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Abstract
Large context problems (LCP) are useful in teaching the history of science. In this article we consider the brachistochrone problem in a context stretching from Euclid through the Bernoullis. We highlight a variety of results understandable by students without a background in analytic geometry. By a judicious choice of methods and themes, large parts of the history of calculus can be made accessible to students in Humanities or Education.

Keywords: Brachistrochrone problem; Large context problems (LCP); history of science; history of mathematics

Introduction

Each year the University of Winnipeg offers several sections of an undergraduate mathematics course entitled MATH-32.2901/3 *History of Calculus* (Babb, 2005). This course examines the main ideas of calculus and surveys the historical development of these ideas and related concepts from ancient to modern times. Students of Mathematics or Physics may take the course for Humanities credit; the course surveys a significant portion of the history of ideas and in fact is cross-listed with Philosophy. On the other hand, many students in Education take *History of Calculus (H of C)* to fulfill their Mathematics requirement; it is therefore necessary that the course offer solid mathematical content. Unfortunately, a significant fraction of these latter students are weak in pre-calculus material such as analytic geometry. Nevertheless, H of C welcomes the weak and the strong students together, and covers technical as well as historical themes.

In Stinner and Williams (1998) the authors enumerate the benefits of studying **large context problems (LCP)** in making science interesting and accessible. In their words 'the LCP approach provides a vehicle for traversing what Whitehead (1967) refers to as "the path from romance to precision to generalization"(p. 19).' For H of C, a useful LCP is the **Brachistochrone Problem:** the

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solution history of finding the curve of quickest descent. A focus on the brachistochrone motivates results ranging from Greek geometry, past the kinematics of Oresme and Galileo, through Fermat and Roberval to the Bernoullis, and the birth of the calculus of variations. By careful selection of material, it is possible to find proofs accessible even to weak students, while still stimulating mathematically strong students with new content.

Quickest Descent in Galileo

One of the topics considered by Galileo Galilei in his 1638 masterpiece, *Dialogues Concerning Two New Sciences,* is rates of descent along certain curves. In Proposition V of "Naturally Accelerated Motion", he proved that descent time of a body on an inclined plane is proportional to the length of the plane, and inversely proportional to the square root of its height (Galilei 1638/1952, p.212). Denoting height by H, length by L and time by T, we would write

$$T = kL/\sqrt{H} \qquad (1)$$

where k is a constant of proportionality. In fact, as we point out to students, this formulation is slightly foreign to the thought of Galileo; for reasons of homogeneity, he only forms ratios of time to time, length to length, etc. He therefore says that the ratio of times (a dimensionless quantity) is proportional to the ratio of lengths, inversely proportional to the ratio of square roots of heights. Galileo proves (1) in a series of propositions starting with the "mean speed rule":

> 'The time in which any space is traversed by a body starting from rest and uniformly accelerated is equal to the time in which that same space would be traversed by the same body moving at a uniform speed whose value is the mean of the highest speed and the speed just before acceleration began.' (Proposition I, Galilei 1638/1952, p.205).

It is insufficiently well-known that this rule had been proven geometrically by Nicole Oresme three centuries earlier! Oresme's very accessible geometric derivation is presented early in H of C. (Babb, 2005)

Moving closer to the question of quickest descent, with his Proposition VI, Galileo established the law of chords:

> 'If from the highest or lowest point in a vertical circle there be drawn any inclined planes meeting the circumference, the times of descent along these chords are each equal to the other'. (Galilei 1638/1952, p.212)

Galileo's proofs are given in a series of geometric propositions. Unfortunately, many of our students would identify with the complaint that Galileo places in the mouth of Simplicio:

> 'Your demonstration proceeds too rapidly and, it seems to me, you keep on assuming that all of Euclid's theorems are as familiar and available to me as his first axioms, which is far from true.' (Galilei 1638/1952, p. 239)

Happily, in an earlier part of H of C dealing with Greek mathematics, some geometrical rudiments are established. In particular, students see a proof of Thales' theorem – an angle inscribed in a semicircle is a right angle – using similar triangles. This allows the following demonstration of the law of chords:

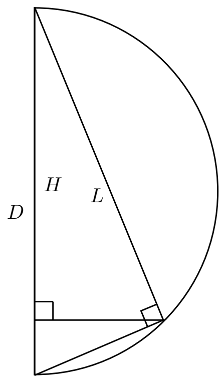

Figure 1: The law of chords

Let a circle of diameter D have a chord of length L and height H inscribed as shown in Figure 1. Let the descent time along the chord be T. By similar triangles L/H = D/L, so that L^2 /H = D. Then by (1),

$$T = kL/\sqrt{H} = k\sqrt{D} \qquad (2)$$

which is indeed constant. Q.E.D.

