after finding the counterexample in which part of a face was shared by two figures, that the justification of Euler's theorem depended on whether to consider the edge divided by a point as one or two.



(a) (b) (c)

Figure 28

Participant B compared the results when the edge (line A in Fig. 28(a)) divided by a point was counted as one (A=1) and when it was counted as two (A=2). Then, B, Episode 5, explained the reason why the edge divided by a point in this solid figure should be counted as two.

Episode 5:

Interviewer: Is the solid a polyhedron?

Participant B: It is a polyhedron.

Interviewer: Then, what can we do?

(Participant B is writing)

Participant B: If there is a vertex in the middle of an edge (even when it is not on the exact center), the left and right sides of the vertex should be separately counted… It is absolutely necessary to separately count this part (left part of line A) and that part (right part of line A). In the case of plane figures, we count any line between two points separately… In solids, to make it (the value of $V-E+F$) become 2, you need to count the left and right sides of the point separately.

Two polyhedra that completely share a face, with one inside the other (Fig. 25) were also considered not to be a counterexample by one of the participants after the method of monster-adjustment was used. Participant D claimed that the figure was not a counterexample because it was considered a sunken solid without a lid, rather than two solids sharing one face.

For the ring-shaped face (Fig. 20), some participants preferred to use the method of monster-barring by not considering it as a face, and subsequently, employed the method of monster-adjustment by not considering the solid figures with ring-shaped faces as counterexamples (e.g. $V=16$, $E=24$, $F=10$, and thus, $V-E+F=2$ in Fig. 19). Participant I, who used the method of exception-barring for cylinders and spheres, used the method of monster-adjustment for the cone, considering the polyhedral theorem to be justified under the condition of $V=1$, $E=1$, and $F=2$.

4.2.3.4. New Conjectures

The participants' approaches were not limited to monster-barring, exception-barring, and monster-adjustment which are similar in the sense that they are used to support the

formula of $V-E+F=2$. Participants also suggested two types of new conjectures. One involved the participants searching for a new formula about the value of $V-E+F$ with which to express the relationships between the points, lines, and faces in solid figures including the counterexamples they found. New conjectures suggested by participants are summarized in Table 1.

$V-E+F$	Conditions	Participants
0	If the ring-shaped face is not a face, in polyhedra with hole(s)	G
1	In polyhedra including a circle	I
3	In polyhedra that completely share either a point or a line with other polyhedra	E and F
	If solid figures are attached at a vertex, edge, or face	H
4	In polyhedra which contained other polyhedra such as a hollow cube	F

Table 1 Summary of Participants Conjectures

The other type of conjecture, suggested by participant A, relates to the necessity of considering new elements other than points, lines, and faces. He proposed, Episode 6, that a formula that including three-dimensional elements be developed.

Episode 6:

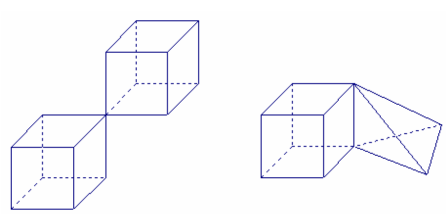
Participant A: In the two-dimensional circumstance, a rule can be easily found using just V, E and F, but in the three dimensions, a new element of space is added. So, if Euler's theorem is a formula established using two-dimensional elements, I guess we can make a new formula that exclusively applies to the third dimension including space, can't we?

Interviewer: The new element of three dimensions. Can we really do it if we consider that?

Participant A: Yes, I think so.

Interviewer: Then how can we determine the numbers in the three dimensions?

Participant A: Space.



(a) (b)

Figure 29

After that, $V-E+F-S=1$ and $V-E+F+S=3$ were proposed as new conjectures, and it was confirmed that $V-E+F-S=1$ is justified with the type of solid figures in Fig. 29 ((a) : $V=15$, $E=24$, $F=12$, $S=2$, $V-E+F-S=1$, (b) : $V=10$, $E=17$, $F=10$, $S=2$, $V-E+F-S=1$). This conjecture led participant A to think that the polyhedral theorem could be expanded to four-dimensional solids.

5. Discussion

Polyhedra, which the participants studied prior to the research, were limited to the category of regular polyhedra, prisms, pyramids, prismoids, and semi-regular polyhedra such as soccer balls, all satisfied Euler's theorem. Nevertheless, the participants thought that there must be some polyhedra for which the theorem was not valid. This belief appears to resulted from the method of justification that the majority of participants used. The value of $V-E+F$ can be obtained by counting the numbers of points, edges, and faces in the case of prisms, pyramids, and prismoids (e.g. in n-angle prism, $V=2n$, $E=3n$, $F=n+2$, and thus, $V-E+F=2$). However, this justification fails to provide information about new kinds of solids that have yet to pass this test. The participants' view that there must be polyhedra with which the polyhedral theorem was not valid indicates they belive that the scope of polyhedra is extensive. This view is supported by the various types of solids that the participants presented as counterexamples.

A strong similarity exists between solid figures suggested by the participants as counterexamples and those discussed by Lakatos (1976). The first type of counterexample that participants found, solids with curved surfaces, appeared as cylinders in Lakatos (p.22). The second type, two or more polyhedra that shares points, lines, or faces, was discovered by mathematicians Hessel (figures that share points or lines) and Lhulier (cube with crest) in 1832 and 1813, respectively (p.15, p.34). The third type of counterexample was first discovered by Lhuilier (p.19). In addition to the tunnel and picture frame mentioned in Lakatos, participants also found a polyhedron which is not completely penetrated. The fourth type, polyhedra within polyhedra, was discovered by Lhuilier and Hessel based on the idea obtained by observing the crystalloid of mineralogic collection enclosed within a translucent crystalloid (p.13).

Counterexamples can be used to help students develop their mathematical reasoning (Lakatos, 1976; Boats, et al., 2003). In this study participants examined concepts such as polyhedron and face and created new definitions. The counterexamples discovered by participants also encouraged them to examine more closely the definition of terms. The ring-shaped face in particular prompted some participants to reconsider the definition of polygon. They asserted that it could not be called a polygon, because the figure did not comply with the sum of interior angles of n-polygon $180\times(n-2)$. This suggests that the formula for the sum of interior angles of a polygon was seen as a definitive property that determines whether the figure was a polygon or not. This method of defining a polygon is

similar to the definition of polyhedron stated by Baltzer (Lakatos, 1976, p.16), "polygon system with which the equation of $V - E + F = 2$"

In Reid (2002) and Athins (1997), the method of monster-barring and the method of exception-barring were observed among elementary students. The method of monster-barring, the method of exception-barring, the method of monster-adjustment, and new conjectures were observed among the participants in this study. The participants did not reject the original theorem and attempted to develop new conjectures that comprised counterexamples, and of the five participants who used the method of exception-barring, four developed new conjectures. In the past, there have been cases in which counterexamples were first recognized as monsters and excluded, but later reintroduced and accepted as examples (e.g. Lakatos, 1976, p.31). This ability to review and change a position was also demonstrated by the participants. Initially, they used the monster-barring method or exception-barring method for the counterexamples they identified, but they attempted to include the counterexamples within the scope of examples through monster adjustment or new conjecture. Krutetskii, (1976) Sriraman, (2004) point out that this flexibility of thinking is an attribute of mathematically gifted.

Lakatos (1976) argues that the method of lemma-incorporation is a productive way of refining conjecture based on the proof. Proof-analysis is a prerequisite to this method and, as Nunokawa points out (1996) proof-analysis is an important component of proofs and refutations. However, in this study, the method of lemma-incorporation and proof-analysis was not observed. When participant B, provided proof of increasing the elements of polyhedra, was encouraged considering the validity of his proof for a counterexample (Fig. 18), he provided a monster adjustment solution stating, "It's not the proof that's wrong, but there is a problem with this solid."

