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Comparison of Geometric Figures

Spyros Glenis[1]
***Department of Mathematics*,**
University of Athens, Greece

Abstract

Although the geometric equality of figures has already been studied thoroughly, little work has been done about the comparison of unequal figures. We are used to compare only similar figures but would it be meaningful to compare non similar ones? In this paper we attempt to build a context where it is possible to compare even non similar figures. Adopting Klein's view for the Euclidean Geometry, we defined a relation "$\leq$" as: $S_1 \leq S_2$ whenever there is a Euclidean isometry $f:\mathbb{R}^2 \to \mathbb{R}^2$, so that $f\left(S_1\right) \subseteq S_2$. This relation is not an order because there are figures (subsets of $\mathbb{R}^2$) S_1, S_2 so that $S_1 \leq S_2$, $S_2 \leq S_1$ and S_1, S_2 not geometrically equal. Our goal is to avoid this paradox and to track down non-trivial classes of figures where the relation "$\leq$" becomes, at least, a partial order. For example there is no paradox if we restrict our attention just to compact figures; thus, we can compare a closed disc with a closed triangular region. Further we present some other "good" classes of figures and we extend our study to the Hyperbolic and to the Elliptic geometry. Eventually, there are still some open and quite challenging issues, which we present at the last part of the paper.

Keywords: Euclidean geometry; Isometries; Klein; Relations

[1] University of Athens – Dept. of Mathematics
E-mail: spyros_glenis@sch.gr
Tel. +316945375647

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1. INTRODUCTION

In Euclid, the geometric equality depends on the capability of superposition of the figures:

Common notion 4

Things which coincide with one another are equal to one another. ([12])

The geometric equality, with respect to Klein's view, depends on the group theory as well as on the set theory:

Definition

Let a set $X \neq \varnothing$, G a subgroup of Aut(X) and the figures $S_1, S_2 \subseteq X$. We shall say that these figures are $G-$geometrically equal if and only if there is an $f \in G$ so that $f(S_1) = S_2$.

The equality, indirectly defines the inequality of geometric figures. Euclid considers that a figure is smaller than another one if with an appropriate rigid motion the first coincides with part of the second. Although for any two figures S_1, S_2 it is easy to decide whether they are equal or not, however it is not that simple to decide if one of them is "smaller" than the other. Obviously a triangular region is never equal to a circular disk, but can we say that a triangular region is smaller than a circular disc if the radius of the disc is greater than or equal to the radius of the circumscribed circle of the triangle? In Euclid, the comparison involves only "similar" figures. On the contrary, Klein's view of equality, prompts us to define a geometric inequality using the notion of "being subset" and enables us to compare even non-similar figures:

We will say that S_1 is equal to or smaller than S_2 whenever there is a euclidean rigid motion f so that $f(S_1) \subseteq S_2$. Then we will write $S_1 \leq S_2$.

This "natural" definition of inequality provides a paradox as we will immediately illustrate using the following example given by the professor V. Nestorides:
Let us consider a closed half plane *A* and let *B* be the half plane *A* with a line segment attached vertically to the edge of the half plane and pointing outside A. Since $A \subseteq B$ we can say that $A \leq B$. Moreover, there is a translation of *B*, so that it is fully covered by *A* and in this case we may write $B \leq A$. It seems logical to assume that A and *B* must be geometrically equal, in other words, that they can coincide if we apply a certain rigid motion. But this is impossible to happen, because every half plane remains half plane whenever we apply a rigid motion to it and obviously it can't coincide with a geometric figure that is not a half plane.

Since the geometric relation "$\leq$" is not antisymmetric it is necessary to restrict the comparison to certain classes of geometric figures. We already know that in the class of the line segments or in the class of the arcs of a circle, the relation "$\leq$" is a total order. Therefore the question is, if there are other classes of figures where the relation "$\leq$" is a total or a partial order.

We shall call *good classes* (of geometric figures) those that among the figures they contain we can't find a paradox like the one mentioned above. A good class, but not the only one, is that of the compact figures (sets). In fact, compact figures have the property not to generate paradox with any other geometric figure whether compact or not. Those figures will be called *good figures*. Besides the compacts, good figures are also the open-and-bounded sets. On the contrary, just bounded figures may not be good as we will prove later using a counterexample, given again by professor V. Nestorides.