The law of chords can also be demonstrated by deriving expressions for the velocities and descent time, and noting that the expression for descent time is independent of the upper point of the chord along the circular arc. The expression for descent time may be obtained by considering the component of the force of gravity along the inclined chord. Alternatively, Nahin (2004, pp. 202-206) derives the descent time by applying the principle of conservation of mechanical energy. Our brief geometrical derivation is rather close to the spirit of Galileo's proof, and is accessible to students without a physics background.

Lattery discusses an interesting approach suggested by Matthews in 1994 for leading students towards a derivation of the[1] law of chords. Students are asked to consider the following thought experiment:

> 'Suppose a ball is released at some point A on the perimeter of a vertical circle and rolls down a ramp to point B, the lowest point on the circle … The ramp may be rotated about point B. For what angle will the time of descent along chord AB be the least? '
> (Lattery, 2001, p. 485)

This way of posing the problem helps students to greater appreciate the surprising result. It also allows the option of discussing the distinction between sliding and rolling motion. H of C is a mathematics course and has no laboratory component. Nevertheless, (particularly for students with weak physical intuition) it is useful to be able to observe the results studied in various descent problems. As experimentalists know, however, the design and operation of physics demonstration apparatus can be as much art as science. For this reason, we have chosen to use physically realistic computer simulations to illustrate various theorems. A computer program in MAPLE allows freedom in the choice of curves studied, as well as the possibility of speeding, slowing or freezing demonstrations. Figure 2 shows two screen snapshots from the program. Portability is one more advantage of this approach, to go along with flexibility of use.

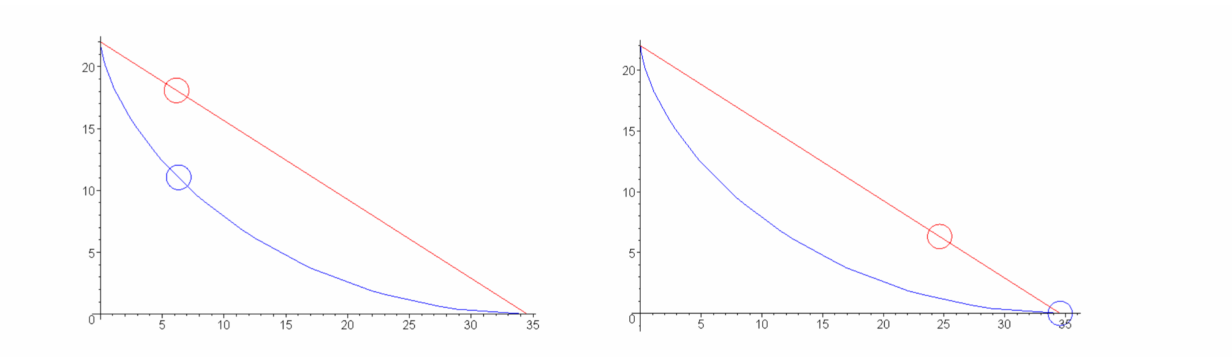

Figure 2: Screen snapshots from *MAPLE* race: cycloid vs. straight-line ramp

Details of the simulation are closely related to Jakob Bernoulli's solution of the brachistochrone problem, and are detailed in a later section.

With his Proposition XXXVI, Galileo proved that the descent time from a point on the lower quadrant of a circle to the bottom is quicker along two consecutive chords than along a direct chord. He began his proof by a clever application of the law of chords, but then completed it by a fairly involved geometric argument. In H of C, the proof is completed using a shorter method proposed by Erlichson (1998), based on conservation of mechanical energy.

Three Curves

Three curves of major interest to the mathematicians of the seventeenth century were the cycloid, the isochrone and the brachistochrone. (See, for example, Eves, 1990, p. 426.) The definitions of these curves are *kinematic*; as students learn in H of C, the acceptance of curves defined via motion was part of a mathematical revolution in the seventeenth century. A *cycloid* is the curve traced by a point on the circumference of a circle as the circle rolls, without slipping, along a straight line. A *brachistochrone* from point A to point B is a curve along which a free-sliding particle will descend more quickly than on any other AB-curve. (It is thus an optimal shape for components of a slide or roller coaster, as we inform our students.) An *isochrone* is a curve along which a particle always has the same descent time, regardless of its starting point. A surprising discovery was that these three curves are one and the same!

Galileo may have been the first to consider the problem of finding the path of quickest descent. This is suggested by the initial statement of his Scholium to Proposition XXXVI:

> 'From the preceding it is possible to infer that the path of quickest descent from one point to another is not the shortest path, namely, a straight line, but the arc of a circle'.
> (Galilei 1638/1952, p.234)

Many researchers, such as Stillman Drake and Herman Goldstine, have concluded that Galileo incorrectly claimed that a circular arc is the general curve of quickest descent (Erlichson 1998, Erlichson 1999). However, Erlichson (1998, p.344), argues that Galileo restricted himself to descent paths that used points along a circle. Galileo's claim is based on an argument that the descent time for a particle along a twice-broken path is less than along a twice-broken path, and that the descent time along a multiply-broken path would be even less. According to Nahin (2004, pp. 208-209), Galileo's claim is correct, but his reasoning was flawed; Erlichson (1998) noted that Galileo's method of proving Proposition XXXVI holds for descent from rest, but fails to generalize to situations in which a particle is initially moving.