6. Conclusion

This study focuses on the constructions of mathematically gifted fifth or sixth-grade students in solving tasks related to Euler's polyhedron theorem and compares them to those of mathematicians discussed by Lakatos (1976). By analyzing ninth grader students notion of proof, Sriraman (2004) reports that the processes used by gifted students demonstrate remarkable isomorphism to those employed by professional mathematicians, This study also shows parallels in constructions of mathematically gifted fifth and sixth grade student and mathematicians discussed by Lakatos. With the exception of the method of lemma incorporation and proof-analysis, counterexamples and the method for solving conflicts between the theorem and counterexamples suggested by the participants demonstrated remarkable similarities to those presented in the history of mathematics.

References

Athins, S. (1997). Lakatos' proofs and refutations comes alive in an elementary classroom. *School Science and Mathematics*, 97(3). 150-154.

Bluton, C. (1983). Science talent: The elusive gift. *School Science and Mathematics*, 83(8). 654-664.

Boats, J. J., Dwyer, N. K., Laing, S. & Fratella, M. P. (2003). Geometric conjectures: The importance of counterexamples. *Mathematics Teaching in the Middle School*, 9(4). 210-215.

Borasi, R. (1992). *Learning mathematics through inquiry*. Portsmouth, NH : Heinemann.

Branford, B. (1908). *A study of mathematical education*. Oxford: Clarendon Press.

Brousseau, G. (1997). *Theory of didactical situations in mathematics* (N. Balacheff, M. Cooper, R. Sutherland & V. Warfield , Ed. and Trans.). Dordrecht: Kluwer academic publishers.

Chazan, D. (1990). Quasi-empirical views of mathematics and mathematics teaching. *Interchange*, 20(1), 14-23.

Clairaut, A. C. (1741). *Éléments de géométrie*. Paris : Gauthier-Villars.

Clairaut, A. C. (1746). *Éléments de algébre*. Paris : Rue Saint Jacques.

Cox, R. (2004). Using conjectures to teach sutdents the role of proof. *Mathematics Teacher*, 97(1). 48-52.

Freudenthal, H. (1983). *Didactical phenomenology of mathematical structures*. Dordrecht: D. Reidel Publishing Company.

Freudenthal, H. (1991). *Revisiting mathematics education*. Dordrecht: Kluwer Academic Publishers.

Gagne, F. (1991). Toward a differentiated model of gifted and talent. In N. Colangelo & G.A. Davis (Eds.). *Handbook of gifted education*. Boston: Allyn and Bacon, 65-80.

Klein, F. (1948). *Elementary mathematics from an advanced standpoint: Arithmatic, algebra, analysis*. (E. R. Hedrick & C. A.. Noble, Trans.). New York: Dover publications. (Original work published 1924)

Krutetskii, V. A. (1976). *The psychology of mathematical abilities in school children*. Chicago: The University of Chicago Press.

Lakatos, I. (1976). *Proofs and refutations: The logic of mathematical discovery*. Cambridge: Cambridge university press.

Lakatos, I. (1986). A renaissance of empiricism in the recent philosophy of mathematics? In T. Tymoczko (Ed.), *New directions in the philosophy of mathematics* (pp. 29-48). Boston: Birkhauser.

Lee, K. H. (2005). Mathematically Gifted Students' Geometrical Reasoning and Informal Proof, In Helen L. C. & Jill, L. V. (Eds.), *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education,* (Vol 3. pp.241-248), Melbourn, Australia : PME.

Mason, J. & Pimm, D. (1984). Generic examples: Seeing the general in the particular. *Educational Studies in Mathematics,* 15, 277-289.

Merriam, S. B. (1998). *Qualitative research and case study applications in education*. John Wiley & Sons, Inc.

Miller, R. C. (1990). *Discovering mathematical talent.* (ERIC Digest No. E482).

Nunokawa, K. (1996). Applying Lakatos' theory to the theory of mathematical problem solving. *Educational Studies in Mathematics*, 31, 269-293.

Reid, D. (2002). Conjectures and refutations in grade 5 mathematics. *Journal for Research in Mathematics Education*, 33(1), 5-29.

Sriraman, B. (2003). Mathematical giftedness, problem solving, and the ability to formulate generalizations: The problem-solving experiences of four gifted students. *Journal of Secondary Gifted Education*, 14(3), 151-165.

Sriraman, B. (2004). Gifted ninth graders' notions of proof: Investigating parallels in approaches of mathematically gifted students and professional mathematicians. *Journal for the Education of the Gifted*, 27(4), 267-292.

Sriraman, B. (2006). An ode to Imre Lakatos: Quasi-thought experiments to bridge the Ideal and actual mathematics classrooms. *Interchange*, 37(1-2), 151-178.

Strauss, A. & Corbin, J. (1998). *Basics of qualitative research* (2nd ed.). Thousand Oaks, CA: Sage Publications.

Toeplitz, O. (1963). *The calculus – A genetic approach* (L. Lange, Trans.). Chicago: The University of Chicago Press. (Original work published 1949)

Yin, R. K. (2003). *Case study research: Design and methods* (3rd ed.). Thousand Oaks, CA: Sage Publications.

How is Mathematics Education Philosophy Reflected in the Math Wars?

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Abstract: Throughout the duration of what has been termed the "math wars", many overly-generalized statements by both sides have detracted from the quest for a solution to the conflict between conceptual and procedural approaches to mathematics study. In terms of philosophies of mathematics education, the absolutist view posits that mathematical knowledge is certain and unchallengeable while the fallibilist view is that mathematical knowledge is never beyond revision and correction. We suggest that the major mathematics education reforms have been absolutist in focus and have not reflected the changing nature of the discipline. Thus we believe that true reform will reflect changing perceptions in mathematics education along with changes in American culture and its expectations of mathematics education.

Keywords: absolutism; curriculum change; fallibilism; Math wars; Philosophy of mathematics; Philosophy of mathematics education; math reforms

"The math wars are over!" "The National Council of Teachers of Mathematics (NCTM) has come to its senses." Such catch-cries abounded in response to the NCTM's publication of *Curriculum Focal Points for Pre-kindergarten through Grade 8 Mathematics* (2006a). This document represents a break with tradition in that it focuses attention on a limited number of significant mathematical goals at each grade level. In the September 12, 2006 press release, NCTM President Francis (Skip) Fennell asserted that "The Curriculum Focal Points present a vision for the design of the next generation of state curriculum standards and state tests" (NCTM, 2006b, para. 3). He sees the focal points as the first step to a national discussion on bringing consistency and coherence to United States mathematics curricula. This document is seen as one response to the "mile wide, inch deep" criticism of United States mathematics instruction, a criticism that most typically arises in the context of comparing the performance of American students with international students on tests of mathematics achievement.

The assertion by some critics that the Focal Points document represents a shift in the NCTM position on basic skills is challenged by Skip Fennell (NCTM, 2006c), who claims that the NCTM has always recognized the importance of building students' ability to memorize certain basic math facts and procedures. These critics have claimed that the NCTM is doing an about-face. They cite such instances as calculator use replacing mastery of basic skills, that having students describe in writing the reasoning behind their answers meant that students were writing about math instead

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of doing it, and that having students discover their own methods to perform math operations could not lead to mastery (Hechinger, 2006). Another critic, Tamar Lewin (2006, November 14), asserted that planned changes in teaching math in American schools were driven by students' lagging performance on international tests and parents' dissatisfaction with their children's performance in mathematics.

The NCTM's recent involvement in mathematics education reform began with the publication of *An Agenda for Action* (1980) which focused on the need for students to learn how to solve problems. More recently, the following documents have caused substantial public reaction: *Curriculum and Evaluation Standards for School Mathematics* (CESSM) (1989), *Professional Standards for Teaching Mathematics* (1991), *Assessment Standards for School Mathematics* (1995), and the *Principles and Standards for School Mathematics* (PSSM) (2000). Certainly, these documents have focused attention on a standards-based approach to mathematics instruction, described by its critics as "fuzzy math" where individual accountability was replaced by group work, proficiency in basic skills was replaced by reliance on calculators, and serious attention to algorithmic thinking was replaced by "real-life problems." At the time of the publication of PSSM, then NCTM president Glenda Lappan asserted that the original standards (CESSM) (NCTM, 1989) put too much emphasis on new ideas such as teaching conceptual understanding over basic skill development, so that teachers missed the main goal that students become highly skilled in using mathematics (Hoff, 2000). Accordingly, critics of these reform efforts appear to have misrepresented the initiatives of the last two decades.

