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Abstract:
Logarithms are an integral part of many forms of technology, and their history and development help to see their importance and relevance. This paper surveys the origins of logarithms and their usefulness both in ancient and modern times.

Keywords: Computation; Logarithms; History of Logarithms; History of Mathematics; The number "e"; Napier logarithms

1. Background

Logarithms have been a part of mathematics for several centuries, but the concept of a logarithm has changed notably over the years. The origins of logarithms date back to the year 1614, with John Napier[2]. Born near Edinburgh, Scotland, Napier was an avid mathematician who was known for his contributions to spherical geometry, and for designing a mechanical calculator (Smith, 2000). In addition, Napier was first to make use of (and popularize) the decimal point as a means to separate the whole from the fractional part in a number. Napier was also very much interested in astronomy and made many calculations with his observations and research. The calculations he carried out were lengthy and many times involved trigonometric functions (RM, 2007). After many years of slowly building up the concept, he finally developed the invention for which he is most known: logarithms (Smith, 2000).

In his book (published in 1614) *Mirifici Logarithmorum Canonis Descriptio* (Description of the wonderful canon of logarithms), Napier explained why there was a need for logarithms,

> Seeing there is nothing…that is so troublesome to Mathematicall practise, nor that doth more modest and hinder Calculators, than the Multiplications, Divisions, square and cubical Extractions of great numbers, which besides the tedious expense of time, are for the most part subject to many errors, I began therefore to consider in my minde by what certaine and ready Art I might remove those hindrances. (Smith, 2000)

During Napier's time, many astronomical calculations required raw multiplication and division of very large numbers. Sixteenth-century astronomers often used prosthaphaeresis, a method of obtaining products by using trigonometric identities like $\sin\alpha \cdot \sin\beta = \frac{1}{2}[\cos(\alpha - \beta) - \cos(\alpha + \beta)]$ and other similar ones that required simple addition and subtraction (Katz, 2004). For example, if one

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[2] The word "logarithm" was coined by Napier from the Greek "logos" (ratio) and "arithmos" (number) [(9)].

were to multiply 2994 by 3562, then sinα would be 0.2994 (the decimal is placed so that the value of α can be used later) and sinβ would be 0.3562—these would make $\alpha \approx 17.42$ and $\beta \approx 20.87$ (values obtainable in a table). Next, α and β values would be inserted into the equation, again a table would be used, and simple subtraction and division by 2 would occur—the result would yield ~0.10665158. By moving the decimal the same number of times that it was moved in order to accommodate the trigonometric equation (eight places to the right), the answer becomes 10,665,158 (approximating the actual 10,664,628). Because this answer is an estimate, the desired number of accurate digits would be dependent on the values initially given to α and β.
Performing such calculation tricks, astronomers could reduce errors and save time (Katz, 2004).

In addition to prosthaphaeresis, Napier also knew about other methods for simplifying calculations. Michael Stifel, a German mathematician, developed in 1544 a relationship between arithmetic sequences of integers and corresponding geometric sequences of 2 raised to those integers (Smith, 2000): $\{1, 2, 3, 4, \ldots, n\}$ and $\{2^1, 2^2, 2^3, 2^4, \ldots, 2^n\}$. Stifel wrote tables in which he showed that the multiplication of terms in one table correlated with addition in the other (Katz, 2004). For example, to find $2^3 \cdot 2^5$, one would add 3+5 (terms in the arithmetic sequence) and the answer could then be inserted back into the geometric sequence to obtain $2^3 \cdot 2^5 = 2^{3+5} = 2^8 = 256$. These tables were limited, however, in their calculating ability; Napier's approach to using logarithms, on the other hand, allowed the multiplication of any numbers through the use of addition (Katz, 2004).