In fact, as we have mentioned, the brachistochrone is not the circle; it is a segment of an inverted cycloid. The cycloid was discovered in the early sixteenth century by the mathematician Charles Bouvelles (Cooke, 1997, p. 331). In the 1590s, Galileo conjectured and empirically demonstrated that the area under one arch of the cycloid is approximately three times the area of the generating circle. That the area is exactly three times that of the generating circle was proven by Roberval in 1634 and by Torricelli in 1644 (Boyer and Merzbach, 1991, p. 356). Roberval constructed the tangent to the cycloid in 1634 (Struik, 1969, p. 232). According to Cooke (1997, p. 331), constructions of the tangent to the cycloid were independently discovered circa 1638 by Descartes, Fermat and, and slightly later by Torricelli. In 1659, Christopher Wren also determined the length of a cycloidal arch, showing it to be exactly four times the diameter of the generating circle (Stillwell, 2002, p. 318). In 1659, Huygens discovered that the cycloid is a solution to the isochrone or tautochrone problem; he showed that a particle sliding on a cycloid will exhibit simple harmonic motion with period independent of the starting point (Stillwell, 2002, p. 238). Huygens published his discovery of the cycloidal pendulum in *Horologium oscillatorium* in 1673 (Boyer and Merzbach, 1991, p. 379). The cycloidal pendulum also features in Newton's Principia (Gauld, 2005). In 1696, Johann Bernoulli demonstrated that a brachistochrone is a cycloid (Erlichson, 1999).

Most of these discoveries concerning the cycloid are inaccessible to students with no calculus background. Remarkably, Roberval's historical construction (Struik, 1969, pp. 234-235) of a tangent to the cycloid is quite accessible to students, as it uses only the parallelogram law for vector addition. The result is presented in H of C:

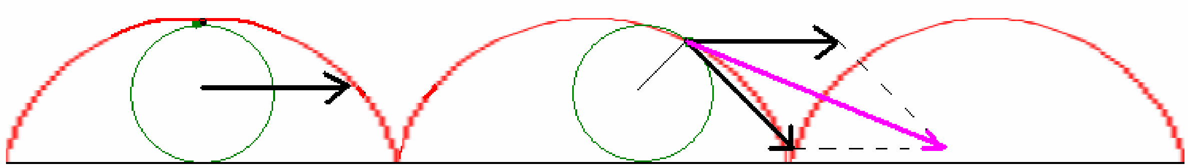

Figure 3. Finding a tangent to the cycloid

The result can almost be presented "without words". (See Figure 3.) Follow the path of the "tracing" point on the generating circle of a cycloid. The motion of this point at a given instant has a horizontal component, corresponding to the horizontal motion of the center of the circle; it also has a component normal to a radius of the circle, since the circle rolls. These components are of equal magnitude, since the circle rolls without slipping. The resultant of the motions is found by the parallelogram law, and is the tangent to the cycloid.

It was early in 1696 that Johann Bernoulli solved the problem of finding the curve of quickest descent; he showed that the brachistochrone was a cycloid. Later, in June of that year, he posed the problem in the journal *Acta Eruditorum*

> ' PROBLEMA NOVUM, ad cujus Solutione Mathematici invitantur.
> " Datis in plano verticali duobus punctis *A* et *B,* assignare mobili *M* viam *AMB,* per quam gravitate sua descendens, et moveri incipiens a puncto *A,* brevissimo tempore perveniat ad alterum punctum *B.*"' (Woodhouse 1810, pp.2 – 3)

An English translation is as follows:

> 'If two points A and B are given in a vertical plane, to assign to a mobile particle M the path AMB along which, descending under its own weight, it passes from the point A to the point B in the briefest time.' (Smith, D.E. 1929, p.644)

The problem was also solved by Jakob Bernoulli, Leibniz, L'Hôpital and Newton. Newton's solution was published anonymously in the *Philosophical Transactions of the Royal Society* in January, 1697. Solutions by Johann Bernoulli, Jakob Bernoulli, Leibniz and Newton were published in *Acta Eruditorum* in May, 1697. According to Stillwell (2002, p. 239), the most profound was Jakob Bernoulli's solution, which represented a key step in the development of the calculus of variations. The historical development of what became the calculus of variations is closely linked to certain minimization principles in physics, namely, **the principle of least distance**, **the principle of least time**, and ultimately, **the principle of least action**. (See Kline, 1972, pp. 572-582.) To understand Johann Bernoulli's solution of the brachistochrone problem, students in H of C are led through Fermat's **principle of least time**: light always takes a path that minimizes travel time.