The study, concerns not only the Euclidean Geometry, but it is also expanded into the Hyperbolic and the Elliptic Geometry and some parts may be formulated in a pure algebraic language so that they cover uniformly all three geometries. The conclusions we have reached, are fully compatible with our previous knowledge about the comparison of geometric figures. In the special case of the Euclidean Geometry we proved that there is a good class, containing all the fundamental geometric figures, where we can compare even non-similar ones. Therefore a comparison between a circular disc and a triangular region is meaningful in the new context.

2. COMPARISON OF FIGURES IN THE EUCLIDEAN GEOMETRY

2.1 *Basic definitions*

We adopt Klein's view for the Euclidean Geometry. Our space is $\mathbb{R}^2$ endowed with the euclidean metric and the group acting on $\mathbb{R}^2$ is $ISO(\mathbb{R}^2)$, the group of euclidean isometries. The couple $\left(ISO(\mathbb{R}^2),\mathbb{R}^2\right)$ generates the euclidean geometric space where we will develop our study.

Definition 2.1

Two figures S_1 and S_2 are geometrically equal when there is a euclidean isometry[2] $f:\mathbb{R}^2\to\mathbb{R}^2$ *so that* $f(S_1)=S_2$. *In that case we will write* $S_1\approx S_2$.

Remarks

I. Figure is any subset of $\mathbb{R}^2$. From now on we will not distinguish the terms "subset of $\mathbb{R}^2$" and "figure".

II. We use the terms "rigid motion" and "isometry" synonymously.

Definition 2.2

For any two figures S_1 and S_2 we shall say that S_1 is equal to or smaller than S_2 when there is a euclidean rigid motion f so that $f(S_1)\subseteq S_2$. *Then we will write* $S_1\leq S_2$.

This "natural" definition does not satisfy in general the antisymmetric property, as we will prove later.

Proposition 2.1

The relation "$\leq$" is a pre-order of figures i.e. it is reflexive and transitive, and the reflex ion is meant in the sense of the geometric equality defined in 1.1

Proof

Let A and B two geometrically equal figures. Then, by definition, there is a euclidean isometry *f* so that $f(A)=B$. Then $f(A)\subseteq B$ also holds and we conclude that the relation $\leq$ is reflexive with respect to the geometric equality.

[2] based on the euclidean metric ρ of $\mathbb{R}^2$ where $\rho\left((x,y),(a,b)\right)=\sqrt{(x-a)^2+(y-b)^2}$

If $A \leq B$ and $B \leq C$ then there are isometries f, g so that $f(A) \subseteq B$ and $g(B) \subseteq C$. Then for the isometry $g \circ f$ holds $g \circ f(A) \subseteq C$ i.e. $A \leq C$. Therefore the relation is transitive ∎

In the following examples we shall prove that "$\leq$" does not satisfy in general the antisymmetric property, with respect to the geometric equality of definition 1.1.

Example 2.1

Let the half lines $A = \{(x,0) \in \mathbb{R}^2 : x \geq 0\}$, $B = \{(x,0) \in \mathbb{R}^2 : x > 0\}$. Since A is a closed subset of $\mathbb{R}^2$ while B is not, **there is not** an isometry f so that $f(A) = B$ [3] thus $A \not\approx B$.

For the isometry $f(x,y) = (x+1, y)$, $f(A) \subseteq B$ holds. But it is also obvious that $B \subseteq A$, so we have both $A \leq B$ and $B \leq A$ while $A \not\approx B$.

Example 2.2

Let the figure $A = \{(x,y) \in \mathbb{R}^2 : x \leq 0\} \cup \{(x,2) \in \mathbb{R}^2 : 0 \leq x \leq 1\}$ and the half plane $B = \{(x,y) \in \mathbb{R}^2 : x \leq 2\}$. Obviously $A \subseteq B$ therefore $A \leq B$ .Since every isometry maps half planes into half planes there is not an isometry f such that $f(A) = B$, so $A \not\approx B$.