To define logarithms, Napier used a concept that is rather different from today's perception of a logarithm.[2] Since astronomers at the time often handled calculations requiring trigonometric functions (particularly sines), Napier's goal was to make a table in which the multiplication of sines could be done by addition instead (Katz, 2004). The process consisted of having a line segment and a ray, where a point was made to move on each (from one extreme end to the other). The starting "velocity" for both points was the same, but the difference began as one point moved uniformly (arithmetically on the ray) and the other moved geometrically such that its velocity would be proportional to the distance left to travel to the endpoint of the line segment. Using this mental model, Napier defined the distance traveled by the arithmetically moving point as the logarithm of the distance remaining to be traveled by the point moving geometrically (Cajori, 1893). In Napier's words, "the logarithm of a given sine is that number which has increased arithmetically with the same velocity throughout as that with which radius began to decrease geometrically, and in the same time as radius has decreased to the given [number]" (Katz, 2004). A detailed account of the process can be seen below in the *Calculation Techniques* section. Clearly, Napier's definition differed from the modern concept of just having a base raised to the corresponding exponent.

It took Napier about 20 years to actually assemble his table of logarithms (Katz, 2004), but shortly after publishing his book, Napier was visited by the English mathematician Henry Briggs (Smith, 2000). A professor of geometry in London, Briggs was impressed with Napier's work,

> My lord, I have undertaken this long journey purposely to see your person, and to know by what engine of wit or ingenuity you came first to thing of this most excellent help in astronomy, viz. the logarithms; but, my lord, being by you found out, I wonder nobody found it out before, when now known it is so easy. (Cajori, 1893)

They both discussed the convenience of setting the logarithm of 1 equal to 0 (rather than the original 10,000,000) and setting the logarithm of 10 at 1. In this way, the more familiar form of the logarithm was born, and a common property like log (xy) = log x + log y could be used to make a new table. Napier died in 1617, so Briggs began to do the calculations to construct the table (Katz, 2004). Briggs did not convert Napier's logarithms to the new common logarithms, however. Instead he set out to calculate successive square roots to obtain the logarithms of prime numbers, and used these to calculate the logarithms of all natural numbers from 1 to 20,000 and from 90,000 to 100,000. Although he did use algorithms to obtain the roots, the amount of work needed to calculate all those logarithms is nonetheless astounding. To calculate the logarithm of 2, for instance, he carried out forty-seven successive square roots (Smith, 2000). In addition, all of the calculations for logs were carried out to 14 decimal places (Cajori, 1893). An example of the calculations needed for this task is shown below in the *Calculation Techniques* section. Finally, in 1624, Briggs published his tables in his *Arithmetica Logarithmica*. The logarithms of the numbers between 20,000 and 90,000 were calculated by the Dutchman Adrian Vlacq, who published the complete table from 1 to 100,000 in 1628 (Cajori, 1893).

Below is a page from Briggs's Arithmetica Logarithmica (MatematikSider, 2007).