Principle of Least Time

An accessible application of the principle of least time is in deriving the **law of reflection**: if a ray of light strikes a mirror, then the angle of incidence equals the angle of reflection. This law was first noted by Euclid in the fourth century BCE (Ronchi, 1957, p. 11), and was explained using a principle of least distance by Heron of Alexandria in the first century CE (Cooke, 1997, p. 149). In the H of C course, the law of reflection is derived geometrically, as per Heron. Since the speed of light (in a fixed medium) is constant, this is equivalent to a derivation from the principle of least time.

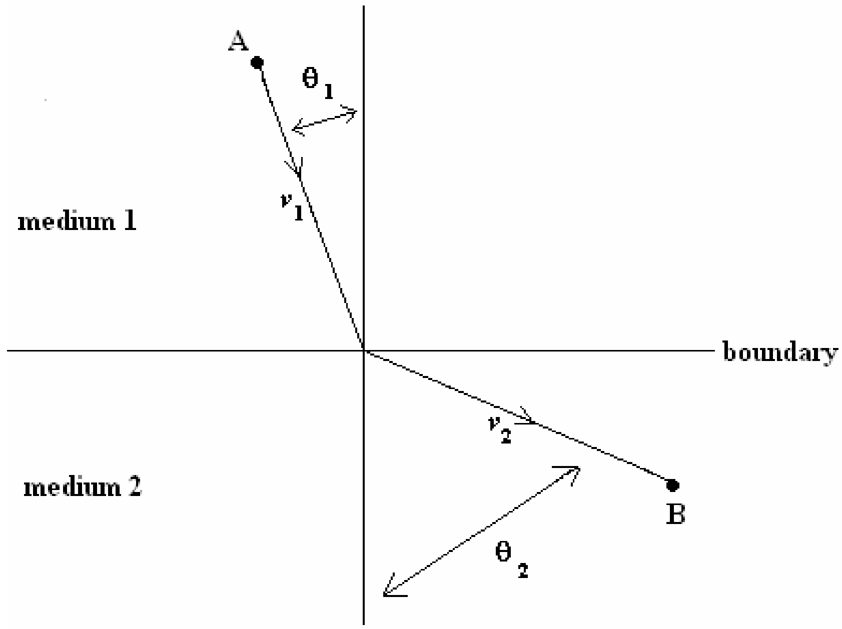

Figure 4: Snell's law of refraction

The **law of refraction** states that when a ray of light crosses the boundary between transparent media, it experiences a change in direction characterized by the relation

$$\sin \theta_1 / \sin \theta_2 = k$$

in which θ_1 is the angle of incidence, θ_2 is the angle of refraction and k is a constant dependent on the nature of the two media. (See Figure 4.) This law was discovered experimentally by the Dutch physicist Willebrord Snel circa 1621; in English it is known as **Snell's law**. Snel noticed that if the first medium is less dense than the second, then $k > 1$; that is, upon entering the second medium, the light ray bends toward the normal to the boundary. (Nahin, 2004, p. 103)

In the mid-seventeenth century, Fermat demonstrated that Snell's law of refraction may be derived from the principle of least time. In the H of C course, such a derivation is given using an abbreviation of the approach outlined by Nahin: Consider a ray of light crossing the boundary between transparent media. For $i = 1, 2$ let v_i denote the speed of light in medium i. Referring to Figure 5, let T denote the transit time for a light ray travelling from point A in medium 1, through point B at the boundary, to point C in medium 2. Then

$$T = (\textit{length of } AB)/v_1 + (\textit{length of } BC)/v_2 = \sqrt{h_1^{\,2} + x^2} \Big/ v_1 + \sqrt{h_2^{\,2} + (d-x)^2} \Big/ v_2$$

To obtain the path of least time, it is necessary to determine x so that the transit time T is minimized.

$$dT/dx = x\big/v_1\sqrt{h_1^{\,2}+x^2} - (d-x)\big/v_2\sqrt{h_2^{\,2}+(d-x)^2} = \sin\theta_1/v_1 - \sin\vartheta_2/v_2$$

The necessary condition for a minimum, namely that $dT\,/\,dx\ = 0$, yields the requirement that

$$\sin\theta_1\,/\,\sin\theta_2 = v_1\,/\,v_2$$

Thus, Snell's media-dependent constant is $k = v_1\,/\,v_2$.