But for the isometry $f(x,y) = (x-3, y)$, $f(B) \subseteq A$ holds. So we conclude that $A \leq B$ and $B \leq A$ while $A \not\approx B$.

Example 2.3

Let the figure $\mathrm{A} = \{(x,y) \in \mathbb{R}^2 : x \geq 0, y \geq 0\}$ which is the right angle $\widehat{xOy}$ and the figure B produced by A when we subtract the inner part of the isosceles right triangle $S = \{(x,y) \in \mathbb{R}^2 : x \geq 0, y \geq 0, x+y < 1\}$. Obviously $B \subseteq A$ so $B \leq A$ holds.

By translating A parallel to the axis $x'x$ by two units we also have that $f(A) \subseteq B$, where $f(x,y) = (x+2, y)$ is an isometry. Thus $A \leq B$.

But A and B are not geometrically equal, because in case there is an isometry g such that $g(A) = B$ and since isometries preserve the angles then B should also be a right angle, which is absurd.

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[3] Every isometry maps closed sets into closed sets.

Example 2.4

Let an angle ω so that $\frac{\omega}{\pi}$ is irrational, for instance we can choose $\omega = \pi\sqrt{2}$. We set $a_k\left(\cos k\omega, \sin k\omega\right)$ a sequence of points lying on the circumference of the unit circle. By definition $a_k \neq a_m$ for $k \neq m$ otherwise we would have integers n, λ so that $\frac{\omega}{\pi} = \frac{\lambda}{n}$.

The set $A = \{a_k : k = 0,1,2,3,...\}$ is a dense and equally distributed subset of the circumference. We also consider the set $B = A \setminus \{a_1\} \subseteq A$, therefore $B \leq A$. If there is an isometry f of the plane such that $f(A) = B$ then we will arrive at a contradiction. For every $a \in A$ there is at least one $a' \in A$ so that the distance $d(a, a') = r$ where $r = 2\sin(\omega/2)$. Since $f(A) = B$ then and for every $b \in B$ there is at least one $b' \in B$ so that the distance $d(b, b') = r$. But this does not hold for $a_o \in B$ and we arrived at a contradiction.

Let T be a rotation by 2ω with center the origin of the axes. Then T is an isometry and $T(A) = \{a_k : k = 2,3,...\} \subseteq B$. So $A \leq B$ and $B \leq A$ hold, while $A \not\approx B$.

We modify the definition of the pre-order $"\leq"$ so that we arrive at an order relation:

Definition 2.3

In the set of figures we define a relation λ such that:

$$A\lambda B \Leftrightarrow \{A \approx B\} \text{ or } \{A \leq B \text{ and not } B \leq A\}$$

Proposition 2.2

λ is an order relation.

Proof

simple

Definition 2.4

A class $\mathcal{E}$ of figures is said to be **good** *when there are not any figures A, B in the class $\mathcal{E}$ so that $A \leq B, B \leq A$ and $A \not\approx B$.*

Remark

The relation "λ" is the relation "≤" without the pathological cases where the antisymmetric property does not hold true. But the definition of "λ" is quite barren and gives no information to the question:

It seems to be wiser to concentrate our study on those sets that satisfy the antisymmetric property of "$\leq$".

Definition 2.5

We will say that a figure A is **good** *when for every figure B, if $A \leq B$ and $B \leq A$ hold, then $A \approx B$ also holds.*

Obviously a class consisting only of good sets is a good class. The converse does not hold true. A trivial case is a class consisting of only one figure (and all the geometric equals) that is not good. Since there is no other figure in the class to provide a counterexample then the class is good.

A non-trivial example is the class of the open or closed angles. We proved in example 1.2 that an angle is not a good set but it is quite easy to verify that using open or closed angles only, we can not provide a counterexample.

2.2 *Quest for good classes of figures*

Proposition 2.3

If A is a good set and $A \approx B$ then B is also a good set.

Proof

Let $C \leq B$ and $B \leq C$. Since $A \approx B$ then $A \leq C$ and $C \leq A$ hold. As A is a good set then there is an isometry f of the plane such that $f(A) = C$. There is also an isometry g of the plane such that $g(B) = A$. Then $g \circ f(B) = C$ i.e. $B \approx C$ ∎

Proposition 2.4

If A is a good set then its complement is also a good set.

