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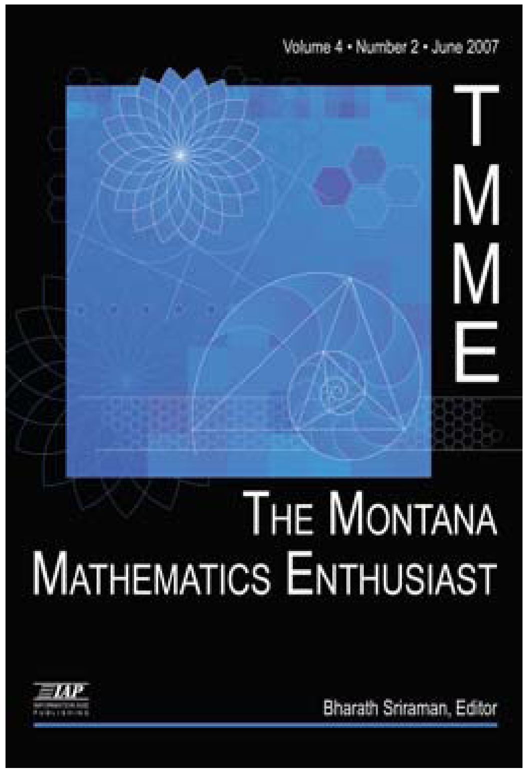

[1] Michael Pyryt passed away in an unfortunate manner on Jan 15, 2008. The journal expresses its condolences to the Pyryt family in Calgary, and regrets the loss of an excellent statistician, scholar and gentleman.

The Montana Mathematics Enthusiast is an eclectic internationally circulated peer reviewed journal which focuses on mathematics content, mathematics education research, innovation, interdisciplinary issues and pedagogy. The journal is published by Information Age Publishing and the electronic version is hosted jointly by IAP and the Department of Mathematical Sciences- The University of Montana, on behalf of MCTM. Articles appearing in the journal address issues related to mathematical thinking, teaching and learning at all levels. The focus includes specific mathematics content and advances in that area accessible to readers, as well as political, social and cultural issues related to mathematics education. Journal articles cover a wide spectrum of topics such as mathematics content (including advanced mathematics), educational studies related to mathematics, and reports of innovative pedagogical practices with the hope of stimulating dialogue between pre-service and practicing teachers, university educators and mathematicians. The journal is interested in research based articles as well as historical, philosophical, political, cross-cultural and systems perspectives on mathematics content, its teaching and learning.

The journal also includes a monograph series on special topics of interest to the community of readers The journal is accessed from 110+ countries and its readers include students of mathematics, future and practicing teachers, mathematicians, cognitive psychologists, critical theorists, mathematics/science educators, historians and philosophers of mathematics and science as well as those who pursue mathematics recreationally. The 40 member editorial board reflects this diversity. The journal exists to create a forum for argumentative and critical positions on mathematics education, and especially welcomes articles which challenge commonly held assumptions about the nature and purpose of mathematics and mathematics education. Reactions or commentaries on previously published articles are welcomed. Manuscripts are to be submitted in electronic format to the editor in APA style. The typical time period from submission to publication is 8-11 months. Please visit the journal website at http://www.montanamath.org/TMME or http://www.math.umt.edu/TMME/

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[1] The opinions expressed in this dialogue are those of the two authors and do not necessarily represent the views of the Montana Council of Teachers of Mathematics nor the position of the journal or its editorial board.

CRITICAL NOTICE ON *THE NATIONAL MATHEMATICS ADVISORY PANEL REPORT*[2]
Guest Edited by Brian Greer (Portland, USA)

FEATURED BOOK EXCERPT

In Sfard, A. (2008). *Thinking as communicating: Human development, the growth of discourses, and mathematizing.* (pp. xiii - xix) Cambridge, UK: Cambridge University Press.

Also available at
http://www.cambridge.org/us/catalogue/catalogue.asp?isbn=9780521867375&ss=fro

FORTHCOMING ARTICLES [VOL6, NOS1&2, January 2009]

[2] The positions taken in this Critical Notice on the National Mathematics Advisory Panel report are those of the authors and do not necessarily represent the views of the Montana Council of Teachers of Mathematics nor the position of the journal or its editorial board.

EDITORIAL
2 to 3- An Omnibus of features & Policy issues in mathematics education

Bharath Sriraman
The University of Montana

Six months have rolled by like a blink of the geologic eye. This simile is translatable into a good Fermi problem involving units of time and proportional reasoning. As the reader will note the journal has moved from 2 to 3 issues a year, a simple arithmetical incremental for various reasons. Submissions to the journal have steadily increased. About a year or so ago, in my editorial in vol4.no2 [June 2007], I reported receipt of 86 manuscripts in 16 months. In the ensuing 12 months the journal received 107 submissions including several from angle trisectors, circle squarers and relativity debunkers, and 10 or so manuscripts in advanced pure mathematics that were beyond the scope of the journal. Sieving these submissions aside, the journal is averaging 90 manuscripts a year, with an acceptance rate of ~30%. The increase in the flow of manuscripts has created a severe backlog of articles necessitating an increase in frequency of issues per year. Henceforth the journal will move to 3 issues per year, one double issue and one normal issue per volume.

Vol5, nos2&3 [July 2008] is the first double issue of *The Montana Mathematics Enthusiast*, consisting of ~300 pages of interesting features, dialogues and a critical notice. The feature articles cover the entire scope of topics the journal purports to address, and the diversity of the authors reveals the geographic reach of the journal. This double issue concludes with a preview of articles in the pipeline for vol6,nos 1 & 2 [January 2009] focused on ***Statistics Education and mathematics education research in South America***. We apologize to authors that will have waited for nearly 10 months to see their articles in print but as this editorial indicates, the backlog will soon be cleared up with the double issues. The journal does not have the myriad options and resources that are available through large publishing companies such as making articles available online immediately after acceptance, however the minor discomfort of waiting comes with the benefit of being unshackled to a business corporation, and having a journal that is openly accessible.

The omnibus of feature articles cover topics from the history of mathematics and science, the teaching and learning of specific and general mathematical topics from the middle school onto the university level and for the math enthusiasts three interesting math articles in the domains of geometry (Spyros, article 3), abstract algebra (Diego and Jónsdóttir, article 8), and recreational mathematics (Humble, article 13).

The opening article by Babb and Currie looks at the famous Brachistochrone Problem popularized in Polya's (1954) classic treatise on the role that analogies play in the discovery of, (or for the non-platonists) the creation of solutions to troubling problems. One may recall the elegant solution illustrated by Polya in this book, namely Bernoulli's solution to the Brachistochrone problem by constructing the appropriate analogy with the path of light in the atmosphere. Babb and Currie use the same problem "as a context stretching from Euclid through the Bernoullis" to highlight the variety of results understandable to students without a background in analytic geometry, as well as make ideas from the history of mathematics accessible to students via the use of technological tools. Michael Fried makes a more critical argument on the difficulty of using the history of mathematics in mathematics education, and examines semiotics as a useful bridge to link the two domains. In articles 5 and 6, the teaching and learning of differentiability is addressed from a historical viewpoint (Mayrargue, article 5), and from a cognitive viewpoint (Viholainen, article 6) respectively. This double issue also includes several articles on key mathematical concepts such as inverses (Lim, article 12), foundations (Bagni, article 4), pre-service and in-service mathematics education (articles 10, 11), problem-solving (Ferreira & Palhares article 7), and assessment (Warwick ,article 9). The two Montana feature articles examine the history of logarithms and the birth of insurance mathematics. These pieces may be of interest to high school teachers who are asked by students about the significance of learning logarithms.

***The Montana Mathematics Enthusiast*, ISSN 1551-3440, Vol. 5, nos.2&3**, pp.167-168

Mathematics education as a field of inquiry has a long history of intertwinement with psychology. In fact one of its early identities was as a happy marriage between mathematics (specific content) and psychology (cognition, learning and pedagogy). However the field has not only grown rapidly in the last three decades but has been heavily influenced and shaped by the social, cultural and political dimensions of education, thinking and learning. To some, these developments are a source of discomfort because they force one to re-examine the fundamental nature and purpose of mathematics education in relation to society. The social, cultural and political nature of mathematics education is undeniably important for a host of reasons such as: Why do school mathematics and the curricula repeatedly fail minorities and first peoples in numerous parts of world? Why is mathematics viewed as an irrelevant and insignificant school subject by some disadvantaged inner city youth? Why do reform efforts in mathematics curricula repeatedly fail in schools? Why are minorities and women under-represented in mathematics and science related fields? Why is mathematics education the target of so much political/policy attention?

The traditional knowledge of cultures that have managed to adapt, survive and even thrive in the harshest of environments (e.g., Inuits in Alaska/Nunavut; Aboriginals in Australia, etc) are today sought by environmental biologists and ecologists. The historical fact that numerous cultures successfully transmitted traditional knowledge to new generations suggests that teaching and learning were an integral part of these societies, yet these learners today do not succeed in the school and examination system. If these cultures seem distant, we can examine our own backyards, in the underachievement of African- Americans, Latino, Native American and socio-economically disadvantaged groups in mathematics and science. It is easy to blame these failures on the inadequacy of teachers, neglectful parents or the school system itself, and rationalize school advantage to successful/dominant socio-economic groups by appealing to concepts like special education programs, equity and meritocracy (see Brantlinger, 2003). The Dialogue included in this issue of the journal examines the No Child Left Behind Act of 2001, the political panacea which was meant to cure the ills of the American public school system and raise student achievement. The Critical Notice of the National Mathematics Advisory Panel Report guest edited by Brian Greer, includes 5 articles which analyze the findings and recommendations of the recently released report (see Greer, guest editorial). The journal welcomes reactions of the readers to the critical notice articles.

In the second edition of the *Handbook of Educational Psychology*, Calfee called for a broadening of horizons for future generations of educational psychologists with a wider exposure to theories and methodologies, instead of the traditional approach of introducing researchers to narrow theories that jive with specialized quantitative (experimental) methodologies that restrict communication among researchers within the field. Calfee also concluded the chapter with a remark that is applicable to mathematics education:

> "Barriers to fundamental change appear substantial, but the potential is intriguing. Technology brings the sparkle of innovation and opportunity but more significant are the social dimensions- the Really Important Problems (RIP's) mentioned earlier are grounded in the quest of equity and social justice, ethical dimensions perhaps voiced infrequently but fundamental to the disciple. Perhaps the third edition of the handbook will contain an entry for the topic." (Calfee, in *Handbook of Educational Psychology*, pp.39-40).

On a concluding note, I am pleased to include in this issue the Introduction of Anna Sfard's *Thinking as communicating: Human development, the growth of discourses, and mathematizing*. The book holds the promise of removing existing dichotomies in the current discourses on thinking, and may well serve as a common theoretical framework for researchers in mathematics education. Again, readers that have or will read the book are urged to submit a reaction to the interesting arguments made by Sfard. Thank you for your support of the journal. I hope you enjoy this issue!

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The Brachistochrone Problem: Mathematics for a Broad Audience via a Large Context Problem

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Abstract
Large context problems (LCP) are useful in teaching the history of science. In this article we consider the brachistochrone problem in a context stretching from Euclid through the Bernoullis. We highlight a variety of results understandable by students without a background in analytic geometry. By a judicious choice of methods and themes, large parts of the history of calculus can be made accessible to students in Humanities or Education.

Keywords: Brachistrochrone problem; Large context problems (LCP); history of science; history of mathematics

Introduction

Each year the University of Winnipeg offers several sections of an undergraduate mathematics course entitled MATH-32.2901/3 *History of Calculus* (Babb, 2005). This course examines the main ideas of calculus and surveys the historical development of these ideas and related concepts from ancient to modern times. Students of Mathematics or Physics may take the course for Humanities credit; the course surveys a significant portion of the history of ideas and in fact is cross-listed with Philosophy. On the other hand, many students in Education take *History of Calculus (H of C)* to fulfill their Mathematics requirement; it is therefore necessary that the course offer solid mathematical content. Unfortunately, a significant fraction of these latter students are weak in pre-calculus material such as analytic geometry. Nevertheless, H of C welcomes the weak and the strong students together, and covers technical as well as historical themes.

In Stinner and Williams (1998) the authors enumerate the benefits of studying **large context problems (LCP)** in making science interesting and accessible. In their words 'the LCP approach provides a vehicle for traversing what Whitehead (1967) refers to as "the path from romance to precision to generalization"(p. 19).' For H of C, a useful LCP is the **Brachistochrone Problem:** the

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solution history of finding the curve of quickest descent. A focus on the brachistochrone motivates results ranging from Greek geometry, past the kinematics of Oresme and Galileo, through Fermat and Roberval to the Bernoullis, and the birth of the calculus of variations. By careful selection of material, it is possible to find proofs accessible even to weak students, while still stimulating mathematically strong students with new content.

Quickest Descent in Galileo

One of the topics considered by Galileo Galilei in his 1638 masterpiece, *Dialogues Concerning Two New Sciences,* is rates of descent along certain curves. In Proposition V of "Naturally Accelerated Motion", he proved that descent time of a body on an inclined plane is proportional to the length of the plane, and inversely proportional to the square root of its height (Galilei 1638/1952, p.212). Denoting height by H, length by L and time by T, we would write

$$T = kL/\sqrt{H} \qquad (1)$$

where k is a constant of proportionality. In fact, as we point out to students, this formulation is slightly foreign to the thought of Galileo; for reasons of homogeneity, he only forms ratios of time to time, length to length, etc. He therefore says that the ratio of times (a dimensionless quantity) is proportional to the ratio of lengths, inversely proportional to the ratio of square roots of heights. Galileo proves (1) in a series of propositions starting with the "mean speed rule":

> 'The time in which any space is traversed by a body starting from rest and uniformly accelerated is equal to the time in which that same space would be traversed by the same body moving at a uniform speed whose value is the mean of the highest speed and the speed just before acceleration began.' (Proposition I, Galilei 1638/1952, p.205).

It is insufficiently well-known that this rule had been proven geometrically by Nicole Oresme three centuries earlier! Oresme's very accessible geometric derivation is presented early in H of C. (Babb, 2005)

Moving closer to the question of quickest descent, with his Proposition VI, Galileo established the law of chords:

> 'If from the highest or lowest point in a vertical circle there be drawn any inclined planes meeting the circumference, the times of descent along these chords are each equal to the other'. (Galilei 1638/1952, p.212)

Galileo's proofs are given in a series of geometric propositions. Unfortunately, many of our students would identify with the complaint that Galileo places in the mouth of Simplicio:

> 'Your demonstration proceeds too rapidly and, it seems to me, you keep on assuming that all of Euclid's theorems are as familiar and available to me as his first axioms, which is far from true.' (Galilei 1638/1952, p. 239)

Happily, in an earlier part of H of C dealing with Greek mathematics, some geometrical rudiments are established. In particular, students see a proof of Thales' theorem – an angle inscribed in a semi-

circle is a right angle – using similar triangles. This allows the following demonstration of the law of chords:

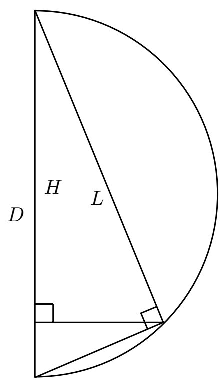

Figure 1: The law of chords

Let a circle of diameter D have a chord of length L and height H inscribed as shown in Figure 1. Let the descent time along the chord be T. By similar triangles L/H = D/L, so that L^2 /H = D. Then by (1),

$$T = kL/\sqrt{H} = k\sqrt{D} \qquad (2)$$

which is indeed constant. Q.E.D.

The law of chords can also be demonstrated by deriving expressions for the velocities and descent time, and noting that the expression for descent time is independent of the upper point of the chord along the circular arc. The expression for descent time may be obtained by considering the component of the force of gravity along the inclined chord. Alternatively, Nahin (2004, pp. 202-206) derives the descent time by applying the principle of conservation of mechanical energy. Our brief geometrical derivation is rather close to the spirit of Galileo's proof, and is accessible to students without a physics background.

Lattery discusses an interesting approach suggested by Matthews in 1994 for leading students towards a derivation of the[1] law of chords. Students are asked to consider the following thought experiment:

> 'Suppose a ball is released at some point A on the perimeter of a vertical circle and rolls down a ramp to point B, the lowest point on the circle … The ramp may be rotated about point B. For what angle will the time of descent along chord AB be the least? '
> (Lattery, 2001, p. 485)

This way of posing the problem helps students to greater appreciate the surprising result. It also allows the option of discussing the distinction between sliding and rolling motion. H of C is a mathematics course and has no laboratory component. Nevertheless, (particularly for students with weak physical intuition) it is useful to be able to observe the results studied in various descent problems. As experimentalists know, however, the design and operation of physics demonstration

apparatus can be as much art as science. For this reason, we have chosen to use physically realistic computer simulations to illustrate various theorems. A computer program in MAPLE allows freedom in the choice of curves studied, as well as the possibility of speeding, slowing or freezing demonstrations. Figure 2 shows two screen snapshots from the program. Portability is one more advantage of this approach, to go along with flexibility of use.

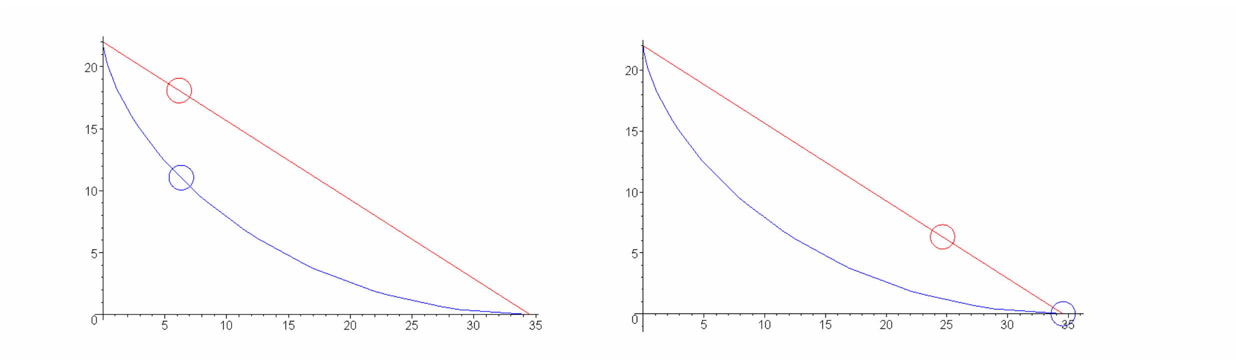

Figure 2: Screen snapshots from *MAPLE* race: cycloid vs. straight-line ramp

Details of the simulation are closely related to Jakob Bernoulli's solution of the brachistochrone problem, and are detailed in a later section.

With his Proposition XXXVI, Galileo proved that the descent time from a point on the lower quadrant of a circle to the bottom is quicker along two consecutive chords than along a direct chord. He began his proof by a clever application of the law of chords, but then completed it by a fairly involved geometric argument. In H of C, the proof is completed using a shorter method proposed by Erlichson (1998), based on conservation of mechanical energy.

Three Curves

Three curves of major interest to the mathematicians of the seventeenth century were the cycloid, the isochrone and the brachistochrone. (See, for example, Eves, 1990, p. 426.) The definitions of these curves are *kinematic*; as students learn in H of C, the acceptance of curves defined via motion was part of a mathematical revolution in the seventeenth century. A *cycloid* is the curve traced by a point on the circumference of a circle as the circle rolls, without slipping, along a straight line. A *brachistochrone* from point A to point B is a curve along which a free-sliding particle will descend more quickly than on any other AB-curve. (It is thus an optimal shape for components of a slide or roller coaster, as we inform our students.) An *isochrone* is a curve along which a particle always has the same descent time, regardless of its starting point. A surprising discovery was that these three curves are one and the same!

Galileo may have been the first to consider the problem of finding the path of quickest descent. This is suggested by the initial statement of his Scholium to Proposition XXXVI:

> 'From the preceding it is possible to infer that the path of quickest descent from one point to another is not the shortest path, namely, a straight line, but the arc of a circle'.
> (Galilei 1638/1952, p.234)

Many researchers, such as Stillman Drake and Herman Goldstine, have concluded that Galileo incorrectly claimed that a circular arc is the general curve of quickest descent (Erlichson 1998, Erlichson 1999). However, Erlichson (1998, p.344), argues that Galileo restricted himself to descent paths that used points along a circle. Galileo's claim is based on an argument that the descent time for a particle along a twice-broken path is less than along a twice-broken path, and that the descent time along a multiply-broken path would be even less. According to Nahin (2004, pp. 208-209), Galileo's claim is correct, but his reasoning was flawed; Erlichson (1998) noted that Galileo's method of proving Proposition XXXVI holds for descent from rest, but fails to generalize to situations in which a particle is initially moving.

In fact, as we have mentioned, the brachistochrone is not the circle; it is a segment of an inverted cycloid. The cycloid was discovered in the early sixteenth century by the mathematician Charles Bouvelles (Cooke, 1997, p. 331). In the 1590s, Galileo conjectured and empirically demonstrated that the area under one arch of the cycloid is approximately three times the area of the generating circle. That the area is exactly three times that of the generating circle was proven by Roberval in 1634 and by Torricelli in 1644 (Boyer and Merzbach, 1991, p. 356). Roberval constructed the tangent to the cycloid in 1634 (Struik, 1969, p. 232). According to Cooke (1997, p. 331), constructions of the tangent to the cycloid were independently discovered circa 1638 by Descartes, Fermat and, and slightly later by Torricelli. In 1659, Christopher Wren also determined the length of a cycloidal arch, showing it to be exactly four times the diameter of the generating circle (Stillwell, 2002, p. 318). In 1659, Huygens discovered that the cycloid is a solution to the isochrone or tautochrone problem; he showed that a particle sliding on a cycloid will exhibit simple harmonic motion with period independent of the starting point (Stillwell, 2002, p. 238). Huygens published his discovery of the cycloidal pendulum in *Horologium oscillatorium* in 1673 (Boyer and Merzbach, 1991, p. 379). The cycloidal pendulum also features in Newton's Principia (Gauld, 2005). In 1696, Johann Bernoulli demonstrated that a brachistochrone is a cycloid (Erlichson, 1999).

Most of these discoveries concerning the cycloid are inaccessible to students with no calculus background. Remarkably, Roberval's historical construction (Struik, 1969, pp. 234-235) of a tangent to the cycloid is quite accessible to students, as it uses only the parallelogram law for vector addition. The result is presented in H of C:

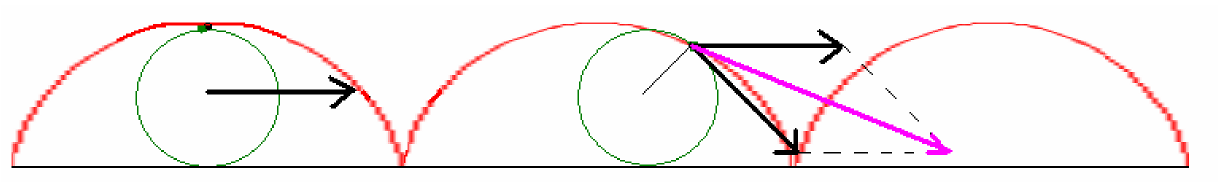

Figure 3. Finding a tangent to the cycloid

The result can almost be presented "without words". (See Figure 3.) Follow the path of the "tracing" point on the generating circle of a cycloid. The motion of this point at a given instant has a horizontal component, corresponding to the horizontal motion of the center of the circle; it also has a component normal to a radius of the circle, since the circle rolls. These components are of equal magnitude, since the circle rolls without slipping. The resultant of the motions is found by the parallelogram law, and is the tangent to the cycloid.

It was early in 1696 that Johann Bernoulli solved the problem of finding the curve of quickest descent; he showed that the brachistochrone was a cycloid. Later, in June of that year, he posed the problem in the journal *Acta Eruditorum*

> ' PROBLEMA NOVUM, ad cujus Solutione Mathematici invitantur.
> " Datis in plano verticali duobus punctis *A* et *B,* assignare mobili *M* viam *AMB,* per quam gravitate sua descendens, et moveri incipiens a puncto *A,* brevissimo tempore perveniat ad alterum punctum *B.*"' (Woodhouse 1810, pp.2 – 3)

An English translation is as follows:

> 'If two points A and B are given in a vertical plane, to assign to a mobile particle M the path AMB along which, descending under its own weight, it passes from the point A to the point B in the briefest time.' (Smith, D.E. 1929, p.644)

The problem was also solved by Jakob Bernoulli, Leibniz, L'Hôpital and Newton. Newton's solution was published anonymously in the *Philosophical Transactions of the Royal Society* in January, 1697. Solutions by Johann Bernoulli, Jakob Bernoulli, Leibniz and Newton were published in *Acta Eruditorum* in May, 1697. According to Stillwell (2002, p. 239), the most profound was Jakob Bernoulli's solution, which represented a key step in the development of the calculus of variations. The historical development of what became the calculus of variations is closely linked to certain minimization principles in physics, namely, **the principle of least distance**, **the principle of least time**, and ultimately, **the principle of least action**. (See Kline, 1972, pp. 572-582.) To understand Johann Bernoulli's solution of the brachistochrone problem, students in H of C are led through Fermat's **principle of least time**: light always takes a path that minimizes travel time.

Principle of Least Time

An accessible application of the principle of least time is in deriving the **law of reflection**: if a ray of light strikes a mirror, then the angle of incidence equals the angle of reflection. This law was first noted by Euclid in the fourth century BCE (Ronchi, 1957, p. 11), and was explained using a principle of least distance by Heron of Alexandria in the first century CE (Cooke, 1997, p. 149). In the H of C course, the law of reflection is derived geometrically, as per Heron. Since the speed of light (in a fixed medium) is constant, this is equivalent to a derivation from the principle of least time.

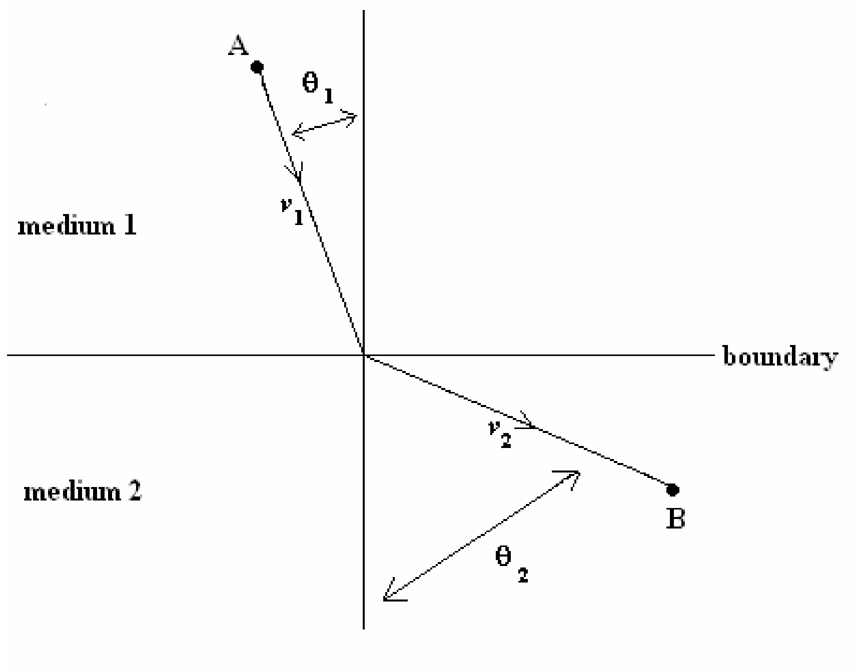

Figure 4: Snell's law of refraction

The **law of refraction** states that when a ray of light crosses the boundary between transparent media, it experiences a change in direction characterized by the relation

$$\sin \theta_1 / \sin \theta_2 = k$$

in which θ_1 is the angle of incidence, θ_2 is the angle of refraction and k is a constant dependent on the nature of the two media. (See Figure 4.) This law was discovered experimentally by the Dutch physicist Willebrord Snel circa 1621; in English it is known as **Snell's law**. Snel noticed that if the first medium is less dense than the second, then $k > 1$; that is, upon entering the second medium, the light ray bends toward the normal to the boundary. (Nahin, 2004, p. 103)

In the mid-seventeenth century, Fermat demonstrated that Snell's law of refraction may be derived from the principle of least time. In the H of C course, such a derivation is given using an abbreviation of the approach outlined by Nahin: Consider a ray of light crossing the boundary between transparent media. For $i = 1, 2$ let v_i denote the speed of light in medium i. Referring to Figure 5, let T denote the transit time for a light ray travelling from point A in medium 1, through point B at the boundary, to point C in medium 2. Then

$$T = (length\ of\ AB)/v_1 + (length\ of\ BC)/v_2 = \sqrt{h_1^{\ 2} + x^2} \Big/ v_1 + \sqrt{h_2^{\ 2} + (d-x)^2} \Big/ v_2$$

To obtain the path of least time, it is necessary to determine x so that the transit time T is minimized.

$$dT/dx = x\Big/v_1\sqrt{h_1^{\,2}+x^2} - (d-x)\Big/v_2\sqrt{h_2^{\,2}+(d-x)^2} = \sin\theta_1/v_1 - \sin\vartheta_2/v_2$$

The necessary condition for a minimum, namely that $dT / dx = 0$, yields the requirement that

$$\sin\theta_1 / \sin\theta_2 = v_1 / v_2$$

Thus, Snell's media-dependent constant is $k = v_1 / v_2$.

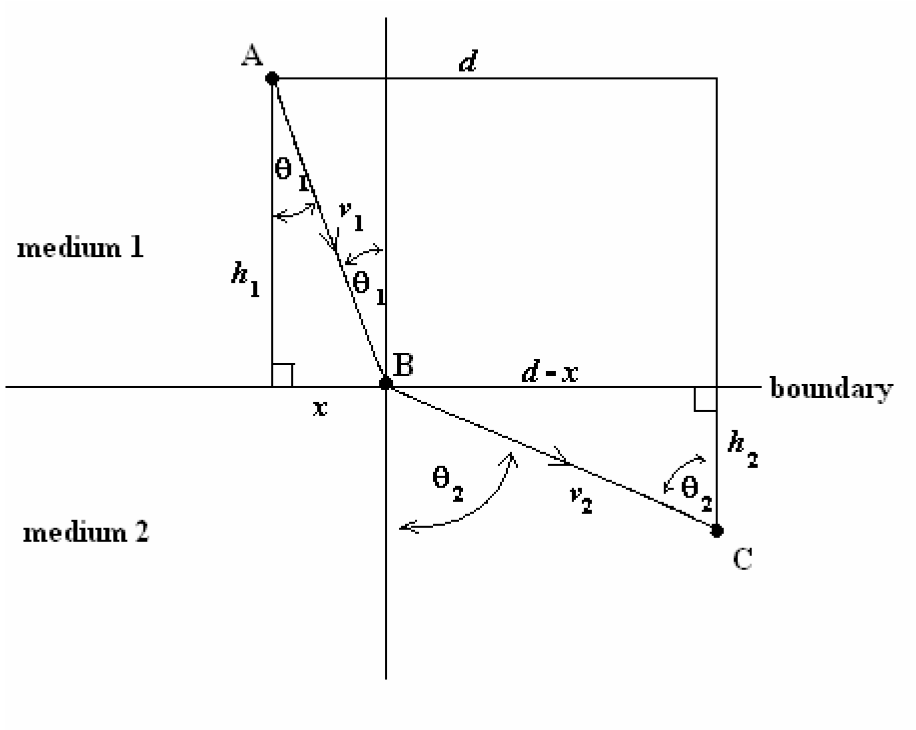

Figure 5: Derivation of Snell's law

It should be noted that Fermat achieved the minimization using his **method of adequality**, which is comparable to differentiation; however, he had to introduce some approximations, since he could not apply his adequality method directly to expressions involving square roots. Nahin (2004, p. 127-134) gives a detailed presentation of Fermat's solution.

Johann Bernoulli's Solution to the Brachistochrone Problem

In the brachistochrone problem, an ideal particle traverses an AB-curve under gravity. The traversal time will be determined if we can fix the speed of the particle at each point along its path. The principle of conservation of mechanical energy implies that if the particle starts at rest, and the vertical drop from A to a point is y, then the particle will acquire a speed at the given point of

$$v = (2\,g\,y)^{1/2} \qquad (3)$$

This speed is independent of whether the particle has dropped vertically, moved along an inclined line, or followed some more complicated path. The brachistochrone problem thus becomes the following:

> A particle moves from A to B in such a way that whenever its vertical drop from A is y, its speed is given by (3). Find the AB-curve with the shortest traversal time.

Johann Bernoulli solved the problem via a brilliant thought experiment. Consider a non-uniform optical medium which becomes increasingly less dense from top to bottom. If light enters from above, its speed becomes faster and faster as it moves down. By a judicious varying of the density, light may be constrained to travel through this medium in a manner satisfying (3). However, by the principle of least time, in any situation, light will always travel along a path with the shortest traversal time. We therefore see that if A is located at the top of this non-uniform medium, and B at the bottom, the path taken by light travelling from A to B is the brachistochrone!

This leaves the question of how light will travel through our non-uniform medium. Consider a light ray travelling through two transparent media, from point A in medium 1 (upper) to point B in medium 2 (lower). Let θ_1 denote the angle of incidence and θ_2 the angle of refraction. Let v_1 and v_2 denote the speed of light in the respective media. Suppose that medium 2 is less dense than medium 1, so that $v_2 > v_1$. Then, since

$$\sin\theta_1 / \sin\theta_2 = v_1 / v_2 < 1$$

$\theta_2 > \theta_1$, and the light ray bends away from the normal to the boundary. (See Figure 4.) Note, also, that

$$\sin\theta_1 / v_1 = \sin\theta_2 / v_2 = \text{constant}$$

Now, consider a similar situation with a light ray traveling downward through many layered transparent media, with each medium less dense than the layer above it. The speed of light increases in the successive media as it progresses through deeper layers and the ray of light bends further away from each successive normal to the boundary at point of contact. (See Figure 6.)

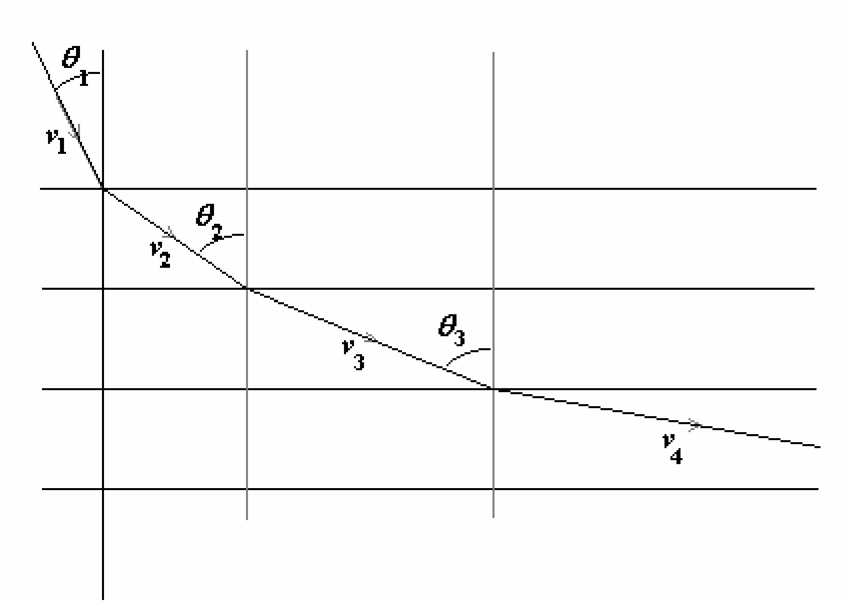

Figure 6: Refraction through multiple layers

Note also, that:

$$\sin \theta_1 / v_1 = \sin \theta_2 / v_2 = \sin \theta_3 / v_3 = \ldots = \text{constant}$$

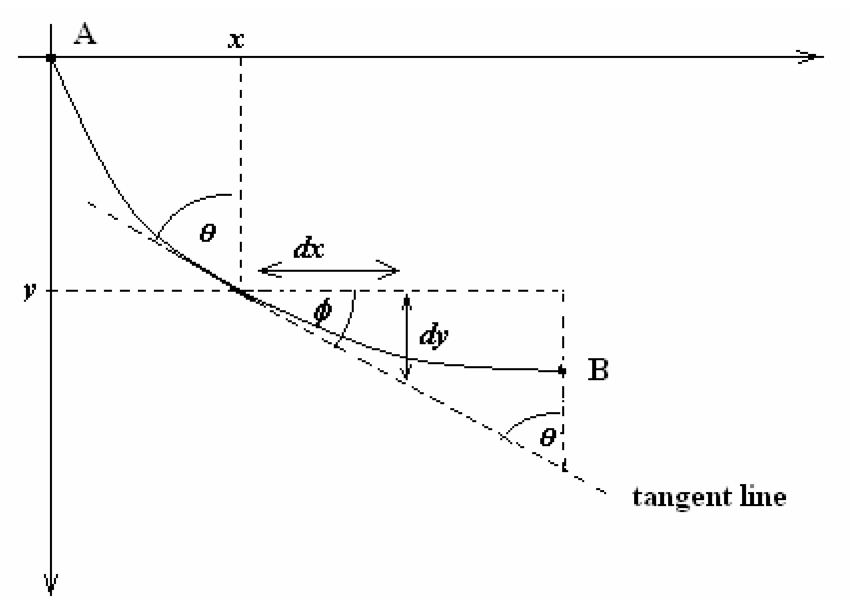

Figure 7: Johann Bernoulli's proof

Letting the number of layers increase without bound and the thickness of each layer decrease towards zero, the path of the light ray becomes a smooth curve. (See Figure 7.) At each point along the curve

$$\sin \theta / v = c \tag{4}$$

Bernoulli thus realized that a particle falling along the curve of quickest descent from A to B must satisfy both equations (3) and (4). From the triangle in Figure 7

$$\sin\theta = \cos\phi = 1 / \sec\phi = 1 / [1 + \tan^2\phi]^{1/2} = 1 / [1 + (dy/dx)^2]^{1/2}$$

Thus

$$\sin\theta / v = \text{constant} = 1 / (v\,[1 + (dy / dx)^2]^{1/2})$$

where $v = (2gy)^{1/2}$. This yields the following nonlinear differential equation:

$$y\,[\,1 + (dy / dx)^2\,] = k \qquad (5)$$

where k is some constant. Algebraic manipulation of the differential quantities, dx and dy, yields the equation

$$dx = dy\,[\,y / (k - y)\,]^{1/2} \qquad (6)$$

which Bernoulli recognized as a differential equation describing a cycloid. In the translated words of Johann Bernoulli:

> 'from which I conclude that the Brachystochrone is the ordinary Cycloid'
> (Struik 1969, p .394 translation of Johann Bernoulli 1697)

Note that in Bernoulli's May 1697 paper in *Acta Eruditorum*, the usual labelling of x and y coordinates is reversed. Equation (6) may be further manipulated to obtain parametric equations for x and y; for details, see the excellent accounts by Simmons (1972), Erlichson (1999) and Nahin (2004).

Jakob Bernoulli's Solution to the Brachistochrone Problem

Although Johann Bernoulli's solution to the brachistochrone problem impresses us with its elegance, it is tailored to a very specific application. Jakob Bernoulli's more methodical approach generalizes, and in fact became the basis of the calculus of variations. In fact, neither of the Bernoullis' solutions uses calculus explicitly in the foreground. Each of the brothers, by a different method, sets up a differential equation, and having found this equation, declares the problem solved. (Struik, 1969, pp. 392-399). Their quickness at this early date to recognize a differential equation of the cycloid is striking!

Since the calculus is in the background, Jakob Bernoulli's solution to the cycloid may be outlined to H of C students. We give our abbreviated presentation of his proof below. It relies on the use of similar triangles, some (typical) hand-waving regarding infinitesimals, and the concept of *stationary points* of functions.

A *stationary point* of a function is one at which the function's rate of change is zero. In the ordinary calculus, we recognize that local extrema of a function occur at stationary points. Jakob Bernoulli

extended this idea to the brachistochrone problem. Since the brachistochrone minimizes descent time, the rate of change of descent time must be zero with respect to infinitesimal variation of the brachistochrone path. Consider Figure 8.

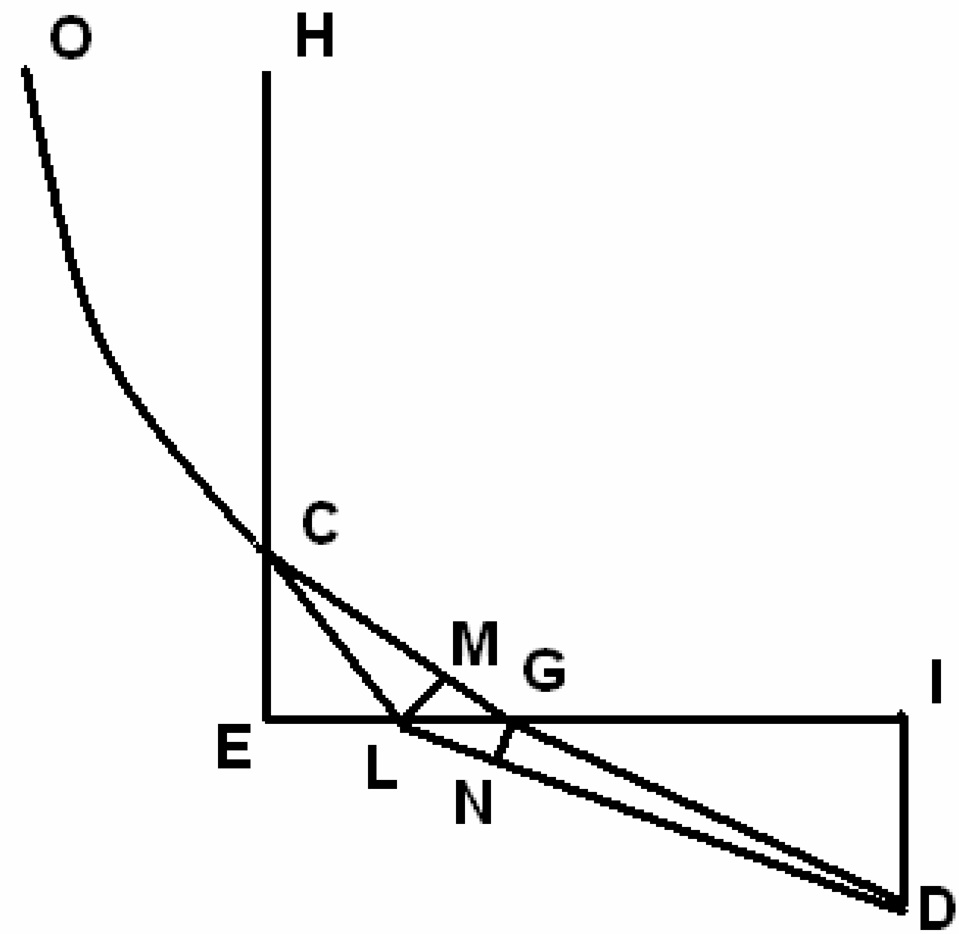

Figure 8: Jakob Bernoulli's solution of the brachistochrone problem

Let curve *OCGD* be a small section of the brachistochrone. Letting y measure vertical drop from *O*, choose units so that a particle moves along the curve with instantaneous speed $\sqrt{y}$ at any point. We consider *CG* to be so short that a particle moves along *CG* with constant speed $\sqrt{|HC|}$ where $|HC|$ denotes the length of *HC*. Similarly, we assume a constant speed $\sqrt{|HE|}$ on *GD*.

Vary the path by moving *G* an infinitesimal distance horizontally, to *L*. As the brachistochrone is stationary, the descent time along *OCLD* must also be minimal. Add construction lines *ML* and *NG* such that triangles ΔCML and ΔDNG are isosceles. The descent times along *CM* and *CL* are thus equal, as are descent times along *GD* and *ND*. As the total descent times along *OCGD* and *OCLD* are to be equal, the descent time along *MG* must equal that along *LN,* and

$$|MG|/\sqrt{|HC|} = |LN|/\sqrt{|HE|}. \tag{7}$$

As we are dealing with infinitesimal distances, we may consider *ML* to be an arc of a circle centered at *C*, and *LMG* to be a right angle. By similar triangles, then, $|MG| \ / \ |LG| = |EG| \ / \ |CG|$. If we let *x* measure horizontal distance, and *s* arc length along the brachistochrone, this can be rewritten as

$|MG| \ / \ |LG| = dx \ / \ ds$ on segment *CG*.

By an analogous argument,

$|LN| \ / \ |LG| = dx \ / \ ds$ on segment *GD*.

Dividing by $\sqrt{y}$ on each of segments *CG* and *GD*, and applying equation (7), we find that

$$dx/\sqrt{y}ds = k \tag{8}$$

with the same constant *k* on both segment *CG* and segment *GD*. We conclude that equation (8) holds everywhere on the brachistochrone. Jakob Bernoulli recognized this as a differential equation of the cycloid.

Note that various mysteries involving infinitesimals take place; segment *CG* is only short, while *LG* is infinitesimal. Also, isosceles triangle ΔCML contains two right angles, and this is essential to the argument with similar triangles. Again, Bernoulli interchanged *x* and *y*, which can be confusing to a modern reader.

Computerized Demonstration of the Brachistochrone

It may be difficult for students to grasp the nature of the minimization problem involved in finding the brachistochrone. We are not finding the tangent or area of some particular *given* curve, as is usually the case in calculus. Instead, we must search among *all possible hypothetical curves* to find that which allows least time descent. The nature of the problem also precludes physical demonstration; we may build an apparatus to demonstrate sliding descent on a particular curve, or we may "race" physical beads on two or three particular curves; however, it is hard to conceive how one would physically demonstrate beads descending on enough curves to allow students to conceptualize descent on an *arbitrary* curve. Here the computer comes to the rescue:

Using the symbolic programming language *MAPLE*, we simulate the descent under gravity of a particle along a curve as follows:

1. Curves are given parametrically: $x = x(s), y = y(s), a \le s \le b$.
2. A given curve is systematically sampled at $n + 1$ points $s_i = a + i\,(b - a)/n$, $i = 0, 1, \ldots, n$.
3. The curve is modeled by straight-line segments, from (x_i, y_i) to (x_{i+1}, y_{i+1}), $i = 0, 1, \ldots, n - 1$.
4. The motion of a particle under gravity down the digitized curve is modeled in a straightforward way: If the particle enters the straight-line segment from (x_i, y_i) to (x_{i+1}, y_{i+1})

moving at speed v_0, its acceleration under gravity will be $a = g \sin \theta$, where g is the acceleration (downward) due to gravity, and $\sin \theta = (y_i - y_{i+1}) / \sqrt{(x_i - x_{i+1})^2 + (y_i - y_{i+1})^2}$. The length of the segment is $D = \sqrt{(x_i - x_{i+1})^2 + (y_i - y_{i+1})^2}$. At time t after entering the segment, the particle has moved distance $d = v_0 t + \frac{1}{2} a t^2$ along the segment. The time T spent traversing the segment is then the solution of $D = v_0 T + \frac{1}{2} a T^2$, so that the particle enters the next segment with a speed of $v_0 + aT$.

5. The positions of particles on several digitized curves can thus be worked out as parametric functions of time. The *MAPLE* command *animate*() is then capable of presenting trajectories of two or more of these particles simultaneously, as they each move along their underlying curves. Our code as currently implemented contrasts the motion of particles along two desired curves.

We have found that $n = 11$ already gives very smooth-looking approximations. Using our software, we "race", for example, a particle on a cycloid arc against a particle on a straight-line ramp (see Figure 2), or a particle on a circular arc. We can in fact race along any (parametric) curve suggested by students. The same software will illustrate Galileo's law of chords. After students have seen Johann Bernoulli's solution to the brachistochrone, they can appreciate the analogy between modern "digitization" and Bernoulli's layers.

Conclusion

We have traced the thread of quickest descent problems and the brachistochrone from Galileo, through Fermat and Roberval, to the Bernoullis and the dawn of the calculus of variations. We have spelled out in detail a selection of mathematical results which we have presented to our H of C students. These results include mathematical content ranging in difficulty from geometry, through vectors, to differential equations. A themed unit on the cycloid shows students at very different levels the strong interplay between mathematics and physics: Geometry informs optics (Fermat), optics informs kinematics (Johann Bernoulli), and kinematics informs geometry (Roberval).

Of particular interest is Johann Bernoulli's beautiful *Gedankenexperiment*, whereby a falling particle becomes a ray of light, moving through media arranged without regard for the possibility of actual physical construction. Again, the purely mathematical and hypothetical nature of the frictionless bead in the brachistochrone problem motivates our software demonstrations to students. The digital sampling of curves in our *MAPLE* code echoes the (finite) layering of media by Johann Bernoulli. A careful examination of the arguments of the Bernoullis introduces students to the interesting philosophical and technical issues related to infinitesimals/differentials in mathematics and physics.

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History of Mathematics in Mathematics Education: a Saussurean Perspective

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For Y. Tobin

INTRODUCTION

It is not only because of a certain eclecticism in mathematics education research that semiotic ideas have begun to take root there: it is also because of the dawning recognition that key areas of interest in mathematics education genuinely have a semiotic nature. For this, one need only point to research focused on meaning, communication, language, and culture (*e.g.*, Presmeg, 1997; Ernest,1997; Radford, 2001; Brown, 2001). The ways in which semiotics informs cultural aspects of mathematics education, particularly those connected with the history of mathematics, were highlighted in a discussion session at the 2003 meeting of the International Group for the Psychology of Mathematics Education (PME27). The present paper continues the theme of the PME27 semiotics session by considering how semiotics can help clarify a problematic aspect of combining history of mathematics and mathematics education.

Before outlining that particular problem, I need to say in what corner of the semiotic field I plan to dwell. For semiotics has become a very large and many-colored thing. Indeed, the history of semiotics, the study of signs, can be traced quite far back into the past—as far back even as Classical times. At the end of the 17th century, Locke made the study of signs one of the three basic sciences "within the compass of human understanding," and he gave this study the name, *Semeoitica* (Locke, 1694, Book IV, chapter xx).[2] In the 18th and 19th centuries, figures (of equal or greater importance in the history of mathematics!) such as Lambert and Bolzano also made serious studies of semiotics (see Jakobson, 1980). But the figures we most associate with the modern study of semiotics are, surely, Charles Sanders Peirce (1839-1914) and Ferdinand de Saussure (1857-1913). Peirce's semiotic ideas have been quite influential, and, perhaps because of his exhaustive taxonomic approach to the subject, they have found broad application far beyond logic and linguistics. Saussure's semiological[3] ideas seem to have had somewhat less influence than Peirce's, even in

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[2] Locke describes the this science as follows: ...the third branch [of science] may be called *Semioitica*, or the *doctrine of signs*; the most usual whereof being words, it is aptly enough termed also logic: the business whereof is to consider the nature of signs, the mind makes use of for the understanding of things, or conveying its knowledge to others. For, since the things the mind contemplates are none of them, besides itself, present to the understanding, it is necessary that something else, as a sign or representation of the thing it considers, should be present to it: and these are ideas…(Locke, 1694, Book IV, chapter xx).

[3] Whereas Peirce called the general study of signs, *semiotic*, Saussure called it *sémiologie*—both, as was Locke's term, were based on the Greek word for 'sign' (which, incidentally, is also the word for 'point' in mathematical contexts) ⧫♒❍♏♓☃❑■.

linguistics where Saussure thought they should have their greatest impact.[4] In thinking about mathematics as a cultural system, however, Saussure's ideas have a cogency, which, to my mind, has not been sufficiently appreciated. Thus, in this paper, I will show how a specifically Saussurean approach to semiotics provides insight into the problem of history of mathematics and mathematics education.

The paper will comprise four parts. In the first part I shall present the theoretical difficulty in combining the history of mathematics and mathematics education. Next, I shall review some of Saussure's semiotic ideas, particularly, his notion of a sign and his notions of synchrony and diachrony. In the third part I shall show how these ideas are apposite to the difficulty presented in the first part. Finally, I shall consider how the semiological outlook discussed in the paper may serve as a foundation for a more humanistically oriented mathematics education.

1. THE PROBLEM OF HISTORY OF MATHEMATICS AND MATHEMATICS EDUCATION[5]

The main problem to be addressed in this section is whether there really is a problem at all in combining the history of mathematics and mathematics education. If there is a problem, most educators, I think it is fair to say, would consider it more practical than theoretical. In other words, what one has to worry about in combining history and mathematics education is chiefly how to do it: What examples should one chose for what material? What kind of history of mathematics activities can be incorporated into the ordinary mathematics curriculum? How does one find time for such activities? How does one find a place for history of mathematics in teacher training? Yet, when one follows these questions to their theoretical end, one begins to see a theoretical problem.

Take, for example, the problem of time, for this would appear the most practical of these practical problems. Avital (1995, p.7) says in this connection, "Teachers may ask 'Where do I find the time to teach history?' The best answer is: 'You do not need any extra time'. Just give an historical problem directly related to the topic you are teaching; tell where it comes from; and send the students to read up its history on their own". Whether or not Avital's solution can truly be called a solution is moot; nevertheless, it illustrates a general approach for combining history of mathematics and mathematics education, a strategy called elsewhere (Fried, 2001) the strategy of addition, namely, a strategy whereby history of mathematics is added to the curriculum by means of historical anecdotes, short biographies, isolated problems, and so on. Another general strategy which also answers the teachers' question "Where do I find the time to teach history?" is the strategy of accommodation (Fried, 2001), namely, using an historical development in one's explanation of a technique or idea or organizing subject matter according to an historical scheme. This strategy finds time by economizing, that is, teaching history and the obligatory classroom material at one stroke. Accordingly, Katz (1995), for example, suggests introducing the logarithm following Napier's geometric-kinematic scheme, arguing that it brings out functional properties of the logarithm important for precalculus students.

What is essential to observe is that however one solves the problem of time the obligatory classroom material must not be neglected. For this reason, in Avital's scheme, it is history of mathematics that must be pursued after school; history of mathematics is not "the topic you are teaching." For Katz, on the other hand, history can be taught as part of the lesson because it brings out ideas important for precalculus students—presumably, ideas belonging to the non-history-oriented curriculum. The point is that the pressure of a practical constraint, like time, forces one to subordinate history to the essential mathematics which teachers are committed to teach, that is, to algebra, geometry, calculus and the other subjects students need for more advanced study of mathematics as well as for engineering and the exact sciences. The mathematics educator must filter out from the history of mathematics what it is relevant from what is irrelevant and what is useful

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[4] In this connection, the linguist Yishai Tobin refers to the Saussurean revolution that never took place! (Tobin, 1990, p.13).

[5] The ideas in this section are explored in much greater detail in Fried (2000, 2001).

from what is not. Indeed, a sign of this subordination is the more than occasional references to history of mathematics as something to be *used*,[6] rather than, I would add, as something to be studied in its own right.

The situation in which history is used to promote modern ends (in our case, the teaching of modern mathematics) is recognizable to the historian; historiographers call this way of doing history 'anachronical' (Kragh, 1987, p.89) or 'Whiggish' (Butterfield, (1931/1951). Herbert Butterfield, who invented the latter term,[7] says that "The study of the past with one eye, so to speak, upon the present is the source of all sins and sophistries in history, starting with the simplest of them, the anachronism...And it is the essence of what we mean by the word 'unhistorical'" (Butterfield, (1931/1951, pp.30-31). That statement may seem extreme, yet it is crucial to emphasize that the vigor of Butterfield's objection is not merely the vigor of a purist; it is a statement of what it means to do history at all. The commitment to avoid anachronism, to avoid measuring the past according to a modern scale of values, goes to the very heart of the historical enterprise. The history of ideas, which includes the history of mathematics, is dedicated to understanding how ideas change and, therefore, must begin by viewing past thought as truly different from modern thought. Writing about the difficulty of understanding Greek thought, W. C. Guthrie makes the nature of this task clear. He say, "…to get inside their minds requires a real effort, for it means unthinking much that has become part and parcel of our mental equipment so that we carry it about with us unquestioningly and for the most part unconsciously" (Guthrie, 1975, p.3).[8]

Using history of mathematics in mathematics education thus entails a clash of commitments. Mathematics educators are committed to presenting *modern* mathematics[9]; they are committed to students' understanding of the mathematical techniques and concepts used so powerfully in so many applications and fields of study; they are committed to students feeling at home with modern mathematics and recognizing it as a tool within their power to use. Historians of mathematics are interested in *shaking off* modern mathematics, "unthinking much that has become part and parcel of [their] mental equipment," as Guthrie put it; they are committed to seeing how the mathematics of the past diverges from mathematics as it is understood today; for them, mathematics is never just mathematics, a coherent body of knowledge that changes only by accumulation. If it were not for this clash of commitments, one might well argue that using the history of mathematics in the service of mathematics education is no different than, say, using examples from physics to illustrate mathematical concepts. As it is, it appears that the difference between history of mathematics and the ends of mathematics education reflects two opposing sets of norms—in fact, two different ways of looking at mathematics.

Thus, in Fried (2001), the theoretical problem was presented as a dilemma: if one is true to the commitments of mathematics education one is forced to adopt a Whiggist brand of history, *i.e.* history which is not history, whereas if one is true to the commitments of the history of mathematics one is forced to spend time on mathematical and philosophical ideas which may not be relevant to the general mathematics curriculum. While the argument may be sound, it is also deeply dissatisfying. One reason this is so is that it pits mathematics education against the history of mathematics—one discipline against another—instead of seeing two disciplines both focused on mathematics. Taking up this latter position one is led away from a conflict between disciplines and towards a view of mathematics itself as thing with two aspects: on the one hand, a

[6] Indeed, one often sees the word "use" in the literature on the subject as in the titles "The Use of History of Mathematics In Teaching", "Using History in Mathematics Education", "Using the History of Calculus to Teach Calculus", "Using Problems from the History of Mathematics in Classroom Instruction", "Improved Teaching of the Calculus Through the Use of Historical Materials", to name a few.

[7] The term refers to a stream of British political history in which events of the past were conceived as steps progressing inexorably towards the democratic ideals that the Whigs held dear—as if the Whig party was the true *telos* of English history!

[8] This is also implicit in Collingwood's famous dictum that "the historian must re-enact the past in his own mind" (Collingwood, 1993, p.282ff).

[9] Counterexamples may exist, but judging from the great majority of mathematics curricula this does not seem to me a naïve generalization.

beautiful and coherent system of ideas, and, on the other hand, a living human creation whose ideas and ends are always changing and being redefined.

It is in clarifying this view of mathematics, as well as clarifying the basic dilemma, that Saussure's semiological ideas are helpful. For, to the extent that mathematics is a sign system, or at least exists within a sign system, Saussure would say that mathematics *must* have this double nature. Certainly, this will not solve all the difficulties in combining mathematics education and history of mathematics, but it points the way to a view of mathematics that makes an understanding of the history of mathematics as well as the usual school subjects *essential* to understanding mathematics as a whole. So, let us turn now to Saussure's theory of signs.

2. SAUSSURE'S SEMIOLOGY

The chief work by which we know Saussure's thought is the *Cours de Linguistique Générale* (Saussure, 1974).[10] The name of the work already hints at its revolutionary character. For it was not a work about a particular language, like Greek or Latin, nor some generalization from a particular language, like classical studies of grammar, nor yet a comparison between languages; for Saussure, linguistics meant *general* linguistics and as such it aimed to understand, in the words which end the book, "…language, considered in and for its own sake." Ironically, this end could not be attained without seeing language within an domain going beyond language, a science which "studies the life of signs at the heart of social life" (Saussure, 1974, p.33). This science of signs Saussure called *semiology*. More specifically, he writes:

> [Semiology] would teach us in what signs consist and what laws rule them. Since it does not yet exists, one cannot say what it will be; but it has the right to be, its place has been determined in advance. Linguistics is only a part of this general science; the laws that semiology will discover will be applicable in linguistics, and linguistics will thus find itself linked to a domain well defined within the ensemble of human realities (faits humains)" (*ibid*)

So, what is a sign? A linguistic sign is the result of coupling a concept with an 'acoustic image' (*ibid*. p.98). Schematically, this coupling can be represented by the following figure given in the *Cours*:

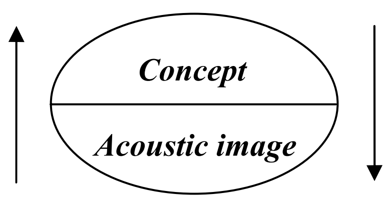


Fig.1

To understand the figure, it must be stressed, to start, that the 'acoustic image' is not itself a sound but a mental pattern of a sound.[11] Once formed, then, a sign exists as a 'psychological entity', with the 'acoustic image' no less in the mind than the concept. Moreover, as the arrows in Saussure's figure show, concepts and 'acoustic images' are mutually formative: concepts are not preexisting things *named* by means of sounds, and sound patterns *by themselves* are not embodiments of concepts; each justifies the other.[12] It may have been to highlight this fact that Saussure, almost immediately after introducing the notion of a sign, shifted his

[10] The work was not written by Saussure himself but was a compilation of notes by two students, Charles Bally and Albert Sechehaye, who attended Saussure's lectures in general linguistics at the University of Geneva during the years 1906-1911. The book was published in 1916, three years after Saussure's death.

Unless noted otherwise, the translations from the *Cours* are my own.

[11] It must be understood, also, that while the 'acoustic image' may be a word, it does not have to be; it may be, for example, an intonation, or part of a word, or a phrase.

[12] It is Saussure's view that sound and thought are as inseparable in language as are the two sides of a single piece of paper, to use his metaphor (*Cours*, p.157).

terminology from concept and 'acoustic image' to 'signified' (the *signifiè*) and 'signifier' (the *signifiant*).[13] It is clear, though, that the generality of the latter terms also serves to bring out the generality of the sign and distinguish it from the specifically linguistic sign.

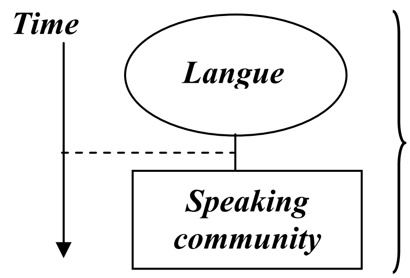
For Saussure, the crucial and most fundamental property of signs is that they are *arbitrary*, that is, no natural, logical, or any other universal law determines what signifier will be coupled with what thing signified. The reason this is so important for Saussure is that if it were not so one would have to admit that there is some kind of *natural* language; one could then identify that language as Greek or Latin or Sanskrit, and linguistics would again be the study of a particular language instead of language in general. The arbitrariness of signs, therefore, is not only the condition for a true diversity of languages, but also, paradoxically, for Saussure's dream of a general linguistics.
But this arbitrariness, as Saussure points out more than once, is not capriciousness; the sign is not subject to anybody's whim. Why is this? It is because of the way signs rest on a social foundation, or, in the case of linguistic signs, on a community of speakers. Thus Sausssure says, "Language at no moment, and contrary to appearances, exists except as a social fact, for it is a semiological phenomenon" (p.112). Indeed, because there is no determinative law of correspondence that joins signifier to signified the formation of the sign can only be a social act, just as the cementing of a tradition is. An individual can change a sign no more easily than an individual can alter a tradition; signs like traditions are made immutable by the "collective inertia" of society; they become, after they have been socially formed and internalized, social and psychological facts.

And yet societies change and languages change; signs too must change, for, like society itself, signs and the systems which comprise them exist in time. Time plays a double role: the semiological choice, the social coupling of signified and signifier, is fixed over time, but within time the relationship between signified and signifier shifts[14] and new signs are formed. Thus Saussure initially tempts the reader with a picture (illustrated by a diagram to that effect) of language as an entity resting on a social foundation, but he then shows that, without time, that picture is incomplete. He summarizes his position as follows:

> Since the linguistic sign is arbitrary, it may seem that language [here and throughout this passage the word is *la langue*], thus defined, would be a free system, organizible by will, dependant only on a rational principle. Its social character, considered in itself, is not directly opposed to this point of view. Without doubt, collective psychology does not operate on purely logical material; it must take into account everything which bends thinking (*fléchire la raison*) in practical relations between individuals. And yet it is not this that prevents us from viewing language as a simple convention, able to be modified to the liking of those interested; it is the action of time combined with that of social force; no conclusion is possible except that outside of the passage of time (*la durée*) linguistic reality is not complete.
>
> If one takes language in time, without the community of speakers (*la masse parlante*)—say, an isolated individual living over the course of many centuries—one could perhaps have no change; time would have no effect on it [language]. On the other hand, if one considered the community of speakers without time, one would not see the effect of social forces at work on language. In order to keep things close to reality we must add to our first scheme [captured by the diagram alluded to above] some sign indicating the passing of time:

[13] Initially, Saussure says only that the shift is to prevent confusion between the 'acoustic image', or whatever else calls the concept to mind, and the sign itself; however, subsequently he says that the new terms have the advantage of showing the difference between the parts of the sign and how they fit together into a unity. This supports my claim that in changing his terminology Saussure had something more fundamental in mind than convenience.

[14] Semiological change, for Saussure is always of this form, always a *shift* in the relationship between signified and signifier ("un déplacement du rapport entre le signifié et le significant" (p.109)).



(Fig. 2)

> So, language is not free because time allows social forces to exercise their effect on language, and one comes to the principle of continuity, which precludes freedom. But continuity necessarily implies change, a shift of relations of greater or lesser extent. (p.113)

The passage just quoted ends a chapter in the *Cours* entitled "Invariability and Variability of the Sign" ("Immutabilité et Mutabilité du Signe"). But how can a sign be *both* invariable and variable? It looks as if the notion of a sign leads us into a contradiction. To find our way out of the contradiction, Saussure's remark in the passage above that "if one considered the community of speakers without time, one would not see the effect of social forces at work on language" is helpful. An a-historical view of language, that is, a view of language at a single moment in the life of a society, presents language as a static perfectly coherent system of signs[15]; the effect of social forces is not evident because only the interrelations among signs, and not their formation, is apparent. When we allow history to enter the picture, the picture completely changes. Language appears from the historical point of view always in flux; we perceive that the "river of language flows without interruption" (p.193); it is in this view that we see society *making* its semiological choices, as opposed to the previous view in which we see the result of the choices having been made. So, it is not that language is characterized by contradictory properties but that it can be seen from two points of view. Saussure calls these, respectively, the *synchronic* and *diachronic* viewpoints.

Saussure pictures these viewpoints as two axes: the synchronic viewpoint is indicated by the axis of simultaneity (AB in fig. 3) and the diachronic view point by the axis of succession (CD in fig. 3).

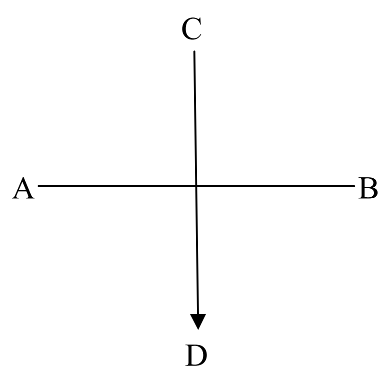


Fig. 3

Naturally, it is only the axis of succession which has an arrow attached to it; every line perpendicular to the axis of succession, that is, every line parallel to the axis of simultaneity, is a *static* linguistic state. The essential characteristic of this scheme is that, by having two perpendicular axes, it is made graphically clear that the

[15] By 'language' I am referring to what Saussure means by *la langue*, the 'language system', as opposed to the set of language acts, *la parole*.

viewpoints are irreducible.[16] But other aspects of the analogy are also suggestive. For example, note that different states, in this representation, are lines with no points in common. According to Saussure, a sign system changes sign by sign. Change, in other words, is never initiated from the system itself but from an individual sign. However, because the signs form a system, a change in a single sign must cause the entire system to change. He says, by comparison, that if the weight of a planet in the solar system were to change, the entire solar system would have make a corresponding adjustment (p.121). Saussure's favorite analogy in this regard, however, is that comparing the succession of synchronic states of a sign system with the succession of positions in a chess game (pp.125-127).[17] The transition in a chess game from position to position occurs as single moves of the chess pieces are made. However, with each move the entire position changes, that is, the relative value of each piece changes: what was safe is now in danger, what was insignificant now threatens to win the game, what was protected as essential is now sacrificed, and so on. So, the lines representing successive states of the sign system have no points in common because although most of the signs in different states may *look* the same, the change in state being initiated by changes only in individual signs, the relative values of the signs within the system have all shifted[18]; in this sense, *all* of the signs change with every change of the sign system.

The representation of the synchronic and diachronic viewpoints as pair of axes brings out, as I have said, the important recognition that the one viewpoint cannot be reduced to the other. It must also be said, however, that language, or any other sign system, cannot be understood entirely by means of one viewpoint or the other alone; the diachronic viewpoint, say, may take on a subsidiary role to the synchronic in one's study of language, but it can never be completely eliminated from one's considerations. Saussure makes this clear with yet another beautiful analogy. He says that the relationship between the two different viewpoints is like two different sections of a plant stem, one crosswise and one longitudinal.[19] Neither section could ever look like the other, yet they are interdependent. Moreover, neither section alone shows the plant in its entirety (nor could any other single section); it could never said that one section is the 'true' section, that is, that one section more truly represents the plant.

Synchrony and diachrony are essential semiological facts of life; without them one would be left with a contradictory notion of invariable signs that nevertheless evolve. But one must not get the impression that Saussure introduced synchrony and diachrony for the sake of logical consistency alone. On the contrary, that there be perfectly coherent semiotic states which are at the same time products of history is an unassailable requirement of any semiological system. For if anything at all is true about such systems it is that they are meant for communication and that they are produced by intelligent beings within a society, particularly, by human beings in human society; but where there is communication there must be coherence, and where there is human production there is history. With that, we must return now to our main concern, the problem of history of mathematics and mathematics education.

3. MATHEMATICS EDUCATION AND THE HISTORY OF MATHEMATICS IN THE LIGHT OF SAUSSURE'S NOTIONS OF SYNCHRONY AND DIACHRONY

[16] By contrast, Jakobson, it should be pointed out, did not see the division between the two points of view as absolute, but always intermingling and interacting (see Holenstein, 1976, pp.25-33).

[17] Saussure points out one weakness in the analogy, namely, that the changes in a chess game, unlike those in language, follow the *intentions* of the players. Harris (1987) points out further weaknesses in the analogy, the most serious of which, in my opinion, is that the "possible states of the board and possible moves are alike governed by the rules of chess, which exist independently of the course of any particular game" (Harris, 1987, p.93). But this does not effect the main points of the analogy: 1) that the state of the board changes move by move and 2) that with every move the entire position changes.

[18] It ought to be mentioned that the word 'value' here is carefully chosen by Saussure. He calls language "a system of pure values" (p.116). One wants to replace the word 'value' with 'meaning', but 'meaning' hints at something that stands outside the system; 'value' is always something relative to other elements of the system; one might say the 'value' of a sign is its 'relative meaning'.

[19] Not only a description but also a drawing of the stem is included in the *Cours* (p.125).

By now the application of Saussure's ideas to the problem of mathematics education and history of mathematics ought to be fairly obvious. In teaching mathematics we present a synchronic view of the subject. We teach concepts, techniques, and procedures as parts of a fixed coherent system: when we teach the meanings of words such as 'function', 'derivative' or 'continuity' or use certain kinds of figures or representations in the classroom we are engaged in a semiological activity in which we have firmly in mind how these signs—for they *are* all signs—are to be placed in a system of mutual relationships. It is not at all surprising, therefore, that when speaking about mathematical understanding, Skemp (1978), for example, emphasized relational understanding and compared it to learning a map; I would stress too that the image implies that there is a genuine unchanging map to be learned.[20] There is nothing reproachable about this. The problem arises when we take up the history of mathematics, that is, a diachronic view of mathematics, without shifting ourselves away from the synchronic view which guided us previously. The result of this is that the history of mathematics becomes viewed not as an account of different ways of thinking, of truly distinct systems of thought, but as an account of a single expansive non-temporal system whose ideas are linked by immutable relations.

Saussure was well aware of the distortions that can arise from failing to distinguish adequately the synchronic from the diachronic viewpoints, but he was also aware of how easy it is to fall into the trap of *not* distinguishing between them. He writes: "Synchronic truth so agrees with diachronic truth that one may confound them or consider it superfluous to make a distinction between them" (p.136). To show what can go wrong, he takes as an example a common explanation for how the Latin verb, *faciō* ('I make') is related to the Latin verb *conficiō* ('I bring about, complete, make ready'). The relationship is explained by saying that the short *a* in *faciō* becomes *i* in *conficiō* since the position of the *a* in the compound *conficiō* is not in the first syllable. Thus, the relationship is understood as a direct one:

$$\textit{faciō} \longrightarrow \textit{conficiō}$$

But this is, indeed, a distortion of the relationship: the *a* in *faciō*, Saussure maintains, never *became* the *i* in *conficiō*. The way the situation must be understood is that there were "two historical periods and four terms" (pp.136-137): in one period, period A, the language community employed a pair of related signs, *faciō* and *confaciō*, while a second period, period B, it employed a second pair of related signs, *faciō* and *conficiō*. In the passage from period A to period B, the sign *faciō* remained the same while *confaciō* was transformed into *conficiō*. Saussure represents the situation schematically as follows:

$$\begin{array}{ccccl} \textit{faciō} & \longleftrightarrow & \textit{confaciō} & & \text{Period A} \\ \downarrow & & \downarrow & & \\ \textit{faciō} & \longleftrightarrow & \textit{conficiō} & & \text{Period B} \end{array}$$

Fig. 4

Is it is important to add, however, that although *faciō* has remained the same in the passage from period A to period B it has a new identity[21] within the system of signs; for this reason, I believe, Saussure speaks of *four* terms and not three. The point is important for it reemphasizes Saussure's position that while the sign system changes sign by sign each such change brings with it a transformation of the entire system.

Mathematics is particularly prone to the confusion Saussure speaks of; it is not at all uncommon in mathematical contexts that "synchronic truth so agrees with diachronic truth" that one is tempted to account

[20] This comment could also be made with respect to Davis's analogy between understanding in mathematics and the piecing together of a puzzle (Davis, 1992)—one must assume that the puzzle is a fixed, though perhaps, ultimately unattainable whole. Incidentally, Toulmin (1960) similarly likened the building of a scientific theory to the making of a map. From which it follows that to the extent that we present students with scientific theories (and *a fortiori* mathematical theories) we are presenting them with a map to be grasped.

[21] Saussure would say it has a different 'value' within the system, as described above.

for mathematical change by quasi-logical explanations befitting terms in a single synchronic plane. To make the comparison with Saussure's discussion clearer, take the example of *faciō* and *conficiō* as a model and imagine there to be a synchronic plane representing the mathematics of some period in the past, historical period A, and another plane representing the mathematics of the present. In the first plane, two signs, *a* and *b*, exists within the mathematical community of period A (see fig. 5); the signs contrast with one another and, therefore, define one another. In the second plane, two signs, a' and b', similarly exist within the present mathematical community. With *a* and a' as corresponding terms (*b* and b' may or may not correspond), like *faciō* in period A and *faciō* in period B above, a connection is made between *a* and b' by projecting these terms, as it were, into an ideal a-temporal plane, so that the sign *a* from the past appears to be related *directly* to the present sign b':

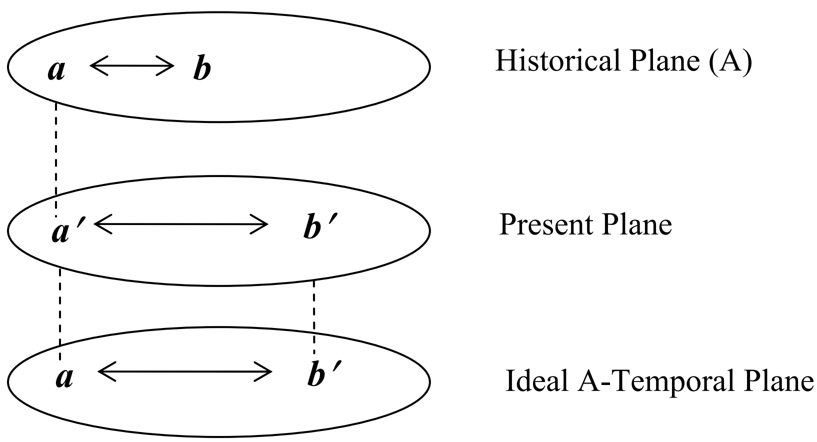


Fig. 5

Projecting the historical and present planes onto one ideal plane produces not one but two distortions. First, as in Saussure's discussion, it distorts the process by which the sign b' is born, that is, it produces a distortion from the point of view of diachrony. However, it also distorts the *synchronic* relationships in the historic plane, for it contrives a direct relationship between sign *a*, belonging to period A, and a sign b′, not belonging to period A. This is, in fact, none other than what was described above as 'anachronism'. For this reason, although we have referred to the diachronic viewpoint as the historical viewpoint, it is this second distortion which is the most serious one for the historian. Indeed, in trying to understand the processes by which mathematical ideas grow, the historian of mathematics must understand, first of all, how such ideas appeared to the mathematical communities using them; one might say the historian of mathematics, or any other historian of ideas, is interested in synchronies of the past.[22]

The problem is that to justify the introduction of history of mathematics into the classroom mathematics educators teaching a standard curriculum must be sure that the historical material is *relevant* to the curricular subjects. In effect, they are *forced* to relate *a* to b' and to ignore the relationship between *a* and *b*. Think of the difficulty of speaking about Euler's notion of a 'function' in a modern classroom studying functions.