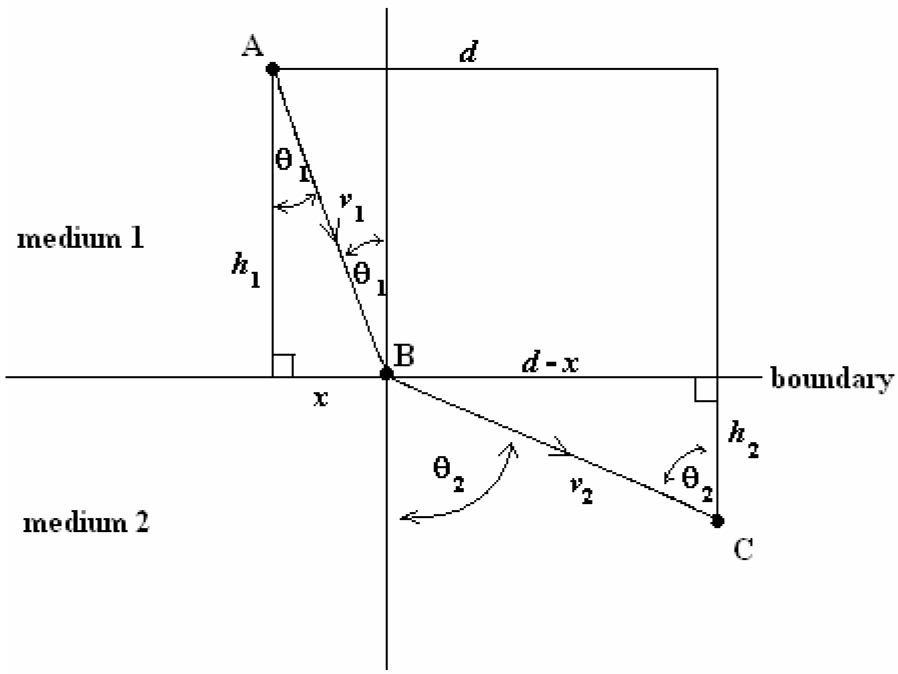

Figure 5: Derivation of Snell's law

It should be noted that Fermat achieved the minimization using his **method of adequality**, which is comparable to differentiation; however, he had to introduce some approximations, since he could not apply his adequality method directly to expressions involving square roots. Nahin (2004, p. 127-134) gives a detailed presentation of Fermat's solution.

Johann Bernoulli's Solution to the Brachistochrone Problem

In the brachistochrone problem, an ideal particle traverses an AB-curve under gravity. The traversal time will be determined if we can fix the speed of the particle at each point along its path. The principle of conservation of mechanical energy implies that if the particle starts at rest, and the vertical drop from A to a point is y, then the particle will acquire a speed at the given point of

$$v = (2\,g\,y)^{1/2} \qquad (3)$$

This speed is independent of whether the particle has dropped vertically, moved along an inclined line, or followed some more complicated path. The brachistochrone problem thus becomes the following:

> A particle moves from A to B in such a way that whenever its vertical drop from A is y, its speed is given by (3). Find the AB-curve with the shortest traversal time.

Johann Bernoulli solved the problem via a brilliant thought experiment. Consider a non-uniform optical medium which becomes increasingly less dense from top to bottom. If light enters from above, its speed becomes faster and faster as it moves down. By a judicious varying of the density, light may be constrained to travel through this medium in a manner satisfying (3). However, by the principle of least time, in any situation, light will always travel along a path with the shortest traversal time. We therefore see that if A is located at the top of this non-uniform medium, and B at the bottom, the path taken by light travelling from A to B is the brachistochrone!

This leaves the question of how light will travel through our non-uniform medium. Consider a light ray travelling through two transparent media, from point A in medium 1 (upper) to point B in medium 2 (lower). Let θ_1 denote the angle of incidence and θ_2 the angle of refraction. Let v_1 and v_2 denote the speed of light in the respective media. Suppose that medium 2 is less dense than medium 1, so that $v_2 > v_1$. Then, since

$$\sin \theta_1 / \sin \theta_2 = v_1 / v_2 < 1$$

$\theta_2 > \theta_1$, and the light ray bends away from the normal to the boundary. (See Figure 4.) Note, also, that

$$\sin \theta_1 / v_1 = \sin \theta_2 / v_2 = \text{constant}$$

Now, consider a similar situation with a light ray traveling downward through many layered transparent media, with each medium less dense than the layer above it. The speed of light increases in the successive media as it progresses through deeper layers and the ray of light bends further away from each successive normal to the boundary at point of contact. (See Figure 6.)