Proof

Let $B \leq A^c$ and $A^c \leq B$. Then there are isometries f, g, of the plane, so that $f(B) \subseteq A^c$ and $g(A^c) \subseteq B$. But then $f(B)^c \supseteq A$ and $g(A^c)^c \supseteq B^c$ hold.

Since f, g are 1-1 and onto, $f(B)^c = f(B^c)$ and $g(A^c)^c = g(A)$ hold. Therefore $f(B^c) \supseteq A$ and $g(A) \supseteq B^c$ which is equivalent to $A \leq B^c$ and $B^c \leq A$. As A is good, there is an isometry h of the plane such that $h(A) = B^c$. Then $h(A)^c = B$ and as h is 1-1 and onto $h(A^c) = B$ i.e. $A^c \approx B$ ∎

Theorem 2.1

Let $\langle X, d \rangle$ *be a compact metric space. If* $f: X \to X$ *is an isometry then* $f(X) = X$.

<u>Proof</u>

Well known

Proposition 2.5

Every compact subset of $\mathbb{R}^2$ *is a good set.*

<u>Proof</u>

Let A compact and B an arbitrary set so that $A \le B$ and $B \le A$. Then there are isometries f, g so that $f(A) \subseteq B$ and $g(B) \subseteq A$.

For the isometry $g \circ f: A \to A$ we have already seen that $g \circ f(A) = A$ because A is compact (theorem 1.1).

But then $g(B) \subseteq A = g \circ f(A)$ which implies that $B \subseteq f(A)$. So $f(A) = B$ holds and $A \approx B$. Therefore A is a good set î

We will introduce now, a new definition that will be particularly useful.

Definition 2.6

A figure $A \subseteq \mathbb{R}^2$ *will be called* **strongly good** *if for every isometry* $f: \mathbb{R}^2 \to \mathbb{R}^2$ *that satisfies* $f(A) \subseteq A$ *, then the equality* $f(A) = A$ *holds true.*

Proposition 2.6

Every compact subset of $\mathbb{R}^2$ *is strongly good.*

<u>Proof</u>

Direct from definition 1.6 and theorem 1.1 î

Proposition 2.7

Every strongly good set is also a good set.

<u>Proof</u>

Let A a strongly good set and B an arbitrary set such that $A \le B$ and $B \le A$. Then there are isometries f, g of the plane so that $f(A) \subseteq B$ and $g(B) \subseteq A$.

For the isometry $g \circ f : A \to A$, $g \circ f(A) = A$ holds since A is strongly good. Then $g(B) \subseteq A = g \circ f(A)$ and therefore $B \subseteq f(A)$.

Finally we conclude that $f(A) = B$ so $A \approx B$ and A is a good set ∎

Remark

The definition of the good figure is difficult to handle, as it depends on the "interaction" with all the other figures. On the contrary the definition of the strongly good figure is intrinsic, because, in simple words, strongly good is any figure that does not fit (without decomposition) into part of itself.

Proposition 2.8

Every open and bounded subset of $\mathbb{R}^2$ *is strongly good.*

Proof

Let A open and bounded and f an isometry of the plane such that $f(A) \subseteq A$. As A is open then

$A \cap \partial A = \varnothing$ and $\overline{A} = A \cup \partial A$. Thus $A = \overline{A} \setminus \partial A$

Also $diam(\overline{A}) = diam(A) < \infty$ so $\overline{A}$ is closed and bounded subset of $\mathbb{R}^2$, hence $\overline{A}$ is compact. Also $\partial A \subseteq \overline{A}$ and $diam(\partial A) \leq diam(\overline{A})$ so the boundary is closed and bounded hence compact subset of $\mathbb{R}^2$.

$f(A) \subseteq A \Rightarrow \overline{f(A)} \subseteq \overline{A} \Rightarrow f(\overline{A}) \subseteq \overline{A}$ and from T*heorem 3.1* $f(\overline{A}) = \overline{A}$ holds.