[22] It should be kept in mind that I am using the words 'synchrony' and 'diachrony' as Saussure uses them. In historiography, by contrast, a 'diachronic' approach indicates something closer to what I have described here, namely, an approach directed towards "synchronies of the past." Thus, Kragh (1987, p.90) writes, "…in the diachronical perspective one imagines oneself to be an observer *in* the past, not just *of* the past. This fictitious journey backwards in time has the result that the memory of the historian-observer is cleansed of all knowledge that comes from later periods."

The modern notion of a function entails any pairing between all the elements of one set and the elements of another with the sole condition that any given element of the first set is paired with only one element of the second. To illustrate the generality of this formulation, it is common for teachers and textbooks to present functions having graphs not only like fig.6a but also like fig. 6b and 6c:

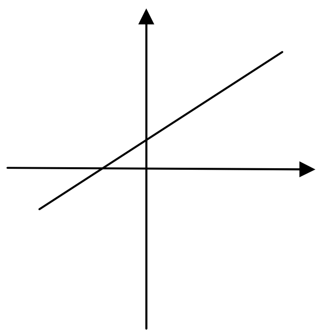


Fig.6a

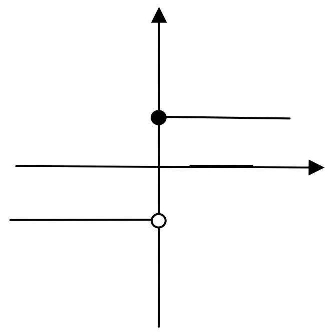


Fig.6b

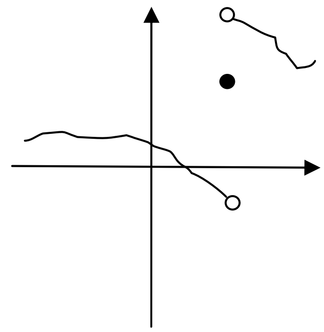


Fig.6c

To bring out this point further, the teacher might want to mention that a function, as Euler viewed understood it, had to be given by a single analytic rule,[23] so that, for him, fig. 6b and fig. 6c would not be considered true functions; in fact, functions defined on a split-domain would not have been considered true functions, nor, in general, would discontinuous functions have been (see Sui, 1995; Kleiner, 1993; Youschkevitch, 1976). The pedagogical strategy here is obvious: students have to expand their view of a function to include functions such as those represented by fig. 6b and 6c; this process is eased and made tangible by showing that the history of mathematics itself followed the same course having similarly to expand its view of what could be called a function. There is some truth in saying that the history of mathematics followed this course and some sense in the general pedagogical strategy. But this apparent correspondence between what happens in history and what steps students must follow to learn some mathematical concept in fact only demonstrates Saussure's observation that "Synchronic truth agrees to such a degree with diachronic truth that one may confound them or consider it superfluous to make a distinction between them."

The problem is, again, one of forgetting that our 'function', 'continuity', 'discontinuity' and Euler's 'function', 'continuity', 'discontinuity' are *different* signs in separate synchronous sign systems. Consider the statement "Euler's notion of function was such that discontinuous functions, in general, were not considered by him to be true functions." An 18^{th} century mathematician would agree with us that this is a true statement. But what does it mean? What *we* mean by it is that functions like those in fig. 6b and 6c were, in general, excluded by Euler. For Euler, however, 'function' signifies a single analytic expression, a single rule, involving a variable and numbers; 'continuous' signifies the continuous application of the rule to the variable throughout the domain of the variable. Thus, the relationship between 'function' and 'continuous' is not one of genus and species as it is for us but almost one of definition. For the same reason, the 'discontinuity' of the function represented by fig. 6b has not to do with the break in the graph but with the break and change in the rule at $x=0$. Hence, for Euler, the function $f(x)=1/x$, for example, is 'continuous' (Kleiner, 1993). The trouble with functions on a split-domain, on the other hand, is that they are 'discontinuous'.[24] Moreover, it

[23] Euler's 1748 definition of a function in his *Introductio in analysin infinitorum* was: "Functio quantitatis variabilis est expressio analytica quomodocunque composita ex illa quantitate variabili et numeris seu quantitibus constantibus" ("A function of a variable quantity is an analytic expression composed in any way from this variable quantity and numbers or constant quantities")

[24] More precisely, Euler called such functions 'mixed'; a 'discontinuous' function was function that was not governed by a rule, and, for this reason, problematic as a 'function' at all.

cannot be said strictly that the notion of function *expanded* to include discontinuous functions; it ought to be said, rather, that 'continuity' has shifted its meaning.[25]

It if all this were only a matter of historical accuracy or completeness, one could suggest simply that teachers 'learn the facts' or, since there is no end to accuracy, that they 'just do their best' and not worry too much about satisfying everyone's standards of accuracy. But the difficulty lies deeper than this. Whether or not one recalls Euler's notion of a 'mixed function' *is* a matter of historical completeness; however, the failure to recognize 'function', 'continuity', and so on as signs in a synchronous sign system different from ours is in some sense to miss the entire historical picture. But this, in its turn, means not understanding present day mathematics as itself set within a specific synchronous sign system existing at a certain moment in time. Why this is important is connected with what it means for there to be such sign systems to begin with: to reiterate, it means that there is a coherent and functioning system of communication produced and used by a community of human beings. It is because of this that studying history of mathematics, in the sense of studying past synchronies (and, accordingly, modern ones as well), can humanize mathematics while providing insight into mathematics as a system of ideas. And this being one of the central arguments adduced for incorporating history of mathematics in mathematics education, one begins to see what is at stake in grasping fully the diachrony/synchrony distinction.

4. IMPLICATIONS FOR TEACHING: THE CHALLENGE OF A HUMANISTICALLY ORIENTED MATHEMATICS EDUCATION

Despite the discussion above, one might still ask whether Saussure's semiological ideas have really placed us in a better position to confront the problem of history of mathematics and mathematics education, or have they only suggested a restatement of the problem? Have we escaped the dilemma between adopting a synchronic view of mathematics where we ignore or, worse, distort history and adopting a diachronic view where we risk leading students away from the main tasks and goals of the standard curriculum? Has not turning to Saussure made the dilemma even more trenchant? After all, it is Saussure who says that "The opposition between the two points of view—the synchronic and the diachronic—is absolute and brooks no compromise" (p.119) and that in studying language one must make a choice between the synchronic and diachronic approaches. Must a mathematics teacher, then, choose between 'mathematics' and 'history of mathematics'? To conclude that from Saussure would be to misunderstand his intention.

Saussure, as discussed in part 2 of this paper, took great pains to show that neither the diachronic nor synchronic view is alone the true view of language, that they are complementary views of language. When Saussure said that one must make a choice between the synchronic and diachronic approaches he only meant that when linguists take up linguistic questions they cannot disregard the point of view from which they frame and investigate them. I do not believe he thought linguists must commit themselves entirely to one or the other approach. But what about *teachers* of linguistics? For this, we have Saussure himself as an example, for we must not forget that the *Cours* records the work of a teacher. In view of this, it is obvious that Saussure believed that understanding 'language' entails understanding both the diachronic and the synchronic aspects of language. So, if the application of Saussure's ideas to the teaching of mathematics is truly valid, we must conclude that teaching 'mathematics' also demands presenting both its diachronic and synchronic aspects; far

[25] Cauchy, for example, pointed out in 1844 that 'mixed' functions such as

$$y = \begin{cases} = x, & x \geq 0 \\ = -x, & x < 0 \end{cases}$$

could be written as a single analytic expression, in this case $y = \sqrt{x^2}$. Hence, as Youschkevitch (1976, pp.72) put it, "the discrimination between *mixed* and *continuous* functions proved theoretically untenable" (Youschkevitch, 1976, pp.72). Therefore, attaching to' function' the idea of a 'continuous' rule became itself problematic.

from having to choose between 'mathematics' and 'history of mathematics' the teacher must give attention to both. This, then, is the first lesson we learn from Saussure.

The second lesson, which is deeply entwined with the first, concerns how teachers ought to treat mathematics as a subject. Recall, Saussure's semiological approach to language provided two important insights: 1) To the extent that the sign is arbitrary, the semiological process is cultural or social; 2) To the extent that, synchronically, signs form a system, semiotic systems have coherence and logic. Applied to mathematics, the second insight offers no great surprise; the first, however, tells us that mathematics, as a product of human activity, is a humanistic subject. Furthermore, Saussure makes us realize that these two insights are linked, for it is within the social framework that signs become fixed, that they become 'immutable'. The application of Saussure's thought to mathematics education suggests, then, not only that mathematics *is* a humanistic subject, but also that it *ought* to be viewed that way.[26] This, then, is the second lesson we learn from Saussure.

Of course, viewing mathematics education in a humanistic vein should not necessarily help students solve any specific mathematical problem. It should, however, inculcate in them a sense that words like 'creativity', 'inventiveness', 'vision' are appropriate for thinking about mathematics. It should show students that mathematics is something that human beings *do*—and this doing includes more than solving problems or proving theorems; it is also includes creating concepts and formulating ideas, that is, as human acts. This was not the approach taken in the example given above concerning functions and continuity. True, an historical figure, Euler, was introduced into the lesson; however, he was not portrayed as formulating a concept which he called a 'function', but only as taking a step towards the modern notion of 'function', as if the latter were somehow a 'natural' concept.[27] We should rather ask, what is the system of ideas that forces Euler to use the sign 'function' as he does? By doing so, we encourage students to ask what is the system of ideas that makes our use of the sign 'function' connected with the sort of representations in figs. 6a-c. In this way, we do not present a 'function' as some object given in advance, but as a sign produced by human beings and possessing a meaning to be interpreted.

The move to a humanistically oriented mathematics education is one answer to the problem of history of mathematics and mathematics education. Is it a radical answer? To answer this question recall that the dilemma faced by teachers of mathematics with regards to the history of mathematics took the form of a clash of commitments. If the move to a humanistically oriented mathematics education requires teachers' changing their commitments as mathematics teachers or altering the framework of standard curricula which express those commitments, then the answer is radical. But if we consider what has just been said about how historical ideas might have been brought into a discussion of functions without history becoming a-historical, we realize that history can play a part in the classroom without the material and focus of our mathematics teaching becoming radically altered. What is altered is a kind of background sense of the mathematical subjects we are teaching; the human origin of mathematical ideas, which the serious study of history brings out supremely, becomes *subsidiary*, to use Polanyi's term (Polanyi, 1958), to the study of the usual topics. Thus, a humanistic mathematics education will not deprive students of the knowledge of the 'state of the art' but will make them realize that the art is, indeed, in a certain, though not necessarily permanent, state.

[26] Somewhat different arguments for a mathematics as a humanistic subject are given in Fried (2003), as well as in the many papers found in White (1993).

[27] To say that there are 'natural' concepts (or, as Saussure says, 'panchronic' concepts), *i.e*. concepts which are not the result of human making, is to adopt a version of Platonism. As a philosophy of mathematics, Platonism is legitimate of course; however, since it assumes that mathematical concepts exist *independently* of human activity, Platonism cannot be consistent with the history of mathematics *qua* history.

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Comparison of Geometric Figures

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Abstract

Although the geometric equality of figures has already been studied thoroughly, little work has been done about the comparison of unequal figures. We are used to compare only similar figures but would it be meaningful to compare non similar ones? In this paper we attempt to build a context where it is possible to compare even non similar figures. Adopting Klein's view for the Euclidean Geometry, we defined a relation "$\leq$" as: $S_1 \leq S_2$ whenever there is a Euclidean isometry $f:\mathbb{R}^2 \to \mathbb{R}^2$, so that $f(S_1) \subseteq S_2$. This relation is not an order because there are figures (subsets of $\mathbb{R}^2$) S_1, S_2 so that $S_1 \leq S_2$, $S_2 \leq S_1$ and S_1, S_2 not geometrically equal. Our goal is to avoid this paradox and to track down non-trivial classes of figures where the relation "$\leq$" becomes, at least, a partial order. For example there is no paradox if we restrict our attention just to compact figures; thus, we can compare a closed disc with a closed triangular region. Further we present some other "good" classes of figures and we extend our study to the Hyperbolic and to the Elliptic geometry. Eventually, there are still some open and quite challenging issues, which we present at the last part of the paper.

Keywords: Euclidean geometry; Isometries; Klein; Relations

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1. INTRODUCTION

In Euclid, the geometric equality depends on the capability of superposition of the figures:

Common notion 4

Things which coincide with one another are equal to one another. ([12])

The geometric equality, with respect to Klein's view, depends on the group theory as well as on the set theory:

Definition

Let a set $X \neq \varnothing$, G a subgroup of Aut(X) and the figures $S_1, S_2 \subseteq X$. We shall say that these figures are $G-$geometrically equal if and only if there is an $f \in G$ so that $f(S_1) = S_2$.

The equality, indirectly defines the inequality of geometric figures. Euclid considers that a figure is smaller than another one if with an appropriate rigid motion the first coincides with part of the second. Although for any two figures S_1, S_2 it is easy to decide whether they are equal or not, however it is not that simple to decide if one of them is "smaller" than the other. Obviously a triangular region is never equal to a circular disk, but can we say that a triangular region is smaller than a circular disc if the radius of the disc is greater than or equal to the radius of the circumscribed circle of the triangle? In Euclid, the comparison involves only "similar" figures. On the contrary, Klein's view of equality, prompts us to define a geometric inequality using the notion of "being subset" and enables us to compare even non-similar figures:

We will say that S_1 is equal to or smaller than S_2 whenever there is a euclidean rigid motion f so that $f(S_1) \subseteq S_2$. Then we will write $S_1 \leq S_2$.

This "natural" definition of inequality provides a paradox as we will immediately illustrate using the following example given by the professor V. Nestorides:
Let us consider a closed half plane *A* and let *B* be the half plane *A* with a line segment attached vertically to the edge of the half plane and pointing outside A. Since $A \subseteq B$ we can say that $A \leq B$. Moreover, there is a translation of *B*, so that it is fully covered by *A* and in this case we may write $B \leq A$. It seems logical to assume that A and *B* must be geometrically equal, in other words, that they can coincide if we apply a certain rigid motion. But this is impossible to happen, because every half plane remains half plane whenever we apply a rigid motion to it and obviously it can't coincide with a geometric figure that is not a half plane.

Since the geometric relation "$\leq$" is not antisymmetric it is necessary to restrict the comparison to certain classes of geometric figures. We already know that in the class of the line segments or in the class of the arcs of a circle, the relation "$\leq$" is a total order. Therefore the question is, if there are other classes of figures where the relation "$\leq$" is a total or a partial order.

We shall call *good classes* (of geometric figures) those that among the figures they contain we can't find a paradox like the one mentioned above. A good class, but not the only one, is that of the compact figures (sets). In fact, compact figures have the property not to generate paradox with any other geometric figure whether compact or not. Those figures will be called *good figures*. Besides the compacts, good figures are also the open-and-bounded sets. On the contrary, just bounded figures may not be good as we will prove later using a counterexample, given again by professor V. Nestorides.

The study, concerns not only the Euclidean Geometry, but it is also expanded into the Hyperbolic and the Elliptic Geometry and some parts may be formulated in a pure algebraic language so that they cover uniformly all three geometries. The conclusions we have reached, are fully compatible with our previous knowledge about the comparison of geometric figures. In the special case of the Euclidean Geometry we proved that there is a good class, containing all the fundamental geometric figures, where we can compare even non-similar ones. Therefore a comparison between a circular disc and a triangular region is meaningful in the new context.

2. COMPARISON OF FIGURES IN THE EUCLIDEAN GEOMETRY

2.1 *Basic definitions*

We adopt Klein's view for the Euclidean Geometry. Our space is $\mathbb{R}^2$ endowed with the euclidean metric and the group acting on $\mathbb{R}^2$ is $ISO\left(\mathbb{R}^2\right)$, the group of euclidean isometries. The couple $\left(ISO\left(\mathbb{R}^2\right),\mathbb{R}^2\right)$ generates the euclidean geometric space where we will develop our study.

Definition 2.1

Two figures S_1 and S_2 are geometrically equal when there is a euclidean isometry[2] $f:\mathbb{R}^2\to\mathbb{R}^2$ so that $f\left(S_1\right)=S_2$. In that case we will write $S_1\approx S_2$.

Remarks

I. Figure is any subset of $\mathbb{R}^2$. From now on we will not distinguish the terms "subset of $\mathbb{R}^2$" and "figure".

II. We use the terms "rigid motion" and "isometry" synonymously.

Definition 2.2

For any two figures S_1 and S_2 we shall say that S_1 is equal to or smaller than S_2 when there is a euclidean rigid motion f so that $f\left(S_1\right)\subseteq S_2$. Then we will write $S_1\leq S_2$.

This "natural" definition does not satisfy in general the antisymmetric property, as we will prove later.

Proposition 2.1

The relation "$\leq$" is a pre-order of figures i.e. it is reflexive and transitive, and the reflex ion is meant in the sense of the geometric equality defined in 1.1

Proof

Let A and B two geometrically equal figures. Then, by definition, there is a euclidean isometry *f* so that $f\left(A\right)=B$. Then $f\left(A\right)\subseteq B$ also holds and we conclude that the relation $\leq$ is reflexive with respect to the geometric equality.

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[2] based on the euclidean metric ρ of $\mathbb{R}^2$ where $\rho\left(\left(x,y\right),\left(a,b\right)\right)=\sqrt{\left(x-a\right)^2+\left(y-b\right)^2}$

If $A \leq B$ and $B \leq C$ then there are isometries f, g so that $f(A) \subseteq B$ and $g(B) \subseteq C$. Then for the isometry $g \circ f$ holds $g \circ f(A) \subseteq C$ i.e. $A \leq C$. Therefore the relation is transitive

In the following examples we shall prove that $"\leq"$ does not satisfy in general the antisymmetric property, with respect to the geometric equality of definition 1.1.

Example 2.1

Let the half lines $A = \{(x,0) \in \mathbb{R}^2 : x \geq 0\}$, $B = \{(x,0) \in \mathbb{R}^2 : x > 0\}$. Since A is a closed subset of $\mathbb{R}^2$ while B is not, **there is not** an isometry f so that $f(A) = B$[3] thus $A \not\approx B$.

For the isometry $f(x,y) = (x+1, y)$, $f(A) \subseteq B$ holds. But it is also obvious that $B \subseteq A$, so we have both $A \leq B$ and $B \leq A$ while $A \not\approx B$.

Example 2.2

Let the figure $A = \{(x,y) \in \mathbb{R}^2 : x \leq 0\} \cup \{(x,2) \in \mathbb{R}^2 : 0 \leq x \leq 1\}$ and the half plane $B = \{(x,y) \in \mathbb{R}^2 : x \leq 2\}$. Obviously $A \subseteq B$ therefore $A \leq B$ .Since every isometry maps half planes into half planes there is not an isometry f such that $f(A) = B$, so $A \not\approx B$.

But for the isometry $f(x,y) = (x-3, y)$, $f(B) \subseteq A$ holds. So we conclude that $A \leq B$ and $B \leq A$ while $A \not\approx B$.

Example 2.3

Let the figure $\mathrm{A} = \{(x,y) \in \mathbb{R}^2 : x \geq 0, y \geq 0\}$ which is the right angle $\widehat{xOy}$ and the figure B produced by A when we subtract the inner part of the isosceles right triangle $S = \{(x,y) \in \mathbb{R}^2 : x \geq 0, y \geq 0, x + y < 1\}$.

Obviously $B \subseteq A$ so $B \leq A$ holds.

By translating A parallel to the axis $x'x$ by two units we also have that $f(A) \subseteq B$, where $f(x,y) = (x+2, y)$ is an isometry. Thus $A \leq B$.

But A and B are not geometrically equal, because in case there is an isometry g such that $g(A) = B$ and since isometries preserve the angles then B should also be a right angle, which is absurd.

[3] Every isometry maps closed sets into closed sets.

Example 2.4

Let an angle ω so that $\frac{\omega}{\pi}$ is irrational, for instance we can choose $\omega = \pi\sqrt{2}$. We set $a_k\left(\cos k\omega, \sin k\omega\right)$ a sequence of points lying on the circumference of the unit circle. By definition $a_k \neq a_m$ for $k \neq m$ otherwise we would have integers n, λ so that $\frac{\omega}{\pi} = \frac{\lambda}{n}$.

The set $A = \{a_k : k = 0,1,2,3,...\}$ is a dense and equally distributed subset of the circumference. We also consider the set $B = A \setminus \{a_1\} \subseteq A$, therefore $B \leq A$. If there is an isometry f of the plane such that $f(A) = B$ then we will arrive at a contradiction. For every $a \in A$ there is at least one $a' \in A$ so that the distance $d(a, a') = r$ where $r = 2\sin(\omega/2)$. Since $f(A) = B$ then and for every $b \in B$ there is at least one $b' \in B$ so that the distance $d(b, b') = r$. But this does not hold for $a_o \in B$ and we arrived at a contradiction.

Let T be a rotation by 2ω with center the origin of the axes. Then T is an isometry and $T(A) = \{a_k : k = 2,3,...\} \subseteq B$. So $A \leq B$ and $B \leq A$ hold, while $A \not\approx B$.

We modify the definition of the pre-order $"\leq"$ so that we arrive at an order relation:

Definition 2.3

In the set of figures we define a relation λ such that:

$$A\lambda B \Leftrightarrow \{A \approx B\} \text{ or } \{A \leq B \text{ and not } B \leq A\}$$

Proposition 2.2

λ is an order relation.

Proof

simple

Definition 2.4

A class $\mathcal{E}$ of figures is said to be **good** *when there are not any figures A, B in the class $\mathcal{E}$ so that $A \leq B, B \leq A$ and $A \not\approx B$.*

Remark

The relation "λ" is the relation "$\leq$" without the pathological cases where the antisymmetric property does not hold true. But the definition of "λ" is quite barren and gives no information to the question:

Which figures form a good class?

It seems to be wiser to concentrate our study on those sets that satisfy the antisymmetric property of "$\leq$".

Definition 2.5

We will say that a figure A is **good** *when for every figure B, if $A \leq B$ and $B \leq A$ hold, then $A \approx B$ also holds.*

Obviously a class consisting only of good sets is a good class. The converse does not hold true. A trivial case is a class consisting of only one figure (and all the geometric equals) that is not good. Since there is no other figure in the class to provide a counterexample then the class is good.

A non-trivial example is the class of the open or closed angles. We proved in example 1.2 that an angle is not a good set but it is quite easy to verify that using open or closed angles only, we can not provide a counterexample.

2.2 *Quest for good classes of figures*

Proposition 2.3

If A is a good set and $A \approx B$ then B is also a good set.

Proof

Let $C \leq B$ and $B \leq C$. Since $A \approx B$ then $A \leq C$ and $C \leq A$ hold. As A is a good set then there is an isometry f of the plane such that $f(A) = C$. There is also an isometry g of the plane such that $g(B) = A$. Then $g \circ f(B) = C$ i.e. $B \approx C$ î

Proposition 2.4

If A is a good set then its complement is also a good set.

Proof

Let $B \leq A^c$ and $A^c \leq B$. Then there are isometries f, g, of the plane, so that $f(B) \subseteq A^c$ and $g(A^c) \subseteq B$. But then $f(B)^c \supseteq A$ and $g(A^c)^c \supseteq B^c$ hold.

Since f, g are 1-1 and onto, $f(B)^c = f(B^c)$ and $g(A^c)^c = g(A)$ hold. Therefore $f(B^c) \supseteq A$ and $g(A) \supseteq B^c$ which is equivalent to $A \leq B^c$ and $B^c \leq A$. As A is good, there is an isometry h of the plane such that $h(A) = B^c$. Then $h(A)^c = B$ and as h is 1-1 and onto $h(A^c) = B$ i.e. $A^c \approx B$ î

Theorem 2.1

Let $\langle X, d \rangle$ *be a compact metric space. If* $f : X \to X$ *is an isometry then* $f(X) = X$.

Proof

Well known

Proposition 2.5

Every compact subset of $\mathbb{R}^2$ *is a good set.*

Proof

Let A compact and B an arbitrary set so that $A \leq B$ and $B \leq A$. Then there are isometries f, g so that $f(A) \subseteq B$ and $g(B) \subseteq A$.

For the isometry $g \circ f : A \to A$ we have already seen that $g \circ f(A) = A$ because A is compact (theorem 1.1).

But then $g(B) \subseteq A = g \circ f(A)$ which implies that $B \subseteq f(A)$. So $f(A) = B$ holds and $A \approx B$. Therefore A is a good set ◾

We will introduce now, a new definition that will be particularly useful.

Definition 2.6

A figure $A \subseteq \mathbb{R}^2$ *will be called* **strongly good** *if for every isometry* $f : \mathbb{R}^2 \to \mathbb{R}^2$ *that satisfies* $f(A) \subseteq A$ *, then the equality* $f(A) = A$ *holds true.*

Proposition 2.6

Every compact subset of $\mathbb{R}^2$ *is strongly good.*

Proof

Direct from definition 1.6 and theorem 1.1 ◾

Proposition 2.7

Every strongly good set is also a good set.

Proof

Let A a strongly good set and B an arbitrary set such that $A \leq B$ and $B \leq A$. Then there are isometries f, g of the plane so that $f(A) \subseteq B$ and $g(B) \subseteq A$.

For the isometry $g \circ f : A \to A$, $g \circ f(A) = A$ holds since A is strongly good. Then $g(B) \subseteq A = g \circ f(A)$ and therefore $B \subseteq f(A)$.

Finally we conclude that $f(A) = B$ so $A \approx B$ and A is a good set ⌷

Remark

The definition of the good figure is difficult to handle, as it depends on the "interaction" with all the other figures. On the contrary the definition of the strongly good figure is intrinsic, because, in simple words, strongly good is any figure that does not fit (without decomposition) into part of itself.

Proposition 2.8

Every open and bounded subset of $\mathbb{R}^2$ *is strongly good.*

Proof

Let A open and bounded and f an isometry of the plane such that $f(A) \subseteq A$. As A is open then

$A \cap \partial A = \varnothing$ and $\overline{A} = A \cup \partial A$. Thus $A = \overline{A} \setminus \partial A$

Also $diam(\overline{A}) = diam(A) < \infty$ so $\overline{A}$ is closed and bounded subset of $\mathbb{R}^2$, hence $\overline{A}$ is compact. Also

$\partial A \subseteq \overline{A}$ and $diam(\partial A) \leq diam(\overline{A})$ so the boundary is closed and bounded hence compact subset of

$\mathbb{R}^2$.

$f(A) \subseteq A \Rightarrow \overline{f(A)} \subseteq \overline{A} \Rightarrow f(\overline{A}) \subseteq \overline{A}$ and from T*heorem 3.1* $f(\overline{A}) = \overline{A}$ holds.

$f(\partial A) = f(\overline{A} \setminus A) = f(\overline{A}) \setminus f(A) = \overline{A} \setminus f(A) \supseteq \overline{A} \setminus A = \partial A$

Hence $f^{-1}(\partial A) \subseteq \partial A$ and as f^{-1} is an isometry then from Theorem *3.1* again, we conclude that

$f^{-1}(\partial A) = \partial A \Leftrightarrow \partial A = f(\partial A)$. Therefore

$f(A) = f(\overline{A} \setminus \partial A) = f(\overline{A}) \setminus f(\partial A) = \overline{A} \setminus \partial A = A$, so A is strongly good ⌷

Proposition 2.9

In $\mathbb{R}^2$*, the union of a compact with an open bounded set is a strongly good set.*

Proof

Let K compact and A open and bounded subsets of $\mathbb{R}^2$, and the isometry f of the plane such that $f(A \cup K) \subseteq A \cup K$.

The set $V=\left(A\cup K\right)^{o}\subseteq A\cup K$ is also open and bounded.

$W=K\setminus V=K\cap V^{c}$ is compact as an intersection of a compact with a closed set.

Obviously $V\cap W=\varnothing$

Since A is open subset of $A\cup K$ then $A\subseteq V$ and

$A\cup K\subseteq V\cup K=V\cup\left(K\setminus V\right)=V\cup W\subseteq A\cup K$. Hence $V\cup W=A\cup K$

Also $V\cap W=\varnothing\Rightarrow f\left(V\cap W\right)=\varnothing\Rightarrow f\left(V\right)\cap f\left(W\right)=\varnothing$

Consequently $f\left(V\cup W\right)\subseteq V\cup W\Rightarrow f\left(V\right)\cup f\left(W\right)\subseteq V\cup W$

$f\left(V\right)$ is an open and bounded subset of $A\cup K$ therefore $f\left(V\right)\subseteq V$ since $V=\left(A\cup K\right)^{o}$. From *proposition 1.8* we conclude that $f\left(V\right)=V$.

As $f\left(V\right)\cup f\left(W\right)\subseteq V\cup W$ and $V\cap W=\varnothing, f\left(V\right)\cap f\left(W\right)=\varnothing$ then $f\left(W\right)\subseteq W$ But W is compact and from *Theorem 3.1* $f\left(W\right)=W$. Therefore

$f\left(V\right)\cup f\left(W\right)=V\cup W\Rightarrow f\left(V\cup W\right)=A\cup K\Rightarrow f\left(A\cup K\right)=A\cup K$ ∎

Proposition 2.10

In $\mathbb{R}^2$, the intersection of a compact with an open bounded set is a strongly good set.

<u>Proof</u>

Let K compact, A open and bounded and f an isometry of the plane such that $f\left(A\cap K\right)\subseteq A\cap K$

We set $X=\overline{A\cap K}$ which is a compact subset of K.

Then $f\left(A\cap K\right)\subseteq A\cap K\Rightarrow f\left(\overline{A\cap K}\right)\subseteq X\Rightarrow f\left(X\right)\subseteq X$. According to *Theorem 1.1* we conclude that $f\left(X\right)=X$.

Since $A\cap K$ is open in K, then it is also open in every closed subset of K. Therefore $A\cap K$ is open in X, so $X\setminus\left(A\cap K\right)$ is compact and it is obvious that

$f^{-1}\left(X\setminus\left(A\cap K\right)\right)\subseteq X\setminus\left(A\cap K\right)$. According to *Theorem 1.1* we conclude that

$f^{-1}\left(X\setminus\left(A\cap K\right)\right)=X\setminus\left(A\cap K\right)$.

But then $f\left(X\setminus\left(A\cap K\right)\right)=X\setminus\left(A\cap K\right)$ and since $f\left(X\right)=X$ and $A\cap K\subseteq X$, we conclude that $f\left(A\cap K\right)=A\cap K$ i.e. $A\cap K$ is strongly good ∎

Remark

The proposition holds even if A is not bounded.

Proposition 2.11

The classes $\mathcal{X} = \{ K \cup A$ *where K compact and* A *open bounded subsets of the plane*$\}$, $\mathcal{Y} = \{ K \cap A$ *where K compact and* A *open bounded subsets of the plane*$\}$, $\mathcal{E} = \mathcal{X} \cup \mathcal{Y}$ *and* $\mathcal{F} = \mathcal{E} \cup \left\{ S \subseteq \mathbb{R}^2 : S^c \in \mathcal{E} \right\}$ *are all good classes.*

Proof

The classes above consist of strongly good sets. Then from definition 1.4 and the propositions 1.7, 1.8, 1.9, 1.10 the conclusion is obvious ▮

The class $\mathcal{F}$ includes almost all the fundamental figures of Euclidean Geometry: line segments, triangles, polygons, circles, arcs etc but not the open or closed angles which, as we have already mentioned, form a good class.

Proposition 2.12

If $\mathcal{W}$ is the class of open or closed angles then the class $\mathcal{F} \cup \mathcal{W}$ is good.

Proof

Since we already know that the classes $\mathcal{F}, \mathcal{W}$ are good, then it is sufficient to examine whether a set from one class provides a counterexample to the other class.

Let A a set of the class $\mathcal{F}$. Then $A \in \mathcal{E}$ or $A \in \mathcal{F} \backslash \mathcal{E}$.

If $A \in \mathcal{E}$ then it is bounded and it cannot provide a counter example with any angle of $\mathcal{W}$.

If $A \in \mathcal{F} \backslash \mathcal{E}$ and provides a counterexample with an angle then its complement $A^c \in \mathcal{E}$, will also provide a counterexample with the complement of the angle (which is also an angle). But this contradicts what we have already proved about the elements of $\mathcal{E}$ which give no counterexamples with the elements of $\mathcal{W}$.

So the class $\mathcal{F} \cup \mathcal{W}$ is a good one and includes the closed and open angles ▮

Remarks

1. The **closed sets** are neither good nor form a good class according to the examples 1.1 and 1.2
2. The **connected sets** are neither good nor form a good class according to the example 1.1
3. The **bounded sets** are neither good nor form a good class according to the example 1.4
4. The **connected and bounded sets** are neither good nor form a good class. This can be easily proved if in the set A of the example 1.4 we attach the inner points of the unit disc.
5. The **convex sets** are neither good nor form a good class according to the example 1.3

6. The **convex and bounded sets** are neither good nor form a good class. This can be easily proved if in the set A of the example 1.4 we attach the inner points of the unit disc.

If two figures A, B are good we have proved that their complements are also good. However the union or intersection of (strongly) good sets is not necessarily a good set as we will illustrate in the following counterexamples:

Example 2.5

Let the strongly good figures $L = \{(x,0) \in \mathbb{R}^2 : x \geq 0\} \cup \{(0,1)\}$ and $M = \{(x,0) \in \mathbb{R}^2 : x \geq 0\} \cup \{(x,1) \in \mathbb{R}^2 : x < 0\}$ [4] then the set $L \cap M = \{(x,0) \in \mathbb{R}^2 : x \geq 0\}$ is not good according to the example 1.1.

Example 2.6

We also use here the previously defined sets L and M.

If g is a reflection with respect to the $y'y$ axis then $g(M)$ is strongly good but the set $G = L \cup g(M) = \{(x,0) \in \mathbb{R}^2 : x \in \mathbb{R}\} \cup \{(x,1) \in \mathbb{R}^2 : x \geq 0\}$ is not good because for the set $V = \{(x,0) \in \mathbb{R}^2 : x \in \mathbb{R}\} \cup \{(x,1) \in \mathbb{R}^2 : x > 0\}$ we have that $V \subseteq G$ and $h(G) \subseteq V$, where $h(x,y) = (x+1,y)$. Therefore $V \leq G$ and $G \leq V$. But G is a closed set and V is not closed so there is no isometry f such that $V = f(G)$, hence $V \not\approx G$.

It is interesting that the new context is also applied, with trivial modifications, into the Hyperbolic and into the Elliptic Geometry. Fot further details see [15].

[4] For a detailed proof see [15]

3. OPEN ISSUES

A fundamental question is whether the definitions of the good set and the strongly good set are equivalent or there is a counterexample of a good set that is not strongly good. In the appendix it is proved that in $\mathbb{R}$ all good sets are also strongly good. So it is our belief that the definitions are also equivalent on the plane.

Another question is whether the algebra produced by $X = \{A \subseteq \mathbb{R}^2 : A$ compact or open and bounded$\}$ consists only of strongly good sets.

Finally, as in $\Omega =$ set of good classes the assumptions of Zorn's[5] lemma are satisfied it would be quite interesting to find maximal classes within the set Ω.

[5] Every totally ordered subset of Ω is defined to be a set of good classes $Y = \{ F_i ,\ i \in I \}$ so that for every $i, j \in I$ it is true that $F_i \subseteq F_j$ or $F_j \subseteq F_i$. Obviously the **good** class $\bigcup_{i \in I} F_i$ is an upper bound of Y.

Appendix

In the appendix we prove that good sets coincide with strongly good sets in $\mathbb{R}$. We do not know whether the same holds or not holds in $\mathbb{R}^2$.

Lemma

Let a set $A \subseteq \mathbb{R}$, $a \in A$ and an isometry $f : \mathbb{R} \to \mathbb{R}$ such that $f(A) = A \setminus \{a\}$ i.e. $A \approx A \setminus \{a\}$. Then A is not a good set.

Proof

If $f : \mathbb{R} \to \mathbb{R}$ is an isometry then $f(x) = x$ or $f(x) = -x$ or $f(x) = -x + c$, $c \neq 0$ or $f(x) = x + c$, $c \neq 0$.

- If $f(x) = x$ then $f(A) = A \setminus \{a\} \Leftrightarrow A = A \setminus \{a\}$ i.e. $a \notin A$, absurd.
- Both $f(x) = -x$ and $f(x) = -x + c$ have the property $f^2 = id_{\mathbb{R}}$.

  From the assumption we have that $f(A) = A \setminus \{a\}$ therefore

  $f^2(A) = A \setminus \{a, f(a)\} \Leftrightarrow A = A \setminus \{a, f(a)\}$ i.e. $a \notin A$, absurd.
- If $f(x) = x + c$, $c \neq 0$.

  Then $f(a - c) = a$, so $a - c \notin A$

  Also $f(a) = a + c \in f(A) = A \setminus \{a\}$.

  Let the set $B = A \setminus \{a + c\}$. Then $a + c \notin B$ and $a - c \notin B$ since $a - c \notin A$ and $B \subseteq A$.

  Obviously $B \subseteq A$ and $f^2(A) = A \setminus \{a, a + c\} \subseteq B$

  Let us assume that there is an isometry $g : \mathbb{R} \to \mathbb{R}$ so that $g(A) = B$

  For every $x \in A$ we have that $f(x) = x + c \in A \setminus \{a\}$ therefore there is at least one $x' = x'(x) \in A$ so that $d(x, x') = c$ (for instance $x' = x + c$).

  But then we will also have that $d(g(x), g(x')) = c$.

  Since $g(A) = B$ there is some $x_o \in A$ such that $g(x_o) = a \in B$.

  Then there is also some $x'' = x''(x_o) \in A$ so that $d(x_o, x'') = d(x_o, x'(x_o)) = c$

  It follows that $c = d(x_o, x'') = d(g(x_o), g(x'')) = d(a, g(x''))$.

We may say that there is some $b \in B = g(A)$, where $b = g(x'')$ so that $d(a,b) = c$.

But then $b = a - c$ or $b = a + c$, which in either case are not points of $B = g(A)$.

This is a contradiction and we conclude that A is not a good set.

Proposition

Let $A \subseteq \mathbb{R}$ be a good set, then A is strongly good.

Proof

We assume that A is not strongly good.

Then there will be an isometry $T : \mathbb{R} \to \mathbb{R}$ so that $T(A) \subsetneq A$.

Therefore there is $a \in A \setminus T(A)$ and $T(A) \subseteq A \setminus \{a\} \subseteq A$ i.e. $T(A) \leq A$

Since $A \subseteq T^{-1}(A) \Leftrightarrow A \subseteq T^{-1}\left(T^{-1}\left(T(A)\right)\right)$ then $A \leq T(A)$ and as A is a good set then $A \approx T(A)$.

We can also prove that $A \approx A \setminus \{a\}$ because:

$A \setminus \{a\} \leq A$ and $A \approx T(A) \leq A \setminus \{a\}$ since A is a good set we conclude that $A \setminus \{a\} \approx A$.

But from the previous lemma such a set A is never a good set , absurd! ⌷

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"Obeying a rule"
Ludwig Wittgenstein and the foundations of Set Theory

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Abstract:
In this paper we propose some reflections on Wittgenstein's ideas about grammar and rules; then we shall consider some consequences of these for the foundations of set theory and, in particular, for the introduction of major concepts of set theory in education. For instance, a community of practice can decide to follow a particular rule that forbids the derivation of arbitrary sentences from a contradiction: since, according to Radford's perspective, knowledge is the result of thinking, and thinking is a cognitive social praxis, the mentioned choice can be considered as a form of real and effective knowledge.

Keywords: Foundations of mathematics; thinking; Set theory; Wittgenstein

1. Introduction

In this paper we shall propose some reflections on the main ideas of Ludwig Wittgenstein (1889-1951) about grammar and rules; then we shall consider some consequences for the foundations of set theory, with regard to mathematics education too.

Wittgenstein's fundamental reflections on the "grammar" and on the meaning of "following a rule" will allow us to approach the focus of this paper. Of course we shall not try to expound "what Wittgenstein really meant", but rather we shall try to see some implications of Wittgenstein's views, in particular for mathematical and educational practice. In order to do this, we are going to refer both to some Wittgensteinian ideas and to some well known interpretations.

First of all, it is necessary to highlight the important sense of the term "grammar": according to Wittgenstein, the grammar is a particular philosophical discipline by which it is possible to describe the use of words in a language (Wittgenstein, 1969, § 23). The importance of the grammar is crucial in Wittgenstein with regard to reflections on mathematics, too: as a matter of fact, a theorem, like every other analytical (true) statement, expresses a grammatical rule (Gargani, 1993, p. 99).

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Let us now consider a rule (in particular, a grammatical rule, with reference to "grammar" in the aforementioned sense) and let us analyse the practice that we identify as "obeying a rule". Wittgenstein notices: «And hence 'obeying a rule' is a practice. And to *think* one is obeying a rule is not to obey a rule. Hence it is not possible to obey a rule 'privately': otherwise thinking one is obeying a rule would be the same thing as obeying it» (Wittgenstein, 1953, § 202). These words, frequently quoted, are very important: as a matter of fact, the distinction between thinking one is obeying a rule and obeying it is the difference between the behaviour that a person recognises to be in accordance, by definition, with the considered rule, and the corresponding grammatical decisions of the community (Frascolla, 2000, pp. 135-136).

It follows, once again, that the collective aspect clearly assumes a primary importance (Wittgenstein, 1953, § 206), although different interpretations attribute different roles to this aspect: for instance, according to C. McGinn, in order to follow a rule a public reference is not strictly necessary; the main point is the presence of support of behaviour that can be observed from outside (Messeri, 2000, pp. 184-185; McGinn, 1984, pp. 43-45).

According to S. Kripke (1982), Wittgenstein's position must be interpreted from a sceptical viewpoint: there is no such fact as "obeying a rule" and Wittgenstein's sceptical solution can be summarized in the (common) possibility of ascribing to someone the behaviour indicated by the expression "obeying a rule" (Messeri, 2000, p. 174). Of course, once again, a public reference is unavoidable (Kripke, 1982, pp. 27-49). According to C. Wright (1980), Wittgenstein suggests that the use of a concept cannot be completely fixed by one's past experience; hence who learns a language does not try to acquire an objective system of applications defined in the teacher's mind; better, the learner looks for the teacher's approval (as stated in: Messeri, 2000, p. 176; this points us towards an interesting educational reflection on the didactical contract).

An important element to be highlighted is that in order to allow comprehension of the expressions employed in a language, it is necessary to describe "obeying a rule" as a fact that can be acknowledged from outside, so a community is the only background that can give sense to the habit according to which we consider and treat individual answers as correct or wrong (Messeri, 2000, pp. 176-177). In addition, a very important position in both Kripke and Wright is worth mentioning: both of them place great emphasis on Wittgenstein's idea, defined as *community view*, according to which the sense of the discussion about "obeying a rule" is to be framed within a collective practice (Messeri, 2000, p. 177).

2. Frege's system and Russell's paradox

The original "ingenuous set theory" is based upon the so-called Comprehension Principle of Gottlob Frege (1848-1925), according to which, if we consider any property $P(x)$, we can define the set: $\{x \mid P(x)\}$ (and it is unique, according to another Fregean principle, the Extensionality Principle).

The subject leads us to remember a very important historical reference: on 16 June 1902, in a celebrated letter, Bertrand Russell (1872-1970) communicated to Frege that he found a contradiction in the theoretical system proposed by the great German logician. The Comprehension Principle, as noted, states the possibility of assigning a set by a characteristic property: given the property $P(x)$ there is a set I such that $x \in I$ if and only if $P(x)$. Nevertheless, if we consider the property $P(x)$ as $x \notin x$ and define the set $H = \{x \mid x \notin x\}$ (and this is allowed by

the aforementioned Principle), then we are dealing with the famous Russell's paradox: from $H \in H$ it follows $H \notin H$ and from $H \notin H$ it follows $H \in H$.

From the logical viewpoint, as it is well known, the presence of a contradiction in a theory causes a lot of problems. *Ex falso quodlibet*, so from a false proposition (absurd) proposition, e.g. "$H \in H \wedge H \notin H$", we can derive any proposition Q:

- $H \in H \wedge H \notin H$, hence $H \in H$;
- $H \in H \wedge H \notin H$, hence $H \notin H$;
- from $H \notin H$ it follows $H \notin H \vee Q$;
- $H \notin H \vee Q$, and being $H \in H$, it follows Q.

Many 20th-century logicians worked to obtain a new theoretical framework in order to avoid contradictions: Frege's Comprehension Principle has been weakened by Ernst Zermelo (1871-1953), who replaced it with a new axiom allowing the introduction of sets by elements such that they are characterised *both* by a given property $P(x)$ *and* by belonging to another (existing) set. As regards Russell's paradox, given a set A we can consider the set of the sets I belonging to A and not belonging to themselves, but it is impossible to consider the set of "all" the sets I not belonging to themselves (Zermelo, 1908). Some years later, Fraenkel, Skolem and Von Neumann revised Zermelo's system and proposed a new version, the Zermelo-Fraenkel theory, ZF (Fraenkel, 1922).

It can then be seen that according to Wittgenstein an is different from a natural law (also) because of the different role of the experience («Does experience tell us that a straight line is possible between any two points?»: Wittgenstein, 1956, III, § 4); he notices that «by accepting a proposition as a matter of course, we also release it from all responsibility in face of experience» (Wittgenstein, 1956, III, § 30). He underlines moreover that «it is quite indifferent why it is evident. It is enough that we accept it. All that is important is how we use it» (Wittgenstein, 1956, III, § 2). Hence the crucial point we must take into account when we consider two different theoretical approaches (e.g. the approaches based upon Frege's Principle or upon Zermelo's) is "how we use" them: and the considered «proposition is not a mathematical axiom if we do not employ it precisely *for that purpose*» (Wittgenstein, 1956, III, § 3).

But what do we mean by that? What do we mean when we say that a proposition is employed "precisely" for the purpose of being a mathematical axiom? Let us quote Wittgenstein once again: «It is not our finding the proposition self-evidently true, but our making the self-evidence count, that makes it into a mathematical proposition» (Wittgenstein, 1956, III, § 3).

So when we accept an axiom we really make "the self-evidence count" and recognise implicitly the character of the considered proposition: in that moment «we have already chosen a definite kind of employment for the proposition» (Wittgenstein, 1956, III, § 5). As we shall see, the main problem is to decide if a particular use is included in this "definite kind of employment".

Let us now turn back to our old "ingenuous set theory" in order to highlight how mathematics, for instance from the educational viewpoint, takes into account the famous letter by Russell. Theoretically speaking, ZF axioms can be effectively considered as the basis of the introduction of the concept of set (see for instance: Drake, 1974); but from the educational point of view, generally the concept of set is not introduced by an axiomatic presentation: it may seem that the concepts of set and belonging are easy concepts to learn. Although it is true that they bear an intuitive meaning, their corresponding mathematical meanings entail a precise conceptualization (Bagni, 2006-a).

These concepts attained a mathematical formulation only in the 19th century (Kline, 1972, p. 995), when Georg Cantor (1845-1918) introduced the concept of set through some synonyms (in *Über unendliche lineare Punktmannigfaltigkeiten*, 1879-1884); Frege also made reference to the concept of set through verbal descriptions (van Heijenort, 1967, pp. 126-128). As a consequence, we can say that introductions of the concept of set in education are still rather close to the aforementioned "ingenuous theory".

Now one can propose a problematic issue (Bagni, 2006-b): *what* are we going to teach when we introduce major concepts of set theory according to an ingenuous perspective? If we introduce sets through general descriptions that cannot be considered "rigorous" (e.g. based upon Frege's Comprehension Principle), we provide our pupils with a potentially dangerous mathematical tool: for instance, a pupil could decide to derive the Russell's contradiction and, from that (the ancient *ex falso quodlibet* should once again be remembered) he or she could state an arbitrary sequence or proposition trivially "true".

Nevertheless, another quotation of Wittgenstein is very relevant to our problem: «'Contradiction destroys the Calculus' – what gives it this special position? With a little imagination, I believe, it can certainly be demolished. (…) Let us suppose that the Russellian contradiction had never been found. Now – is it quite clear that in that case we should had possessed a false calculus? For are theren't various possibilities here? And suppose the contradiction had been discovered but we were not excited about it, and had settled e.g. that no conclusions were to be drawn from it. (As no one does draw conclusions from the 'Liar'.) Would this have been an obvious mistake?» (Wittgenstein, 1956, V, § 12).

Wittgenstein's reference to the Liar is interesting: although using the verb "to lie" can lead us to (unavoidable) contradictions, our language *works* (Wittgenstein, 1956, III, § 3). No one "draw conclusions from the 'Liar'", i.e. no one uses this celebrated contradiction to produce arbitrary results (*ex falso quodlibet*). Wittgenstein recognises that «if it is consistently applied, i.e. applied to produce arbitrary results», a contradiction «makes the application of mathematics into a farce, or some kind of superfluous ceremony» (Wittgenstein, 1956, V, § 12). But it is important to underline this issue: a contradiction would be "consistently applied" (consistently, we mean, with respect to logical calculus) when it is applied "to produce arbitrary results"; this is just *one* possible choice: perhaps it is a "consistent" choice, but it is not the one and only.

Of course it is possible to object that the rigor of our logical calculus (or, better: the rigor we traditionally ascribe to our logical calculus) is very different from the features of our everyday language: «'But in that case it isn't a proper calculus! It loses all *strictness*!' Well, not *all*. And it is only lacking in full strictness, if one has a particular idea of strictness, wants a particular style in mathematics» (Wittgenstein, 1956, V, § 12).

This is the point: does an "ingenuous set theory" lose "all strictness"? As a matter of fact the presence of a contradiction does not block the *construction* of a grammar completely (let us remember once again that according to Wittgenstein every Platonic approach is excluded). Clearly Fregean logical calculus has been put into a very critical position by Russell's paradox: but this fact must be considered with reference to a "particular idea of strictness", and hence to a "particular style in mathematics" (related to a "proper calculus"): «'But didn't the contradiction make Frege's logic useless for giving a foundation to arithmetic?' Yes, it did. But then, who said that it had to be useful for this purpose?» (Wittgenstein, 1956, V, § 13).

So Frege's logic is quite useless for giving a foundation to arithmetic: but if we consider it from an educational perspective we must reconsider our judgement. As a matter of fact the contradiction

could have very different consequences and a very different influence upon language games: firstly, it is simply possible that no one derives the contradiction itself; secondly, Russell's paradox can be considered by our students (looking for teacher's approval: Wright, 1980) as a strange, bizarre statement, and never used in order to derive arbitrary propositions: «We shall see the contradiction in a quite different light if we look at its occurrence and its consequences as it were anthropologically – and when we look at it with a mathematician's exasperation. That is to say, we shall look at it differently, if we try merely to *describe* how the contradiction influences language-games, and if we look at it from the point of view of the mathematical law-giver» (Wittgenstein, 1956, II, § 87).

We can suggest, following R. Rorty, that the major point is related to «the attempt to model knowledge of perception and to treat 'knowledge of' grounding 'knowledge that'» (Rorty, 1979, p. 316); more precisely, we could say that according to a new perspective our perception of Russell's paradox will not be based just upon *knowledge of* the paradox itself (and its potentially destructive use): «our certainty will be a matter of conversation between persons, rather than a matter of interactions with nonhuman reality» (Rorty, 1979, p. 318). And, as previously noticed, this conversation does not lead speakers to the derivation of the contradiction.

3. Concluding remarks

Previous considerations suggest us to turn briefly to a more general issue. According to Luis Radford's perspective, knowledge is linked to activities of individuals and this is essentially related to cultural institutions (Radford, 1997), so knowledge is built in a wide social context. Moreover Radford states: «Drawing from the epistemologists Wartofsky and Ilyenkov, I have suggested that knowledge is the product of a specific type of human activity – namely *thinking*. And thinking is a mode of social *praxis*, a form of reflection of the world in accordance to conceptual, ethical, aesthetic and other cultural conceptual categories» (D'Amore, Radford & Bagni, 2006).

This connection knowledge-social *praxis* is a crucial point, from the educational point of view, too, and several issues ought to be considered: for instance, what do we mean by "pupils' minds"? More generally, can we still consider our mind as a "mirror of nature" (following Rorty, 1979), and make reference to our "inner representations" uncritically? R. Rorty underlines the crucial importance of "the community as source of epistemic authority" (Rorty, 1979, p. 380), and states: «We need to turn outward rather then inward, toward the social context of justification rather than to the relations between inner representations» (Rorty, 1979, p. 424).

Let us turn back to our previous remarks. We can state that surely the presence of a contradiction in a logical theory, and particularly its possible destructive use, is a potentially dangerous trap: but in the examined case it is a well-marked one. Of course it is important to keep such a danger in mind, and this is, in a certain sense, what happens in didactical practice: «The pernicious thing is not, to produce a contradiction in the region in which neither the consistent nor the contradictory proposition has any kind of work to accomplish; no, what *is* pernicious is: not to know how one reached the place where contradiction no longer does any harm» (Wittgenstein, 1956, III, § 60).

We previously noticed that a mathematical proposition expresses a grammatical rule and is necessarily connected to a decision to be taken into a community. Now, in a community of practice (e.g. in the classroom or, more generally, in the educational "custom", defined as a set of compulsory practices induced by habit, often implicitly: Balacheff, 1988, p. 21) we can decide to follow a particular rule that forbids the derivation of arbitrary sentences from a contradiction (or,

according to Kripke's perspective, we say someone is obeying the considered rule when he/she does not actually derive arbitrary sentences from the contradiction); so, by that, we really "reached the place where contradiction no longer does any harm". Now we must know "how" we reached this place. For instance, our decision must be framed within a social and cultural context, whose features can have a relevant role (let us notice that Wittgenstein himself explicitly states that a conceptual system must be considered with reference to a particular context: Wittgenstein, 1953, XII). Since, in Radford's words, «knowledge is the result of thinking, and thinking is a cognitive social *praxis*» (D'Amore, Radford & Bagni, 2006), our choice can be considered as a form of real and effective knowledge. According to Habermas (1999), the rationality itself has three different roots, closely related the one to the others: the predicative structure of knowledge at an institutional level, the teleological structure of the action and the communicative structure of the discourse. We can state that the rule that forbids the derivation of arbitrary sentences from contradiction is mainly related to the second root.

We previously underlined that a *community view* is fundamental in Wittgenstein's approach. So any "ingenuous set theory" would risk a future change, in the community of practice, of the aforementioned rule that (nowadays) forbids the derivation of arbitrary sentences from a contradiction, with fatal consequences for our theory: «'Up to now a good angel has preserved us from going *this* way'. Well, what more do you want? One might say, I believe: a good angel will always be necessary, whatever you do» (Wittgenstein, 1956, V, § 13).

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How can science history contribute to the development of new proposals in the teaching of the notion of derivatives?

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Abstract:
The 18th century was a milestone for the incorporation of mathematics into physics. By this time already seen in Newton's work, we know that a great deal of progress had found the light of day concerning the relationship between physics and mathematics, particularly as the latter began to deal with universal gravitation and optics. Based on his theory of monochromatic light, which used mathematics to describe optical phenomena, Newton became interested in the behavior of the various colors which, according to him, compose white light. Furthermore, toward the end of the 17^{th} century, scientists had a remarkable mathematical tool at their disposal, differential and integral calculus, both of which were probably developed independently by Newton in *Principia*, and Leibniz in *Acta Eruditorum*, respectively. What interests us today is the fact that the utilization of these tools—or the lack thereof—was the object of many rich debates, even controversies, the nature of which partly depended on the scientific, cultural and political environment, not to mention the various ways of studying physical phenomena.

Keywords: history of science; historical mathematical methods; Newton; Leibniz; teaching of derivatives

Introduction

First, I will demonstrate the influence scientific and political circles had on the development, popularization and use of this method, followed by the impact of this new mathematical process in light of results obtained both in France and in Great Britain, the latter having taken into account and put forward various forms of theoretical expression for physical phenomena. To this end, I will concentrate on a case involving the phenomenon of astronomical refraction, about which Pierre Bouguer offered an explanation having relied on the use of infinitesimal calculation. I will then highlight the results obtained by Pierre Bouguer through a comparative study between both his, as well as the English approach to the same phenomenon. Finally, I will explore the possibilities of using this historical and epistemological perspective in the modern-day teaching of the notion of derivative function.

Published in 1729—time at which links were made between astronomy and optics—these works initiated the study of the optical aspect of bodies in movement, and later the works of Albert

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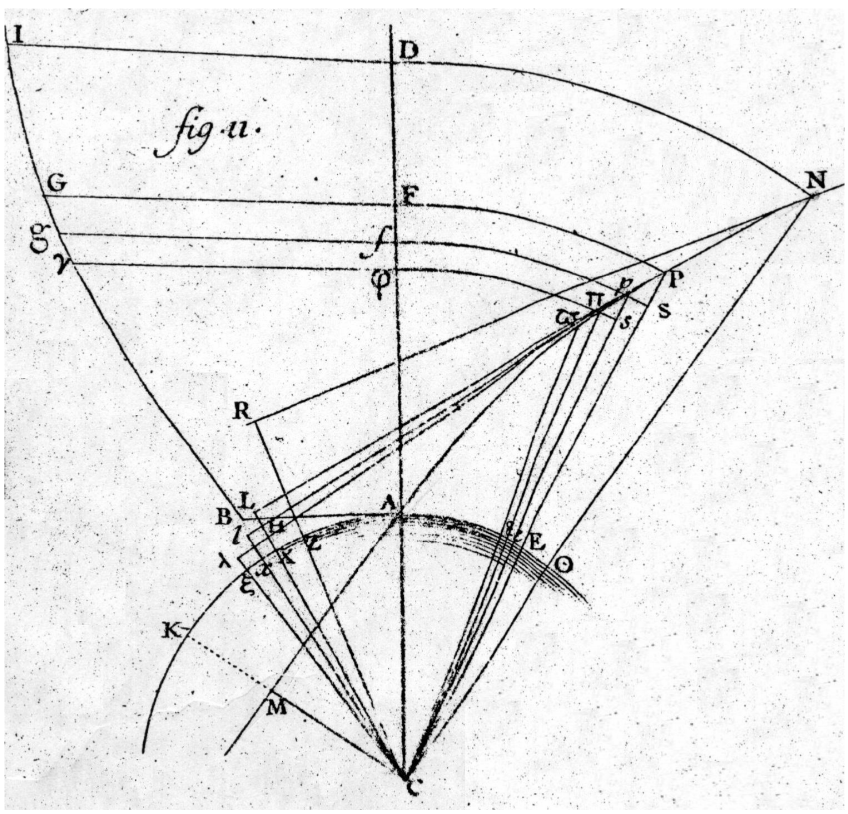
Einstein; this is also a time at which the soaring use of the mathematical method can be observed. This is the context in which we intend to pursue our study.

Context

He who wishes to find his way on land—or particularly at sea—by means of heavenly observation is obligated to take into account two domains which are intimately linked: astronomy and optics. This question was of special importance in 1729, year when Pierre Bouguer wrote a thesis, *On the best method of observing the altitude of stars at sea* having won a prize conferred by the French Science Academy, and in which its author addressed a long studied phenomenon, astronomical refraction. We know that the light that reaches us from a star will be more or less purloined depending on the medium through which it travels. This phenomenon—called astronomical refraction consequently prevents us from seeing where a star really is.



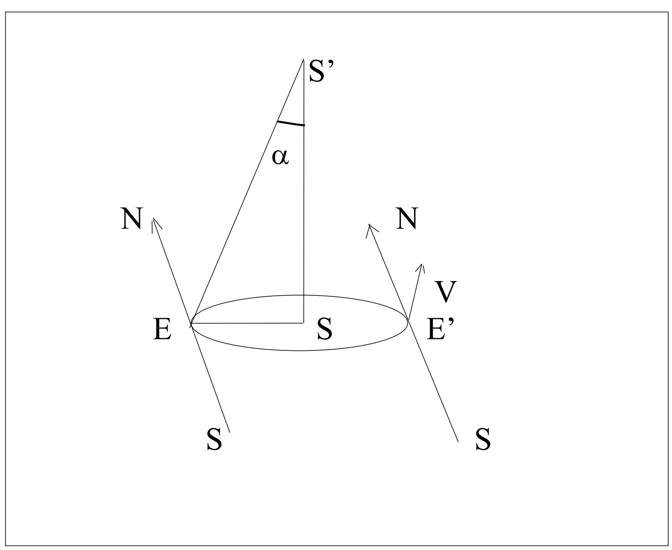
Pierre Bouguer, *De la méthode d'observer exactement sur mer la hauteur des astres, 1729*

Instead of being located in such or such position, it is elsewhere. Therefore, stars are not observed in their actual position. Hence the necessity to make corrections with the help of tables that can be adjusted according to one's location in order to find the real position of the stars. Long studied by Ibn Al Haïtham, Roger Bacon, or Tycho Brahé, the investigation of this phenomenon was resumed by Pierre Bouguer at a time when there were a great deal of breakthroughs bringing both astronomy

and optics into play. Fifty years prior in France, Rømer brought the idea of the finiteness of the speed of light to the fore through his observation of the emergence of Jupiter's moons. From 1725 to 1728 in England, James Bradley continued with some of the astronomical observations made by Jean Picard in 1671 and had noticed some anomalies that he was able to render public. His text was published in the December 1728 issue of Philosophical Transactions.

I am going to expose Bradley's results here, which are interesting because he worked both at the same time, and in the same field as Bouguer—that is to say at the forefront of astronomy and optics—but also because he belonged to a very different school than that of Bouguer, allowing new insight into the uniqueness of his methods, as much in form as in substance, and in particular concerning the use of mathematical analysis mentioned above. Indeed, Bradley exposed his work using a narrative style in a letter addressed to Edmond Halley whereby he presented events in the form of suspenseful riddles to be solved. He focused firstly on curiosity, surprises and questions evoked, and only afterwards on any observations he made. The implementation of experimental procedures can be seen here, and which one may consider having been inspired largely by Francis Bacon. Bradley took care to highlight the importance of questioning oneself about what pitfalls to avoid, if any, the attention to be brought to the handling of instruments of observation, and the necessity to reproduce observations. Nothing has been found, however, regarding approaches to the differential and integral calculus.

From the beginning of his letter, he indicates that his initial project was to determine parallaxes of several different fixed stars. Measuring these parallaxes revealed very interesting for astronomers since it allowed them to determine interstellar distances. There was also another more important reason for this—a lot less technical this time—and that was that the parallax would definitively prove the Copernicus' heliocentric theory. To measure this parallax, one preceded in this way: the letters S', S, E, respectively stand for a star, the Sun and the Earth; α is the angle that makes ES' and SS'



It was a matter of measuring the value of the α angle, which represents the parallax, where E and E' are the positions of Earth at six-month intervals. Knowledge of the α angle permitted the

measurement of the star's distance (SS'), since SS'=a/tg α, where *a* is the distance from the Earth to the Sun.

Bradley carried out his observations with the purpose of finding an eventual parallax, and in so doing, he noticed an anomaly. Indeed, the star under observation moves, but not in the direction expected, especially having only considered spatial relationships, as happened to be the case. From this point of view, the star should, in fact, move continuously in one direction on the ESS' plane. The star seems to move around perpendicularly on the ESS' plane.

This anomaly drove Bradley to implement what Bacon had called "the negative path", according to the framework of induction. Therefore, instruments were not brought into question, since—as indicated by Bradley— a great deal of regularity can be seen in the observations. Besides, the anomaly's regularity was such that it allowed for the prediction of cyclic phenomena which took place as predicted. Bradley successively eliminated, one after another, other causes of the anomaly, such as in the example of nutation. He then took on an affirmative procedure, and concentrated on the results of Rømer's observations, taking into account both the Earth's rotation and the finiteness of the speed of light: "…the phenomenon "proceeded from the progressive motion of light and the earth's annual motion on its orbit".was due to light's continuous movement and to the Earth's yearly orbit around the Sun."[2]

James Bradley is therefore aware of the cyclic phenomenon known as the aberration of fixed stars; this is important, notably because it foreshadows the development of the study of the optics of bodies in movement, relying on both astronomy and optics. Founded on the use of inductive reasoning, his approach was rather influenced by the ideas of Francis Bacon. To this end, he differentiated his approach from that of Bouguer's by studying astronomical refraction at the same time. Furthermore, we shall come to see that even if he had come to this very significant conclusion, he ended up not stating his them, for which he was blamed by Clairaut, because these conclusions would have had practical applications. The fact that Bradley was unable to further elaborate on these useful conclusions by finding practical applications for his discoveries is probably explained by the fact that he doesn't use the differential and integral calculus.

The development of the method

Probably developed double-handedly by both Newton in *Principia* and Liebniz in *Acta Eruditorum* toward the end of the 17th century, the differential and integral calculus elicited varying reactions from the scientific community. It is known that debate concerning its application in France was the source of conflict at the turn of the 18th century at the French Science Academy, where its members were, to say the least, divided and even opposed to its use. La Hire, Abbey Bignon, the Academy's president, Gallois and Rolle all expressed strong opposition to the new method at the time of its publication in France in spite of enthusiasm of such academics as L'Hospital or Varignon, both of whom had already been initiated to methods put forth by Leibniz.

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[2] James Bradley, A Letter from the Reverend Mr. James Bradley Savilian Professor of Astronomy at Oxford, and F.R.S. to Dr.Edmond Halley Astronom. Reg. &c. Giving an Account of a New Discovered Motion of the Fix'd Stars, *Philosophical Transactions of the Royal Society* , N°406, Vol.35, (1729) 637-661 ; 646. For a detailed analyze, see Mayrargue, *L'aberration des étoiles, et l'éther de Fresnel (1729-1851)*, (thèse Université Paris 7, 1991).

The method's dissemination was nevertheless made possible through the publication of several works, such as *L'Analyse démontrée,* published in 1708 by Père Reyneau. Furthermore, it was probably thanks to Reyneau that Bouguer had been initiated into differential and integral calculus.

The English scientific community also seemed to have experienced similar reservations regarding the use of the differential and integral calculus —even in the 1730's—, notably among astronomers working on particular sectors of the celestial sky. Evidence of preferences can be found among scientists who were concerned about both the reach of their findings, as well as how their work was organized, a fact to which supporters of the new method didn't hesitate to bring attention. The works of Bouguer in 1729, or those of Taylor who had preceded him in 1715, testify to this fact, not to mention the works of Clairaut, who, in 1737, and then again in 1746, read two theses at the Academy criticizing the Bradley's aforementioned works by distinguishing his approach from that of the latter's, highlighting the importance of using the differential and integral calculus.

"I believe that the Academy would approve of the goal that I have set for myself, to clarify this Theory in the Thesis that I am currently submitting. I will demonstrate the Methods for which Mr. Bradley hasn't yet given the results and I am adding several new studies on the same subject. I have found two solutions, one by way of this Method, and the other by way of synthesis; I have separated them so that those who only wish to study one may do so without studying the other."

Clairaut also referred to a thesis on the same subject by Manfredi, and offered a simple argument asserting his own results after having used the method: "I believe that [my rules]—in practice—would appear a lot simpler and more precise"

We have highlighted here tensions born between scientists on the basis of their preference for, or against the Method. Here, however, no consensus regarding these points of view can be observed. It can be said that although Taylor, mentioned above, used the differential and integral calculus just like Bouguer did, his style was different. His references were Newtonian analysis, as much in his notation as in his numerous references to *Principia*, not to mention his frequent necessity to exploit the concept of Newtonian attraction in his studies of atmospheric density variations and astronomical refraction. Whereas Bouguer speaks of the movement of light, Taylor reasons in terms of *vis gravitatis*. The former falls back on Newton, while the later—at the very beginning of his thesis—announces his reservations about Newton's preferences regarding the ways in which he studies phenomena, and therefore turns to Leibniz.

The political situation

Through this contextual analysis, it is also necessary to take into account the influence of the political situation. Works dealing with the Method gained popularity in France at a time when the Academy finds itself stuck in a period of decline after the reform of 1699. Its membership dropped sharply, and between 1699 and 1720 there was a 24 % fall, probably due to difficulties in replacing first generation scholars. At the beginning of the 18th century, established institutions seem to continue to demonstrate a sort of ostracism toward scientists who work within this new framework. Research dealing with infinitesimal analysis was actually quite rare. However the situation was divaricatious since at this same time, both Leibniz's and Bernoulli's work elicited numerous debates. Research on the subject was published by L'Hospital (1696), Carré (1700) and Reyneau (1702). While reading a thesis on air density variation, Varignon, in 1716, makes use of differential and integral calculus. The situation unfolds toward the second half of the 1720's, when Bouguer—who

had been working on astronomical refraction since 1727, year in which he supposedly wrote the first draft of *Essai d'optique sur la gradation de la lumière* which was published in 1729—was conferred a prize by the Academy, this time in a different context in view of the fact that there was increasing interest in infinitesimal calculus, and in mathematics, more generally. This political situation is evidenced in relationships that French scientists maintained with their English and German counterparts. After his election to the Royal Society (1728), and his meeting in Basel (1729) with Jean Bernoulli, it is known that Maupertuis began to play a role in the dissemination of infinitesimal calculus in France.

In the thesis defended by Pierre Bouguer at the Paris Academy of Sciences in 1729, the articulation of several ideas originally broached by Leibniz, L'Hospital, and Reyneau can be seen. It is all the more noteworthy that research on astronomical refraction is considered more commonly along the lines of theories originated by La Hire, Cassini, or Maraldi, all of whom Bouguer refers to. Because they ignored expansion of the Leibnizian as much as the Newtonian method, their style is distinct from one another. Apart from the frequent use of differential equations in his calculations, Bouguer was intent on founding his reasoning with the help of serial methods, the Inversion Formula, and the arithmetical triangle in order to establish tables showing astronomic refraction. This particular mathematical practice puts Bouguer in line with scientists such as Taylor in England or Varignon in France, who made use of results produced by Newtonian or Leibnizian mathematical methods. This is what we will examine here.

What path does the light coming from a star follow when it is observed from Earth? This is the question that Bouguer attempted to answer in his quest to establish an equation with a function designed to solve this problem. To this end, he concentrates on an idea introduced by Leibnizian formalism, and later researched by the Marquis de l'Hospital and Father Reyneau: the arithmetical—or differential—triangle, which allowed him to discriminate infinitely minuscule quantities. After having declared, "I call the algorithm of this calculation 'differential'", it is known that Leibniz's 1684 article had substantiated a relationship of proportionality between finite and infinitely small quantities with the help of congruent triangles. He formalized this idea in 1686, "In all these graphs, I imagined […] a triangle that I called 'arithmetical' whose sides were indivisible (rather, infinitely small), that is to say, differential quantities." The Marquis de l'Hospital had addressed this same idea again at the beginning of the second section of his work. Bouguer proceeded in the same way, and could thus, through geometric considerations of congruent triangles, establish a solaire? differential equation, based on which it was possible, we are told, "to easily construct the solaire?" Finally, having relied on these critical results revealed by Leibniz, he was therefore able to connect infinitely small quantities with finite quantities, consequently coming to the desired result after application.

From this example, the sheer muscle of the infinitesimal method in finding solutions to problems can be observed. In France, the concept of "derivative" is first introduced in the sixth form (*première et terminale*). There, the mathematics program asks that preference be given to work on a collection of historical texts linked to a common theme, for example, the notion of "derivative"; the objective of using epistemology and science history is to give insight into the nature of questions at the root of concepts as well as the language in which questions are formulated and discussed. It seems particularly welcome there to call on examples backed by historical accounts to convey notions studied. The example mentioned earlier concerning the phenomenon of astronomical refraction, borrowed from Physics, could be chosen for a sixth-form class, in which a degree of interdisciplinary studies is recommended, the goal being to demonstrate how mathematics has the power to establish patterns.

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Incoherence of a concept image and erroneous conclusions in the case of differentiability

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Abstract:

The level of the coherence of a concept image conveys how well the cognitive structure concerning the concept is organized. This study considers the relationship between deficiencies in the coherence of the concept image and erroneous conclusions in the case of differentiability. The study is based on an interview where the student made conclusions contradictory to the formal theory of mathematics. He used an erroneous method to study the differentiability of piecewise defined functions. This method became the key factor which maintained the internal coherence of the concept image. It made it possible to build a cognitive structure whose basis was erroneous.

Keywords: Cognitive structure, Coherence of a concept image, Concept image, Definition, Derivative, Differentiability, Erroneous conclusions, Mathematical reasoning, Representation

1. Introduction

During the academic year 2004-2005, a grand total of 146 subject-teacher students in mathematics from six universities in Finland participated in a written test. Typically, 150-250 subject-teacher students in mathematics graduate in Finland annually. Most of the participants were at the final phase of their studies. In addition, 20 subject-teacher students from one university in Sweden participated in the test. The test contained a task where the students had to determine which of the given functions (both the graphs and the formulas were given) were continuous and/or differentiable. One of the functions was the function f_3 in Figure 1.

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$$f_3(x) = \begin{cases} x^2 - 4x + 3, & x \neq 4, \\ 1, & x = 4. \end{cases}$$

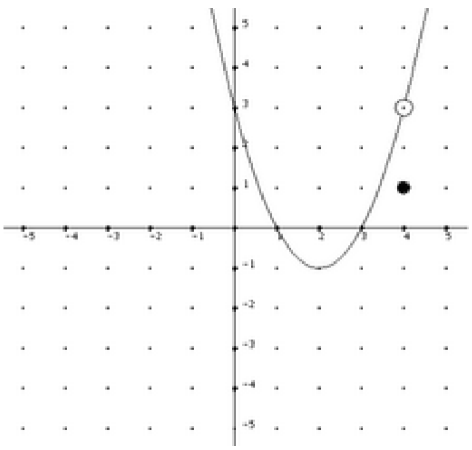

Figure 1: The graph and the formula of the function f_3 used in the test.

The starting point of the study presented in this paper was the observation that 38 Finnish and four Swedish students answered that the function f_3 was differentiable but not continuous. This outcome was very surprising and demanded an explanation. All of the students had during their studies encountered a theorem of calculus according to which continuity is a necessary but not sufficient condition for differentiability.

This study is based on the analysis of an interview of a student who had in the test answered that the function f_3 is not continuous but differentiable. In the interview, this student made several other erroneous conclusions. The goal of the study is to analyse how the conclusions in this interview were created and to discuss why some of them were erroneous with respect to the formal theory of mathematics. Several deficiencies were found in the knowledge structure concerning the concepts of derivative and differentiability. In the analysis, the theory about the concept image is applied. The term "coherence of a concept image" refers to the internal organisation of the knowledge structure concerning a certain concept. In order to clarify this term, a list of characteristics of a highly coherent concept image is presented in Section 2.2. In the analysis the coherence of the interviewee's concept image is evaluated on the basis of these characteristics.

In this paper, the term "erroneous conclusion" stands for a result of a concluding process which is in contradiction with the formal theory of mathematics. It does not primarily refer to the illogicality of the concluding process. "Resulting misconception" could be an alternative expression for it.

2. Concept image and its coherence

2.1. Structure of the concept image

Tall and Vinner have defined the term *concept image* to describe the total cognitive structure that is associated with a concept (Tall & Vinner, 1981). According to them, a concept image includes all the mental pictures and associated properties and processes relating to the concept, and it is built up through experience during one's lifetime. For clarity, it is reasonable to define that every concept has only one concept image in an individual's mind. Different portions of the concept image can be activated in different situations, but as a whole, the concept image of one concept is an entity.

Tall and Vinner have defined the *concept definition* to be a form of words used to specify the concept (ibid). The concept definition generates its own concept image, which Tall and Vinner call the *concept definition image*. They have also separated a *personal concept definition* from a *formal concept definition;* the former means an individual's personal way to define the concept in practice, whereas the latter is part of the formal axiomatic system of mathematics. This system consists of axioms, definitions, undefined elementary concepts (e.g., a point and a line in geometry), rules of logic, and mathematical language, and it forms an *institutionalized way of understanding mathematics* (Harel, in press).