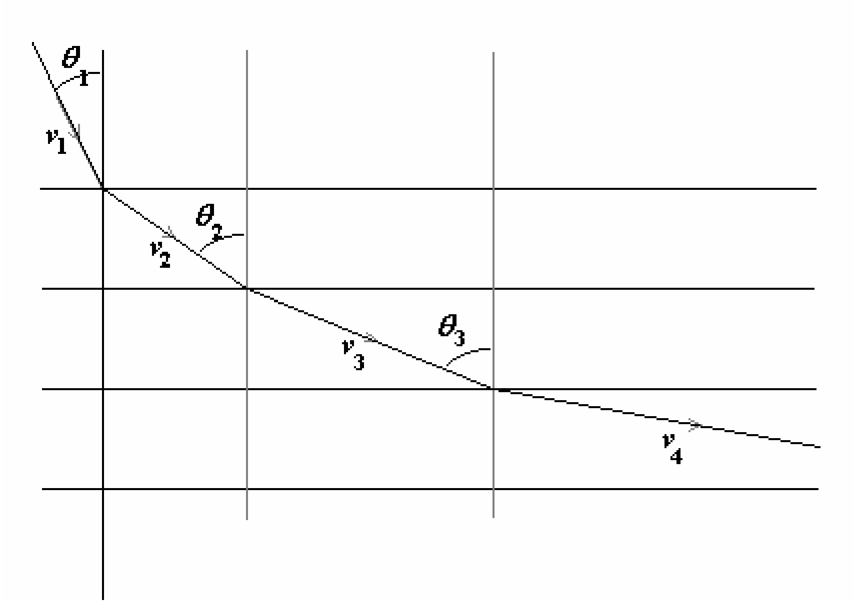

Figure 6: Refraction through multiple layers

Note also, that:

$$\sin\,\theta_1 \,/\, v_1 \;=\; \sin\,\theta_2 \,/\, v_2 \;=\; \sin\,\theta_3 \,/\, v_3 \;=\; \ldots \;=\; \text{constant}$$

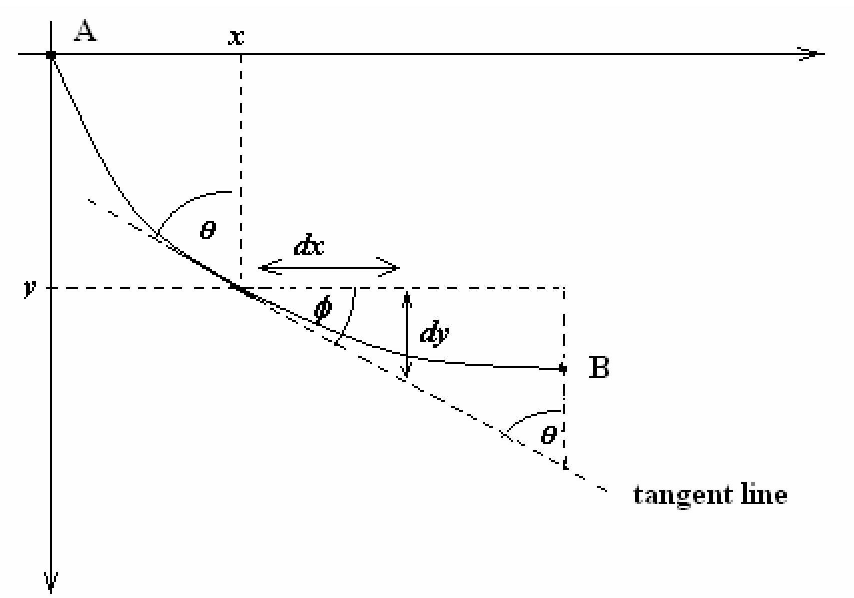

Figure 7: Johann Bernoulli's proof

Letting the number of layers increase without bound and the thickness of each layer decrease towards zero, the path of the light ray becomes a smooth curve. (See Figure 7.) At each point along the curve

$$\sin\,\theta \,/\, v = c \tag{4}$$

Bernoulli thus realized that a particle falling along the curve of quickest descent from A to B must satisfy both equations (3) and (4). From the triangle in Figure 7

$$\sin\theta = \cos\phi = 1 / \sec\phi = 1 / [1 + \tan^2\phi]^{1/2} = 1 / [1 + (dy/dx)^2]^{1/2}$$

Thus

$$\sin\theta / v = \text{constant} = 1 / (v\,[1 + (dy / dx)^2]^{1/2})$$

where $v = (2gy)^{1/2}$. This yields the following nonlinear differential equation:

$$y\,[1 + (dy / dx)^2] = k \tag{5}$$

where k is some constant. Algebraic manipulation of the differential quantities, dx and dy, yields the equation

$$dx = dy\,[y / (k - y)]^{1/2} \tag{6}$$

which Bernoulli recognized as a differential equation describing a cycloid. In the translated words of Johann Bernoulli:

> 'from which I conclude that the Brachystochrone is the ordinary Cycloid'
> (Struik 1969, p .394 translation of Johann Bernoulli 1697)

Note that in Bernoulli's May 1697 paper in *Acta Eruditorum*, the usual labelling of x and y coordinates is reversed. Equation (6) may be further manipulated to obtain parametric equations for x and y; for details, see the excellent accounts by Simmons (1972), Erlichson (1999) and Nahin (2004).

Jakob Bernoulli's Solution to the Brachistochrone Problem

Although Johann Bernoulli's solution to the brachistochrone problem impresses us with its elegance, it is tailored to a very specific application. Jakob Bernoulli's more methodical approach generalizes, and in fact became the basis of the calculus of variations. In fact, neither of the Bernoullis' solutions uses calculus explicitly in the foreground. Each of the brothers, by a different method, sets up a differential equation, and having found this equation, declares the problem solved. (Struik, 1969, pp. 392-399). Their quickness at this early date to recognize a differential equation of the cycloid is striking!