What tends to be overlooked in these exchanges is how these accusations began. Ever since the First International Evaluation of Achievement (Husen, 1967), we have heard charges of the American mathematics curriculum being a mile wide and an inch deep. Then, as now, one explanation of the relatively poor performance of American students in international tests of comparison was the large number of topics covered at the secondary level, many of them topics that had been introduced previously.

The purpose of this paper is not to join sides in the "Math Wars," or to attribute blame to one side or the other. In a recent commentary, Wallis (2006, November 19) claimed that American education is every bit as polarized, red and blue, as American politics. However, the real issue is not this polarization. Rather, it is to try to understand why different groups within our society have different expectations of a school mathematics curriculum and how they can be explained by differences in philosophy of mathematics education.

Connections between math education philosophy and curriculum change

There is evidence that the two sides of these math wars are not attending to the points made by the other side. McKeown (Hoff, 2000) claimed that the NCTM and its critics agree on "platitudes", but disagree about how much emphasis to put on them. Thus the real issue seems to hinge on different philosophical considerations about the nature of mathematics education. Paul Ernest (1991) describes the two opposing perspectives as "[t]he absolutist view of mathematical knowledge" which "consists of certain and unchallengeable truths" (p. 7), and the "fallibilist view", which "is the view that mathematical truth is fallible and corrigible, and can never be regarded as beyond revision and correction" (p. 18). He maintains that the rejection of the absolutist view "leads to the acceptance of the opposing fallibilist view" (p. 18). Ernest (1994) sees the "central problem of the philosophy of mathematics education" as "the issue of the relationship between philosophies of mathematics and mathematics education" (p. 4).

Although it is generally accepted that Ernest's classification is a dichotomous one, we claim that there exists a continuum of positions between these extremes. To support our claim, we would draw attention to the philosophy implied when observing a math teacher in action. There should be little disagreement when observing a teacher's math lesson that it be categorized as either absolutist or fallibilist. Does that teacher manifest the same implicit philosophy in all lessons he/she teaches? Could it not be that this teacher may demonstrate an absolutist philosophy when teaching math skills, but a fallibilist philosophy when engaging the students in bona fide problem solving? These questions become important when the NCTM is accused of a change of heart. Has the organization (or its leadership) really converted from a fallibilist to an absolutist position?

Most mathematics educators would agree with Fennell's (2006) assertion that the NCTM position has in essence not changed. While the *Curriculum and Evaluation Standards for School Mathematics* (NCTM, 1989) made the case that memorization of basic facts and traditional factual tests were being de-emphasized (p. xx), it was not claimed that traditional approaches would disappear. Moreover, articles published in the NCTM journals (*Teaching Children Mathematics*, *Teaching Mathematics in the Middle School*, and *The Mathematics Teacher*) during the past 17 years lean heavily in the direction of student involved approaches. Could we infer from this that such approaches were typical of the mathematics teaching force?

The most succinct answer to this question is provided by an analysis of the *Third International Mathematics and Science Study* (TIMSS) Grade 8 videotapes. We are told (Stigler, Gonzales, Kawanaka, Kroll, & Serrano, 1999; Stigler & Hiebert, 1999) there was uniformity of style among American eighth grade teachers, and the focus clearly indicated an absolutist view of mathematics. How can we explain the evidence that American teachers consistently design lessons grounded in an absolutist perspective despite the existence of a hierarchy of perspectives that includes fallibilism?

Although Ernest (1991) asserts that teachers' strategies in the classroom depend upon their philosophical perspective, he emphasizes the importance of social context. That is, teachers who have different philosophical perspectives may still teach in similar ways and adopt similar classroom practices depending upon the socialization effect of the context. Implementing a fallibilist view in practice, for instance, is far less likely if a teacher's peers and school climate support the absolutist perspective. Teachers may "shift their pedagogical intentions and practices away from their espoused theories" (p. 289) when faced with constraints created by the social context.

Social context, then, is a significant determining factor with respect to educational practice and policy. Educational historian, David Tyack (1974), writes that the central reason for reform failures "has been precisely that they called for a change of philosophy or tactics on the part of the individual school employee rather than systemic change" (p. 10). Although Tyack refers to social injustice and attempted social reform through education policy and practice, the argument can be applied to mathematics reform initiatives. Systemic change may include, in the case of mathematics education, the transformation of the school climate from one emphasizing training for occupation to one enriched with philosophical discourse and the concomitant empowerment of teachers and students to create change. Systemic change, however, has not been typically the focus of recent reform efforts, and that fact may provide a partial explanation for their failures.

History of Mathematics Education Reform

The history of education reform in the twentieth century documents one failure followed by another. Curricular and pedagogical change have been the objects of extensive efforts by authorities in all fields, only to result in very little visible change in the basic curriculum (Kliebard, 1987) or classroom practice (Cuban, 1993). If hundreds of educational reform reports and policy initiatives have had little to no effect with respect to educational changes, then why have policy and reform efforts failed so miserably? Have these policies been poorly designed or designed by individuals and organizations that have little working knowledge of what really goes on in schools? Are schools simply tools of a power elite or dominant class, and curricular and pedagogical reform efforts are designed to maintain the extant power/class structure? Does society need to change before classroom practice will change? Is it possible that teachers' conceptions of mathematics need to undergo significant revision before the teaching of mathematics can be revised? Ultimately, is the problem one of the construction of policy, a socio-cultural context, or philosophically-bound practitioners?

The tension between two almost diametrically-opposed philosophies is visible throughout educational history in the United States. Beginning with the early reform movements that were focused on compulsory education and proceeding through contemporary policy such as the No Child Left Behind Act (NCLB) of 2002, claims about the purpose and means of education have been contentious. The prominent educational philosopher, John Dewey, supported a vision of education as one connected with experience and real-life. Dewey (1936a) argued that mathematics education should be integrated: "Number arises in connection with the measuring of things in constructive activities; hence arithmetic should be so taught and not in connection with figures or the observation of objects" (pp. 213-214). Dewey's critique of the absolutist perspective in mathematics education found expression in his Chicago Laboratory School where mathematics experiences were integrated with other disciplines. His contention was that "Until educators have faced the problem and made an intelligent choice between the contrasting conceptions…I see no great hope for unified progress in the reorganization of studies and methods in the schools" (1936b, p. 396).

Lawrence Cremin (1961) points out that Dewey recognized the importance of learning the basics before proceeding to interpretation and extension. "To recognize opportunities for early mathematical learning, one must know mathematics" (p. 138). According to Cremin, Dewey attempted throughout his life to emphasize the necessity of teachers thoroughly comprehending the disciplines without creating a strictly utilitarian curriculum that accommodated students to existing conditions without enhancing their ability to envision and create change. Cremin writes: "In short, the demand on the teacher is twofold: thorough knowledge of the disciplines and an awareness of those common experiences of childhood that can be utilized to lead children toward the understandings represented by this knowledge" (p. 138).

Visualizing the range of perspectives as an arena rather than as a continuum with absolutist at one end and fallibilist at the other, one might conclude that a melding of the absolutist and fallibilist perspectives, or philosophical movement within the arena of mathematics perspectives, may be essential for authentic learning to occur. The melding of the perspectives, however, has not been visible in the history of American mathematics education. E. L. Thorndike's emphases on recitation and rote memorization followed by measurement of outcomes through achievement testing, advocated in the early years of the twentieth century, have determined the

state of affairs in mathematics education. Students trained in an absolutist philosophy who become teachers will likely teach from the absolutist standpoint.