In order to understand the formal concept definition, which is presented, for instance, in a textbook or in a lecture, an individual has to interpret the expression in the definition: He/She has to create a *personal interpretation of the definition*. These interpretations are essential factors in the *personal way of understanding* mathematics (Harel, in press). The formal concept definition is usually unambiguous, but the personal interpretations of the definition may vary between individuals, and they may also depend on the context (Pinto, 1998; Pinto & Tall, 1999; Pinto & Tall, 2002). The personal interpretation of the definition does not mean the same as the personal concept definition: The latter is not necessarily based on the formal definition at all. It can be thought that every time when an individual applies a formal definition in a reasoning, he/she in fact applies his/her personal interpretation of the definition. According to the definition of Tall and Vinner, the concept definition image includes the total cognitive structure that is associated with the concept definition. Thus, it is very natural to think that the personal interpretation of the concept definition is part of the concept definition image, which, for one, is part of the whole concept image. A diagram describing the internal structure of the concept image is presented in Figure 2. The personal concept definition is not included in this diagram, because its location is not unambiguous: It may be equal with the personal interpretation of the formal concept definition, but it may also lie outside the concept definition image. In the latter case the individual's own way to define the concept is not based on the formal definition.

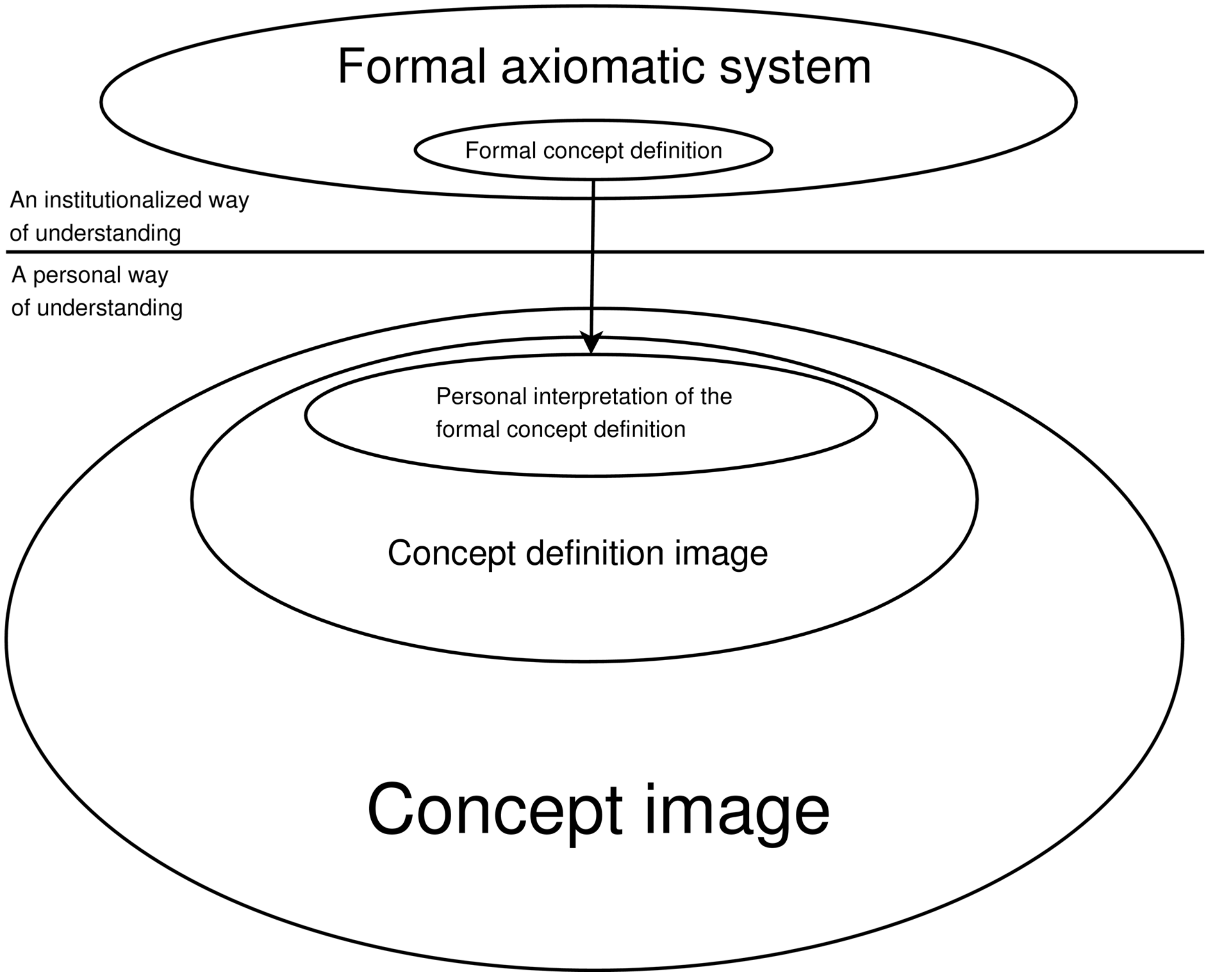

Figure 2: A diagram about the relationship between the concept image and the concept definition.

One reason for creating the terms concept image and concept definition was to separate reasoning based on the definition from reasoning based on other conceptions, representations and mental images. Thus the concept image and the concept definition were originally seen as opposite entities. In this way these terms are used, for example, in Tall's and Vinner's original paper (1981) and in Vinner (1991). Later Tall (2003; 2005) has reported about differences between him and Vinner regarding the view about the location of the concept definition: Tall considers the concept definition rather as a part of the concept image whereas Vinner has emphasized the distinction of them. In my model the formal concept definition is located outside the concept image, but through its personal interpretation it has a notable effect on the concept image.

My model may seem to have a positivistic or naturalistic basis: There is a system outside the mind of human which one attempts to understood. However, this model does not describe the whole process of learning or thinking, but only the role of the definition in the concept image and the process of understanding a given definition. The model in this form is not suitable for situations where the definition is not static but it is created or reconstructed.

2.2. Coherence of a concept image

Creative mathematical thinking requires that the concept image includes a variety of multifaceted conceptions, representations and mental images concerning the meaning and properties of the concept and relationships to other concepts. Representations and mental images[2] may be, for example, verbal, symbolic, visual, spatial or kinesthetic. It is also important for the concept image to be well ordered. The term *coherence of a concept image* is used to refer to the level of organization of various elements in the concept image. In practice, the concept images are hardly ever fully coherent or fully incoherent, but their level of coherence varies. To clarify the term, some criteria for a high level of coherence of a concept image are presented in the following list:

1. An individual has a clear conception about the concept.
2. All conceptions, cognitive representations and mental images concerning the concept are connected to each other.
3. A concept image does not include internal contradictions, like contradictory conceptions about the concept.
4. A concept image does not include conceptions which are in contradiction with the formal axiomatic system of mathematics.

Like the structure of a concept image as a whole, the level of coherence is not static but it changes all the time during mental activities concerning the concept.

The way the person views the concept (cf. criterion 1) may vary depending on the context. However, usually one of these conceptions is above the others, and it can thus be considered, according to Tall's and Vinner's terminology, as a personal concept definition. If the concept image is highly coherent, the different conceptions are mentally connected to each other (cf. criterion 2), but they are not in contradiction with each other (cf. criterion 3) or with the formal theory of mathematics (cf. criterion 4).

The criterion 2 means that there exist mental connections between the elements of the concept image. Goldin and Kaput (1996) have defined that the connection between two representations is *weak* if an individual is able to predict, identify or produce one representation from the other, and the connection is *strong* if an individual is, from a given action upon one representation, able to predict, identify or produce the results of the corresponding action on the other representation. Hähkiöniemi (2006a, 2006b) has defined that a person makes an *associative* connection between two representations if he or she changes from one representation to another and that a person makes a *reflective* connection between two representations if he or she uses one representation to explain the other. These are examples of potential types of the connections between the elements of the concept image. A highly coherent concept image makes possible both weak and strong connections (cf. Goldin & Kaput) and, respectively, associative as well as reflective connections (cf. Hähkiöniemi) between representations concerning the concept. Observed strong and reflective connections can be considered as stronger indications of the coherence than weak and associative connections.

For example, the concept of the derivative is according to its formal definition a limit of a difference quotient. On the other hand, the derivative can be visually interpreted as a slope of the

2 These both are widely used terms in the discipline of mathematics education, but their meaning is not unambiguous. In this connection these (both) are considered as mental configurations which represent (corresponds, associates, stands for symbolizes etc.) something else. This view is in accordance with Goldin's and Kaput's (1996) traditional definitions for the concept of representation.

tangent line, or it can be understood as a measure of an instantaneous rate of change. These are three different interpretations concerning the meaning of the concept of the derivative. If an individual has a highly coherent concept image, he/she is able to utilize all these interpretations in a problem solving process regardless the original form of the problem. If needed, he/she is able to change interpretation (weak or associative connection), and if changes in a system based on one interpretation happen, he/she is able to see the corresponding changes in another system which is based on another interpretation (strong connection). He/she is also able to explain, for example, on the basis of the definition why it is justified to consider the derivative as a slope of the tangent line (reflective connection).

The connections between elements of the concept image are important for preventing internal contradictions (cf. criterion 3). For preventing contradiction with the formal theory (cf. criterion 4), it is important that the elements of the concept image have connections also to the formal axiomatic system. This requires that the personal interpretation of the formal definition is correct and it has a central role in the concept images: Other conceptions, representations and mental images concerning the concept should be *reflectively connected* (cf. terminology by Hähkiöniemi) to this interpretation, in other words, they should be justifiable on the basis of this interpretation.

If the coherence of a concept image has some deficiencies with respect to criterion 3, it is very probable that it has deficiencies also with respect to criterion 4, because the formal axiomatic system of mathematics is (at least it should be) consistent[3]. On the other hand, it is possible that a concept image includes entities which are internally coherent, but which are in contradiction with the formal theory. These kind of entities may be based, for instance, on one or more misconception, misinterpretation or erroneous conclusion.

To some extent, the coherence of concept images and the *conceptual knowledge* mean the same thing. The term conceptual knowledge has been defined as a knowledge of relationships between pieces of information (Hiebert & Lefevre, 1986) or as a knowledge of particular networks and a skilful drive along them (Haapasalo & Kadijevich, 2000). The network consists of elements (concepts, rules, problems, and so on) given in various representation forms (ibid.). Thus, if a concept image of a certain concept includes a variety of knowledge and the structure is highly coherent, the level of the conceptual knowledge regarding this concept can be considered to be high. However, the high level of conceptual knowledge in a broader sense provides rich connections between the concept images of various concepts. It can be assumed that the conceptual knowledge, on one hand, provides networks consisting of connections between elements of knowledge inside each concept image and, on the other hand, a network consisting of connections between various concept images. However, the theories about concept image are not very usable in analyzing the whole structure of knowledge, but they are useful when concentrating on the knowledge concerning one concept at a time. In the latter case the conceptions about the relationships between a concept under analysis and other concepts can be considered as elements inside the concept image of the concept under analysis. For example, let us assume that an individual has a conception that continuity is a necessary but not sufficient condition for differentiability. When analyzing the knowledge structure of differentiability, this conception can be considered as an element of the concept image of differentiability, and, respectively, it can also be seen as an element of the concept image of continuity when concentrating on continuity. So this element, which is common for both concept images, forms a link between the

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3 According to Gödel's incompleteness theorem, proven in 1930, an axiomatic system cannot be proven consistent. However, the consistency is a fundamental goal in building the theory of mathematics.

concept images.

The level of coherence of a concept image can be examined by finding from an individual's behavior indications referring either to coherence or incoherence of the concept image. In the analysis presented in this paper, we have mainly concentrated on finding indications of incoherence of an interviewee's concept image of differentiability.

2.3. Previous studies

Several studies concerning mathematics students' reasoning have shown significant indications of a low level of coherence of concept images. In the following, some studies concerning this in the area of calculus or basic analysis are briefly reviewed. In some cases, the results of the study are related to the above criteria of the coherence of a concept image.

With respect to the limit concept, Juter (2005) has shown that students can have contradictory conceptions about the attainability of a limit value of a function so that conceptions which come up in a theoretical discussion differ from conceptions used in problem solving situations. For instance, some students in Juter's study said in a theoretical discussion that a function cannot attain its limit values, but in a problem solving situation they considered it to be possible. This indicates contradictory conceptions inside the concept image (cf. criterion 3) and with respect to the formal theory (cf. criterion 4). One reason for the conception about the unattainability of the limit was an erroneous interpretation of the definition of the limit.

Zandieh's study (1998) considered high-achieving high school students' abilities to relate the formal definition of the derivative to other aspects of their understanding. The results varied between students, and, according to Zandieh, the crucial factors in this were ability to understand mathematical objects and processes also in other contexts than in the symbolic one and, on the other hand, ability to use mathematical symbols as a language to express knowledge in other contexts. This result indicates the importance of connections inside the concept image (cf. criterion 2).

Aspinwall et. al. (1997) have shown how, in the case of the derivative, an uncontrollable use of visual images may become a source of conflicts. In their study a student reasoned on the basis of a graph that a parabola presenting the function x^2 approaches asymptotically a vertical straight line when x increases or decreases enough. On the other hand, he reasoned that the graph of the derivative of this function is a straight line. He regarded these conclusions contradictory to each other. According to the interpretation of Aspinwall et. al., this conflict was caused by the inadequate control in using the visual image. It can also be interpreted that the connections inside the concept images of the function and the derivative were inadequate. A more thorough consideration of connections between the graph and symbolic expression of the function might have made the needed control possible and thus prevented the erroneous conclusion about the asymptotic approach.

Aspinwall and Miller (2001) have discussed possible methods to explore and to improve the coherence of the concept image in the case of the derivative. They have analyzed possible conflicts concerning, for example, the interpretation of the derivative as the slope of the tangent and the relationship between the average and instantaneous rate of change. They have also studied the coherence of the concept image in the case the concept of definite integrals.

It is natural to assume that, compared to students, mathematicians have more coherent concept images of mathematical concepts and thus a more coherent knowledge structure with respect to mathematics as a whole. Some studies have considered this issue. For instance, according to Raman

(2002; 2003), mathematicians are able to see connections, *the key idea*, between heuristic ideas and formal proofs, but students consider these arguments separately, without seeing the connections. Stylianou (2002) has shown that mathematicians, during problem solving processes, very systematically take turns between visual and analytical steps, but many students cannot utilize visual representations in analytical problems at all. These findings have a significant role in explaining the differences in performances between mathematicians and students.

Even though concepts are determined by definitions in the formal axiomatic system, in students' concept images they tend to stay as isolated cells. It has been shown that students, in connection with basic analysis, have difficulties in consulting definitions and that they often avoid using them (Cornu, 1991; Pinto, 1998; Vinner, 1991). This may be an essential problem causing incoherence of a concept image. Pinto's study (1998) revealed that students also have different modes to work with definitions and to deal with a formal theory: *Formal thinkers* attempt to base their reasoning on the definitions, while *natural thinkers* reconstruct new knowledge from their whole concept image (Pinto & Tall, 2001). Both modes have their own advantages and disadvantages with respect to the coherence of a concept image: For instance, if the meaniong to the concept is extracted from the formal defintion, the concept image is well tied to the formal theory, but, on the other hand, the informal imagery may leave poorly connected. However, both modes can lead to a success or a failure (ibid; Pinto, 1998; Pinto & Tall, 1999).

3. Methodology

The student, whose interview is thoroughly analysed in this paper, was selected among eight interviewed students, who had in the written test answered that the function f_3 in Figure 1 was differentiable but discontinuous. This particular interview seemed to offer usable data concerning concluding processes, erroneous conclusions, and coherence of the concept image. The interviewee, called Mark in this paper, was majoring in mathematics. He had studied five years at university, and, according to his own estimate, his success in studies had been on the average level. Mark told that in the future he would like to teach mathematics, by choice, at a lower secondary school.

The main goal of the interviews was to study the students' conceptions about the meaning of the derivative and differentiability and their abilities to understand relationships between the formal definitions and some visual interpretations of these concepts. In the interview, the interviewee was asked to justify the differentiability or nondifferentiability of the functions presented in Figure 3. In some cases continuity was also considered. The functions were given to the interviewee by showing both the symbolic expression of the formulas and the graphs on paper. The interview also included a discussion about the visual meaning of the derivative and differentiability and the relationships between continuity and differentiability.

The functions f_1, f_2 and f_3 were used also in a task of the written test. In this task, the students were asked to determine which of the functions were continuous and which of them were differentiable.

$$f_1(x) = \begin{cases} x+1, & x<1, \\ -2x+6, & x \geqslant 1. \end{cases} \qquad f_3(x) = \begin{cases} x^2-4x+3, & x \neq 4, \\ 1, & x = 4. \end{cases}$$

$$f_2(x) = \begin{cases} x+2, & x<1, \\ -2x+5, & x \geqslant 1. \end{cases} \qquad f_4(x) = \begin{cases} x, & x<1, \\ x+1, & x \geqslant 1. \end{cases}$$

Figure 3: The formulas of the functions used in the interview.

The interviews were semistructured: The main questions were planned in advance, and many additional questions emerged during the interview. The formal definitions of continuity, derivative and differentiability were given to the participants both in the written test and in the interview. The forms of the given definitions are presented in Figure 4. The only tools allowed in the interview were pen and paper. The interviews were videotaped so that the video camera was focused on the paper.

Definitions

Continuity
A function $f : \mathbb{R} \to \mathbb{R}$ is *continuous at a point* $x_0 \in \mathbb{R}$, if and only if the limit $\lim_{x\to x_0} f(x)$ exists and

$$\lim_{x\to x_0} f(x) = f(x_0).$$

A function is *continuous*, if and only if it is continuous at all points in the domain of the function.

Derivative and differentiability
A function $f : \mathbb{R} \to \mathbb{R}$ has a *derivative* or it is *differentiable at a point* $x_0 \in \mathbb{R}$, if and only if the limit

$$\lim_{h\to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

exists. Then the derivative of the function f at the point x_0 is equal to the value of this limit.

A function is *differentiable*, if and only if it is differentiable at all points in the domain of the function.

Figure 4: The definitions of continuity, derivative and differentiability given to the participants in the written test and in the interview (translated from Finnish).

The thorough analysis of Mark's interview mainly applied the principles of the video data analysis procedures presented by Powell et al. (2003). The interview was first transcribed from the video. Then, the transcribed data was divided into episodes so that every episode included reasoning concerning one question, for example, the question of differentiability of one of the presented functions or the question concerning the relationship between differentiability and continuity. After that, the reasoning during the episodes was described and *critical events* (ibid.) with respect to the progress of the reasoning were identified. By comparing episodes and critical events of the interview and by searching common features between them, it was possible to find some features of the interviewee's thinking which were typical during the whole interview.

4. Results concerning erroneous conclusions and the structure of the concept image

4.1. A fundamental change of view

Mark's conceptions about the relationship between continuity and differentiability changed due to the conclusions which he made during the interview. Unfortunately, the change happened to a more erroneous direction.

At the beginning of the interview Mark believed that differentiability presumed continuity. In fact, continuity was the first property which Mark mentioned when the interviewer asked him about the meaning of differentiability (see excerpts 10-11 in Section 4.2.). However, during the interview, Mark reasoned that the function f_4 was differentiable but not continuous. This forced him to change his view, although it was not too easy for him.

(1) Interviewer: Then, how about the question whether differentiability presumes continuity? How do you respond to that?

(2) Mark: If I now here claim that this function is differentiable, it means that it does not presume continuity.

(3) Interviewer: You had a memory that it would presume.

(4) Mark: Yes, I did. This appears to be contradictory…

[...]

(5) Mark: This time, I say that it does not presume continuity!

Observable erroneous conclusions –as well as erroneous conceptions– with respect to the formal theory concerning some concept can be regarded as indicators of the incoherence of the concept image of that concept. Thus, the above-presented conclusion as such reveals that something was wrong with respect to the coherence of the concept image of differentiability. However, in the following analysis the viewpoint is in the opposite direction: The goal is to analyse issues relating to the coherence of the concept image and from this basis to discuss how the above erroneous conclusion was built up. The main attention is focused on Mark's study about the differentiability of the functions f_1–f_4. This process is described in Section 4.3. In Section 4.2 some indications of incoherence of the concept image of differentiability, which came out before Mark started this process, are described.

4.2. Indications of incoherence of the concept image before the study of differentiability of the given functions

The discussion with Mark about the meaning of continuity and differentiability revealed some indications of incoherence in his concept images of these concepts. The clearest indication was his explicit uncertainty about whether the cornerlessness of a graph was a property of continuity or a property of differentiability. This came out when the interviewer asked Mark to explain what the continuity of a function means in practice. Mark began to think about properties of a continuous function:

(6) Mark: A connected graph, a graph of a function which does not jump and... There are no sharp corners... Or does this belong to differentiability?

Mark seemed to be unsure if it is possible to have corners in a graph of a continuous function. At first he guessed that it is not possible:

(7) Mark: There can't be, I guess.

Then Mark tried to argue this by characterizing the continuity on the basis of his subjective, everyday life associations of the word continue:

(8) Interviewer: Why not?

(9) Mark: Why not? It does not continue then. If you drive a car suddenly to a sharp corner, then... it seems not to continue.

Mark did not seem to have a particularly clear conception about the meaning of differentiability, either. First, it was difficult for him to mention any other property of a differentiable function than continuity:

(10) Interviewer: What kind of function is differentiable? What should it be like in order to be differentiable?

(11) Mark: Continuous.

(12) Interviewer: Continuous? Does it have to be something else?

(13) Mark: Can I resort to the definition? If it helped me in some way...

However, Mark did not say anything explicit about the definition. Finally, he mentioned cornerlessness:

(14) Mark: There cannot be (in the graph of a differentiable function) these [...] corners, because we cannot draw a tangent to a sharp corner. Or, in fact, we can draw the tangent almost anyhow we like.

After that the discussion moved to continuity of function f_2. In this discussion Mark changed his view regarding continuity and cornerlessness. In the written test Mark had answered that the function f_2, whose graph includes a corner, is continuous. This is against the view that he presented above (cf. excerpts 6-9). When the interviewer presented this answer to him, he was ready, again using his subjective associations of the word continue, to argue that a graph of a continuous function can include corners:

(15) Mark: It does not break the function, its graph. If we look at this, we can see that it continues. (*He traces the graph with a pen.)*

Above presented hesitances show that in the beginning of the interview Mark did not have very clear conception about the meaning of continuity and differentiability (cf. criteria 1).

Another contradiction between test answers and the conceptions which came out in the interview was following: In the test Mark had answered that function f_3 was discontinuous but differentiable, whereas in the interview Mark considered continuity as a prerequisite of differentiability (cf. excerpts 10-11). The interviewer asked Mark to explain why he had thought function f_3 to be differentiable:

(16) Mark: At the point four the derivative is zero.

(17) Interviewer: Why?

(18) Mark: Because it (value of the function) is a constant!

This excerpt shows that Mark was aware that the derivative of a constant function is zero and that differentiability means the existence of the derivative. However, he applied these facts in an erroneous way for the function f_3, assuming that the above-presented reasoning was really his argument for differentiability during the test. It can be interpreted so that, in the test situation, these

pieces of knowledge had erroneous connections between them in the concept image (cf. criterion 2), and due to this Mark considered the rule concerning the derivative of a constant function to be applicable. Of course, it cannot be claimed that Mark did not know the prerequisites of this rule, but at least in this situation he did not take them into account correctly.

As a whole, the above-presented observations reveal that significant deficiencies with respect to coherence appeared in Mark's concept image of differentiability at the beginning of the interview, before he began to study the differentiability of the functions f_1–f_4. The observed deficiencies mainly concerned the meaning of the concept and the connections between elements of knowledge inside the concept image. However, it has to be noticed that the deficiencies might not have been permanent: The structure of the concept image may have changed already in the situations where the deficiencies appeared.

4.3. Vitality of a method based on the differentiation rules

When solving problems concerning differentiability of piecewise defined functions, Mark in several cases first differentiated both expressions used in the definition of the function by using differentiation rules, and then checked if both expressions obtained an equal value at the point where the expression is changed.

Mark was told to begin by considering the differentiability of the function f_2. First, Mark explained visually, by using tangents, why this function was not differentiable at the point x=1. He explained that it was not possible to draw an unambiguous tangent line at the corner. Then the interviewer asked Mark to calculate the right-hand and left-hand limits for the difference quotient of the function f_2 at the point x=1, when h in the definition of the derivative approaches 0. Mark differentiated the expressions x+2 and -2x+5 and gave the answers 1 and -2. He said:

(19) Mark: The use of the difference quotient would lead to the same result.

Furthermore, the interviewer asked him to calculate this by using the definition. Mark calculated the limit of the difference quotient for the function x+2 and came up with 1:

(20) Mark: h is negative. [...] It becomes h/h, and it is one. And if h approaches zero...

In this calculation Mark did not specify the point x=1 but performed the calculation generally for the function x+2 at a point $x=x_0$. (See Figure 5.) In fact, this is not a correct way to calculate the left-hand limit at the point x=1, because according to the definition of the function f_2 the expression x+2 is not in force at this point.

$$\lim_{h \to 0_-} \frac{x_0 + h + 2 \quad -(x_0 + 2)}{h} = \lim_{h \to 0_-} \frac{h}{h} = \lim_{h \to 0_-} 1 = 1$$

Figure 5: Mark's way to use the definition of the derivative in calculating the left-hand limit of the difference quotient of the function f_2 at the point x=1.

After that he said that the right-hand limit of the difference quotient could be found similarly.

In this way Mark introduced his differentiation method to study differentiability. Excerpt 19 reveals that already before the use of the definition Mark had a clear view that his method was compatible with the definition. It does not explicitly come out from the data why he believed so, but

a possible explanation could be that very often in practice, especially at high school, the use of the definition of the derivative is replaced by the differentiation rules. This perhaps was the origin of Mark's differentiating method. The calculation with the definition offered Mark an additional confirmation for the correctness of his method.

For the function f_1, Mark applied the same method, and it seemed to work very well:

> (21) Mark: This is a similar situation... at x=1... Let's consider the lower part *(refers to the expression)* first. The derivative is minus two for the lower part, and then, if we try when x<1, the derivative approaches one, or it equals one.

Because the derivatives of the expressions x+1 and -2x+6 were not equal, Mark's conclusion was that the function f_1 was not differentiable.

As mentioned above, in the written test Mark had answered that the function f_3 was differentiable (cf. excerpts 16-18). However, in the interview he wanted to apply his differentiation method also to this function. Thus he differentiated the expressions x^2-4x+3 and 1 and came up with the expressions 2x-4 and 0. Because these did not take the equal value at the point x=4, Mark concluded that the function f_3 was not differentiable. He rejected his previous conclusion and believed that the conclusion obtained by applying the differentiation method was the right one. He commented:

> (22) Mark: My view changes when I think over these things more and more.

In this situation Mark met an obvious conflict. He had used two different methods for examining the differentiability of the function f_3, and these methods led to opposite conclusions. In fact, according to the formal theory, both methods were erroneous. Mark, however, believed that the result obtained by the differentiation method was the right one, and he was ready to reject the result obtained by the other method. This can be interpreted as a sign of confidence in the differentiation method: At least, it shows that in this situation his confidence in it was stronger than in the other method. However, Mark did not consider what was wrong in the other method, and thus, this part of the conflict was left unsolved.

The question about the differentiability of the function f_4 was hard for Mark. As before, he started by calculating the derivatives of the expressions used in the definition of the function (the expressions x and x+1) and noticed that these were equal at the point x=1. It is notable that when calculating these, he spoke about the difference quotients:

> (23) Mark: The difference quotients are equal in both domains.

This again suggests that Mark believed that the use of his differentiation method could be substituted for the explicit use of the definition of the derivative. According to Mark, the result forced the function f_4 to be differentiable. However, Mark immediately saw from the graph that the function f_4 was not continuous and he remembered very clearly that continuity is a necessary condition for differentiability. This caused a serious conflict for Mark, but in spite of that, his confidence in his method was strong:

> (24) Mark: Yes, both derivatives are equal if we come either from left or from right. [...] If we think only that they are equal... then it has to be differentiable... But it is not continuous at that point!

In this excerpt, Mark spoke about the equality of "both derivatives", even though, according to the formal theory, differentiability requires the equality of the both-hand limits of the difference quotient. This indicates a confusion between concepts: Perhaps Mark was not able to recognize the difference between the limit of the derivative and the limit of the difference quotient in this

situation.

Then Mark began to doubt his memory. He tried to find another differentiable but discontinuous function. He wondered if the tangent-function (the function $f(x) = tan\ x$) could be one example. Finally, he decided to explore the differentiability of the function f_4 by using the definition of the derivative explicitly. However, he made a mistake in this: He calculated the difference quotients generally at the point $x=x_0$ and separately for the expressions x and x+1 (see Figure 6). This is not the correct method to study differentiability at a point where the defining expression of the function changes. Using the terminology presented in Section 2.1, it can be said that Mark's personal interpretation of the formal concept definition was not consistent with the formal theory in this situation.

$$\frac{x_0 + h - x_0}{h} = 1$$

$$\frac{x_0 + h + 1 - (x_0 + 1)}{h} = 1$$

Figure 6: Mark's way to calculate the left-hand and right-hand derivatives of the function f_4 at the point $x=1$.

Because Mark got equal results from both of these calculations, he concluded that the function f_4 was differentiable:

(25) Mark: One comes from both. It could be reasoned from this that it is differentiable.

(26) Interviewer: Is this your answer?

(27) Mark: Ok, let it be my answer, this time!

Finally, he was ready to break his strong conception that continuity is a necessary condition for differentiability (cf. excerpts 1-5).

Like with the function f_3, Mark again met an obvious conflict with function f_4. Now there were, against each other, his very strong memory that differentiability presumes continuity and the result based on the differentiation method, on which he had relied in the three previous cases. In the case of the function f_3, it was not difficult for Mark to reject the other conclusion, but in the case of function f_4 he felt he could not reject either of the results, even if they were contradictory. In the latter case he was convinced of both results. It seems that the explicit -indeed, erroneous (cf. Figure 6)- use of the definition had a crucial role in the solution of this conflict, but, furthermore, after using the definition, it was not easy for Mark to reject his memory concerning the continuity of a differentiable function (cf. excerpts 25-27 and 1-5).

After the interview, the interviewer gave a brief feedback for Mark about his performance. He, among others, revealed that continuity is a necessary condition for differentiability. In this situation there was not time for an extensive discussion, and Mark's reactions for the feedback were not taped.

5. Discussion

Above we have presented how Mark's interview revealed several indications of the incoherence of his concept image of differentiability, and described his process to study the differentiability of the given four functions. During this process, Mark made many erroneous conclusions. In the following we will discuss which matters can be learnt about mathematical thinking and learning through this analysis which could help us to improve teaching practises.

The above analysis reveals several cases in which **an erroneous conclusion was a consequence of an erroneous way to connect the pieces of knowledge**. In these situations single pieces of knowledge, as such, were correct, but they were connected in an erroneous way. This suggests that the knowledge about relationships is deficient. The explanation of the test answer regarding differentiability of function f_3 (cf. excerpts 16-18) is an illustrating single example of erroneous connections. Mark's differentiation method, also, can be seen to be based on erroneous connections between pieces of knowledge concerning existence of the derivative and differentiation rules.

This analysis shows also that **misconceptions and erroneous conclusions may lead to cognitive structures, which are, at least in some extent, internally coherent, but whose basis is erroneous**. In this study, Mark constructed a structure which was based on the conception that differentiability of a piecewise defined function can be studied by checking if derivatives of the expressions used in the definition of the function obtain an equal value at the point where the expression is changed. This method seemed to work very well in the cases of the functions f_1 and f_2, and in the cases of the functions f_3 and f_4 he rejected results which were in contradiction with it. Mark also became convinced that this method was compatible with the formal definition. With a strong confidence on this method, Mark changed his previous conception about the continuity of a differentiable function. In this way the differentiation method became a key factor for the internal coherence of the concept image of differentiability. Mark reconciled the other conceptions and results with the differentiation method, and the confidence in it maintained the internal coherence of the concept image. In fact, it would have been interesting to continue discussion by revealing for Mark that differentiability really presumes continuity but not giving any other feedback in this phase. This would have broken the internal coherence, and an extensive reconstruction would have been needed to repair it. Therefore, this study also illustrates that **sometimes the fundamental reason for erroneous conceptions can lie deep in the knowledge structure**: The conception that a differentiable function can be discontinuous was a result of a quite extensive reasoning process which was based on an erroneous method to study differentiability of piecewise defined functions. In practice, discussion with other people, the use of literature or another kind of social interaction often influences the process of constructing the knowledge structure and prevents the development of very wide-ranging erroneous structures. This is one reason why the social interaction in its different forms is important in the learning of mathematics. It contribute to recognizing misconceptions even by judging some conceptions directly erroneous or by bringing out situations where conflicts might be created. In this way the misconceptions are probably recognized earlier than it may be happened in an individual study.

This study also brings out some issues about the role of the definition in constructing the concept image. First, **it is important that the personal interpretation of the formal concept definition is correct**. If Mark had used the definition of the derivative in a correct way when he studied differentiability of the function f_4 (cf. Figure 6; excerpts 25-27), he would have met a conflict which could have forced him to re-examine his differentiation method. Already when calculating the left-hand-limit for the difference quotient in the case of the function f_2 (cf. Figure 5; excerpts 19-20),

a more careful examination of the definition might have had a similar effect. Another conclusion which comes out from this study is that **the definition −or, in fact, its personal interpretation− should have a central role in the reasoning concerning the concept in question**. Reasoning concerning the concept should be based on the definition, or, at least, an individual should be aware why the reasoning is in accordance with the definition. In Mark's reasoning the definition appeared to have only a minor role. The only situation where Mark without the interviewer's intervention used the definition was the conflict in the case of the function f_4, but even then the definition was not the primary method to resolve the conflict. Thus, it seems that Mark had a tendency to avoid using the definition. Instead the definition, the differentiation method became a crucial criterion for the differentiability in his reasoning. When derivative is considered for the first time in mathematics education, for example, in upper secondary school, the definition is usually left to the background, and the use of the differentiation rules are emphasized. This may be one reason why Mark avoided the use of the definition. However, as discussed above, Mark probably believed that his method was compatible with the definition. This maybe the reason why he did not feel a need to use the definition. On the other hand, the data does not explicitly show whether Mark had understood the crucial role of the definition: Did he understand that the definition determines the final truth regarding the concept, or did he consider the definition only as one description of the meaning of the concept among others? The observation that the definition-based argument seemed to resolve the conflict in the case of the function f_4 (cf. Figure 6; excerpts 25-27) defends the former view.

The observations concerning students' study of differentiability made by Tsamir et. al. (2006) are quite similar to the results of this study. In their study, three prospective teachers were able to give a correct definition for the derivative, but, in spite of that, they did not use it in the problem solving, and they reached erroneous conclusions. When studying the differentiability of the absolute value function ($f(x)=|x|$), one of these students used the same kind of a method as Mark.

This study shows that there exists a notable interaction between the structure of the concept image and conclusions which are attained by reasoning. Single conclusions may have a wide-ranging influence on the structure of the concept image, and on the other hand, conclusions depend on the existing structure of the concept image. The list of criteria for coherence of a concept image offers a framework for analysing mathematical reasoning, especially reasoning concerning one concept. It could also be interesting to analyse a longer-term learning process by using this framework and in this way study how the coherence of a concept image is developed.

Almost every day mathematics teachers meet in their work erroneous conclusions made by students. Many of these are random careless mistakes, but others are based on a deliberate reasoning. In the latter case a careful personal discussion with a student may be needed in order to find out how deep in the knowledge structure the problems lie. In the learning process it is important that the pieces of knowledge which an individual learns and which he/she already knows form coherent entities. How could teachers and designers of textbooks and curricula take this goal into account? This study highlights two factors: First, the fundamental role of definition should be emphasized. Therefore, in teaching tasks in which the definition is really needed as a central part of reasoning should be used. Second, there should be tasks which lead students to critically reflect qualities of mathematical concepts and relationships between concepts. Especially, controlled conflict situations may offer fruitful starting points for this kind of reflection.

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Chess and problem solving involving patterns

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Abstract: In this paper we present the context and results from a study, with 3rd to 6th grades children, about the relationship between chess and problem solving involving geometric and numeric patterns. The main result of this study is the existence of a relation between strength of play and patterns involving problem solving. We have included in the beginning an analysis of chess as a context for elementary mathematics problems, also showing its richness historically.

Keywords Strategy games; Chess; Problem solving, Patterns; Primary and middle school; Correlational study.

Strategic games and patterns in the curriculum of elementary teaching

The Principles and Standards for School Mathematics point to the identification of patterns and the use of strategic games, in the mathematics teaching (NCTM, 1991). Keith Devlin (2002, p.12) defines mathematics as being "the science of patterns". In addition, the document of the fundamental competences for Elementary School published by the Department of Basic Education in Portugal defines mathematics as the science of regularities (DEB, 2001). This document stresses the identification and exploration of patterns, as we can see from its continuous allusion in several topics of mathematics curriculum: numbers and operations, geometry algebra and functions. For each of these fields, mathematical abilities to develop in the Elementary School are explicit: "the predisposition to recognize numerical patterns in mathematical and not-mathematical situations (…) the aptitude to recognize and to explore geometric patterns (…) the predisposition to recognise patterns and regularities and to formulate generalizations in different situations, in numeric and geometric contexts".

The Curriculum of Elementary Teaching Mathematics points to the use of strategic games in problem solving context. Children like to play games and teachers must make use of the benefits of games environment to promote mathematics education. Chess is pointed as one of the games that increase "the capacity to accept and to follow a rule; the development of the memory; the agility of the way of thinking; the aim for challenge; the construction of personal strategies" (DEB, 1998). The curriculum also stresses the importance of the strategy games in the development of problem

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solving abilities. And it also points that strategy games contribute for the development of mathematical capacities, connecting reasoning, strategy and reflection with challenge and competition in a very rich and playful form (DEB, 1998).

Chess studies: some conclusions

There are several studies about chess and its implications on children education. In those studies the main conclusions are that chess promotes academic performance, especially problem solving strategies, increases memory, concentration, scores in IQ tests, critical thinking, and develops visual and spatial abilities and the capacity to identify patterns (Liptrap, 1998; Dauvergne, 2000 Thompson, 2003; Stefurak, 2003; Brenda, 2003). Studies focusing on the effect of children playing chess, disclose that chess players develop critical thinking, self-confidence, self-respect, concentration (Stefurak, 2003), and problem solving skills (Dauvergne, 2000).

The mathematics curriculum in Canada make use of chess to teach logic from grades 2 to 7 and with this curriculum problem solving scores improved from 62% to 81% (Liptrap, 1998).

Fergunson describes a study from Venezuela that relates improvement in most students IQ scores after only 4.5 months of systematic chess study. The results of this study led the Venezuelan government to introduce chess lessons at schools, since 1988/89 (Dauvergne, 2000).

A study by Murray Thompsom (2003) disclosed a significant effect between playing competitive chess and better academic performance, relating that best students tend to have also better IQ levels. However, it relates the possibility of playing chess contributing to students IQ, being the benefit of chess playing absorbed by the variable IQ. This researcher claims that playing competitive chess demands great skills of concentration, logical thinking and of projecting possible positions of pieces, helping to develop visual and spatial abilities. The van Hiele Model of Geometry Thought asserts that children learn geometry sequentially through five levels of understanding, being the first one, the level of visualization, where children understand the shapes for its appearance, as total entities. In elementary school teachers must help students to move from the visual level to the level of analysis (according to van Hiele Model), carrying on activities that develop the capacity of visualization (Ponte & Serrazina, 2000). A way for this development is the game of chess.

The speed of visual perception requires multiple and codified settings, being important the capacity to codify information and to identify significant places to focus attention. And it happens that experienced chess players memorize positions with a bigger number of pieces than less experienced players (Charness, Reingold, Pomplun & Stampe, 2001).

The study

The goal of this study concerns the verification of a relationship between the game of chess and patterns. More specifically, it is intended to identify the capacity to solve problems based on patterns, of chess players and of non chess players, to verify the relation between this capacity and the capacity to play chess. We also intend to identify the relation between these capacities and age, schooling years, gender and mathematics grades.

Methodology was based on a quantitative paradigm, with a correlational design. According to Cohen and Manion (1989), correlational studies are appropriate in educational research when there is a need to discover or clarify relationships and little or no previous research has been undertaken.

In fact, "*the investigation and its outcomes may then be used as a basis for further research or as a source of additional hypotheses*" (Cohen & Manion, 1989, p.161).

The sample of this study was constituted by 437 students from 3rd to 6th grades. To collect data the following instruments were used: a questionnaire and one test on problem solving based on geometric and numerical patterns that was constructed and validated for this study.

The following research questions have been followed:

1. Is there a relationship between playing chess and solving problems involving patterns?
2. Is there a relationship between solving problems with geometric patterns or with numeric patterns and playing chess?
3. Is there a relationship between solving problems based on patterns and age, schooling year, gender and mathematical levels of achievement?

Now we are going to identify the variables used in the research and explain how they have been implemented. Problem solving capacities that involve patterns have been investigated using a test constructed and validated for this study. The capacity to play chess was measured through the ELO rating of the players, as published by the chess federation, at the time when the test was implemented. ELO rating is a "quantitative system based on exponential smoothing of a player's rating depending on the actual proportion of victory compared with that expected given the rating of the opponents. (…) there is a direct relationship in the difference between two players' rating and their chance of victory – irrespective of the magnitude of the players' ratings. Thus a player who is ranked 100 points above an opponent will have a 64% chance of victory" (Clarke & Dyte, 2000, p. 586)

To collect data for the variables play chess, age, schooling year and gender we used a survey. The school grades in mathematics information were measured using the 1st assessment of the year.

The population of this study consisted of students from 3rd to 6th grades, chess players and students with school chess. This population was organized in the following way: students from 3rd to 6th grades from Braga; chess players and students with school chess from clubs of several areas of Portugal. There were involved 380 students from 3rd and 4th grades and 368 students from 5th to 6th grades.

The option for this selection was based on the proximity of schools to the residence of the researcher together with the opportunity to find chess players in national competitions. This extensive population contributed to a sample with a significant number of students. It must be clarified that chess players also included students with chess in school who participated in chess competitions. As instruments to collect data we used a survey and a test. The test included problems that included numeric and geometric patterns.

In the elaboration of the questions the following structure was used:

- identification of the following element of a pattern;
- identification of the element that doesn't fit in the pattern;
- producing patterns.

This structure was based on the structure of similar questions used by other authors, such as Krutetskii (1976). It is also based on the conclusions of Krutetskii's research, stating the existence of three types of mathematical ability: analytical, geometric and harmonic (combining the other two). The test was validated by a panel constituted by two university teachers of mathematics, one teacher

of mathematics of the 2nd cycle and one 1st cycle teacher specialized in mathematics. From the analysis of the test by the elements of the panel we have selected 26 questions.

A pilot application of the test was made on a sample of 105 students: 20 from 2nd grade, 23 from 3rd grade, 22 from 4th grade, 23 from 5th grade and 17 from 6th grade. The lesser number of pupils from 6th grade is explained by the fact that three pupils have missed classes on that day. The test has been implemented by the researcher.

The elaboration of the test correction criteria was based on the principles reported by Charles, Lester and O'Daffer (1992) in the point “Analytic Scoring Scale”. The scoring of the test was a very difficult moment that demanded organization and persistence due to the great number of questions: 437 tests with 24 questions, totalizing 10488 questions to score (excluding the pilot test). To ascertain test reliability we used Cronbach's Alpha, which measures the internal consistency of items. Cronbach’s Alpha must be greater than 0.70. However, there are some references accepting values lower than 0.70 (Santos, 1999).

Initially, with 105 pupils and 26 questions, Cronbach's Alpha was 0.835. However, considering school grades, it was verified that for the 2nd year Cronbach's Alpha was just 0.217. The value of the Cronbach's Alpha has to be at least 0.70 (Fraenkel & Wallen, 1990) and the value for the 2nd grade would be far too below of the recommended value. Removing this grade, Cronbach's Alpha got a value of 0.756. To improve the reliability level we decided to remove two questions: question 5a) of the first part (P5a) and question 2 of the second part (S2). Removed these two questions, we analyzed the value of Cronbach's Alpha. The Cronbach's Alpha established was 0.763, a value appropriate to start the study.

To test the reliability of the scorer we used 30 tests. After the interval of one month between ratings the correlation coefficient was 0.99, significant at the 0.01 level. With this value we had good conditions to continue scoring tests.

The statistical treatment was done using SPSS for Windows, version 13.0. In the analysis, different statistical procedures had been used, adjusted to each case. Cronbach's Alpha was used to measure internal consistency. To test normality, that is, to verify if the distribution of data was parametric, we used the Kolmogorov-Smirnov test. To observe the correlation between problem solving involving patterns and the ELO of chess players we used the Pearson (r) coefficient, when the data was parametric, using the square of this coefficient (R^2) for interpretation (Field, 2000). R^2 can be interpreted as a ratio (Chen & Popovich, 2002). When one of the variables was dichotomic, as in gender, we used the point-biserial correlation (rpb) coefficient (Field, 2000). The Spearman coefficient was used when the distribution of data was non parametric, since it is not affected by the asymmetry of the distribution. Kendall's Tau (τ) coefficient was used for the variables school year and levels of achievement in mathematics, as they contain a considerable amount of ties.

The partial coefficient correlation was used to verify the correlation between the total classification obtained on the test and ELO, controlled by age, school year, gender and levels of achievement in mathematics. To make the interpretation of the correlation coefficients we used the following boundaries:

- Correlations between 0.2 and 0.35 reveal a small relationship between variables, too small to make predictions;
- Correlations between 0.35 a 0.65 are often found in educational research. They may have theoretical and practical importance depending on context. They allow for group predictions (Cohen & Manion, 1989; Fraenkel & Wallen, 1990).

Results of the study

In this study we intended to investigate the existence of a relation between a number of research variables. Now we are going to answer to each of the research questions. Nevertheless, it is also important to describe in more depth some results out of the scores students have obtained in the test.

Test results

The capacity to identify patterns was measured after one test that was constructed and validated for this study. It was verified that pupils were able to identify patterns, according to the test average. Concerning each of the parts of the test (geometric and numerical) we could notice that students, in general, had no difficulty in answering to the first part, and the pupils of 3rd grade exhibited great difficulties in the second part of the test. We have also verified that the score on the test in average increases as the school year increases. Analyzing the test scores in function of playing chess, we have verified that chess players had better scores in the test, being more evident using the scores of the second part of the test. Therefore we can mainly conclude that students that play chess appear to be the ones that better identify patterns. And more precisely, students that are chess players do identify numerical patterns better than those that do not play chess. In turn, we conclude that differences in the identification of patterns between players and students that have school chess are not significant. In this research we have also verified that most students discover geometric patterns more easily than numeric patterns. Inversely, chess players find more easily numerical patterns.

Playing chess and solving problems involving patterns

As to the relationship between the capacity to play chess and solving problems involving patterns, some conclusions were drawn:

a) Strength of play is positively related to problem solving involving patterns with a coefficient of correlation $r = 0.458$ (table 1);
b) School grade affects the relationship between strength of play and problem solving based on patterns. However when we exclude its effects, still the relationship is above 0.38;
c) Age and gender affect slightly the relation between strength of play and problem solving involving patterns. But its effects are not significant.

		Total	EloTeste
Total	Pearson Correlation	1	,458**
	Sig. (1-tailed)		,000
	N	437	65
EloTeste	Pearson Correlation	,458**	1
	Sig. (1-tailed)	,000	
	N	65	65

**. Correlation is significant at the 0.01 level (1-tailed).

Table 1: Correlation between test scores and ELO rating

Taking into account these results we can conclude that playing chess well seems to constitute a good foundation to identify patterns. This is in conformity with the recommended to use strategy games in the curriculum.

Playing chess and solving problems involving numeric and geometric patterns

The capacity to identify geometric patterns was measured using the first part of the test and the capacity to identify numerical patterns using the second part. As to the capacity to identify geometric patterns we can conclude that it was positively related with strength of play. However it is a not too strong relation. As we can observe (table 2) the correlation coefficient is $r = 0.320$.

		somaP	EloTeste
somaP	Pearson Correlation	1	,320**
	Sig. (1-tailed)		,005
	N	437	65
EloTeste	Pearson Correlation	,320**	1
	Sig. (1-tailed)	,005	
	N	65	65

**. Correlation is significant at the 0.01 level (1-tailed).

Table 2: Correlation between the scores of the first part of the test and ELO rating

Concerning the capacity to identify numerical patterns, we can conclude that it is also positively related to strength of play and this relation is stronger than the preceding. As we can see in table 3, a correlation coefficient of $r = 0.463$ between strength of play and the capacity to identify numerical patterns was obtained.

		somaS	EloTeste
somaS	Pearson Correlation	1	,463**
	Sig. (1-tailed)		,000
	N	437	65
EloTeste	Pearson Correlation	,463**	1
	Sig. (1-tailed)	,000	
	N	65	65

**. Correlation is significant at the 0.01 level (1-tailed).

Table 3: Correlation between scores of the second part of the test and ELO rating

Based on these results we can conclude that there is a relationship between the ability to solve problems involving numeric or geometric patterns and the ability to play chess, being stronger in the case of numerical patterns.

Relation between solving problems based on patterns and age, school grade, gender and mathematics levels of achievement

Concerning the third research question, we are now going to put forward some conclusions on the relations between solving problems based on patterns and age, school grade, gender and mathematics levels of achievement. As a result, we can conclude that:

a) Playing or not playing chess has no relation with problem solving involving patterns ($r = 0.13$);
b) There's a weak negative relationship between the ability to solve problems involving patterns and students date of birth ($r = -0.25$);
c) There's a weak positive relationship between the ability to solve problems involving patterns and school grade ($r = 0.23$);
d) Belonging to feminine or masculine gender is not related to the ability to solve problems involving patterns ($r = 0.03$);
e) There's a weak relationship between the ability to solve problems involving patterns and mathematics levels of achievement ($r = 0.22$).

Conclusions

The results of this study do not allow us to go outside the population of elementary school students, being pertinent for the studied population. Teaching students to play chess well may constitute a strategy to help students to identify patterns. Therefore we think that it would be desirable that teachers invest on chess systematic teaching so that their students become better players, and in order to respect curriculum guidelines. However we are aware that no implications can be set between the two, and more research should be developed in order to construct such implication.

The test reveals that, inversely to others students, chess players perform better on numerical patterns rather than on geometric patterns. We think it has become relevant to look for the reasons inherent to this difference. Why good chess players identify numerical patterns better than others students? The answer to this and other questions could be the aim of new research. Finally, chess is not the only strategy game. And the Curriculum of Elementary Teaching also refers to others games like draughts and mastermind. Would these games have the same results as we had with chess? We recommend more research in order to find analogous relations between other strategy games and problem solving involving patterns.

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Associative Operations on a Three-Element Set

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Abstract: A set with a binary operation is a fundamental concept in algebra and one of the most fundamental properties of a binary operation is associativity. In this paper the authors discuss binary operations on a three-element set and show, by an inclusion-exclusion argument, that exactly 113 operations out of the 19,683 existing operations on the set are associative. Moreover these 113 associative operations are accounted for by means of their operation tables.

Keywords: binary operations; associativity; inclusion-exclusion principle; recreational mathematics

1. Introduction

It may catch someone by surprise that the number of distinct binary operations on a set of only three elements is as large as 19,683. To prove this is an easy calculation; there are 3 different answers for each of the 9 seats of a 3×3 operation table so the number of distinct operations is 3^9. The number of commutative operations on the set can be calculated in a similar way, resulting in $3^6 = 729$ operations. The number of operations for which there exists an identity in the set is $3 \cdot 3^4 = 3^5 = 243$ and for the same number of operations there exists a zero. This calculation is straightforward, but no easy calculation and no educated guess, seems to give the answer to the following question:

"How many binary operations on a three element set are associative?"

The objective of this paper is to answer this question. In other words to decide how many of the 19,683 different binary operations on a three-element set are associative. The authors have deliberately chosen to use algebraic concepts and tools to arrive at their conclusion rather than programming a computer to do the task. Overall the argument is inclusive-exclusive and operations are grouped together according to the algebraic properties they share. Initially the 243 operations for which there exists an identity in the set are studied, followed by the 243 operations where there exists a zero and finally the remaining 19,215 operations for which there neither

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exists an identity nor a zero (18 contain both) are studied. For the last mentioned operations further classification is needed.

We point out that the number of operations, the number of commutative operations and the number of operations containing an identity/zero is no harder to calculate for an n-element set than for a set of three elements. These numbers are $n^{(n^2)}$, $n^{\frac{n(n+1)}{2}}$ and $n^{(n-1)^2+1}$ respectively. To the interested reader we recommend the following references for further reading: [**1**], [**2**].

2. General Concepts

A *binary operation* (hereafter referred to only as an *operation*) on a set S is a rule that assigns to each ordered pair (a, b), where a and b are elements of S, exactly one element, denoted by ab, in S. A set S with an operation is said to be *closed* under the operation and in general if H is a subset of S then H is *closed* if ab is in H for all a and b in H. Quite often an operation on a set is referred to as multiplication. Throughout this section each of the sets S and S' will be closed under operation.

An operation on S is *commutative* if $xy = yx$ for every x and y in S. An operation on S is *associative* if $x(yz)= (xy)z$ for every x, y and z in S. A *semigroup* is a set S with an associative operation. If a is an element of a semigroup and n is a natural number then a^n is defined to be the product $aaa...a$, of n factors. An element x of S is said to be an *idempotent* if $xx = x$, an element e of S is said to be an *identity* if $ex = x$ and $xe = x$ for all x in S, and an element z of S is said to be a *zero* if $zx = z$ and $xz = z$ for all x in S.

An *isomorphism* between S and S' is a one-to-one function φ mapping S onto S' such that $\varphi(xy) = \varphi(x)\varphi(y)$ for all x and y in S. If there exists an isomorphism between S and S', then S and S' are said to be *isomorphic*, denoted $S \approx S'$.

An *anti-isomorphism* between S and S' is a one-to-one function φ mapping S onto S' such that $\varphi(xy) = \varphi(y)\varphi(x)$ for all x and y in S. If there exists an anti-isomorphism between S and S', then S and S' are said to be *anti-isomorphic*, denoted $S \approx a\ S'$.

3. Useful Theorems

The following theorems about sets, S and S', closed under operations are well known and easy to prove:

Theorem 1 If an identity exists in S then it is unique.
Theorem 2 If a zero exists in S then it is unique.
Theorem 3 In a set S of more than one element, an identity and a zero have to be distinct.
Theorem 4 If there exists an isomorphism between S and S' and the operation on S is associative then the operation on S' is also associative.
Theorem 5 If there exists an anti-isomorphism between S and S' and the operation on S is associative then the operation on S' is also associative.

Theorem 6 If S is a finite set, the operation on S is defined by an operation table A and A^{T} (the transposition of A) defines an operation on the set $S' = S$, then there exists an anti-isomorphism between S and S'.

4. Example

Consider a set $S = \{a, b\}$ with two elements. The number of different binary operations on this set is 16. The corresponding operation tables are listed below:

$$\begin{array}{c|cc} 1 & a & b \\ \hline a & a & a \\ b & a & a \end{array} \quad \begin{array}{c|cc} 2 & a & b \\ \hline a & a & a \\ b & a & b \end{array} \quad \begin{array}{c|cc} 3 & a & b \\ \hline a & a & a \\ b & b & a \end{array} \quad \begin{array}{c|cc} 4 & a & b \\ \hline a & a & a \\ b & b & b \end{array} \quad \begin{array}{c|cc} 5 & a & b \\ \hline a & a & b \\ b & a & a \end{array} \quad \begin{array}{c|cc} 6 & a & b \\ \hline a & a & b \\ b & a & b \end{array}$$

$$\begin{array}{c|cc} 7 & a & b \\ \hline a & a & b \\ b & b & a \end{array} \quad \begin{array}{c|cc} 8 & a & b \\ \hline a & a & b \\ b & b & b \end{array} \quad \begin{array}{c|cc} 9 & a & b \\ \hline a & b & a \\ b & a & a \end{array} \quad \begin{array}{c|cc} 10 & a & b \\ \hline a & b & a \\ b & a & b \end{array} \quad \begin{array}{c|cc} 11 & a & b \\ \hline a & b & a \\ b & b & a \end{array} \quad \begin{array}{c|cc} 12 & a & b \\ \hline a & b & a \\ b & b & b \end{array}$$

$$\begin{array}{c|cc} 13 & a & b \\ \hline a & b & b \\ b & a & a \end{array} \quad \begin{array}{c|cc} 14 & a & b \\ \hline a & b & b \\ b & a & b \end{array} \quad \begin{array}{c|cc} 15 & a & b \\ \hline a & b & b \\ b & b & a \end{array} \quad \begin{array}{c|cc} 16 & a & b \\ \hline a & b & b \\ b & b & b \end{array}$$

By studying the operation tables we come to the following conclusions:
Eight operations are commutative. See tables 1, 2, 7, 8, 9, 10, 15 and 16.
Eight operations are associative. See tables 1, 2, 4, 6, 7, 8, 10 and 16.
Six operations are both commutative and associative. See tables 1, 2, 7, 8, 10 and 16.
For four operations there exists an identity in S. See tables 2, 7, 8 and 10.
For four operations there exists a zero in S. See tables 1, 2, 8 and 16.
For two operations there exist both an identity and a zero in S. See tables 2 and 8.

For the purpose of finding the number of associative operations it is not necessary to study all sixteen tables, we can group together isomorphic and anti-isomorphic tables (see Section 3). For this purpose we use the one-to-one function from S onto S' that interchanges a and b. The result is the following: $1 \approx 16$, $2 \approx 8$, $3 \approx a\,5$, $3 \approx 12$, $3 \approx a\,14$, $4 \approx a\,6$, $5 \approx a\,12$, $5 \approx 14$, $7 \approx 10$, $9 \approx 15$, $11 \approx a\,13$, $12 \approx a\,14$ which cuts the number of tables to be studied down to seven. See tables 1, 2, 3, 4. 7, 9 and 11.

5. Operations on a Three-Element Set

As mentioned in the introduction, the number of possible binary operations on a set of three elements is 19,683. Of these 729 are commutative and for 243 different operations there exists an identity, and likewise for 243 operations there exists a zero. We now proceed to answer the question: How many associative operations exist on a set of three elements?

For a three-element set S proving associativity for a given operation amounts to verifying 27 different equations:

$$(xy)z = x(yz), \text{ where } x, y \text{ and } z \text{ are elements of } S.$$

A single counterexample suffices to show that a given operation is not associative. Clearly counterexamples need not to be unique. When referring to a particular equation involving the elements x, y and z of S, we shall simply refer to it as xyz. It can easily be seen that if one of the elements x, y and z is an identity or a zero then the equation $(xy)z = x(yz)$ holds.

The analysis of the associative operations on a three-element set $S = \{a, b, c\}$ will now be divided into two steps. First we assume the existence of an identity or a zero, since this considerably increases the likelihood of an associative operation. Subsequently operations for which there neither exists an identity nor a zero will be discussed.

5.1. Operations for which there exists an identity or a zero

5.1.1. Operations for which there exists an identity

Let us assume that a is an identity. This determines 5 of the 9 places in a 3×3 operation table and leaves 81 different possibilities of filling in the remaining 2×2 subtable. For 16 of those subtables the subset $\{b, c\}$ is closed with respect to the operation. According to previous discussion of associative operations in a two element set (Section 4) one can find a counterexample to associativity for 8 of those 16 operations and such an example will also give a counterexample to the associativity of the 3×3 operation table. The 8 remaining 2×2 operation tables show an associative operation and a being an identity assures associativity of the 3×3 table. These 8 associative operations are shown below:

	a	b	c
a	a	b	c
b	b	b	b
c	c	b	b

	a	b	c
a	a	b	c
b	b	b	b
c	c	b	c

	a	b	c
a	a	b	c
b	b	b	b
c	c	c	c

	a	b	c
a	a	b	c
b	b	b	c
c	c	b	c

	a	b	c
a	a	b	c
b	b	b	c
c	c	c	b

	a	b	c
a	a	b	c
b	b	b	c
c	c	c	c

	a	b	c
a	a	b	c
b	b	c	b
c	c	b	c

	a	b	c
a	a	b	c
b	b	c	c
c	c	c	c

Now let us consider the operations where the 2×2 subtable is not closed, in other words the subset $\{b, c\}$ is not closed. There are 4 possibilities: a can appear exactly once, exactly two times, exactly three times or four times in the subtable. We investigate each case separately.

i) a appearing exactly once

The number of corresponding 3×3 tables is 32 and only two of them turn out to be associative:

	a	b	c
a	a	b	c
b	b	a	c
c	c	c	c

	a	b	c
a	a	b	c
b	b	b	b
c	c	b	a

In both tables a is on the diagonal which must be the case since otherwise *cbc* or *bcb* give a counterexample to associativity. In the table on the left, c is a zero, and in the other one, b is a zero. Again this must be the case since otherwise counterexamples to associativity can be found among: *bcc*, *ccb*, *cbb*, *bbc*.

ii) a appearing exactly two times
The corresponding 3×3 tables are 24 and only one of them turns out to be associative:

	a	b	c
a	a	b	c
b	b	c	a
c	c	a	b

In all other cases a counterexample may be found among: *bbc*, *ccb*, *cbb*, *bcc*.

iii) a appearing exactly three times
The corresponding 3×3 tables are 8 and none of them associative. Counterexamples may be found among: *bbc*, *ccb*.

iv) a appearing four times
The corresponding 3×3 table is unique and not associative. Counterexample *bbc*.

It has now been shown that exactly 11 operations on S for which a is an identity, are associative and the respective operation tables have been listed. The corresponding operation tables where b is an identity are likewise 11, and additional 11 tables have c as an identity. One can conclude (see Section 3) that the number of associative operations on S for which there exists an identity (a, b or c) amounts to 33.

5.1.2. Operations for which there exists a zero

Let us assume that a is a zero. We first consider operations where the subset $\{b, c\}$ is closed and as before, when a was assumed to be an identity, this gives exactly 8 associative operations:

	a	b	c
a	a	a	a
b	a	b	b
c	a	b	b

	a	b	c
a	a	a	a
b	a	b	b
c	a	b	c

	a	b	c
a	a	a	a
b	a	b	b
c	a	c	c

	a	b	c
a	a	a	a
b	a	b	c
c	a	b	c

	a	b	c
a	a	a	a
b	a	b	c
c	a	c	b

	a	b	c
a	a	a	a
b	a	b	c
c	a	c	c

	a	b	c
a	a	a	a
b	a	c	b
c	a	b	c

	a	b	c
a	a	a	a
b	a	c	c
c	a	c	c

Next let us consider the operations where the 2×2 subtable is not closed with respect to $\{b, c\}$. As before there are 4 possibilities: a can appear exactly once, exactly two times, exactly three

times or four times in the subtable. We again investigate each case separately.

i) *a* appearing exactly once
The corresponding 3×3 tables are 32 and only two of them turn out to be associative. These are the tables:

	a	*b*	*c*
a	*a*	*a*	*a*
b	*a*	*a*	*b*
c	*a*	*b*	*c*

	a	*b*	*c*
a	*a*	*a*	*a*
b	*a*	*b*	*c*
c	*a*	*c*	*a*

In all other cases counterexamples to associativity can be found among: *bbb*, *bbc*, *bcb*, *bcc*, *cbb*, *cbc*, *ccb*, *ccc*.

ii) *a* appearing exactly two times
The corresponding 3×3 tables are 24 and 5 of them turn out to be associative:

	a	*b*	*c*
a	*a*	*a*	*a*
b	*a*	*b*	*a*
c	*a*	*a*	*c*

	a	*b*	*c*
a	*a*	*a*	*a*
b	*a*	*a*	*a*
c	*a*	*b*	*c*

	a	*b*	*c*
a	*a*	*a*	*a*
b	*a*	*b*	*c*
c	*a*	*a*	*a*

	a	*b*	*c*
a	*a*	*a*	*a*
b	*a*	*a*	*b*
c	*a*	*a*	*c*

	a	*b*	*c*
a	*a*	*a*	*a*
b	*a*	*b*	*a*
c	*a*	*c*	*a*

In other cases counterexamples can be found among the same conditions as given in i).

iii) *a* appearing exactly three times
The corresponding 3×3 tables are 8 and 4 turn out to be associative:

	a	*b*	*c*
a	*a*	*a*	*a*
b	*a*	*b*	*a*
c	*a*	*a*	*a*

	a	*b*	*c*
a	*a*	*a*	*a*
b	*a*	*c*	*a*
c	*a*	*a*	*a*

	a	*b*	*c*
a	*a*	*a*	*a*
b	*a*	*a*	*a*
c	*a*	*a*	*b*

	a	*b*	*c*
a	*a*	*a*	*a*
b	*a*	*a*	*a*
c	*a*	*a*	*c*

Counterexamples for the other 4 tables may be found among: *bbc*, *bcc*, *cbb*, *ccb*.

iv) *a* appearing four times
The corresponding 3×3 table is unique and clearly associative:

	a	*b*	*c*
a	*a*	*a*	*a*
b	*a*	*a*	*a*
c	*a*	*a*	*a*

It has been shown that exactly 20 operations on *S*, where *a* is a zero, are associative. The respective operation tables have been listed. The corresponding operation tables where *b* is a zero are likewise 20 and additional 20 tables have *c* as a zero. One can conclude (see Section 3) that the number of associative operations on *S* for which there exists a zero (*a, b* or *c*) amounts to 60.

5.1.3. Operations for which there exists an identity and a zero

The number of operations on S having both an identity and a zero is readily seen to be 18; there are 3 possibilities for choosing the identity (a, b or c) for each one of these, 2 possibilities for choosing the zero (see Section 3) and finally 3 possibilities (a, b or c) to fill in the remaining seat of the operation table. To show that each one of these 18 operations is associative we assume, without loss of generality, that a is the identity and b the zero. Then the only condition for associativity that needs to be checked is ccc which holds in all three cases; $cc = a$, $cc = b$, $cc = c$:

$$\begin{array}{c|ccc} & a & b & c \\ \hline a & a & b & c \\ b & b & b & b \\ c & c & b & a \end{array} \qquad \begin{array}{c|ccc} & a & b & c \\ \hline a & a & b & c \\ b & b & b & b \\ c & c & b & b \end{array} \qquad \begin{array}{c|ccc} & a & b & c \\ \hline a & a & b & c \\ b & b & b & b \\ c & c & b & c \end{array}$$

Summary of Section 5.1.

Section 5.1.1 is devoted to operations where there exists an identity in S. The total number of these is 243; 81 operation for each of a, b or c being the identity. The number of associative operations having a for an identity is shown to be 11 and accordingly the total number of associative operations with an identity (a, b or c) is 33. Section 5.1.2 is devoted to the case where there exists a zero in S. As for the identity the total number of operations with a zero is 243; 81 for each of a, b or c being the zero. The number of associative operations having a for a zero is shown to be 20 and hence the total number of associative operations on S with a zero is 60. In Section 5.1.3 the number of associative operations with both an identity and a zero is counted as 18 and an argument given that each one of these operations is associative. We summarize our results of Section 5.1: The number of associative operations on S with an identity, a zero or both an identity and a zero in S is 75:

$$33 + 60 - 18 = 75.$$

5.2. Operations for which there neither exists an identity nor a zero

Having considered all possible operations for which there exists an identity or a zero we will from now on, assume that there neither exists an identity nor a zero in S. The following theorem is of great help.

Theorem If an operation on $S = \{a, b, c\}$ is associative then S contains an idempotent.

Proof: Consider the elements a, a^2, a^3 and a^4 of S. These, of course, are at most three different elements of S so two of them are equal and one of the following six cases must hold true:

i) $a^2 = a$ and a is an idempotent.

ii) $a^3 = a$ and a^2 is an idempotent as $(a^2)^2 = a^4 = a^3a = aa = a^2$.

iii) $a^4 = a$ and a^3 is an idempotent as $(a^3)^2 = a^6 = a^4a^2 = aa^2 = a^3$.

iv) $a^3 = a^2$ and a^2 is an idempotent as $(a^2)^2 = a^4 = a^3a = a^2a = a^3 = a^2$.
v) $a^4 = a^2$ and a^2 is an idempotent
vi) $a^4 = a^3$ and a^3 is an idempotent as $(a^3)^2 = a^6 = a^4a^2 = a^3a^2 = a^4a = a^3a = a^4 = a^3$.

This theorem on the existence of an idempotent in a three-element semigroup is a special case of a theorem proven by Frobenius in an article published 1895 [**3**]. In the article Frobenius shows that if S is a semigroup, a an element of S and the subsemigroup $\{a, a^2, a^3, a^4 \ldots\}$ is finite then this subsemigroup will contain exactly one idempotent. In 1902 E. H. Moore showed that some power of each element in a finite semigroup is an idempotent. See [**4**].

According to the theorem above one can, when looking for associative operations in a three-element set $S = \{a, b, c\}$, assume the existence of an idempotent in S and we begin by assuming that a is an idempotent.

Given the condition $aa = a$ there are 9 possibilities for the first line of an operation table: *a a a*, *a a b*, *a a c*, *a b a*, *a b b*, *a b c*, *a c a*, *a c b*, *a c c*. Three of these, *a a b*, *a c b*, *a c a*, can be immediately eliminated by counterexamples to associativity found among *aac*, *aab* so we are left with 6 possibilities: *a a a*, *a a c*, *a b a*, *a b b*, *a b c*, *a c c* and we will refer to those by numbers 1 to 6.

Similarly, the corresponding 6 columns are the only possibilities for the first column of an operation table, and we will also refer to those by numbers 1 to 6. We have thus at most $6 \cdot 6 = 36$ possibilities for the first line and first column of an operation table, if the operation is to be associative and we assume $aa = a$. We refer to these partial tables by the numbers given to their line and column, for example partial table 2-4, which is shown below, consists of line no. 2 and column no. 4:

2-4	*a*	*b*	*c*
a	*a*	*a*	*c*
b	*b*		
c	*b*		

For simplification, rather than study these possibilities one by one, the possibilities that would lead to the same number of associative operation tables are grouped together and a representative chosen for each group.

Let f denote the isomorphism from S to S, that interchanges b and c. This isomorphism relates a table starting with line no. 2 (*a a c*) with a table starting with line no. 3 (*a b a*) and therefore each operation table starting with line no. 2 corresponds to a table that starts with line no. 3 under this isomorphism. A similar relation holds true for tables that start with lines no. 4 and no. 6, *a b b* and *a c c*. On the other hand lines no. 1 and no. 5, *a a a* and *a b c*, will be mapped onto themselves by this isomorphism. The same holds true for columns, so for the six lines (columns) and this isomorphism f one can list the relationship between lines (columns) in the following way referring to the lines (columns) by their numbers: $f(1) = 1$, $f(2) = 3$, $f(3) = 2$, $f(4) = 6$, $f(5) = 5$,

$f(6) = 4$. In general, the isomorphism f relates the partial table denoted x-y to the partial table $f(x)$-$f(y)$.

In this way f gives several examples of different partial tables that give rise to the same number of associative operations, for example 1-2 and 1-3, 1-4 and 1-6, 2-1 and 3-1, 2-2 and 3-3. Furthermore partial tables x-y and y-x will always result in the same number of associative operations since the partial table y-x is the transposition of the partial table x-y (see Section 3). According to these observations we can now classify the previously mentioned 36 possibilities into the following 13 classes which we label C1-C13:

C1	C2	C3	C4	C5	C6	C7	C8	C9	C10	C11	C12	C13
1-1	1-2	1-4	1-5	2-2	2-3	2-4	2-5	2-6	4-4	4-5	4-6	5-5
	1-3	1-6	5-1	3-3	3-2	3-6	3-5	3-4	6-6	5-4	6-4	
	2-1	4-1				4-2	5-2	4-3		5-6		
	3-1	6-1				6-3	5-3	6-2		6-5		

Each class could be characterized by its algebraic properties, and partial tables of the same class will result in the same number of associative operation tables. For further inspection it is therefore sufficient to choose one representative of each class, and we choose the first partial table listed in each class. Indeed, since operations for which there exists an identity or a zero have been accounted for, one can skip class C1 (where a is a zero) and class C13 (where a is an idempotent).

Now consider classes C7, C9 and C12.The partial tables of their representatives are:

2-4	a	b	c
a	a	a	c
b	b		
c	b		

2-6	a	b	c
a	a	a	c
b	c		
c	c		

4-6	a	b	c
a	a	b	b
b	c		
c	c		

It is easy to see that aca or aba gives a counterexample for each one of those partial tables so there are no associative operations to be found in these classes.

There are 8 classes left, each containing two or four partial tables:

	C2	C3	C4	C5	C6		C8		C10	C11		
	1-2	1-4	1-5	2-2	2-3		2-5		4-4	4-5		
	1-3	1-6	5-1	3-3	3-2		3-5		6-6	5-4		
	2-1	4-1					5-2			5-6		
	3-1	6-1					5-3			6-5		

As mentioned before, the following partial tables have been chosen to represent these classes:

1-2	*a*	*b*	*c*
a	*a*	*a*	*a*
b	*a*		
c	*c*		

1-4	*a*	*b*	*c*
a	*a*	*a*	*a*
b	*b*		
c	*b*		

1-5	*a*	*b*	*c*
a	*a*	*a*	*a*
b	*b*		
c	*c*		

2-2	*a*	*b*	*c*
a	*a*	*a*	*c*
b	*a*		
c	*c*		

2-3	*a*	*b*	*c*
a	*a*	*a*	*c*
b	*b*		
c	*a*		

2-5	*a*	*b*	*c*
a	*a*	*a*	*c*
b	*b*		
c	*c*		

4-4	*a*	*b*	*c*
a	*a*	*b*	*b*
b	*b*		
c	*b*		

4-5	*a*	*b*	*c*
a	*a*	*b*	*b*
b	*b*		
c	*c*		

Let us consider representative 1-2

For the partial table 1-2 one can choose *bb* and *bc* in the following *9* different ways which we number *1-9*:

1	*a*	*b*	*c*
a	*a*	*a*	*a*
b	*a*	*a*	*a*
c	*c*		

2	*a*	*b*	*c*
a	*a*	*a*	*a*
b	*a*	*a*	*b*
c	*c*		

3	*a*	*b*	*c*
a	*a*	*a*	*a*
b	*a*	*a*	*c*
c	*c*		

4	*a*	*b*	*c*
a	*a*	*a*	*a*
b	*a*	*b*	*a*
c	*c*		

5	*a*	*b*	*c*
a	*a*	*a*	*a*
b	*a*	*b*	*b*
c	*c*		

6	*a*	*b*	*c*
a	*a*	*a*	*a*
b	*a*	*b*	*c*
c	*c*		

7	*a*	*b*	*c*
a	*a*	*a*	*a*
b	*a*	*c*	*a*
c	*c*		

8	*a*	*b*	*c*
a	*a*	*a*	*a*
b	*a*	*c*	*b*
c	*c*		

9	*a*	*b*	*c*
a	*a*	*a*	*a*
b	*a*	*c*	*c*
c	*c*		

Counterexamples to associativity can be found among *bba*, *bbc*, *bca* for all tables other than *1*, *4*, *6*. For these tables we point out that if the sought operation is to be associative one can use *cab* and *cac* to calculate *cb* = *c* and *cc* = *c*. This makes *b* an identity for table *6* and we refer to Section 5.1. Tables *1* and *4* turn out to give associative operations with neither an identity nor a zero:

	a	*b*	*c*
a	*a*	*a*	*a*
b	*a*	*a*	*a*
c	*c*	*c*	*c*

	a	*b*	*c*
a	*a*	*a*	*a*
b	*a*	*b*	*a*
c	*c*	*c*	*c*

Next consider representative 1-4

We use *bab* and *bac* to calculate *bb* = *b* and *bc* = *b* which gives the table:

	a	*b*	*c*
a	*a*	*a*	*a*
b	*b*	*b*	*b*
c	*b*		

This partial table can be finished in two ways to give an associative operation:

	a	*b*	*c*
a	*a*	*a*	*a*
b	*b*	*b*	*b*
c	*b*	*b*	*b*

	a	*b*	*c*
a	*a*	*a*	*a*
b	*b*	*b*	*b*
c	*b*	*b*	*c*

There are no other possibilities to fill in partial table 1-4 without getting a counterexample to associativity among *cba*, *cca*, *cba*.