Since the calculus is in the background, Jakob Bernoulli's solution to the cycloid may be outlined to H of C students. We give our abbreviated presentation of his proof below. It relies on the use of similar triangles, some (typical) hand-waving regarding infinitesimals, and the concept of *stationary points* of functions.

A *stationary point* of a function is one at which the function's rate of change is zero. In the ordinary calculus, we recognize that local extrema of a function occur at stationary points. Jakob Bernoulli extended this idea to the brachistochrone problem. Since the brachistochrone minimizes descent time, the rate of change of descent time must be zero with respect to infinitesimal variation of the brachistochrone path. Consider Figure 8.

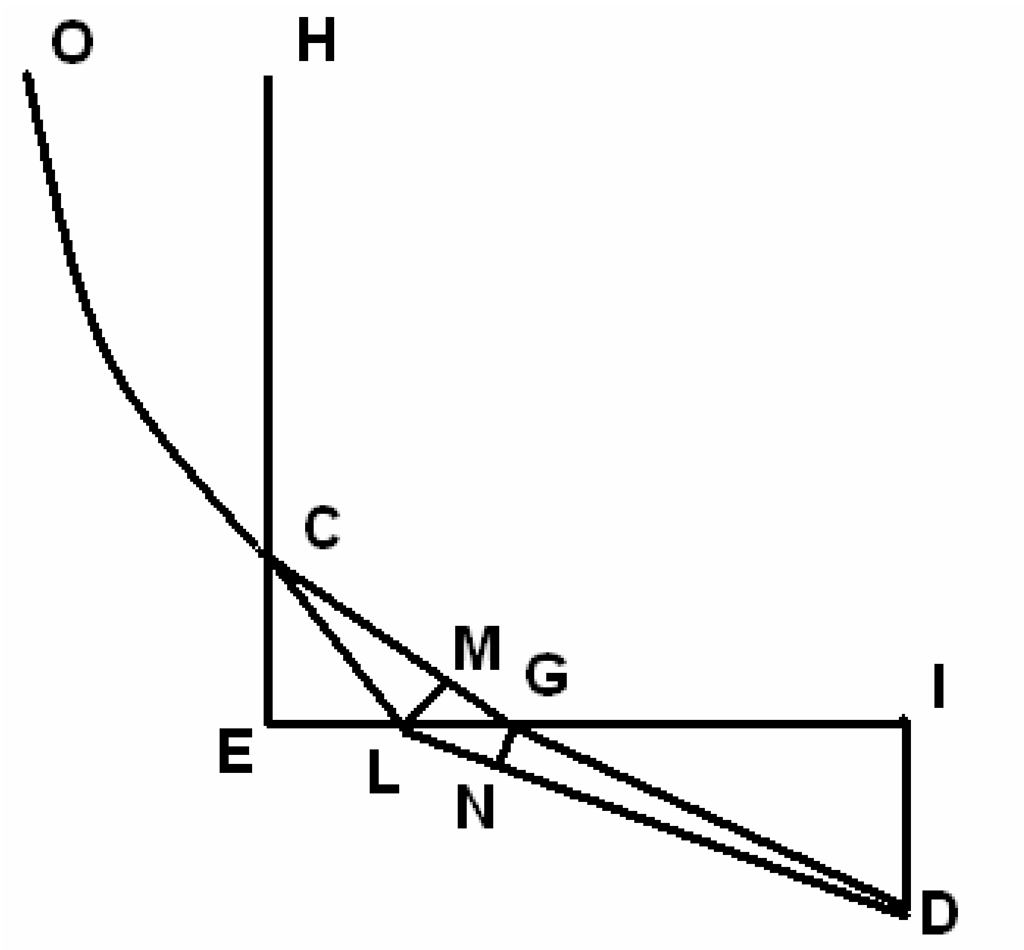

Figure 8: Jakob Bernoulli's solution of the brachistochrone problem

Let curve $OCGD$ be a small section of the brachistochrone. Letting y measure vertical drop from O, choose units so that a particle moves along the curve with instantaneous speed $\sqrt{y}$ at any point. We consider CG to be so short that a particle moves along CG with constant speed $\sqrt{|HC|}$ where $|HC|$ denotes the length of HC. Similarly, we assume a constant speed $\sqrt{|HE|}$ on GD.

Vary the path by moving G an infinitesimal distance horizontally, to L. As the brachistochrone is stationary, the descent time along $OCLD$ must also be minimal. Add construction lines ML and NG such that triangles ΔCML and ΔDNG are isosceles. The descent times along CM and CL are thus equal, as are descent times along GD and ND. As the total descent times along $OCGD$ and $OCLD$ are to be equal, the descent time along MG must equal that along LN, and

$$|MG|/\sqrt{|HC|}=|LN|/\sqrt{|HE|}. \qquad (7)$$

As we are dealing with infinitesimal distances, we may consider *ML* to be an arc of a circle centered at *C*, and *LMG* to be a right angle. By similar triangles, then, $|MG|\ /\ |LG| = |EG|\ /\ |CG|$. If we let *x* measure horizontal distance, and *s* arc length along the brachistochrone, this can be rewritten as

$|MG|\ /\ |LG| = dx\ /\ ds$ on segment *CG*.