$f(\partial A) = f(\overline{A} \setminus A) = f(\overline{A}) \setminus f(A) = \overline{A} \setminus f(A) \supseteq \overline{A} \setminus A = \partial A$

Hence $f^{-1}(\partial A) \subseteq \partial A$ and as f^{-1} is an isometry then from Theorem *3.1* again, we conclude that $f^{-1}(\partial A) = \partial A \Leftrightarrow \partial A = f(\partial A)$. Therefore

$f(A) = f(\overline{A} \setminus \partial A) = f(\overline{A}) \setminus f(\partial A) = \overline{A} \setminus \partial A = A$, so A is strongly good ∎

Proposition 2.9

In $\mathbb{R}^2$*, the union of a compact with an open bounded set is a strongly good set.*

Proof

Let K compact and A open and bounded subsets of $\mathbb{R}^2$, and the isometry f of the plane such that $f(A \cup K) \subseteq A \cup K$.

The set $V=(A\cup K)^{o}\subseteq A\cup K$ is also open and bounded.

$W=K\setminus V=K\cap V^{c}$ is compact as an intersection of a compact with a closed set.

Obviously $V\cap W=\varnothing$

Since A is open subset of $A\cup K$ then $A\subseteq V$ and

$A\cup K\subseteq V\cup K=V\cup(K\setminus V)=V\cup W\subseteq A\cup K$. Hence $V\cup W=A\cup K$

Also $V\cap W=\varnothing\Rightarrow f(V\cap W)=\varnothing\Rightarrow f(V)\cap f(W)=\varnothing$

Consequently $f(V\cup W)\subseteq V\cup W\Rightarrow f(V)\cup f(W)\subseteq V\cup W$

$f(V)$ is an open and bounded subset of $A\cup K$ therefore $f(V)\subseteq V$ since $V=(A\cup K)^{o}$. From *proposition 1.8* we conclude that $f(V)=V$.

As $f(V)\cup f(W)\subseteq V\cup W$ and $V\cap W=\varnothing, f(V)\cap f(W)=\varnothing$ then $f(W)\subseteq W$ But W is compact and from *Theorem 3.1* $f(W)=W$. Therefore

$f(V)\cup f(W)=V\cup W\Rightarrow f(V\cup W)=A\cup K\Rightarrow f(A\cup K)=A\cup K$ ∎

Proposition 2.10

In $\mathbb{R}^{2}$*, the intersection of a compact with an open bounded set is a strongly good set.*

Proof

Let K compact, A open and bounded and f an isometry of the plane such that $f(A\cap K)\subseteq A\cap K$

We set $X=\overline{A\cap K}$ which is a compact subset of K.

Then $f(A\cap K)\subseteq A\cap K\Rightarrow f(\overline{A\cap K})\subseteq X\Rightarrow f(X)\subseteq X$. According to *Theorem 1.1* we conclude that $f(X)=X$.

Since $A\cap K$ is open in K, then it is also open in every closed subset of K. Therefore $A\cap K$ is open in X, so $X\setminus(A\cap K)$ is compact and it is obvious that

$f^{-1}(X\setminus(A\cap K))\subseteq X\setminus(A\cap K)$. According to *Theorem 1.1* we conclude that

$f^{-1}(X\setminus(A\cap K))=X\setminus(A\cap K)$.

But then $f(X\setminus(A\cap K))=X\setminus(A\cap K)$ and since $f(X)=X$ and $A\cap K\subseteq X$, we conclude that $f(A\cap K)=A\cap K$ i.e. $A\cap K$ is strongly good ∎

Remark

The proposition holds even if A is not bounded.

Proposition 2.11

The classes $\mathcal{X} = \{ K \cup A$ *where K compact and* A *open bounded subsets of the plane*$\}$, $\mathcal{Y} = \{ K \cap A$ *where K compact and* A *open bounded subsets of the plane*$\}$, $\mathcal{E} = \mathcal{X} \cup \mathcal{Y}$ *and* $\mathcal{F} = \mathcal{E} \cup \left\{ S \subseteq \mathbb{R}^2 : S^c \in \mathcal{E} \right\}$ *are all good classes.*

Proof

The classes above consist of strongly good sets. Then from definition 1.4 and the propositions 1.7, 1.8, 1.9, 1.10 the conclusion is obvious ◊

The class $\mathcal{F}$ includes almost all the fundamental figures of Euclidean Geometry: line segments, triangles, polygons, circles, arcs etc but not the open or closed angles which, as we have already mentioned, form a good class.