10 D *ARITHMETICA* E

	Numeri continue Medij inter Denarium & Unitatem.	*Logarithmi Rationales.*
	10	1,000
1	31622,77660,16837,93319,98893,54	0,50
2	17782,79410,03892,28011,97304,13	0,25
3	13335,21432,16332,40256,65389,308	0,125
4	11547,81984,68945,81796,61918,213	0,0625
5	10746,07828,32131,74972,13817,6538	0,03125
6	10366,32928,43769,79972,90627,3131	0,01562,5
7	10181,51721,71818,18414,73723,8144	0,00781,25
8	10090,35044,84144,74377,59005,1391	0,00390,625
9	10045,07364,25446,25156,64670,6113	0,00195,3125
10	10022,51148,29291,29154,65611,7367	0,00097,65625
11	10011,24941,39987,98759,85395,51805	0,00048,82812,5
12	10005,62312,60220,86366,18495,91839	0,00024,41406,25
13	10002,81116,78778,01323,99249,64325	0,00012,20703,125
14	10001,40548,51694,72581,62767,32715	0,00006,10351,5625
15	10000,70271,78941,14355,38811,70845	0,00003,05175,78125
16	10000,35135,27746,18565,08581,37077	0,00001,52587,89062,5
17	10000,17567,48442,25738,33846,78274	0,00000,76293,94531,25
18	10000,08783,70363,46121,46574,07431	0,00000,38146,97265,625
19	10000,04392,84217,31672,36281,88083	0,00000,19073,48632,8125
20	10000,02195,91867,55542,03317,07719	0,00000,09536,74316,40625
21	10000,01097,95873,50204,09754,72940	0,00000,04768,37158,20312,5
22	10000,00548,97921,68211,14626,60250,4	0,00000,02384,18579,10156,25
23	10000,00274,48957,07382,95091,25449,9	0,00000,01192,09289,55078,125
24	10000,00137,24477,59510,83282,69572,5	0,00000,00596,04644,77539,0625
25	10000,00068,62238,56210,25737,18748,2	0,00000,00298,02322,38769,53125
26	10000,00034,31119,22218,83912,75020,8	0,00000,00149,01161,19384,76562,5
27	10000,00017,15559,59637,84719,93879,1	0,00000,00074,50580,59692,38281,25
28	10000,00008,57779,79451,03051,17588,8	0,00000,00037,25290,29846,19140,625
29	10000,00004,28889,89633,54198,42901,3	0,00000,00018,62645,14923,09570,3125
30	10000,00002,14444,94793,77767,42970,4	0,00000,00009,31322,57461,54785,15625
31	10000,00001,07222,47391,14050,76926,8	0,00000,00004,65661,28730,77392,57812,5
32	10000,00000,53611,23594,13317,14831,4	0,00000,00002,32830,64365,38696,28906,25
33	10000,00000,26805,61846,70731,51508,7	0,00000,00001,16415,32182,69348,14453,125
34	10000,00000,13402,80923,26383,99277,7	0,00000,00000,58207,66091,34674,07226,5625
35	10000,00000,06701,40461,60946,55519,6	0,00000,00000,29103,83045,67337,03613,28125
36	10000,00000,03350,70230,79911,91730,0	0,00000,00000,14551,91522,83668,51806,64062,5
37	10000,00000,01675,35115,39815,61857,6	0,00000,00000,07275,95761,41834,25903,32031,25
38	10000,00000,00837,67557,69872,72426,9	0,00000,00000,03637,97880,70917,12951,66015,625
39	10000,00000,00418,83778,84927,59087,9	0,00000,00000,01818,98940,35458,56475,83007,8125
40	10000,00000,00209,41889,42461,60262,5	0,00000,00000,00909,49470,17729,28237,91503,90625
41	10000,00000,00104,70944,71230,25311,0	0,00000,00000,00454,74735,08864,64118,95751,95312
42	10000,00000,00052,35472,35614,98950,4	0,00000,00000,00227,37367,54432,32059,47875,97656
43	10000,00000,00026,17736,17807,46048,9	0,00000,00000,00113,68683,77216,16029,73937,98828
44	10000,00000,00013,08868,08903,72167,8	0,00000,00000,00056,84341,88608,08014,86968,99414
45	10000,00000,00006,54434,04451,85869,75	0,00000,00000,00028,42170,94304,04007,43484,49707
46	10000,00000,00003,27217,02225,92881,337	0,00000,00000,00014,21085,47152,02003,71742,24853
47	10000,00000,00001,63608,51112,96427,283	0,00000,00000,00007,10542,73576,01001,85871,12426
48	10000,00000,00000,81804,25556,48210,295	0,00000,00000,00003,55271,36788,00500,92935,56213
49	10000,00000,00000,40902,12778,24104,311	0,00000,00000,00001,77635,68394,00250,46467,78106
50	10000,00000,00000,20451,06389,12051,946	0,00000,00000,00000,88817,84197,00125,23233,89053
51	10000,00000,00000,10225,53194,56025,921 *L*	0,00000,00000,00000,44408,92098,50062,61616,94526
52	10000,00000,00000,05112,76597,28012,947 *M*	0,00000,00000,00000,22204,46049,25031,30808,47263
53	10000,00000,00000,02556,38298,64006,470 *N*	0,00000,00000,00000,11102,23024,62515,65404,23631
54	10000,00000,00000,01278,19149,32003,235 *P*	0,00000,00000,00000,05551,11512,31257,82702,11815