By an analogous argument,

$|LN|\ /\ |LG| = dx\ /\ ds$ on segment *GD*.

Dividing by $\sqrt{y}$ on each of segments *CG* and *GD*, and applying equation (7), we find that

$$dx/\sqrt{y}ds = k \qquad (8)$$

with the same constant *k* on both segment *CG* and segment *GD*. We conclude that equation (8) holds everywhere on the brachistochrone. Jakob Bernoulli recognized this as a differential equation of the cycloid.

Note that various mysteries involving infinitesimals take place; segment *CG* is only short, while *LG* is infinitesimal. Also, isosceles triangle ΔCML contains two right angles, and this is essential to the argument with similar triangles. Again, Bernoulli interchanged *x* and *y*, which can be confusing to a modern reader.

Computerized Demonstration of the Brachistochrone

It may be difficult for students to grasp the nature of the minimization problem involved in finding the brachistochrone. We are not finding the tangent or area of some particular *given* curve, as is usually the case in calculus. Instead, we must search among *all possible hypothetical curves* to find that which allows least time descent. The nature of the problem also precludes physical demonstration; we may build an apparatus to demonstrate sliding descent on a particular curve, or we may "race" physical beads on two or three particular curves; however, it is hard to conceive how one would physically demonstrate beads descending on enough curves to allow students to conceptualize descent on an *arbitrary* curve. Here the computer comes to the rescue:

Using the symbolic programming language *MAPLE*, we simulate the descent under gravity of a particle along a curve as follows:

1. Curves are given parametrically: $x = x(s)$, $y = y(s)$, $a \le s \le b$.
2. A given curve is systematically sampled at $n + 1$ points $s_i = a + i\,(b - a)/n$, $i = 0, 1, \ldots, n$.
3. The curve is modeled by straight-line segments, from (x_i, y_i) to (x_{i+1}, y_{i+1}), $i = 0, 1, \ldots, n - 1$.
4. The motion of a particle under gravity down the digitized curve is modeled in a straight-forward way: If the particle enters the straight-line segment from (x_i, y_i) to (x_{i+1}, y_{i+1})

moving at speed v_0, its acceleration under gravity will be $a = g \sin \theta$, where g is the acceleration (downward) due to gravity, and $\sin \theta = (y_i - y_{i+1}) / \sqrt{(x_i - x_{i+1})^2 + (y_i - y_{i+1})^2}$. The length of the segment is $D = \sqrt{(x_i - x_{i+1})^2 + (y_i - y_{i+1})^2}$. At time t after entering the segment, the particle has moved distance $d = v_0 t + \frac{1}{2} at^2$ along the segment. The time T spent traversing the segment is then the solution of $D = v_0 T + \frac{1}{2} aT^2$, so that the particle enters the next segment with a speed of $v_0 + aT$.

5. The positions of particles on several digitized curves can thus be worked out as parametric functions of time. The *MAPLE* command *animate*() is then capable of presenting trajectories of two or more of these particles simultaneously, as they each move along their underlying curves. Our code as currently implemented contrasts the motion of particles along two desired curves.

We have found that $n = 11$ already gives very smooth-looking approximations. Using our software, we "race", for example, a particle on a cycloid arc against a particle on a straight-line ramp (see Figure 2), or a particle on a circular arc. We can in fact race along any (parametric) curve suggested by students. The same software will illustrate Galileo's law of chords. After students have seen Johann Bernoulli's solution to the brachistochrone, they can appreciate the analogy between modern "digitization" and Bernoulli's layers.

Conclusion

We have traced the thread of quickest descent problems and the brachistochrone from Galileo, through Fermat and Roberval, to the Bernoullis and the dawn of the calculus of variations. We have spelled out in detail a selection of mathematical results which we have presented to our H of C students. These results include mathematical content ranging in difficulty from geometry, through vectors, to differential equations. A themed unit on the cycloid shows students at very different levels the strong interplay between mathematics and physics: Geometry informs optics (Fermat), optics informs kinematics (Johann Bernoulli), and kinematics informs geometry (Roberval).

Of particular interest is Johann Bernoulli's beautiful *Gedankenexperiment*, whereby a falling particle becomes a ray of light, moving through media arranged without regard for the possibility of actual physical construction. Again, the purely mathematical and hypothetical nature of the frictionless bead in the brachistochrone problem motivates our software demonstrations to students. The digital sampling of curves in our *MAPLE* code echoes the (finite) layering of media by Johann Bernoulli. A careful examination of the arguments of the Bernoullis introduces students to the interesting philosophical and technical issues related to infinitesimals/differentials in mathematics and physics.

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