New Math Reforms

The new math reforms took on different formats in various countries. In the United States and Europe, there was concern that an insufficient number of well-qualified students would be proceeding to post-secondary mathematics. There was a belief among university mathematicians that the secondary school mathematics syllabus needed to be reformed. Accordingly, groups dominated by university mathematicians set out to determine what mathematics should be studied in school to prepare students for university level mathematics. In 1952, the University of Illinois Commission on School Mathematics (UICSM) was formed under the leadership of Max Beberman and its focus was on the structure of mathematical thinking. The School Mathematics Study Group (SMSG) was started in 1958 under the leadership of Edward Begle at Yale (later at Stanford) University. Like UICSM, SMSG was university dominated, and focused on the development of abstract mathematical ideas.

Likewise, in France, Lucienne Felix (1961), a close associate of the Bourbaki group, characterized the revolution there as a response to the need to replenish the supply of potential mathematicians, so many of whom were victims of the war. By contrast, in the United Kingdom, there was considerable involvement by the teaching community in the reforms undertaken. Bryan Thwaites of the University of Southampton headed up the School Mathematics Project where teachers did the writing, and the funding came from industry, not the government. In both settings, the claim was made that there was little change in the mathematics that was to be taught (although that little was quite substantial in the United States and Europe), but that there was to be considerable change in how the mathematics would be taught. However, as Beeby (as cited in Griffiths & Howson, 1974) points out about the role of teachers in curriculum reform, "the average teacher has a very great capacity for going on doing the same thing under a different name" (p. 143).

Not all mathematicians in the United States were supportive of the direction their colleagues had chosen. Morris Kline (1958) deplored the university domination of the reform efforts which he accused of being more interested in training a new generation of mathematicians than in providing mathematics courses for all. In particular, he found the SMSG program to be prematurely abstract and lacking in links to science.

Why did the "new math" fail? One perspective is offered by Amit and Fried (2002):

> In considering why a reform is desirable one looks at the needs and nature of mathematics, the needs and values of society, and the relationship between them; in considering who brings a reform about, one discovers that a balanced participation of mathematicians, teachers, politicians, parents, and students is necessary and that all must somehow work together and come to understand their different motivations and ways of thinking. (p. 365)

Logistically, it would appear that the "new math" failed to meet the criteria identified by Amit and Fried. It was a top-down reform, initiated by the mathematical community, without buy-in from teachers or the public. Griffiths and Howson (1974) discuss two approaches to curriculum change, the first being a reorganization of the mathematics itself, which tends to be teacher-centered, discipline-based and authoritarian. The second is described as an open tendency, one

that reflects liberal attitudes in schools and society at large, and tends to be pupil-centered. These approaches closely parallel the absolutist and fallibilist philosophical positions as described by Ernest. Thus we can infer that one cause of failure of the "new math" was that it represented an absolutist philosophy in what was emerging as a fallibilist world.

The "new math" movement in the USA in the early 1960s reflected the point made above by Amit and Fried (2002). Larry Cuban (1993) discusses the constancy in mathematics instruction even after the "new math" or "Post-Sputnik" reform initiatives were supposedly instituted. Cuban contends that if significant change in schooling is to occur, "then organizational conditions, the forms of education that teachers receive, and the occupational cultures within which they work must also change" (p. 290). The "new math" curriculum changes were not accompanied by larger systemic changes, and were thus judged as failures.

Usiskin (see Amit & Fried, 2002) pointed out that the new math movement was judged a failure by a public misled by the results of such examinations as the SAT and NAEP. SAT scores were low in the new math era, but so were verbal scores. There is a more crucial point, however: there is no evidence that the SAT reflected the values and goals of the new math. Again, as Usiskin points out, the new math did nothing to help slower students–nor was this its intent. Thus, on this score, it would have been legitimate to challenge the values and goals of the new math. We contend, and the history of mathematics education reveals, that a reform movement or new policy or practice cannot be evaluated solely in terms of its own criteria. The reform process cannot be isolated from the educational system as a whole.

Recent Reform Endeavors

The above point is also being made by Stigler and Hiebert (1999) when they assert that "Educational reforms in this country often have been driven by an effort to change our performance on quantifiable indicators" (p. 98). For example, a major thrust of the new math reform was changing the textbooks. Thus one might expect, because American teachers typically rely on the textbook, that the teaching would change accordingly. The National Advisory Committee on Mathematical Education found: "Teachers are essentially teaching the same way they were taught in school. Almost none of the concepts, methods, or big ideas of modern mathematics have appeared" (Conference Board of the Mathematical Sciences, 1975, p. 77). One explanation offered by Stigler and Hiebert is: "The widely shared cultural beliefs and expectations that underlie teaching are so fully integrated into teachers' worldviews that they fail to see them as mutable…teachers fail to see alternatives to what they are doing in the classroom" (p. 100).

Many reports have been written in recent decades attesting to deficiencies in American mathematics education and suggesting reform initiatives to correct the problems. Those reports and the subsequent failure of suggested reforms provide evidence for the necessity for systemic change. The problem in mathematics education gained national attention in 1983 with the federal government issuance of *A Nation at Risk* (ANAR) (National Commission on Excellence in Education (NCEE), 1983). This report situated blame on education for a perceived diminishing economic vitality in the Unites States. The central concern of the report was that the nation was falling behind other countries in student achievement which led to a decline in the production of automobiles, steel, and tools. Written by a "blue ribbon" commission appointed by the Secretary of Education, T. H. Bell, the report expressed concerns about improving the quality of schools across the nation. It contained general discussions of the "rising tide of mediocrity" and our threatened future if higher standards were not imposed (p. 5). The report contained a great deal

of inflammatory language that stressed the necessity of raising achievement test scores and increasing the amount of homework required of students. It also contained an interpretation of achievement test scores that has since been repudiated by educational researchers (Berliner & Biddle, 1997).

Bell expressed a personal hope that the commission would emphasize the mastery of mathematics along with other subject areas. The central concern of the final report, however, was not mathematics. The report itself was a brief pamphlet that contained vague recommendations but no suggestions for federal funding to help in carrying out the recommendations contained. It was claimed that the NCEE represented a cross section of the American public, and that the concerns expressed were shared by many. Power structure and content analyses, however, demonstrate that the commission members did not represent the majority of Americans and the recommended reforms would serve specific interests and not others (Hadden, 2003). Despite rhetoric to the contrary, it would appear that the objectives of ANAR were not necessarily in line with those of mathematics teachers or classroom teachers in general, parents, or students. The report, which largely represented the concerns of a section of the American power structure that was highly and intricately connected to the federal government, was likely intended to serve other interests while creating a climate of fear about education.

Establishing the climate of fear was one of the successes to which ANAR may lay claim. That report was quickly followed by hundreds of similar reports nationwide issued by corporations, foundations and think tanks, and school districts. All were concerned with issues of technology and achievement in mathematics and science. They seemed to have little effect, however, on mathematics instruction and learning. Indeed, the ultimate results of ANAR and the other reports did not create significant change in mathematics education. Real results, accompanied by additional legislation, did not even become visible until much later when the No Child Left Behind Act (NCLB) was signed into law in 2002. Between ANAR and NCLB, however, other legislation concerning the state of mathematics education appeared. *America 2000 Excellence in Education Act* (1991) and *Goals 2000: Educate America Act* (1993) established more plans to increase student achievement. Evidence for the failure of reform initiatives, however, continued to mount as demonstrated by the recent math wars. Thus if reforms are to be successful, the culture of both education and our society must be changed.

NCTM Standards

The National Council of Teachers of Mathematics has spearheaded the effort to focus attention on a standards-based curriculum. The most pertinent publications manifesting this attention have been the recent NCTM documents identified above. Support for the NCTM *Standards* came not only from the scientific and professional communities, but also from a diverse group of community organizations.

In considering why a reform is desirable one looks at the needs and nature of mathematics, the needs and values of society, and the relationship between them. In considering who brings a reform about, one discovers that a balanced participation of mathematicians, teachers, politicians, parents, and students is necessary and that all must somehow work together and come to understand their different motivations and ways of thinking. Perhaps this explains why the *Standards* documents have been more generally discussed than previous reform efforts. Whereas, by its very nature, the "new math" was a product of the world of mathematicians, the reform efforts sponsored by the NCTM have had a broad-based buy-in: "The content and processes

emphasized in *Principles and Standards* also reflects society's needs for mathematical literacy, past practice in mathematics education, and the values and expectations held by teachers and the general public" (NCTM, 2000, p. xii).