The remaining six representatives, 1-5, 2-2, 2-3, 2-5, 4-4, 4-5, can all be dealt with in a similar manner and we state the results:

Each of the following representatives lead to one associative operation: 1-5, 2-2, 4-4, 4-5. The corresponding operation tables are shown below:

	a	*b*	*c*
a	*a*	*a*	*a*
b	*b*	*b*	*b*
c	*c*	*c*	*c*

	a	*b*	*c*
a	*a*	*a*	*c*
b	*a*	*a*	*c*
c	*c*	*c*	*a*

	a	*b*	*c*
a	*a*	*b*	*b*
b	*b*	*a*	*a*
c	*b*	*a*	*a*

	a	*b*	*c*
a	*a*	*b*	*b*
b	*b*	*b*	*b*
c	*c*	*c*	*c*

Representative 2-3 does not lead to an associative operation and representative 2-5 does not lead to an associative operation without an identity or a zero.

Summary of Section 5.2.

Section 5.2 is devoted to finding the associative operations on *S* where there neither exists an identity nor a zero. First a theorem is proved which states the existence of an idempotent in *S* if the operation is associative, and thereafter *a* is assumed to be idempotent. Possible operation tables are grouped into 13 classes according to the first line and first column of each table and one representative chosen for each class. Two of the classes have been accounted for in Section 5.1 (existence of an identity/zero), and three classes are immediately shown to give no associative operations and thereby excluded. The remaining eight classes are studied, case by case, by considering elements of the four remaining seats of each operation table and the tables either included as associative or excluded. It turns out that associative operations are found in six of the eight classes and the result is shown below:

Representative of class	1-2	1-4	1-5	2-2	4-4	4-5
Associative operations for the representative	2	2	1	1	1	1
Number of partial tables in class	4	4	2	2	2	4
Total of associative operations	8	8	2	2	2	4

There are 26 associative operations altogether. Let us recall that a was assumed to be an idempotent ($aa = a$) and that neither an identity nor a zero exist. Obviously we get the same number of associative operations assuming that b is an idempotent and likewise by assuming that c is an idempotent. By writing down the 26 tables where a is an idempotent we can count that among them there are 18 tables where $bb = b$ and of these 18 there are 14 with $cc = c$. We summarize our results of Section 5.2: The number of associative operations on a three-element set for which there neither exist an identity nor a zero is 38:

$$26 + 26 + 26 - (18 + 18 + 18) + 14 = 38.$$

6. Conclusion

The conclusion of this article is that among the 19,683 different operations on a three-element set, $S = \{a, b, c\}$, there are exactly 113 operations which are associative, in other words there exist exactly 113 three-element semigroups. The article also reveals that 75 of these semigroups have an identity or a zero, but 38 have neither an identity nor a zero. Out of the 75 that have an identity or a zero, 18 have both, 15 have an identity but not a zero and 42 have a zero but not an identity.

The article lists 42 of the 113 operation tables for associative operations in a three-element set and points out how each of the remaining associative operations corresponds to one of the 42 operations given.

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A Case Study Using Soft Systems Methodology in the Evolution of a Mathematics Module

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Abstract
This paper describes the application of Soft Systems Methodology as a tool for facilitating the review of a taught mathematics module so that the views of those engaged with the module could be captured and conflicting expectations and views highlighted. Checkland's Soft Systems Methodology is used since it enables the capture of stakeholder views and addresses both 'hard' and 'soft' aspects of the learning experience. Stages in the application of Soft Systems Methodology are illustrated including the development of a rich picture and conceptual models and the work was conducted using a stakeholder group that included students taking the module (surveyed by questionnaire) and discussion with staff involved in the design and delivery of the material. Changes made to the delivery of the module are described particularly in the way that student support is delivered. The benefits derived from the application of this methodology are illustrated together with module monitoring and control mechanisms that help to trace the development of students. The paper argues that the application of this approach can enhance the understanding that faculty members have of student perceptions of a module, allows individual views to be understood and challenged and that this type of learning cycle undertaken periodically can be used to structure the evolution of a taught module.

Keywords: assessment of instructional modules; beliefs; student perceptions; soft systems methodology; mathematical modules; module monitoring; stake holders

Introduction

A recent article in The Montana Mathematics Enthusiast (Latterell, 2007) it was argued that undergraduate mathematics professors need to have a better understanding of their students' perceptions, the pressures on their lives and their preferred learning styles as each of

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these are subject to change over time. This paper fully supports that argument and extends it further to suggest that professors need to understand not just their students' views and expectations, but also those of their fellow professors and the external pressures from government or relevant professional bodies that may impinge on the design and delivery of a mathematics module.

The teaching of mathematics at university level, either as a single subject or as support for another subject area such as computing, requires a good deal of reflection on the part of the teaching staff as there are changing and challenging external influences at work which require corresponding change to mathematics curricula and to learning and teaching strategies. As we describe below, the need for reflection is probably as acute in mathematics and the related disciplines as in any other subject area. Unfortunately, there is often little time within a crowded academic calendar for such periods of reflection to be regularly undertaken. What seems to be more likely is that little change occurs until there is some sort of 'Kuhnian crisis' (Kuhn, 1996) with the module which is to say that the existing learning and teaching philosophy is recognised as critically failing to engage and inform the students and a radical re-think is required as opposed to a gradual evolution of current thinking. Rather than lurch into such radical change, it would be advantageous to set in place a framework which could be used to structure reflective thinking about a module on a regular basis. There are a number of ways in which we might approach this but a key underlying characteristic is that the process must be able to capture the views of all participants if it is to be comprehensive.

This paper describes a reflective review of a mathematics module which forms part of the first year curriculum of an undergraduate computing studies programme at London South Bank University (LSBU). In order to ensure that the views of all participants were captured the review incorporated the use of Soft Systems Methodology and this paper describes some of the findings of the review generated from two contrasting perspectives – that of the lecturer and the student.

Some Current Issues in the Teaching of Mathematics in UK Universities

A wide range of UK commentators have expressed concern about the decline, over the last 15 or so years, in the numbers of students who are taking mathematics (and other related subjects such as Physics and Engineering) at university level. This is coupled with concern over the general mathematical abilities of students at school level and those arriving at university to undertake mathematically related courses or courses requiring mathematics as a support topic (Goodfellow, 2006). The mathematical knowledge and skills exhibited by students entering the UK higher education sector has been a matter of some concern and debate for a number of years (Henry, 2004) and this concern has been felt not only in terms of the mathematics required for general university entrance (usually GCSE mathematics at grade C or better) but also on courses for which mathematics is a primary requirement (Engineeringtalk, 2001).

In 2001 the UK Government commissioned a review of the development of science and engineering skills in the UK (Roberts, 2002) and this review reported that among young people many have a poor experience in science and engineering education and that many have the impression that mathematics is boring and irrelevant. These concerns regarding mathematics gave rise to a further study (the Smith Report) which this time concentrated on post-14 mathematics education (Smith, 2004) and among the many issues raised by the report was that there was a " … failure of the current curriculum and qualifications framework to meet the requirements of learners, higher education and employers …" (Smith, 2004, p. 9). Following this a number of other studies have attempted to audit the skills of school leavers and those applying for university places and it has been reported that among UK higher education lecturers and admissions staff there are " … underlying concerns about basic numeracy and literacy, and

perceived problems with higher level mathematical skills, essay writing and independent learning skills." (Wilde et al, 2006, p. 6).

The Smith Report is by far the most comprehensive recent examination of the UK mathematics education system and it concluded that there are a number of possible reasons why students choose not to continue studying mathematics after the age of 16. These included the impression that students have of mathematics being more difficult a subject than others (although perceptions of difficulty can have many causes and indeed manifestations) and that the mathematics curriculum lacked the ability to interest and motivate students.

Turning now to my own institution, London South Bank University is an inner-city university and it has a truly cosmopolitan student body (London South Bank University, 2006). For example, of its 21,000 current students 60% come from black and minority ethnic backgrounds, 66% are aged 25 or over and only 20% arrive at the university with what would be regarded as 'traditional' entry qualifications ('A' level qualifications gained in the UK for example). Representing over 80 different countries, students arrive at the university with an eclectic mix of qualifications from both the UK and overseas and a variety of educational experiences in their previous schooling. Those students who enrol onto courses with a significant mathematical content do so with sometimes very mixed mathematical experiences and degrees of success. As a colleague has written "The main problem we have at LSBU is that student's previous mathematical ability is varied, and the mathematical skills that the student enters university with are extremely diverse." (Starkings, 2004, p. 22).

It is clear that UK institutions at all levels face a number of challenges in devising mathematical curricula and associated learning and teaching strategies, and in providing academic support for those students who find the transition to higher levels of study difficult.

The Need for Review

The teaching of mathematics as part of the first year computing studies curriculum at LSBU has been subject to periodic review for a number of years. Although the mathematics modules in their various forms have had pass rates that were viewed as acceptable by the faculty in comparison with other first year modules, there is no doubt that many students struggle with the subject material and frequently require additional support, over and above the normal timetabled tutorial sessions. Many universities provide mathematics support to students in addition to their timetabled class time and there have been a number of models of support provision tried (Perkin and Croft, 2004). At LSBU, students who had performed badly in their first mathematics assignment were routinely directed to the support sessions run by the university's Centre for Learning Support and Development (CLSD) and were able to get group or individual help from CLSD support staff. These were, however, voluntary on the part of the student and monitoring of the usage of support sessions produced evidence that the take-up was generally poor. Although acceptable numbers of students were passing the mathematics module and student evaluation of the module was generally good (so that traditional quality measures of the performance of the module were indicating no problems) there were a number of issues that teaching staff felt needed to be addressed or at least examined:

- Some students failed to engage with the modules at all, attended poorly and would then fail the module;
- Students were identified as needing support but then declined to take up support when it was made available;

- Although pass rates were acceptable, average scores on assignments were low. It was felt that students could do better if they were better motivated and took advantage of support. Motivating students was seen as an issue;
- It was felt that students did not want to study mathematics. They were here to study computing and did not see the need for mathematics;
- On the other hand, employers have bemoaned the lack of quantitative skills exhibited by many job applicants from both schools (Smithers, 2006) and universities (Blair, 2006) across the country;
- It was difficult to cater for the mixed abilities of students being taught, good students were finding the work easy and were also then de-motivated.

These views were due partly to the literature already described providing a rather negative environment in which to teach mathematics and partly were the result of teaching experience over the years with the students on the computing programme. It was clear that in addition to the views already expressed, student views needed to be captured if a complete picture of the module and its operation was to be seen, and in order to accommodate all these views, Peter Checkland's Soft Systems Methodology (SSM) was used as a tool with which to try and capture some of the problems and issues associated with the teaching of these mathematics modules.

Soft Systems Methodology

Scientists and engineers have traditionally been raised on the principle of reductionism so that analysis of a problem focuses on structure and decomposition that reveals how things work. The process focuses on decomposition, explanation and finally synthesis (Hitchens, 1992). Whilst it is true that certain types of investigation can be undertaken by the application of this 'scientific method', the desire to study observed phenomena that extend beyond the foundation sciences of physics, chemistry and biology into psychology and the social sciences has caused researchers to question whether existing modes of thinking are appropriate in capturing the influences and interactions that underpin some of these phenomena (Rosenhead and Mingers, 2001).

The systems movement, on the other hand, contends that system ideas can provide a source of explanation for many kinds of observed phenomena which are beyond the reach of reductionist science. Checkland views systems thinking as a holistic reaction against the reductionism of natural science. This has led to the manifestation of Systems Thinking which tackles the issues of irreducible complexity through a form of thinking based on wholes. Furthermore, systems thinking is based on two pairs of ideas: emergence and hierarchy as one pair, and communication and control as the other (Checkland, 1981; Checkland and Poulter, 2006).

Soft Systems Methodology places an emphasis on human activity systems i.e. humans involved in purposeful activity within an organisation of some sort. The methodology provides a window through which the complexity of such human interaction can be investigated, described and hopefully understood. Once an understanding of the situation under study has been achieved then the methodology allows the identification of change that is both systemically desirable (in that it will alleviate some of the problems and issues) and culturally feasible (in that actors within the system will be inclined to engage with the changes proposed and the change process itself). SSM encourages learning and understanding which will hopefully lead to agreed change and the resolution of problems.

The need to make sense of the complex and dynamic interacting web of ideas, issues and views that characterise many social problems has seen the emergence of SSM through 30 years of reflective intervention experiences – experience dealing with what Ackoff termed 'messes'

(Ackoff, 1974). During this period of evolution, the process model of SSM has emerged and the main stages of the process are described in Table 1.

Stage	Stage Objective
1 and 2	Attempt to build the richest possible picture of the situation.
3	Aims to describe the nature of the chosen system.
4	Produces conceptual models of the defined system.
5	Compares conceptual model with actual situation in order to generated debate with the stakeholders.
6	Outline possible changes that are desirable and feasible.
7	Involves taking action based on stage 6.

Table 1. Key stages of Soft Systems Methodology

Some applications of systems thinking to educational development have been made (Ison, 1999) although SSM has not, seemingly, found wide acceptance. The benefits that can be derived from its use are primarily that with its ability to focus on 'soft' issues, a systems view is generated that contrasts nicely with the rather more quantitative results-driven and analytical quality assurance processes that are traditionally used to assess the effectiveness of a module. As stated by Patel (Patel, 1995, p. 13):

> … the methodology is unique because it enables the analyst to embark on a process of learning about the real world situation being investigated, while simultaneously seeking to improve it by analysing the situation … and suggesting recommendations for further action to improve the problem situation.

The systems approach has also been applied to more general issues related educational management (Bell and Warwick, 2007) but here we restrict ourselves to thinking about a single taught mathematics module. In this paper we contrast two key views of the situation – that of the lecturer (influenced by the published views of colleagues, Government sponsored reports and personal experience) who takes responsibility for the design, delivery and assessment of the module and that of the student who has to engage with the unit and, hopefully, pass.

The Application of Soft Systems Methodology

We now describe the stages of SSM in terms of the work undertaken on the review of the mathematics module and the outputs produced.

<u>Stages 1 and 2</u>

In order to develop a rich picture of the situation under study, a number of sources of information were utilised to capture views of the module from the perspective of the university, the faculty and the lecturer. These included government and university documents that describe the requirements of module design at various levels of study (SEEC, 2003), discussion with colleagues as to the required syllabus (this module was providing support for studies in computing) and documents already referred to that describe the general environment of mathematics teaching and possible remedies. In order to capture the student perspective, it was

necessary to try and elicit student views both through questionnaire and then follow-up discussion with a smaller group of students to confirm findings.

A questionnaire was distributed to a sample of 62 new students joining the computing programme and who would be taking the mathematics module. Since the objective of the module was to provide a relevant and useful syllabus that students were equipped to study and for which additional support could be provided, it was decided to structure the questionnaire around three factors. The first was 'mathematical self-efficacy', the second was 'previous educational experience in mathematics' and the third was 'the perceived relevance of mathematics as part of the course of study'.

The first of these factors relates to an individual's self-efficacy beliefs and these are conjectured to be oriented around four core concepts: 'performance experiences', 'vicarious experiences', 'verbal feedback', and finally 'physiological and affective states'. Each of these contributes to the individual's ability to organise and execute effective learning and can be tailored to specific subject domains. To give a little more detail we can turn to descriptions taken from the literature (Phan and Walker, 2000). 'Performance experience' relates to indications of capability based on performance in past assessments that the student may have undertaken, or performance on previous courses etc.; 'Vicarious experiences' relates to evidence based on competencies and informative comparison with the attainment of others i.e. the student's performance in relation to their peers; 'Verbal persuasion', as its name suggests, refers to the student's response to verbal feedback from those in a position of greater authority such as teachers or adults; 'Physiological and affective states' are judgements of capability, strength and vulnerability to dysfunction.

The second factor, 'Previous educational experience in mathematics', was related to how the students perceived their past education in mathematics and so had a broader context than the self-efficacy criteria described above.

The third factor, 'The perceived relevance of mathematics', was included since it was identified in recent studies as a reason why students were 'turned off' mathematics. The perceived relevance of the subject could well effect the degree of motivation and time allocated to study of the module by students.

Thus the questionnaire was constructed to elicit views across six criteria – four relating to mathematical self-efficacy together with previous educational experience and perceived relevance of mathematics. Each student was given a questionnaire consisting of 24 statements relating to the six criteria and asked to indicate the extent to which they agreed with the statement on a 7-point Likert scale ranging from 1 (not true) through to 7 (very true). There were four questions relating to each of the six criteria with some expressing a positive sentiment and some a negative sentiment.

In the questionnaire results shown in Table 2, the statements have been sorted by average (arithmetic mean) Likert score and the middle third of the statements are those for which there was neither strong agreement nor disagreement having average scores in the range 3.5 to 4.5 approximately.

	Text of the Statement	**Average Score**	
1	Mathematics is useful for anyone's life	6.27	General to strong agreement
2	I find it useful to be able to improve my mathematics	6.11	
3	I like to get verbal feedback from my teacher	6.00	
4	Mathematics is important in studying IT	5.92	
5	I enjoy learning new mathematical facts and ideas	5.66	
6	Mathematics is interesting	5.47	
7	When my teacher praises me I want to do well in mathematics	5.23	
8	It has been a long time since I studied mathematics	4.68	
9	I was always encouraged to improve my mathematics	4.61	Neither agreement or disagreement
10	I am always worried about mathematics	4.21	
11	I don't have anyone to help me with mathematics	4.15	
12	My classmates have generally been good in mathematics	4.15	
13	I have a close friend who is good in mathematics	3.92	
14	I have been able to access good mathematics resources	3.76	
15	I struggle to pass mathematics assessments	3.66	
16	I am generally pleased with the mathematics results I get	3.53	
17	Compared with other students I am weak in mathematics	3.35	General to strong disagreement
18	I always get good marks in mathematics	3.26	
19	My friends tell me I am good in maths	3.21	
20	I am not good in mathematics	3.21	
21	I've never had a good mathematics teacher	2.81	
22	I get put off when I am told I am wrong in mathematics	2.69	
23	I hate mathematics	2.66	
24	I know enough mathematics without studying more on this course	1.84	

Table 2. Average Questionnaire Results for Each Statement

An examination of this sorted list of statements suggested a number of interesting observations regarding student views.

- In terms of perceived relevance of mathematics, questions 1, 4, 5 and 24 gave a strong indication that students did see mathematics as useful, that they accepted the requirement to study mathematics as part of their course and that they accepted the limitations of their current knowledge. This was unexpected as it was felt by staff that students did not see the module as adding much to their course of study i.e. that it was largely irrelevant to their study of computing.
- Questions 3, 7, 19 and 22 related to students' reaction to positive and negative feedback. Again there was an unexpectedly strong reaction to these questions and the responses emphasised the

importance of giving feedback regularly (this could be summative or formative) and that students were not averse to receiving negative feedback.

- Questions 11,12, 13 and 17 reflected the students' vicarious experiences and these responses were grouped in the middle of the table. On initial inspection this seemed to indicate that students had no strong feelings either way but in fact the scores for these questions were clearly bi-modal with some students strongly agreeing and others strongly disagreeing and producing a rather deceptive average result. For example, in response to the statement "I don't have anyone to help me with mathematics" 34% of the sample strongly disagreed (indicated 1 or 2 on the Likert scale) and 39% strongly agreed (indicated 6 or 7) so there were clearly groups of students who have support available among their friends but a larger group who do not and therefore would require support.
- Finally, in terms of previous educational experience, questions 15, 16, 18, 20 indicated that although the students generally acknowledged that they were not good in mathematics (40% indicating 1 or 2) there was a reluctance blame this on poor mathematics teaching since 57% strongly disagreed (indicating 1 or 2) with the statement that they had never had a good mathematics teacher.

Much of this information was represented as a rich picture which is one output from stages 1 and 2 of SSM. The rich picture gives a pictorial description of the situation under investigation and provides a focal point for further discussion and analysis. A rich picture was constructed and is shown in figure 1. From this rich picture we can began to draw out some issues and problems which seemed to be emerging. In our case study, three issues seemed to be particularly key:

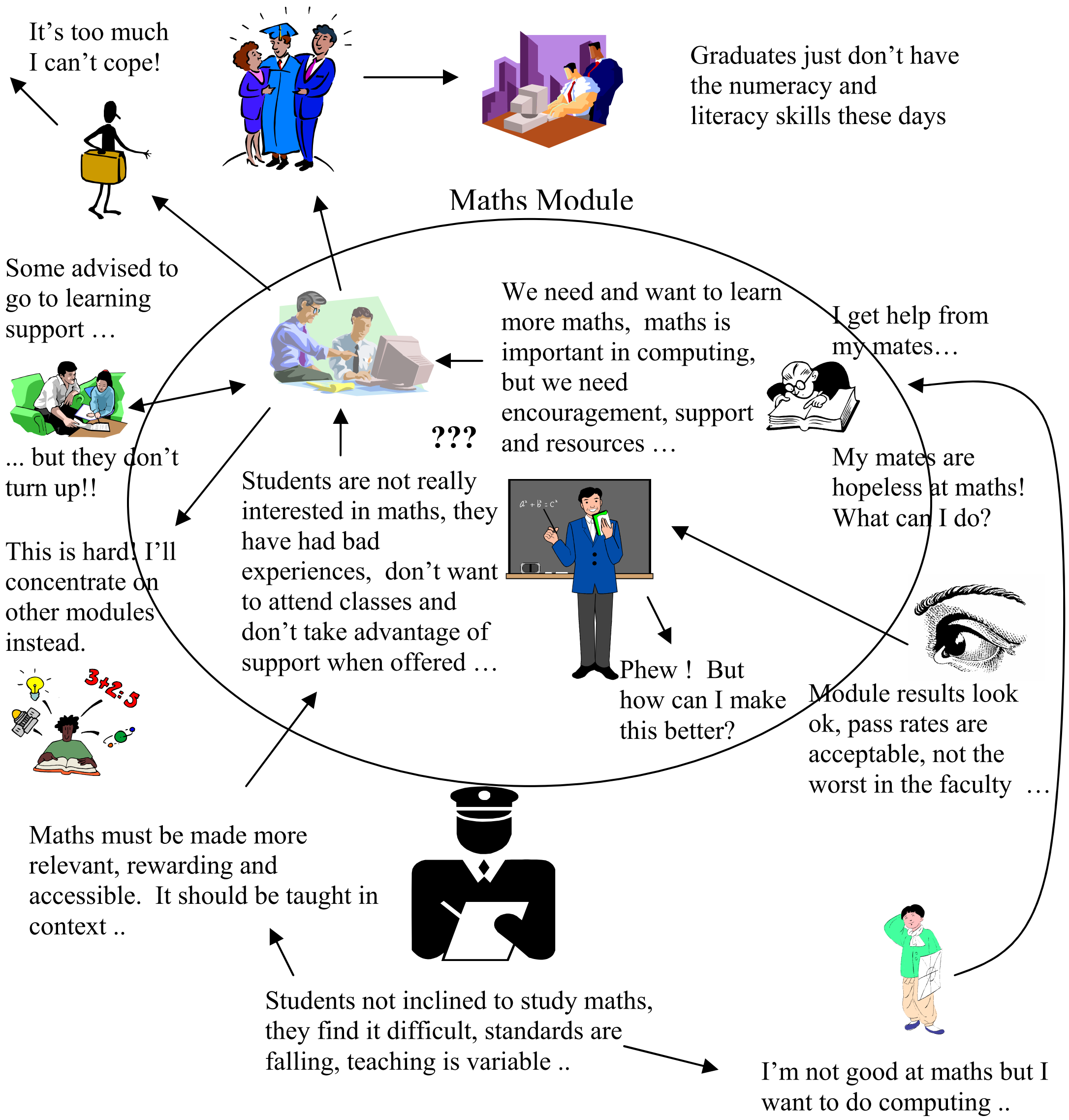

Figure 1. A Rich Picture

- There was a clear perception among staff that students were not using the support services offered by the university and evidence seemed to indicate that this was true. How could support be structured to make it more effective?
- There was an issue with the motivation of students. Attendance often dropped during the module yet students claimed to be motivated to learn mathematics on entry and saw it as important to their studies;
- Many students, by their own admission, were not good in mathematics and did not have access to resources outside the classroom. How could this be improved?

Having identified some key issues, we now began to think in soft systems terms about how we could resolve these issues i.e. we began to develop relevant systems that would address these issues. Here we give examples of three relevant systems which related to: how the module was formally described, how additional support and tuition for students could be provided and how best to motivate and encourage student engagement with the module.

Stage 3

Having established from the rich picture a number of relevant systems that need investigation, stage 3 required a root definition of each system to be constructed. The root definition should contain sufficient information for a conceptual model to built of the system based on the root definition alone. The well-known mnemonic CATWOE was used to identify elements of the root definition.

Element of CATWOE	**Description**
Customers	Who are the victims or beneficiaries of the transformation?
Actors	Who makes the transformation happen?
Transformation	What are the inputs and (transformed) outputs?
Weltanschauung	What makes the transformation meaningful in context?
Owners	Who could stop the transformation process?
Environmental Constraints	Which elements outside the system are taken as given?

Table 3. The Elements of the Root Definition

We began by developing a root definition for three relevant systems each described using CATWOE: a system to deliver and assess a university approved module; a system to provide additional support and tuition for students who require it; a system to motivate and encourage

student engagement with the module. Tables 4, 5 and 6 below illustrate the three root definitions.

Element of CATWOE	**Module Description**
Customers	The students enrolled onto the module.
Actors	The module lecturer(s).
Transformation	The need for students to be able to {insert aims of the module} transformed to the need met by attendance at a series of lectures and the successful completion of assessments designed to test achievement of {insert learning outcomes}.
Weltanschauung	The further study of mathematics in year 1 is essential for students entering with GCSE mathematics, with module content designed to mesh with studies in computer hardware, software and business applications as specified by programme tutors, employers and professional associations.
Owners	Dean of Faculty or Head of Department
Environmental Constraints	Library, web and other online learning resources specifically required by the module, other general physical and human resources required for effective learning and student support.

Table 4. System 1 Root Definition

Element of CATWOE	**Module Description**
Customers	The students enrolled onto the module.
Actors	The module lecturer(s) or university support staff
Transformation	The need for students to be identified who have specific weaknesses in core mathematics which must be remedied transformed to the need met by appropriate student evaluation and support organised during the running of the module, the provision of resources and plans to address the students' specific weaknesses. Student attendance must be ensured to enable the transformation
Weltanschauung	Although students meet general entry requirements, many have specific weaknesses in mathematics, lack confidence in the use of mathematics and need to strengthen their core mathematics skills to increase their likelihood of passing the module.
Owners	Dean of Faculty or Head of Department, Module Leader.
Environmental Constraints	There are limited resources available for additional support provision – some may be provided by the university and others may be local to the faculty.

Table 5. System 2 Root Definition

Element of CATWOE	**Module Description**
Customers	The students enrolled onto the module.
Actors	The module lecturer(s).
Transformation	The need for students to remain motivated and to attend and engage with the mathematics module transformed to the need met by appropriate content, delivery processes and assessment regimes.
Weltanschauung	Student willingness to study mathematics must be nurtured by appropriate content and assessment that gives continual feedback and support to all students.
Owners	Dean of Faculty or Head of Department, Module Leader.
Environmental Constraints	Students need to acclimatise to university life and deal with a range of studies in their first year. Assessment demands of other modules, the schedule of work and outside commitments (part-time work, family etc) often restrict time for study and force compromises.

Table 6. System 3 Root Definition

Stage 4

Once the root definition for a system had been established then stage 4 required the construction of a conceptual model which described the activities that must take place in order to achieve the transformation and also how the operation of the system was to be monitored and controlled. Monitoring and control activities usually revolve around the three Es of efficacy, effectiveness and efficiency. Efficacy requires that the system has a purpose to fulfil (i.e. that the transformation is still necessary within the broader view), effectiveness requires that the system is designed correctly to fulfil its purpose (carry out the transformation) and efficiency requires that the system carries out the transformation with efficient use of resources. Conceptual models are generated with reference only to the root definition and not to activities taking place in the real world. They are, then, theoretical models of systems that can bring about the stated transitions and their value lies in comparison with the real world activities.

Of the three root definitions presented above, the second and third are illustrated with conceptual models. The conceptual model derived for system 2 is shown below in Figure 2, and for system 3 in Figure 3.

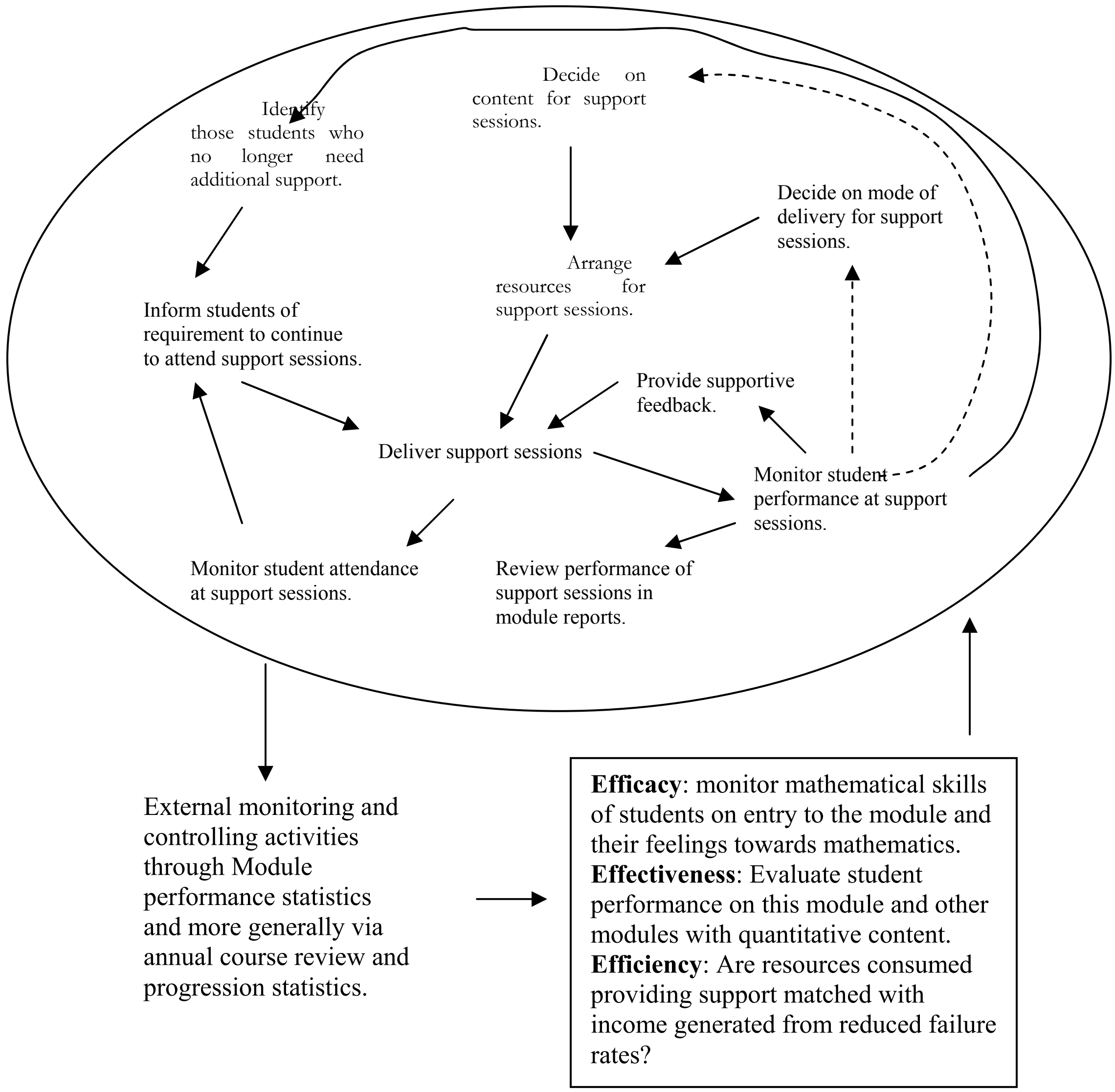

Figure 2. Conceptual Model for System 2

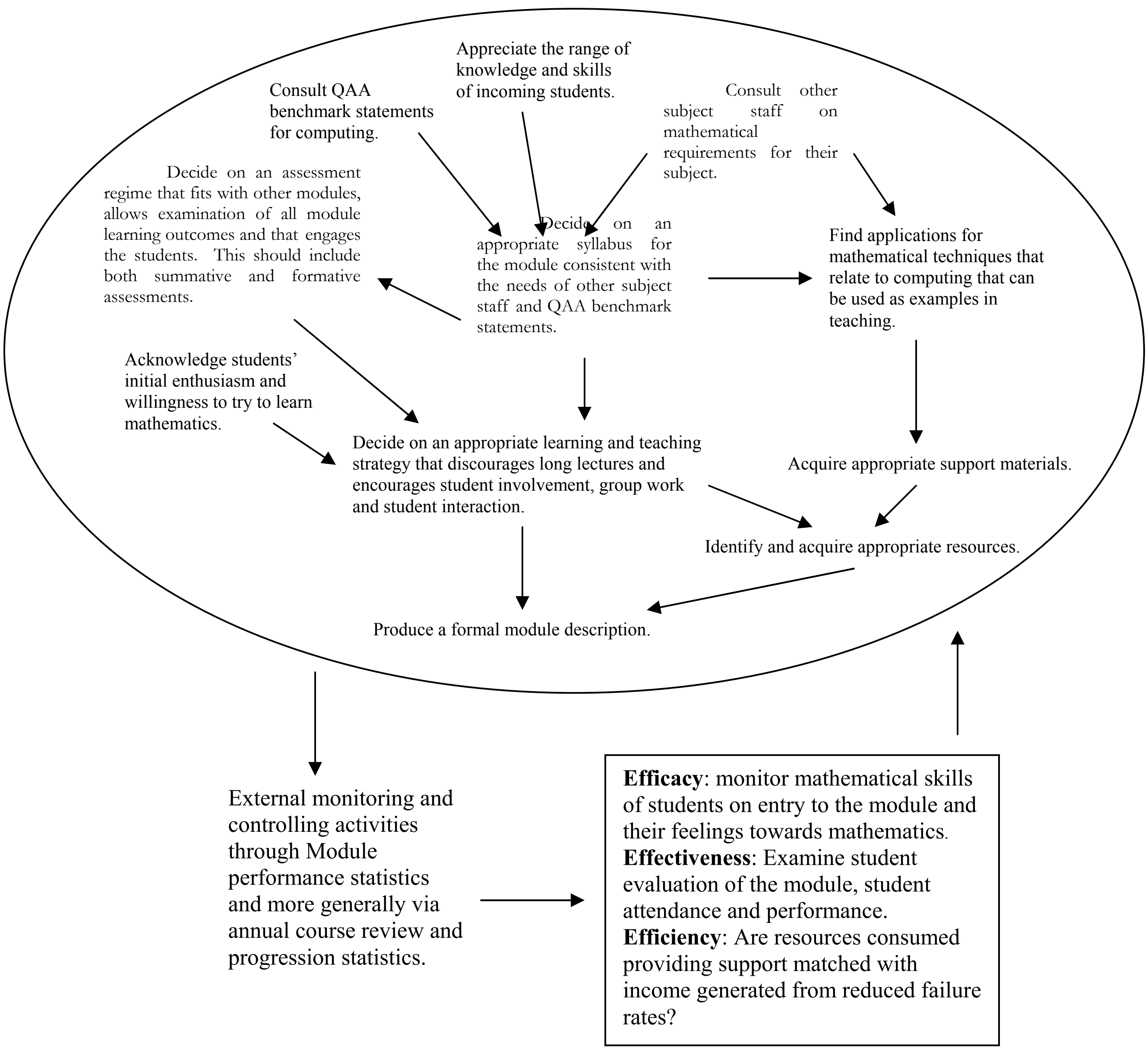

Figure 3. System 3 Conceptual Model

Restricting our discussion to system 2 only, we consider Figure 2. In order to address the issue of attendance at support sessions it was decided that all students should attend support sessions at least initially but that students with the necessary mathematical competencies would be later excused attendance at the sessions. It was also desirable to provide students with regular feedback on their progress and areas of continued weakness and improvement. The dotted lines indicate some feedback links which could be possible during the running of the module but these would be minor alterations only. The activities (or indeed groups of activities) described in these conceptual models were themselves explored more deeply by considering them as sub-systems and developing lower level conceptual models. For example, the attendance and feedback actions were further developed as shown in Figure 4. This now gave a more detailed picture of the operation of these inter-related actions. This conceptual model included the notion of a series of student self-tests which could be used to monitor performance and provide rapid feedback to students.

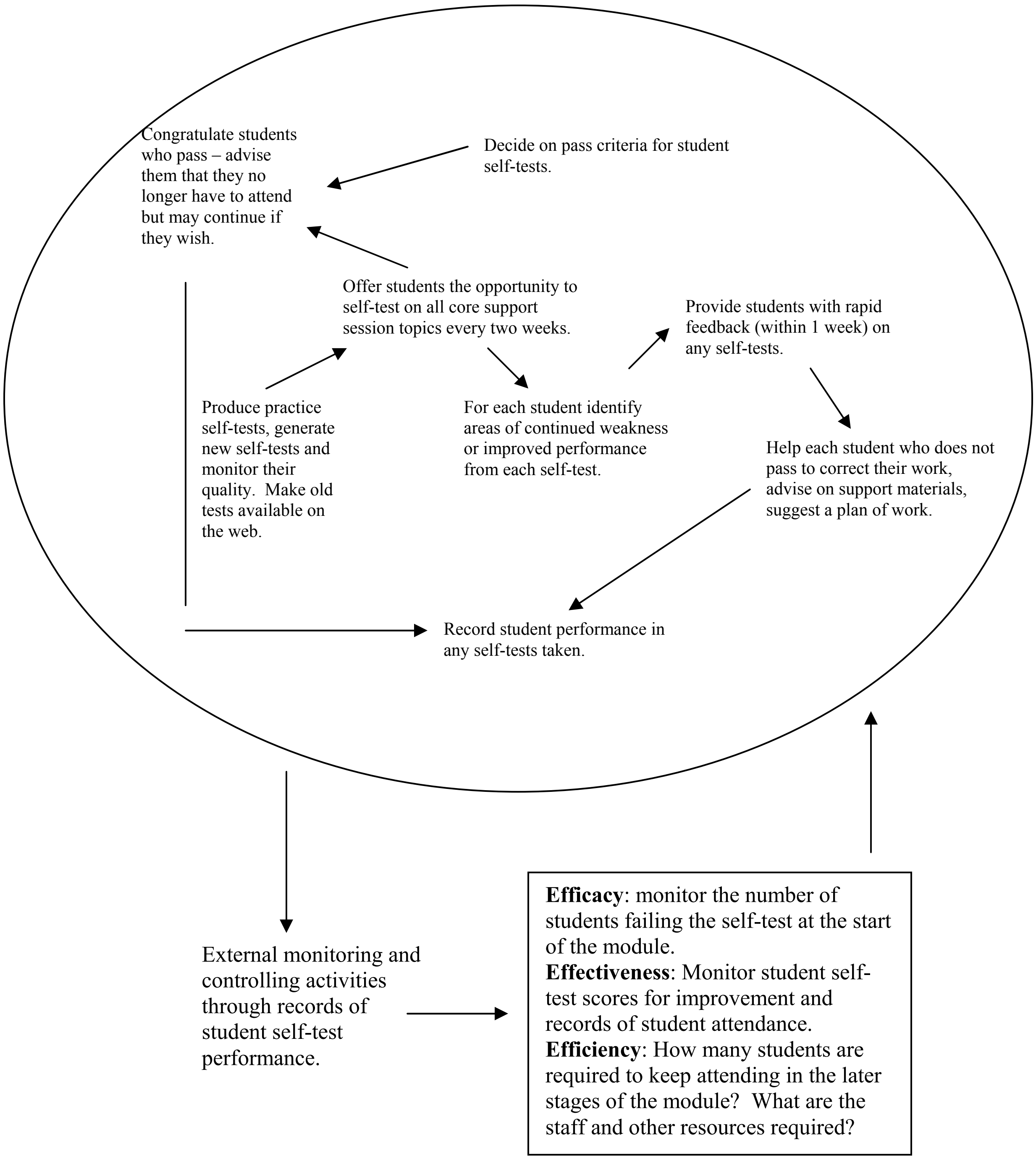

Figure 4. 'Monitoring Student Performance' Conceptual Model

Stages 5 and 6

These stages of the process involved comparing the conceptual models with reality and using any observed differences (or indeed similarities) to generate discussion and debate among the reviewers as to why the differences had occurred. This helped to identify desirable and feasible change. In terms of the evolution of the mathematics module, these stages are illustrated here in terms of the System 2 conceptual models. The module in its original form had a formal set of lectures and tutorials but mathematics support was organised on a voluntary basis via CLSD. The development of conceptual models caused a re-think of the support process and in its new and revised form the module now has a timetabled support session which all students must attend until they can pass a self-test which covers all the support topics and which is offered every two weeks. Students who take a self-test but do not pass have the test returned by the tutor and are given individual feedback on their performance, their strengths and weaknesses. Suggestions for further work and suitable resources are also made by the tutor.

Stage 7

This final stage involves implementation of the changes identified. The redesign of the mathematics module was accomplished over the summer of 2006 and ran for the first time in September 2006 having been formally approved by the university. Changes made to the content and running of support sessions were all adopted within the new version of the module as were a number of other changes not referred to in this paper.

The Validity of Conceptual Models

As we should with any investigation involving modelling, we now turn to the question of validity of the models developed as part of an SSM enquiry. The requirements for establishing the validity of any model depend on the type of model being constructed and the use that is to be made of it. Validity is commonly described as the extent to which the model can be said to be an adequate representation of reality, but in the case of SSM, the conceptual models built may be of systems that are not actually in existence at all. Thus conformance to reality is not an appropriate question to consider. Examining the validity of any models generated as part of a soft systems enquiry is difficult and Checkland (1995) suggests that there are really only two aspects that can help differentiate a good model from a bad one and these relate to whether the models as developed are in any sense relevant and whether the models are competently built. The question of competence relates to ensuring that the root definitions and conceptual models have been derived systematically from the rich picture and the issues identified within it and also that the conceptual models are built only from the root definition. The relevance of the models is a matter for the participants to determine and is related to the extent to which the models generated improve the understanding of issues and the generation of subsequent actions.

This investigation resulted in a number of changes to the module particularly to the way in which support sessions are organised. We feel that we now have a far firmer understanding of the needs and expectations of our students which we feel gives considerable validity to the models that have been developed. Furthermore, the issues described within the rich picture were agreed with a small selection of students from the original sample and the subsequent changes made to the module were formally validated through the university quality assurance process. It is too early to tell whether the overall performance of students has been enhanced by the changes made to the module, but anecdotal evidence from students seems to confirm that the support sessions are appreciated by the students and there is a gradual flow of students who

are improving their self-test scores and passing out of the support sessions. Full evaluation will only be possible at the end of the academic year in July 2007 when further module reflection and evolution using another iteration of SSM will be possible.

Conclusion

Most educators would acknowledge the importance of exploring the views and opinions of professional colleagues and students in order to improve the design and delivery of mathematics modules. Unfortunately, the crowded academic calendar can leave little time for reflective thinking on the learning and teaching process and although academic staff do make changes to modules from time to time such changes are generally rather ad hoc, restricted in scope and often reflect only the immediate experiences of the teaching staff.

As we have seen in this paper, the opinions of staff can often be incorrect, particularly when staff make assumptions about the views and expectations of their students. By employing a systems based methodology such as SSM we have been able to see the value of trying to capture the different perceptions of all participants in a module's design and delivery. The particular strengths of the approach that this case study has demonstrated are:

- Academics are often guided by the findings of professional body or governmental reports and pay too little attention to the views of their students. When student views are elicited, it is often at the end of a module for quality monitoring purposes and they do not reflect the views and expectations of students starting a module;
- Comparing the views of all participants reveals inconsistencies between students' expectations of a module and the real experience of a module. In addition, when students fail to engage with parts of the system (such as centralised support services) then it is important to determine why this is happening;
- The root definitions allow a description of the defining features of a system, in particular the transformation which defines the purpose of the system. Building a conceptual model based only on this root definition allows the modeller to view the system from differing viewpoints and to describe system processes that achieve the desired transformation. This gives a clarity of thinking to the modelling process which can be quite revealing when compared to the real systems which have evolved over time. Furthermore, the hierarchical structure of the conceptual models allows the emergent properties of sub-systems to be seen as they contribute to the emergent properties of larger systems (the 'emergence and hierarchy' stream of SSM activity). For example, the system for monitoring student performance (Figure 4) had emergent properties that allowed students to know their self-test scores, get rapid feedback on their performance and a plan of future work. This contributed to the wider system for the provision of a support service (Figure 3);
- SSM is often described as a process of investigation and learning and in reality continually loops around the seven stages. Thus the effect of implementation of system changes will generate new insights and promote further investigation and change. As an example, in the module review described it has been interesting to witness student responses to the support sessions. Some have just wanted to attend without taking the self-tests, others have passed a self-test but still want to continue attending. A further group are gradually improving their scores and will pass a self-test soon and yet another (smaller) group are not making significant improvement. All of these observations will feed into further review so that the support sessions can be modified next time the module is run to give greater support to this last group;

- For each of the systems described by conceptual models, thought had to be given to external monitoring and control from the perspective of the three 'e's of efficacy, effectiveness and efficiency. This resulted in the description of monitoring activities through which student progress and module performance could be monitored rather more closely than just with assessment results. The rapid and continual feedback given to students through the self-tests was felt to be key in helping students to concentrate on their weaknesses and recognise their strengths. This is typical of the 'communication and control' ideas of SSM.

Any change to a system requires that changes be desirable from the systems perspective but also culturally feasible. In this application, academic staff have been required to work closely with students in the support sessions which is quite a different skill to lecturing to larger groups. This will require staff development sessions to be organised through which such skills can be enhanced so that this way of working truly complements the more formal lectures and is of value to the students who have less confidence in the application of mathematics.

It is hoped that this type of review will continue on a yearly basis and that the module will continue to evolve under the influence of all participants – including the students.

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Mathematics and the World: What do Teachers Recognize as Mathematics in Real World Practice

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Abstract
Elementary school teachers are encouraged to better integrate appropriate mathematics pedagogy with deeper, more relevant mathematics content. However, many teach a mathematics they do not fully understand to students who see, recognize, and use less mathematics than ever before. Both teachers and students struggle to articulate the role mathematics plays in society as mathematics becomes more embedded into our technology. In this study, we asked teachers to record the mathematics they used on a daily basis during a 1-week period. Their responses indicate that they do not recognize that mathematics plays any important role in technological and professional practices. This negatively impacts their ability to effectively teach mathematics in the elementary classroom because they cannot make connections between classroom practices and real-world uses of mathematics.

Keywords: mathematical beliefs; mathematics pedagogy; mathematics content; mathematical literacy; pre-service teacher education; real world mathematics

Introduction

As American society becomes more technologically reliant, the actual, day-to-day use of mathematics diminishes (Noss, 2001; Skovsmose, 2005). The application of mathematics is less obvious (Skovsmose, 2005) and less understood (Friedman, 2005; Schiesel, 2005) by K-8 classroom teachers and their students. Thus, many teachers teach a mathematics they do not fully understand to students who see, recognize and use less mathematics in their lives than ever before (Hastings, 2007, February 2).

"School mathematics" (Gerofsky, 2004) is defined as a interconnected set of content knowledge (including numbers and operations, algebra, geometry, measurement, and data analysis)

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and cognitive process skills (including the ability to use content knowledge and conceptual understanding to reason, solve routine problems, develop proofs, and effectively communicate, represent, and model mathematical ideas) (Mulls, Martin, Gonzalez, & Chrostowski, 2004; National Council of Teachers of Mathematics, 2000; 2006). In practice, mathematics curricula stress the importance of linking school mathematics, often presented as a static, predetermined body of knowledge, to the flexible and changing realities of students' daily lives, as a way of connecting academic mathematics to contextual and/or practical realties and understandings (Desimone, Smith, Baker, & Ueno, 2005; Graeber, 1999; Hannaford, 1998; Nasser, 2005; National Council of Teachers of Mathematics, 2000; 2006). K-6 teachers try to explicitly tie mathematics to the world of their students (Delpit, 2006; National Council of Teachers of Mathematics, 2000; Reys, Lindquist, Lambdin, & Smith, 2007) as they look to make realistic connections between mathematical rules and algorithms and the events children participate in on a daily basis.

Yet teachers often struggle to respond to students' (and even some parents') frustrations that they rarely, if ever, use the mathematics they learn in school. The assumption that mathematics is fixed body of knowledge that offers clear-cut answers to numerically-based problems precludes recognition that mathematics is a creative and experimental tool that explicitly informs planning, organizing, and ethical decision making within specific constructs (Bakalar, 2006; Bishop, Clarke, Corrigan, & Gunstone, 2006; Delpit, 2006; National Council of Teachers of Mathematics, 2000). Formal mathematics is associated with scientific, technological, and engineering practices and there is little understanding that important mathematics is embedded in many professions not usually associated with mathematical understanding (Barton & Frank, 2001; Lesser & Nordenhaug, 2004; Masingila, 1996; Mewborn, 1999; Nicol, 2002; Rauff, 1996; Sithole, 2004; Zlotnik & Galambos, 2004). Therefore, many teachers do not appreciate the practical utility of many topics they teach (FitzSimons, 2002; Gutstein, 2006; Mukhopadhuyay & Greer, 2007) and are ambivalent about the necessity of teaching mathematics (Mewborn, 1999): teachers neither understand how mathematics serves professional practices nor do they recognize that the ability to conceptualize mathematical thinking outside of the classroom is an important skill for their students (Gainsburg, 2006). Thus, the students themselves have a limited appreciation of the intersection of mathematical understanding and ability with the other areas of curriculum or real-world tasks (Iverson, 2006; Mudaly, 2007).

Mathematics in Society

Certainly, the role of mathematics in society is changing. The more that technology impacts and influences our daily lives, the less mathematics is visible (Iverson, 2006; Noss, 2001; Oers, 2001; Skovsmose, 2005). While mathematicians, scientists, and engineers recognize that technological advances require a deep understanding of mathematics (Tate & Malancharuvil-Berkes, 2006), societally, we do not explicitly "see" the mathematics nor do we perceive when mathematics is used on a daily basis (Bishop et al., 2006; Empson, 2002; Gainsburg, 2006; Mudaly, 2007). The implicit use of mathematics is ubiquitous in the United States (e.g., bar codes that monitor inventory, global positioning systems, fast food restaurant cashier counters that display pictures of food items instead of numerals), yet these embedded uses of mathematics obscure explicit uses of mathematics. Even the recent NCTM *Curriculum Focal Points* (National Council of Teachers of Mathematics, 2006) fails to directly address this. While calculator and computer use is encouraged, to help students visualize, explore, and manipulate a variety of mathematical ideas and representations, there is no discussion regarding helping students recognize the important mathematics that underlies the design, development, and maintenance of the technological supports they are using. Therefore, it becomes difficult to explain to children (and, often, to teachers themselves) that the mathematics that is responsible for innovations, advances, and creative technological practices depends on the

elementary concepts and building blocks of basic mathematics and arithmetic (Hastings, 2007, February 2). Additionally,even when mathematics is explicitly used professionally or vocationally, the mathematics taught in the K-8 classroom ("school math") often does not mirror the math used in occupational practices (Gerofsky, 2006; Masingila, 1996; Shockey, 2006; Tate & Malancharuvil-Berkes, 2006). Oers (2001) suggests that school mathematics is the activity of participating in a mathematical practice. What happens, then, when students and their teachers do not recognize that they are participating in mathematical practices?

The goal of mathematics education for pre-service teachers focuses on ensuring that they understand the basic mathematics concepts they will teach (Dahl, 2005; Graeber, 1999; Hagedorn, Siadat, Fogel, Amaury, & Pascarella, 1999; Hannaford, 1998; Hill, Rowan, & Ball, 2005) and have access to developmentally appropriate pedagogy and practices (Dahl, 2005; Donnell & Harper, 2005; Gerofsky, 2004). Additionally, it is hoped that they recognize connections between "school math" and daily practices (e.g., calculating unit cost or interpreting a graph or chart in the newspaper) (National Council of Teachers of Mathematics, 2006). Yet little attention is paid to ensure that educators acknowledge implicit and/or embedded mathematical practices that are part of daily life, professional practices, and technological underpinnings beyond the connections made in textbooks (Reys et al., 2007; Sheffield & Cruikshank, 2005).

It is recognized that equitable societies ensure that all students have appropriate, high level mathematical knowledge (Empson, 2002; Hannaford, 1998) because it is this knowledge that allows citizens to participate effectively in the democratic, decision-making process that guides the future of the nation (National Council of Teachers of Mathematics, 2006). Also, students need an explicit mathematical vocabulary and a philosophical framework within which mathematical knowledge can be questioned and understood in order for the mathematics to have deep meaning (Iverson, 2006; National Council of Teachers of Mathematics, 2006). In the United States, some curricula have been developed that tie classroom mathematics to social justice issues (National Council of Teachers of Mathematics, 2006) and concerns that reflect the lives of students, their families, and their communities (Braver et al., 2005; Gutstein, 2006; Gutstein & Peterson, 2005; National Council of Teachers of Mathematics, 2006); these are to be applauded. These curricula and lesson plans illustrate how mathematics is embedded into the political and economic fabric of our society, such as issues that perpetuate social injustices in terms of immigration policy, inequitable taxation, and gentrification of inner-city neighborhoods (Grant, Kline, & Weinhold, 2002; Gutstein, 2006; Gutstein & Peterson, 2005; Lesser & Nordenhaug, 2004; Mukhopadhuyay & Greer, 2007). However, they do not routinely explore the "hidden" mathematics included in, for example, computer design and architecture, product standardization, advertising graphics, organizing seasonal game schedules for a sports leagues, and health policy decision-making. While social justice education in the mathematics classroom helps students consider vocational and professional options, neither students nor most teachers are able to articulate how the "school mathematics" taught in elementary and middle school translates into important, implicit, and embedded mathematical knowledge that is used in professional practice in technical and non-technical fields. Yet, when teachers are able to make these connections, there is evidence that students 1). begin to recognize the role of mathematics in technology, innovation, planning, and decision-making (FitzSimons, 2002; Hannaford, 1998); 2). recognize the social justice impacts of mathematical knowledge (Braver et al., 2005); and 3). understand that mathematics is more than just a "right answer" (Gerofsky, 2004; Gutstein, 2006; Mudaly, 2007; Mukhopadhuyay & Greer, 2007).

Thus, K-6 teachers may limit or omit discussions of embedded mathematics because 1). they may be unable to recognize these implicit practices; 2). they may not even be aware that such mathematical practices exist; 3). their teacher preparation programs did not stress these connections; 4). they may be mirroring the practices of their own K-6 mathematics education; and/or 5). the

textbooks they use in their classrooms do not focus on these connections. Yet if the teachers themselves were able to reflect on implicit or embedded uses of mathematics, whether or not they fully understand how the mathematics is implemented, it is arguable that they could discuss the breadth of usage of mathematics in our society. If teachers do not recognize the many ways that mathematics is embedded into our daily lives, then, regardless of the depth of their mathematical content knowledge, they may be unable to help students make connections between school mathematics and the reasons for studying the mathematics.

While much literature has addressed teachers' mathematics content knowledge (Empson, 2002; Gerofsky, 2006; Graeber, 1999; Hill et al., 2005) and their understanding and utilization of appropriate pedagogical practices (Iverson, 2006; Lesser & Nordenhaug, 2004; Mewborn, 1999), little if any work has explicitly explored what mathematics K-6 teachers recognize as inherently mathematical outside of the K-6 classroom. While several reports have speculated about teachers' inability to connect classwork to actual practice (e.g., FitzSimons, 2002; Gerofksy, 2004; Gutstein, 2006), there is no empirical evidence supporting this contention. This study addressed those concerns directly in an initial attempt to understand what teachers recognize as mathematics. Specifically, preservice and practicing teachers (collectively referred to as "teachers" in this paper) were asked to identify and articulate their recognition of mathematics usage in their daily lives in a typical week. This analysis of the mathematics they reported addresses the main areas of their mathematical recognition and acknowledgment: how much and what types of implicit or embedded (not readily visible) mathematics they acknowledge; and what does their recognition of specific types of mathematics implies about their understanding of the need for and utilty of mathematics in the K-6 mathematics classroom.

Methods and Data Sources

Participants:

Participants were teachers (n=28) enrolled in one of two Introduction to Research courses as part of a graduate-level Masters of Education program at a regional university in the northeastern United States during the Spring of 2006. Eleven students were licensed and certified teachers and had been or currently were elementary school teachers; seventeen were completing initial licensure and certification. All held a Bachelors degree in Social Science (n=16), Science or Engineering (n=6), Education (n=4) or Accounting (n=2). Twenty-three had completed three years of high school math, including Algebra, Geometry, Trigonometry and Advanced Algebra. Sixteen had continued their high school mathematics for another year to include pre-calculus or calculus. Only two reported no high school coursework in mathematics. Fifteen completed college level calculus courses (n=9) or algebra/statistics courses (n=6). One calculus student and one algebra student also completed a mathematics methods course designed for prospective K-6 mathematics specialists. Four others completed at least one of two mathematics content courses designed for prospective teachers during which they developed an understanding of the NCTM mathematics curriculum.

Data Collection:

We began with an in-class discussion of overt, explicit, covert, implicit, and embedded mathematics that we use on a daily basis to ensure that all participants shared a common understanding of "mathematics" and "mathematical encounter." Teachers shared examples of the explicit and overt mathematics they used and recognized, such as balancing checkbooks and measuring recipe quantities. They also discussed how mathematics is implicit and embedded in much of today's technology. Many straightforward uses of technology were identified (e.g., how computers translate bar codes and magnetic stripes to digital and electronic pulses that are recognized by electronic circuitry). Teachers also recognized less common uses of mathematics embedded in modern technology. These included traffic management protocols, digital

communication optimization, and many manufacturing techniques. The conversation also included discussion of the content and process standards that are incorporated in school mathematics (National Council of Teachers of Mathematics, 2000). These examples were not meant to be inclusive; teachers were encouraged to use these examples as models and exemplars to guide their identification and recognition of mathematical practices. Thus, "mathematical encounters" were defined as any recognized, concrete, mathematical event that the teacher participated in (e.g., preparing a budget) or observed (e.g., watching a cashier make change or a carpenter review a blueprint). Teachers were asked to also report their own thoughts and questions about mathematics and mathematical practices. Both the concrete events and the mathematical speculations were defined as "mathematical encounters." Teachers were asked to carry a notebook and record all mathematical encounters, including repetitive mathematical events, during the data collection period.

Using the classroom discussion as a starting point, the teachers spent seven days monitoring their recognition, practice, and use of mathematics. This allowed them to report the mathematics used on their jobs as well as mathematics that they identified during their personal time, as well. They were able to capture mathematics used professionally, vocationally, and avocationally, as well as mathematics used to maintain their household and mathematics used during leisure time activities. It was recognized that students' daily lives and activities were different and would not produce similar lists of mathematical encounters. To control for these possibilities, the researchers examined the individual scenarios presented by the teachers in their journals to identify what other mathematics could have been reported in the individual scenario. For example, a student discussed the gas pump at the gas station that keeps track of the gallons pumped and the total cost of the gas yet did not mention the pumps capacity to maintain a constant inventory or the ability of the pump to monitor the flow of the gas to the car.

Data Analysis:

This study was a qualitative analysis of the mathematics that the practicing and preservice teachers recognized and recorded in their journals. We explored the following questions:

1. What types of mathematics did the practicing and preservice teachers recognize and/or use in their daily lives?
2. What types of mathematics were not recognized and/or acknowledged?
3. What are the implications of teacher mathematical recognition for K-6 teacher education?

Journals were collected and entered into a data base that allowed us to manipulate and categorized their thoughts and ideas utilizing NVIVO (QSR International, 2006). Constant comparison, as a basis for theory development, was employed to identify core properties of mathematical descriptors and mathematical understanding (Charmaz, 2006; Creswell, 2007) to allow us to create matrix that illustrates teachers' understanding of the interplay between school mathematics and real world practices.

Within each journal entry, individual mathematical items identifying content and/or process were identified and entered into the database as individual records. Entries that illustrated similar ideas were collected into *topic groups*. Topic groups were sorted and organized into one of nine shared *categories*. Categories were classified into one of four overarching *classes*. Topic groups, categories, and classes were all generated from the journal entry data and were not pre-identified or pre-conceived.

Results

In order to articulate a coherent model of teacher understanding of the connection between K-6 mathematics teaching expectations and mathematical practices in real world contexts, a multilayered data structure was employed (Bazeley, 2007; Charmaz, 2006). Individual mathematics

encounters were organized into 30 different *topic groups*; each topic group described similar events and observations, such as driving (distance, speed, mileage), price comparisons, and budgeting, spatial relationships. Topic groups were collected into nine *categories*. Members of each category shared an underlying utilization of mathematics in practice (e.g., use of algorithms, decision making). Finally, four *classes* were developed from the categories, reflecting a hierarchical model of mathematics use and recognition (see Table 1). All individual mathematical encounters reported in the journals were included in the analyses. Mathematical encounters that reflected more than one mathematical encounters were placed in the "highest" level of mathematics reported.

Table 1
Categories and Classes of Mathematics

CLASS	Non-Mathematics	Counting and Calculation	Estimation and Planning	Embedded (Implicit) Mathematics
CATEGORIES	1. Number Recognition 2. Dialing Telephone	1. Counting 2. Algorithms 3. Allocation (Budgeting)	1. Comparisons and Decision Making 2. Logistics (including Spatial Relationships)	1. Mathematics /Technology Interactions 2. Pattern Recognition

Identification of Mathematics:

Student journals included bulleted lists, individual sentences, or short paragraphs that outlined a mathematical description or speculation about mathematical practices. These constituted the individual "mathematical encounters" that were categorized. In the seven-day period during which teachers recorded their recognition of mathematics-related phenomenon, based on the definition of mathematics that they developed in their class, 27 teachers ($N_{(male)} = 9$, $N_{(female)} = 18$, one student did not complete the mathematics diary)) identified 695 mathematical encounters (Table 2). Women reported 2.3 times as many mathematical encounters as men, which is consistent with the fact that twice as many women as men participated in this study.

Table 2
Mathematical Encounters

	Male **# of reference (%)**	**Female** **# of references (%)**	**Total** **# of reference (%)**
• **Non Mathematics**	16 (7.6%)	44 (9.1%)	60 (8.6%)
• **Explicit Mathematical Relationships** (including Calculations and Algorithms; Implicit Mathematics; and Embedded Mathematics	194 (92.4%)	441 (90.9%)	635 (91.4%)
TOTAL	210 (100%)	485 (100%)	695 (100%)

Non-mathematical encounters were defined as purely nominative uses of numbers. Examples of nominative mathematics include dialing telephone numbers, identifying a room number, or tracking a basketball player by his jersey number. While this type of use is technically related to mathematical ideas, they were identified by teachers as mathematical because, as one student stated "numbers means you're dealing with mathematics." However, for the purposes of this study, nominative identification of numbers, which constituted less than 9% of the reported mathematical encounters, were omitted from the final analysis.

Explicit mathematical encounters (n=635) were recognized in all spheres of life activities, including work (accounting, allied medical services, business, teaching), home (budget and planning, cooking, scheduling, child care activities, transportation), and recreation (sports, interactive gaming). Most of the reported mathematics included explicit use of numbers or formulas, although many of the journal entries reflected uses of mathematics as a tool for logic and decision-making that was not reliant on explicit calculations. Very few entries reflected or described implicit or embedded uses of mathematics that were not readily visible, such as traffic signal efficiency or automatic inventory control associated with self-checkout counters at supermarkets.

Explicit mathematical encounters (Table 3) are defined as activities that require use of mathematical strategizing beyond the simple recognition of numbers. Explicit mathematical relationships often involved numbers (e.g., budget planning, bill paying, calculating sports statistics) but were not limited to the numeric manipulations (e.g., reading maps, choreographing a dance).

Table 3
Explicit Mathematical Relationships

	Male **# of reference (%)**	**Female** **# of references (%)**	**Total** **# of reference (%)**
• **Measurement, Calculations and Algorithms**	143 73.7%	329 (74.6%)	472 (74.3%)
• **Estimation and Planning**	48 (24.7%)	109 (24.7%)	157 (24.7%)
• **Embedded Math (Implicit Mathematics)**	3 (1.5%)	3 (0.7%)	6 (0.9%)
TOTAL	194 99.9%	442 100.1%	635 100.1%

Measurement, Calculations and Algorithms, which account for over 70% of the mathematical encounters, represent the most straightforward uses of mathematics. Teachers recognized this type of mathematics both at home and at work, for recreational, administrative, and professional purposes. This type of mathematical enterprise closely mirrored school uses of mathematics to solve problems that were easily described. Nearly 70% of these explicit calculations involved home and work finances, including bill paying, making change, and calculating tips (n=172, 37.8%), and calculations of elapsed time, expected time constraints, and time-distance calculations (n=138, 30.3%).

> Purchased the carpet and provided the figures to the sales person and made the purchase. Purchased groceries while in Auburn. Kept track of selected items in my head to be sure that I had enough cash for the purchase as I do not like to use a credit card for this type of purchase. Again this is math as I used addition and subtraction. I did this while I was in the grocery store [Male 21]
>
> Considered if I could drive to work (number of miles) on the amount of gas (% of tank, fraction of gas in tank). Considered cost of gas vs. cost of running out of gas. Decided to get gas later. [Female 4]
>
> Filling out my tax forms- I add up my yearly income at various jobs I've held. I subtract the sale price from the buying price of stocks I've sold this year (these calculations were done on a calculator, and are very practical for life). [Male 18]

Other calculations reported included revising recipes to increase or decrease serving sizes and calculating room areas to buy paint and carpeting. Several teachers discussed how mathematics is used to calculate scores during sports and games (n=10, 2.2%): Six teachers reported calculating scores while playing games or watching sports, four reported using mathematics to calculate gambling odds or payouts.

> I used math last night while watching the season's finale of *The Gauntlet II* on MTV because the team of 8 won 250, 000 to be divided equally. Worked out to be $ 31,250.00 each. At first I estimated the amount to be $ 30,000 something each and then to be exact I used a calculator. [Female 12]

Almost 25% of the reported uses of mathematics recognized the mathematics as a tool for estimation and planning. Within this category, formulae and algorithms were not explicitly discussed. Logical understandings described nearly half of the reported entries in this category (n=73, 46.5%) and were invoked to make purchases ("Mentally calculated how much wood we'd need to make a bookshelf at home depot" [Female 6]), plan a project ("Create a portfolio at a glance. Must estimate how much information I will need to fill a tri-fold brochure" [Female 4]) and drink tea:

> I sip tea- this is the first time I realized that I use math when I drink or eat something hot! I realize that, subconsciously, I feel the radiant heat on my lips of a hot beverage or food. Depending n how far away it is before I feel its heat, I can judge how hot it is. Based on the temperature of the air surrounding me, I make a judgment of how much time it needs to cool off enough so that I won't burn myself [Male 18].

Logical understandings of mathematics also included the use of spatial relationships. These were recognized in sports, driving, and art:

> Played racquetball at local YMCA; this sport is all about math. Reading angles of the ball to know where it is going to go. If you do not read the angle of the shot and just try to react, you will generally be too slow. It's not an exact math, but an estimation done in my head. [Male 25]
>
> Driving anywhere – distance needed to pull out in front of car or to make a u-turn or k-turn, Find a parking space, keeping speed constant, increasing pressure on gas to go up a hill, decreasing when going down. [Female 2]
>
> Creating formations for dancers to stand- dancers have to be a safe distance apart while looking visually appealing and symmetrical [Female 11]

Mathematics as a decision making tool accounted for the other half of journal entries in the Estimation and Planning category (n=84, 53.5%). This was described in terms of approximation, comparing/contrasting, and probabilistic estimation. Teachers described using mathematical ideas to interpret charts and graphs, identify best value for money, and make game and gambling decisions.

> Figured the cost of moving the trailer and deck from Union Springs to Dexter (Watertown) myself or having someone move it for me. I received a quote from the person who moved the trailer a few years ago and determined that I will move it myself. This is relevant due to the cost of gas and labor involved and I now have a vehicle that I can move it with. All of the figures I used if I move the trailer myself were estimates and the final figure was quite a bit lower if we complete the move on our own. [Male 21]
>
> Playing games: I have a group of friends that I play some obscure games with, but all of them involve some math. First is Bonanza, which involves a lot of probability. Knowing the number of each type of card that is left and playing the odds is an important part of the game. It is math that is done in my head, but can be difficult to

> track because there are different amounts of each type of card and the more rare they are the more they are worth. [Male 25]

During their week of data collection, teachers were especially encouraged to identify embedded mathematics, such as implicit uses of technology and hidden mathematics, in their daily mathematics encounters. Based on the in-class discussion prior to data collection, teachers acknowledged many implicit uses of mathematics, including bar-code technology, traffic management, and manufacturing. However, less than 1% of the responses identified such embedded or implicit mathematics. Of the six responses in this category, half discussed how a computer translates keypad instructions to electronic impulses:

> I use a computer. When I use a computer, I press a symbol on the keypad. The computer uses binary math (ones and zeros) to perform a specific operation and display an output (mathematics is being used here because the computer does not have the ability to speak English, rather each symbol on the keypad has its own mathematical formula understood by the computer [Male 18].

The other three responses focused on issues of pattern recognition and encryption (e.g., "Open door with code" [Female 9]) with limited discussion of the connection between the mathematics involved in the technological enterprise.

Discussion

This study has important implications in light of the mathematics education offered to preservice K-6 teachers. While pedagogy and content knowledge are important, this work suggests that K-6 teachers may not value the mathematics they teach. They fail to connect the mathematics and mathematical thinking they teach to mathematical practices outside their classrooms, from explicit calculations to logical and organized thought to the modeling protocols that inform and influence technological design and decision-making. While this disconnection may be acceptable for a layperson, it is worrying when observed in a teacher population. Thus, teachers need support in identifying where mathematical is located and how it is used in society, beyond superficial and explicit calculations and algorithms. As teachers become better able to identify and articulate mathematical thinking in non-mathematical contexts (National Council of Teachers of Mathematics, 2006) they will be better able to help students recognize mathematics across the curriculum,(i.e., cartographic understanding in social studies, mixing paint in art class). Mathematics education for preservice teachers must incorporate the exploration of professional, vocational, and avocations contexts of mathematics into the discussion of pedagogy and content in order to ensure that K-6 teachers can introduce students to true mathematical contexts outside of the mathematics classroom.

Teachers did overtly acknowledge that mathematical ideas underlie much of the technology that they encounter, however they did not report such embedded mathematics. While not mentioning the implicit mathematics of technology is not the same as not recognizing this mathematics, the lack of such mentions is disturbing, given that teachers were specifically asked to mention any mathematics they did recognize. Responses in a variety of areas (e.g., choreography, logical decision making, planning) suggest that teachers do identify less visible, less common uses of mathematical ideas and practices. Thus, the lack of technology related mathematical encounters suggests that such encounters were, indeed, unidentified. In fact, Fitzsimons (2002) contends that except in specific technologically-based vocational education classrooms, most teachers, educators, and students fail to grasp such connections. The thrust of K-6 teachers' practice in the mathematics classroom focuses on algorithmically-based mathematics and logical strategies to solve explicit

problems and make straight-forward decisions (National Council of Teachers of Mathematics, 2006; Reys et al., 2007; van de Walle, 2001). Calculating distance traveled per gallon gas used, making change, and scheduling and organizing events were seen as mathematical because the teachers recognized that mathematics embodies both algorithmic understanding and logical planning and organization. However, less tangible uses of mathematics and mathematical ideas that are not easily visible – such as the mathematics that underlies the technology that is used daily or mathematical modeling protocols to compare and contrast solutions and to explore the possible impacts of various decisions – were rarely mentioned. Several teachers described buying gas and paying for it, recognizing that the calculations of miles per gallon and the cost of the gas were mathematical. None mentioned the mathematics involved in maintaining the embedded computer in the gas pump that monitors the fuel flow from the nozzle to the car tank, automatically shuts off the pumping mechanism if problems arise, displays the gallons sold and the price to be paid, transmits that information to the clerk at another cash register, and tracks the total gas inventory of the gas station.

Similarly, the mathematics identified reflected the teachers' ambivalence about mathematics. When the mathematics they attempted to describe veered away from common and/or recognizable calculations and/or explicit organization and planning routines, they often wondered if what they were doing was mathematical at all or just common sense. Others teachers questioned whether they should include mathematics when they observed things they tended to take for granted, such as the geometrical shapes and sizes of buildings or the choreography of a dance routine. This suggests that the teachers are not accustomed to thinking broadly about mathematics and reflects Iverson's (2006) suggestion that mathematical curriculum infused with philosophical discussion would give teachers and students a vocabulary with which to speculate about mathematics use outside of the classroom environment. Simultaneously, teachers' journals also indicated a lack of confidence in their knowledge of what is mathematics and what should be labeled as mathematics in daily life. Teachers recognize mathematics as important, in that on its most basic level, as an algorithmic tool, mathematics is used on a daily basis. On a somewhat more theoretical level, mathematics as an instrument for logical, organized thinking and planning is also part of teachers' daily lexicon. Yet teachers are unable to fully articulate more complex uses of mathematics, which models possible, theoretical, and creative solutions to problems, responds to a variety of real-time variables, and anticipates that which may be imagined.

In-class conversation suggested that the teachers could articulate broad mathematical recognition. However, the mathematics that the teachers reported verified and corroborated this "school mathematics" perspective. This is troubling because sixteen of the teachers had completed advanced mathematics in high school and/or college level mathematics, and had been exposed to more abstract understandings of mathematical thought, yet they did not appear to have internalized this understanding as inherently "mathematical."

This failure to give mathematical credence to the underlying technologies and implicit uses of mathematics in their daily lives raises questions about what is valued about mathematical understanding and utilization. It is not clear that these teachers recognize that the basic mathematics taught in most K-6 classrooms, as suggested and defined by the NCTM (2000, 2006) standards, is an important precursor to the implicit mathematics that governs the technology that we rely on and the political, economic, social, and technological decision making that impacts our lives. . Perhaps teachers' inability to explain the mathematics embedded in technology (for example) may also make them shy away from acknowledging it. If that is not recognized, their ability to teach this mathematics with meaning and understanding may be compromised. Although nearly everyone reported using calculators to complete various calculations, very few people recognized that the calculator itself relied on mathematical algorithms translated into a mathematically-based computer processing language that allows communication between people and the machines. Similarly,

although several people reported using computer programs to calculate budgets and prepare tax returns, no one mentioned the mathematical-logical processes required to write the software itself or the mathematically-based computer architecture design decisions that allowed the computer itself to run the program and solve the problems. As Mewborn (1999) suggested, this raises questions about what should we expect teachers (and students) to recognize, understand, and value in terms of mathematics and mathematics education. If technology – be it a calculator, computer, automatic teller machine, or set of switching relays monitoring traffic patterns – solves problems accurately and efficiently, why should teachers and students try to solve those problems manually? If technology does the job well, what is the value of understanding how the machine "does it?" Or, more philosophically, how can teachers and students acknowledge and value the underlying mathematics if they do not recognize, acknowledge, and/or understand that mathematics exists in these locations?

This is the question that must be faced in mathematics education today. Formal definitions of mathematics (Mulls et al., 2004; National Council of Teachers of Mathematics, 2000; 2006) strive to help teachers create a classroom environment that allows students to explore mathematics itself. What is missing, however, is the link that helps teachers and students connect the important mathematics that is part of the K-12 curriculum to the less visible mathematics that undergirds the technological supports of our society. Ultimately, this is to our disadvantage: fewer students are studying mathematics at the university level (National Center for Education Statistics, 2005). As a nation, we are not supporting the mathematical background needed to maintain the structure of our current technology nor the development of needed technologies or new uses of existing technologies.

If we are teaching mathematics as an arcane set of skills that helps students hone their abilities to think, organize, and solve straightforward problems, then the mathematics curriculum we are teaching today may be appropriate. However, we must clearly articulate to teachers and students that that is the goal of mathematics education. Many have suggested, however, that mathematical understanding is the key to creating the future that we envision (Empson, 2002; FitzSimons, 2002; Gutstein, 2006; Hannaford, 1998; Lesser & Nordenhaug, 2004; Mukhopadhuyay & Greer, 2007; Nicol, 2002; Noss, 2001). If teachers do not recognize the many uses of mathematics in our lives, then they cannot be expected to prepare students for using mathematics to build a viable tomorrow.

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Mathematics for Middle School Teachers: Choices, Successes, and Challenges

How do we get there from here?