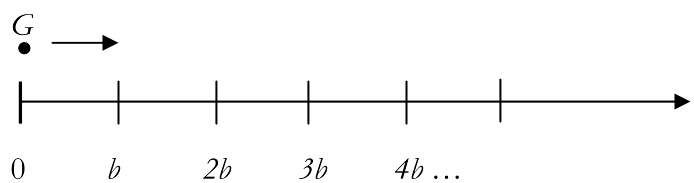
The way logarithms were viewed changed over time, and today's notation for a logarithm was developed by Leonhard Euler in the late 1700s. He related exponential and logarithmic functions by defining $\log_x y = z$ to hold true when $x^z = y$ (Smith, 2000). This definition proved very useful and found multiple applications. A classic example of a practical application of logarithms is the slide rule. In 1622 the Englishman William Oughtred made a slide rule by placing two sliding logarithmic scales next to each other. The slide rule could replace the need to look up values in a logarithm table by instead requiring values to be aligned in order to perform the multiplication, division, and many other operations (depending on the model). Up until the 1970s, with the incoming of electronic calculators, the slide rule was widely used in the fields of science and engineering (Stoll, 2006). A look at how the slide rule could be used for calculations is shown below in the *Calculation Techniques* section.

Although the common logarithm has many practical uses, another logarithm is widely used in fields ranging from calculus to biology. The natural logarithm is of the form $\log_e a = n$. The base of a logarithm could be any number larger than 1, but the use of *e* brings on various advantages (Lowan, 2002). The definition of *e*, the limit of $(1+1/n)^n$ as n approaches infinity, might seem a bit awkward at first, but it turns out that *e* not only turns up frequently in nature, but it also makes natural logarithms have the simplest derivatives of all logarithmic systems (Evans, 1939). Various solutions to applied mathematical problems can be expressed as powers of *e*: the flow of electricity through a circuit, radioactive decay, bacterial growth, etc. (Lowan, 2002). The natural logarithm arose from modifications of Napier's logarithms made by John Speidell, a mathematics teacher from England. In 1622 he published the book *New Logarith* with logarithms of tangents, sines, and secants in a format that showed natural logarithms (except that he had omitted decimal points). For example, he gave log 10= 2302584, which would be written today as $\log_e 10= 2.302584$ (Cajori, 1893). As an interesting note, the Napier log of x would be equivalent to the expression $10^7\log_{1/e}(x/10^7)$ in modern terms (Smith, 2000).

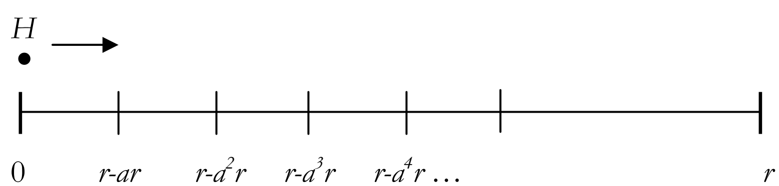
2. Calculation Techniques

- *Napier's definition of logarithms* (see Cairns, 1928, and Cajori, 1893, and Katz, 2004, and Pierce,1977)*:*

Given a ray and a line segment, the point *G* moves along the ray and the point *H* moves along the line segment.



G moves at a constant velocity by traveling *b* distance in equal time intervals (along an increasing arithmetic sequence).



H moves towards *r* in equal time intervals from 0 to *r-ar*, *r-ar* to $r-a^2r$, $r-a^2r$ to $r-a^3r$, etc. Napier made r = 10,000,000 and *a* be less than 1 (but very close to 1).

He made the line segment (from 0 to *r*) be the "sine of 90º", and the distance from *r* to *H* the sine of the arc with the distance traveled by *G* as its logarithm. Thus, Napier had $\log 10^7 = 0$.

Under this system, the notion of using bases with corresponding exponents did not apply.

In a calculus sense, Napier's logarithms could be seen as measures of "instantaneous" velocities. For example, the velocity of H could be $V_H = \Delta d/\Delta t = d(r-x)/dt$, where x is the distance remaining to be traveled by H to reach r. Similarly, the velocity of G would be $V_G = dy/dt$, where y is the distance traveled by G (this velocity is constant).