The *Principles and Standards* document begins by outlining a vision which, from a philosophical standpoint, is clearly fallibilist in nature–constructivist learning where the mathematics is substantive and the learning is student-oriented. Not only did that goal fail to materialize, but concerns about the state of mathematics and science education have become even more vocal in recent years. In the Glenn (2001) report, *Before It's too Late*, we are told that we have only a small window before the intellectual damage to our youth cannot be reversed. Other reports such as *Adding It Up* (National Research Council, 2001) also call for an overhaul of instruction, curricula, and testing for elementary and middle school students in the United States.

Nowhere is this problem with mathematics education made clearer than in the results of TIMSS. This study has led to hand-wringing, finger-pointing, and considerable discussion about what can be done to reverse this trend. One potentially rich explanation emerged from the study of the videotaped lessons at the eighth grade level. These results are described in Stigler et al. (1999). Looking specifically at the United States and Japan, we note that their educational contexts are based on very different cultural traditions. Any attempt to improve mathematics teaching in the United States without changing its culture will prove fruitless. The work of Stigler and Hiebert, along with that of Fernandez and Yoshida (2004), has led to considerable interest in Lesson Study in the United States. (An indicator of this interest is the number of workshops conducted at recent meetings of the National Council of Teachers of Mathematics.) Ultimately, the issue is whether a program designed to improve mathematics instruction in Japan can be transplanted to the United States and prove successful. We can transpose that question and ask: Is it possible to change the prevailing educational philosophy in the United States?

Stigler and Hiebert (1999) believe it is possible by working within the existing educational system. These researchers caution, however, that even in the best of circumstances, change will be gradual and incremental. The stage must be set, they believe, by first building a consensus for continuous improvement; second, set clear learning goals for students and align assessments with these goals; and, third, restructure schools as places where teachers can learn (pp. 138-142). It must be emphasized, at this point in the discussion, that Stigler and Hiebert recommend professionalizing teachers and the occupation of teaching and building infrastructure for support of professionalization. These researchers see Lesson Study and professionalization as answers to achievement problems.

Implications of Math Curriculum Reform

One stated purpose of the NCLB legislation was to provide every child (including the disadvantaged) with the opportunity to succeed in mathematics. In fact, the law indicates that there will be consequences for schools if all students do not succeed. We have observed that students are tested annually, that failure to perform adequately on these tests has serious consequences, and that because of these consequences, many teachers are "teaching to the test" and ignoring other educational goals. Considering the influence of the NCLB legislation on the teaching and learning of mathematics in the United States, and given the nature of these reports

and this legislation, it is pertinent to ask whether the conflicting philosophies of mathematics education implied by the absolutist/fallibilist dichotomy are rigid or somewhat pliant.

When Stigler and Hiebert (1999) advocate the use of Lesson Study in United States schools, they are well aware that its implementation cannot be successful unless it is accompanied by ideological and cultural change within schools. We cannot expect this to happen in the short-term. It will take sustained, deliberate, long-term efforts resulting in small incremental changes. It remains to be seen whether this can happen in the USA, or whether the culture is so ingrained that change will not readily occur. If this is the case, what *will* it take to bring about real change in mathematics education in the United States? In the midst of this philosophical debate, it is pertinent to think about the different sides in the discussion and whether we are dealing with "essentially contested concepts" that are so ideologically laden that they do not allow for resolution (Gallie, 1955-1956).

Our concern is that one side's "eternal truths" will be viewed by the other side as "stagnant dogma". Unless the supporters of both perspectives attempt the development of some common ground, the future of mathematics education may lead only to the recyling of time-worn arguments. If this is the case, elementary and secondary students may continue to ask why this or that skill or concept must be learned. The models employed will continue to reduce mathematics to a set of measurable objectives, narrowly defined, tested and then discarded because they are applicable only in rare circumstances.

We are uneasy about the possibility that mathematics may not be seen as a fruitful avenue for solution to real social problems. Perhaps some of our students may continue to believe that as long as one can balance the checkbook and employ third grade addition and subtraction skills, there is no reason to learn algebra. The models employed will remain constraining and confining. They will limit the vision of what is possible. If mathematics is taught as a body of knowledge to be memorized and regurgitated, it may lose its dynamic character and become a set of painful mind games. If educators continue to plan and teach lessons based on the absolutist philosophical position without taking account of history, context, and culture, some students will continue to see mathematics as stagnant rather than dynamic–something to be gotten through. The end becomes the goal, rather than the goal being a joyful journey of learning. If the fallibilist continues to espouse a position that amelioration between philosophical positions is not possible, then perhaps it *will not be* possible.

For all students to see the intrinsic joyfulness and usefulness of mathematics, the subject areas must be integrated with the physical, biological, and social sciences. We must teach our students that the mathematics are and always have been tied up with power and social relations; and, we should encourage our students to study those relations. Students must be allowed the freedom to shape curriculum in ways that are meaningful to them. Studying statistics, for instance, without the vigor of interpretation may cause the field to become theoretically bankrupt. The question of causes and alternative perspectives will enrich that study. Integrated with the study of history and economics, the study of statistics becomes a way to reveal underlying power relations.

Teachers, as well, should be encouraged to develop professionally through philosophical discourse with their peers, to plan and teach together, and engage in peer evaluation that is truly collaborative and non-threatening. It is our concern that mathematics may not be seen by students or teachers as a spirited discipline with a vivacious past and robust future unless and

until it can be seen as interdependent and integrated with other disciplines. On the other hand, there are some concepts that must be learned, that appear to us now as eternal and unchanging. Those must be taught, as well, with the knowledge that at this time and in this place this concept is the one that informs our thinking on this matter. The point is that those "eternal" concepts are readily taught and learned. The real work comes in constructing learning that reflects changing conceptions and the social circumstances in which they occur.

Our society is undergoing continual transformation. The education system, however, tends to be in a reactive mode. The community at large has something of an industrial age expectation of mathematics curricula as evidenced by the absolutist perspective and its emphasis on procedural competence. We seek, instead, an information age perspective based on a combination of absolutist and fallibilist conceptions. We are looking for reforms in mathematics education that reflect the changing nature of American society, but incorporate a vision of cultural change.

References

Amit, M., & Fried, M. N. (2002). Research, reform and times of change. In L. D. English (ed.) *Handbook of international research in mathematics education* (pp. 355–381). Mahweh, NJ: Lawrence Erlbaum.

Berliner, D. C. & Biddle, B. J. (1997). *The manufactured crisis: Myths, fraud, and the attack on America's public schools.* White Plains, NJ: BasicBooks.

Conference Board of the Mathematical Sciences, National Advisory Council on Mathematical Learning (1975). *Overview and analysis of school mathematics, grades K-12*. Washington, D. C.: author.

Cremin, L. (1961). *The transformation of the school: Progressivism in American education, 1876-1957*. New York, NY: Vintage.

Cuban, L. (1993). *How teachers taught: Constancy and change in American classrooms, 1880-1990.* New York, NY: Teachers College Record.

Dewey, J. (1936a). The Dewey school. In J. A. Boydston (ed.) *John Dewey: The later works, 1925-1953*, (pp. 202-216). Carbondale, IL: Southern Illinois University Press.

Dewey, J. (1936b). Rationality in education. In J. A. Boydston (ed.) *John Dewey: The later works, 1925-1953*, (pp. 391-396). Carbondale, IL: Southern Illinois University Press.

Ernest, P. (1991). *The philosophy of mathematics education*. Bristol, PA: Falmer.

Ernest, P. (1994). Introduction. *Mathematics, education, and philosophy: An international perspective*. (pp. 1-8). P. Ernest (ed.) London: Falmer.

Felix, L. (1961). *The modern aspect of mathematics.* New York, NY: Basic Books.

Fernandez, C., & Yoshida, M. (2004). Lesson study: A Japanese approach to improving mathematics teaching and learning. Mahweh, NJ: Lawrence Erlbaum.