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Abstract:
The opportunities for mathematics department faculty from institutions of higher education (IHEs) to work with middle school mathematics teachers are on the rise; the U.S. Department of Education has allocated almost $800 million for mathematics and science partnerships since 2004 that require collaboration between faculty from IHEs departments of arts and sciences and school districts. Changes in credentialing requirements due to the No Child Left Behind Act of 2001 mean that middle school mathematics teachers must take more mathematics content courses, yet current offerings for teachers in many mathematics departments typically focus on elementary or high school mathematics and often neglect the needs of middle school teachers. This article talks about the efforts of a group of mathematics faculty and school district personnel to develop mathematics courses that would help middle school teachers develop a Profound Understanding of Middle School Mathematics. An explanation of the curriculum structure – highlighted by the teachers' responses – is given here.

Keywords: middle school mathematics; in-service teacher training; mathematics content; mathematics pedagogy; No Child Left Behind;

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Introduction

Amelie,[3] a middle school teacher, was reflecting back on a week of intensive algebra. We asked her to comment on the themes of the course, and she wrote, "I feel these themes have helped me to make many connections that I have never made before. It is fascinating to see how interrelated math really is, like how completing the square is really doing the steps of the quadratic formula, or how eliminating and substituting are really two different ways of arriving at the same solution." A day earlier, the class had struggled to complete the square for the equation $ax^2 + bx + c = 0$; several participants were able to do this for themselves, and most of the rest were able to follow the class demonstration by one of the instructors. At the end, the class erupted in applause—they got it, and in the words of another participant, "It was so helpful to see the relationship between completing the square and the quadratic formula. That was so cool."

In November 2006, our third cohort of middle-grade teachers finished the third of three math courses specifically designed for them. The courses were taught by different combinations of instructors, each adapting and refining the materials in an effort to improve them. But for each iteration of each course, our goal was always been the same: to help lay the foundations for what we call here a Profound Understanding of Middle School Mathematics (a specialization of Liping Ma's notion of a Profound Understanding of Fundamental Mathematics, or PUFM). Ma states:

> A teacher with PUFM is aware of the "simple but powerful" basic ideas of mathematics and tends to revisit and reinforce them. He or she has a fundamental understanding of the whole elementary mathematics curriculum, thus is ready to exploit an opportunity to review concepts that students have previously studied or to lay the groundwork for a concept to be studied later. (1999, p. 124)

It is interesting that Ma's notion of PUFM resonates with both research mathematicians and mathematics educators. Her book provides an access point for mathematicians into the issues of teacher knowledge for teaching precisely because they are able to recognize in her work a view of mathematics that reflects their own. And mathematics educators see that Ma's description of PUFM includes other critical components of mathematics learning, for instance, how mathematical topics should be organized in order to be best learned. In PUFM, mathematical and pedagogical knowledge are entwined. In his review of Ma's book, Roger Howe said,

> It seems that successful completion of college course work is not evidence of thorough understanding of elementary mathematics. Most university mathematicians see much of advanced mathematics as a deepening and broadening, a refinement and clarification, an extension and fulfillment of elementary mathematics. However, it seems that it is possible to take and pass advanced courses without understanding how they illuminate more elementary material, particularly if one's understanding of that material is superficial…. [I]t seems also that the kind of knowledge that is

[3] Names have been changed where we know them—in this case, these comments were made on an anonymous end-of-course evaluation, so we don't know which of our participants wrote it (but probabilistically speaking, it was a woman).

> needed is different from what most U.S. teacher preparation schemes provide, and we have currently hardly any institutional structures for fostering the appropriate kind of understanding. (1999)

While many teacher education programs do have mathematics courses for elementary teachers, courses specifically for middle school teachers are relatively uncommon. Richard Askey elaborates:

> Courses for prospective elementary school teachers, for example, frequently slight material dealing with fractions since whole number arithmetic is the main focus in our elementary schools. Middle school teachers frequently fall between the cracks. The material they will be teaching is not taught in detail to either prospective elementary school teachers or to prospective high school teachers; there are no courses specifically for middle school teachers. (1999)

Since he wrote this, new courses for middle school teachers have sprung up around the US, but the national discussion of what these courses should look like is still quite new. This was the context in which we initially set out to design our mathematics courses for middle school teachers.

When we were designing and refining our courses, we were aware of several resources, some of which directly address the mathematical preparation of teachers such as Conference Board (2001), Howe (2006), Madden (n.d.), Milgram (2005), Wu (1999), and some of which addressed the middle school curriculum more directly, such as the NCTM Standards (2000) and the state standards that our teachers were required to address in their own teaching. But these resources range from philosophical considerations to encyclopedic coverage of vast areas of mathematics[4], and there are many decisions that need to be made when creating an extended curriculum. For those of us who have had the opportunity to design mathematics courses for middle school teachers, we have had to make concrete and immediate decisions about what mathematical topics to cover and how to address them so that teachers have a chance of developing PUMM. Our challenge was to find a pathway that leads toward a deeper understanding of middle school mathematics, and to identify the curricular and instructional components necessary to achieve this.

Like many other mathematicians and mathematics educators, we struggled with these issues. Before settling on course materials, we considered the backgrounds of the teachers, how to balance pedagogy and mathematical content, and how to present the content in a way that we hoped would lead to substantial development of PUMM. Our results were encouraging. What follows is the story of these decisions and the lessons we learned in the process.

Guiding principles

In the summer of 2003, one of us visited the Vermont Mathematics Initiative (VMI) summer institute for elementary teachers at the invitation of its founder, Dr. Ken Gross. Dr. Gross said that the two guiding principles for the VMI were to

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[4] The two exceptions that we know of are the set of curriculum materials developed specifically for middle school mathematics teachers by Ira Papick and his colleagues on the Connecting Middle School and College Mathematics Project, see Papick (n.d.), and the set of curriculum materials developed by Jim Lewis and his colleagues at the Mathematics in the Middle project, see Lewis (n.d.), but these were being developed or just being published around the time we were creating our own materials and were not readily accessible to us at that time.

1. Be true to the mathematics, and
2. Treat teachers as professionals with their own intellectual needs.

This philosophical stance struck a deep chord with us. More importantly, the clear success of the VMI approach was apparent in the mathematical work of the participating teachers. These elementary teachers were working on challenging mathematical problems and using mathematical language more sophisticated than many university calculus students use. Inspired, we held our first summer institute the very next year.

Because we were working primarily with middle school teachers, we still had a lot of work to do developing the curriculum, but the pedagogical approach (from team-teaching down to paper covering the tables so that participants and instructors could write mathematics on every nearby surface) was very much inspired by what we saw at the VMI.

It was important to us to develop courses specifically designed for middle school teachers. We wanted to address the topics that form the core of the middle school curriculum, such as fractions, pre-algebra, basic geometry, and topics in data analysis. Culling this list of topics to a two-week institute (and even the extended 6 course version of our program) required making many choices. We chose topics that were prominent in the middle school curriculum and that were highly connected to each other and to topics that anchor the high school curriculum.

Teaching teachers requires different strategies than teaching the same material to children. Adult learners have a broader range of mathematical and life experiences that shape what they currently know. One can be quite certain that they have all seen integer arithmetic and linear equations, but many may have only a partial understanding or hold misconceptions about the mathematics that they have studied. One must consider how to build upon adult learner's previous experience and structure the work so that they see that the mathematics is deep and important.

Most of our participants had a solid ability to do basic arithmetic and to solve problems using a guess-and-check strategy; there were very few other mathematical topics with which all participants were equally comfortable. Developing materials for such an audience requires careful planning, but also offers certain advantages. On the one hand, it requires beginning with material that is easily accessible but can be developed at deeper levels. On the other hand, capitalizing on and connecting these different levels can help bootstrap teachers who may struggle with more advanced material.

Building on prior knowledge

"Build on prior knowledge" is almost a mantra in mathematics education. In this section we elaborate on how we attempted to do this in the context mathematics courses for middle school teachers.

First, we introduced many topics by using a contextual problem. These are not meant to be "real-world problems," but are situated in an idealized context that our participants can readily understand yet are naturally represented by the mathematical objects that we are studying. This allowed our participants to see that the abstractions of mathematics are related to things they already know and understand. There is another reason that we chose to use contexts extensively throughout the curriculum. People often have good intuition and common sense in everyday quantitative situations, but then fail to apply those reasoning skills in more formal mathematical situations. The contexts we provide help them to face this conflict, and ultimately lead them to apply what they already understand to the mathematical abstractions. In other words, the context is not just meant to help them

appreciate the usefulness of the mathematics, but is designed to lead them to a deeper understanding of the mathematics itself.

Certain problems and contexts were chosen because they can be used to illustrate several different important topics. For example, the first problem that participants tackled on the first day of class was used to illustrate the idea that different approaches can (and should) lead to the same solution, the value of different mathematical representations, the importance of taking one's time to understand a problem thoroughly, and later to introduce arithmetic sequences, the notion of a function, and systems of equations. Some of our problems span multiple courses. For example, one problem that participants solved in the algebra course was revisited in both the geometry and calculus courses as well. In this way, we are assured that all of our participants have a common experience on which to draw when learning new mathematics. As one participant said at the end of one of the summer institute, "I believe that taking the same problems and using them throughout the two weeks, but in different ways, really helped me see some of the concepts differently."

Let us illustrate this principle with an example. Our first summer institute began with the following problem, borrowed from the VMI. It is important to note that this problem is given to the participants with no mathematical introduction, so from their perspective, it is open-ended.

1. Photo Developing.[5]

There are two photography stores in town, Perfect Picture (abbreviated PP) and Dynamic Developers (abbreviated DD) that do custom film developing. At PP the cost to develop one roll of specialty film is $12 but any additional rolls of film cost only $10. At DD the cost of developing one roll of film is $24 but each additional roll is developed at a cost of only $8.

For what number of rolls of film is the cost of developing the same at PP and DD?

In our experience, almost everyone gets an answer, either by doing some systematic guessing or by setting up a system of equations. For participants who are more comfortable with systematic guessing, their understanding of the problem and its solution will help them to understand the equations that represent the problem equally well. For participants who are comfortable with the algebraic representation of the problem, it is instructive for them to see how others struggle with writing equations.

The next question, while related in its mathematical structure (a system of linear equations or a linear inequality will represent it quite nicely), requires a more substantial mathematical understanding in order to interpret the result.

2. More Photo Developing.

Suppose you feel strongly that PP does a much better job of developing your film; in fact, so much better that you are willing to develop the film at PP unless the cost is more that double the cost at DD. How many rolls of film do you have to develop in order for the

> cost at PP to be more than double the cost at DD? What do you think of this "double-cost" rule?

Here the solution is, "If you are developing the same number of rolls, the cost at PP will never be more than twice the cost at DD." So in this case, the problem cannot be solved by systematic guessing (although this can help with understanding the solution). Furthermore, even participants who solve it algebraically have a difficult time interpreting the answer they get (since the value where one is double the other is negative), and typically they see it best if they also see an alternate representation for the problem (such as a table of positive values that "never work out" or a graph of a related system of equations). Seeing that the equations and graphs which represent the problem are much more powerful tools than systematic guessing helps to motivate the purposeful study of these mathematical structures later in the course.

In summary, there are two strategic ways we attempt to build on our participants' prior knowledge when introducing new mathematical ideas. The first is to initially set problems in contexts that are likely to be familiar and to let them solve it in any way they see fit. The second is to strategically include certain problems early in the curriculum that they can then build on later. An example of the first situation is the initial experience with the photo developing problem. After participants have solved this problem, however, the solution methods we subsequently discuss, such as using equations and graphs, become part of the participants' knowledge base on which we were able to build later in the program. This is a key to the success of the curriculum.

Increasing the intellectual load

For many people, the photo developing problems look like they belong in a long set of exercises at the end of a section on systems of equations. But it is the context in which they are given that creates the appropriate *intellectual load*. The intellectual load can be very different for the same problem depending on when it is given and who is "doing the work." Most participants (and even some university faculty) have considered these problems to be substantial open-ended problems because of the context in which they were considering it. Initially, some participants were frustrated with this approach, and sometimes complained that we shouldn't start things out with a problem that "doesn't work." But the solidity of understanding that comes with struggling through it themselves soon becomes more important to the participants. For example, in the first follow-up meeting after the first summer institute, we gave them a typical problem involving common factors of two whole numbers. Given with no context, however, it becomes an open-ended problem:

> Iggy is filling a bulletin board with pictures. The bulletin board measures 144 cm by 96 cm. If Iggy wants to hang square pictures of the same size (and that measure a whole number of cm on each side), what size picture can Iggy use? (Assume he doesn't want the pictures to overlap or have gaps between them.)

An interesting natural experiment occurred the first time we gave this problem. It was intended to be given with no introduction so that it might serve as a common context for a discussion of common factors, but because of miscommunication between the instructors, one gave a mini-lecture on common factors right before the participants were given the problem. During the class period, one of the participants commented, "I knew I was supposed to use the greatest common factor, but I didn't know why." (In fact, the

intention was to discuss all common factors, not just the greatest common factor.) On the daily evaluation, another said, "I'm not sure telling what we are going to see was effective. I think the discovery approach will bring it out better." Still another said, "I really like to have more time to work on the problems and a little less explanation. It was interesting to realize that I often want to explain too much before allowing my students to struggle." This was one of the days we felt our greatest sense of accomplishment. Our participants had gone from wanting us to give them answers more directly to wanting more time to think about it for themselves.

How can we improve?

During the coursework, we did not discuss pedagogy directly; we assumed that teachers would learn best how to teach mathematics by learning it themselves in a deep, coherent way. We hoped that they would then be able to adapt the strategies for use in their own classrooms. In fact, when asked what has benefited them most about their experiences in the program, teachers were almost as likely to cite knowledge of effective teaching as content knowledge gains[6]. For example, at the end of the 2004 summer institute, one of the participants wrote:

> I never knew quite where to start teaching. I see how the text books are organized, but when my students have such a hard time with abstract thought, I've had a hard time prioritizing. I mean, place value is just a lot of talk unless it can be used. Now that I see how concepts can be developed slowly and shown by illustration and with manipulatives, I see that I can make something like place value important. I need patience and a slow pace for something to sink in, but it will be worth it…. I've learned a lot more about teaching and my own math skill has improved greatly."

While we feel that the program had a very positive impact on teachers' attitudes about teaching, it is an open question to what extent it impacts their actual practice. Informal observations of their classrooms have shown us that the extent to which their experiences in our program have actually impacted their practice in easily observable ways varies greatly from teacher to teacher. This has caused us to consider how we can improve our program so that teachers make more frequent and direct connections with their own teaching practice. Recent conversations with Mark Thames have broadened our thinking about this subject. He states, "Studies attempting to link students' achievement to teachers' content knowledge consistently suggest that mathematical knowledge more closely related to practice–for instance, related to specific curricula or to the work teachers do–is more likely to have a positive effect on teaching and learning." (2008) Deborah Ball and her colleagues have elaborated this view of "mathematical knowledge for teaching," and their work provides us with deep insights into the nature of such knowledge (see Ball, 2002, Ball, Hill, Bass, 2005). We feel that our next step is for us to find or create materials that help teachers develop this kind of knowledge as well.

The intellectual needs of teachers

What is deep or challenging depends on what you already know and how it is presented. For many teachers, an opportunity to look more deeply into the content that they teach and the mathematics of nearby grades can be very intellectually stimulating. As one

[6] They also often cited content knowledge gains, and their scores on content pre- and post-tests confirmed this, see Umland (2006).

participant said, "I am overjoyed that it was understandable for me, and yet it was also very stretching and challenging." It is also important to provide materials that are clearly about real mathematics. Another participant said, "I have greatly improved my conceptual knowledge…We didn't get any 'cutsie' activities… It's that 'give them a fish… teach them to fish' thing."

The participants in our program had a wide range of backgrounds and mathematical expertise. Our goal was to create a course that would contribute to the growth of all of these teachers. Below are excerpts from the essays of two participants, one who was very anxious about mathematics and another who was very confident on entering the program (their pretest scores showed that they were indeed at the bottom and top in terms of entering knowledge, respectively).

> [H]ow did I end up in [the program]?... [It] seemed suited to helping me grow in my understanding of the why [of math]. Yet I was very apprehensive in signing up because I sensed I would be very much out of my comfort zone. And was I ever!... [Yet] I experienced a great deal of satisfaction getting through the two weeks. The morning after the last day, I woke up with a new insight about slope and how it is used!
>
> When I read the description of [the program], I was really skeptical about the program. I thought I knew everything I needed to know about math…. I decided to try it mostly because I was very interested in being a part of a lesson study. I now know that I made one of the best decisions of my teaching career. Not only have I learned so much more about the basics and foundations of algebra, I have also learned the importance and validity of multiple approaches to the same problem…. After spending a good part of my life focusing on getting the right answers quickly, I now know that to be a better teacher I need to help my students go beyond learning how to just get the right answer.

Carefully chosen problems and content can benefit teachers at all places along the mathematical spectrum, as long as one balances challenge and support. Maintaining high standards while giving the conceptual and emotional support that teachers (and probably most math students) need is the key to starting where teachers are and helping them reach for PUMM. As one participant said, "What I kept thinking was why didn't they teach it this way when I was in school. I have a much better understanding of how everything fits together."

We often hear people talk about teaching mathematics in a "connected, coherent way." But this is a very abstract idea. How does this translate into choices for actual topics and problems? Our answer was to take the content of middle school mathematics and a couple of layers up and to show, in very specific ways, how these ideas fit together. So when Amelie talked about her insights into the relationship between completing the square and the quadratic formula or between different algebraic methods of solving systems of linear equations, she was seeing the important structures of middle school mathematics. Our next challenge is to help teachers like Amelie provide the same kinds of experiences for their own students.

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Inverses – why we teach and why we need talk more about it more often!

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Abstract: This article examines the key role that the notion of inverses plays in numerous mathematical concepts.

Keywords: Inverse, Group, Ring, Field, Binary Operators

Introduction

The point of "inverse" gets down to "the inverse of which operation?". Different operations lead different answers about the inverse. The main reason why students are confused or make mistakes are that they are not clear which operation teachers refer to. In this writing, I first discuss the inverse of three types of operations: addition (its inverse is known as "opposite"), multiplication (its inverse is known as "reciprocal") and composition of maps or functions (its inverse is known as "inverse"). Then I introduce the concept of group, ring, and the field in order to challenge students to utilize their cognitive understanding of inverses in solving equations.

It was not until I had begun to teach trigonometry that I realized the students' understanding of inverse is not so concrete. Given $\sin 30° = \frac{1}{2}$, most students knew $\csc 30° = 2$; some shouted "you flip it! That's how you got it." It appeared that they understood the reciprocal of $\sin\theta$ is $\csc\theta$. A few days later, when asked the same group about the value of $\sec 60°$, a number of students responded $\sec 60° = \cos\left(\frac{1}{60}°\right)$. Intrigued by this, I did a survey of asking a sophomore class about the inverse of 2. Most students answered $\frac{1}{2}$. When asked the same question, a number of senior students, however, responded, "where is x and y?"

How many inverses are there?

When it comes to inverses, there seems to be a sequential understanding of inverse as students progress from pre-algebra to algebra and pre-calculus. Why are they confused and

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inadequate in doing inverse questions? The unfamiliar notation itself is a reason and they have not learned inverses in a more systematic way where each inverse is presented under different operation.

In the beginning, they learn the opposite of 3 is -3, the reciprocal of 3 is $\frac{1}{3}$, then ultimately the inverse of $f(x)=2x+1$ is $f^{-1}(x)=\frac{1}{2}x-\frac{1}{2}$. Still, a math teacher encounters a student who mistake $f^{-1}(x)$ for $\frac{1}{f(x)}$ on a daily basis. The trouble continues to be epidemic in trigonometry where many students thinks that $\sin^{-1}\theta=\frac{1}{\sin\theta}$. The following was one of my pop quiz questions.

[Example 1]

6. Find a correct statement.

(A) $\tan^{-1}x=\cot x$ (B) $\sin\frac{1}{15}=\frac{1}{\sin 15}$ (C) $5\sin\theta=\sin 5\theta$

(D) $\sin\frac{2}{17}=\csc\frac{17}{2}$ (E) $\cos\frac{1}{15}=\frac{1}{\sec\left(\frac{1}{15}\right)}$

I was surprised only one third of the class in fact identified the answer choice (E) correctly. One may dismiss this problem as unfamiliarity with notations. In fact, if a teacher focuses on using correct notations, students began to do well with questions like example 1. Also, teachers are encouraged to discusses that the meaning of $f^{-1}(x)$ is the composition inverse of $f(x)$ as a function, not the reciprocal inverse of $f(x)$. Additionally, a number of students also find it more effective and consistent for a teacher to use notations such as $\arcsin\theta$, $\arccos\theta$ and $\arctan\theta$, instead of $\sin^{-1}\theta$, $\cos^{-1}\theta$ and $\tan^{-1}\theta$.

The followings (example 2; example 3) are is my lecture notes I used particularly with introduction of inverse functions because I thought that there should be an emphasis on a more concrete understanding of inverses under different operations.

[Example 2]

Quantity	Operation	Inverse
2	*Addition*	-2 (opposite)
2	*Multiplication*	½ (reciprocal)
y = 2x + 1	*Composition*	½ x – ½ (inverse)

[Example 3]

The inverse of $\tan\theta=\arctan\theta$.

(The inverse of $\tan\frac{1}{23}=\arctan\left(\frac{1}{23}\right)$)

The reciprocal of $\tan\theta = \dfrac{1}{\tan\theta} = \cot\theta$

(The multiplicative inverse of $\tan\theta = \dfrac{1}{\tan\theta} = \cot\theta$)

The opposite of $\tan\theta = -\tan\theta$

(The additive inverse of $\tan\theta = -\tan\theta$)

After emphasizing this and repeating the notations during a number of class activities, then any math teacher would realize students begin to pick up a correct notation and can develop the idea that inverse is a more comprehensive and a practical concept. Furthermore, the following example [example 4] is also a powerful case in point where students appreciate their understanding of inverse.

[Example 4]

$\sin x = 0.3$

$\sin^{-1}(\sin x) = \sin^{-1}(0.3)$

$x \approx 17.5°$

In fact, there is more to think about in example 4. In the beginning, teaching to a test, I experienced a huge success when I take an easy way out by just telling the students to use a short-cut key in a graphing calculator whenever they needed to find a measure of the angle. There is no doubt that most students easily solved a question like example 4. However, I soon found that the students still struggle with grasping the concept of inverse in the context of composite functions. Not surprisingly, they had a difficulty in finding an answer to the following question [Example 5].

[Example 5]

Find x and y.

$\cos^{-1}(3x) = 20°$

$\cot(x^2+1) = 3$

Although it appeared to be time-consuming to walk through the process with students by discussing the inverse functions, it turned out to be more effective in the long term. The following [Example 6] is an example of a discussion modeling how to solve questions like the second question in example 5.

[Example 6]

Question: Solve for x. Given $\cot(x^2+1) = 3$

Teacher: *It is an equation with a variable x. How would you isolate x?*

Student: *Take arc co-tangent of both sides. That gives back* (x^2+1) *with the variable x.*

Teacher: *If you're not familiar with arc co-tangent, how would you change the equation.*

Student: *Rewrite the equation* $\tan(x^2+1) = \dfrac{1}{3}$ *because the reciprocal of* $\cot\theta$ *is* $\tan\theta$. *Then, take the inverse of tangent of both sides so that it returns* x^2+1, *which equals* $\tan^{-1}\left(\dfrac{1}{3}\right)$.

Teacher: *Try to show give the exact value of x.*

Student: $\left(x^2+1\right)=\tan^{-1}\left(\frac{1}{3}\right)$. *Then,* $x=\pm\sqrt{\tan^{-1}\left(\frac{1}{3}\right)-1}$

At the end of the day, I concluded that the students need to be presented with a variety of contexts where inverses plays important roles and that high school math teachers might need to spend more time discussing inverses and their roles in both arithmetic and advanced mathematics.

For example, in solving a simple linear equation, students need to be reminded of the fact that they use the concept of inverse everyday.

[Example 7]

4x – 5 = x + 7	
3x – 5 = 7	**cancel out x; additive inverse of x is -x**
3x = 12	**cancel out 5; additive inverse of -5 is +5**
x = 4	**cancel out 3; multiplicative inverse of 3 is 1/3**

In example 7, often we see a student saying "move -5 to the other side and in fact I used to explaining that way because that's how my math teacher taught me in Korea. But that explanation doesn't reveal much about the idea of inverse. In stead of instructing the student to switch the sign if a number needs to move the other side of the equation, a teacher should specifically mention that to cancel out a number, they need to the reciprocal or the opposite or the inverse function depending on the operations. That is, though informally, students need to understand that inverse is the quantity which cancels out a given quantity and that as a result, the operation gives the identity. Additionally, there are different kinds of inverses for different operations (i.e., addition, multiplication, composite, matrix multiplication, etc.)

Furthermore, a math teacher should be willing to re-teach the following [Example 8; Example 9] if necessary.

[Example 8]

Given 5

5 + 0 = 5 ; 0 + 5 = 5

Then, 0 is the identity under the additive operation because the identity returns the value.

Therefore, to formulate this idea, a + a^{-1} (=inverse of a) = 0

a^{-1} = –a (opposite or additive inverse of A)

It is also important to note that a^{-1} is being exclusively used for the multiplicative inverse.

[Example 9]

Given 5

$5\times1=5$; $1\times5=5$

Then, 1 is the identity under the multiplicative operation because the identity returns the value.

Therefore, to formulate this idea, $a\times a^{-1}=1$.

a^{-1} (=inverse of a) = $\frac{1}{a}$ (reciprocal or multiplicative inverse of A)

Sooner than later, all the efforts and hard work to introduce inverses, a group of math enthusiastic students may emerge with more fundamental questions. "What would happen if we get rid of inverses in math?" "How do inverses help to do mathematics? A succinct answer could be:

"To solve an equation." Ideally, it is the best time to introduce a basic group theory in terms of inverses.

The following is a collection of informal definitions of key terminologies in group, ring and field. (Woldfram MathWorld Websites)

A group G is a finite or infinite set of elements together with a binary operation (called the group operation) that satisfy the four fundamental properties of closure, associativity, the identity property, and the inverse property. There must be an inverse of each element in the group.

A ring is a set S with binary operators $+$ and $\times$, satisfying the six conditions: Additive associativity, Additive commutativity, Additive identity, Additive inverse, Left and right distributivity, and Multiplicative associativity. Note that it is not necessarily that there is an inverse of reciprocal of nonzero element in the ring. Example of rings are integers without multiplicative inverse and even-valued integers without multiplicative identity.

A field is a special ring with additional three conditions: Multiplicative commutativity, Multiplicative identity, and Multiplicative inverse. Most notable is that there must be a multiplicative inverse for each non-zero element. Examples of fields is the complex numbers, rational numbers, and real numbers. (Dummit 2003)

The following is an example of an activity where students create a system of numbers such as a group. In the process, the students see the significance of inverse in constructing a number system.

[Assignment 1]

Create a binary operation using addition and multiplication in $\Re$.

(a) Define the binary.

(b) Determine whether there is an identity.

(c) If there is an identity, examine whether it is a group under this binary operation (i.e. whether the binary operation satisfies closure, associativity, identity and inverses.)

[Student Work Sample 1]

Define $A\#B = 3(A+B) - A \cdot B$ where $A, B \in \Re$

$A\#e = 3(A+e) - A \cdot e = A$

Then $e = \dfrac{2A}{(A-3)}$

Teacher's comments: How about 3? You must disallow 3, but then it contradicts your set, which is $\Re$. Also, e is an identity which must be independent of any elements. Here, $e = \dfrac{2A}{(A-3)}$ depends on A.

[Student Work Sample 2]

Define $A\#B = 7(A-B)$ where $A, B \in \Re$

$A\#e = 7(A-e) = A$

Then $e = \dfrac{6A}{7}$; $A^{-1} = \dfrac{43}{49}A$

To solve, $e\#A = 7(e-A) = A$

Then $e = \dfrac{8A}{7}$; $A^{-1} = \dfrac{41}{49}A$

Student comments: This group doesn't have the identity because A#e = e#A.

<u>*Teacher comments*</u>: e can't be the identity because it is dependent of elements. Also show that A # B = B # A

[Student Work Sample 3]
$A\Diamond B = A + B - A \cdot B$ where $A, B \in \Re$
$A\Diamond e = A + e - A \cdot e = A$
Then, $e = \dfrac{0}{1-A} = 0$, $A \neq 1$
$e\Diamond A = e + A - e \cdot A = A$
Then, $e = \dfrac{0}{1-A} = 0$, $A \neq 1$

This group is closed; if A$\in \Re$, B$\in \Re$, then $(A + B - A \cdot B) \in \Re$.
The identity is 0.
The inverse is $\dfrac{A}{A-1}$ where $A \neq 1$
Associaitivity stands ; $(A\Diamond B)\Diamond C = A\Diamond(B\Diamond C) = A + B + C - AB - BC - CA + ABC$

<u>*Teacher's comments:*</u> Note that $A \neq 1$. Since A = 1 does not have inverse, $A\Diamond B = A + B - A \cdot B$ is not a group.

It is important to discuss why it is important to note that $A \neq 1$. A teacher needs to show (See Teacher Demonstration 1) what would happen if they allow A=1.

[Teacher Demonstration 1]
Solve it.
$1\Diamond x = 7$
Then, 1 + x – x = 7
1 = 7, which is not possible.

Solve it.
$2\Diamond x = 7$
2 + x – 2x = 7
-x = 5
x = -5

The students find it more interesting to see that using 1 in an equation gives an empty set as a solution. This demonstration is more valuable than dismiss the number 1 simply because $\dfrac{A}{A-1} = \phi$ if A = 1. Then it would be interesting to see some students modify their group like the following:

[Student Work Sample 3 *Modified*]
$A\Diamond B = A + B - A \cdot B$ where $A, B \in \Re - \{1\}$

$e = 0$; $A^{-1} = \dfrac{A}{A-1}$

Note that if A and B are not 1, then A + B – AB is not 1 either.

[Teacher Demonstration 2]

Solve it. $5 \Diamond x = 8$

(Solution A)

$5^{-1} \Diamond 5 \Diamond x = 5^{-1} \Diamond 8$; $5^{-1} = \dfrac{5}{5-1} = \dfrac{5}{4}$

$x = \dfrac{5}{4} \Diamond 8$

$x = \dfrac{5}{4} + 8 - \dfrac{5}{4} \cdot 8 = \dfrac{-3}{4}$

(Solution B)

5 + x – 5x = 8

-4x = 3

x = $-\dfrac{3}{4}$

The Solution B is quick and more concise. The students solve it by following the definition of the operation, which is an important habit to keep in learning mathematics. The Solution A is, however, more intriguing to the students because they witness how the inverse plays out in solving an equation. Then it is also important to ask students in the Solution A whether it would be OK to do $5^{-1} \Diamond 5 \Diamond x = 8 \Diamond 5^{-1}$ instead of $5^{-1} \Diamond 5 \Diamond x = 5^{-1} \Diamond 8$.

[Student Work Sample 4]

$5^{-1} \Diamond 5 \Diamond x = 8 \Diamond 5^{-1}$

$x = 8 \Diamond \dfrac{5}{4} = 8 + \dfrac{5}{4} - 8 \cdot \dfrac{5}{4} = -\dfrac{3}{4}$

Students then understand that why it is important to have the property of commutative. It is also valuable to mention that the number system we are using is a group of operations such as addition and multiplication that satisfies some basic rules (i.e., axioms). And it is a good idea to pose them a question such as what would happen if a number system doesn't obey commutative rules (see Student Work Sample 5]. Additionally, a teacher needs to note that if no identity is found, then consequently no inverse would exist either. (Ash 2006) If no inverse would exist, then there would be no way to solve an equation. (Mirman 1997) For example, for a new operation @ defined as A@B = 7(A – B). Then one figures out that there is no identity. Hence, no inverse. Then given 8@x = 2, no one would be able to solve this equation using the operation, @ because the inverse of 8 doesn't exist.

[Student Work Sample 5]

8@x = 2

7(8 – x) = 2

$$x = \frac{52}{7}$$

To solve, x@8 = 2
7(x – 8) = 2

$$x = \frac{58}{7}$$

To conclude, 8@x is not equivalent to x@8.

For more motivated students, the modular arithmetic is a good topic to be introduced after a math teacher helped the students build strong foundation in the previous topics. Zn is the algebraic structure of the set { 0, 1, 2, 3, ..n-1 } with the operations of addition and multiplication. For example, our clock system: Z_{12} has elements {0, 1, 2, 3, 4, ….11} which is equivalent to {12 o'clock, 1 o'clock, ….11 o'clock}. The following table gives the set of possible elements for each operation.

[Table 1; addition]

+	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**
0	0	1	2	3	4	5	6	7	8	9	10	11
1	1	2	3	4	5	6	7	8	9	10	11	0
2	2	3	4	5	6	7	8	9	10	11	0	1
3	3	4	5	6	7	8	9	10	11	0	1	2
4	4	5	6	7	8	9	10	11	0	1	2	3
5	5	6	7	8	9	10	11	0	1	2	3	4
6	6	7	8	9	10	11	0	1	2	3	4	5
7	7	8	9	10	11	0	1	2	3	4	5	6
8	8	9	10	11	0	1	2	3	4	5	6	7
9	9	10	11	0	1	2	3	4	5	6	7	8
10	10	11	0	1	2	3	4	5	6	7	8	9
11	11	0	1	2	3	4	5	6	7	8	9	10

[Table 2; multiplication]

X	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**
0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	**1**	2	3	4	5	6	7	8	9	10	11
2	0	2	4	6	8	10	0	2	4	6	8	10
3	0	3	6	9	0	3	6	9	0	3	6	9
4	0	4	8	0	4	8	0	4	8	0	4	8
5	0	5	10	3	8	**1**	6	11	4	9	2	7
6	0	6	0	6	0	6	0	6	0	6	0	6
7	0	7	2	9	4	11	6	**1**	8	3	10	5
8	0	8	4	0	8	4	0	8	4	0	8	4
9	0	9	6	3	0	9	6	3	0	9	6	3
10	0	10	8	6	4	2	0	10	8	6	4	2
11	0	11	10	9	8	7	6	5	4	3	2	**1**

Well prepared by the previous topics, most students can find that Z_{12} an additive group with the identity, 0 and the inverses after they verified the axioms. Likewise, the students would try to verify Z_{12} is a multiplicative group. But soon they found out that only the elements 1, 5, 7 and 11 have their inverses.

[Student Work Sample 6]

Question (A): Is Z_{12} a group?

Solution: Z_{12} an additive group, but Z_{12} is not a multiplicative group.

Question (B): Solve x + 8 = 1 ($x \in Z_{12}$) using the table 1 of addition.

Solution: $8^{-1} = 4$

x = 1 + 4

x = 5

Question (C): Solve 5x = 4.

Solution: $5^{-1} = 5$

x = 5*4

x = 8

The Student Work Sample 7 shows the students' investigation on Z_5.

[Student Work Sample 7]

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

Closure OK
Associativity OK
Identity = 0
Inverses OK

X	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

Closure OK
Associativity OK
Identity = 1
Inverses OK!
(Note: 0^{-1} doesn't exist, but the requirement is every non-zero element has an inverse.)

Summary: Z_5 is an additive group; Z_5 is a multiplicative group; Z_5 is a ring; Z_5 is also a field.

The following is student generated question and solutions written by them.

[Student Work Sample 8]

Question (A)
Solve it. $4x \equiv 2 \pmod 5$
$4^{-1} \equiv 4$
$4^{-1}\, 4\, x \equiv 4^{-1}\, 2$
$x \equiv 3 \pmod 5$

Quesition (B)
Solve it. $3\,(4x + 1) \equiv 2 \pmod 5$
$3^{-1}\,3(4x + 1) \equiv 3^{-1}\,2$
$4x + 1 \equiv 4$
$4x + 1 +$ (add.inverse of 1) $\equiv 4 +$ (add.inverse of 1)
$4x \equiv 3$
$4^{-1}\,4\,x \equiv 4^{-1}\,3$
$x \equiv 2 \pmod 5$

Conclusion

These additional activities would enhance the students understanding of inverses and appreciation of its crucial role in establishing a number system. If a math teacher only teaches that the opposite of 4 is -4, the reciprocal of 3 is $\frac{1}{3}$, and the inverse of $f(x) = x + 1$ is $f^{-1}(x) = x - 1$, undoubtedly a student would struggle with conceptualizing inverses in more advanced mathematics such as modular functions. There is no doubt that the students would get lost and have weak understanding because there are so many notions of inverses. This in fact calls for a need for a theory that unifies all these notions. It is one of the reasons why people want to introduce the concept of group, ring and the field. The abstract math concepts are very powerful, it could be applied to different concrete cases although it pays the price of being "abstract." It is the beauty of mathematics.

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Magic Card Maths

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Starting lessons with a good story, captures students imagination. The story then leads nicely into a magic card experiment which can be used to help teach the topic. I have used magic with a variety of age groups and found that they ask different questions of the task, and search for different levels of answers.

John Scarne (1903-1985)

John Scarne, who was one of the world's foremost gambling authorities, was not a gambler but has published some of the best and most comprehensive information on gambling and cheating during his reign. His most well known titles include Scarne on Dice, Scarne's Guide to Modern Poker and Scarne's New Complete Guide to Gambling. He was an outstanding and highly skilled magician who specialized in card magic and sleight-of-hand.

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Scarne had a sharp mathematical mind, but never finished high school. His education came from observing card sharps in local carnival grounds and from a novelty shop owner who made imperfect dice and dubious decks of playing cards.

As well as advising casinos about security Scarne was the technical advisor and doubled for Paul Newman in the classic movie "The Sting" in 1973. In the movie most of the sleight-of-hand demonstrated by Newman actually featured Scarne's hands. He also appeared in the short film "Dark Magic" as well as several commercials.

Scarne's most famous card trick was appropriately titled "Scarne's Aces". The trick involved taking a spectators shuffled deck of cards, performing a series of shuffles himself and then cutting to all four aces.

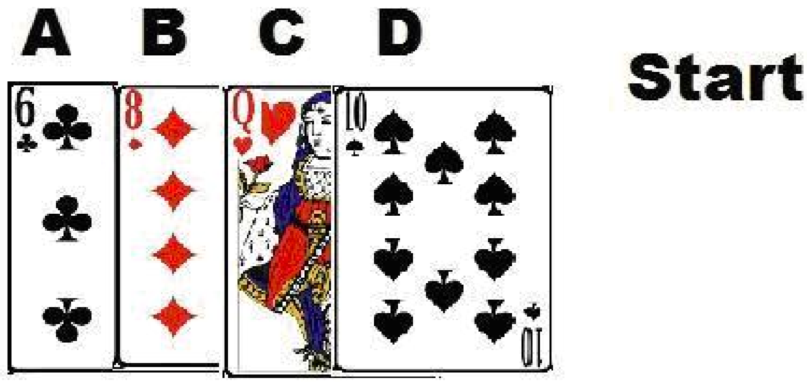
He knew the importance of the long history of magic in relation to his own life and is quoted to have said "All of my adventures and exploits . . . will, of course, be forgotten soon enough," He wrote this at the conclusion of his autobiography The Odds Against Me. "Gamblers and magicians come and go but their tricks stay forever"

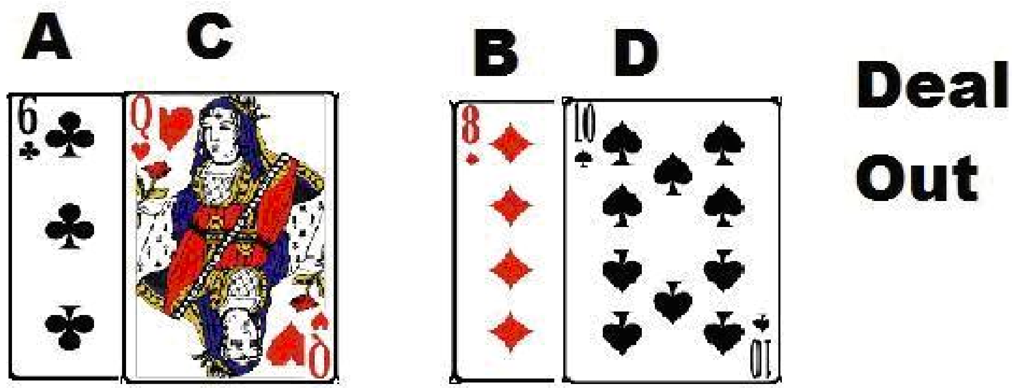
Several techniques are used to shuffle a deck of cards. While some achieve a better randomization, others are difficult to learn and handle or are better suited for special decks of cards. A procedure called Slide Shuffle is where small groups of cards are removed from the top or bottom of a deck and are replaced on the opposite side. Another common shuffling technique is called a Riffle, in which half of the deck is held in each hand with the thumbs inward, then cards are released by the thumbs so that they fall to the table intertwined. The list of various shuffle types goes on with Hindu, Pile, Chemmy, Mongean to name but a few. See the further references at the end of the chapter for more information.

Lets look at one particular shuffle called the Faro Shuffle or sometimes called the Perfect Shuffle. According to John Scarne this is the most popular shuffle favoured by magicians for its strange properties.

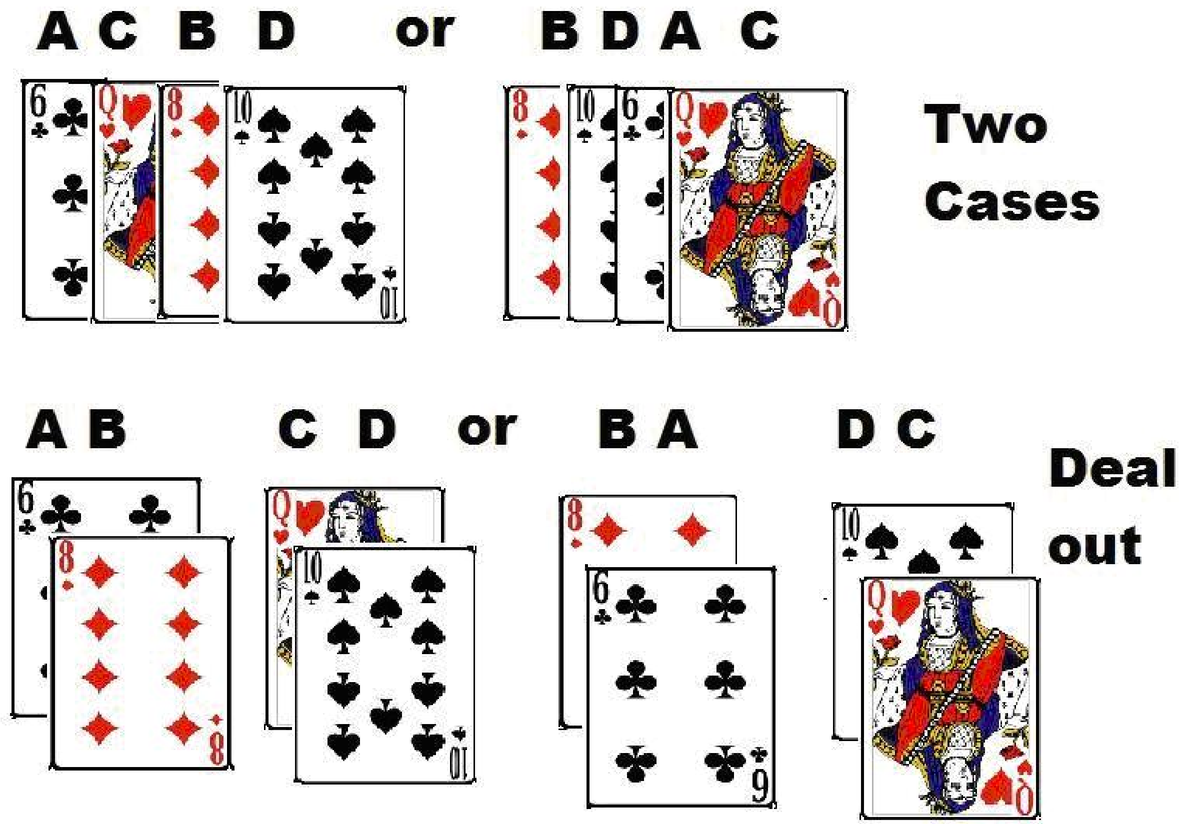
For a Perfect Shuffle the deck of cards is cut precisely in half and interleaved alternately with one card from each half. If you take a normal deck of 52 cards and perform a Perfect Shuffle 8 times you will find that the deck is returned to its original order. You can use the maths behind the Perfect Shuffle to perform the following card trick. It is best to do this trick using 8 or 16 cards, but to explain how it works I will only use 4 cards.

Without telling you, a spectator thinks of a number from 1 to 4, or 1 to 8 if using eight cards, and then counts from card A to that card. Make a mental note of card A, then turns the cards face down and deal alternate cards into two piles. The spectator is then asked which pile their card is in and these cards are put underneath the other two cards.

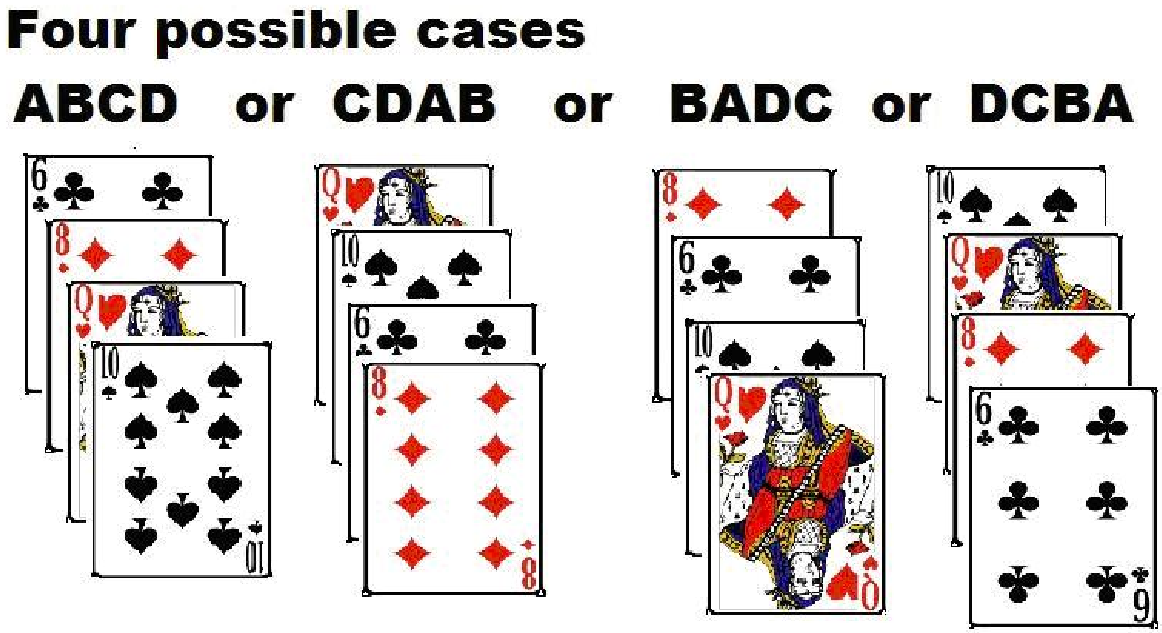




Turn the 4 cards face down and deal alternately into two piles. Repeat the process of asking which pile the spectator's card is in, then put these cards underneath the other two cards and turn them face down.



Their chosen card is revealed when you turn over the top card. No matter which card they have picked this card will always be at the top. Here are the four possible cases showing each card at the top.



If you now count to card A (6 of clubs in this example), which is the card you made a mental note of, you will find that this is their original number 1 to 4. Maths or magic?
To find the spectators chosen card when you do this trick with 8 cards, you will need to do 3 "deal outs" (2^3), with 16 cards 4 "deal outs" (2^4), etc…

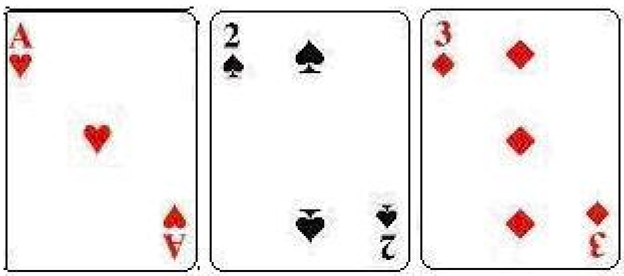
To add extra magic to the trick, place the value of card A in an envelope before you start. This will leave the audience stunned, as you have not only found their card and their number, but predicted the value of the card at their number within the deck.
The reason for the number of "deal outs" can be seen by looking at all the possible combinations needed to obtain each card at the top of a pile. From the illustration with four cards you can see you get
ABCD or BADC and CDAB or DCBA

With eight cards this becomes ABCDEFGH or DCBAHGFE or BADCFEHG and so on generating all eight possible variations of each card in position one and the A card in all possible places.

The maths of the *Perfect Shuffle* is used in everyday life, such as with our home PCs in dynamic computer memories to help improve the storage of data and the interconnections of networks. If you want to find out more about this then see further reading at the end of the chapter.

The Perfect Shuffle card trick relies on the mathematics of arrangements. The next shuffle trick relies on a set of transformations on a small group of cards which result in it returning to its initial form. In mathematics we call this type of transformation invariant, and a number of famous magic tricks use this idea. Bob Hummer was one of the first innovators in this area in the 1940s, and here is one of his card tricks.

Humble



Place any Ace, two and three face up in a row on a table. Turn away and invite the spectator to choose one of the cards and turn it over. They then turn over the other two cards having first switched their positions. You then pick up the cards so that from top to bottom you have the card to your right, the middle card, and the card to your left. Shuffle the cards by cutting the cards to the bottom, one or two at a time. To add to the effect shuffle two cards or one randomly, but keep a secret count and stop once you have moved 4 or 7 or 10 cards, or any addition of three on from 10.
Once you have stopped shuffling deal out, face down, the top card to the middle, the second card to your right and the third to your left. In your head imagine these cards are 3, 2 and Ace from left to right. A reversal of the initial set up.
The spectator guesses which card they think is theirs and turns it over, giving no hint as to whether they have guessed right. You either congratulate them on picking their card or say that's not your card and turn over the correct one.

This tricks works via these transformations:

Starting with 1 2 3
the spectator picks a card and swaps the other two. You do not know which card is which so lets call them

A B C

You pick up the cards in the reverse order

C B A

Shuffle once, four times, seven times etc and this will always have this effect

B A C

Put the cards back on the table

C B A

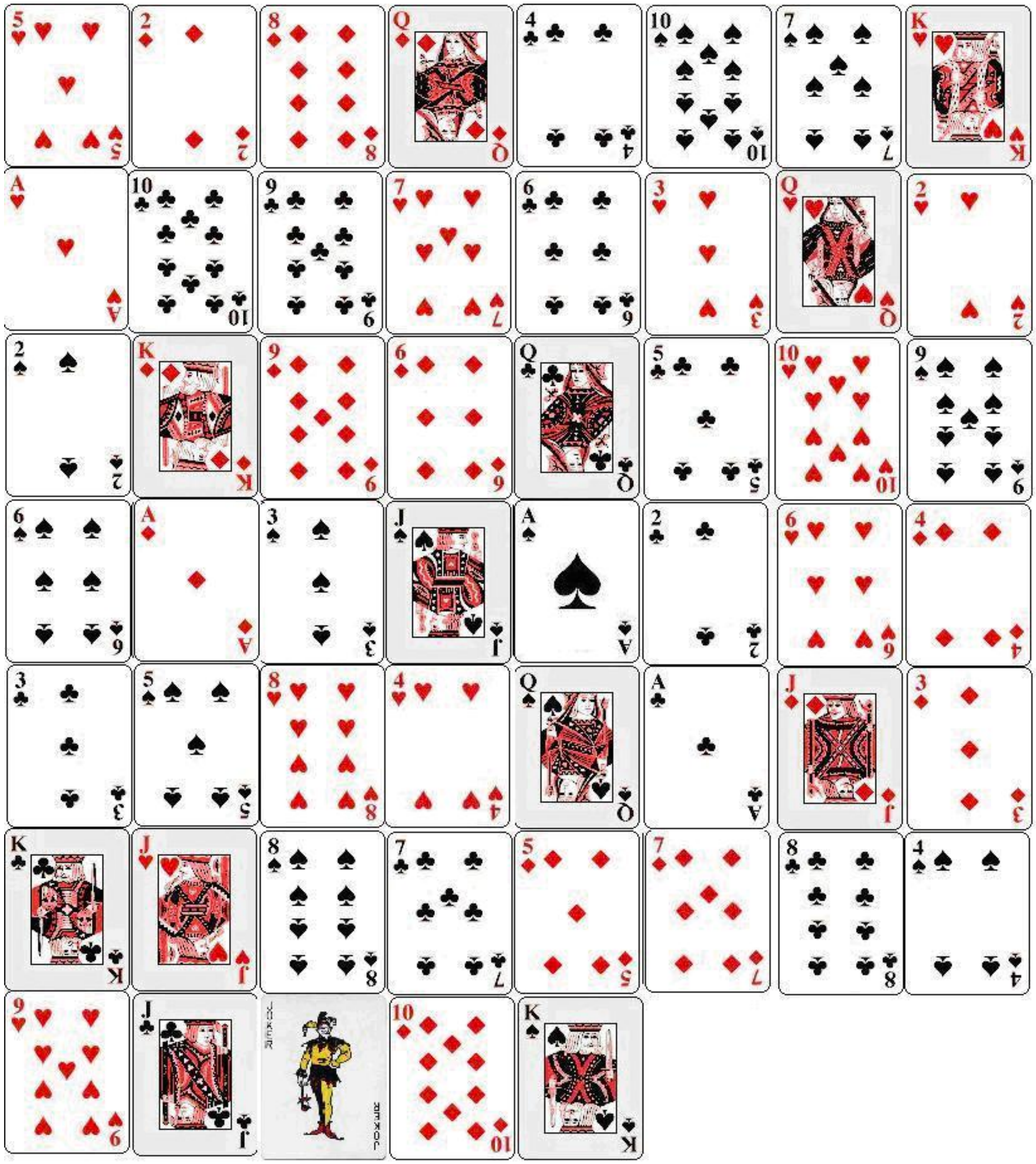
3 2 1

If the spectator turns over A, B or C and it matches 1, 2 or 3 then this is their card.
If the spectator turns over A, B or C and it does not match then that cards valve tells you what to do:

If C was an Ace then this means the correct card cannot be A, so it must be B.
If C was a 2 then this means the correct card cannot be B, so it must be A.
If B was an Ace then this means the correct card cannot be A, so it must be C.
If B was a 3 then this means the correct card cannot be C, so it must be A.
If A was a 2 then this means the correct card cannot be B, so it must be C.
If A was a 3 then this means the correct card cannot be C, so it must be B.

Bob Hummer's prediction trick has little to do with reading the mind of the spectator or seeing the future, but as with all good magic it can show us how we can be confused by what we see. The next card trick has its roots in Probability, and wants us to believe that apparently unrelated chains of events can be doomed to link into sync after a while. Synchronicity is a word coined by the psychiatrist C. G. Jung in his 1973 paper "Synchronicity: An Acausal Connecting Principle". Jung argued that meaningful coincidences occur far more frequently than chance allows for. A great deal of work has now been done in this area. One factor which effects coincidence is the way we experience it. It has been found that the way a story is told can change the degree of surprise. Surprise ratings increase if the same story was told as a potential future event, as opposed to things which have just happened.
So with this in mind, how strange would it be if you and a friend started at two unrelated random points and ended up at the same place? What if you and seven friends started at eight random points and all ended up at the same place? Synchronicity, magic or maths?

Here is a variation on a card trick called Kruskal's Count invented by Martin Kruskal. Ask a spectator to shuffle a deck of cards thoroughly, and then pick a number from 1 to 8. Deal out all the 52 cards, face up, with eight in each of the first 6 rows and four in the last.



The spectator then starts at the number they have chosen in advance on the top row. So if in the example above they had picked the number 5 in this case it would be the four of clubs. They then count to the next card using the value of the card, e.g count 4 cards along to the two of hearts. In this example they would then count 2 along to the three of hearts and 3 along to the nine of clubs and so on until they can go no further. Counting any court card they land on as one.
In the above example they will eventually end up at the Joker on the bottom row. No matter which card they start with on the top row they will always end up at the Joker.

Synchronicity, the fact that you and seven friends started at eight random points and all ended up at the same place or is it maths?

This magic trick of randomly dealing out the pack and then picking random starting points will (nearly) always result in one final destination. Once you find this final destination card then mark it with the Joker and ask other spectators to try starting at different points on the top row.
The mathematics for this amazing trick has to do with the fact that the various routes you take to get to the end, cross each other along the way. When these routes have the same cards in their pattern they become one route. As they cross a number of times before getting to the bottom row, the chance is high that they will obtain the same cards. The more cards in the pattern the greater the chance that the routes will join.
Probability can show how high this chance is. With eight cards in the top row, you have a 1 in 8 chance of someone picking the same card as you. The average position of this card in the top row is (1+2+3+….+8)/8=4.5

The average length of each subsequent jump you make, moving from one card to the next is

(1+2+3+….+10+1+1+1)/13=4.46

The number of expected cards which will be chosen after the first card is
(52-4.5)/4.46=10.65, which is about 10. The chance you will land on the same card is approximately 1 /4.46

Therefore the chance of missing all the cards is $$\left(\frac{7}{8}\right)\left(1-\frac{1}{4.46}\right)^{10} \approx 0.07$$

This means that you have around 7 chances in 100 that the trick will not work for all eight cards on the top row, but 93 times out of 100 that it will.

Make your own magic
In this section we will look at the art of making your own card tricks using the mathematics of algebra. Once you have learnt this method you will find a number of other card tricks that follow this pattern.

Take an ordinary pack of cards. Select nine Hearts numbered one to nine and nine Spades numbered one to nine. Ask a spectator to pick one Heart and one Spade, without you seeing which they have chosen. Then ask them to do the following, without disclosing any of the answers:
Take the Heart card, double its value, add one and then multiply the result by five. Add on the total value of the Spade card and ask them to tell you the final answer.
For example if the spectator picked the four of Hearts and seven of Spades, he would have doubled the four to get eight, then added one and multiplied by five to get 45. Finally by adding seven, a total of 52 is reached.

When you are told the final answer, in your head, always subtract 5. The number you have left will show you the two card values. In this case 52 take 5 is 47. So you can give the answer of four of Hearts and seven of Spades.

Looking at the Algebra shows how this trick works and also lets you make your own personal tricks. Lets say the red Heart card is R and the black Spade card is B.

Double gives	$2R$
Add 1	$2R + 1$
Multiply by 5	$5(2R + 1)$
Add Black Card	$5(2R + 1) + B$
Multiply out the brackets	$10R + 5 + B$

It can be seen that once 5 is taken from this you are left with $10R + B$
The unit digit is the black card and the tens digit is the red card.

Using this idea you can make your own personal card tricks, remembering to end up with tens and units.

For example $2(5R+3) + B$
Five times the red card, add 3 and then multiply the result by 2. Finally add on the total value of the black card.
You would have to take 6 off in your head to find the answer.

Or with a harder trick you could involve division, $3(20R + 18)/6 + B$
Twenty times the red card, add eighteen, multiply by 3 and then divide by six. Finally add on the total value of the black card.
You would have to take 9 off in your head

Why not try and make you own tricks following this idea?

Quick Magic
Here is a simple self working card trick using four Aces in memory of the great John Scarne.

Ask a spectator to pick a number from 14 to 25. Then count out that number of cards from the deck into a pile. Now pick up this pile of cards and deal out four hands as you would if playing a game of cards.
Reveal in whatever way you think best, the amazing chance that four Aces are on the top of each of these piles. What is the chance of that?

The only preparation required for this trick is before you start is to put the four Aces on the top of the deck.

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Chopping Logs: A Look at the History and Uses of Logarithms

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Abstract:
Logarithms are an integral part of many forms of technology, and their history and development help to see their importance and relevance. This paper surveys the origins of logarithms and their usefulness both in ancient and modern times.

Keywords: Computation; Logarithms; History of Logarithms; History of Mathematics; The number "e"; Napier logarithms

1. Background

Logarithms have been a part of mathematics for several centuries, but the concept of a logarithm has changed notably over the years. The origins of logarithms date back to the year 1614, with John Napier[2]. Born near Edinburgh, Scotland, Napier was an avid mathematician who was known for his contributions to spherical geometry, and for designing a mechanical calculator (Smith, 2000). In addition, Napier was first to make use of (and popularize) the decimal point as a means to separate the whole from the fractional part in a number. Napier was also very much interested in astronomy and made many calculations with his observations and research. The calculations he carried out were lengthy and many times involved trigonometric functions (RM, 2007). After many years of slowly building up the concept, he finally developed the invention for which he is most known: logarithms (Smith, 2000).

In his book (published in 1614) *Mirifici Logarithmorum Canonis Descriptio* (Description of the wonderful canon of logarithms), Napier explained why there was a need for logarithms,

> Seeing there is nothing…that is so troublesome to Mathematicall practise, nor that doth more modest and hinder Calculators, than the Multiplications, Divisions, square and cubical Extractions of great numbers, which besides the tedious expense of time, are for the most part subject to many errors, I began therefore to consider in my minde by what certaine and ready Art I might remove those hindrances. (Smith, 2000)

During Napier's time, many astronomical calculations required raw multiplication and division of very large numbers. Sixteenth-century astronomers often used prosthaphaeresis, a method of obtaining products by using trigonometric identities like $\sin\alpha \cdot \sin\beta = \frac{1}{2}[\cos(\alpha - \beta) - \cos(\alpha + \beta)]$ and other similar ones that required simple addition and subtraction (Katz, 2004). For example, if one

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[2] The word "logarithm" was coined by Napier from the Greek "logos" (ratio) and "arithmos" (number) [(9)].

were to multiply 2994 by 3562, then sinα would be 0.2994 (the decimal is placed so that the value of α can be used later) and sinβ would be 0.3562—these would make $\alpha \approx 17.42$ and $\beta \approx 20.87$ (values obtainable in a table). Next, α and β values would be inserted into the equation, again a table would be used, and simple subtraction and division by 2 would occur—the result would yield ~0.10665158. By moving the decimal the same number of times that it was moved in order to accommodate the trigonometric equation (eight places to the right), the answer becomes 10,665,158 (approximating the actual 10,664,628). Because this answer is an estimate, the desired number of accurate digits would be dependent on the values initially given to α and β.
Performing such calculation tricks, astronomers could reduce errors and save time (Katz, 2004).

In addition to prosthaphaeresis, Napier also knew about other methods for simplifying calculations. Michael Stifel, a German mathematician, developed in 1544 a relationship between arithmetic sequences of integers and corresponding geometric sequences of 2 raised to those integers (Smith, 2000): $\{1, 2, 3, 4,\ldots, n\}$ and $\{2^1, 2^2, 2^3, 2^4,\ldots, 2^n\}$. Stifel wrote tables in which he showed that the multiplication of terms in one table correlated with addition in the other (Katz, 2004). For example, to find $2^3 \cdot 2^5$, one would add 3+5 (terms in the arithmetic sequence) and the answer could then be inserted back into the geometric sequence to obtain $2^3 \cdot 2^5 = 2^{3+5} = 2^8 = 256$. These tables were limited, however, in their calculating ability; Napier's approach to using logarithms, on the other hand, allowed the multiplication of any numbers through the use of addition (Katz, 2004).

To define logarithms, Napier used a concept that is rather different from today's perception of a logarithm.[2] Since astronomers at the time often handled calculations requiring trigonometric functions (particularly sines), Napier's goal was to make a table in which the multiplication of sines could be done by addition instead (Katz, 2004). The process consisted of having a line segment and a ray, where a point was made to move on each (from one extreme end to the other). The starting "velocity" for both points was the same, but the difference began as one point moved uniformly (arithmetically on the ray) and the other moved geometrically such that its velocity would be proportional to the distance left to travel to the endpoint of the line segment. Using this mental model, Napier defined the distance traveled by the arithmetically moving point as the logarithm of the distance remaining to be traveled by the point moving geometrically (Cajori, 1893). In Napier's words, "the logarithm of a given sine is that number which has increased arithmetically with the same velocity throughout as that with which radius began to decrease geometrically, and in the same time as radius has decreased to the given [number]" (Katz, 2004). A detailed account of the process can be seen below in the *Calculation Techniques* section. Clearly, Napier's definition differed from the modern concept of just having a base raised to the corresponding exponent.

It took Napier about 20 years to actually assemble his table of logarithms (Katz, 2004), but shortly after publishing his book, Napier was visited by the English mathematician Henry Briggs (Smith, 2000). A professor of geometry in London, Briggs was impressed with Napier's work,

> My lord, I have undertaken this long journey purposely to see your person, and to know by what engine of wit or ingenuity you came first to thing of this most excellent help in astronomy, viz. the logarithms; but, my lord, being by you found out, I wonder nobody found it out before, when now known it is so easy. (Cajori, 1893)

They both discussed the convenience of setting the logarithm of 1 equal to 0 (rather than the original 10,000,000) and setting the logarithm of 10 at 1. In this way, the more familiar form of the logarithm was born, and a common property like $\log(xy) = \log x + \log y$ could be used to make a new table. Napier died in 1617, so Briggs began to do the calculations to construct the table (Katz, 2004). Briggs did not convert Napier's logarithms to the new common logarithms, however. Instead

he set out to calculate successive square roots to obtain the logarithms of prime numbers, and used these to calculate the logarithms of all natural numbers from 1 to 20,000 and from 90,000 to 100,000. Although he did use algorithms to obtain the roots, the amount of work needed to calculate all those logarithms is nonetheless astounding. To calculate the logarithm of 2, for instance, he carried out forty-seven successive square roots (Smith, 2000). In addition, all of the calculations for logs were carried out to 14 decimal places (Cajori, 1893). An example of the calculations needed for this task is shown below in the *Calculation Techniques* section. Finally, in 1624, Briggs published his tables in his *Arithmetica Logarithmica*. The logarithms of the numbers between 20,000 and 90,000 were calculated by the Dutchman Adrian Vlacq, who published the complete table from 1 to 100,000 in 1628 (Cajori, 1893).

Below is a page from Briggs's Arithmetica Logarithmica (MatematikSider, 2007).

10 D ARITHMETICA E

	Numeri continue Medij inter Denarium & Unitatem.	Logarithmi Rationales.
	10	1,000
1	31622,77660,16837,93319,98893,54	0,50
2	17782,79410,03892,28011,97304,13	0,25
3	13335,21432,16332,40256,65389,308	0,125
4	11547,81984,68945,81796,61918,213	0,0625
5	10746,07828,32131,74972,13817,6538	0,03125
6	10366,32928,43769,79972,90627,3131	0,01562,5
7	10181,51721,71818,18414,73723,8144	0,00781,25
8	10090,35044,84144,74377,59005,1391	0,00390,625
9	10045,07364,25446,25156,64670,6113	0,00195,3125
10	10022,51148,29291,29154,65611,7367	0,00097,65625
11	10011,24941,39987,98758,85395,51805	0,00048,82812,5
12	10005,62312,60220,86366,18495,91839	0,00024,41406,25
13	10002,81116,78778,01323,99249,64325	0,00012,20703,125
14	10001,40548,51694,72581,62767,32715	0,00006,10351,5625
15	10000,70271,78941,14355,38811,70845	0,00003,05175,78125
16	10000,35135,27746,18565,08581,37077	0,00001,52587,89062,5
17	10000,17567,48442,26738,33846,78274	0,00000,76293,94531,25
18	10000,08783,70363,46121,46574,07431	0,00000,38146,97265,625
19	10000,04392,84217,31672,36281,88083	0,00000,19073,48632,8125
20	10000,02195,91867,55542,03317,07719	0,00000,09536,74316,40625
21	10000,01097,95873,50204,09754,72940	0,00000,04768,37158,20312,5
22	10000,00548,97921,68211,14626,60250,4	0,00000,02384,18579,10156,25
23	10000,00274,48957,07382,95091,25449,9	0,00000,01192,09289,55078,125
24	10000,00137,24477,59510,83282,69572,5	0,00000,00596,04644,77539,0625
25	10000,00068,62238,56210,25737,18748,2	0,00000,00298,02322,38769,53125
26	10000,00034,31119,22218,83912,75020,8	0,00000,00149,01161,19384,76562,5
27	10000,00017,15559,59537,84719,93879,1	0,00000,00074,50580,59692,38281,25
28	10000,00008,57779,79451,03051,17588,8	0,00000,00037,25290,29846,19140,625
29	10000,00004,28889,89633,54198,42901,3	0,00000,00018,62645,14923,09570,3125
30	10000,00002,14444,94793,77767,42970,4	0,00000,00009,31322,57461,54785,15625
31	10000,00001,07222,47391,14050,76926,8	0,00000,00004,65661,28730,77392,57812,5
32	10000,00000,53611,23594,13317,14831,4	0,00000,00002,32830,64365,38696,28906,25
33	10000,00000,26805,61846,70731,51508,7	0,00000,00001,16415,32182,69348,14453,125
34	10000,00000,13402,80923,26383,99277,7	0,00000,00000,58207,66091,34674,07226,5625
35	10000,00000,06701,40461,60946,55519,6	0,00000,00000,29103,83045,67337,03613,28125
36	10000,00000,03350,70230,79911,91730,0	0,00000,00000,14551,91522,83668,51806,64062,5
37	10000,00000,01675,35115,39815,61857,6	0,00000,00000,07275,95761,41834,25903,32031,25
38	10000,00000,00837,67557,69872,72426,9	0,00000,00000,03637,97880,70917,12951,66015,625
39	10000,00000,00418,83778,84927,59087,9	0,00000,00000,01818,98940,35458,56475,83007,8125
40	10000,00000,00209,41889,42461,60262,5	0,00000,00000,00909,49470,17729,28237,91503,90625
41	10000,00000,00104,70944,71230,25311,0	0,00000,00000,00454,74735,08864,64118,95751,95312
42	10000,00000,00052,35472,35614,98950,4	0,00000,00000,00227,37367,54432,32059,47875,97656
43	10000,00000,00026,17736,17807,46048,9	0,00000,00000,00113,68683,77216,16029,73937,98828
44	10000,00000,00013,08868,08903,72167,8	0,00000,00000,00056,84341,88608,08014,86968,99414
45	10000,00000,00006,54434,04451,85869,75	0,00000,00000,00028,42170,94304,04007,43484,49707
46	10000,00000,00003,27217,02225,92881,337	0,00000,00000,00014,21085,47152,02003,71742,24853
47	10000,00000,00001,63608,51112,96427,283	0,00000,00000,00007,10542,73576,01001,85871,12426
48	10000,00000,00000,81804,25556,48210,295	0,00000,00000,00003,55271,36788,00500,92935,56213
49	10000,00000,00000,40902,12778,24104,311	0,00000,00000,00001,77635,68394,00250,46467,78106
50	10000,00000,00000,20451,06389,12051,946	0,00000,00000,00000,88817,84197,00125,23233,89053
51	10000,00000,00000,10225,53194,56025,921 L	0,00000,00000,00000,44408,92098,50062,61616,94526
52	10000,00000,00000,05112,76597,28012,947 M	0,00000,00000,00000,22204,46049,25031,30808,47263
53	10000,00000,00000,02556,38298,64006,470 N	0,00000,00000,00000,11102,23024,62515,65404,23631
54	10000,00000,00000,01278,19149,32003,235 P	0,00000,00000,00000,05551,11512,31257,82702,11815

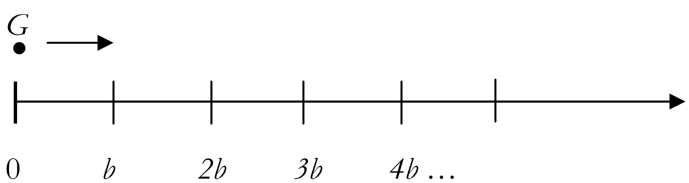
The way logarithms were viewed changed over time, and today's notation for a logarithm was developed by Leonhard Euler in the late 1700s. He related exponential and logarithmic functions by defining $\log_x y = z$ to hold true when $x^z = y$ (Smith, 2000). This definition proved very useful and found multiple applications. A classic example of a practical application of logarithms is the slide rule. In 1622 the Englishman William Oughtred made a slide rule by placing two sliding logarithmic scales next to each other. The slide rule could replace the need to look up values in a logarithm table by instead requiring values to be aligned in order to perform the multiplication, division, and many other operations (depending on the model). Up until the 1970s, with the incoming of electronic calculators, the slide rule was widely used in the fields of science and engineering (Stoll, 2006). A look at how the slide rule could be used for calculations is shown below in the *Calculation Techniques* section.

Although the common logarithm has many practical uses, another logarithm is widely used in fields ranging from calculus to biology. The natural logarithm is of the form $\log_e a = n$. The base of a logarithm could be any number larger than 1, but the use of *e* brings on various advantages (Lowan, 2002). The definition of *e*, the limit of $(1+1/n)^n$ as n approaches infinity, might seem a bit awkward at first, but it turns out that *e* not only turns up frequently in nature, but it also makes natural logarithms have the simplest derivatives of all logarithmic systems (Evans, 1939). Various solutions to applied mathematical problems can be expressed as powers of *e*: the flow of electricity through a circuit, radioactive decay, bacterial growth, etc. (Lowan, 2002). The natural logarithm arose from modifications of Napier's logarithms made by John Speidell, a mathematics teacher from England. In 1622 he published the book *New Logarith* with logarithms of tangents, sines, and secants in a format that showed natural logarithms (except that he had omitted decimal points). For example, he gave log 10= 2302584, which would be written today as $\log_e 10 = 2.302584$ (Cajori, 1893). As an interesting note, the Napier log of x would be equivalent to the expression $10^7 \log_{1/e}(x/10^7)$ in modern terms (Smith, 2000).

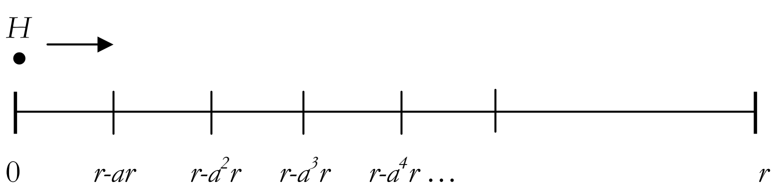
2. Calculation Techniques

- *Napier's definition of logarithms* (see Cairns, 1928, and Cajori, 1893, and Katz, 2004, and Pierce,1977)*:*

  Given a ray and a line segment, the point *G* moves along the ray and the point *H* moves along the line segment.



  G moves at a constant velocity by traveling *b* distance in equal time intervals (along an increasing arithmetic sequence).



  H moves towards *r* in equal time intervals from 0 to *r-ar*, *r-ar* to $r\text{-}a^2r$, $r\text{-}a^2r$ to $r\text{-}a^3r$, etc. Napier made *r* = 10,000,000 and *a* be less than 1 (but very close to 1).

  He made the line segment (from 0 to *r*) be the "sine of 90º", and the distance from *r* to *H* the sine of the arc with the distance traveled by *G* as its logarithm. Thus, Napier had $\log 10^7 = 0$.

  Under this system, the notion of using bases with corresponding exponents did not apply.

In a calculus sense, Napier's logarithms could be seen as measures of "instantaneous" velocities. For example, the velocity of H could be $V_H = \Delta d/\Delta t = d(r\text{-}x)/dt$, where x is the distance remaining to be traveled by H to reach r. Similarly, the velocity of G would be $V_G = dy/dt$, where y is the distance traveled by G (this velocity is constant).

To obtain the definition of a Napier logarithm in modern calculus terms:
$d(r\text{-}x)/dt = x$, since the velocity of H is proportional to the distance remaining to be traveled by H to reach r. So, $dr/dt - dx/dt = x$, and since r is a constant (10^7):
$0 - dx/dt = x \rightarrow 1/(-dx/dt) = 1/x \rightarrow -dt/dx = 1/x \rightarrow \int -dt = \int 1/x\, dx \rightarrow -t = \ln x + c$.
Since both G and H start at the same velocity, when $t = 0$ then $x = r$, thus
$0 = \ln r + c \rightarrow c = -\ln r$, therefore, $-t = \ln x - \ln r$.

Point G progresses in an arithmetical fashion, and its velocity is dy/dt. Having established that its velocity is constant and that it is equal with H's velocity at $t = 0$, then $dy/dt = r$ so $dy = rdt \rightarrow \int dy = \int r\, dt \rightarrow y = rt$.

Finally, to relate x and y:
$-t = \ln x - \ln r \rightarrow t = \ln r - \ln x \rightarrow t = \ln (r/x) \rightarrow y = r \ln (r/x)$

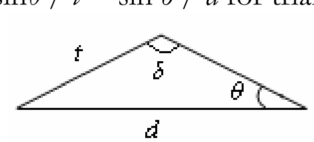
By his definition, the Napier log $x = y$ is Naplog $x = r \ln(r/x) = 10^7 \ln(10^7/x)$

Napier did not use the notion of e in calculating his logarithms, but this perspective helps to see the connection between logarithms, calculus, and the usefulness of e and the natural logarithm.