To obtain the definition of a Napier logarithm in modern calculus terms:
$d(r-x)/dt = x$, since the velocity of H is proportional to the distance remaining to be traveled by H to reach r. So, $dr/dt - dx/dt = x$, and since r is a constant (10^7):
$0 - dx/dt = x \rightarrow 1/(-dx/dt) = 1/x \rightarrow -dt/dx = 1/x \rightarrow \int -dt = \int 1/x\, dx \rightarrow -t = \ln x + c$.
Since both G and H start at the same velocity, when $t = 0$ then $x = r$, thus
$0 = \ln r + c \rightarrow c = -\ln r$, therefore, $-t = \ln x - \ln r$.

Point G progresses in an arithmetical fashion, and its velocity is dy/dt. Having established that its velocity is constant and that it is equal with H's velocity at $t = 0$, then $dy/dt = r$ so $dy = rdt \rightarrow \int dy = \int r\, dt \rightarrow y = rt$.

Finally, to relate x and y:
$-t = \ln x - \ln r \rightarrow t = \ln r - \ln x \rightarrow t = \ln (r/x) \rightarrow y = r \ln (r/x)$

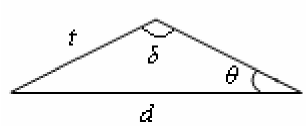
By his definition, the Napier log $x = y$ is Naplog $x = r \ln(r/x) = 10^7 \ln(10^7/x)$

Napier did not use the notion of e in calculating his logarithms, but this perspective helps to see the connection between logarithms, calculus, and the usefulness of e and the natural logarithm.

- Using Napier's Logs in calculations (see Katz, 2004):

To use his logs in calculations, Napier had to note that Naplog $10^7 = 0$.
If $j/p = w/z$, then Naplog (j) – Naplog (p) = Naplog (w) – Naplog (z).
If $f/q = q/m$, then Naplog (f) – Naplog (q) = Naplog (q) – Naplog (m) and
2Naplog (q) = Naplog (f) + Naplog (m)
And if $f/q = m/k$, then Naplog (f) + Naplog (k) = Naplog (q) + Naplog (m).

Using these properties he established, conforming to his logarithms, a triangle could be solved by reference to his tables.
Example: using the law of sines, $\sin\theta \,/\, t = \sin \delta \,/\, d$ for triangle



So to find δ the properties are applied,
Naplog ($\sin\delta$) = Naplog ($\sin\theta$) + Naplog (t) – Naplog (d)
Referring back to his tables, Napier could calculate δ by simple addition and subtraction.

- Briggs's logarithms (see Cairns, 1928 and Henderson, 1930):

Briggs adapted Napier's logs to fit log 10 = 1 instead, thus giving birth to today's common logarithms. By taking successive square roots, Briggs concluded, for example, that if

$\sqrt{10} \approx 3.162277$, then $\log 3.162277 = 0.5$
$\sqrt{\sqrt{10}} \approx 1.77828$, then $\log 1.77828 = 0.25$
$\sqrt{\sqrt{\sqrt{10}}} \approx 1.33352$, then $\log 1.33352 = 0.125$, etc.

To find the logarithms of prime numbers Briggs used the following method:
To find log 2, he noticed that if he raised 2 to a certain power, the number of digits in the result gave an approximation for log 2 (because of the properties of using logarithms with base 10); the log of a number with x number of digits is between $x - 1$ and x. For example, $2^8 = 256 \rightarrow 2 < \log 256 < 3$.
He then noted that x and $x - 1$ could be divided by the exponent to which 2 was raised to get an approximation of the log of 2:

$2^{10} = 1024$	$\rightarrow 3 < \log 1024 < 4$	so $0.3 < \log 2 < 0.4$
$2^{20} = 1048576$	$\rightarrow 6 < \log 1048576 < 7$	so $0.3 < \log 2 < 0.35$
$2^{40} \approx 1.1 \times 10^{12}$	$\rightarrow 12 < \log 2^{40} < 13$	so $0.3 < \log 2 < 0.325$
$2^{60} \approx 1.2 \times 10^{18}$	$\rightarrow 18 < \log 2^{60} < 19$	so $0.3 < \log 2 < 0.3167$
$2^{80} \approx 1.2 \times 10^{24}$	$\rightarrow 24 < \log 2^{80} < 25$	so $0.3 < \log 2 < 0.3125$
$2^{100} \approx 1.3 \times 10^{30}$	$\rightarrow 30 < \log 2^{100} < 31$	so $0.3 < \log 2 < 0.31$