Gallie, W. B. (1955-1956). Essentially contested concepts. *Proceedings of the Aristotelian Society, 56*. pp. 167-198.

Glenn, J. (2001). *Before it's too late.* Washington, D. C: National Commission on Mathematics and Science Teaching for the 21st Century.

Griffiths, H. B., & Howson, A. G. (1974). *Mathematics: Society and curricula.* London: Cambridge University Press.

Hadden, J. (2003*). The power structure behind educational policy: The 1983 reports*. Unpublished Ph. D. dissertation, University of Utah.

Hechinger, J. (2006, Sept 12). Reports and challenges in the public media. *Wall Street Journal* (Eastern Edition), p. A1.

Hoff, D. J. (2000, April 12). Math revisions add emphasis on basic skills. *Education Week, 19*(31), pp. 1, 19.

Husen, T. (ed.). (1967). *International study of achievement in mathematics*. New York, NY: Wiley.

Kliebard, H. (1987). *The struggle for the American curriculum: 1893-1958*. New York, NY: Routledge.

Kline, M. (1958). The ancients versus the moderns, a new battle of the books. *Mathematics Teacher, 51*, 418–427.

Lewin, T. (2006, November 14). As math scores lag, a new push for the basics. *New York Times*. Retrieved from

www.nytimes.com/2006/11/14/education/14math.html

National Advisory Committee on Mathematical Education (1975). Conference Board of the Mathematical Sciences.

National Commission on Excellence in Education (1983). *A nation at risk: The imperative for educational reform*. Washington, D. C.: US Department of Education.

National Council of Teachers of Mathematics (1980). *An agenda for action*. Reston, VA: NCTM.

National Council of Teachers of Mathematics (1989). *Curriculum and evaluation standards for school mathematics.* Reston, VA: NCTM.

National Council of Teachers of Mathematics (1991). *Professional standards for teaching mathematics.* Reston, VA: NCTM.

National Council of Teachers of Mathematics (1995). *Assessment standards for school mathematics.* Reston, VA: NCTM.

National Council of Teachers of Mathematics (2000). *Principles and standards for school mathematics.* Reston, VA: NCTM.

National Council of Teachers of Mathematics (2006a). *Curriculum focal points for prekindergarten through grade 8 mathematics.* Reston, VA: NCTM.

National Council of Teachers of Mathematics (2006b). NCTM releases *Curriculum Focal Points* to focus math curricula. Reston, VA: NCTM. Retrieved 9/27/07 from

www.nctm.org.standards/focalpoints.aspx?id=7760

National Council of Teachers of Mathematics (2006c). All eyes are on NCTM's Curriculum Focal Points. *NCTM News Bulletin*, 43(4), 1, 6.

National Research Council. (2001). *Adding it up: Helping children learn mathematics*. J. Kilpatrick, J. Swafford, & B. Findell (eds.). Washington, DC: National Academy Press.

Stigler, J. W., Gonzales, P., Kawanaka, T., Kroll, S., & Serrano, A. (1999*). The TIMSS videotape classroom study: Methods and findings from an exploratory research project on eighth-grade mathematics instruction in Germany, Japan, and the United States*. U.S. Department of Education, Washington, DC: National Center for Educational Statistics.

Stigler, J. W. & Hiebert, J. (1999). *The teaching gap: Best ideas from the world's teachers for improving education in the classroom*. New York: NY: The Free Press.

Tyack, D. (1974*). The one best system: A history of American urban education.* Cambridge: Harvard University Press. United States Department of Education (1991). *America 2000: An education strategy*. Washington, D. C.: author.

United States Department of Education (1993). *Goals 2000: Reforming education to improve student achievement.* Washington, D. C.: author.

Wallis, C. (2006, November 19). How to end the math wars. *Time*. Retrieved from
www.time.com/time/magazine/article/0,9171,1561144,00.html

Teaching Proof *at KS4*

Sue Waring (UK)

Proof is a fundamental component of mathematics and so, in my opinion, it should be part of mathematical education in schools. It is an important link between topics and so can help pupils achieve a deeper understanding of the "wholeness" of mathematics and, moreover, it is vital for able pupils if they are to be inspired by mathematics and given a firm basis for future mathematical studies.

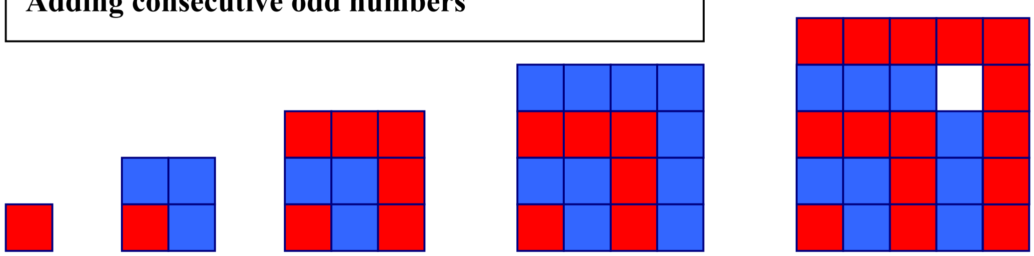
Ideally, proof in primary schools would take the form of explanations of (mainly) number properties and patterns and the language used could be diagrams, or even pictures. Even in secondary schools much proof would be informal but older, able pupils should be exposed to formal proofs, probably including some from Euclid, and also be made aware of different proof methods. Because this ideal has not always been achieved I have found it necessary to introduce some older pupils to ideas about proof during the latter part of their schooling.

I have provided proofs, mainly deductive, for nearly all the traditional mathematics they have learned and I have also explained why they cannot yet prove the few exceptions, like the formula for the volume of a sphere (*they do not study calculus*). To cater for the needs of pupils who prefer a visual analysis I have used proofs based only on diagrams as proposed by Skemp (1971, p 99), alongside the conventional verbal-algebraic forms, of some geometric theorems. Diagrams have also been used to explain some numerical and algebraic relationships and some of these are shown below.

Diagrams as proof

Pupils who fail to recognise a sequence of square numbers frequently resort to "adding the next odd number" and the first diagram explains why this works.

Adding consecutive odd numbers



1   1 + 3 = 5   1 + 3 + 5 = 9   1 + 3 + 5 + 7 = 16   1 + 3 + 5 + 7 + 9 = 25

Many pupils learn algebraic identities by rote and often use them with little real understanding and so the next diagram can help to increase insight.

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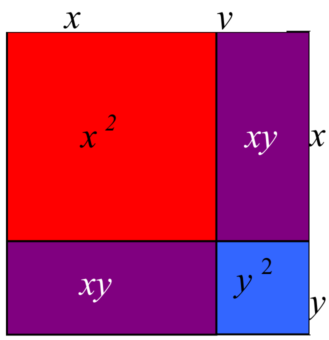


$(x + y)^2 = x^2 + 2xy + y^2$

Proof and pattern

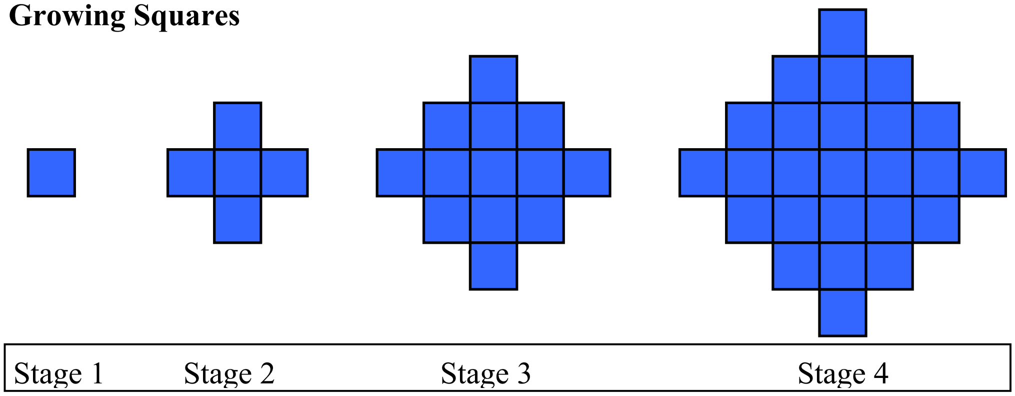
Although traditional proofs and "proofs without words" can deepen understanding of the mathematical concepts they explain they do not necessarily enhance appreciation of the need for and the nature of proof. Even some able pupils do not have an intuitive grasp of this. Using the context of pattern separates proof from the constraints of formal mathematics, such as symbolic language and unknown varying lengths and angles, and so makes it more accessible to more pupils. Patterns are either visual or composed of discrete quantities, which can be represented by concrete apparatus or diagrammatically, and explaining them does not necessarily involve formal mathematical language.