- Using Napier's Logs in calculations (see Katz, 2004):

To use his logs in calculations, Napier had to note that Naplog $10^7 = 0$.
If $j/p = w/z$, then Naplog (j) – Naplog (p) = Naplog (w) – Naplog (z).
If $f/q = q/m$, then Naplog (f) – Naplog (q) = Naplog (q) – Naplog (m) and
2Naplog (q) = Naplog (f) + Naplog (m)
And if $f/q = m/k$, then Naplog (f) + Naplog (k) = Naplog (q) + Naplog (m).

Using these properties he established, conforming to his logarithms, a triangle could be solved by reference to his tables.
Example: using the law of sines, $\sin\theta \,/\, t = \sin \delta \,/\, d$ for triangle



So to find δ the properties are applied,
Naplog ($\sin\delta$) = Naplog ($\sin\theta$) + Naplog (t) – Naplog (d)
Referring back to his tables, Napier could calculate δ by simple addition and subtraction.

- Briggs's logarithms (see Cairns, 1928 and Henderson, 1930):

Briggs adapted Napier's logs to fit log 10 = 1 instead, thus giving birth to today's common logarithms. By taking successive square roots, Briggs concluded, for example, that if

$\sqrt{10} \approx 3.162277$, then $\log 3.162277 = 0.5$
$\sqrt{\sqrt{10}} \approx 1.77828$, then $\log 1.77828 = 0.25$
$\sqrt{\sqrt{\sqrt{10}}} \approx 1.33352$, then $\log 1.33352 = 0.125$, etc.

To find the logarithms of prime numbers Briggs used the following method:
To find log 2, he noticed that if he raised 2 to a certain power, the number of digits in the result gave an approximation for log 2 (because of the properties of using logarithms with base 10); the log of a number with x number of digits is between $x - 1$ and x. For example, $2^8 = 256 \rightarrow 2 < \log 256 < 3$.
He then noted that x and $x - 1$ could be divided by the exponent to which 2 was raised to get an approximation of the log of 2:

$2^{10} = 1024$	$\rightarrow 3 < \log 1024 < 4$	so $0.3 < \log 2 < 0.4$
$2^{20} = 1048576$	$\rightarrow 6 < \log 1048576 < 7$	so $0.3 < \log 2 < 0.35$
$2^{40} \approx 1.1 \times 10^{12}$	$\rightarrow 12 < \log 2^{40} < 13$	so $0.3 < \log 2 < 0.325$
$2^{60} \approx 1.2 \times 10^{18}$	$\rightarrow 18 < \log 2^{60} < 19$	so $0.3 < \log 2 < 0.3167$
$2^{80} \approx 1.2 \times 10^{24}$	$\rightarrow 24 < \log 2^{80} < 25$	so $0.3 < \log 2 < 0.3125$
$2^{100} \approx 1.3 \times 10^{30}$	$\rightarrow 30 < \log 2^{100} < 31$	so $0.3 < \log 2 < 0.31$

…and so forth until Briggs obtained log 2 to 14 decimal places. Once he calculated the logs for other prime numbers, he followed the rules of logarithms: for example, $\log 10 = \log (2 \cdot 5) = \log 2 + \log 5$. Until his tables covered the logarithms of 1-20,000 and 90,000-100,000.

- Slide rule calculations (see Stoll, 2006):

The slide rule works by simplifying multiplications and divisions into logarithmic scale additions or subtractions. Slide rules basically print fit scales into a ruler-type setup and by just sliding a cursor against another scale, long operations can be done quickly. One could get away with using a slide rule without really understanding logarithms, but to make one, the following rules are essential:

$\log xy = \log x + \log y$ $\qquad$ $\log (x/y) = \log x - \log y$
$\log x^y = y \log x$ $\qquad$ etc.…

Briggs's logarithms allowed long operations like $10478 \cdot 97503$ to become
$\log 10478 + \log 97503 = 4.020278 + 4.989018 = 9.009296$, then
antilog $9.009296 = 10478(97503) \approx 1{,}021{,}636{,}000$.

- Natural logarithms:

Logarithms with base e unavoidably spring up in calculus (which was developed a little after Napier's death). To see how these logs are essential to obtain certain integrals: let $f(x) = \log_e x = \ln x$

$$f'(x) = \lim_{h\to 0} [f(x+h) - f(x)] \,/\, h = \lim_{h\to 0}[\ln(x+h) - \ln(x)] \,/\, h = \lim_{h\to 0}[\ln((x+h)/x)] \,/\, h$$

$$= \lim_{h\to 0} [\ln(1+ h/x) \,/\, h\,] \,[(1/x)/(1/x)] = \lim_{h\to 0} \quad [1/x][\ln(1+ h/x)] \,/\, [h/x]$$

$$= \lim_{h\to 0} [1/x][x/h][\ln(1+ h/x)] = \lim_{h\to 0} \quad [1/x][\ln(1+ h/x)^{x/h}] = [1/x]\lim_{h\to 0} \quad \ln (1+ h/x)^{x/h}$$

$$\lim_{x/h\to \infty}$$

$= [1/x] \qquad \ln(1+ h/x)^{x/h} = [1/x] \ln e = 1/x$

Thus, $d(\ln x)/dx = 1/x$, and $\int 1/x\, dx = \ln x + c$.

The definition of e, $\lim_{n \to \infty}(1+ 1/n)^n = e$, allows the above demonstration to hold.

3. Conclusions & Implications

Today's concept of logarithms might make it seem strange that logarithms really developed out of comparing velocities of arithmetically and geometrically moving points. Napier's idea took him decades to fully develop and conclude, and the work of Briggs helped simplify and enhance a useful mathematical invention. What today seems like a simple base to exponent relationship really has a long history of work and improvements. The natural logarithm further helps us see the connection between the labors of a Scottish mathematician (and many others) with calculus and all its modern applications in math, science, and technology.

Napier's invention of the logarithm has surely left an important mark in the history of mathematics. The applications derived from the calculations he and others developed, still have relevance today. Although slide rules are now obsolete, the principles that allow them to work are not. The story of the development of logarithms is a good example of the effects that mathematical discoveries and inventions can have on society and the technological world.

In writing this paper I have learned a great deal about these calculation aids. But perhaps more importantly, I have realized that figuring out mathematical operations and tricks certainly takes significant amounts of effort, time, and devotion. Today, we often take for granted those symbols and explanations that are neatly compiled into math and science textbooks. It is easy to forget that every equation encases a story: frustration, fascination, arduous work, friendly collaborations, disappointment, and the occasional serendipity. Mathematics is not just about numbers, but it is also about the people whose work gives us the luxury and pleasure of understanding.

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Early Insurance Mechanisms and Their Mathematical Foundations[1]

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Abstract:
This article gives a historical survey of early insurance mathematics and its development in relation to the maritime industry and the origins of the life insurance industry. The history of the probability and statistics and its intricate connection to the actuarial sciences and modern insurance can be traced to the time period described in this paper

Keywords: Actuarial sciences; Anders Hald (1913-2007); Insurance mathematics; Life insurance; Logarithms; risk management; history of probability; history of mathematics

Background

Popular history tells the story of how insurance began at a coffeehouse in London called Lloyds, where merchant shipmen sat around the table drinking coffee and discussing their future voyages. One merchant explained that he was afraid that he would lose everything if his next ship was lost at sea, and another merchant offered to share in his risk – for a fee. Around the table they went, dividing up the risk of loss and the profits of the voyage, and giving birth to insurance. Or, so the story goes. In truth, insurance existed in many early forms long before that conversation could have taken place at Lloyds coffeehouse. This article considers the factual basis for the history of early insurance mechanisms, provides a survey of their general forms, and discusses their foundations in mathematics. Particular emphasis will be placed on the insurance mechanisms developed and popularized during the 13th through the 17th centuries.

In today's world insurance is a part of everyday life. Modern insurance relies heavily on sophisticated mathematics. Greene, in his 1961 article highlights some applications of mathematics to the insurance field. His examples include utility analysis, game theory, statistical decision making, and actuarial science. In actuarial science in particular, the ties between insurance and advanced mathematics are very tight.

This paper was born out of a curiosity to know whether the strong ties of mathematics and insurance trace back to the industry's origins. The study was separated into two parts to consider the earliest examples of Life Insurance and Property Casualty Insurance. Both areas are deserving of analysis, although the depth of previous study on each varies greatly.

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[1] **In memoriam Anders Hald (3 July, 1913 – 11 November 2007).**

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Life Insurance

The earliest ancestors of life insurance as it exists today are annuities, which have existed since ancient times. Annuities are life incomes given or sold by one party to another party called the annuitant. An annuitant would be paid a certain sum yearly until his death, but as some men lived longer than others the value of an annuity could vary greatly. The earliest documentation of attempts to valuate annuities can be traced to Ancient Rome. Rules were developed in 40 B.C. for the handling of annuities in the event of the death of the testator, or grantor of the annuity. In order to settle the estate and disburse the inheritance among heirs, the expected value of outstanding annuities needed to be ascertained. Hald (1990) provides translation of tables developed by Roman jurist D. Ulpian for the calculation of the expected or maximum value of the annuity based on the annuitant's age. These tables present our earliest known records of the consideration and calculation of life expectancies. While it is unlikely that they are mathematically sophisticated or represent relevant estimations, they are important as the foundation of an idea that would later become the basis of life insurance mathematics.

Annuities and "bequests of maintenances" (Hald 1990) continued as a general practice throughout history. However, no documented references are available to indicate further attempts to quantify the value of an annuity until the plague years. It was during that time that mathematicians laid their hands on mortality records as a data source through which to explore the emerging field of probability theory.

The Black Death is the largest known outbreak of plague, but plague was fairly common in Europe throughout the 1300s-1700s. It was fear of plague, and action of the Church of England that can be credited with the collection and compilation of mortality data in a manner that allowed mathematicians to analyze the data. In the city of London in the 1530s the church instituted measures to record the number of weddings, deaths and christening that took place within the various parishes, including the categorization of deaths by cause. This data was used to tell where plague was occurring, and give warning as to where it might reach epidemic threat. Beginning in 1604 (Hald, 1990) the Company of Parish Clerks began publishing and selling these statistics weekly as bills of mortality. The demand for these bills is truly a testament to the impact plague had on society.

John Graunt (1620-1674), a London city politician and businessman was the first person to realize the potential of the data provided in the bills of mortality. In 1662 he published his first edition of *Natural and Political Observations Made Upon the Bills of Mortality*. This publication is credited as being the first statistical analysis of demographic data. Graunt's genius must be understood in the context of his environment; as the son of a tradesman, he received an ordinary education, certainly nothing that would foreshadow his mathematical achievement. Following his father in trade he developed a keen mind for the simple mathematics of running a business. From such simple mathematics he devised the advanced idea of statistical analysis. The significance of his contribution and the high regard with which it was viewed is evidenced by his eventually membership in The Royal Society.

Graunt also fundamentally contributed to the development of statistics and probability by his thorough critical analysis of the validity of his data. His *Observations* clearly highlighted flaws he saw in the bills of mortality, and showed the assumptions and methods he applied to mitigate those flaws. Among the most significant weaknesses Graunt saw in his data was the lack of total

population numbers. Graunt held that it was likely that the population was unstable due to movement of people into and out of the city as a result of plague scares.

Hald, in both his 1990 book and 1987 article, gives an excellent detailed account of how Gaunt devised his various observations, including his work to estimate the life expectancy of the London population. It is with this particular part of Graunt's Observations that we concern our discussion of the development of life insurance mathematics. For centuries to follow, mathematicians and politicians would refer to Graunt's tables and comment on their implications. Among the great mathematicians to take up Graunt's work were the Bernoulli family, the brothers Huygens, and fellow Royal Society member Edmond Halley.

Graunt's invention was immediately put to use by his friend Sir William Petty, a fellow politician. Petty led a varied and educated life, and explored several intellectual subjects. He is best known for his work on statistics and political economics, and is credited with being the first to coin the term "political arithmetik." An innovator, Petty was also a founding member of the Royal Society in 1662, and his work and Graunt's are credited with having influenced the establishment of statistical offices.

Independent of Graunt, mathematician Christiaan Huygens had taken up the study of probability theory in the Netherlands. A talented scholar, Huygens spent much of his life engaged in research in mathematics and physics as well as other areas of scientific enquiry. In 1656 he published his first work on probability, the title of which translates to *On Games of Chance*. In the following years he continued his exploration of probability theory and the work of fellow mathematicians Pascal and Fermat. In 1657 he published his work *De Ratiociniis in Ludo Aleae*, a text on the theory of probability.

Living as he did so far from London, Huygen's first knowledge of Graunt's invention came from his friend Sir Robert Moray in 1662. He does not appear to have concerned himself overly with the development until 1667, when his brother Ludwig wrote to him to initiate discussion on the usefulness of Graunt's tables to the application of annuities. The correspondence of the brothers is well documented among the twenty-two volumes of Huygens collected writings, the *Oeuvres Completes* (Hald 1990). Despite Ludwig's interest in life annuities, the brothers' correspondence centered on the broader probabilistic interpretations of Graunt's tables. Huygen's major contribution to the field of insurance mathematics came in the interpretation of Graunt's tables as a continuous distribution, and is credited as being the first to produce a graph of statistical data as a continuous distribution function. This step would lead to developments in curve fitting of statistical data to generate fully predictive functions.

Jan DeWitt, a contemporary of Christian Huygens who studied under the same mathematics professor, made additional contributions to the development of early life insurance mathematics. DeWitt was well educated in mathematics and law, and became the prime minister of the Dutch Republic in 1653 at the tender age of twenty-eight. It was a combination of both his education and political knowledge that made his contributions the life insurance mathematics possible. As a political leader in times of war DeWitt understood that governments need to raise capital to finance their armed forces. The sale of annuities was a popular method employed by governments for that purpose. In 1671, based on the work of Graunt and Huygens, he wrote the *Value of Life Annuities in Proportion to Redeemable Annuities*. Rather than a paper of solely mathematical interest, this was an application of mathematics to a particular problem facing the Dutch Republic. The government

sorely needed to raise funds, and DeWitt sought to assure that annuities would be priced in a mathematically accurate manner.

DeWitt's probabilistic analysis of the distribution of the number of deaths divided the data into four segments, beginning at age three which would be the youngest age at which an annuity could be purchased. He further considered that annuities were paid twice annually, and for the purposes of his analysis regarded that a man was no more likely to die in the first half year than the second. Through his mathematical computations, which were reviewed and approved by fellow Dutch statistician Jan Hudde, DeWitt calculated that previous annuities had been sold at an undue discount of two years' purchase.

Under DeWitt's leadership, the Dutch Republic became the first country to offer annuities based on the annuitants age. However, despite his valuations, the annuities were offered at significantly less than his indications. As a political economist, DeWitt was acutely aware that an annuity offered at a price unpalatable to the market would be of no value to the government. Despite its remarkable contribution to the field of insurance mathematics, DeWitt's work is not believed to have been widely distributed outside the political environment of the Dutch Republic. This lack of distribution may be a result of his fall from power and execution by a mob a year later (1672) after the French invaded the Dutch Republic.

The next great mathematician to take up the topic of valuation of life annuities was Edmond Halley, better known for his work on astronomy and in defense of Newton. In something of a departure from his traditional fields of inquiry, Halley undertook an analysis of the valuation of annuities at the behest of the Royal Society which had just received new mortality data. This new mortality data, the first since the London Bills used by Graunt, had been compiled in the city of Breslau. Halley took up the challenge and in1693 presented his paper *An Estimate of the Degrees of the Mortality of Mankind, drawn from curious Tables of the Births and Funerals at the City of Breslaw; with an Attempt to ascertain the Price of Annuities upon Lives*.

Halley utilized Graunt's work extensively in establishing the distribution of mortality of persons aged six and younger. Further, he created a table of the number of deaths at reported ages and, like DeWitt before him, divided the data into four age intervals. For each age interval he established a minimum and a maximum number of deaths to occur per year. Like Graunt, Halley was highly analytical and critical of the validity of his data. Halley addressed the questionable areas of the data through comparison to other sources of comparable data. This practice is heavily utilized by actuaries today as a means of shoring up the statistical significance of an individual insurer's proprietary data. Hald (1990) provides clear reproductions of Halley's tables. Through manipulation of the data, Halley developed the distribution of deaths as a piecewise function calculated on age intervals.

Members of the famed Bernoulli family of mathematicians were also engaged in the study of probability. James (or Jacob) Bernoulli had already demonstrated his mathematical prowess through his work on infinite series and the new calculus of Liebnitz. Turning to probability, he began work on his masterpiece *Artis Conjecturi,* which would not be published during his life. He made intervening publications on the topic, including one in 1686 in which he refers to Graunt's tables. Nicholas Bernoulli, James' nephew and student, worked closely with his uncle on the study of probability theory and produced results similar to those generated by Huygens which he published

in his thesis on probability and law, *De Usu Artis Conjecturi in Jure* (The Art of Conjecture in Law) in 1711.

Three decades later the topic would be further developed by a pair of controversial mathematicians. Thomas Simpson was a self-taught mathematician, sometime teacher, and author of texts on mathematical concepts. Unfortunately, it is this latter occupation that got Simpson in trouble. Simpson's first genius was not in the development of new mathematical ideas, but in his ability to clarify and simplify the work of those before him. He published eleven text books, but was criticized for having grossly plagiarized most of them. One example is his book *The Nature and Laws of Chance. The Whole After a new, general and conspicuous Manner, And illustrated with A great Variety of Examples*, published in 1740 of which Hald (1990) states "There is, however, nothing 'new, general and conspicuous' in Simpson's book; it is simply plagiarism of the mathematical parts of the Doctrine," (p 414) in reference to Abraham de Moivre's groundbreaking Doctrine of Chances, originally published in 1718.

De Moivre was, understandably, outraged at Simpson's appropriation of his work In 1742 when Simpson published a text book on life insurance mathematics, de Moivre was convinced that Simpson had once again copied his own work, from his *Annuities upon Lives: or, The Valuations of Annuities upon any Number of Lives; as also of Reversions. To which is added, An Appendix concerning the Expectation of Life, and Probabilities of Survivorship*, published in 1725. While it is generally held that Simpson's work on life insurance mathematics was indeed founded upon the work of de Moivre, it is in this field that Simpson truly found his own unique genius and made original contributions.

Both de Moivre and Simpson delved into great detail of the intricate valuation methodologies of variations on annuities including reversions at marriage and succession, and the valuation of annuities upon multiple lives. These topics had been left unresolved by Halley, and it was de Moivre who is credited with having identified that Halley's distribution could be linearly approximated as a piecewise function. This presented to de Moivre a means of approximating the more complicated annuities which he had been otherwise unable to estimate. De Moivre advocated the use of linear approximation to calculate multi-life annuities.

Simpson had an advantage on de Moivre when he published his *The Doctrine of Annuities and Reversions, Deduced from General and Evident Principles: With Useful Tables, Shewing the Values of Single and Joint Lives, etc. at different Rates of Interest* in 1742. De Moivre had developed his calculation on Halley's data on the mortality rates of Breslau, and had not had access to any new mortality data. Simpson, on the other hand, had data that became available after DeMoivre's treatise, the newly published London Bills of Mortality. He fundamentally disagreed with de Moivre's linear approximation method and instead advocated the use of tables for deriving the value of annuities on multiple lives. His tables were far more advanced than any previously produced, and it is with Simpson's tables that the insurance industry sided. For hundreds of years to follow, mortality tables would be produced in similar form.

These advances in the understanding of the financial mathematics of life insurance helped fuel the generation of private life insurers. While previously annuities had only been offered for sale by governments, now the burgeoning capitalism of Europe created an environment ripe for privatization. This shift was not without troubles. Many insurance companies formed in the early 1700s failed, to the financial detriment of their annuitants. Insurance companies came and went for the next several centuries, and were held in great suspicion by the populace. So much so, in fact that

in 1843 author Charles Dickens created a fictitious and villainous life insurance company, the Anglo-Bengalee Disinterested Loan and Life Assurance Company in his serialized novel *The Life and Adventures of Martin Chuzzlewit* (Hickman, 2004). Dickens is not the only literati to turn his pen to topics of insurance. A century later, great English playwright George Bernard Shaw would write an essay entitled *The Vice of Gambling and the Virtue of Insurance*. Though not, by his own estimation, a talented mathematician, Shaw proved to have an excellent grasp of the vagaries of insurance, and its mathematical precepts.

The privatization of insurance and general suspicion of insurance companies, lead to the creation of the actuarial profession, one of many new professions that were born during the Victorian Era. In his *History of Actuarial Profession*, James Hickman gives an excellent account of the specifics of the requirements of the new profession. By Hickman's account, the actuarial profession was created to serve a public purpose - the reasonable protection of purchasers of life insurance products. Actuaries were changed with assuring adequacy of assets for the fulfillment of life insurer obligations. While the actuarial profession's origins trace to life insurance, the role of actuaries has expanded to included property and casualty insurance. Actuaries now work publicly and privately, both at the side of the insurance companies, and with the governments that regulate insurance.

Marine Insurance

Like life insurance, the precursors to modern property insurance can be traced back to ancient times. The concept of spreading risk and providing for the assurance of monetary aid in the event of loss are common to many cultures. The first written records of what could be considered early insurance law are found in the Code of Hammurabi (circa 1760 B.C.) as promulgated by Hammurabi, the sixth King of Babylonia. Guilds also played a significant role in the evolution of modern property insurance. Membership in a guild brought protection; it also meant mandatory participation in the funding of this protection. Early caravan traders are also purported to have developed schemes of protection of property that was at risk to over-land travel.

These early ancestors of insurance as we know it today demonstrate the concept of risk sharing or risk distribution, one of the fundamental concepts underlying modern insurance. In many ways, the agreements among guilds and traders for the protection of assets under threat of risk during transport bear simplistic similarity to such modern insurance forms as Risk Retention Groups. In both cases, the risk is transferred and distributed among parties of interest who all participate in the same business (affinity groups) and each participant in the agreement not only agrees to pay losses incurred by another, but also contributes their own risk to the pool.

The risks faced by early overland traders were minimal compared to the risks faced by merchants of seafaring trade in the 13th and 14th centuries. While land traders were exposed to risk of loss from theft and spoilage, seafaring traders faced all the same risks, plus the increased risk of piracy and general sea risk. These risks required more sophisticated insurance mechanisms. It is in these early years of ocean trade that we find the first examples of property insurance as we consider it in modern times. In early marine insurance mechanisms we see some signs of the types of risk-sharing exhibited by earlier insurance ideas. The changing nature of the trade soon resulted in the emergence of true risk transfer, or the insurer as unaffiliated third party.

The exact origin of marine insurance is a subject of some disagreement among scholars. Most properly, the evolution of the industry should be credited to medieval Italy, and its ports of trade. The early Italian insurance mechanisms are thought to have found their roots in the contracting of loans, specifically the sea loan.

Another form of loan that is sometimes considered to be an ancestor of marine insurance is the loan of bottomry. A loan of bottomry was an agreement between the owner of a ship and the ship's master, in the event that the two parties were different. A loan of monies was made by the ships owner to pay for costs of repair to the ship and its components in its destination port. The reason loans of bottomry are considered to be a form of early insurance is that the agreement would generally stipulate that the amounts were only repayable upon the safe transit of the ship and its cargo. However, if the party loaning the money (the ship owner) is also the party that would stand the risk of financial detriment in the event of loss of the ship there is no true risk transfer, without which it does not meet the standard of a true scheme of insurance.

The sea loan, on the other hand, exhibits risk transfer and is therefore a monre proper ancestor of marine insurance. These loans faced significant challenge from the canonical prohibition on usury. To try to skirt the prohibition, loans were falsely recorded as having been made *gratis et amore* (without interest). In fact, the amount repaid would be in excess of what was actual received by the borrower, the additional payment constituting both interest on the loan and payment for the risk taken by the lender. Such practices are believed to have been common for many financial transactions during the prohibition on usury. Regardless of the misleading records, the sea loan was condemned as usurious in 1236 by Pope Gregory IX (De Roover, 1945). De Roover (1945) indentifies the *cambium nauticum* and the *foenus nauticum* as two early forms of insurance loans practiced in medieval Italy.

The *foenus nauticum* was a loan made by a lender to the seafaring merchant which was repayable only if the ship completed its journey. Merchants often used the borrowed money to purchase cargo, and their financial success was therefore dependant on their ability to sell the cargo at a price greater than the repayment. Under such loans, the lender may be regarded as the actually owner of the cargo (by virtue of its use as collateral), which brings into question whether or not it truly constitutes risk transfer. Perhaps due to fear of condemnation of usury, such loans were not terribly popular, and were replaced in practice by *cambium nauticum*.

The *cambium nauticum* was likely a more successful instrument for avoiding accusations of usury. *Cambium*, in general, were contracts of exchange made through the sale of bills of exchange. Money was advanced in one currency which would be repaid by the borrower in the currency of a different port. In the case of cambium nauticum, this sum was only payable if the ship reached its foreign port safely. Critically speaking, such agreements simultaneously reduced a merchants risk while exposing him to the additional exchange risk. While protected in the event of loss of cargo or ship, the merchant risked loss from a change in the value of the terminal port's currency from that assumed upon the issuance of the bill of exchange. Scholars believe that lenders selected high estimations of the port currency's value to ensure that exchange risk was one-sided and born exclusively by the merchant.

The 13th century saw the advent of partnerships among two or more cooperating merchants. These partnerships can be identified as the roots of the modern multi-national corporation, and like their modern descendents required the division of duties due to geographic factors. One party, generally

a person possessing greater capital and familial ties, acted as the "headquarters" of the partnership; he would not travel aboard the ship, but would remain on land and manage the relationship of the partnership with suppliers and creditors. The other partner would act as the managing partner abroad, accompanying their ship and cargo, and negotiating contracts of sale and resupply at the destination port.

Demand for insurance increases when the party to be insured possesses additional assets that would be exposed to the risk of a financial loss. Within the new partnerships, the loss of a ship at sea (and potentially one of the partners) could expose the sedentary merchant's personal assets to being seized by creditors. The sedentary partner most likely also possessed greater financial decision-making authority within the partnership. The development of partnerships created a condition of increased demand for marine insurance.

Despite their obvious drawbacks, *cambium nauticum* persisted as the most common form of marine insurance until they were replaced by standard *cambium* contracts in the 1300s. Perhaps the lenders became unwilling to accept the risk of loss at sea. A standard *cambium* was repayable regardless of the possible loss of the ship or cargo. Without the previously available form of insurance, a new market was required to help merchants mitigate their sea risk. This vacuum of coverage, coupled with the increasing demand, provided the right conditions for the development of the first true contracts of insurance.

A great body of research has been conducted by legal scholars on the earliest known true contracts of insurance. This research has provided no common opinion as to the exact date of the first contract. Because the modern contract of insurance evolved from various risk-sharing instruments such as sea loans, it is difficult to pinpoint when exactly insurance as a separate financial contract truly developed. Scholars generally agree that in order to be considered insurance in the modern context, the contract must contain risk transfer, risk distribution, and the provision of premiums. While the early loan-based mechanisms provided for transfer of risk to the lender, they failed to meet the threshold of risk distribution or payment of premium and are therefore not considered true contracts of insurance from a legal perspective. De Roover (1945) attributes the earliest contract of premium insurance to certain texts, records of account, and statutes originating in the period of 1318-1320. Holdsworth (1917) credits the first contract as one existing in the archives of Genoa dating from 1347

Within the history of the contract of marine insurance we also find reference to the origins of the concept of insurable interest. Holdsworth (1917) discusses the common abuse of insurance as a vehicle for gambling during the 15th century. Parties otherwise un-related to the business of a ship could use an insurance policy to wager on whether or not a ship would be lost at sea. This clearly violates the intent of the contract of insurance to provide for indemnity upon a loss. We find in the actions of the legislature at Genoa in the late 15th century, and the statutes of Barcelona in 1484 the first attempt to create in law the provision that contracts of insurance are only binding when there exists an insurable interest. These laws became the foundation of insurance law, and provisions for insurable interest are still found today in both insurance regulation and insurance contract language.

While the histories of marine insurance provide ample discussion on the origins of early insurance mechanisms, they do not address their foundation in mathematics. Very little research is available to indicate the particulars of how premiums were calculated prior to the creation of Lloyd's of London. Straus (1938), describes Lloyds as "the most famous single name in all the world." Lloyds' history is

well documented, from its beginnings as a London coffee house where merchants met to discuss business, to its emergence as the world's largest private insurer. The history of Lloyds also tells of the role of the insurance underwriter in the establishment of insurance premiums.

The term underwriter was first coined to describe the individual who agreed to insure a risk. He agreed to the contract by writing his name *under* the terms of the contract, thus the name underwriter. It was the underwriter who determined the value of the potential loss, and the corresponding premium to charge for insuring it. In the early days of marine insurance the risks to be insured were very large and possessed unique characteristics such as trade route, crew experience, and cargo. This made the job of pricing marine insurance particularly varied and subjective to the individual determinations of the underwriter.

Unfortunately, though exhaustive, the histories of Lloyds fail to document the mathematics used by early underwriters. The likelihood that underwriters employed formal, consistent mathematical models to evaluate risk seems unlikely. In the absence of any evidence to the contrary, it is easy to speculate that premium determination in early marine insurance was far more like gambling than any type of actuarial science as known today. Such speculation is born up by Lloyds's early organizational ties to wealthy British gentlemen for whom gambling was reported to be a common pastime.

Early marine underwriting of a small number of individually large risks contrasts with the majority of modern insurance rating which involves risks that tend to be larger in number, smaller in relative value, and generally homogenous within each line or type of insurance. Further, in modern insurance, the process of determining insurance premiums is far more complicated than was encountered by early underwriters. In the U.S. and other countries, the process is heavily regulated. Insurance today would not be possible without the contribution of actuaries, insurance mathematicians who quantify the value of risk. Actuaries utilize a vast array of advanced mathematical tools including calculus, probability theory and statistics to model data and analyze probable value of losses an insurer might expect to pay on a risk. Underwriting of modern insurance involves considering the individual variations of a particular risk and selection of an actual premium based on the analysis of actuaries.

Observations

This brief survey of insurance mathematics highlights an interesting difference between the level of sophistication of early life insurance mathematics and that of early marine insurance. There are several probable explanations for this disparity. First, the respective eras that saw their development were dramatically different. The mathematics of the 13th and 14th centuries were far less advanced than the mathematics of the 17th and 18th centuries. The 17th and 18th centuries were a time of immense expansion of mathematical knowledge. From Napier's development of logarithms to Newton and Leibnitz and their work on the development of the Calculus, the 1600s were an exciting time for mathematics. It certainly makes sense that life insurance, which came into being during and after this scientific renaissance, would have more advanced mathematics attributed to it than earlier insurance mechanisms.

Further, the types of insurance considered in this paper are in and of themselves very different. As previously mentioned, early marine insurance involved individual risks that were generally large, but

restricted in number. Life insurance, even early annuities, on the other hand, involved a larger number of risks that were comparatively smaller in individual value. The development of life insurance mathematics also hinged significantly on the availability and value of mortality data. While records of ships lost at sea were also kept, their smaller number may have made the data less adaptable to probabilistic interpretation.

In this context we consider one more fundamental contribution to insurance mathematics, James Bernoulli's posthumously published *The Art of Conjecture*. Within this seminal work of probability theory, Bernoulli identifies and proves what has come to be known as the Law of Large Numbers, or Bernoulli's theorem. Bernoulli's theorem is of paramount importance to the practice of modern insurance mathematics. Through the collection, compilation, and analysis of large amounts of data, actuaries are able to develop far more accurate probabilistic interpretations of insurer's losses. This creates a more scientific basis for the industry of insurance, and further distances it from its gambling-like origins.

The impact of the quantity and relative size of the risk on the analysis of data warrants further research. Additional analysis of the evolution of property insurance, and an identification of when it became more scientific in application would be particularly valuable. Research into the origins of fire and personal property insurance might lend a greater understanding of this stage in property insurance's evolution, and create a more thorough understanding of similarities and differences between the mathematical origins of modern life and health versus property and casualty insurance. Further research into the evolution of the actuarial profession, and its transition from life insurance to all lines of insurance might also provide valuable insight into this topic. Such research however is outside the scope of this paper.

In the study of the development of Life Insurance Mathematics, we see an interesting phenomenon in the interplay of contributions made by practical versus theoretical mathematicians. There is little question that the development of the mathematics of annuities played a significant part in the greater picture of the study and development of proability theory and statistics. It is intriguing that while important contributions were made by famous figures from the mathematical world (the Bernoullis, de Moivre, Halley, et al.), contributions of at least equal import were made by practical men with little formal mathematical training. Examples of these practical mathematicians are John Graunt and Thomas Simpson. We recall that neither Graunt nor Simpson received formal mathematics training. Graunt appears to have developed his genius through practical inspirations and his knowledge of basic bookkeeping, while Simpson's knowledge was almost exclusively self-taught. Both men came from humble, working-class origins. Neither supported himself primarily through his mathematical endeavors (although Simpson would be able to do so later in life).

Conversely, we consider the giants of mathematical theory from the same era: Halley, Huygens, and the Bernoulli's. These men were true geniuses of academia, raised by families of generous means, and educated in the finest of classical and mathematical pedagogy. In Huygens case, we know that he lived for many years supported by his father while he dedicated his time to intellectual pursuit. Later, he would move to Paris and become a member of the Academie de Royales des Sciences where he was supported by Louis XIV. Halley was the son of a very wealthy London family. His father provided him not only with an extensive education, but also the resources and instruments he needed to isolate himself in his scientific observations.

The Bernoulli's are well known for their success in capitalizing on their mathematical developments. Acting as private tutors and professors, they were able to carve out a comfortable place for themselves entirely upon their intellectual pursuits. Nicholas Bernoulli, the nephew of James (Jacob) and John (Johann) would also turn to the study of law, but even there his pursuits were of a far more theoretical than practical bent. The 16th through the 18th centuries were not an era of intellectual popularism – advanced education of the type enjoyed by our giants of mathematical theory was reserved for young men of a certain degree of privilege. The humble nature of Simpson and Graunt's origins and education make their contributions to probability and statistics even more remarkable. These practical mathematicians were instrumental in the development of what is the practical mathematics of Probability and Statistics.

Probability and Statistics are only limited in their application to insurance by their supposition that past behavior is indicative and predictive of future behavior. In our ever changing soci-political and increasingly globalized society, this fundamental precept of probability is sometimes open to challenge. In the following centuries insurance mathematics will further evolve as application of game theory and other advanced mathematics are developed to continue the traditions detailed in this paper. The influence of mathematics on insurance is firmly established. Only time will tell if insurance will again provide the type of influence on the development of new mathematics that it did in the early days of life insurance mathematics.

Acknowledgements

This paper was inspired by my experience in the insurance industry and my academic forays into advanced mathematics, during which I became drawn to the history of how the advanced mathematical concepts were devised. My curiosity as to the origins of insurance mathematics began with references to the applicability of Napier's logarithms to finance mathematics and eventually led to my discovery of the work of Anders Hald, formerly Professor of Statistics at the University of Copenhagen in Denmark. My humble survey of life insurance mathematics pales in comparison to the depth of detail exhibited in Professor Hald's work. His book *A History of Probability and Statistics and Their Applications Before 1750* is a must read for anyone interested in developing a greater understanding of the technical and computational aspects of the mathematics referenced in this paper. Hald, in turn, makes acknowledgement to other authors whose work has made our understanding of this topic possible.

It is with great sadness that I note that my discovery of Professor Hald's work coincides to the month with his passing at the age of ninety-four. I dedicate this paper to his memory and the hope that future generations of mathematicians and insurance academics will know and honor his contributions to their fields.

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DIALOGUE[1]

Main Points of Objections to the "No Child Left Behind" (NCLB) Law

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[*Taken from testimony given in Congress by the AFT President in spring, 07*]

Adequate Yearly Progress (AYP):

--The guidelines fail to distinguish between successful schools and unsuccessful schools
--Between schools under-performing in just one area vs. schools likely to need a complete overhaul
--Does not give credit for student growth toward a high standard
--Need to establish AYP levels which make a distinction between struggling schools and those needing only limited assistance

Mike: I think that the most important of those points above is that credit for growth, no matter how incremental, is a rewardable goal, not just reward when reaching some plateau goal. As educators know, success in true learning is measured by positively sloped accomplishment, not attainment of some arbitrary goal.

Bettina: I would particularly like to address the third point as well as add to what Mike says above. I agree with this point. One of the purposes behind NCLB is for instance that all students should be able to read and do mathematics at or above grade level, closing for good the

[1] The opinions expressed in this dialogue are those of the two authors and do not represent the views of the Montana Council of Teachers of Mathematics nor the position of the journal or its editorial board.

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achievement gap between disadvantaged and minority students and their peers. This is a very admirable goal but I would argue that there are even more important than closing the gap. Creating a school where each child has the possibility to reach his or her full potential, below, at or (far) above the given grade level is a better goal. All children can learn, but they are not going to achieve the same, learn at the same pace, in the same way or at the same time. There is also a danger that watering down standards would in itself create a smaller gap. In relation to achieving below the grade level, there ought to be different grades of failing so that failing students might still experience a progress and so that it becomes clearer how much below the student is. For instance the European ECTS (European Credit Transfer System) grading scale has two failing grades: 'Fx', meaning 'some more work required' and 'F' meaning 'considerable further work required'. Also the gifted students need to be able to take tests that shows how much ahead they are.

Mike: In thinking about Bettina's response on this question, I was impressed with two points made: [1] There should be more than one purpose for NCLB, namely, that of closing the gap between stated English and math norms and disadvantaged/minority children. We should also be matching each child to his/her potential, whatever level of achievement that is. [2] The European (ECTS) grades distinguish "mild F's" and "hard F's".

Based upon these observations, I propose the following 2 proposals to the heads of state with NCLB. [1] NCLB should consider going from one goal, that of closing the gap between math/English standards and disadvantaged and minority children, to 2 goals. The second goal is to simultaneously challenge all students of all ability levels to achieve their academic potential, and assess this challenge in a quantitative way. [2] We should change the grading system from the current A, B, C, D, F to a two dimensional rubric, as pictured in the table below.

Achievement	**A**	**A-**	**B**	**B-**	**C**	**C-**	**D**	**D-**	**F**	**F-**
Effort										
1										
2										
3										

Effort assessment would be a quantitative way we could evaluate how much the student is living up to his/her potential. The F would correspond to the "mild F" of the Europeans with the F- corresponding to the "hard F". The number 1 represents good to excellent effort, striving to potential, 2 represents average effort, with room to increase the striving, and 3 represents insufficient effort.

Students would now get grades like the following: an A-2 in English composition, A-1 in English Literature, and B- -- 1 in Algebra II. Students would take tests (pre and post) to show their potential. This would show how far ahead the gifted are, as well as where every student is, in the academic "scale". The pre and post testing would indicate effort and school system effectiveness.

One size fits all:

--The law recommends one size fits all solutions for schools needing intense multiple interventions as well as those needing only limited help

--"Assistance" given is punitive and based upon ideology rather than being based on evidence

--Lawmakers should provide resources and flexibility so schools and districts can implement research-based interventions

--Also change the law so interventions are targeted to the students who are not proficient

Mike: I think that "one size fits all" is not a solution to the problem, but only an ineffective "fix". I agree that that approach should be purged from NCLB laws. I also agree that some specific (funded) provisions should be targeted for those students "not proficient" at various skills. However, providing resources/flexibility based on research is too monumental a task, and will only neutralize the NCLB movement, and should be left for another committee or time. I have no problem with assistance being given punitively or positively—my theory is that either approach will get targeted results. I do think some sort of flexibility should be given schools and districts, but that it should be simple and rudimentary to define/implement.

Bettina: I agree that a "one size fits all" is not a solution. Students and schools are very different with unique needs requiring different approaches. Individually tailored approaches are needed. A way to do this might be to allow states to use the educational funding for any educational purpose they wish. At the same time states, local communities, and schools should be free to implement whatever practice works best for their students while still receiving federal support. The state and local communities and schools should be accountable to the public and the parents for the academic achievement of the students. This would give less bureaucracy as well as free the initiative and inventiveness of the local teachers, principals, and parents in improving the learning of students who are not proficient as well as (hopefully) fostering and furthering talents in the students who are able to achieve above the grade level – even if the latter means an increasing achievement gap. As long as an increasing number of students perform at grade level, I do not see a problem in that at the same time, an increasing number of students perform (far) above grade level. A "gap" and a "gap" is not the same, and the USA needs highly education people

Mike: I especially like how Bettina described the benefits and overall value of local control, and "individualization" of student academic needs at the K-12 level. The writer is correct and eloquent on this. From a practical standpoint, however, I don't believe that the Federal funds will be poured into educating our kids in a consistent basis, unless they retain some control over the uses of these funds. I don't believe that tax payers will stand for "philanthropic financial offerings" from the federal govt. to local school systems without labeling these items as "special interest", or "pork". No elected federal official in these times wants to go through defending accusations of pork spending.

Therefore, I propose that the Feds. keep their stated goals for academic competency, as well as generate some for the gifted and high-end students. They will keep feeding sufficient federal funds to local school systems, as long as the locals are drifting toward those goals on a steady basis.

Now that I think about it, I ***prefer*** having the Feds. generate these goals—but not be so strict on enforcing them. I don't think they should cut funding because a school missed a goal in the federal time frame, but only if measurable progress is ***not*** displayed. I think local systems should be a bit more uniform in their learning outcomes, much more so than they are now. Note that I am ***not*** advocating only ***one*** set of goals nationwide, but rather proposing more consistency among local systems.

Finally, is it possible that school systems can be more easily accountable to the federal government and their demands than to the parents/public with their demands? I don't know. I just throw this question out for discussion.

Testing:

--State tests must be aligned to state standards and curricula used in the classroom
--Only 11 states have tests aligned with strong content standards
--Instruction time is being replaced by "drill and kill" test preparation
----the added focus on math and reading has resulted in lessening of other important subjects
--the current law uses tests primarily to sort students and rate schools
--more emphasis should be to focus on providing teachers with specific timely test results so they can improve instruction

Mike: I strongly agree that tests must be aligned to state standards, otherwise the teachers and students won't have any chance of attaining NCLB goals and will be a setup for certain failure. I believe that some class time (not all) should be spent on "drill and kill" work. What percentage should go to this skills acquiring is up for debate in journal research. Test results should be an important, though not exclusive, way to sort students and rate schools. Refer to the "one size fits all" points above for more input factors. The best suggestion is the last one about emphasizing improved instruction with timely test results (and other dynamic classroom assessments).

Bettina: I would like to make four points: First, it is common that the format of a test has a "backwash" effect on the teaching, i.e. "teaching to the test". This often means that if a test merely tests abilities to perform certain operations, this will usually be what the teacher will focus on in the classroom. One could certainly argue that this discourages creativity, curiosity, and thinking. However, it does not have to be this way – the individual teacher and school leadership can decide to also teach higher order thinking and creativity – even if these competencies are not directly being tested. I know several teachers and schools that do that. It is not an excuse to simply blame bad teaching on the tests. It is the responsibility of the teachers and principals to provide good teaching. Furthermore, it is the states that create both the state standards and the state tests, not NCLB. An approach might be to seek influence on the creation of the state test so that the tests not only test "drill" operations but also higher order thinking and creativity. I would argue that it would be useful for teachers

to be able perform formative assessment such as for instance diagnostic testing, which during the school year can help the teachers determine which concepts the students have developed properly as well as the ones they has not understood well enough.

Second, about "drill and kill", I agree with Mike above that some portion of "drill and kill" is necessary. Actually I am not fun of the term "drill and kill" since both experienced teachers and well respected and established theorists argue for the necessity of automatic and routine manipulations for problem solving activity since it is otherwise not possible to concentrate successfully on the difficulties. Naturally, in for instance mathematics, automatic manipulation does not form the whole of mathematics. However, it is a necessary, but not sufficient, part of teaching and learning mathematics.

Third, if subjects other than mathematics and reading, such as history and music, are not receiving the attention they ought to, it is definitely not acceptable.

Fourth, I also agree to a certain extent that state tests should be aligned to state standards, if the goal is to achieve the NCLB goals. But perhaps this is here one of the problems lie. This is a contradiction to the benefits from flexibility and local control.

Mike: I agree with Bettina about tests/assessments including higher order thinking and creativity as well as the "drill". Possibly, state standards and NCLB should both get on board with this dual assessment, and realize that **both** drill and creative/higher order skills are needed for mathematical literacy in today's citizens. Much friction could be avoided and smoother "teaching to the test" could be accomplished by the states and federal entities having more of a "meeting of the minds" regarding academic mathematical goals.

Possibly a third dimension could be added to my aforementioned rubric, which would assess the math task as heavily skill oriented, heavily creative, or heavily higher order thinking.

I apologize for "drill and kill" and agree that it is too negative a term. I should simply call it "drill".

A possible solution to the contradiction between NCLB and state standards mentioned in Bettina's "fourth point" could be for the two governments (state and federal) to come more together in educational goals. Our 21st century school systems need both state and federal dollars, so we must have both as masters.

Private Tutoring For Hire:

--Private tutoring firms are not cost efficient and are not held accountable for their results

--Money should be used to help students catch up, not make adults rich

Mike: I think that the second point, about making adults rich, is not called for an inappropriate in a discussion like this one. It resounds of unionism, cliques, and pettiness. I also believe that private tutoring firms must of necessity be cost efficient, or at least be effective and efficient, because they are under the survival of the fittest criteria which the public institutions are not

subject to. So, by the very nature of business competition, they must be relevant. In overview, I believe that the AFT president should have steered well clear of these 2 comments altogether.

Bettina: I think that the second comment is unserious, I agree completely with Mike.
Also I think that that these private firms in fact are held accountable. If they are not doing their job, parents, teachers and principals will soon discover this, and the firms will have to close down.

Mike: No comment on the comments!!

"Highly Qualified" Teachers:

--"Highly qualified teacher status" is unfairly tied to student test results based upon statistically unreliable measures.

Mike: I believe this is an important point. What makes a good teacher? One who is liked by students, one who consistently remains true to the decided upon curriculum, one who empowers students the most, one who gets the best test scores out of students? Or some other qualities. I've heard of all of these. If I was pressed to decide which one, I would say it entails all of these items (and a few more) but the greatest of these is the one who empowers students most consistently in his/her career. This determination can only be accurately be made, I believe, at some time frame AFTER the student has left the teacher's classroom, and been removed from the particular influence of the teacher for a time. Only then can the student assess which instructors or teachers empowered him/her to the degree to be labeled as "highly qualified".

Bettina: I agree to a certain extent. The real good teachers are not (always) the ones who are immediate popular, friendly and relaxed, which students sometimes realize many years after school. One could argue that the teachers do not determine their "material"; i.e. if a teacher has a class of students who were already good, motivated, disciplined, bright etc. at the beginning of the school year; the job is so much easier than if this is not the case. This is true. However, I would argue that if a teacher consistently over a number of years has students who do very well at the tests, s/he must be doing something right – nobody is that lucky with the quality of students. Generally I believe that the discussion about if we can 'measure' who is a good teacher rest on one's belief about whether or not the quality of teaching in some way can be quantified. This is a question of one's philosophy of methodology and social science. Regardless of one's views on this question I think that, firstly, if we can state what a good teacher is not, which we can, the opposite should not be that difficult, which for instance Mike gives good examples of. Perhaps it is worth the exercise to try to verbalise and explicitly state this into some qualitative and/or quantitative measures. Then we would be a long way. I would also like to add to what Mike above writes about the importance of the time span AFTER the students have left the school. Perhaps questionnaires could be sent out to students and their parents some years after graduation. I know that some university departments send out such questionnaires to their former students a year after graduation.

Mike: After reading Bettina's comments about "good teaching", I sense that s/he is much wiser and

has done much more pondering on this subject than I have. I would challenge Bettina to next respond on **just this topic** alone, I, for one, would be very excited to read his/her words on this subject.

Poignant Quote:

"The entire reputation of our school hangs on one test. It's not about balanced curriculum, enrichment, or learning any more. It's all about avoiding the failing school label."

Mike: This quote was made by a grammar school teacher, and its clarity, poignancy, and correctness to the de-facto NCLB movement to date is stunning.

Bettina: This is very sad.

Other solutions to the NCLB testimony by the AFT President was located at

www.aft.org/nclbrees.pdf[4]

Mike: Things forgotten, I believe, in this testimony. Include:

--nothing was said about the bright/gifted students in this NCLB movement. Most of the discussion was about getting the underachievers up to par (both students and teachers). However, I believe that some of the flexibility mentioned previously should be targeted to the bright and gifted, those who will lead our society in the future. It seems that the bright and gifted, when noticing that most of the attention is given the poorer academic students, tend to "kick back" and coast with their natural ability, rather than motivate or challenge themselves to heights. They learn, it seems, not to work hard to attain something academically, but rather to just live off of their academic ability to quickly attain easier concepts. These people will be our leaders, and if they learn that coasting, quick fixes, and lack of elbow grease is fashionable or "cool", they will take these attitudes into their leadership roles, and make a mess of our society in the future. Bottom line, I believe it is a fallacy to believe that it is OK to leave the bright and gifted unattended, because they will "teach themselves" the more advanced concepts and challenge themselves. These kids need our leadership (as teachers) at least as much as the under achievers.

--A comment about "drill and kill". Values of doing this activity include: having kids experience the effort during boredom needed as part of the work ethic in the workplace; this is the mechanism where kids acquire the baseline skills to become problem solvers and "educated" citizens; this activity will guarantee attainment of the minimum level of academic competency; there seems to be some emotional satisfaction from all students when they accomplish the tasks involved in these drills (the success is starkly attained and quickly assessed).

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[4] For some apparent reason this testimony seems to have been moved from this webpage. Contact Mike O'Lear for a hard copy

Bettina: A thing that has surprised me is how little emphasis there is on free school choice. In Denmark, where I am from and currently live and work, public money follows the child, also into private schools. Denmark has a tradition for private schools with a substantial government subsidy. On average public money pays for 75% of the cost of going to private school. There are also grants available. This means that most families have the financial means to send their children to private school, if they want. The Danish democratic Constitution adopted in 1849 stipulates general compulsory education - not compulsory school attendance. Around 13% of the Danish K10 students attend a private school. There are private schools within the entire religious, political and educational philosophy spectrum. The schools are run by a board elected by the parents. The board hires the teachers (who do not have to have the formal teacher education) and it organises the teaching according to their own beliefs. All that the law demands is that the teaching measures up to that of the municipal schools. The school can choose not to use the municipal schools' final examination. It is up to the parents of each private school to check that its performance measures up to the demands of the municipal schools. The free school choice also goes for the municipality schools. Parents are free to choose another municipality school than the one in their school district (provided it has space). Within a municipality, all public funds for children are the same per child no matter what school district they live in. Public money follows the child. I would argue that these very different (and often smaller) schools have a variety of approaches and pedagogies that other things equal will help reaching more children. I believe that flexibility in how the schools, and the teachers, teach produces the best educational results; students are different and individually tailored academic programs and policies are needed. If the money follows the children and the parents have a real free choice in deciding their children's school, the children are much more likely to receive an education tailored to their specific needs.

Mike: It seems that Europe may have "better ideas" than the United States on doing many things: e.g., in educational ideas, in economic structure (Euros) and possibly in approaches to health care cost structure!!

Guest Editorial
Reaction to the Final Report of the National Mathematics Advisory Panel

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The panel and the report

In 2006, President Bush appointed the National Mathematics Advisory Panel (NMAP henceforth) which issued its Final Report on March 13, 2008. The Final Report (summary), plus reports of three subcommittees and five Task Groups, together with two appendices containing the Presidential Executive Order setting up the panel, and details of the panel membership and other personnel, can be downloaded from www.ed.gov/about/bdscomm/list/mathpanel/index.html

The subcommittees dealt with Standards of Evidence, Instructional Materials, and a National Survey of Algebra Teachers and the task groups with Conceptual Knowledge and Skills, Learning Processes, Teachers and Teacher Education, Instructional Practices, and Assessment. Henceforth "the report" refers to these documents collectively, and "Final Report" to the summary document.

Practitioners, scholars, and researchers within the field of mathematics education were under-represented on the panel, and accorded surprisingly little input to the style and content of the report. In this collection of papers, some of us raise a number of issues that we find troublesome in the report. Many others issues, notably practices of assessment in school mathematics, are equally deserving of scrutiny.

Taking aim

> Consider the way in which the National Council of Teachers of Mathematics (NCTM) begins its draft of Standards 2000. No Socrates-like character asks “And shall we teach mathematics?” Even if the answer is a preordained “Of course, Socrates,” asking the question raises a host of others: To whom shall we teach mathematics? For what ends? Mathematics of what sort?” In what relation to students’s expressed needs? In what relation to our primary aims? And what are these aims?
> (Noddings, 2003, p. 87).

In the NMAP Final Report, the main aim is clearly stated. Mathematics (and science) education is seen as key to economic competitiveness, with implications, moreover, for national security. Thus, it is declared (p. xi) that "the safety of the nation and the quality of life – not just the prosperity of the nation – are at issue" (and see Gutstein, this issue).

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Nationalistic motivations of this kind are by no means confined to the United States, but are most strongly expressed in this country. Yet, there is an alternative worldview in which mathematics and science are seen as having a central role in solving the problems of humankind in general. D'Ambrosio (2003) has written passionately about the ethical responsibilities of mathematicians and mathematics educators:

> It is clear that Mathematics is well integrated into the technological, industrial, military, economic and political systems and that Mathematics has been relying on these systems for the material bases of its continuing progress. It is important to look into the role of mathematicians and mathematics educators in the evolution of mankind. … It is appropriate to ask what the *most universal mode of thought* – Mathematics – has to do with the *most universal problem* – survival with dignity.
> I believe that to find the relation between these two universals is an inescapable result of the claim of the universality of Mathematics. Consequently, as mathematicians and mathematics educators, we have to reflect upon our personal role in reversing the situation. (Emphasis in original).

The task that the panel was charged with was daunting, as can be seen by reading the President's Executive Order that launched the endeavor. As pointed out by O'Brien and Smith (this issue), surprisingly few of the panel were researchers in mathematics education. Members from outside the field (including mathematicians) could not be expected to begin the process with the width and depth of multidisciplinary knowledge that those within the field attempt to achieve.

In order to make the task manageable, it was therefore necessary to simplify it. I believe that the choice of means for simplification was, in many respects, inappropriate, leading to simplism, incoherence, imbalance, and to a number of startling lacunae, the most glaring of which are characterized below.

In particular, the members (with commendable, but arguably misplaced, diligence) plowed through (and regurgitated) huge masses of empirical work, preselected according to strict criteria that excluded most of the deepest work in the field. In my opinion, a lot of their time and intellectual energy would have been much better spent on reading rich reflections on mathematics education, such as the work of Hans Freudenthal (1983, 1981).

The Executive Order from President Bush required that the report should be "based on the best scientific evidence available" and this requirement was interpreted rigidly. As a result, the literature cited is heavily tilted towards psychology journals (psychologists were heavily represented on the panel), with relatively sparse citation from the mathematics education literature. According to De Corte, Greer, & Verschaffel (1996, p. 492):

> A further source of tension between mathematics educators and psychological researcher is the balance among, in Bishop's words, "what is", "what might be", and "what should be" (Bishop, 1992, p. 714). Psychologists who take mathematics as an area of application tend to investigate the situation as they find it or perceive it, to take mathematics as an [uncontroversial given]. In the course of their research they may identify problems and make suggestions for improvements within the existing framework, but without questioning fundamental goals. Mathematics educators, by contrast, are more likely to call for radical change …

Hence the preponderance of psychogical research contributes to the maintenance of the status quo as described by Roth (this issue).

According to Kilpatrick (1981, p. 22) research in mathematics education exhibits "reasonable ineffectiveness". He quotes Kristol (1973, p. 62):

> Everywhere we hear the refrain: "We can go to the moon, can't we? Well, why can't we do something equally marvelous about the ghettos or education or whatever?
> The answer is, of course, that going to the moon is easy whereas improving our system of education is hard. The one is nothing but a technological problem, the other is everything but a technological problem.
> Doing something about education means doing something about people – teachers, students, parents, politicians – and people are just not that manipulable.

Berliner (2002 p. 18) made similar points, arguing that educational research is "the hardest science of all".

What was the purpose of NMAP? On one level, it can be seen as a reaction to the perceived crisis that Gutstein (this volume) describes, as a number of factors combine to threaten the economic dominance of the United States. On another level, it can be seen as another engagement in the Math Wars (Schoenfeld, 2004), and this is reflected in the unbalanced make-up of the panel (O'Brien & Smith, this volume).

Missing disciplines

The field of mathematics education – at least within the circles I move in – has developed very remarkably in the last few decades in moving beyond psychology and mathematics as the predominant disciplinary influences to embrace many others, such as sociology, anthropology, sociolinguistics, cultural-historical activity theory (Roth, this volume) and many others (De Corte, Greer, & Verschaffel, 1996). That the report does not reflect this rich diversity is perhaps not surprising. What is undoubtedly strange, though, is the exclusion of the work of leading thinkers, practitioners, and researchers *from the field of mathematics education itself*, reflected in the make-up of the panel (O'Brien & Smith, this volume), in uncited literature, and in the lack of reference to the most prominent scholars in the field.

It is also notable that the Task Group on Learning Processes produced a report of 263 pages (for whom to read?) with some 500 references that reads like a textbook on the cognitive psychology of learning/teaching mathematics. Let me stress that I am not questioning the relevance of such work, rather the disproportionate status accorded it in the report relative to other work in the field. Neurological studies (albeit with appropriate disclaimers) enjoy undue prominence. (Maybe 10 years ago it would have been neural networks, and 20 years ago Artificial Intelligence).

Excluded research

Only certain types of research, meeting rigid methodological criteria, were considered to meet the requirement of "best available scientific evidence". If the goal is to improve mathematics education, there are many important questions – perhaps the most important – that are not amenable to such forms of enquiry. For example, as Martin (this volume) points out :

> The imposition of these standards assuredly eliminated a host of qualitative, ethnographic, case-study, and descriptive studies that are commonly used to examine the experience of

> students of color in school settings, including experiences with race and racism. These criteria also minimize the important of studies that have situated schools in their larger sociopolitical context.

For example, how do you try to answer the assuredly relevant question "What are the cross-generational effects on populations against whom, historically, education has been used as a means of oppression?" through the kinds of experimental methods sanctioned? (Ladson-Billings, 2007).

As the panel was doing its work, the second NCTM Handbook of Research on Mathematics Teaching and Learning (Lester, 2007) was nearing completion and Lester wrote as follows to the Chair and Vice-Chair of NMAP, beginning as follows:

> Dear Drs. Faulkner and Benbow,
>
> I am contacting the two of you in your roles as chair and vice-chair of the National Mathematics Advisory Panel to inform you of a resource that may be of value to the Panel in it deliberations. Specifically, I am editing a revision of the Handbook of Research on Mathematics Teaching and Learning that was originally published in 1992 by Macmillan. The 1992 handbook has been the most widely cited reference on research in mathematics education in the world and the second edition is likely to be just as valuable to the research community.
>
> (Lester, 2008).

He received only a formal response from an administrator. The panel showed no interest in receiving the resource, and it is not cited anywhere in the report.

Excluded mathematics

The instruction by President Bush that directed the panel towards a focus on algebra may have partially contributed to incoherence – perhaps it would have been more appropriate to call it the National Algebra Advisory Panel. In any case, it is very noticeable that geometry, probability, and data handling receive little attention. Moreover, geometry (in some cases) and combinatorics (throughout) are accorded attention only insofar as they feed into algebra.

More serious yet, in my opinion, there is the almost total inattention to two fundamental aspects of doing mathematics, namely (a) applications and modeling, including data handling and statistical modeling, (b) problem solving (O'Brien & Smith, this issue). The first of these I consider of much more importance for responsible citizenship than a knowledge of algebra.

Ignoring the real world of schools in the United States

Curiously, the report makes almost no mention of *No Child Left Behind*, despite its evident effects on mathematics education and education in general, and research on those effects (Nichols & Berliner, 2007). This lack is particularly odd for the Task Group on Assessment, where the act is scarcely mentioned.

Achievement gaps are mentioned at several points, but there is no attempt to place them in the context of "the education debt" (Darling-Hammond, 2006) and resources gaps. Here is a reality check from Darling-Hammond (2007, pp. 247-248):

> At Luther Burbank school, students cannot take textbooks home for homework in any core subject because their teachers have enough textbooks for use in class only … Some math, science, and other core classes do not have even enough textbooks for all the students in a

> single class to use during the school day …Luther Burbank is infested with vermin and roaches, and students routinely see mice in their classrooms …
> … Eleven of the thirty-five teachers at Luther Burbank have not yet obtained regular, nonemergency credentials, and seventeen of the thirty-five teachers only began teaching at Luther Burbank this school year (Williams v. State of California, Superior Court of the State of CA for the County of San Francisco, 2001, Complaint 58-66).

The composition of the panel and the contents of the report markedly fail to reflect demographic trends within the United States and the constellation of problems that arise from class and racial inequities (Martin, this issue).

Lack of appreciation of mathematics as a human activity
The report also gives no inkling that mathematics has a long, multicultural, intellectual and social history, one that is of interest in its own right and also offers insights into the cognitive obstacles that students face. An extra-terrestrial reading the report might reach the conclusion that the people of this planet recently received mathematics as a complete body of knowledge.

Many of the developments within the field of mathematics education manifest aspects of the general theme of mathematics as a human activity, including ethnomathematical research, studies of the history of mathematics, studies of people behaving mathematically in natural settings such as work, new directions in the philosophy of mathematics, attempts to establish connections between school mathematics and the lived experiences of the students. Within our field, it is being questioned whether mathematics as a school subject should continue to be dominated by mathematics as an academic discipline or should reflect more fully the range of mathematical activities in which humans engage. The report doesn't go there.

Curious lack of intellectual excitement
Perhaps the strongest impression that the report leaves me with is a lack of any sense that mathematics is intellectually exciting, and could be taught in a way that makes it so, for at least a majority of students. Instead, what the report brings to mind is the satirical depiction of education by Charles Dickens in "Hard Times", in particularly the second chapter which is entitled "Murdering the Innocents".

Among the pancultural well-springs of mathematics as a human endeavor are a sense of aesthetic and pattern, and a delight in puzzles and games. As mentioned above, the report scarcely deals with problem solving, in the sense that Polya wrote about. Polya (1945, p. v) pointed out that just as "he cannot know that he likes raspberry pie if he has never tasted raspberry pie" a student who has not experienced the tension and triumph of discovery in mathematics is unlikely to develop a taste for mathematics. In a powerful short piece, Noddings (2007) stated that:

> Intellectual life is challenging, enormously diverse, and rewarding. It requires initiative and independent thinking, not the tedious following of orders. It should not be reduced to mental drudgery.

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Mathematical Cognition and the *Final Report* of the *National Mathematics Advisory Panel*: A Critical, Cultural-historical Activity Theoretic Analysis

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Abstract
In its *Final Report*, the *National Mathematics Advisory Panel* has depicted a stark image of mathematical competencies and achievement among U.S. students. The *Advisory Panel* notes the lack of research with "truly scientific" rigor and, mentioning Vygotsky's cultural-historical activity theory in passing, suggests its utility to be untested. In reading the report, I noted the limited understanding of the mathematics education literature it articulates and a complete failure to draw on established, "tried and proven," theory and practice in mathematics education founded upon an encompassing cultural-historical activity theory. This theory is comprehensive and encompassing, because it retains activity in its entirety as the unit of analysis, which leads to an integrated and integral consideration of those "factors" that are taken to be external to mathematical cognition in the *Final Report*. In this article, I articulate the current state-of-the-art understanding of cultural-historical activity theory and then use it to provide a critical perspective on the report, its recommendations, and its conclusions as these pertain to the learning of mathematics.

Keywords: Cultural-historical activity theory; National Mathematics Advisory Panel; mathematics education research; socio-cultural perspectives; Vygotsky

One of the first thoughts that arose in my mind while reading the *Final Report* of the *National Mathematics Advisory Panel* was the French saying "Plus ça change, plus ça change [pas]" (the more it changes, the more it doesn't change).[2] The more educational research finds out, the less educational policy changes, as it plays up to the powerful who tend to desire the reproduction of the status quo rather than to bring about changes of life conditions that lead to differences that make a difference.[3] Because the *Final Report* does not articulate a coherent position on what constitutes mathematical knowing, there is little that holds together the statements and recommendations—the report constitutes a disconnected collection of sentences that play into the hands of (simple-minded?) politicians and policymakers rather than providing a comprehensive framework for the mathematical

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[2] Perhaps more frequently, the saying may be "Plus ça change, plus c'est la même chose" (the more it changes, the more it remains the same).

[3] Locking thieves up in prisons is easier than to change society so that the poverty from which many thieves issue becomes a thing of the past. Changing society would mean that those who have will have to share with those of have not. Ideologically, this is not where an individualist society wants to head.

education of current and future generations. A comprehensive framework would have take into account research evidence—such as that issuing from the many ethnographic studies of mathematical knowing in the everyday world—that the panel oh so lightheartedly disqualified as non- or insufficiently scientific. More so, the *Final Report* provides considerable evidence that its findings are not in themselves legitimate but that to the contrary, its legitimacy has to be articulated from the outside by means of statements about its own scientificity and scientific rigor. The report further justifies its own conclusion by taking a deficit perspective on mathematical knowing generally and the mathematical knowing of certain parts of society more specifically, failing to take into account the research on the knowing and learning of mathematics in everyday life and therefore failing according to its own criteria of scientific and scholarly rigor. Notably, the *Final Report* does not interrogate the political dimension of schooling and therefore remains oblivious to the fact that schooling produces deficits, hierarchies, and inequities that differentially lead the children of specific classes into a hierarchy of career and into the for a globalized capitalist economy necessary structural unemployment. The *Final Report* thereby plays into the hand of a particular individualist ideology of the worst kind, continually produced and reproduced by and under the current U.S. administration.

In this essay, I take up a number of problematic points concerning mathematical learning processes as these become apparent throughout the report. Perhaps the greatest problem with the *Final Report* is that it leaves unstated its theoretical framework for mathematical knowing and learning, which, inherently, constitutes the framework within which the value statements and deficit perspectives are framed (e.g., "Far too many middle and high school students lack the ability to accurately compare the magnitudes of such numbers" [p. 27]). The authors of the *Final Report* do not have or articulate a theoretical framework that would allow us to understand why, according to their own account, the students coming from "low parental education levels, low incomes, and single parents" (p. 25) end up bringing "less foundational knowledge for learning school mathematics" (p. 25) and, as a result of lower achievement at the end of schooling, end up in the same societal classes as their parents. I begin by articulating a comprehensive theoretical framework of knowing and learning that does take into account why human beings do what they do and how learning accrues from their actions. Such a comprehensive framework is necessary, because only then can we answer questions about why and how students actually take up and identify with the motives of school mathematics and therefore what the grounds of reason they have for doing what they do in school generally and school mathematics more specifically. The theoretical framework I outline is grounded in a lineage of social psychology that begins with Lev S. Vygotsky, who is mentioned in the report but only briefly but whose work and legacy remains unappreciated and misunderstood in the text of the *Final Report*. The framework I develop generally is referred to as cultural-historical activity theory and I show how it already provides answers to questions that the *Final Report*—I wish it were the final, that is, last of its kinds—can only recommend for further study. I then take up a number of claims and issues made in the *Final Report*, making explicit what it leaves out, especially the research on knowing and learning in mathematics that we are familiar with and that does not seem to have been accounted for in the writing of the report.

Cultural-Historical Activity Theory

Cultural-historical activity theory provides a framework for understanding human activities (Roth & Lee, 2007). In English, the term "activity" is used to render two very different concepts from the Russian and German languages in which much of the theoretical framework was developed: The term aktivnost'/Aktivität means being busy, doing something, whereas the term deiatel'nost'/Tätigkeit means societally motivated and society sustaining human endeavors. Thus,

children doing school tasks engage in the former, whereas farming, manufacturing tools, and schooling constitute activities in the second sense. Tasks do not require knowing *why* one does what one does, whereas participation in societal activity does. In fact, students *do* participate in an activity, but it is, as I show below, concerned with *schooling* rather than with education and knowing.

Cultural-historical activity theory fundamentally assumes the dialectical relationship between agency and structure. On the one hand, agency mobilizes material and social structures, but it is itself enabled by and the product of these structures. It therefore does not make sense (a) to speak of agency independent of the enabling and mobilized structures or (b) to speak of structures independent of the agency that picks out and therefore defines relevant structure. Following one interpretation of cultural-historical activity theory (Engeström, 1987), many researchers draw on one particular, triangular representation to make salient the structural moments and mediational effects of human activity systems (Figure 1).

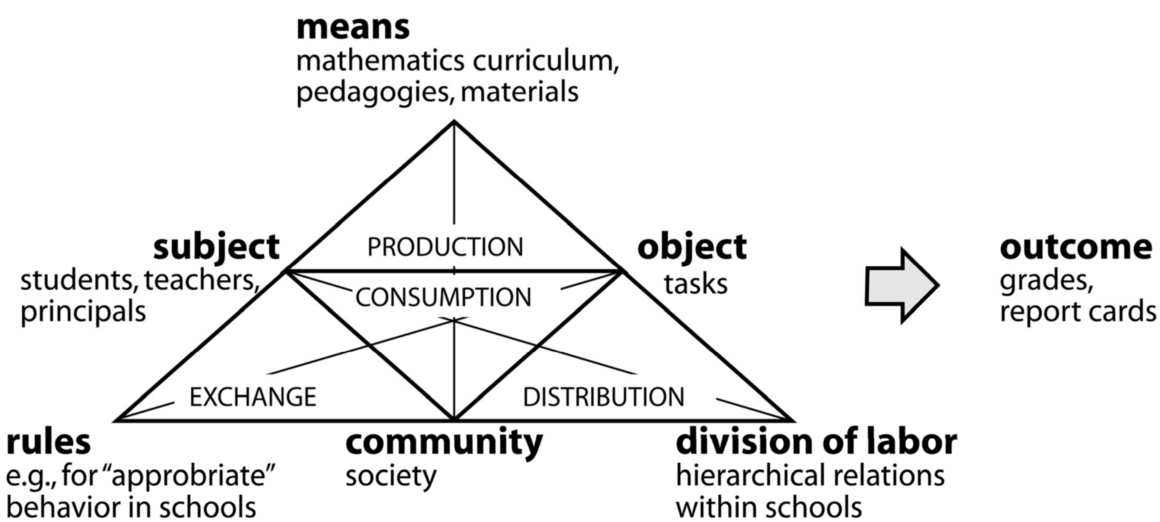

Figure 1. Structural perspective on the cultural-historically evolved activity of schooling, which contributes to the production and reproduction of society, including its hierarchical relations of ruling.

In Figure 1, the structural *moments* of schooling are articulated. (A different example I could have used is the production of the *Final Report*, which would have included its authors as the "subject," existing research, computers, libraries, and meeting facilities as the "means," and the U.S. president, his policymakers, and other politicians as part of the relevant "community.") The activity system as a whole and the different *inner* structures identified—subject, object, means, rules, community, division of labor, and outcome—cannot be reduced to one another. In other words, a structural component such as the "subject" does not constitute an *element* but an irreducible *moment* so that the relevant subject is constituted by the activity as a whole generally and its other moments more specifically. The object of activity cannot be reduced to the intentions (motives, goals) of the acting subject, but rather, subject and object mutually constitute (determine, imply) each other. Each moment denoted in Figure 1 is constitutive of the activity as a whole, but it is only as part of an existing activity that these moments become relevant in the first place. Each moment only constitutes a one-sided image of the activity as a whole, though it may be used, in synecdochical (metonymic) fashion to denote the whole.

Cultural-historical activity theory organizes our gaze concerning human activities by highlighting outcomes of activity. What do students walk away with from their school experience? The answer is: grades and report cards. Some readers may contest this answer and argue that "knowledge" is the true outcome of schooling. We can, however, immediately see that the latter answer is in contradiction with observations according to the *Advisory Panel*'s own account: American students generally and those from low-income, African American, and single parent families specifically do not come away from schooling with the forms of knowledge that the *Advisory Panel* desires them to come away with. More so, research on mathematics in the everyday life shows that mathematical performance of supermarket shoppers (Lave, 1988) or among child street vendors of candy (Saxe, 1991) is unrelated to the number of years someone has studied mathematics or the grades someone has achieved.

Another argument against knowledge being the outcome of schooling can be gleaned from the generally observed fact that students at all levels memorize some facts for exams, which they are frequently proud of having forgotten only hours or days later. It is not knowing something that is valued but "what'ya get?," that is, the grade that the instructor provides in exchange for the exam or assignments handed in on the part of the students. Whatever the student does and learns is less important and in fact irrelevant; what counts are the individual grades, the course grades, and the grade point averages. "Cheating" and "plagiarism" are typical forms of actions that are enabled by a system focusing on grades rather than on knowledge. If learning, that is, the expansion of action possibilities and room to maneuver were the outcome, it is hard to understand why students would "cheat" and "plagiarize" given that they would go against their own interests. "Cheating" and "plagiarizing" work *in their interest* of getting higher grades—at least as long as they do not get caught. Cheating and plagiarizing therefore are action possibilities in and made available by a system rather than deviancies located in the characters and minds of specific individuals.

Further structure for our gaze is provided in the focus that cultural-historical activity theory has on the usage of the outcomes of an activity system within the system or in other systems to which they are "traded" (exchanged) and how the outcomes are distributed within the relevant community. Thus, it is immediately evident that high school graduates use their grades and report cards to garner access to universities; it is common knowledge that higher grade point averages provide graduates with more choices and access to more prestigious institutions. Grades, report cards, and GPAs constitute a form of (symbolic) capital that students can use to open doors; and entry through these doors provides them with many more opportunities to gain cultural, symbolic, and financial capital. Those students with the lowest grade point averages, on the other hand, might find themselves in manual labor and service profession where the results of schooling do not matter. Interestingly, as the *Final Report* shows, children from low-income and single parent families, for example, come with "less knowledge" to school and end up with less, because "the mathematical knowledge that children bring to school influences their math learning for many years thereafter, and probably throughout their education" (p. 25). Why might this be? The simple and unsophisticated answer is that prior knowledge determines future learning. This answer is unsophisticated, because it does not take into account the ways in which culture and language at school and the cultures and languages from which children issue mediate one another: there is research that does show how schools and schooling embody middle class culture leading to an enhancing resonance phenomenon for children and students from the middle class and negatively mediating enculturation, socialization, and learning of working class students (Eckert, 1989). The *Final Report* does not provide evidence about whether panel members are aware of such research or whether, if they are, simply discounted it.

Cultural-historical activity theory is capable of accounting for both phenomena, the inequitable *distribution* of grades and grade point averages available from the students' report cards and the resulting *exchange* relations in which students trade their these, their symbolic capital, against access (Figure 1). These grades allow a hierarchical ordering of students, provide them with differential access to further resources and opportunities, and lead to the production and reproduction of social stratification (Foucault, 1975). Another example may further show the power of cultural-historical activity theory in articulating structures that mediate the attainment of grades. Thus, the *rules* (Figure 1, lower left) regulating "appropriate behavior" in schools are modeled on those that are characteristic of the middle class, including those mediating interactions with teachers, the use of language, and time and temporality. Students who come from poor African American produce time and temporalities very different from those of middle class students and they end up being punished (e.g., excluded from accessing schools) for being late; or they use "foul" language that leads teachers and administrators to suspend students (Roth, 2005). Here, then, the normative rules teachers, administrators, and often police enforce in schools mitigates against the engagement with school tasks, resulting in lower grades, lower achievement, more years in school, and so forth.

Production is a function of the degree to which the subject in an activity system takes up the collective motive of activity. For example, the intended outcome of taking an algebra class supposedly is mathematical knowledge. However, many students do not take up this intention: they are said to be unmotivated. Or they fake motivation but resort to all sorts of learning other than mathematical learning that promises them not to be punished or penalized—such learning is defensive, for it is motivated by avoiding negative repercussions. To make up for this chasm between intentions and their uptake, schools and teachers are supposed to provide incentives that motivate students. Now tested widely, some school districts in the US provide cash incentives for students to work for getting higher grades (Ash, 2008). Why would students who see their room to maneuver and action possibilities increase *not engage* in learning? Why would students act against their own interests not engaging in an activity that expands their control over their life conditions and enhances their opportunities for acting? Not surprisingly, the *Final Report* does not articulate a comprehensive model in which motivation and emotions are *internal* moments of cognition but that are mere factors—which, as all factors, inherently are outside and therefore *indifferent* to the phenomenon (Hegel, 1806/1977).