Proposition 2.12

If $\mathcal{W}$ is the class of open or closed angles then the class $\mathcal{F} \cup \mathcal{W}$ is good.

Proof

Since we already know that the classes $\mathcal{F}, \mathcal{W}$ are good, then it is sufficient to examine whether a set from one class provides a counterexample to the other class.

Let A a set of the class $\mathcal{F}$. Then $A \in \mathcal{E}$ or $A \in \mathcal{F} \setminus \mathcal{E}$.

If $A \in \mathcal{E}$ then it is bounded and it cannot provide a counter example with any angle of $\mathcal{W}$.

If $A \in \mathcal{F} \setminus \mathcal{E}$ and provides a counterexample with an angle then its complement $A^c \in \mathcal{E}$, will also provide a counterexample with the complement of the angle (which is also an angle). But this contradicts what we have already proved about the elements of $\mathcal{E}$ which give no counterexamples with the elements of $\mathcal{W}$.

So the class $\mathcal{F} \cup \mathcal{W}$ is a good one and includes the closed and open angles ◊

Remarks

1. The **closed sets** are neither good nor form a good class according to the examples 1.1 and 1.2
2. The **connected sets** are neither good nor form a good class according to the example 1.1
3. The **bounded sets** are neither good nor form a good class according to the example 1.4
4. The **connected and bounded sets** are neither good nor form a good class. This can be easily proved if in the set A of the example 1.4 we attach the inner points of the unit disc.
5. The **convex sets** are neither good nor form a good class according to the example 1.3

6. The **convex and bounded sets** are neither good nor form a good class. This can be easily proved if in the set A of the example 1.4 we attach the inner points of the unit disc.

If two figures A, B are good we have proved that their complements are also good. However the union or intersection of (strongly) good sets is not necessarily a good set as we will illustrate in the following counterexamples:

Example 2.5

Let the strongly good figures $L=\{(x,0)\in\mathbb{R}^2 : x\geq 0\}\cup\{(0,1)\}$ and $M=\{(x,0)\in\mathbb{R}^2 : x\geq 0\}\cup\{(x,1)\in\mathbb{R}^2 : x<0\}$ [4] then the set $L\cap M=\{(x,0)\in\mathbb{R}^2 : x\geq 0\}$ is not good according to the example 1.1.

Example 2.6

We also use here the previously defined sets L and M.

If g is a reflection with respect to the $y'y$ axis then $g(M)$ is strongly good but the set $G=L\cup g(M)=\{(x,0)\in\mathbb{R}^2 : x\in\mathbb{R}\}\cup\{(x,1)\in\mathbb{R}^2 : x\geq 0\}$ is not good because for the set $V=\{(x,0)\in\mathbb{R}^2 : x\in\mathbb{R}\}\cup\{(x,1)\in\mathbb{R}^2 : x>0\}$ we have that $V\subseteq G$ and $h(G)\subseteq V$, where $h(x,y)=(x+1,y)$. Therefore $V\leq G$ and $G\leq V$. But G is a closed set and V is not closed so there is no isometry f such that $V=f(G)$, hence $V\not\approx G$.

It is interesting that the new context is also applied, with trivial modifications, into the Hyperbolic and into the Elliptic Geometry. Fot further details see [15].

[4] For a detailed proof see [15]

3. OPEN ISSUES

A fundamental question is whether the definitions of the good set and the strongly good set are equivalent or there is a counterexample of a good set that is not strongly good. In the appendix it is proved that in $\mathbb{R}$ all good sets are also strongly good. So it is our belief that the definitions are also equivalent on the plane.

Another question is whether the algebra produced by $X = \{A \subseteq \mathbb{R}^2 : A$ compact or open and bounded$\}$ consists only of strongly good sets.