Pupils can often analyse a pattern and work out the next three terms and most able pupils can also work out a general rule. In order to do this they assume that the numerical relationships observed in the first few terms continue. However, even amongst able pupils, there are many who are taken by surprise when asked to explain why the rule works. To alert pupils to the fact that not all patterns continue as expected and to emphasise that the purpose of proof is to provide confidence in the truth of a claim it is wise to use at least one example where an apparent pattern does not continue.

The investigation "Regions in a Circle", described in my book "Can you prove it?" (Waring, *published by the Mathematical Association*, 2000, p157), uses a counter-example to demonstrate this. Pupils count the number of regions (*r*) formed by joining with straight lines all possible pairs of dots around the circumference of a circle. Most pupils describe the pattern observed in the first five circles as "doubling" and some can also express it algebraically as $r = 2^{d-1}$. Pupils expect that there will be 32 regions for six dots and are surprised that 31 is correct and that the apparent pattern fails. Pupils are not expected to find the correct complex pattern, but should be aware that one exists and offered copies of a proof (e.g. Beevers, 1994, p10) to read

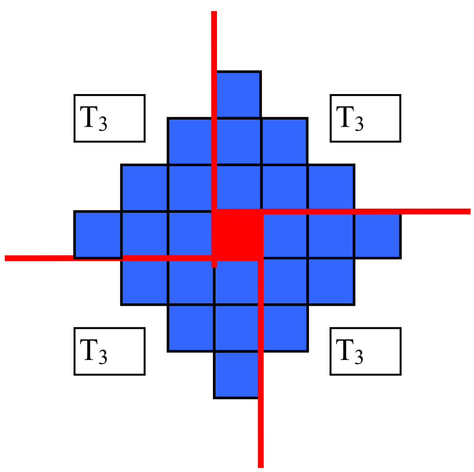
There are many other examples of patterns to explain in my book and they are accompanied by details of how they have been used in the classroom. One example (part of a pupil worksheet), which has the advantage of having several alternative proofs, is given below.

Growing Squares

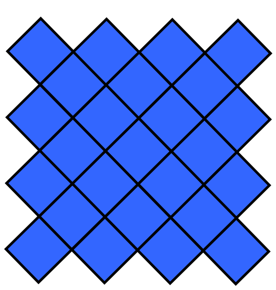


Find a formula for the number of squares (s) at the nth stage.
Show how you derived it and explain why it works.

This task was set for homework after a lesson on a similar growing pattern of triangles. This lesson established that for t triangles the result of dividing $(t - 1)$ by 3 produced a triangle number. Although pupils were encouraged to tabulate numerical values it was recognised that some pupils would analyse the structure of the pattern.
For the homework assignment one pupil produced the following diagrammatic proof:

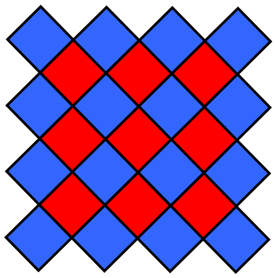


Two pupils produced the following diagram and formula

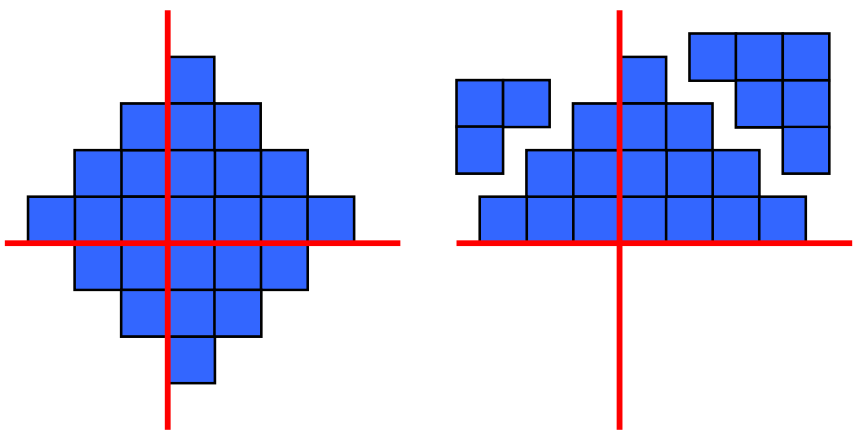


$$s = n^2 + (n - 1)^2$$

but did not explain the link. Neither they nor their class mates could see the connection when shown the above diagram in the next lesson, until the diagram below was shown to the class.



When this was done the pupils who had produced the original diagram said that it explained what they meant. Most of the pupils in this able group understood that this was a specific example of a result which could be generalised since they could also follow the algebraic proof referred to in the introductory lesson. On another occasion a pupil produced the following diagrammatic analysis of the pattern as the sum of consecutive square numbers.



Exposure to activities of this kind shows pupils that patterns can be explained, and sometimes in more than one way. Exhortation by the teacher that patterns, like all mathematics, should be explained and provision of regular practice helps pupils learn to construct their own, albeit informal, proofs.

Proof methods

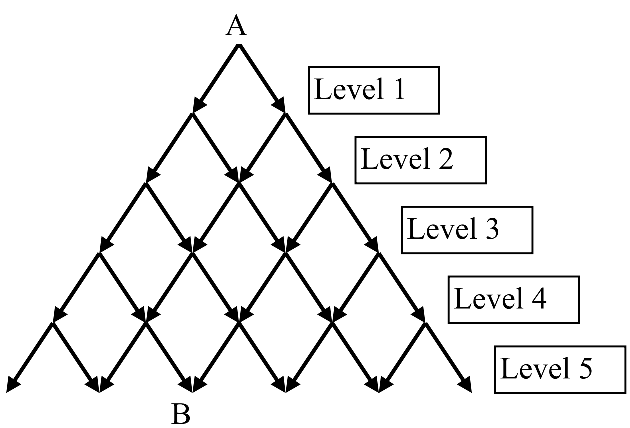
When pupils have been introduced to such proofs of patterns and also some formal traditional proofs it is appropriate to highlight the existence of different proof methods. Another series of lessons hoped to achieve this by considering apparently different problems and using different proof methods to establish confidence in the findings. The mathematics underlying all the problems, namely "Combinations", has not been studied by the pupils and so they do not have access to rote learned responses.

The first problem – "In how many ways can two colours for a team strip be selected from a range of five colours" – is appropriate for younger children but is used here as an easy

introduction to the main task, and to highlight the method of "proof by exhaustion". Although younger pupils need guidance in how to list possible outcomes systematically older pupils can do this quickly and thus are sure that there are ten choices. However, the idea that they have used "proof by exhaustion" and the fact that it is appropriate because there are few cases to consider are new.

The second problem – "How many different triangles can be formed by joining three dots in a set of five dots (no three collinear)" – is included because it is mathematically similar, although this is not recognised by pupils. Younger pupils could do this with rubber bands on a pin board but older pupils use pencil and paper. They are expected to understand the link with the previous problem, that choosing three out of five is equivalent to choosing two, and the fact that proof by exhaustion is still feasible.