To understand the for task relevant moments of goal formation and meaning, another dimension of cultural-historical activity theory needs to be articulated. Thus, whereas activity characterizes a societal formation, any activity is concretely realized in and through the goal-directed actions of the subject. But the (individual, collective) subject acts because of the activity. Thus, what teachers in schools are oriented to and do, and what makes their orientations and doings meaningful, *is* the activity of schooling. Lecturing or asking students to complete an algebra task is meaningful *because* teachers find themselves in schools, with their students, and the process of accomplishing the overarching motive—the production of grades and, hopefully though not necessary, the production of some mathematical knowledge. But precisely the same actions realize schooling in a concrete way. The actions are a condition for schooling to emerge and exist. The activity that organizes and motivates goals and the actions that realize it therefore are mutually constitutive. Actions are meaningful to the extent that they are constitutive moments of activity; but activities only exist because of the meaningfulness of actions.

Not every aspect of an action is conscious like the goals. Rather, unconscious operations realize actions and their goals, which nevertheless have occasioned the production of the operations. Thus, we may have the intent to contribute to a conversation, but we do not have to choose the words—the production of words is something unconscious, unless we struggle and actively search for words in some instances. Similarly, writing an algebra test, we do not have to think about writing the numbers, letters, and symbols in the equation $5a + 3a = ?$. Rather we think and the letters, symbols, and numbers are produced in a process that itself, as its objects, remains unconscious. That is, operations are conditioned by the context, including the current state of the action with respect to the ultimate goal.

All three levels are involved when it comes to emotions and affect. The motive of the activity and the goals subservient to them are mediated by short-term costs and long-term payoffs, evaluated in emotional-volitional terms (Bakhtin, 1993). A person may put up with short-term negative effect on emotion—e.g., studying hard, engaging in repetitive tasks—if these actions promise likely-to-be-attained long-term payoffs that increase the person's room to maneuver (being able to do more, better, receiving money that one can purchase things otherwise unattainable, etc.). The bodily states that contribute to emotion also mediate performance, expressing the irreducible relationship between quality of performance and emotional constitution. In the extreme case, getting a rush from engaging in mathematical tasks increases the likelihood of performing much higher than if the engagement is indifferent and certainly than if the engagement is merely for the purpose of avoiding punishment and other negative effects (e.g., low or failing grades).

In this theoretical framework, learning is actually an epiphenomenon arising from the fact that in pursuing and realizing the current activity, thereby reproducing it to a certain extent in form and content, the subject changes becoming more adept in producing relevant entities. Doing something better and having more alternatives at one's disposition than previously here characterize learning. That is, learning is constitutive and the outcome of an expansion: in control and action possibilities. In the ideal case, students learn one or more alternatives of doing certain algebra tasks. But this does not have to be. It is immediately evident that this framework better than others explains why one possible outcome of schooling is a greater number of increasingly better "cheaters" or why students are focused on getting grades rather than on really knowing whatever the curriculum specifies as that to be learned.

Societal Mediation of Poverty and the Reproduction of Inequity

The *Final Report* takes an approach that tinkers with symptoms rather than with the causes of mathematical knowing and achievement. For example, the panel notes that it is unfortunate that

> most children from low-income backgrounds enter school with far less knowledge than peers from middle-income backgrounds and the achievement gap in mathematical knowledge progressively widens throughout their PreK–12 years. (p. xviii)

The first question we may raise—about this and similar statements—is about the "fortune" that leads children from low-income backgrounds to come with "less knowledge" and to face an increasing rather than decreasing gap. The word fortune derives from and relates to the Latin words *fors*, chance, and *ferre*, to bear. Children from these backgrounds do *bear* poverty as a burden. But this burden does not come by chance (i.e., fortune): it is a direct result of exploitation, societal stratification, and the effects of radically individualist capitalist society. (Sweden, a social democratic

country, has low child poverty rates whereas the US, one of the two richest countries in the world, has among the highest child poverty rates.)
The report does not provide an hypothesis or explanation for the increase in the gap between children from well-to-do and poor families. What lies at the origin of the *widening* gap? Why does a school system supposedly providing equal opportunities to *all* children create increasing inequities? What are the social and cognitive forces that lead to those differences? It is quite clear to me that a model of learning that focuses on the mental storage of procedural and declarative knowledge individuals acquire cannot handle the ways in which the participation in cultural practices, such as schooling, not only produces (changes) the practices but also reproduces them including the structures that enable the practices and that are mobilized in these practices.

The discursive ideology of the report is quite clear and not unlike the medical discourses concerning obesity, coronary heart diseases, or cancer: fix the symptoms rather than enact much less expensive (but less lucrative for multinationals) prevention that get rid of the causes. Thus, rather than putting kids into "Head Start and other programs serving preschoolers from low-income backgrounds" (p. 25), recommendations toward fixing unemployment and poverty, the structural prerequisites and outcomes of extreme capitalist markets, would get closer to the *causes* of "less functional knowledge for learning school mathematics" (p. 25). Rather than dealing with societal problems generally, which lead poor children to eat the worst foods on the market (the cheapest foods are those offered by fast-food chains), they offer "school lunches" and think of it as a true accomplishment. Getting rid of exploitation and poverty would mediate having to hand out school lunches in the first place.

The simple-mindedness of the report clearly is notable in statements such as those about how to deal with the lack in "foundational knowledge for learning school mathematics" that characterizes students from families with low-parental education levels, low incomes, and single parents. Thus, the report recommends "training" teachers in how to implement "a variety of promising instructional programs [that] have been developed to improve the mathematical knowledge of preschoolers and kindergarteners, especially those from at-risk backgrounds" (p. 25). There is much evidence that such programs do not eliminate or even marginally mediate school achievement differences and subsequent differential access to further schooling opportunities, jobs, and other desirable dimensions of society. Thus, in France with a generous pre-school daycare system, the production and reproduction of the cultural elite remains one of the outstanding features of the society. Not only does the father's education and level of job correlate with a child's educational prospects, but also there are also differential effects on the gender of the child (Bourdieu & Passeron, 1979). Single-minded cause–effect reasoning as advocated in the *Final Report* will not lead to changes, and there is (non-scientific?) research that already provides evidence of this.

Model of Cognition: Knowing and Learning

In the cultural-historical activity theoretic model outlined above, the three levels of an activity—motives, goals, conditions—are integrally related and cannot be reduced to one another or to some other simpler element. The activity in its entirety is the appropriate unit of analysis.

In contrast, the *Final Report* addresses cognition in terms of disconnected facets, which raises questions about *why* there is thinking and consciousness in the first place. Thus, for example, the Report states that "computational facility with whole number operations rests on the automatic recall of addition and related subtraction facts, and of multiplication and related division facts" (p. 26). The Report goes on note "that many contemporary U.S. children do not reach the point of fast

and efficient solving of single-digit addition, subtraction, multiplication, and division with whole numbers, much less fluent execution of more complex algorithms" (p. 26). And the Report concludes as a surprise that in the US, "many never gain such proficiency" (p. 26).

Here, the report makes statements about cognition generally and about the relationship between computational facility, on the one hand, and automatic recall of addition and related subtraction facts. This may be the case under certain conditions, for example, when research participants subjected to constraints that do not exist in other everyday situation outside of the psychological experiment; this also may be the case in school classrooms, which decontextualize in a similar way motives, goals, and conditions. But it certainly is not the case in out-of-school everyday contexts where the motives of the activity come to be the overarching leitmotiv that allows the general sense of actions to emerge as well as make the situation meaningful *as an everyday pursuit*. Thus, for example, supermarket shoppers perform very differently on ratio-and-proportion (fraction) requiring best-buy tasks if the situations are changed from their normal shopping *in* the supermarket, *in front of* the supermarket with selected products made available on a table, or on paper-and-pencil tests presenting equivalent problems (Lave, 1988). When persons shopping for groceries were asked while choosing certain products about the "best buys" available, many concerns entered the decision that had very little to do with ratio and proportion, and prices also were evaluated in terms of the shelf life of a product once opened—mitigating against a larger and less-pricy product that would spoil because the family could not consume it quick enough—personal taste, and others. When a range of products was taken from the shelf and presented on a special table in front of the supermarket, the conditions already had changed, where the framing already conditioned like a psychological task rather than one in which the person truly made a choice and, for this, had specific goals to be addressed. Not surprisingly, with changing conditions the (tacit, unconscious) operations enacted also change, just as activity theorists have postulated from the beginning (e.g., Leont'ev, 1978). Finally, the same shoppers' performance dropped from near perfect score in the market to about 50% correct on paper-and-pencil tasks asking for the best buy among to products of which nothing more was known than the price and weight. Again, the operations mobilized in the attempts to provide answers to the researchers' questions differed, conditioned by the different context in which goals were framed.

In contrast to the efficiency and competency on best-buy tasks in everyday out-of-school life, the shoppers did poorly on school-related fraction tasks. The *Advisory Panel* notes difficulties with such tasks among American students: "Difficulty with the learning of fractions is pervasive" (p. 28). The *Advisory Panel* also notes that algebra teachers rated "students as having very poor preparation in 'rational numbers and operations involving fractions and decimals'" (p. 28). If the *Advisory Panel* had taken into account the result of ethnographic work in workplaces, it might have begun to hypothesize that the school and the school-nature of the task that is responsible for producing the results it so deplores. Perhaps the difficulties with learning fractions are endemic to the schooling system guided by inappropriate theoretical framework rather than the result of faulty teaching or unwilling and incapable students? Do the statements about students learning difficulties and lack of knowledge not constitute an indictment of decades of research in the psychology or didactics of mathematics?

In a similar way, a significant number of research scientists did not correctly interpret graphs culled from the materials in an introductory course of their own discipline (Roth, 2003). Here again, the scientists did not mobilize basic graphing skills to be recalled and applied in and to the particular task. Many did not focus on the required graph feature, such as slope or height, and therefore made

the same kinds of "confusions" that high school students enact on traditional paper-and-pencil problems (Leinhardt, Zaslavsky, & Stein, 1990). On the other hand, when asked to read a graph directly related to the topic of their own research, then the scientists not only provided highly sophisticated answers that required the recall of simple facts, but they also mobilized their intimate understanding with the motive of activity (knowledge production versus technological application), the contextual particulars under which the research was conducted, strength and fallibility of the equipment and instrumentation employed in the data collection, and so on. That is, under given conditions, we can make any scientists look bad and exposing, like students, "poor knowledge of the core [. . .] concepts" (p. 26).

The authors of the *Final Report* complain that "[F]ew curricula in the United States provide sufficient practice to ensure fast and efficient solving of basic fact combinations and execution of standard algorithms" (p. 26). Again, the theoretical framework underlying the report treats "basic fact combinations" as elements to be deployed in concrete situation, recalled as declarative or procedural knowledge and applied in the situation at hand. The research in everyday contexts shows however that whatever basic skills are mobilized to solve the problematic issues arising for particular people in particular contexts is a function of the activity as a whole, with its affordances and constraints on action possibilities that emerge in a dialectical fashion. Thus, in everyday situations, it is not the application of a particular skill that is important but arriving at a resolution of the problem—which, when it comes to price comparisons, may be a quick look at the unit price that is frequently printed in almost invisible size at the bottom or in a corner of the shelf price tag.

There is a lot of research on mathematics in a variety of workplaces—among street vendors and bookies, scientists, technicians, dairy factory workers, supermarket shoppers—that underscores the "situated" nature of cognition. By this sometimes-overused term situated I mean here that productive activity always is oriented toward a motive, which is realized in the formation of goals and by means of goal-directed actions all of which are mediated by the activity as a whole. Each action and each operation that contributes to realizing one or more actions is made possible, constrained, and made intelligible by the motive that provides the guiding light for anything that realizes the activity. Any goal-directed action takes into account and is mediated by the available tools, the forms of division of labor available, the community for which the outcomes are produced, and the rules that are to be adhered relative to the different pairwise relation within activity (Figure 1).

This research on knowing and learning mathematics in the everyday life outside of schools also raises questions about the possibility of transfer, thereby questioning the very position that the *Advisory Panel* takes on this issue. Thus, whereas the *Advisory Panel* notes that "people's ability to make links between related domains is limited" (p. 30), it continues to assume that the transfer of "basic skills—the fast, accurate, and effortless processing of content information" (p. 30) is possible and should be fostered. I do agree that we need to support students' *education* (which is not the same as schooling), but the essential question about the role of the collective motive of activity and the personally relevant goals that make for the conditions that determine the mobilization of any unconscious operation ("basic skill") is not posed. More so, cultural-historical activity theory already has a model for the transformation of conscious actions into unconscious operations and the reverse transformation during the critical analysis of actions in conditions of breakdown—i.e., when a familiar way of doing things fails.

"The Sociocultural Perspective of Vygotsky"

The *Final Report* notes that the "sociocultural perspective of Vygotsky also has been influential in education" (p. 30). However, the *Final Report* immediately misrepresents what the sociocultural perspective is about in suggesting that the theory "characterizes learning as a social induction process through which learners become increasingly independent through the tutelage of more knowledgeable peers and adults" (p. 30). The Report concludes in stating that the "utility in mathematics classrooms and mathematics curricula remains to be scientifically tested" (p. 30).

First, there has been more than one study from a Vygotskian perspective, and this existing work has "invariably produced descriptions stressing the ways in developmentally primary and expanded forms of cognition are transformed into new, abbreviated, and more complicated forms" (Davydov & Andronov, 1981, p. 4). Here, we find precisely the kinds of answers that the *Final Report* suggests that little is known about—e.g., to answer how students achieve "fast, accurate, and effortless" automaticity of operations to produce the "more complex aspects of problem solving" (NMAP, p. 30). More so, cultural-historical activity theory has a model what happens to "abbreviated operations" that seem to disappear into automaticity: in fact, they "do not simply disappear. They take on a status in which they are treated as if they had been performed and are hence being 'kept in mind'" (Davydov & Andronov, 1981, p. 5). There is, at least in the Russian literature—and contrary to what the *Final Report* claims ("students must eventually transition from concrete [hands-on] or visual representations to internalized abstract representations. The crucial steps in making such transitions *are not clearly understood at present* and need to be a focus of learning and curriculum research" [NMAP, p. 29])—ample evidence "to describe in detail the external features of abbreviated (ideal, cognitive) acts in contrast to their material, object prototypes" (pp. 5–6). Thus, making reference to some examples concerning addition, the authors summarize that the

> ideal act of addition, based on an objective symbol, takes place in the form of a real <u>motor</u> act (the sweeping movement of the hand as the cardinal number is pronounced in a drawl). Then, in a compact reduced form, this movement itself becomes the symbol of the number, manifested in unique gestures and in the accented pronunciation of the cardinal numbers. (p. 25, original underline)

In this way, internalization is but the other coin of externalization, because "the ideal is immediately realized in a symbol and through a symbol, i.e., through the external, sensuously perceived, visual or audible body of a word" (Il'enkov, 1977, p. 266). In the study of the emergence and transformation of hand movements as a basic component of ideal acts (e.g., mental addition), "initially the symbol was the hand's movements together with articulation, and subsequently only the reduced articulation as a basic component of the word in designating the number" (Davydov & Andronov, 1981, p. 25).

It might have been excusable in the 1970s and early 1980s not to know about the developments of Vygotskian theory and educational practices in mathematics. With the wide availability of original and translated research from the former USSR, today it is inexcusable that the *Advisory Panel* is unaware of the considerable work concerning the development of routine skills in the pursuit of meaningful activity as made available, for example, in *Types of Generalization in Instruction* (Davydov, 1990) published by the National Council of Teachers of Mathematics.

With respect to the development of algorithmic knowledge and skills, Vygotskian theory has considerable explanatory power and supportive evidence. There are curricula not only in Russia (now and during the Soviet Union) but also in the US that implement educational strategies for developing algorithmic knowledge that does take into account the conditions in which these skills are mobilized. *Conditional learning* is an outstanding feature of mathematics curricula based on cultural-historical activity theory. Thus, for example, children are encouraged to solve addition and subtraction tasks in a variety of ways and are then asked to select the ways in which the calculations are done easiest (Schmittau, 12004). The children taught according to Vygotskian principles "find it advantageous to add numbers in column form, applying standard algorithm for multi-digit addition (a method which they themselves have derived)" (p. 25), leading them to employ rounding methods when asked which manner of solving the problems constitute the "easiest way." Similarly, the children develop rounding algorithms when asked to mentally solve orally presented addition tasks such as 242 + 37 + 118. That is, when the task involves "finding the easiest way" as motive for identifying among one's different ways of going about a task, then children begin to develop algorithmic knowledge that allows them to flexibly adapt to the demands of the tasks. Modifying and providing for a motive of activity is precisely the kind of work that cultural-historical activity theory has spawned in the USSR and Russia.

From a Vygotskian perspective, rather than teaching symbols that absorb the genesis of the concept they denote, a genetic analysis is required for the design of appropriate curriculum. A requirement for the curriculum is an understanding of how, ontogenetically, semiotic forms absorb into themselves the experience of objective acts, which they represent in abbreviated form, a process that I have denoted elsewhere as the emergence of signifying moments of a situation that denote the situation as a whole in a (metonymic, synecdochical) pars-pro-toto fashion (Roth, 2004). The construction of curriculum requires the historical study of ontogenetic development because "the basic structure of the concrete, objective acts cannot be reconstructed, if what is given is only their semiotic form" (Davydov, 1988, p. 179). Thus, "It is necessary to follow the whole 'history' of different solutions of one and the same problem, in order to see in the abbreviated forms of thought its original course" (p. 179). Once this development is seen and experienced, we come to understand the ways in which automatization emerges from purposeful activity as a whole because "to see in the abbreviated forms of thought its original course . . . one should uncover the laws and rules of this abbreviating and then to 'recapitulate' the full structure of the processes of thought being analyzed" (p. 179).

According to cultural-historical activity theory—which generations of Soviet psychologists have extensively developed and used to study knowing and learning since Vygotsky initially framed it—the real object-oriented activity of collaborating individuals lies at the heart of a psychological understanding of all mental functions. These mental functions are ideal forms of real human relations, and, in a truly dialectical fashion, these ideal forms are realized in the "material and spiritual products that *objectify* the internal psychological conditions of their realizations" (p. 180). Every psychological structure underlying a mathematical concept is the result of an activity, and students develop the relevant and necessary structure by participating in a relationship with material and social reality that corresponds to the concept. It is this reality, social and material, that comes to be synecdochically denoted by the mathematical sign. That is, by participating in activity with internal structure such as shown in Figure 1, the appropriate structures of consciousness that accompanies and is prerequisite of the activity also develop. This perspective has consequences in the way we understand mental activity and its development:

> If the forms of mental activity—concepts in particular—are regarded as the idealization of certain modes of concrete activity, and if in the products of activity one finds the conditions of their social actualization, that determine the future behavior of man, then such attitudes lead inevitably to discarding of the naturalist conception of acquisition and, in the final analysis, to overcoming of passive sensualism, conceptualism, and associationism. Thereby it also becomes clear that the absolutization of formal generalization is unjustifiable. An alternative to it is the generalization of a *contentful* character. (p. 180)

The development of procedural skills and abstraction from concrete material action is not achieved by leaving behind the concrete and enter the world of the ideal. Rather, the ideal and material are relevant to the development and enactment of higher order concepts because they are *never* separated in the first place but exist in and as of a continual exchange between the continually developing processes of thinking and, at a minimum, speaking (Vygotsky, 1986). Our more recent work shows that not only thinking and speaking but more so thinking and communicating generally are in a continually developing dialectical relationship denoted by the term "meaning."

I now briefly address the conclusion that the *Advisory Panel* arrived at, that is, that the utility of a Vygotskian sociocultural perspective remains to be "scientifically" tested. There is nothing in the *Final Report* that would provide evidence for stating that the *Advisory Panel* was aware of the research on Vygotskian curricula in mathematics education and the extensive amount of "scientific" research that has been conducted. There is nothing that would support the claim that the *Advisory Panel* actually understood Vygotsky's cultural-historical activity theory or that it has informed the panel in making sense of mathematical learning and the design of appropriate curriculum. As pointed out here, given the way in which cultural-historical activity theory articulates consciousness and the material world, it makes no sense to make a distinction between the material (also the objectivity of social relations) and the ideal (mental), the inside/interior and the outside/exterior of the subject. All human activity is understood as a production that is mediated by the social and material relations that embed and contextualize the individual subject, who, in its production, also reproduces the activity itself. Performance is never understood independent of the material, social, and contextual particulars—as illustrated in the notion of *conditional learning,* which allows learners to develop efficient and fast algorithms as they reflect upon alternative actions so that component operations come to achieve automaticity and the actions themselves are "demoted" to become conditioned operations.

Sociality, Motivation, and Emotion: More than (External) Influences

In the previous section, I raise doubt about the *Advisory Panel*'s understanding of Vygotsky's (1986) cultural-historical approach to cognition. This is especially highlighted in the manner in which the *Final Report* separates mathematical cognition from sociality, motivation, and emotion (affect). I write separate, because the *Final Report* denotes sociality, motivation, and affect as factors; and factors are never more than *external* aspects of a phenomenon (Hegel, 1806/1977). They cannot enter the dynamic of the system itself or the internal dynamic of the concept at work. Vygotsky operated differently, theorizing systems as a irreducible wholes, where the "factors" are thought of as mutually constitutive and subordinate moments. Not only has Vygotsky advocated unit analysis over analyses that reduce complex systems into elements that are said to constitute the system, which can be reconstructed by a combination of these elements, which do not loose their basic properties in the process. Whereas Vygotsky was mostly concerned with the role of the word in communication, leading him to identify word meaning as the continually developing process that

mediates the exchange between thinking and speaking, those following him articulated the unit as activity. None of the structures identifiable within an activity—subject and object of production, means of production (tools, instruments), recipient of production (community), and the reigning forms of sociality (division of labor)—can be understood on its own but only as mediated by activity as a whole and by the other moments specifically. Consciousness, realized in the developing relation of developing speaking and thinking, always "is practical consciousness-for others and, consequently, consciousness-for-myself" (p. 256). Thus, consciousness, as the word, is nothing individual, subjective, and singular but always and already "becomes a reality for two" (p. 256). Sociality of human beings is not an external influence enhancing or decreasing the efficiency of thinking and learning. Rather, it is the very condition for thought. In speaking, even the uninstructed student is and expresses sociality that makes anything like a lesson possible in the first place. In speaking, gesturing, using symbols, whether with others or for oneself in private, the inherently societal nature of thinking expresses itself. It is not that students express *their* thinking in words and symbols, but the very fact of using words and symbols is an expression of the *sociality* of thought. Mind is as much in society as society is in the mind: Mind and society are irreducibly the same (Bakhtine/Volochinov, 1977; Vygotsky, 1978). Recent neuroscientific research nicely supports what early on was but a hypothesis and presupposition. Thus, any action makes sense because its production requires recognition as an action of others, and the recognition of an action on the part of others requires competency in executing the action (Gallese, 2003). This includes such phenomena as empathy, which is the largely human capacity to feel what another person is feeling.

Recent neuroscientific work shows that affect generally and emotion specifically, too, cannot be separated from cognition: Affect is the very condition for thought (Damasio, 2000). In considering affect as an influence rather than as something constitutive of thinking, the *Advisory Panel* exhibits a limited understanding and conceptualization of thinking generally and enculturation more specifically. More importantly, the *Advisory Panel* excludes for itself the very possibility to understand the nature of thinking, which is based on the close interrelation of thought and affect. Their separation has been, as Vygotsky (1986) noted, "a major weakness of traditional psychology" (p. 10). Vygotsky wrote this in the early 1930s, and there is little that appears to have been learned since; and the hypothesis lies near that the separation of thought and affect in the design of mathematics curriculum is one of the great hindrances to make mathematics generally and algebra more specifically core competencies that students *want to* and do develop. The separation of the study of affect and thought is a weakness, "since it makes the thought process appear as an autonomous flow of 'thoughts thinking themselves,' segregated from the fullness of life, from the personal needs and interests, the inclinations and impulses, of the thinker" (p. 10). That is, affect is the regulating element that underlies the motive of activity, which orients all forms of activity, and which is regulated by the emotional volitional moments of being (Bakhtin, 1993).

With the role of collective motives as orienting moments of activity, and the emotional-volitional moment that serves in the evaluation of actions that realize the activity in concrete form, there is no longer a need for the concept of motivation—which, in any case, is a pseudo-scientific concept because nothing more than an everyday concept that has been operationalized in the service of a bourgeois psychology serving the interests of an oppressive class (Holzkamp-Osterkamp, 1976). Especially the research in a Western offshoot of cultural-historical activity theory, the Berlin school of social psychology denoted by the name *Critical Psychology*, exhibited the weaknesses of the traditional motivation concept, which they showed to be an instrument in the hands of capitalists and teachers to make others do what they do not normally want or intend to do. When conceptualized within cultural-historical activity theory, however, motivation simply is an expression

of the fact that the acting subject has taken up the collective motive and, to expand his or her action possibilities and room to maneuver, engages in expansive learning. Being able to have more control over one's life conditions and being able to do more *inherently* belong to the motive of activity and therefore do not have to be theorized as outside factors that impact and affect cognition either positively (as belief in one's ability) or negatively (leading to performance-decreasing anxiety). That is, in the cultural-historical activity theoretic model, the emotional and motivational "factors" of mathematical achievement are already inherent and therefore have to be taken into account for to understand all levels of performance. The issue, for me, therefore is not the "development of promising interventions for reducing serious mathematics anxiety" (p. 31) but a change in the very activity that leads to the ontogenetic development of mathematical practice.

In thinking about and asking for the development of interventions that "reduce serious mathematics anxiety," the *Advisory Panel* does what critical psychologists have articulated as the goal of traditional psychology, which is but a means of the bourgeoisie in the control of the working class, its productive means and exploited resource. Thus, rather than considering changing school conditions, task forms, and other aspects of schooling, the *Advisory Panel* recommends the development of interventions that constitute outside influences and forces on the individual rather than creating conditions where the anxiety problem does not even surface. Again, the approach is one of fixing the symptoms, here mathematics anxiety, rather than dealing with the causes of a phenomenon. That the *Advisory Panel* has interests of the type indicated, that is, the modification of others to serve their and their employers goals also can be seen in the way the *Advisory Panel* writes about changing the beliefs of others.

The problem in the mistaken approach to questions of affect in the *Final Report* derives, so Vygotsky, from the failure to recognize it as an internal moment of activity generally and thought particularly. Thus, only when thought and affect are theorized as internally connected can we understand the *mutually constitutive* nature and mediation of thought and affect. "The old approach precludes any fruitful study of the reverse process, the influence of thought on affect and volition" and, thereby, the "door is closed on the issue of the causation and origin of our thoughts, since deterministic analysis would require clarification of the motive forces that direct thought into this or that channel" (p. 10).

Coda

In my reading, the *Final Report* constitutes the worst imaginable scenario, one that plays into the hands of simplistic-slogan-using politicians interested more in election and reelection than in the change of existing, inequitable societal conditions that are produced and reproduced by current forms of schooling. The *Final Report* constitutes a piecemeal approach, presenting what we know about knowing and learning in mathematics in a highly selective and disconnected manner, therefore realizing itself as nothing more than selected "knowledge in pieces." In its excessive, president G. W. Bush-rhetoric-supporting and evidence-eliminating discourse on the nature of true scientific research, the *Advisory Panel* fails to provide directions for truly democratic and democracy-enhancing ways of changing mathematics education. Instead, the *Final Report* plays into the hands of those who advocate *training*—e.g., basic skills, automaticity, rote, and routine learning—over *education*, which constitutes an emancipatory and expansive process of learning that people engage in because it enhances their control and room to maneuver.

A way out of the impasse that the *Final Report* leads us into is one that Vygotsky already recommended for better understanding cognition, knowing, and learning: Unit analysis. This form of analysis is advantageous because

> it demonstrates the existence of a dynamic system of meaning in which the affective and the intellectual unite. It shows that every idea contains a transmuted affective attitude toward the bit of reality to which it refers. It further permits us to trace the path from a person's needs and impulses to the specific direction taken by his thoughts, and the reverse path from his thoughts to his behavior and activity. (pp. 10–11)

In sum, I suggest that the *Final Report* provides us with a rather limited and limiting perspective of what it takes to transform mathematics education. I would say that it is its own utility that ought to be scientifically tested—if it were not for the undesirable consequence that we would actually have to implement its program at least partially to subject it to such tests. From my perspective, the recommendations are more like a curse. I do not wish on anyone the experience and consequences of educational practice that the *Final Report* calls for. As I am writing these lines, I wish this were a *final* report, one never to be repeated, one that we could relegate to oblivion, rotting in the musty cellars of the U.S. Department of Education. I do hope for the nation as a whole and for the students who would be subject to the changes brought about on the basis of the *Final Report*'s recommendation that the winds of change bring about a change not only in the political leadership but also in the national position on education.

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E(race)ing Race from a National Conversation on Mathematics Teaching and Learning: The National Mathematics Advisory Panel as White Institutional Space

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Abstract:
In this paper, I wish to argue that several factors support a characterization of the National Mathematics Advisory Panel as an instantiation of the white institutional space (Moore, 2008) that characterizes mathematics education research and policy contexts more generally. In particular, my analysis highlights that mathematics education research and policy contexts such as the National Mathematics Advisory Panel are not immune to the structural and institutional racism that characterize many other areas of U.S. society.

Keywords: White space; institutional racism; mathematics education; National Mathematics Advisory Panel; policy; racial exclusion

One of the goals of the National Mathematics Advisory Panel—which was commissioned by President George Bush and crisscrossed the country for more than eighteen months—was to foster a national conversation on the teaching and learning of school mathematics. National panels and conversations initiated by a President of the United States are not unprecedented. In 1997, William Clinton launched *One America in the 21st Century: The President's Initiative on Race*. Similar to the National Math Advisory Panel, the Advisory Board on race held a series of public meetings across the country and, by the end of their work, produced a 229-page final report detailing their recommendations. A

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partial summary of the rationale and desired outcomes of the race panel's work is given below:

- The goal of the President's Initiative on Race is to strengthen our shared foundation as Americans so that we can live in an atmosphere of trust and mutual respect.

- The President has asked the Advisory Board to join him in reaching out to local communities and listen to Americans from all different races and backgrounds, so that we can better understand the causes of racial tension.

- The Advisory Board will help foster and participate in constructive dialogues on race that the President has called for in this Initiative. President Clinton has asked the Board to recommend creative ways to resolve these problems with the help and input of the community leaders who are committed to tackling these difficult issues.

- Finally, President Clinton believes that, while thoughtful dialogue will be an enormous benefit, no real progress can be made without specific actions aimed at breaking down the walls that surround the issue of race relations. The Advisory Board will study critical substantive areas in which racial disparities are significant, including *education*, economic opportunity, housing, health care and the administration of justice. (italics added)

By way of comparison, the rationale, goals, and desired outcomes of the National Math Advisory Panel—taken from the U.S. Department of Education's official website documenting the Panel's work and progress—are summarized below:

- The National Mathematics Advisory Panel is part of the President's plan to strengthen math education so that America's students receive the tools and skills necessary for success in the 21st century.

- Based on the influential National Reading Panel, the math advisory board will convene experts to evaluate the effectiveness of various approaches to teaching math and in so doing, create a research base to improve instructional methods for teachers.

- The Panel was charged with providing recommendations to the President and U.S. Secretary of Education Margaret Spellings on the best use of scientifically based research to advance the teaching and learning of mathematics.

- Expert panelists, including a number of leading mathematicians, cognitive psychologists, and educators, reviewed numerous research studies before preparing a final report containing guidance on how to improve mathematics achievement for all students in the United States.

Reading through the history and details of how these two panels came into existence (e.g., Domestic Policy Council, 2006; The White House, 2004), there are several interesting similarities. The underlying concerns—race relations and mathematics education—were both characterized as being in a state of crisis at the time that their respective panels and conversations were started. Progress in both of these areas was deemed a necessary precursor to the well-being and growth of American society. Moreover, both initiatives were backed by the authority of a United States

President, offering strong confirmation that race relations and mathematics education are heavily steeped in the sociopolitical fabric of our society and our everyday lives.

It is noteworthy that while the Advisory Board on race correctly highlighted *education* as a context where racial disparities exist and, simultaneously, as a context where race relations could be improved, *race* is conspicuously absent in the National Mathematics Advisory Panel's final report despite a review of 16,000 research publications and policy reports and testimony from 110 individuals. In fact, the word *race* appears a total of three times, without definition or conceptual development, in the final narrative. In one instance, the same sentence is repeated. In the third instance, there is only a parenthetical reference to race along with gender and language.

The absence of race also occurs despite the claim that the final report would provide policy recommendations on how to improve mathematics achievement for *all* students. This absence clearly ignores the findings of the earlier panel on race as well as a large body of research supporting the fact that schools are highly racialized spaces (e.g., Anderson, 2004; Barajas & Ronnkvist, 2007; DeCuir & Dixson, 2004; Flores-Gonzalez, 2002; Ginwright, 2005; Ladson-Billings & Tate, 1995; Lee, 2005; Lewis, 2003a, 2003b, 2004; Lipman, 1998; Noguera, 2003; Pollock, 2004a, 2004b; Tate, 1994, 1995; Woodson, 1990).

In effect, the panel on race confirmed that teaching and learning within school contexts are not immune to considerations of race and racism while the National Math Panel appeared to minimize this reality. In fact, I would argue that the panel on race offered strong confirmation that schools and their very organizational structures are not race-neutral. Rather, the institutional practices and norms within schools contribute deeply to the sociopolitical construction of race, racial hierarchies, and racial inequality in the larger society (e.g., Barajas and Ronnkvist, 2007; Martin, in press-b).

Although the omission of race—which I will refer to as *e(race)sure*—from the final report of the National Mathematics Advisory Panel is very apparent, it is not surprising. More generally, the omission of race, on one hand, and its conceptual underdevelopment, on the other, are epidemic to mainstream mathematics education research and policy discussions (Martin, in press-b).[2] As a result, both race and racism have remained non-central considerations in mainstream discussions of mathematics teaching and learning despite a growing literature documenting that mathematics teaching and learning are *racialized experiences* for *all* students (e.g., Berry, 2003, 2005; Martin, 2000, 2006a, 2006b, in press-a, in press-b; Moody, 2001; Nasir, 2002; Nasir, Heimlich, & Atukpawu, 2007; Oakes, 1985, 1990; Powell, 2002; Powell & Frankenstein, 1997; Reyes & Stanic, 1988; Spencer, 2006; Stinson, 2004; Tate, 1994, 1995).

In this paper, I wish to argue that several factors support a characterization of the National Math Advisory Panel as an instantiation of the *white institutional space* (Moore, 2008) that characterizes mathematics education research and policy contexts more generally. These factors include (a) the increased politicization of mathematics education for workforce needs (Committee on Science, Engineering, & Public Policy, 2007; Domestic Policy Council, 2006; National Research Council, 1989; National Sciences Board, 2003, 2004; U.S. Department of Education, 1997, 2006) and assimilation purposes (e.g., *Mathematics for All* and *Algebra for All*) and (b) structural and ideological barriers to centering race in both mathematics education research and policy conversations.

The term *white institutional space* comes from the work of sociologists Joe Feagin (1996) and Wendy Moore, who, in her book *Reproducing Racism: White Space, Elite Law Schools, and Racial Inequality* (2008), examined the white space of law schools and how the ideologies and practices in these schools serve to privilege white perspectives, white ideological frames, white power, and white dominance all the while purporting to represent law as neutral and objective.

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[2] Race has received slightly more attention among international math education scholars but even in the international context, white scholars dominate.

Based on her analysis, Moore (2008, p. 27) claimed that the historical development of law schools as white institutional space is characterized by four foundational elements: (1) racist exclusion of people of color from elite law schools and positions of power in legal institutions which results in the accumulation of white economic and political power, (2) the development of a white frame that organizes the logic of these institutions and normalizes white racial superiority, (3) the historical construction of a curricular model based on the thinking of white elites, and (4) the assertion of law as a neutral and impartial body of doctrine unconnected to power relations.

In Martin (in press-b), I characterize the broad contexts of mathematics education research and policy as examples of white institutional space. I liken the white institutional space of mathematics education research and policy to a large projection screen. Although up-close inspection of the screen reveals bits and pieces of color, it is the overall whiteness of the screen that is most apparent and serves as its defining characteristic. In addition, the screen appears neutral yet it has the power to project "meanings and symbols that are associated with the dominant culture, thus reproducing an ideological framework that rationalizes and reproduces structures of inequality" (Moore, 2008, p. 17). More practically, one can also understand mathematics education as white institutional space by considering who is allowed to speak on issues of teaching, learning, curriculum, and assessment and who dominates positions of power in research and policy contexts. In each instance, white scholars disproportionately fill these roles, an important signifier of white institutional space (Moore, 2008).

What is highlighted by this reality is that mathematics education research and policy contexts are not immune to the structural and institutional racism that characterizes many other areas of U.S. society. Structural and institutional racism exist above and beyond the good intentions of individual whites who may advocate for issues of equity. Consideration of structural and institutional racism in mathematics education also helps to avoid the tendency to reduce racism to the level of individuals and individual psychology (Bonilla-Silva, 1997, 2001, 2003, 2005; Moore, 2008).

Within a structural and institutional analysis, it is revealed that even well-intentioned individual whites benefit from the historically contingent constructions of race and racial groups. These constructions serve to produce and reproduce racial boundaries circumscribing *whiteness*, *non-whiteness*, and *blackness* and to create a racial hierarchy that rewards those who are socially constructed as white, that oppresses those who are constructed as black, and that selectively gives honorary white status and privilege to some non-whites (Bonilla-Silva & Glover, 2004). Structural and institutional perpetuation of whiteness and white privilege (McIntosh, 1989) also help to maintain white institutional space. Barajas and Ronnkvist (2007), expanding on the work of Doane (1997) and Lewis (2003), stated that "whiteness is a mechanism of power that allows dominant group ideologies surrounding race to be imposed on other groups, often in subtle ways" (p. 1520).

Moore's (2008) four foundational elements and a consideration of white privilege can be used to analyze the National Mathematics Advisory Panel as an instantiation of the white institutional space of mathematics education research and policy. I argue that the norms characterizing this space—including the almost exclusively white construction of the standards, values, and ideological frameworks that organize mathematics education research and policy—help to explain why the National Mathematics Advisory Panel failed to include race-based considerations in its final report, why no African American, Latino, or Native American mathematics education *researchers* were members of the Panel, and why the Panel failed to actively seek out the opinions of African American mathematics education scholars to provide comments and feedback on relevant topics.

Racial Exclusion[3]

In terms of demographics and racial exclusion, it is notable that only two of the nineteen members of the National Math Panel members, Dr. Wade Boykin and Mr. Vern Williams, are African American. Only Mr. Williams is a mathematics educator and neither Dr. Boykin nor Mr. Williams is a mathematics education researcher. On a panel whose formation was premised on using *research* to make policy recommendations for all students, the research perspectives of African American math education scholars was essentially nullified and they had no voice in setting the direction of the Panel's recommendations. Moreover, based on the Panel's own accounting, the list of experts called on to examine materials, offer opinions on specialized topics, and examine drafts of sections of the final reports included no African American math education scholars.

Although it might be argued that Dr. Boykin and Mr. Williams' African American racial identity and their respective professional backgrounds are sufficient to insure that the perspectives of African American math education scholars would be represented, I would disagree. In the politics and practice of racial exclusion, it is not uncommon for those in power to selectively choose which non-whites will be allowed to speak, particularly when the views of those chosen offer no challenge to the status quo. In an analysis of policy contexts such as the National Math Panel, it is important to consider who is allowed to speak, who gets left out, and why. Significantly, given the absence of African American mathematics education scholars from the Math Panel, there were no loud calls from white mathematics education scholars in positions of power on or off the Panel to rectify this exclusion even if those calls would have amounted to symbolic solidarity gestures.

In addition to the Panel membership, analysis of the final report indicates that, among the studies cited, none were authored by African American math education scholars. Among the reports produced by the task groups on conceptual knowledge and skills, learning processes, instructional practices, teachers, and assessment, my analysis showed that research produced by African American math education scholars was cited only a handful of times.

The absence of African American math education scholars in such a key discussion of mathematics teaching and learning is unacceptable and only serves to preserve and protect the privileged status of white scholars and white perspectives. As a result of this exclusion, the knowledge base in mainstream mathematics education remains largely uninformed by African American perspectives and insights and, thus, limited in its explanatory power with respect to African American children's mathematical experiences and development.

Those who disagree with my analysis would suggest that African American perspectives had equal opportunity to be represented in the various public forums held by the Panel as it moved across the country. My response would be that responding from the audience and speaking from the position of power represented by Panel membership are very different. Moreover, it is very unlikely that the perspectives of speakers in the audience would have carried much weight given that the Panel did not even solicit the perspectives of African American scholars in its requests for feedback.

Those who disagree might also suggest, correctly, that not all African American mathematics education scholars produce research or advocate for policy that is in the best interests of African Americans while some white scholars, in fact, do advocate for African Americans. However, the numerical dominance of white scholars and the subsequent inattention to race, for example, in the vast majority of mainstream mathematics education research and policy serve as evidence that these white scholars are a minority within a much larger majority. While I agree that the voices of critical white scholars are important, they are no substitute for the authentic voices of critical African

[3] In this section, I devote the majority of my discussion to the exclusion of African American mathematics education researchers.

American scholars. The dominant presence of white scholars, whatever their orientation, only insures that white perspectives become the *only* perspectives that matter.

Development of a White Frame: E(race)ing Race from the Conversation

To illustrate how the work of the Math Panel perpetuates the development of a white frame that organizes the logic of mathematics education research and policy, one can examine if, and how, the concept of *race* and the reality of racism were invoked as considerations in mathematics teaching and learning. Traditionally, mathematics learning has been conceptualized as a cognitive (e.g., Schoenfeld, 1987) or sociocultural activity (e.g., Saxe, 1988). However, rarely has it been conceptualized as a *racialized* form of experience (e.g. Martin, 2006a, 2006b). Within mainstream mathematics education research and policy contexts, race has typically been invoked only as a categorical variable used to disaggregate data and to rank students in a racial hierarchy of mathematics ability (Martin, in press-a, in press-b). Racism is rarely invoked.

Consistent with its omission from the final report, the word *race* is also absent in the reports produced by the task groups on conceptual knowledge and skills, learning processes, instructional practices, teachers, and assessment. The word race was not referenced a single time within the text of the reports on conceptual knowledge and teaching. When it was invoked in the other reports, it was only done so for the purpose of reifying the notion of a racial achievement gap in mathematics achievement.

A clear sign that race would not be considered, or taken seriously, in the work of the National Math Panel was the continued reference to *best available scientific evidence*. This phrase and the word *scientific* appear more than two dozen times in the report. As stated in the report's narrative:

> The Panel's strongest confidence will be reserved for studies that test hypotheses, that meet the highest methodological standards (internal validity), and that have been replicated with diverse samples of students under conditions that warrant generalization (external validity)….The final category corresponds to statements based on values or weak evidence; these are essentially unfounded claims and will be designated as opinions as opposed to scientifically justified conclusions…. All of the applicable high quality studies support a conclusion (statistically significant individual effects, significant positive mean effect size, or equivalent consistent positive findings) and they include at least three independent studies with different relevant samples and settings or one large high quality multisite study. (p. 82-83)

The imposition of these standards essentially eliminated a host of qualitative, ethnographic, case-study, and descriptive studies that are commonly used to examine the experiences of students of color in school settings, including their experiences with race and racism. These criteria also minimize the importance of studies that situate schools in their larger sociopolitical contexts.

As an example of the kind of research that *was* given value and which received support for future research involving African American and Latino students, the Panel highlighted issues of motivation, task engagement, and self-efficacy. Although important, these areas focus attention on students as though they are unmotivated, inclined to disengagement, and lacking in agency. No mention was made of the institutional and structural barriers inside and outside of school, including racism, that affect student mathematics achievement, engagement, and motivation.

Construction of a Curricular Model by White Elites

As indicated in the final report of the Panel (U.S. Department of Education, 2008), "While the presidential charge contains many explicit elements, there is a clear emphasis on the preparation of students for entry into, and success in, Algebra" (p. xv). In fact, an entire chapter of the final report is devoted to defining the curricular space called *school algebra*, including a specific list of topics to be covered and a specification of the topics and essential skills that should be covered and learned in the years leading up to algebra. Subsequent chapters discuss learning processes, teaching and teachers, instructional practices, instructional materials, and assessment. I argue that the choice of algebra and the specifications in subsequent chapters amounts to a construction of a curricular model on behalf white elites who will benefit most from the larger goal of U.S. international competitiveness. Viewed in this light, the choice of algebra as a critical area of focus is not politically neutral. Yet, several questions can be raised about this choice. Why algebra? Who decides? Algebra for whom and for what not-so-apparent purposes? Whose interests are served by these choices? Whose interests are not served?

Mathematics Education as Politically Neutral

Moore (2008) suggested that the assertion of law as a neutral and impartial body of doctrine unconnected to power relations is a foundational element contributing to the historical development of law schools as white institutional spaces. Similarly, the minimization of race is indicative of the ways that mathematics education and policy are positioned as neutral in the production of racial disparities and other social inequalities (Apple, 1992; Diversity in Mathematics Education, 2007; Martin, 2007, in press-b). Rather, a color-blind approach to inclusion symbolized by *Mathematics for All* rhetoric, for example, implicitly promotes cultural assimilation and the maintenance of white privilege (i.e., white student achievement, U.S. national interests and competitiveness). Indeed, the rhetoric of the final report of the National Mathematics Advisory Panel indicates that workforce considerations dominate the rationales for students to learn mathematics and for teachers to teach it effectively. According to the report (U.S. Department of Education, 2008):

> During most of the 20th century, the United States possessed peerless mathematical prowess—not just as measured by the depth and number of the mathematical specialists who practiced here but also by the scale and quality of its engineering, science, and financial leadership, and even by the extent of mathematical education in its broad population. But without substantial and sustained changes to its educational system, the United States will relinquish its leadership in the 21st century. This report is about actions that must be taken to strengthen the American people in this central area of learning. Success matters to the nation at large. It matters, too, to individual students and their families, because it opens doors and creates opportunities. Much of the commentary on mathematics and science in the United States focuses on national economic competitiveness and the economic well-being of citizens and enterprises. There is reason enough for concern about these matters, but it is yet more fundamental to recognize that the safety of the nation and the quality of life—not just the prosperity of the nation—are at issue. (p. 1)

The unequivocal advancement of workforce needs and national competitiveness necessarily takes mathematics education out of the sheep's clothing of being politically neutral. As stated by McLaren (2004):

> Clearly, there are vested interests for certain kinds of mathematical knowledges that provide ballast for regnant regimes of capitalist exploitation that give rise to asymmetrical relations of

> power and privilege centered around race, class, and gender relations and affiliations, and hat give license to pursue narrow neoliberal approaches to educational policy and pedagogy…[S]chool mathematical knowledges generated across various institutional bases have become functionally advantageous for particular modes of governmentality and social control. (p. xiii-xiv)

Not discussed by the National Math Panel is what Dowling (1998) and Valero (2004) have called the *myth of participation*, the "conviction that people are handicapped to participate in society if they do not understand and are not able to use mathematics in a critical way" (Valero, p.8). Assimilation-oriented calls for all students, including underrepresented minorities, to participate more fully in mathematics ignores the fact that the nation does not have the capacity, or moral commitment, to absorb all of those who would be trained in mathematics and science. Simple supply and demand would dictate that the overproduction of engineers and scientists would lead to declining wages and standards of living and would put downward pressure on those at the lower rungs of the labor market, creating an even wider gulf between those with high levels of education and those without.

The blind and uncritical advocacy of mathematics education by members of the Panel also provides evidence for Valero's (2004) claim that:

> The unquestioned intrinsic goodness of both mathematics and mathematics education represent the core of its 'political' value: If students and citizens come to learn a considerable amount of mathematics properly, they will be become per se better people and better citizens; that is, mathematics and its education empower or have the capacity of giving power to people…The problem with this kind of assumption is that there is no necessity for a further examination neither of mathematics as a knowledge and of mathematics education as practices, nor of power….Is it possible to assume that mathematics is a knowledge associated exclusively with progress and the well being of humanity? Or do we need to consider the involvement of that knowledge in the creations of both wonders and horrors in our current technological society? (p. 13-15)

Clearly, mathematics education is not politically neutral. Yet, in the context of white institutional space, we are asked to believe that it is so. It appears that the Panel wishes us to believe that because they sought out the best scientific evidence, their advocacy is based on fact and not values or opinion. However, a critical analysis shows that mathematics education, as framed by the Panel in terms of what is important and for what purposes, is part of a larger political agenda focused on national interests in a global world. The question left unanswered by the Panel is who, among us, beyond the white elite will *really* benefit from this agenda?

Conclusion

My goal in this paper was simple. I set out to demonstrate that the National Mathematics Advisory Panel is an instantiation of the white institutional space of mathematics education research and policy. This space is characterized by power differentials, racial exclusion, and the perpetuation of white privilege above and beyond the good intentions of individual white scholars. In particular, I claimed that these forces help explain why there were no African American mathematics education scholars on the National Math Panel, why no research produced by African American scholars was cited in the final report despite key recommendations for future research involving African American children, and why race was essentially e(race)d from the final report.

Perhaps the most interesting aspect of the Panel's insistence on the best available scientific evidence as the preferred criteria for considering some research and dismissing others because it is

based on values and opinion is how this stance diminishes the importance of the Panel's own report. In choosing algebra as the primary curricular focus and emphasizing workforce preparation and national competitiveness, the Panel revealed its own values by indicating what mathematics they considered most important and what purposes mathematics education should serve. In fact, the Panel used the entire report to support those values and invoked its own criteria to eliminate perspectives and research that did not support those values.

The imposition of white elite ideology dictating what mathematics is important and for what purposes as well as the imposition of standards of evidence that silence and minimize research that does not support these ideologies are key factors that help sustain white institutional space. Continued work and interrogation is needed to dismantle this space.

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Three Strikes

***Tom O'Brien*[1] *and Marianne Smith*[2]**

Abstract
The first part of the paper describes and comments upon three aspects of the back-to-basics movement: the make-up and mindset of the National Mathematics Advisory Panel (NMP), the movement's history in California, and recent "grassroots" activities of the movement in the state of Washington. The second part reports and comments on the principal findings of the NMP report.

Keywords: Mathematics education reform; Back to basics movement; California math reform; the National Mathematics Advisory Panel (NMP); policy

The first part of this article was written in advance of the March 2008 release of the final report of the National Math Panel. The observations provide a context for viewing the Panel's work and the various responses to the final report.

In recent years there have been at least three concerted assaults against a problem-solving oriented approach to mathematics education, undermining the *Principles and Standards of School Mathematics* (PSSM) of the National Council of Teachers of Mathematics and the mathematics education programs and initiatives of the National Science Foundation.

Strike 1. The National Mathematics Advisory Panel (NMP) established by the U.S. Department of Education in Spring 2006 is expected to deliver its final report on the present and future of mathematics education in our nation in Spring 2008.

Media coverage of the Panel has ranged from The Washington Post to The New York Times, and these news articles have been posted on Internet web sites, along with a variety of blog commentaries.

The intent of the report is to offer definitive advice leading to competence in algebra and readiness for higher levels of mathematics in terms of "conceptual knowledge and skills, learning processes, instructional ,practices, teachers and assessment." The final report will contain recommendations and "take-away" messages.

From the draft version of the document available to the public in early December 2007, it

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appears that everything is wrong with current mathematics education in the US, and much must be done because we are losing the achievement test score competition to countries in Asia and Europe.

The findings of the Panel are likely to emphasize the importance of learning the "classic" algorithms and procedures of arithmetic and algebra. The assumption is almost certain to be that a back-to-basics approach will lead to improvement. Given the writings and presentations of Panel members, there is likely to be little support for the use of technology in the learning process, save the recommendation that software can be helpful in drill and practice to master procedural skills. There will be a subtle disparagement of the value of using "real world" problem contexts, as well as a skeptical view of the value of integrated high school mathematics curricula.

In considering the mindset of many Panel members, it is worth bearing in mind that:

1. In 1991, almost a generation ago, a study by Educational Testing Service showed that 88.5 per cent of American thirteen year olds owned a calculator. [Educational Testing Service, International Assessment of Educational Progress, Report 13, Background Questions and Proficiency Scores for all Populations, 1992.]

2. The graphics calculator was invented more than two decades ago—1985. [http://www.math.ohio-state.edu/~waitsb/inventor.html]

3. Present-day computers can perform about 500 trillion calculations in one second. [http://www.top500.org/lists/2007/11/performance_development]

The panel is likely to base its findings on what they call "legitimate" research, which is narrowly defined to include only those "scientific" studies conducted using a clinical trial method of random assignment of treatments over experimental groups and control groups. Many studies reviewed by the National Research Council over the past 10 years have shown what is known and regarded as essential in mathematics teaching and learning; it is a good bet that the Panel will largely choose to ignore or to cherry pick among these studies. [See National Research Council. (2001). Adding it up: Helping children learn mathematics. http://www.nap.edu/openbook.php?isbn=0309069955]

The panel (13 men, 6 women) consists largely of partisan appointees from the back-to-basics camp, many of whom have no background in school mathematics, limited and/or no experience in teaching children, or both. Thus it is a reasonable guess that they have little understanding of the spirit of NCTM or NSF priorities, not to say anything of those of the classroom teacher.

When the panel was first appointed in 2006, Education Week predicted,"The panelists' backgrounds suggest they will favor a particular approach to teaching math—generally speaking, one that stresses the need for drill and practice in basic computation at early grade levels, at the expense of problem solving." (Cavanagh, 2006). ["Some Worry About Potential Bias on the National Math Panel," EdWeek, May 19, 2006, By Sean Cavanagh] "Similar charges of bias dogged the National Reading Panel, formed in 1997, which administration officials claimed as a model for the math group.," (Ed Week, 2006). " ("White House

Suggests Model Used in Reading To Elevate Math Skills," Ed Week, Feb. 15, 2006.) The federally championed reading program has had a history of corruption and cronyism. [Justice Dept. Is Asked To Investigate Reading Plan, New York Times, By Diana Jean Schemo, April 21, 2007. [See http://select.nytimes.com/search/restricted/article?res=FA0C16FB3D5A0C728EDDAD0894DF404482]

Yet another question might be raised concerning the selection of panel members: it is not clear that mathematically knowledgeable people from science, industry, technology and the arts were represented. Many members of the panel are academics, the sorts who have intimidated generations of math students. Where were leaders who know about the real world, the economy, the future, applications of mathematics in science and technology? Perhaps most important, where were leaders who know about children and their ways of thinking and learning and mathematizing?

To their credit, the Department of Education has published transcripts of the meetings, including public commentary by non-Panel members. Transcripts are available on the NMP web site. [See http://www.ed.gov/about/bdscomm/list/mathpanel/meetings.html]

Perhaps most important of all, the transcripts contain Public Comment presentations by experts who were not part of the Panel. These provide interesting and revealing news, insights and suggestions can be read in full at the DoE website. The public comments—*not* always reflecting a back-to-basics viewpoint—often shed more light on important issues than the formal transcript of the Panel members' deliberations.

What follows are excerpts from the public's contributions. They often encourage the Panel to consider a mathematics curriculum that is more than a focus on "basic skills."

Patrick Thompson, Professor of Mathematics Education, (Chicago meeting, April 20, 2007), brings challenges to the Panel involving ideas that are likely to be new to them.

> [T]he Panel has the significant task of responding to a list of charges that take "skills" as the primary component of mathematics learning when the notion of skill itself is hardly well defined. Do you take "skill" to mean a child's ability to perform reliably a procedure when told to perform that procedure? Or, do you take "skill" to mean a child's ability to have developed sufficient knowledge and appropriate flexibility of thought to solve most problems of a particular genre of problems, even those that might have subtle and nuanced differences from any the students might have seen?
> For the same reason, one cannot simply look to "research" to answer the question of what policies the nation should follow in preparing students for algebra. Which algebra? Push-button algebra or, as Kaput calls it, the algebra of progressive generalization and symbolization? The two entail different philosophic and intellectual commitments for those who embrace them, and they entail different expectations for students' learning and teachers' knowledge at every grade.

Mathematician Sol Garfunkel pleaded for the Panel's report to be different from what he anticipates.

> My comment to this Panel is don't . . . write the report that we all expect to come out of this Panel, because I think it will set back mathematics education for a number of years. Don't write a report that says there is a lot we don't know . . . and [that] until that research is complete, we should stop innovation in curriculum development, except if we adopt something like the Singapore Program, and that we should cut off funding for that curriculum development, we should cut off funding for the National Science Foundation. I suspect that that's what this report will eventually say and it's a terrible mistake.
> http://mathpanelwatch.blogspot.com/

Mathematics educator Susan Friel addressed the development of mathematical thinking in grades K-5. Her paper included a noteworthy and up-to-date bibliography (Chapel Hill meeting). The bibliography is welcome, especially in jurisdictions where the back-to-basics leaders have marginalized anything connected with the National Council of Teachers of Mathematics.
http://www.ed.gov/about/bdscomm/list/mathpanel/2nd-meeting/presentations/index.html

Classroom teacher Holly Concannon testified about her rescue of a mathematically damaged young girl and then went on to list noteworthy parallel school district successes in the Boston public schools. The curriculum leading to these successes was the reform project (i.e., NSF-supported) TERC's Investigations in Number, Data and Space.

> I am proud to share with you today the gains we have made in math. In 1999, our school had devastating results on the statewide tests. Fifty-four percent 54% of our students landed in the warning category. Six years later, we have just 9% in that same category. The number of students achieving advanced or proficient rose 32 percentage points. These statistics have given the Murphy [School] great reason to celebrate. However, we are not the only school worthy of the celebration. The Boston Public School District as a whole is making progress. Early this year, we made national headlines for having the greatest gains in our NAEP scores, among 11 major urban districts. [Cambridge meeting]
> http://www.ed.gov/about/bdscomm/list/mathpanel/3rd-meeting/presentations/concannon.holly.pdf

From Randy Harter, Mathematics Specialist, Buncombe County Schools, Asheville, North Carolina.

> In 2001, the Mathematics Learning Study Committee stated in Adding It Up that "Mathematics learning has often been more a matter of memorizing than of understanding." My concern is that our longstanding traditions and culturally based instructional practices and the unbalanced emphasis on mathematics as procedures in most K-8 classrooms in this country have inhibited the development of reasoning and problem solving. For most students that come through this system, the result has been that mathematics is merely a set of procedures. A significant study by Jo Boaler, now at Stanford, came to a similar conclusion for students in England. She said, "Students thought that success in mathematics involved learning, rehearsing, and memorizing standard rules and procedures. … They did not regard mathematics to be a thinking

> subject." One student's comment was typical, "In maths you have to remember, in other subjects, you can think about it."
> [Chapel Hill meeting] [See Note 1.]
> http://www.ed.gov/about/bdscomm/list/mathpanel/2nd-meeting/presentations/index.html)

Strike 2. The origins of the most recent swing of the pendulum toward back-to-basics in mathematics education has been documented in books and articles about the trajectory of education policy in the state of California during the 1990s.

By1999, changes were made in the state's mathematics framework and academic standards, in teacher professional development programs, as well as to textbook adoption guidelines. The changes were made under very controversial circumstances. [See Fraser's Panel testimony below.] The rigidity of these changes significantly affected professional development providers, who must sign a loyalty oath— an agreement to follow the back-to-basics California state standards.

The California back-to-basics movement has since metastasized through the years to Massachusetts, New York, Missouri, Washington, New Jersey and Utah.

Despite claims made by California back-to-basics leadership that their Standards are "world-class" and "rigorous," 2007 data show that only 23% of California students are proficient in Algebra I by the end of high school and NAEP data showed 30 and 24 per cent of pupils proficient at grades 4 and 8 respectively. The California grade 4 NAEP results were higher than only 1 of 52 states and other jurisdictions and the grade 8 results were higher than only 4 of 52 jurisdictions. [http://www.cde.ca.gov/ta/tg/nr/caresults.asp]

Testimony by Sherry Fraser at the Palo Alto meeting on November 6, 2006 shed light on many unacknowledged aspects of the decade-long back-to-basics movement in California.

> … It was at the end of that decade that the National Council of Teachers of Mathematics released their Curriculum and Evaluation Standards for School Mathematics (1989). Contrary to what you hear today, they were widely accepted and endorsed. This set of standards had the potential to help the American mathematics educational community begin to address the problems articulated throughout the 1980's.
>
> Shortly after publication, the National Science Foundation began funding the development of large scale, multi-grade instructional materials in mathematics to support the realization of the NCTM Standards in the classroom. Thirteen projects were funded. Each of the projects included updates in content and in the context in which mathematics topics are presented. Each also affected the role of the teacher. Each has been through rigorous development that included design, piloting, redesign, field-testing, redesign, and publication. This amount of careful development and evaluation is rarely seen in textbook production.
>
> … These NSF projects were developed to address the crisis in mathematics education. They did not cause the problem; they were the solution to the problem. Their focus went

beyond memorizing basic skills to include thinking and reasoning mathematically.

… These model curriculum programs show potential for improving school mathematics education. When implemented as intended, research has shown a different picture of mathematics education to be more effective.

…A study by the American Association for the Advancement of Science (AAAS) evaluated 24 algebra textbooks for the potential to help students understand algebra and …, the NSF-funded curriculum programs rated at the top of the list. And in 2004 the National Academy of Sciences released a book, On Evaluating Curricular Effectiveness: Judging the Quality of K-12 Mathematics Programs, which looked at the evaluation studies for the thirteen NSF projects and six commercial textbooks. Based on the 147 research studies accepted it is quite clear which curriculum programs have promise to improve mathematics education in our country. They are the NSF-funded curriculum projects.

You might be asking yourself why hasn't mathematics education improved if we have all this promising data from these promising programs? Let me use California as an example.

In 1997 California was developing a set of mathematics standards for K-12. A State Board member hijacked the process. She gave the standards, which had been developed through a public process, to a group of four mathematicians to fix. She wanted California's standards to address just content and content that was easily measurable by multiple-choice exams. The NCTM standards, which the original California standards were based on, were banned and a new set of California standards was adopted instead. This new set punished students who were in secondary integrated programs and called for Algebra 1 for all 8th grade students, even though the rest of the world, including Singapore, teaches an integrated curriculum in 8th grade and throughout high school. The four mathematicians and a few others called California's standards "world class". But saying something is world class doesn't make it so. In fact, we now have data to show these standards haven't improved mathematics education at all. Most of California's students have had all of their instruction based on these standards since they were adopted almost ten years ago. Yet, if you go to the California Department of Education's website on testing and look at the 2006 data you will find that only 23% of students are proficient in Algebra I by the end of high school, a gain of 2 points over four years….

… Why, then, do you read in newspapers about how terrible the mathematics programs developed in the 1990's are and how successful California is? It has to do with an organization called Mathematically Correct, whose membership and funding is secret. Their goal is to have schools, districts, and states adopt the California standards and they recommend Saxon materials as the answer to today's problems. They are radicals, out of the mainstream, who use fear to get their way. [See http://www.ed.gov/about/bdscomm/list/mathpanel/4th-meeting/presentations/sherry-fraser.pdf]

Strike 3. In October 2006, two of the California back-to-basics leaders conducted a

marathon of presentations in Seattle, Washington. Eyewitness reports say that they called for the dismissal of everyone who has been engaged in supporting Washington's math Standards and the WASL (state tests) from the Superintendent of Public Instruction Terry Bergeson on down.

Their other suggestions included replacing the Standards with California's "world class" standards; purge the state schools of any "reform" curricula; erase the influence of the National Council of Teachers of Mathematics standards; make sure that no decision on math instruction is influenced by any educational research or anyone from a college of education; adopt alternative textbooks, such as those now published in Russia or Singapore; look to mathematicians and "good teachers" while avoiding advice of "mathematics educators" (a rung or two below the night custodian) and teachers whose instruction mirrors constructivist notions, a practice which separates them from the "good teachers."

The back-to-basics incursion was undertaken in support of a new movement in Washington state, alleged to be a grassroots movement, called "Where's the Math,"

Meanwhile, in California, restrictive criteria on the instructional materials that can be purchased with state funding had eliminated all but a few programs in California's K-8 classrooms that could be characterized as "reform" programs. Textbook adoption is administered under the Curriculum Commission, which, observers say, has a long history of cronyism, replete with closed-door deliberations and biased commissioners, all of went unchecked. The net effect of these discriminatory practices was to discourage publishers from submitting for review all but one of the K-8 mathematics programs funded by the National Science Foundation.

In the fall of 2006, California's education policy structure began to weaken.
A court order ruled the state textbook adoption process unfair. For the first time in nearly a decade, the California Mathematics Council, an NCTM affiliate, began a watchdog process. Over the summer and fall of 2007, CMC carefully observed K-8 mathematics textbook adoption deliberations and published a blog as commentary. [See adoption blog at www.cmc-math.org]

With the curriculum review process under scrutiny, and the adoption regulations
under revision, the Commissioners were more circumspect. Classroom teachers, educators, and publishers were openly critical about the limitations of the state's "world class" standards, which are largely computational/procedural, making it difficult to create programs which address conceptual learning of mathematics. By December 2007, 40 out of 54 programs submitted were approved. Three of the K-8 curriculum programs on the new adoption list are materials that not only reflect reform principles, but are now officially approved for districts to purchase with state funds because they are aligned with the California mathematics standards (http://www.cde.ca.gov/ci/ma/im/)

Given the policy developments in California and the fact that California math results are lower than corresponding data in Washington [See http://nces.ed.gov/nationsreportcard/states/profile.asp] [See
also Notes 2 -4] it is ironic that the Washington "Where's the Math?" movement has
increased its membership, enthusiastically waving the California mathematics standards flag

and denouncing Washington state's NCTM standards-based reform policy. The group has aggressively pursued and received extensive media coverage, and organized legislative action campaigns.

The website urges its followers to activism:

> We must make it clear to the Governor and our elected officials that reformed math is not internationally competitive or recognized for its excellence. Reformed programs such as Everyday Math, TERC/Investigation, Connected Math, CMP, IMP, and other NSF-funded programs are not balanced and do not teach computational fluency which are essential to higher-level math and thinking skills.

"Where's the Math?" now has branches in major cities around the state, and, amazingly. its members sit on an official advisory panel to the Washington state board of education, along with pro-reform teachers and academics (see www.washmath.com).

Characterizing the "Where's the Math?" mentality is a video titled "Math Education: An Inconvenient Truth," the video stars a member of the group who is also a Seattle weather broadcaster, M...J. McDermott.

[See http://www.youtube.com/watch?v=Tr1qee-bTZI,)

The speaker conducts a broadcast-quality lecture in which she extols the virtues of the traditional approach to two-digit multiplication "If you think that Washington pupils should perform multiplication and division with mastery by the end of fifth grade you must insist that schools and school districts not use TERC's *Investigations in Number, Data and Space*, and *Everyday Mathematics*." (The texts were shown on camera). (Note: the same TERC program being disparaged has been noted in Holly Concannon's testimony to the Math Panel as contributing to very successful results by Boston schoolchildren.)

McDermott takes us through the computation exercise, step by step.

```
   1
  26
 x 31
1 2 6   1 times 6 is 6 and 1 times 2 is 2
7 8     3 times 6 is 18. Write 8 and we carry the 1.  3 times 2 is 6 plus 1 is 7.
8 0 6   We do the addition.  6. 8 + 2 is 10. We carry the 1.  7+1 is 8,
```

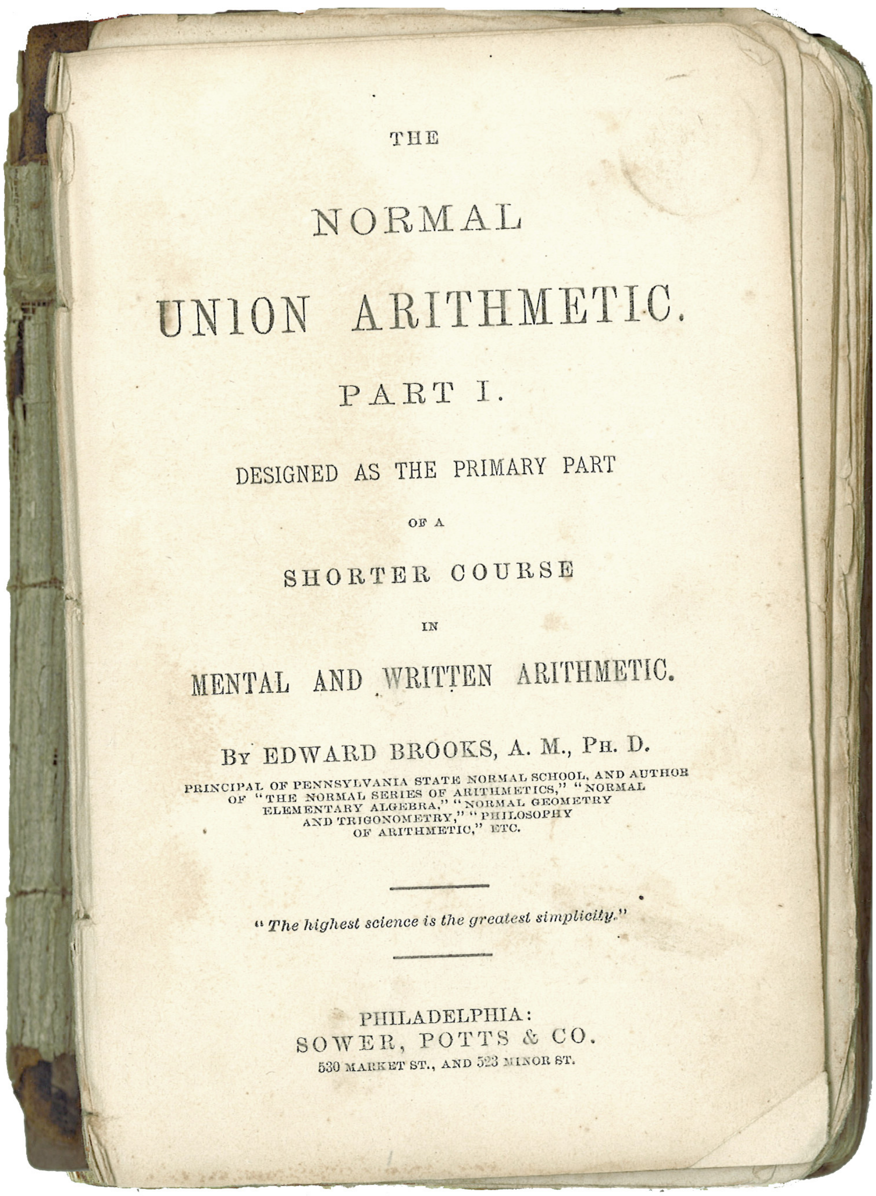
She shows us chalkboard examples from Investigations and the Everyday Math program approaches and claims that children rarely become efficient, confident, and fluent in computation in the first case. In Everyday Math, she admits that partial products method always works, but she often has trouble remembering which bit gets added to which bits. [See Note 4.]

In what direction are we moving in mathematics education? Should we return to the mentality of The Normal Union Arithmetic[3], published in 1878?

[3] The authors thank K. Dulle for loaning us his copy of The Normal Union Arithmetic.

No. [See Note 5.]

The way the game is usually played, it's ***three strikes and you're out.***



1. Multiply 65 by 36.

SOLUTION.—6 times 5 are 30; we write the 0 and carry the 3 to the next product: 6 times 6 are 36 and 3 are 39; we write the 39; 3 times 5 are 15, we write the 5 under the 3 and carry the 1 to the next product: 3 times 6 are 18, and 1 are 19; we write the 19: adding, we have 2340.

OPERATION.

```
  65
  36
 ---
 390
195
----
2340
```

NOTE.—Teach the pupil how to do the work first; when he is old enough, show him the reason for the method.

WRITTEN EXERCISES.

(2)	(3)	(4)	(5)	(6)	(7)
35	46	56	65	75	60
23	24	32	34	45	37
(8)	(9)	(10)	(11)	(12)	(13)
76	43	64	38	85	78
42	38	75	43	76	67
(14)	(15)	(16)	(17)	(18)	(19)
345	463	547	708	825	756
23	43	35	46	64	63
(20)	(21)	(22)	(23)	(24)	(25)
725	817	725	809	728	560
45	65	74	86	67	87
(26)	(27)	(28)	(29)	(30)	(31)
2356	4216	2057	3508	7069	4185
35	43	54	63	37	64
(32)	(33)	(34)	(35)	(36)	(37)
345	372	2184	4725	2057	3608
123	345	416	326	354	436

MENTAL EXERCISES.

1. How many are 10 times 2? 20 times 2? 30 times 3? 40 times 4? 50 times 5?

2. What two numbers multiplied together make 12? 18? 20? 24? 36? 40? 42? 48? 50? 60? 64?

3. How many words will 12 boys spell, if each boy spells 11 words?

Notes:

1. For further background on Boaler's research, see "Creating Mathematical Futures through an Equitable Teaching Approach: The Case of Railside School" by Boaler and Staples [http://www.tcrecord.org/search.asp?kw=Boaler&x=0&y=0] See also http://www.sussex.ac.uk/education/profile205572.html.

2. Of the 52 states and other jurisdictions, Washington fourth graders' NAEP math scores (2005) were lower than those in only 4 jurisdictions. For eighth grade, Washington scores were lower than only 2 jurisdictions nationwide. [See http://nces.ed.gov/nationsreportcard/states/profile.asp]

3. Just after this article was completed, 2007 NAEP data became available. The percentage of eighth grade math students in Washington who performed at or above the NAEP Proficient level was 36%. For California the figure was 24%.[See http://nces.ed.gov/nationsreportcard/states/profile.asp]

4. A parent who viewed the Where's the Math web site captured things well: "So, because a Seattle weather reporter can't remember which bits to add to which bits, the rest of the country's children may be doomed to view math as a set of arbitrary exercises in rote memorization, rather than as an integrated way of understanding the world?"

5. After reading a draft of the present paper, a friend in Hungary writes about authoritarian approaches to mathematics education: "I'm writing a book, and maybe [will] include some arguments about the responsibility of math teaching in the totalitarian madness of the last century. Math teaching was essential in convincing the majority of people that they were good for nothing but [to] obey."

References

["Some Worry About Potential Bias on the National Math Panel," EdWeek, May 19, 2006, By Sean Cavanagh]

["White House Suggests Model Used in Reading To Elevate Math Skills," Ed Week, Feb. 15, 2006.]

EPILOGUE

Now that the Panel's report has been issued, some comments**:**

1. The report can be found at http://www.ed.gov/about/bdscomm/list/mathpanel/index.html.

2. Summaries of the report can be seen at
http://www.washingtonpost.com/wp-dyn/content/article/2008/03/13/AR2008031301492.html

http://www.eschoolnews.com/news/top-news/?i=53070

3. Reactions to the report can be found at these sites, among others.
http://mathpanelwatch.blogspot.com/

http://www.districtadministration.com/pulse/

Anthony Ralston,"A Nation Still at Risk, "FOCUS, July/August 2008, Mathematical Association of America, in press. [See http://www.maa.org/pubs/focus.html]

4. The report can be summarized as "back to basics in elementary school arithmetic (especially fractions) leading to coursework in algebra."

4.1 **Arithmetic.** There should be no argument that proficiency in arithmetic should be a major goal of mathematics education at the elementary school level This is "a good and narrow aim, according to Professor George Polya. [See http://www.mathematicallysane.com/analysis/polya.asp]

> Therefore the good and narrow aim of the primary school [is] to teach the arithmetical skills — addition, subtraction, multiplication, division, perhaps a little more. Also to teach fractions, percentages, rates, and perhaps even a little more. Everybody should have an idea how to measure lengths, areas, volumes. If you don't do that then you are lost in everyday needs of your everyday life. You wish to paint your house. The paint is sold you that is enough, it says on the bottle, that it is enough to paint so and so many square feet. If you don't know what square feet are then you cannot estimate how many square feet of walls you have. Well, you will not buy the right size of bottle. And that is just a very trivial example. Arithmetical skills, some idea about fractions and percentages, some idea about lengths, areas, volumes, everybody must know this. This is a good and narrow aim of the primary schools, to transmit this knowledge, and we shouldn't forget it.