…and so forth until Briggs obtained log 2 to 14 decimal places. Once he calculated the logs for other prime numbers, he followed the rules of logarithms: for example, $\log 10 = \log (2 \cdot 5) = \log 2 + \log 5$. Until his tables covered the logarithms of 1-20,000 and 90,000-100,000.

- Slide rule calculations (see Stoll, 2006):

The slide rule works by simplifying multiplications and divisions into logarithmic scale additions or subtractions. Slide rules basically print fit scales into a ruler-type setup and by just sliding a cursor against another scale, long operations can be done quickly. One could get away with using a slide rule without really understanding logarithms, but to make one, the following rules are essential:

$\log xy = \log x + \log y$ $\qquad$ $\log (x/y) = \log x - \log y$
$\log x^y = y \log x$ $\qquad$ etc.…

Briggs's logarithms allowed long operations like $10478 \cdot 97503$ to become
$\log 10478 + \log 97503 = 4.020278 + 4.989018 = 9.009296$, then
antilog $9.009296 = 10478(97503) \approx 1{,}021{,}636{,}000$.

- Natural logarithms:

Logarithms with base *e* unavoidably spring up in calculus (which was developed a little after Napier's death). To see how these logs are essential to obtain certain integrals: let $f(x) = \log_e x = \ln x$

$$f'(x) = \lim_{h\to 0} [f(x+h) - f(x)] / h = \lim_{h\to 0}[\ln(x+h) - \ln(x)] / h = \lim_{h\to 0}[\ln((x+h)/x)] / h$$

$$= \lim_{h\to 0} [\ln(1+ h/x) / h] [(1/x)/(1/x)] = \lim_{h\to 0} [1/x][\ln(1+ h/x)] / [h/x]$$

$$= \lim_{h\to 0} [1/x][x/h][\ln(1+ h/x)] = \lim_{h\to 0} [1/x][\ln(1+ h/x)^{x/h}] = [1/x]\lim_{h\to 0} \ln (1+ h/x)^{x/h}$$

$$\lim_{x/h\to \infty}$$

$= [1/x] \qquad \ln(1+ h/x)^{x/h} = [1/x] \ln e = 1/x$

Thus, $d(\ln x)/dx = 1/x$, and $\int 1/x \, dx = \ln x + c$.

The definition of e, $\lim_{n\to\infty}(1+ 1/n)^n = e$, allows the above demonstration to hold.

3. Conclusions & Implications

Today's concept of logarithms might make it seem strange that logarithms really developed out of comparing velocities of arithmetically and geometrically moving points. Napier's idea took him decades to fully develop and conclude, and the work of Briggs helped simplify and enhance a useful mathematical invention. What today seems like a simple base to exponent relationship really has a long history of work and improvements. The natural logarithm further helps us see the connection between the labors of a Scottish mathematician (and many others) with calculus and all its modern applications in math, science, and technology.

Napier's invention of the logarithm has surely left an important mark in the history of mathematics. The applications derived from the calculations he and others developed, still have relevance today. Although slide rules are now obsolete, the principles that allow them to work are not. The story of the development of logarithms is a good example of the effects that mathematical discoveries and inventions can have on society and the technological world.

In writing this paper I have learned a great deal about these calculation aids. But perhaps more importantly, I have realized that figuring out mathematical operations and tricks certainly takes significant amounts of effort, time, and devotion. Today, we often take for granted those symbols and explanations that are neatly compiled into math and science textbooks. It is easy to forget that every equation encases a story: frustration, fascination, arduous work, friendly collaborations, disappointment, and the occasional serendipity. Mathematics is not just about numbers, but it is also about the people whose work gives us the luxury and pleasure of understanding.

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