The third problem – "In how many different ways can you get to each junction of this grid"-



is also based on "Combinations" but, because the context is different, this is not recognised initially by pupils. Even if they do not realise selecting left or right moves from the total number of levels moved this can be made clear in the class discussion about the problem. This discussion highlights the fact that the number of ways of getting from A to B is the same as in the previous two problems because it is equivalent to selecting two right (or left) and three left (or right) moves from five moves. The method of proof used is still "exhaustion" but the question of finding and proving results for a larger grid with, say, twenty levels is raised and the need for a more efficient approach appreciated.

The remaining class discussion returns to selecting colours and elicits the facts that there are five (or n) ways of choosing the first colour and 4 (or $n-1$) ways of choosing the second. Consequently there are 5x4 (or $n(n-1)$) ways of selecting a first and second colour and therefore 20/2 (or $\frac{1}{2}n(n-1)$) ways of selecting two colours in either order. Pupils can then be reminded that this is a deductive proof and is more powerful than the proofs by exhaustion because the result can be applied with confidence to the problem of selecting two objects from any number of objects, however large.

The last problem in the series considers the problem of expanding binomials with powers up to five. Introductory class discussion establishes that $(a + b)^0 = 1$ and that the coefficients of a and b in $(a + b)^1$ are both 1 and then reminds pupils of the expansion for $(a + b)^2$. Some pupils need reminding that $(a + b)^3$ can be found by multiplying $(a + b)^2$ by $(a + b)$ and some also need help in starting this. They are then given time to expand $(a + b)^4$ and $(a + b)^5$ and told to examine the coefficients. The summarising discussion establishes that the coefficients are the same numbers as those at Level 5 of the grid above, explains why this is the case and considers the proof method used.

It has also been drawn to the attention of pupils that the pattern in the grid and also formed by the coefficients of binomial expansion form Pascal's Triangle. Pupils are interested in this and can usually see how it can be extended but combinations of more than three from more than five are not discussed at this stage.

To broaden the experience of pupils in mathematics and to introduce a new method of proof, Euler's theorem has been established and its proof discussed with classes of older, able pupils. This involves pupils thinking in three dimensions, provides valuable experience in a previously unfamiliar area of mathematics and a different style of reasoning, and also an unusual example of how changing a problem facilitates its solution. Pupils are given access to a variety of solids and told to count faces (F), vertices (V) and edges (E) and tabulate these in groups – prisms, pyramids and "others". Some pupils are able to find a correct relationship between F, V and E and most recognise that one form is $F + V – E = 2$.

Many able pupils can explain why this is true for a prism whose cross-section is a polygon with n sides – there are $n + 2$ faces (n along the length and one at each end); $2n$ vertices (n at each end) and $3n$ edges (n at each end and a set of n along the length); combining these as $n + 2 + 2n – 3n$ gives the required result of 2. Some able pupils can also analyse pyramids in a similar way to give $(n + 1) + (n + 1) – 2n = 2$ and most can understand this proof. The fact that that both these are deductive proofs is highlighted. Pupils enjoy handling the many solids and meeting new words like "parallelepiped", "trapezoidal prism" and "icosahedron" and so have a positive attitude to this activity.

The proof for other solids cannot be proved in the same way, by deduction. The method used explains how any solid can be transformed into a network, by "squashing" it so that faces, vertices and edges become regions (R), nodes (N) and arcs (A) and then derives the proof that $R + N – A = 2$, through class discussion. The proof is by induction and establishes that $R + N – A = 2$ for the simplest case (two nodes joined by an arc); that $R + N – A$ remains constant if an arc or node is added; and that therefore $R + N – A$ must always be two.

In a group of pupils aged 14-15 years and between the 10^{th} and 20^{th} percentiles of mathematical ability numerical results were collated and the relationship $F + V – F = 2$ discovered and also the need for proof recognised. Pupils took an active part in the class discussions about proofs and were given a printed summary. On a later occasion older (aged 15-16 years), very able (at or above 10^{th} percentile) pupils were instructed to investigate polyhedra and networks simultaneously and search for and prove any connections. Their reports on the first stage of the investigation included proofs of the theorem for prisms and pyramids with little or no teacher intervention. After the discussion of the inductive proof for other solids their written

explanations indicated that they understood the proof of the relationship for networks and the equivalence of the relationship for solids.

Conclusion

Although much of the material outlined above has not been specified in the curriculum these wider and deeper mathematical experiences seem to interest and motivate able pupils so that many of them elect to study mathematics further. I have every confidence that their exposure to proof has given them a firm foundation for more advanced mathematics.

References

Beevers, B. (1994) "Patterns which aren't" *Mathematics in School*, **23**, (5) 10-12
Skemp, R.R. (1971) "*The Psychology of Learning Mathematics*" Pelican
Waring S. (2000) "*Can you prove it? Developing concepts of proof in primary and secondary schools*" The Mathematical Association

Consecutive numbers

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The hidden secrets of our number system can often reveal the magical quality of mathematics. Through the process of discovery and discussion with fellow classmates, the hidden depths of maths takes on new appeal. Consecutive numbers is one such area that gives this excitement.

Starting with the open question:

How many ways can you share out 6 counters?

The children will find cases like these:

1+5, 2+4, 3+3, 1+1+2+2, 1+2+3

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What about 7 counters and other numbers?
Are there any special cases from all these?

Then notice that 6=1+2+3 and 7=3+4 can be written as the sum of consecutive numbers.

Why can 7 be made from two consecutive numbers and 6 from three? Are there patterns like this in other numbers? Can all numbers be written as the sum of consecutive numbers?

After further investigation a proof can be found for all odd numbers.

$$n + (n + 1) = 2n + 1$$

It can be seen that all multiples of 3 can be written as three consecutive numbers from the following proof:

$$(n - 1) + n + (n + 1) = 3n$$

After this the students see the real benefit of using algebra and move to four consecutive numbers.

$$(n - 1) + n + (n + 1) + (n + 2) = 4n + 2 = 2(2n + 1)$$

So if you double any odd number these can be written as four consecutive numbers.

Therefore since we have just proved that we can write all the odd numbers as consecutive numbers, if we double any odd number we can also write this as a consecutive number. Is that therefore a proof that all numbers can be written as the sum of consecutive numbers?

Odds	1	3	5	7	9	11	13	15	17…
Double2	6	10	14	18	22	26	30	34…	

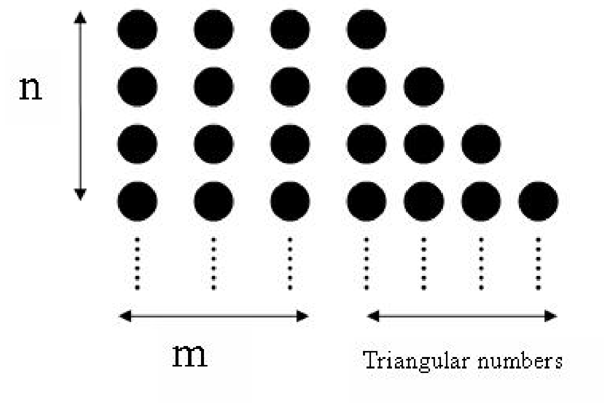
You can see from the previous table that gaps still exist in our logic, for example 4, 8, 12, 16, 20, 24….

We can find more proofs to help solve some of these problems, but 4, 8, 16, 32… still remain. By this time the student may begin to wonder if continuing to prove particular cases will ever prove to be enough!

We need to move to the next level and the crowning glory of this problem. By looking at why 4,8,16,32… cannot be partitioned into the sum of consecutive numbers, we begin to understand the link to triangular numbers and other deeper issues.

Theorem: All consecutive numbers have at least one odd factor.

Proof:



Sum = $nm + \frac{1}{2}n(n+1)$

Sum = $\frac{n}{2}(2m+n+1)$

Either i) n is even, $\therefore$ $2m+n+1$ is odd

ii) $(2m+n+1)$ is even, $\therefore$ n is odd

This problem is an excellent way to motivate thinking about proof and why proof is necessary. It gives students confidence in the use of algebra and the ability to find particular results which can be shown to always be true. Yet its real magic is in this final proof, which shows the need to stand back and look. If we are not careful, sometimes we can lose ourselves in the detail and not see the whole picture.

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