But Polya goes on to say that there are higher aims:

> However, we have a higher aim. We wish to develop all the resources of the growing child. And the part that mathematics plays is mostly about thinking. Mathematics is a good school of thinking. This was said so many times. The point about it is, "what is thinking?" Well, thinking which you can learn in mathematics is for instance to handle abstractions. Mathematics is about numbers. Numbers are an abstraction. When we solve a practical problem, then from this practical problem we must first make an abstract problem. Mathematics applies directly to abstractions. Some mathematics enables a child, finally *should* enable a child, at least to handle abstractions. To handle construct formations, to handle abstract structures. Structure is a fashionable

> word now. It is not a bad word. I am quite for it.
>
> But I think there is one point which is even more important. Mathematics, you see, is not a spectator sport. To understand mathematics means to be able to do mathematics.

One can be certain that the arithmetic espoused by Polya is not mindless back-to-basics spouting of arithmetical nonsense-syllables, but a fabric of ideas and actions held together by the sense-making of the pupil.

The fact that a child can say stuff like, "6 X 3 = 18"without hesitation is not a trustworthy indication that he or she knows arithmetic. In research published in Spring 1983 in School Science and Mathematics O'Brien and Casey asked children who had been on a chant-out-their-arithmetic-facts diet, What is 6 X 3?" Virtually 100% of the answers were correct.

Then children were asked in individual interviews to give a real-life story or a word problem for 6 X 3 = 18.

These kids had been surrounded by real life for their entire schooling, and for virtually all their school lives they had force-fed arithmetic "facts."

Incredibly, a large proportion of the children said something like this: "On Monday I bought 6 doughnuts. On Tuesday I bought 3 doughnuts. How many doughnuts did I buy altogether? 18 because 6 X 3 = 18."

Even though answers involving repeated addition (6 doughnuts on Monday, 6 on Tuesday, 6 on Wednesday) were accepted as multiplicative, more than 75% of the responses at grade 4 and 85% of the responses at grade 5 were incorrect. Worse, half the incorrect responses at grades 4 and 5 were stories for 6 plus 3.

So much for the alleged value of the parrot math approach to instruction and the notion that instant production of arithmetic facts (for example, 6 X 3 = 18). is a sufficient condition for thoughtfulness in arithmetic.

What about Polya's higher aims? What about mathematics beyond arithmetic?
Polya's claim that mathematics is a good school of thinking doesn't get much play in the Panel's report

4.2 **Algebra.** Is it a reasonable guess that the college math teachers on the Panel see lots of weaknesses in fractions and algebra in students' work in the traditional (outdated?) university courses they are presently teaching? Is the Panel's emphasis on fractions and algebra equivalent to an elementary school teacher's saying, "If kids learned to spell—if they'd stop writing gril for GIRL and too for TO and beleive for BELIEVE—they'd have all the essential knowledge to succeed in language and literature/"Is it too optimistic to think that a presidential panel might do more than administer bandaids to a doddering curriculum?

The Panel's recommendation is that algebra is the be-all and end-all of elementary school mathematics. How about statistics and data-handling and data-representation? Problem solving? Proof? Patterns? [See Annie Selden and Kien Lim, "Developing Mathematical

Habits of Mind," MAA FOCUS, March 2008.] Given Polya's call more than 30 years ago for mathematics as a school for thinking, are there not more appropriate goals? Research by Julian Stanley tells us that mathematical talent involves the ability to get to the heart of a situation and to toss away irrelevant attributes. Could this be the heart of a fruitful course for upper elementary and secondary pupils? Is it reasonable that pupils be given a variety of courses to choose from? [See Howard Gardner, "E Pluribus …. A Tale of Three Systems," in EdWeek, April 23, 2008.]

Based on practices of "high performing" countries, the National Math Panel report recommends schools accelerate mathematics learning, including study of algebra at grade 8. The report offers outlines content of an "authentic" algebra course (the question of what constitutes an inauthentic course will not be discussed here).

A bit of history may provide a useful perspective on the wisdom of this practice. For the past five years, algebra has been mandated for high school graduation in California, and the state's ten-year-old standards are designed to encourage teaching algebra in 8th grade. Course content is similar to the panel's syllabus: heavy on symbol manipulation and procedural fluency, light on conceptual understanding. A recent study—the most supportive of the Panel's position we could find— reports that more 8th graders in California schools are now taking algebra; however, the degree of "high performance" appears to be lacking, as only 38% of these 8th grade Algebra I students scored at proficient or advanced levels on the 2006-07 state tests. In the same year, only 17% of those who took Algebra I in 9th grade scored at proficient or advanced. [See "Eighth Graders Score Best on Algebra 1 CST" (February 2008) and "More Students are Taking Algebra 1"(February 2008); See www.edsource.org]

The reports on algebra performance in California are byzantine. State data for 2007 show that only 23 per cent of California pupils were proficient or advanced in Algebra 1. [See http://www.cde.ca.gov/nr/ne/yr07/yr07rel98.asp] For first-time test-takers, the figure was 26%, for repeat test-takers, 15%. And a report by David Foster, presented at the December 2007 meeting of the California Math Council says that (at the end of their junior year) 21% of the graduating class of 2007 met the standard for algebra 1. For algebra 2, the figure was 10%. [David Foster, "The Real Change Agents: *Building a Professional Learning Community*." See www.noycefn.org/math/resources]

It is clear that the data do not give much support for the Panel's embrace of California's "world class" curriculum.

5. **Summary.** We should be concerned with children's construction of thinking. The goal of education should be to enable children (and the adults they will become) to live in equilibrium with their environment and with the demands of new situations, not merely to transmit to children the facts, rules, procedures, conventions, and nomenclature of narrow knowledge. [See T.C. O'Brien, "What's Basic? A Constructivist View," Basic Skills and Choices 1 (Washington, D.C.: National Institute of Education, April 1982), pp. 85-94. Copies available from the author.]

English and Watters say it succinctly:

> *Children today are facing a world that is shaped by increasingly complex, dynamic, and powerful systems of information in a knowledge-based economy (…) Being able to interpret and work with complex systems involves important mathematical processes that are under-emphasized in numerous mathematics curricula, such as constructing, explaining, justifying, predicting, conjecturing and representing, together with quantifying, coordinating, and organising data.*
> English, L & Watters, J., 2005, "Mathematical modelling with young children," Proceedings of Psychology of Mathematics Education 28.

In 1977 the mathematician Hans Freudenthal gave his retirement address at the University of Utrecht after a distinguished career as a mathematician and as a contributor to mathematics education. “What have I tried to do with my career?”, he said. I have tried to make important ideas trivial.”

Puzzled as to how important ideas could be trivial, I [O’Brien] asked Freudenthal to elaborate.

> I have tried to make important ideas known to everyone. For example, there was once a time when only one man in the history of the human race knew the equation $E=mc^2$. As I am sure you know, that was Professor Einstein. Now, not very many years later, even modestly talented secondary students in math and physics all over the world can derive the equation $E=mc^2$.

In contrast with Freudenthal’s goal of making important ideas trivial (i.e., widespread) it is a reasonable summary to say that the National Mathematics Advisory Panel has tried to make trivial ideas seem important.

The Political Context of the National Mathematics Advisory Panel

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Abstract:
The National Mathematics Advisory Panel needs to be situated in its broader political context to more fully understand it. Who created it, for what purpose, and who will (and will not) benefit from it are key questions I address in this article. My argument is that the NMAP, as part of a larger initiative undertaken by the Bush Administration and US financial/corporate elites, serves capital's efforts to shore up the US's weakening economic global position and does not benefit the majority of the US people—particularly marginalized and excluded students of color and low-income students.

Keywords: Politics, policy, economy, capital, racism, mathematics education, economic competition, National Mathematics Advisory Panel

To more fully understand the National Mathematics Advisory Panel (NMAP), like any social phenomenon, one needs to examine its broader political context. Who created it, for what purposes, what agendas does it serve, and who will and will not benefit from it? To look at it critically—the purpose of this brief article—is to consider not only its stated aims but also to investigate the above questions as they affect historically marginalized students and their families/communities, particularly students of color and low-income students in the US. In this article, I do not discuss the specific report that the NMAP recently issued, as the other authors in this issue amply dissect this, but instead focus on its larger context (see Gutstein, in press, for a more comprehensive discussion).

The NMAP is an integral part of the *American Competitiveness Initiative* (ACI) that President Bush unveiled in his 2006 State of the Union Address. Thus, to understand the NMAP, one needs to begin with the ACI. Now partially codified by the *America Competes Act* (signed into law in August 2007), the ACI is a far-reaching endeavor with multiple facets (see Domestic Policy Council [DPC], 2006). The aptly named ACI is designed to deal with what the administration and financial/corporate elites consider to be the untenable situation facing the US on the global scale—that its economy is either already second-rate or is in imminent danger of attaining that status with

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respect to other nations. The initiative is based on a series of highly influential national reports, including *Rising Above the Gathering Storm* (National Academies*,* 2006), *Tough Choices or Tough Times* (the National Center on the Education and the Economy, 2007), *The Looming Work Force Crisis* (National Association of Manufacturers, 2005), and *America's Perfect Storm: Three Forces Changing our Nation's Future* (Educational Testing Service, 2007), among others. All these reports, and US government documents about the ACI, while extolling the virtues of the US economy ("The American economy today is the envy of the world," DPC, 2006, p. 4), have the consistent theme that the economy is in danger of losing its number one position. A consensus position of the documents is that technological innovation is, and has been historically, the engine for US economic growth and global position. However, while the US has not stood still, other countries are working overtime to catch and surpass the US (e.g., "Billions of new competitors are challenging America's economic leadership," Department of Education, 2006a, p. 4). The reports are full of data about the number of engineers being produced by South Korea, China, etc., and how various countries are eclipsing the US in these areas. Furthermore, they claim that US workers are not sufficiently prepared technologically, nor are US students ready for the challenging mathematics and science that they will need to know to help resolve the putative national crisis. As the National Academies (2006) argued:

> Having reviewed trends in the United States and abroad, the committee is deeply concerned that the scientific and technological building blocks critical to our economic leadership are eroding at a time when many other nations are gathering strength…. we are worried about the future prosperity of the United States…. This nation must prepare with great urgency to preserve its strategic and economic security (p. 4).

The documents all frame the concerns as immediate and grave. Along with their disaster-invoking titles, the language portends gloom, using words describing the US global economic position such as ""woeful," "unfortunate," "troubling," "disturbing," "danger," "risk," and "alarming!" (And these were written before the sub-prime mortgage crisis, $4-$5 per gallon gas, and the current economic slowdown/recession.) The *Looming Work Force Crisis* tells us that:

> By itself, this problematic trend [the declining percentage of STEM college degrees in the US versus the increases reported in other countries] should be enough to grab the attention of our nation's leaders and compel them to develop a comprehensive strategy for reinvigorating science and engineering education. Viewed in an international context, the facts should be downright frightening for policymakers. (p. 5)

The report continued:

> These troubling trends can lead our country down a path we do not want to take. Without an educated and highly skilled workforce to drive 21st century innovation, America's capacity to remain the world's most advanced economy is at risk. Other countries are moving fast to educate their workers and to innovate. If we do not implement a concerted national strategy to do the same, we risk America's future (p. 10)

The warning about the risk to the future of the country as a whole is consistent throughout the ACI documents and the various reports which all frame the crisis as one that affects us all. For example, *Tough Choices or Tough Times* (NCEE, 2007) commented on the poor educational showings of US students on international assessments:

> If we continue on our current course, and the number of nations outpacing us in the education race continues to grow at its current rate, the American standard of living will steadily fall relative to those nations, rich and poor, that are doing a better job. If the gap gets to a certain—but unknowable—point, the world's investors will conclude that they can get a greater return on their funds elsewhere, and it will be almost impossible to reverse course. Although it is possible to construct a scenario for improving our standard of living,

> the clear and present danger is that it will fall for most Americans. (p. 8)

In a similar vein, the National Academies (2006) report states: "Without high-quality, knowledge-intensive jobs and the innovative enterprises that lead to discovery and new technology, our economy will suffer and our people will face a lower standard of living" (p. 3). These pronouncements give the impression that the "crisis" will hurt all the US people, and the solution—the ACI—will therefore, by extension, help us all. Improve US mathematics, science, engineering, and technology education (an explicit purpose of the NMAP with respect to mathematics), increase worker output and skill-sets, stimulate investment in technological enterprises, up the number of available slots for highly educated immigrants who can contribute to US scientific and financial enterprises, and increase research into profitable sectors of the economy, all to enhance productivity and benefit the people—so the story goes.

However, the history of productivity increases shows that this boost to us all is a myth, in fact, a profound one. Over the past 40 years, US productivity has increased a great deal, albeit at uneven rates. But for the most part, only the wealthy have benefited, while both income and wealth inequality within the US have also increased, markedly. The *gini coefficient* is a one-number summary of income (or wealth) inequality; the larger the number, the larger is the income (wealth) inequality. The gini index ranges from 0.0 (perfect equality) to 1.0 (maximum inequality). The US gini coefficient (in terms of income) has increased since 1968 when it was 0.386. Ten years later, in 1978, it was 0.402; in 1988, it was 0.427; and in 1998, it was 0.456. By 2001, it had climbed to 0.466, and was 0.470 in 2006 (US Census Bureau, 2007). As a relative increase, income inequality went up 21.8% in under 40 years. During this same period, wealth inequality also climbed, but much more so. In 2003, Edward Wolff, a well-known economist who studies wealth inequality, reported, "We [the US] have had a fairly sharp increase in wealth inequality dating back to 1975 or 1976," and the *wealth* gini index in 2003 was "0.82, which is pretty close to the maximum level of inequality you can have" (Multinational Monitor, 2003)—far more unequal than even income inequality.

Other measures corroborate this. For example, the ratio of mean US CEO annual salary to that of a minimum wage (federal) worker was 51 to 1 in 1965, but by 2005, it had soared to 821 to 1 (Economic Policy Institute, 2006). The ratio of CEO salary to the mean wages of US workers went from 24 to 1 in 1965 to 262 to 1 in 2005 (Mishel, Bernstein, & Allegretto, 2007). In a carefully argued, detailed analysis that directly tied US productivity to income distribution, Dew-Becker & Gordon (2005), stated that from 1966 till 2001, "*nobody below the 90th* [income] *percentile received the average rate of productivity growth*" [emphasis original] (p. 58). Where did the gains in productivity go? Their answer is that, "*only the top 10 percent of the income distribution enjoyed a growth rate of real wage and salary income equal to or above the average rate of economy-wide productivity growth*" [emphasis original] (abstract). They further point out that the skewing of income inequality is more pronounced as one gets richer:

> Another way to state our main results is that the *top 1 percent* of the income distribution accounted for 21.6 percent of real total income gains during 1966-2001 and 21.3 percent during the productivity revival period 1997-2001, again excluding capital gains. Still another and perhaps even more stunning way to describe our results is that the top one-tenth of one percent of the income distribution earned as much of the real 1997-2001 gain in wage and salary income, excluding nonlabor income, *as the bottom 50 percent* [emphasis original] (p. 76).

In other words, during that period, the top 0.1 percent made as much as the bottom 50 percent—which is 500 times larger. Dew-Becker and Gordon further noted: "Not only have the bottom 90 percent of American workers failed to keep up with productivity growth, many have been harmed by it" (p. 77). Others point out the stagnation of US wages: "…from 1980 to 2004, while U.S. gross domestic product per person rose by almost two-thirds, the wages of the average worker

fell after adjusting for inflation" (Tabb, 2007, p. 20). Economist and New York Times columnist Paul Krugman (2004) added, about income inequality in the US more generally:

> According to estimates by the economists Thomas Piketty and Emmanuel Saez—confirmed by data from the Congressional Budget Office—between 1973 and 2000 the average real income of the bottom 90 percent of American taxpayers actually fell by 7 percent. Meanwhile, the income of the top 1 percent rose by 148 percent, the income of the top 0.1 percent rose by 343 percent and the income of the top 0.01 percent rose 599 percent. (Those numbers exclude capital gains, so they're not an artifact of the stock-market bubble.)

If the past is any indication, any productivity boosts the ACI causes will go not to the bottom 90 percent, but rather to the wealthiest. There is nothing in the initiative nor in any of the reports that make visible the skewing of the distributions as they now exist, and more to the point, nothing whatsoever about redressing the past and present inequities or about income or wealth redistribution. While the reality of the US economic woes is clear, for both capital and the rest of us, the ACI is oriented toward the problems of the rich.

What specific role does the National Mathematics Panel play within the larger program of the ACI? The NMAP's purpose was clearly elaborated in President Bush's charge to it. The first words of the executive order that created were: "In order to keep America competitive, support American talent and creativity, encourage innovation throughout the American economy…" Point number one of the order said that its role was to recommend, "[t]he critical skills and skill progressions for students to acquire competence in algebra and readiness for higher levels of mathematics," (Department of Education, 2006b). Thus, the stated purpose of the NMAP is consistent with that of the ACI, and its key role is to help determine what students need to access higher-level mathematics. But the purpose of this knowledge is also clearly stated: to help "keep America competitive," and address the "danger" of the US economy losing its supreme status that was so clearly specified in the ACI descriptions and background reports.

The above suggests who will benefit from the NMAP-ACI goals of increased technological innovation and increased productivity in the US. It does not, however, address who will not, a key question in any critical analysis. The NMAP's (2008) final report stated:

> Success matters to the nation at large. It matters, too, to individual students and their families, because it opens doors and creates opportunities. Much of the commentary on mathematics and science in the United States focuses on national economic competitiveness and the economic well-being of citizens and enterprises. There is reason enough for concern about these matters, but it is yet more fundamental to recognize that the safety of the nation and the quality of life—not just the prosperity of the nation—are at issue. (p. xi).

It also stated:

> Moreover, there are large, persistent disparities in mathematics achievement related to race and income—disparities that are not only devastating for individuals and families but also project poorly for the nation's future, given the youthfulness and high growth rates of the largest minority populations. (p. xii)

From these statements, it might appear that the NMAP's report seriously addressed issues of the quality of life and the life chances and opportunities for low-income students and students of color. However, it did not. Other than a reference to improving the school readiness of low-income children (e.g., by recommending that teachers in Head Start and similar programs better understand children's mathematical knowledge) and one reference that the "achievement gap" between students of color and whites, and that between low-income and wealthier students, can be "reduced or even eliminated" with school mathematics success, nothing in the document addressed the specific *mis-education* (Woodson, 1933/1990) experienced by African American, Latino/a, Native American, and some Asian students, along with low-income students of all races/ethnicities. There was no mention

of the historical legacy of racism, the massive disinvestment in public education, the unequal educational experiences of these students. The NMAP did not, for example, propose that teachers have detailed knowledge of the students they teach, the issues in their communities, or the language and culture of their people. This is despite the vast amount of scholarship both without *and* some within mathematics education documenting the importance of these knowledges (Delpit, 1988; Foster, 1997; Gutstein, Lipman, Hernández, & de los Reyes, 1997; Ladson-Billings, 1994, 1995; Martin, 2007; Tate, 1995). Nor did the NMAP suggest that teachers have a historical role to play, as activists in social movements, committed to working in partnership for the liberation of their students against oppressive regimes that attempt to exclude and marginalize them—as critical educators argue, again, both in and outside of mathematics education (Apple, 1992; Bigelow & Peterson, 2002; Frankenstein, 1987, 1998; Freire, 1970/1998, 1998; Gutstein, 2006, 2008; Martin & McGee, in press).

The ACI is no better with respect to addressing these issues for miseducated students. The college scholarship money for "talented" mathematics and science students is limited to those with at least a 3.0 grade point average who have passed either two AP exams (with a score of 3 or better) or *international baccalaureate* program exams (with a score of 4), or who took a "rigorous" high school course of study. Rigor is defined at the state level but generally includes three years of mathematics, three years of science, four years of English, three years of social studies, and one year of foreign language. Students who have struggled or who had no opportunity to take "rigorous" courses, or those leading to AP classes, are not eligible for the grants. In Illinois, where I live, the state education board mandates less social studies and science than the ACI grants require (ISBE, 2007), so a student with straight A's who followed the "basic" plan of her school is ineligible. In fact, *no* ACI money supports these students for college. Furthermore, the first year ACI grants are only $750, not trivial for working-class and low-income students, but even if they are "successful," it may not be enough to go to college.

The ACI documents proclaim that it is intended for all students, for example, "The expansion of AP-IB programs [proposed by the ACI] will not only benefit students passing the AP exams, but will also serve as a mechanism to upgrade the entire high school curriculum so that other students benefit" (Department of Education, 2006c). The additional AP calculus (and other) teachers and classes proposed in the ACI will indeed reach some students in urban public schools—those youth who have managed to fight through various obstacles and are ready to do the course work, or those lucky enough to be in schools that promote excellence and truly respect, and are integrated with, students and their communities. But the relatively few additional individuals who will take advanced mathematics will not necessarily address the vast problems, educational and social, faced by low-income communities and those of color. Moreover, in a tracked educational system and stratified labor market, it is not clear that the "entire" curriculum will be upgraded so that "other students benefit." The US does not have jobs for a work force that is entirely "highly educated." In fact, according to the Bureau of Labor Statistics (BLS), whose analysts conduct a 10-year occupational employment projection every two years, the *majority* of US workers in 2016 will need at most short- or moderate-term on-the-job training (not college)—including "truckdrivers, heavy and tractor-trailer; and secretaries, except legal, medical, and executive" and "retail salespersons; and waiters and waitresses" (Dohm & Shniper, 2007, p. 104). These 86 million low-skill workers, from capital's and the administration's perspective, have little need to take AP tests because few will likely go to college, if even finish high school, given the state of education in the US and the lack of meaningful, decent-paying employment opportunities. A recent study showed that 25.8% of African American male public high school students in Chicago dropped out in one year (Greater Westtown Community Development Project, 2003). Extrapolating to four years, this suggests that only approximately one third of the Black males entering high school in Chicago will graduate, and that

does not even account for those who leave during middle school or after eighth grade. These young men are slated—and in general, educated (Anyon, 1980)—for those low-skilled jobs, but even that will be a challenge because of structural and institutional racism—Black unemployment rates are double that of whites (9.7% versus 4.9% as of May, 2008, BLS, 2008; see Gutstein, in press for more detail).

Thus, there is little evidence that the ACI, or NMAP, is "for all." Rather, there is a strong historical basis that the proposed "solution" to the "crisis" afflicting the US economy on the global scale will benefit the wealthy rather than marginalized students and their families and neighborhoods. If one attempts to analyze or understand the NMAP without taking into account the larger sociopolitical context, one runs the risk of thinking within a box—of arguing whether or not the NMAP is promoting "good" math, mathematical literacy, and mathematical power. That may or may not be the case, but that is not my argument here. The Enron financial manipulators who bilked vast sums of money from thousands of ordinary people were certainly mathematically literate and had excellent "conceptual understanding, computational fluency, and problem-solving skills"—key attributes that the NMAP (2008) suggests all mathematical learners need (p. 30). Pentagon engineers who design precision missiles that rain "collateral damage" on Iraqi civilians likewise have this mathematical knowledge. But it is the question of the politics and purpose of the knowledge that concerns me here. In the case of the NMAP, as an integral part of the ACI, the goal is to serve the administration and capital's plan to reclaim the planet, markets, and resources for US supremacy. That is the broader political context of the National Mathematics Panel.

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Algebra for all?

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Déjà vu all over again

In 1957, the director of one of the major math curriculum projects in the UK was quoted in a newspaper as saying: "Up went Sputnik and down came all the pure mathematicians saying we must do sets and be saved". The corresponding message from the National Math Panel Final Report is "Up went the economic performance of China and India and down came all the pure mathematicians saying we must do algebra and be saved".

Not quite all the mathematicians, however. Davis (1999) wrote:

> What is necessary is to teach enough so that the commonplace diurnal mathematical demands placed on the population are readily fulfilled. What is also necessary is to infuse sufficient mathematical and historical literacy that people will be able to understand that the mathematizations put in place in society do not come down from the heavens: that they do not operate as pieces of inexplicable ju-ju, that mathematizations are human cultural arrangements and should be subject to the same sort of critical evaluation as all human arrangements.
> At the risk of sounding like a traitor to my profession, I would say that high school algebra or beyond is not necessary to achieve this goal.

What is algebra *for*?

> "Algebra … the intensive study of the last three letters of the alphabet"
> (source unknown)

On a personal level, I have positive memories of algebra at school as something that could almost always be done routinely and the answers checked (as opposed to problems in Euclidean geometry that were more intellectually challenging). Yet, for the majority of students, the experience of "pawing at symbols" is not fulfilling, and leaves no trace beyond a general negative feeling. It is a troubling experience to sit beside an eighth grader who is vainly trying to remember what to do with an algebraic equation and reflect that several more years of frustration lie ahead for that student.

While I derived reinforcement from being competent at doing it, I have no reflection of any of my technically excellent teachers ever discussing what algebra is *for,* except, of course, the passing of exams and progression to higher levels of the same.

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Kaput (1999) presents what I consider a balanced framework for teaching school algebra. He begins (p. 133) by stating:

> The traditional image of algebra, based in more than a century of school algebra, is one of simplifying algebraic expressions, solving equations, learning the rules for manipulating symbols – the algebra that almost everyone, it seems, loves to hate…
>
> …School algebra has traditionally been taught and learned as a set of procedures disconnected both from other mathematical knowledge and from students' real worlds.

The importance of algebra at a societal level is clear (p. 134):

> … algebraic reasoning in its many forms, and the use of algebraic representations such as graphs, tables, spreadsheets, and traditional formulas are among the most powerful intellectual tools that our civilization has developed. Without some form of symbolic algebra, there could be no higher mathematics and no quantitative science; hence no technology and modern life as we know them. Our challenge then is to find ways to find the power of algebra (indeed, all mathematics) available to all students…

I include this citation, with which I agree (who wouldn't), to make the point that the argument in this paper is not against the worth of algebra, nor against making it accessible and intellectually enjoyable to all students who want it, but against the declaration by fiat that all students will learn algebra, will be required to pass courses in algebra to have good educational and economic opportunities, and – above all – will continue to be taught algebra in a boring fashion.

Kaput (p. 134) proposes the following broad outlines for a more productive approach to the teaching of school algebra:

- begin early (in part, by building on students' informal knowledge);
- integrate the learning of algebra with the learning of other subject matter (by extending and applying mathematical knowledge),
- include the several different forms of algebraic thinking (by applying mathematical knowledge);
- build on students' naturally occurring linguistic and cognitive powers (encouraging them at the same time to reflect on what they learn and to articulate what they know), and
- encourage active learning (and the construction of relationships) that puts a premium on sense making and understanding.

Kaput suggests that there are five forms of algebraic reasoning, namely (a) the generalization and formalization of patterns and constraints, (b) syntactically guided manipulation of (opaque) formalisms, (c) the study of structures abstracted from computations and relations (not limited to generalized arithmetic), (d) the study of functions, relations, and joint variation, (e) a cluster of modeling and phenomena-controlling languages.

I consider it bizarre that Kaput's empirical, curricular, and conceptual work on algebra, based on an unmatched knowledge of mathematics, the intellectual history of mathematics, psychology, cognitive and developmental psychology, epistemology, semiotics, and so on, is not represented in the NMAP report. The failure to reference the review chapter by Kieran (2007) and the work of so many other leading researchers working on algebra within mathematics education is likewise part of the pattern of exclusion referred to in this issue's editorial..

Who needs algebra?

Is it not a fundamental characteristic of an advanced society to have differentiation of roles? Instead of demanding "Algebra for all" why not something like "A great deal of algebra for a few, some algebra for the majority" (admittedly inferior as a sound-bite-sized slogan).

McGregor (2001, pp. 405-406) reported a selection of responses to the question "Have you ever made use of any of the algebra you learned at school?":

> House painter: "Algebra? All that *x* and *y* stuff? I had no idea what it was about. I couldn't do maths at all, even in primary".
> Bus driver: "Never. I could understand some of it in my head but I never knew how to write it".
> Primary teacher: "Algebra? Oh, I remember, if you have 2 and then a bracket and then *a* plus *b*, it's 2*a* and 2*b*. Is that right?
> Lecturer in literacy: "Algebra! I never understood it. I hated maths and was no good at it".
> Journalist: "Never, but I liked solving problems with calculus at school".
> Electrical engineer: "Yes, I work with formulas all the time. that's algebra. I always work with spreadsheets. I use Excel. That's all I need. I know what the answers should look like; that's important".
> Financial mathematician consulting to large corporations: "Not really, not that kind of algebra. I write individual programs for each client. I use APL. Many people doing my kind of work use Visual Basic".
> Information systems consultant: "No. To solve a problem I would write a program, probably in J or Java".

The people who developed Excel, Java, and so on, almost certainly knew plenty of algebra and other advanced mathematics. However, the range of quotations above underlines the point alluded to above about the differentiation of roles in contemporary industrialized societies, wherein so much of mathematics is embedded in tools, and essentially invisible to the users of those tools, As expressed by Skovsmose (2006, p. 325):

> The education systems must ensure a supply of people with competencies according to a matrix that represents society's demands for competencies. Some groups must be well-educated in mathematics; some must be able to operate with certain mathematical techniques; some must be able to read diagrams; some must know the mathematics included in instructions; a great many must know the mathematics necessary for shopping and dealing with payment and bank transactions.

As a response to these needs, "Algebra for all" is simplistic and, essentially dishonest. Arguably the most significant problem, with the severest consequences, is when passing an algebra course becomes a barrier to educational and economic advancement. Why should algebra be necessary for someone who wants to become a musician, a lawyer, a counselor, an English teacher, a politician (you can continue the list)? Whether consciously planned or not, erecting algebra at the entrance to a national educational and economic gated community is an extreme form of political/social engineering. This form of "accessment" gives US education "a way to sort children by race and social class, just like the old days, but without the words 'race' and 'class' front and center" (McDermott & Hall, 2007, p. 11).

The implied dominance of the curriculum by algebra

The assignment given to the panel specifically mentioned algebra in the first of 10 requirements for the report: "the critical skills and skills progression for students to acquire competence in algebra and readiness for higher levels of mathematics" (Presidential Executive Order 13398). In response, the report focuses on algebra to the detriment of a balanced view on mathematics. It is striking how emphasis is put on the teaching of arithmetic, geometry, and combinatorics in relation to their importance in laying foundations for algebra. In geometry, for example, teacing about similar triangles is singled out because they enlighten the graphical representation of linear functions, in particular the invariance of the slope (incidentally, it is only the special case of similar right-angled triangles that is needed for this purpose). The subject of combinatorics, apparently, owes its importance to its relationship to the binomial theorem, not its role in developing applications of probability theory. Thus, the final entry in Table 1 of the Final Report (p. 16) is "combinations and permutations, as applications of the binomial theorem and Pascal's Triangle".

Characterization of algebra in the NMP report

A definitive statement is presented in the Final Report (Table 1, p. 16) as a list of topics that is a subset of what I learnt at grammar school in Northern Ireland nearly 50 years ago, with two exceptions, namely (a) fitting simple mathematical models to data, and (b) combinations and permutations, as applications of the binomial theorem and Pascal's Triangle. This specification is based on a number of comparisons, including the algebra standards in Singapore's mathematics curriculum for Grades 7-10. It is intended as "a catalog for coverage, not as a template for how courses should be sequenced or texts should be written" (footnote 9, p. 15). Nevertheless, an overview of school algebra is presented in the report of the Task Group on Conceptual Knowledge and Skills (pp. 1-6 to 1-15) that is offered as a guide for mathematics teachers and textbook publishers. There are many parts within this explanation that I find odd.

First, in the preamble, it is written (p. 1-6):

> … the discussion of word problems will be somewhat abbreviated. In no way, however, should this emphasis be interpreted to mean that problem solving is considered to be less important than [connections between basic concepts and skills]. Indeed, the solution of multistep word problems should be part of students' routine.

Problem solving should not be equated with word problems. There are many fascinating problems in algebra that are not word problems.

Second, it is firmly stated that the most basic protocol in the use of symbols is "Always specify precisely what each symbol stands for" (p. 1-6). Yet at several points thereafter, new usages of symbols are introduced without specifying what they stand for. Most perplexingly, we find a statement (p. 1-14) "Let X be a symbol". I'm no expert on logic, but it seems clear to me that there is a problem here!

Third, in talking about solving a quadratic equation, it is stated (p. 1-9) that "assuming there is a solution … students deduce what it must be", which sounds odd given that, in general a quadratic equation has two solutions.

A further aspect of the outline of school algebra that strikes me as a little odd is the inclusion of the statement (but not proof) of the Fundamental Theorem of Algebra, namely that every polynomial form of positive degree has a complex root (from which it follows that every polynomial form of degree n can be expressed as the product of n linear expressions). It is further stated (p. 1-15) that

"the importance of the theorem justifies that school students learn it and use it even if they will not see how it is proved" (the proof is quite advanced). It is not clear to me how it could be used in the context of high-school algebra.

Assessment

Let me finish with a few comments on the assessment of algebraic thinking, as an area that, above all, shows up the weaknesses of multiple-choice tests, at least as they are currently constructed (they are capable of improvement). For example, consider the following item, cited with approval in the Report of the Task Group on Assessment (p. 6-76):

> Mona counted a total of 56 ducks on the pond in Town Park. The ratio of female ducks to male ducks that Mona counted was 5:3. What was the total number of female ducks Mona counted on the pond?
> A. 15 B. 19 C. 21 D. 35
> Comment: A student has to decide which fractions are relevant.

How might a students solve this problem? The formal approach, as advocated by Wu (2001) is to make use of the property that:

$^a/_b = {}^c/_d \Rightarrow {}^a/_{(a+b)} = {}^c/_{(c+d)}$

where, in this case, $a = 5$, $b = 3$, and $c + d = 56$. That leads to an equation that can be solved, and is, indeed, a way to find the answer. Another way is to think of a grouping of 5 female ducks and 3 males. Dividing 56 by 8 reveals that the ducks may be partitioned into 7 such groupings, hence there are 7 X 5 = 35 female ducks. Even simpler is to realize there are more female than male ducks, so the answer must be more than half of 56, which eliminates all answers but D! As with many multiple-choice items supposedly designed to test algebraic competence, this one is awash with possibilities for false positive responses.

Summary

The main points are easy enough to summarize:

1. I argue against the goal of "algebra for all" on the grounds that, while collectively a cadre of mathematical and scientific specialists is needed for society to operate effectively, most individuals in our society do not need to have studied algebra. This by no means implies that anyone should be denied the opportunity to do so (in an intellectually stimulating way). Moreover, students should be encouraged to study algebra in the spirit of keeping options open, given its status as a gatekeeper to many educational and economic opportunities.
2. There is a need to radically rethink the role of algebra within the mathematics curriculum, as argued by Kaput (1999).

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Introduction to
Thinking as Communication

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> If we see knowing not as having an essence, to be described by scientists or philosophers, but rather as a right, by current standards, to believe, then we are well on the way to seeing conversation as the ultimate context within which knowledge is to be understood. Our focus shifts from the relation between human beings and the objects of their inquiry to the relation between alternative standards of justification, and from there to the actual changes in those standards which make up intellectual history.
>
> Richard Rorty[1]

This book is a result of years-long attempts to change my own thinking about thinking, a task seemingly as improbable as breaking a hammer by hitting it with itself. In this unlikely undertaking, I have been inspired by Lev Vygotsky, the Byelorussian psychologist who devoted his life to "characterizing the uniquely human aspects of behavior,"[2] and by Ludwig Wittgenstein, the Austrian-British philosopher who insisted that no substantial progress can be made in this kind of endeavor unless the ways we talk, and thus think, about uniquely human "forms of life" undergo extensive revisions.

My admittedly ambitious undertaking had modest beginnings. I was initially interested in learning and teaching of mathematics. Like many others before me, I was mystified by what could best be described as vagaries of human mind: whereas some people juggled numbers, polygons and functions effortlessly, some others were petrified at the very mention of numbers or geometric figures. Many of those who erred in their use of mathematical terms and techniques, seemed to err in a systematic, surprisingly similar ways. And then, there was the wonder of little children doing strange things with numbers before gradually becoming able to handle them the standard way. Above all, however, one could not but puzzle over why the persistent attempts to improve mathematics learning that have been lasting for many decades, if not centuries, did not seem to have any sustainable effect. After years of grappling with these and similar phenomena, I realized that one cannot crack the puzzles of mathematical thinking without taking a good look at human thinking at large. I ended up wondering with Vygotsky

In Sfard, A. (2008). *Thinking as communicating: Human development, the growth of discourses, and mathematizing*. (pp. xiii - xix) Cambridge, UK: Cambridge University Press.
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about how the unique human abilities "have been formed in the course of history" and about "the way they develop over an individual lifetime."[3]

I soon discovered that whoever forays into this exciting territory dooms herself to an uneasy life. The first predicament of the student of human development is her being torn between two conflicting wishes: the wish to be scientific, whatever this word means to her, and the desire to capture the gist of those phenomena that are unique to humans. Whenever one of these needs is taken care of, the other one appears to be inherently unsatisfiable. Indeed, across history, the tug-of-war between the two goals, that of scientific reproducibility, rigor and cumulativeness, on the one hand, and that of doing justice to the complexity of the "uniquely human," on the other, resulted in the pendulum-like movement between the reductionist and the "gestaltist" poles. Reductionist theories, of which behaviorism is arguably the most extreme example, can boast the scientific operationality of their vocabulary, but they eventually kill their object by throwing some of its vital parts away. Socioculturally-minded followers of Vygotsky, on the other hand, aware of the futility of the search conducted "under the lamp" rather than in those dark places where answers to their questions may really be hiding, fail to communicate their rich ideas clearly enough to give rise to well-defined programs of study.

Today, our sense of helplessness may well be at its most acute. New technologies afford unprecedented insights into human phenomena and produce high-resolution evidence of the utmost complexity of human forms of life. With audio- and video-recorders as standard ingredients of the researcher's toolkit, the fleeting human action acquires permanence and becomes researchable in ways unknown to our predecessors. When carefully documented and transcribed, even the most common of everyday conversations prove to be a complex, multifaceted phenomenon, and an inexhaustible source of wonderings. This makes us as aware as ever of the fact that our ability to analyze and explain lags behind our ability to observe and to see. In this respect, our current situation is comparable to that of the 17th century scientists just faced with the newly invented microscope: Powerful, high-resolution lenses that reveal what was never noticed before are yet to be matched by an equally powerful analytic apparatus.

Inadequacies of conceptual tools are what Wittgenstein had in mind while complaining, more than half a century ago, about the state of research on human thinking. "[T]he concepts of psychology are just everyday concepts," he said, whereas what we need are "concepts newly fashioned by science for its own purpose."[4] These words seem as much in force today as they were when originally written. Lacking a designated, operationally defined vocabulary, the study of humans remains plagued by resilient dilemmas. Just look at time-honored controversies about human development that recur time and again, alas in different disguises, throughout history.

Take, for example, the famous "nature versus nurture" dilemma, "mind and body" problem or the controversy about the "transfer of learning." All these quandaries have an appearance of disagreements about empirical facts, but may, in reality, be a matter of lexical ambiguities. The blurriness of the vocabulary is the most obvious explanation for our inability to overcome the differences and build on each other's work: Unknown to ourselves, we are likely to be using the same words – *nature*, *nurture*, *mind*, *transfer* – in different ways. Similarly, our inability to capture the complexity of human phenomena may well be a matter of an inadequacy of our analytic methods, the weakness that, in the absence of explicit, operational definitions, seems incurable.

At a closer look, the lack of operationality is only the beginning of the researcher's problem. Without clear definitions, one is left at the mercy of metaphors, that is, of concepts created by transferring familiar words into unfamiliar territories. Indeed, if we are able to use words such as *nurture* or *transfer* in the context of human learning and development, it is because both these terms are known to us from everyday discourse. The services rendered by metaphors, however, are not without a price: together with the unwritten guidelines for how to incorporate the old term into new contexts come hordes of unforeseen metaphorical entailments, some of which may interfere with the task of gaining useful insights into the observed phenomena. Whereas the use of metaphor cannot be barred – after all, this is one of the principal mechanisms of discourse building – the risks of metaphorical projections may be considerably reduced by providing the metaphorically engendered notions with operational definitions.

Being explicit and operational about one's own use of word, however, is not an easy matter. Some people circumvent the challenge by turning to numbers. Precise measurement seems such an obvious antidote to the uncertainties of descriptive narratives! Rather than merely describing what the child does when grappling with mathematical problems, those who speak "numerese" would look at students' solutions, divide them in categories and check distributions. Rather than scrutinizing the utterances of a girl executing an arithmetic operations they would measure her IQ, consider her grades and decide whether the numbers justify labeling her as "learning disabled." Never mind the fact that in the quantitative discourse the numbers may be originating in categorizations as under-defined as those that belong to its "qualitative" counterpart (after all, there is no reason to assume that the words signifying things to be counted, when not defined in operational terms, are more operational than any other.) Forget the fact that in their zeal to bring simplicity, order and unification, the quantitatively minded interlocutors are likely to gloss over potentially significant individual differences. It is only too

tempting to believe that numbers can say it all and that when they speak, there is no need to worry about words.

I do worry about words, though, and this book is the result of this concern. In spite of my liking for numbers – after all, I am the native of mathematics – I am acutely aware of the perils of the purely numerical talk. The uneasy option of operationalizing the discourse about uniquely human forms of life seem the only alternative. On the following pages, I take a close look at the basics terms such as *thinking*, *learning* and *communication* and try to define them with the help of clear, publicly accessible criteria. If this operationalizing effort raises some brows – if somebody protests saying that thinking and communication are natural phenomena and thus not anything that people should bother to define – let me remind that defining regards the ways we talk about the world, not the world as such, and it is up to us, not to the nature, to decide how to match our words with phenomena. And to the readers who feel that I try to tell them how to talk let me explain that this, too, is not the case. All I want is to be understood the way I intended, on my own terms. For me, being explicit about my use of words is simply a matter of "conceptual accountability," of being committed to, and responsible for, effectiveness of my communication with others.

The conceptualization I am about to propose may be regarded as an almost self-imposing entailment of what was explicitly said by Vygotsky and what was implied by Wittgenstein. The point of departure is Vygotsky's claim that historically established, collectively implemented activities are developmentally prior to all our uniquely human skills. Being one of these skills, human thinking must also have a collective predecessor. Obviously, interpersonal communication is the only candidate. In this book, therefore, thinking is defined as the *individualized version of interpersonal communication* – as a communicative interaction in which one person plays the roles of all interlocutors. The term *commognition*, a combination of *communication* and *cognition* comes to stress that inter-personal communication and individual thinking are two varieties of the same phenomenon.

In the nine chapters of this book, the introduction to the commognitive perspective is accompanied by a careful examination of its theoretical consequences and of its implications for research and for educational practice. The task is implemented in two steps. Part I (chapters 1 through 4) is devoted to the double project of telling a story of human thinking and creating a language in which this story may usefully be told. After presenting a number of time-honored controversies regarding human learning and problem solving (Chapter 1), and after tracing the roots of these quandaries to certain linguistic ambiguities (Chapter 2), the commognitive vision is introduced as a possible cure for at least some of the persistent dilemmas and uncertainties

(Chapter 3). Although it is repeatedly stressed that language is not the only medium in which communication may take place, it is now claimed that verbal communication may well be the primary source of the distinctively human forms of life (Chapter 4.) Indeed, if one was to name a single feature that would set human kind apart from all the others in the eyes of a hypothetical extraterrestrial observer, the most likely choice would be our ability to accumulate complexity of action, that is, the fact that our forms of life, unlike those of other species, evolve and grow in intricacy and sophistication from one generation to another, constantly redefining the nature and range of individual development. It may now be argued that this gradual growth is made possible by the fact that our activities are verbally mediated. More specifically, thanks to the special property of human language known as recursivity, the activity-mediating discourses, and the resulting texts, become the primary repository of the gradually increasing complexity. Consistently with this vision, research on human development becomes the study of the growth of discourses.

In Part II I return to the questions that started me on this project: I use the commognitive lens to make sense of one special type of discourse called *mathematical*. By choosing mathematics I hope to be able to illustrate the power of commognitive framework with a particular clarity. Mathematical thinking has been psychologists' favorite object of study ever since the advent of the disciplined inquiry into human cognition. Widely regarded as perhaps the most striking instantiation of the human capacity for abstraction and complexity, mathematics is also a paragon of rigor and clarity: It is decomposable into relatively neatly delineated, hierarchically organized layers that allow for many different levels of engagement and performance. The tradition of using mathematics as a medium within which to address general questions about human thinking goes back to Jean Piaget,[5] and continues with the wide variety of developmental psychologists and misconceptions seekers, ending up, at least for now, with the sociocultural thinkers who vowed to reclaim the place of the social within the time honored trinity world-society-individual.[6] Throughout history, students of human mind were often divided on questions of epistemology, methodology and of the meaning of observed phenomena, but they always agreed that mathematical thinking is a perfect setting for uncovering general truths about human development[7].

In the four chapters devoted to mathematical thinking, I develop the commognitive vision of mathematics as a type of discourse – as a well-defined form of communication, made distinct by its vocabulary, visual mediators, routines and the narratives it produces (Chapter 5). The questions of the nature and origins of the objects of mathematical discourse is then addressed, and the claim is made that mathematics is an autopoietic discourse – one that spurs its

own development and produces its own objects (Chapter 6). I follow with the questions of the uniquely mathematical ways of communicating (Chapter 7) and of the goals and gains of communicating in these special ways (Chapter 8). All along, particular attention is given to the question of how mathematical discourse comes into being and how and why it subsequently evolves. The vision of mathematics as a discourse, and thus as a form of human activity, makes it possible to identify mechanisms that are common to the historical development of mathematics and to its individual learning. Having stated all this, I return to the initial quandaries and ask myself whether the commognitive vision brought the wished-for resolution. At the same time, I wonder about a series of new puzzles, some of them already being taken care of and some others still waiting to be transformed into researcheable questions (Chapter 9.)

All along the book, theoretical musings are interspersed with numerous empirical instantiations. Although the examples are mostly mathematical, they are rather elementary and easily accessible to anybody who knows a thing or two about the basic arithmetic. The mathematical slant, therefore, should not deter non-mathematical readers, not even those who suffer from mathematical anxiety. It is also worth mentioning that the book may be read in different ways, depending on one's needs and foci. Those interested mainly in theorizing about human thinking may satisfy themselves with Part I, where references to mathematics are scarce. Those who reach for this book because of their interest in mathematical thinking, can head directly toward Part II. The glossary in the end of the volume will help them, if necessary, with concise explanations of basic terms and tenets.

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Once we agree that thinking is an individualized form of interpersonal communication, we must also concede that whatever one creates is a product of collective doing. Even when sitting alone at her desk and deeply immersed in thoughts, a person is engaged in a conversation with others. Like any human artifact, this book is thus full of "echoes and reverberations" of conversations that took place at different times and places, involving people whom I never met, and probably many others of whom I haven't even heard. Being "filled with others' words"[8] this text has therefore more contributors than I am aware of. While echoing other peoples' words or when taking exception with what they said I dragged them into this conversation, sometimes intentionally and sometimes unconsciously. If their roles were revealed, not all of these involuntary contributors would agree to take any credit for the final product. Nevertheless, I would dearly like to acknowledge them all. Unfortunately, I can express my gratitude only to those few people of whose contribution I am aware, hoping to be forgiven by all the others.

Let me begin, therefore, with Lev Vygotsky and Ludwig Wittgenstein, two giants whose shoulders proved wide enough to accommodate legions of followers and a wide variety of interpreters. Although libraries have already been filled with exegetic treatises, the Byelorussian psychologist and the Austrian-born philosopher continue to inspire new ideas even as I am writing these lines. This, it seems, is due to one important feature their writings have in common: rather than provide information, the two authors address the reader as a partner in thinking; rather than presenting a completed edifice with all the scaffolding removed, they extend an invitation for a guided tour of the construction site; rather than imposing firm convictions, they share the "doubt that comes *after* belief."[9] These two writers had a major impact on my thinking; I can only hope they had a similar effect on my ability to share it.

[1] Rorty, 1979, pp. 389-390

[2] Vygotsky, 1978, p. 19

[3] ibid

[4] Wittgenstein, 1980, [Remarks on the Philosophy of Psychology Vol. II. 1948, German-English parallel text; Edited by G.H. Wright and H. Nyman, tr. C.G. Luckhardt and M.A.E. Aue. Oxford: Balckwell; §62]

[5] E.g. Piaget, 1952

[6] E.g. Lave, 1988; Walkerdine, 1988

[7] H. J. Reed and J. Lave (1979) make a compelling case for using mathematics as a "laboratory" for studying human thinking in the article with the tale-telling title "Arithmetic as a tool for investigating the relation between culture and cognition" (*American Ethnologist, 6,* 568-582).

[8] Bakhtin, 1999, p. 130 [Bakhtin, M.M. (1999), The problem of speech genres. In A. Jaworski & N. Coupland (Eds), *The discursive reader* (pp. 121-132). London: Routledge.]

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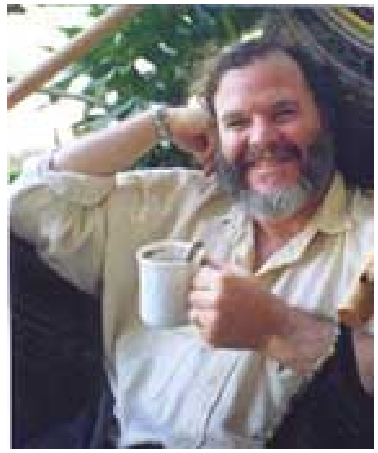

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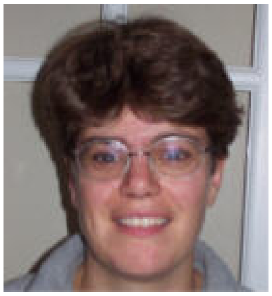
Woong Lim is a doctoral student in the Department of Curriculum and Instruction at University of Houston. He has taught various high school mathematics courses and is currently teaching mathematics at Lee College, Baytown, Texas. He studied applied mathematics at Northwestern University and earned a masters in mathematics from University of Houston. If he is not teaching, he is probably fencing epee, reading or writing.



Linda Martin is currently the chair of the Mathematics Department at Central New Mexico Community college where she has been developing and teaching undergraduate mathematics courses for 11 years.

Bettina Dahl Soendergaard is currently an Assistant Professor of mathematics and higher education at the University of Aarhus, Faculty of Science in Denmark. Prior to returning to her native country, she worked in as an Assistant Professor in Mathematics Education at Virginia Tech, USA. Here she came from a job as an Adviser (senior admin position + academic work) at the Norwegian Centre for Mathematics Education at the Norwegian University of Science and Technology. Before that she was a Research Officer in EU education policy at the University of Oxford, UK. She holds a Ph.D. in mathematics education (Roskilde, DK), a M.Sc. in Educational Research Methodology (Oxon, UK), and a combined M.Sc. and B.Sc. in mathematics and social science (Aalborg, DK).

Anna Sfard is based at the University of Haifa, Israel, and the Institute of Education, University of London, UK, and also affiliated to Michigan State University. With a formal background in mathematics and physics, and with a life-long interest in history, philosophy and language, she focuses her research on issues of mathematical learning and creative thinking. Her research is guided by what she calls "commognitive" approach to cognition, according to which thinking can be regarded as a form of communication (the term *commognition* is a combination of *cognition* and *communication*, meant to remind that thinking and communicating are two manifestations of the same human activity). Currently, she is engaged, together with her students, in research projects in which commognitive framework is applied to the study of development of mathematical discourses and of cultural embeddedness of learning skills, with a special emphasis on the roots of learning difficulties in mathematics. She is the recipient of the 2007 Hans Freudenthal Medal given by the International Commission of Mathematics Instruction. The text featured in this issue of the journal is the introduction to her book *Thinking as Communicating: Human development, the growth of discourses, and mathematizing*, published by Cambridge University Press in 2008.

Kristin Umland is in the Mathematics and Statistics Department at theUniversity of New Mexico where she has been teaching for 11 years. She has worked almost exclusively with pre- and in-service teachers for the past six years.

Rafael Villarreal-Calderon is currently a sophomore at the University of Montana, Missoula. He is majoring in physics and pre-medical sciences and enjoys studying the sciences as they apply to everyday life. He hopes to study medicine in the future.

Antti Viholainen is a Researcher at the Department of Mathematics and Statistics University of Jyväskylä, Finland, nearing the completion of his PhD in mathematics education. He is interested in research on student's understanding of Calculus concepts.

Jon Warwick completed his first degree in Mathematics and Computing at South Bank Polytechnic in 1979 and was awarded a PhD in Operational Research in 1984. He has many years of experience in teaching mathematics, mathematical modelling, and operational research in the higher education sector and his research interests include systems theory and mathematics learning and teaching. He is currently Professor of Educational Development in the Mathematical Sciences at London South Bank University.

AUTHORS OF CRITICAL NOTICE

Brian Greer came to mathematics education with a background in mathematics and psychology, leading to an interest in the relationship between cognitive psychology and mathematics education. After some 30 years in the School of Psychology in Belfast, Ireland, he took a position in mathematics education at San Diego State University, which he left in 2003 to work as an independent scholar in Portland, Oregon. Topics that he has focused on include multiplicative structures, probabilistic thinking, and word problems. More recently, particularly under the influence of Swapna Mukhopadhyay, with whom he collaborates intensively, he characterizes mathematics and mathematics education as human activities that are historically, culturally, socially, and politically situated.

Eric (Rico) Gutstein is Professor of Curriculum and Instruction at University of Illinois-Chicago. His interests include teaching mathematics for social justice, Freirean approaches to teaching and learning, and urban education. He has taught middle and high school mathematics. Rico is a founding member of Teachers for Social Justice (Chicago) and is active in social movements. He is the author of Reading and Writing the World with Mathematics: Toward a Pedagogy for Social Justice (Routledge, 2006) and an editor of Rethinking Mathematics: Teaching Social Justice by the Numbers (Rethinking Schools, 2005).

Danny Martin is Associate professor at the University of Illinois at Chicago, where he holds a joint appointment in the College of Education and the Department of Mathematics, Statistics, and Computer Science. His primary research interest is equity issues in mathematics education, with a focus on mathematics socialization and the construction of mathematics identities among African American adults and adolescents. He is author of the book, *Mathematics Success and Failure Among African Youth*. His recent articles include "Mathematics Learning and Participation as Racialized Forms of Experience: African American Parents Speak on the Struggle for Mathematics Literacy and Mathematics Learning," "Participation in the African American Context: The Co-Construction of Identity in Two Intersecting Realms of Experience," and "Beyond Missionaries or Cannibals: Who Should Teach Mathematics to African American Children?"

Wolff-Michael Roth is Lansdowne Professor of Applied Cognitive Science at the University of Victoria, British Columbia. One of his many research interests constitutes mathematical knowing and learning from early grades to professional practice of academics (e.g., scientists) and non-academics (e.g., electricians, fish culturists, environmentalists). He integrates ideas and practice of embodiment across a variety of intensively pursued activities, including academic research, gardening, gourmet cooking, building and renovating, and cycling. His most recent publications include *Doing Teacher Research: A Handbook for Perplexed Practitioners* (Sense, 2007) and *Toward an Anthropology of Graphing* (Springer, 2003

Tom O'Brien is author, consultant, researcher and professor emeritus in mathematics education, Southern Illinois University at Edwardsville. His work in education is three-fold: teacher education, curriculum development, and research on children's thinking. As a researcher, he has studied the growth of mathematical ideas in subjects from preschool to medical school and law school. As a teacher, he has worked with pupils from preschool through graduate school and for more than twenty years he was the director of the Teachers' Center Project, a project widely regarded as the foremost approach to in-service teacher education in the country. As a curriculum developer, he has authored more than fifty books for children, in addition to having written and edited some eighty papers on children's thinking and education published through the Teachers' Center Project. In addition, he has published and delivered some 450 papers on children's thinking, mathematics education, intellectual development and educational change. His presentations and seminars have taken place in the USA, Canada, Brazil, UK, Italy, France, Holland, Switzerland, Hungary, and the Republic of South Africa. He was named a North Atlantic Treaty Organization (NATO) Senior Research Fellow in Science in 1978. O'Brien received his bachelor's degree from Iona College, his master's degree from Teachers College/Columbia University and his Ph.D. from New York University.

Marianne Smith is an Oakland,CA-based writer, editor and communications professional. Her work includes analysis of education policy dynamics, as in the influence of the media, the blogosphere and policymakers on K-12 mathematics education: assessment, curricula and instruction issues.

NEW RELEASES IN MATHEMATICS EDUCATION

INFORMATION AGE PUBLISHING, INC.
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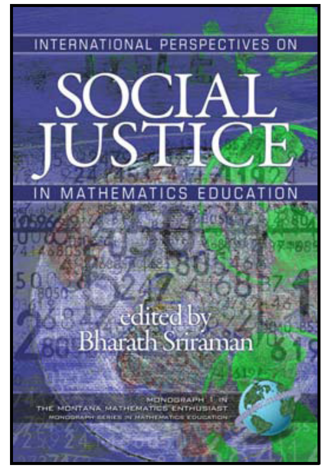
New Book Information

International Perspectives on Social Justice in Mathematics Education

Edited by **Bharath Sriraman,** *The University of Montana*

A Volume in **The Montana Mathematics Enthusiast: Monograph Series in Mathematics Education**

Series Editor **Bharath Sriraman,** *The University of Montana*



International Perspectives and Research on Social Justice in Mathematics Education is the highly acclaimed inaugural monograph of The Montana Mathematics Enthusiast now available through IAP. The book covers prescient social, political and ethical issues for the domain of education in general and mathematics education in particular from the perspectives of critical theory, feminist theory and social justice research. The major themes in the book are (1) relevant mathematics, teaching and learning practices for minority and marginalized students in Australia, Brazil, South Africa, Israel, Palestine, and the United States., (2) closing the achievement gap in the U.K, U.S and Iceland across classes, ethnicities and gender, and (3) the political dimensions of mathematics. The fourteen chapters are written by leading researchers in the international community interested and active in research issues of equity and social justice.

"This book...is an astonishing collection of scholarly articles from all over the world offering a kaleidoscope of perspectives of tremendous importance to mathematics educators interested in our shared concern to create a saner, equitable and more peaceful society."

- Ubiratan D'Ambrosio, Recipient of the 2005 Felix Klein Medal awarded by ICMI

"The monograph...does not just dwell on the past, and it is neither overly theoretical nor esoteric. Its major strength is that it is in touch with the ever-changing conflicts and struggles faced by teachers and students."

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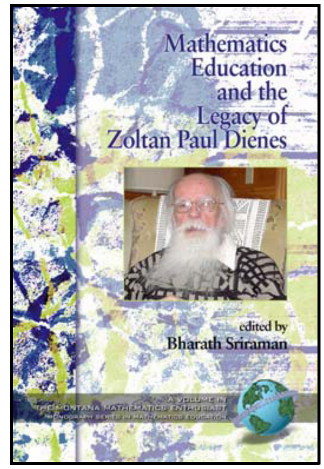
New Book Information

Mathematics Education and the Legacy of Zoltan Paul Dienes

Edited by **Bharath Sriraman,** *The University of Montana*

A Volume in **The Montana Mathematics Enthusiast: Monograph Series in Mathematics Education**
Series Editor **Bharath Sriraman,** *The University of Montana*



The name of Zoltan P. Dienes (1916-) stands with those of Jean Piaget and Jerome Bruner as a legendary figure whose theories of learning have left a lasting impression on the field of mathematics education. Dienes' name is synonymous with the Multi-base blocks (also known as Dienes blocks) which he invented for the teaching of place value. He also is the inventor of Algebraic materials and logic blocks, which sowed the seeds of contemporary uses of manipulative materials in mathematics instruction. Dienes' place is unique in the field of mathematics education because of his theories on how mathematical structures can be taught from the early grades onwards using multiple embodiments through manipulatives, games, stories and dance.

Dienes' notion of embodied knowledge presaged other cognitive scientists who eventually came to recognize the importance of embodied knowledge and situated cognition - where knowledge and abilities are organized around experience as much as they are organized around abstractions. Dienes was an early pioneer in what was later to be called sociocultural perspectives and democratization of learning.

This monograph compiled and edited by Bharath Sriraman honors the seminal contributions of Dienes to mathematics education and includes several recent unpublished articles written by Dienes himself. These articles exemplify his principles of guided discovery learning and reveal the non-trivial mathematical structures that can be made accessible to any student. The monograph also includes a rare interview with Dienes in which he reflects on his life, his work, the role of context, language and technology in mathematics teaching and learning today. The book finds an important place in any mathematics education library and is vital reading for mathematics education researchers, cognitive scientists, prospective teachers, graduate students and teachers of mathematics.

CONTENTS: Preface: The Legacy of Zoltan Paul Dienes, *Bharath Sriraman.* Reflections of Zoltan P. Dienes on Mathematics Education, *Bharath Sriraman and Richard Lesh.* Some Problems with Logic Blocks, *Suggested by Zoltan Dienes.* Logic Blocks to Other Embodiments, *Zoltan Paul Dienes.* Some Reflections on Order and Density: A Child's Path to the Bolzano–Weierstrass Theorem, *Zoltan Paul Dienes.* Games Leading to the Tetrahedron, *Zoltan Paul Dienes.* Cognitive Pyschology and Mathematics Education: Reflections on the Past and the Future, *Lyn D. English.* The Impact of Zoltan Dienes on Mathematics Teaching in the United States, *James Hirstein.*

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Beliefs and Mathematics:

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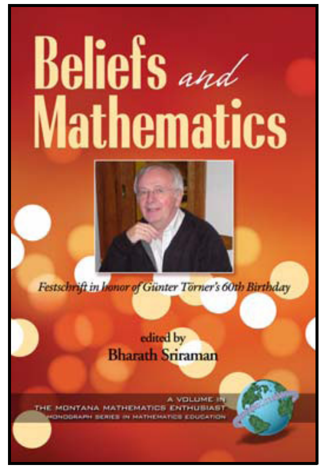

Beliefs and Mathematics is a Festschrift honoring the contributions of Günter Törner to mathematics education and mathematics. Mathematics Education as a legitimate area of research emerged from the initiatives of well known mathematicians of the last century such as Felix Klein and Hans Freudenthal. Today there is an increasing schism between researchers in mathematics education and those in mathematics as evidenced in the Math wars in the U.S and other parts of the world. Günter Törner represents an international voice of reason, well respected and known in both groups, one who has successfully bridged and worked in both domains for three decades. His contributions in the domain of beliefs theory are well known and acknowledged.

The articles in this book are written by many prominent researchers in the area of mathematics education, several of whom are editors of leading journals in the field and have been at the helm of cutting edge advances in research and practice. The contents cover a wide spectrum of research, teaching and learning issues that are relevant for anyone interested in mathematics education and its multifaceted nature of research. The book as a whole also conveys the beauty and relevance of mathematics in societies around the world. It is a must read for anyone interested in mathematics education.

CONTENTS: Preface**,** *Bharath Sriraman.* Pictures of the Projective Plane, *Benno Artmann.* From Morphismsto Matrices, *Klaus Hoechsmann.* A Theory of Teaching and Its Applications, *Alan H. Schoenfeld.* Beliefs: What Lies Behind the Mirror? *Gilah C. Leder.* Political Issues in Mathematics Education, *Ubiratan D'Ambrosio.* Snapshots from the 1960s: Tensions and Synergies in the Emerging of New Trends in Mathematics Education, *Fulvia Furinghetti.* Matrizen Als Diagramme, *Lisa Hefendehl-Hebeker.* Mathematizing Definitions of Beliefs, *Norma Presmeg.* Integrating Oral Presentation into Mathematics Teaching and Learning: An Exploratory Study with Singapore Secondary Students, Lianghuo Fan and Shu Mei Yeo. The Role of Beliefs on Future Teacher's Professional Knowledge, *Gabriele Kaiser, Björn Schwarz, and Sebastian Krackowitz.* Girls' Beliefs about the Learning of Mathematics, *Guðbjörg Pálsdóttir.* Modelling in School: Chances and Obstacles, *Hans-Wolfgang Henn.* Complex Systems in the Elementary and Middle School Mathematics Curriculum: A Focus on Modeling, *Lyn D. English.* Attitude Toward Mathematics: Overcoming the Positive/Negative Dichotomy, *Rosetta Zan and Pietro Di Martino.* Subject-Related Education Management: Framework of a Professional Development Program and First Research, *Konrad Krainer and Florian H. Müller.* Integrating Intuition: The Role of Concept Image and Concept Definition for Students' Learning of Integral Calculus, *Bettina Rösken and Katrin Rolka.*

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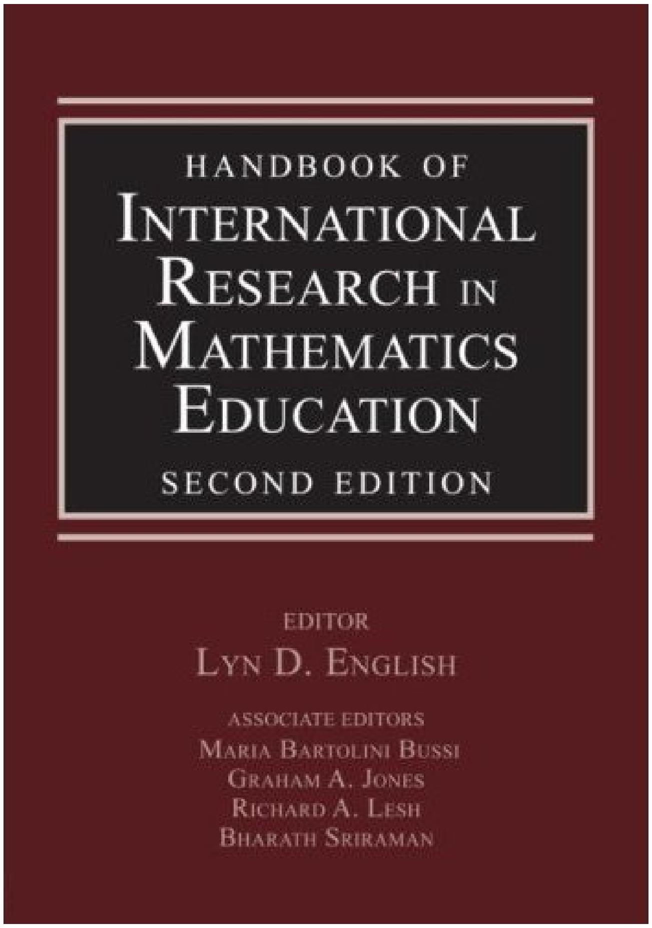
HANDBOOK OF
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LYN D. ENGLISH
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GRAHAM A. JONES
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Handbook of International Research in Mathematics Education

Second Edition

The Second Edition of the *Handbook of International Research in Mathematics Education* continues the mission of bringing together important new mathematics education research that makes a difference in both theory and practice. It updates and extends the *Handbook's* original key themes and issues for international research in mathematics education for the 21st century, namely:

- Priorities in international mathematics education research
- Lifelong democratic access to powerful mathematical ideas
- Advances in research methodologies
- Influences of advanced technologies

Each of these themes is examined in terms of learners, teachers, and learning contexts, with theory development being an important component of all these aspects.

This edition also examines other catalysts that have gained increased import in recent years including a stronger focus on the teacher and teacher practice, a renewed interest in theory development, an increased focus on the mathematics needed in work place settings, and a proliferation of research designs and methodologies that have provided unprecedented opportunities for investigating (and ultimately improving) mathematical teaching and learning.

Advancing the discipline by presenting dynamic, future-oriented works that address new and emerging priorities in mathematics education research, the Second Edition includes ten totally new chapters; all other chapters are thoroughly revised and updated. This is a must-have volume for scholars, professors, and graduate students in the field of mathematics education and in related areas such as educational psychology and educational research.

Lyn D. English is currently professor of mathematics education at the Queensland University of Technology, Australia. She is an elected Fellow of The Academy of the Social Sciences in Australia, and founding editor of *Mathematical Thinking and Learning: An International Journal*. Publications include *Mathematical and Analogical Reasoning of Young Learners*, the first edition of the *Handbook of International Research in Mathematics Education, Classroom Research in Mathematics: A Critical Examination of Methodology* (with Simon Goodchild), *Mathematical Reasoning: Analogies, Metaphors, and Images*, and *Mathematics Education: Models and Processes* (with Graeme Halford), and numerous book chapters, journal articles, conference papers, and special journal issues.

Handbook of International Research in Mathematics Education

Second Edition

Edited by
Lyn D. English

Associate Editors
Maria Bartolini Bussi
Graham A. Jones
Richard A. Lesh
Bharath Sriraman

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Contents

Preface

This second edition of the *Handbook of International Research in Mathematics Education* is intended for those interested in international developments and future directions in educational research, in particular, mathematics education research. The original edition (English, 2002) was prepared in response to a number of major global catalysts for change, including the impact of national and international mathematics comparative assessment studies; the social, cultural, economic, and political influences on mathematics education and research; the influence of enhanced sophistication and availability of technology; and the increased globalization of mathematics education and research.

Although our discipline has made considerable progress in the past decade, with significant theoretical and practical advances in many areas, the foregoing catalysts for change are still pertinent today, but with varying degrees of impact. Other catalysts have gained greater import in recent years including an increased focus on the teacher and teacher practice, a renewed interest in theory development in our discipline (e.g., semiotic mediation, which is the focus of a few chapters here), an increased focus on the mathematics needed in work place settings, and a proliferation of research designs and methodologies that have provided us with unprecedented opportunities for investigating (and ultimately improving) mathematical teaching and learning.

This second edition addresses the original priority themes and issues for international research in mathematics education for the 21st century, namely, life-long democratic access to powerful mathematical ideas, advances in research methodologies, and influences of advanced technologies. Each of these themes is examined in terms of learners, teachers, and learning contexts, with theory development being an important component of all these aspects. Some of the original chapters have been removed from this second edition and new chapters have been added (namely, chapters 4, 11, 18, 20, 25, 27, 28). Remaining original chapters have been updated, many quite substantially.

The volume comprises four sections. The first section, *Priorities in International Mathematics Education Research*, provides important background information on the key themes of the book, and also introduces new and emerging research trends in the field. Following my introductory chapter, Carol Malloy (chapter 2) explores democratic access to mathematics through democratic education, while Dylan Wiliam and Frank Lester (chapter 3) address the purpose of mathematics education research, and Cliff Konold and Richard Lehrer explore technology and mathematics education.

Section 2 focuses on *lifelong democratic access to powerful mathematical ideas* from the perspective of (a) learning and teaching, and (b) learning contexts and policy issues. With respect to learning and teaching, consideration is given to students' learning during the preschool and beginning school years (Bob Perry & Sue Dockett, chapter 5), the elementary and middle school years (Cynthia Langrall, Edward Mooney, Steven Nisbet, and Graham Jones, chapter 6), the secondary school years (Teresa Rojano, chapter 7), and finally, the advanced levels of mathematics education (Joanna Mamona-Downs & Martin Downs, chapter 8). Issues pertaining to representation

in mathematical learning and problem solving are addressed by Gerald Goldin in chapter 9. Research in preservice and inservice teacher education is explored in chapter 10 by Ruhama Even and Dina Tirosh, and in chapter 11 by João Pedro da Ponte and Olive Chapman.

The second component of section 2, namely, *learning contexts and policy issues*, covers a range of globally significant topics such as access and opportunity within the political and social context of mathematics education (Celia Rousseau and William Tate, chapter 13), democratic access to mathematical learning in a developing country (Luis Moreno-Armella and Manual Santos-Trigo, chapter 14), and a cultural psychology perspective on mathematical learning in out-of-school contexts (Guida de Gabreu, chapter 15). The complexities of change in mathematics education reform are addressed by Miriam Amit and Michael Fried in chapter 16, while Ole Skovsmose's and Paolo Valero's analysis of democratic access to powerful mathematical ideas completes the section.

In section 3, the chapters focus on *advances in research methodologies.* In chapter 18 Alan Schoenfeld explores the numerous past, current, and possible future trends in conceptual frameworks and paradigms used in mathematics education research. Margret Hjalmarson and Richard Lesh consider design research with a focus on engineering, systems, products, and processes for innovation. The importance of linking research with practice is also emphasized in this section, in particular, in the chapters by Nicolina Malara and Rosetta Zan (chapter 20), Kenneth Ruthven and Simon (chapter 21), and Douglas Clements (chapter 22). In chapter 25, Fulvia Furinghetti and Luis Radford discuss how the pedagogical use of the history of mathematics can serve as a means to transform teaching. The section concludes with Bharath Sriraman's and Günter Törner's survey and analysis of different traditions in mathematics education research within Europe particularly in Germany, France, and Italy (chapter 25).

In the final section, the *influences of advanced technologies on mathematical learning and teaching are investigated*. These chapters also include substantial theoretical development in relation to technology and mathematics education. Chapter 26, by Jim Kaput, Richard Noss, and Celia Hoyles looks at developments of new notations for mathematics learning in the computational era. This chapter has not been altered from its original version, given the tragic passing of Jim Kaput in 2005. The ideas presented in the chapter are as relevant, powerful, and future-oriented as they were in the first edition. New chapters in this section include those by Ferdinando Arzarello and Ornella Robutti (chapter 27), who address an embodied mind approach within a multimodal paradigm, and Maria Bartolini Bussi and Maria Alessandra Mariotti, (chapter 28), who provide insights into semiotic mediation in the mathematics classroom. In the remaining chapters, Michal Tabach, Rina Hershkowitz, Tommy Dreyfus, and Abraham Arcavi (chapter 29) present a research-design view of computerized environments in mathematics classrooms, Michal Yerushalmy and Danile Chazan (chapter 30) look at technology and curriculum design with a focus on the ordering of discontinuities in school algebra, and Rosa Bottino and Giampaolo Chiappini (chapter 31) explore advanced technology and learning environments with a focus on their relationships within the arithmetic problem-solving domain.

The concluding chapter (32) addresses some of the key 21st century issues in the advancement of mathematics education and mathematics education research. These include, among others, interdisciplinary debates on the powerful mathematical ideas students need to succeed in today's world, calls for research to support more equitable mathematics curriculum and learning access for all students and to find more effective ways of creating learning environments that can increase such learning access, and the need to improve teacher education and development to achieve our goal of powerful mathematics for all. The concluding chapter also reviews the broadening of research designs and methodologies in our discipline, which are providing strong bases for advancing the learning and teaching of mathematics.

Acknowledgments

This second edition of the *Handbook of International Research in Mathematics Education* would not have been possible without the unwavering support of many people. First, I wish to

extend my sincere thanks to all the authors—without their contribution the *Handbook* would not exist. Second, I convey my heartfelt thanks to the associate editors for their continued support, in particular, I wish to note my appreciation of the contribution of the new associate editor, Bharath Sriraman, for his insightful reviews of many of the chapters. Third, I sincerely thank Jo Macri, who has been a wonderful support to me in finalizing the *Handbook*. Her dedication, efficiency, and keen eye for detail have been superb.

Last, but not least, I wish to thank Larry Erlbaum, Naomi Silverman, and Erica Kica from Lawrence Erlbaum Associates for providing me this opportunity to produce a second edition of the Handbook and for their continued support throughout this process. The more recent support I have received from Taylor & Francis, in particular, Mary Hillemeier, is also gratefully acknowledged.

FORTHCOMING ARTICLES
[VOL6, NOS1&2, January 2009]

Focus Issue on Statistics Education and *Research in South America*

The following articles are accepted and scheduled for publication in vol6,nos1&2, January 2009

STATISTICS EDUCATION

1. Teacher Knowledge and Statistics: What types of Knowledge are used in the primary classroom?
Tim Burgess (New Zealand)

2. Undergraduate student difficulties with independent and mutually exclusive events concept
Adriana D'Amelio (Argentina)

3. Difficulties of Teaching Statistics:Two Case Studies from Hungarian Higher Education
Andras Komaromi & Klara Lokos Toth (Hungary)

4. Statistics Teaching in an Agricultural University: A Motivation Problem
Andras Komaromi & Klara Lokos Toth (Hungary)

5. Students' Conceptions About Probability and Accuracy
Ignacio Nemirovsky , Mónica Giuliano , Silvia Pérez , Sonia Concari , Aldo Sacerdoti and Marcelo Alvarez (Argentina)

6. Enhancing Statistics Instruction in elementary schools: Integrating technology in professional development.
Maria Meletiou-Mavrotheris(Cyprus) , Efi Paparistodemou(Cyprus) & Despina Stylianou(USA)

7. What makes a 'good' statistics student and a 'good' statistics teacher in service courses?
Sue Gordon, Peter Petocz and Anna Reid (Australia)

FEATURE ARTICLES

8. Learning, participation and local school mathematics practice
Cristina Frade (Brazil) & Konstantinos Tatsis (Greece)

9. If A.B = 0 then A = 0 or B = 0?
Cristina Ochoviet(Uruguay) & Asuman Oktaç (Mexico)

10. The Origins of the Genus Concept in Quadratic Forms
Mark Beintema & Azar Khosravani (Illinois, USA)

MONTANA FEATURE ARTICLES

BOOK REVIEWS