Finally, as in $\Omega =$ set of good classes the assumptions of Zorn's[5] lemma are satisfied it would be quite interesting to find maximal classes within the set Ω.

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[5] Every totally ordered subset of Ω is defined to be a set of good classes $Y = \{ F_i,\ i \in I \}$ so that for every $i, j \in I$ it is true that $F_i \subseteq F_j$ or $F_j \subseteq F_i$. Obviously the **good** class $\bigcup_{i \in I} F_i$ is an upper bound of Y.

Appendix

In the appendix we prove that good sets coincide with strongly good sets in $\mathbb{R}$. We do not know whether the same holds or not holds in $\mathbb{R}^2$.

Lemma

Let a set $A \subseteq \mathbb{R}$, $a \in A$ and an isometry $f:\mathbb{R} \to \mathbb{R}$ such that $f(A) = A \setminus \{a\}$ i.e. $A \approx A \setminus \{a\}$. Then A is not a good set.

Proof

If $f:\mathbb{R} \to \mathbb{R}$ is an isometry then $f(x) = x$ or $f(x) = -x$ or $f(x) = -x + c$, $c \neq 0$ or $f(x) = x + c$, $c \neq 0$.

- If $f(x) = x$ then $f(A) = A \setminus \{a\} \Leftrightarrow A = A \setminus \{a\}$ i.e. $a \notin A$, absurd.
- Both $f(x) = -x$ and $f(x) = -x + c$ have the property $f^2 = id_{\mathbb{R}}$.

  From the assumption we have that $f(A) = A \setminus \{a\}$ therefore

  $f^2(A) = A \setminus \{a, f(a)\} \Leftrightarrow A = A \setminus \{a, f(a)\}$ i.e. $a \notin A$, absurd.
- If $f(x) = x + c$, $c \neq 0$.

  Then $f(a - c) = a$, so $a - c \notin A$

  Also $f(a) = a + c \in f(A) = A \setminus \{a\}$.

  Let the set $B = A \setminus \{a + c\}$. Then $a + c \notin B$ and $a - c \notin B$ since $a - c \notin A$ and $B \subseteq A$.

  Obviously $B \subseteq A$ and $f^2(A) = A \setminus \{a, a + c\} \subseteq B$

  Let us assume that there is an isometry $g:\mathbb{R} \to \mathbb{R}$ so that $g(A) = B$

  For every $x \in A$ we have that $f(x) = x + c \in A \setminus \{a\}$ therefore there is at least one $x' = x'(x) \in A$ so that $d(x, x') = c$ (for instance $x' = x + c$).

  But then we will also have that $d(g(x), g(x')) = c$.

  Since $g(A) = B$ there is some $x_o \in A$ such that $g(x_o) = a \in B$.

  Then there is also some $x'' = x''(x_o) \in A$ so that $d(x_o, x'') = d(x_o, x'(x_o)) = c$

  It follows that $c = d(x_o, x'') = d(g(x_o), g(x'')) = d(a, g(x''))$.

We may say that there is some $b \in B = g(A)$, where $b = g(x'')$ so that $d(a,b) = c$.

But then $b = a - c$ or $b = a + c$, which in either case are not points of $B = g(A)$.

This is a contradiction and we conclude that A is not a good set.

Proposition

Let $A \subseteq \mathbb{R}$ be a good set, then A is strongly good.

Proof

We assume that A is not strongly good.

Then there will be an isometry $T : \mathbb{R} \to \mathbb{R}$ so that $T(A) \subsetneq A$.

Therefore there is $a \in A \setminus T(A)$ and $T(A) \subseteq A \setminus \{a\} \subseteq A$ i.e. $T(A) \leq A$

Since $A \subseteq T^{-1}(A) \Leftrightarrow A \subseteq T^{-1}\left(T^{-1}\left(T(A)\right)\right)$ then $A \leq T(A)$ and as A is a good set then $A \approx T(A)$.

We can also prove that $A \approx A \setminus \{a\}$ because:

$A \setminus \{a\} \leq A$ and $A \approx T(A) \leq A \setminus \{a\}$ since A is a good set we conclude that $A \setminus \{a\} \approx A$.

But from the previous lemma such a set A is never a good set , absurd! ∎

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