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Editorial: Globalization, History, Technology and Mathematics Education

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Welcome to Vol.2,no.2 of The Montana Mathematics Enthusiast. The Fall 2005 issue is being released a month earlier to coincide with the start of the school year in Montana. As the table of contents will indicate this is the first of (hopefully) many high quality international issues, featuring articles from mathematicians and mathematics educators worldwide. The journal is now indexed in *the Zentralblatt für Didaktik der Mathematik* (ZDM) and articles appearing in the journal are periodically reviewed by ZDM.

The journal is mutating with the changing times, and reveals some of the benefits of globalization and technology. The web-site statistics provided in this issue indicate that TMME is accessed from 30+ countries. This has resulted in a steady flow of high quality manuscripts from across the globe, some which present innovative mathematics content, and others which tackle issues related to classroom pedagogy, such as the use of technology and history to enhance the teaching and learning of mathematics. In this issue, two of the articles provide research based recommendations for the use of Computer Algebra Systems (CAS) in the classroom, whereas one analyzes in depth the use of Dynamic Geometry Software (DGS) for problem solving, posing and facilitating the discovery and generalizations of mathematical results via the use of such software. These three papers also contain non-trivial mathematics relevant for the middle and high school classroom. The other papers of this issue present cross-national curricular comparisons and a glimpse into the genius of John von Neumann.

The first article by Nurit Zehavi and Giora Mann (Israel) builds on a previous study on the use of CAS, and reports on encouraging an awareness of ways in which CAS manipulate symbols algebraically, their corresponding graphical representations and meanings, and its usefulness in fostering connections in analytic geometry. This article makes an interesting comparison of a traditional solution and CAS-based solution of a problem in analytic geometry with recommendations for the teaching and learning of analytic geometry. The use of CAS in the classrooms has engendered criticisms from opponents of the use of such technology in the classroom because it allows students to engage in "button pushing" without understanding the mathematics. Robyn Pierce (Australia) addresses this issue in the third article and argues that CAS can help students develop algebraic insights and facilitate the ability to link different representations. Pierce's article also outlines a framework useable for planning such activities and monitoring student's progress on CAS.

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International studies such as TIMSS and the recently concluded PISA have shown that the U.S is lagging behind many countries in Europe and Asia. While a positive consequence is more collaboration between mathematics educators in the U.S with researchers in countries like Singapore, Japan and the Netherlands to improve school mathematics curricula and teacher education programs with the aim of positively impacting students in the classroom, a negative consequence of international studies is the general “bashing” of the U.S. educational system and blaming school teachers. One of the arguments commonly heard in the U.S is to increase the mathematics content that prospective mathematics teachers are exposed to in schools and universities. The article by Bettina Dahl (USA) compares secondary mathematics teacher programs in Denmark and Virginia and reports on how much mathematics students get in these countries and the different “values” communicated to them by their respective teacher education systems. The article allows readers to draw their own conclusions about the pros and cons of different educational systems.

Steve Humble (England) contributes an historical article on the legendary human computer John von Neumann and touches on one of his numerous seminal contributions to the science of simulations. The mathematics in this paper is very accessible to high school students interested in probability. This article implicitly reveals the value of technology, in the form of freely available JAVA applets on the world wide web for introducing students to beautiful results in probability theory via the use of simulations.

The final article by Constantinos Christou and colleagues (Cyprus) investigates ways in which students engage in problem solving and problem posing in a dynamic geometry environment. Many of today’s hand held technology (or otherwise) typically include software such as Cabri or Geometer’s Sketch Pad. The question is how do we use this powerful technology to our benefit in the classroom to enhance learning? The interesting avenues of mathematical exploration chosen by six pre-service teachers on two geometry problems provides us with research based insights on the mathematical and pedagogical outcomes of DGS.

It is hoped that MCTM members and all our worldwide readers will enjoy this issue. Readers are encouraged to submit manuscripts that critique or provide commentary on previously published manuscripts. Offers for reviewing manuscripts and book reviews are also welcomed.

Instrumented Techniques and Reflective Thinking in Analytic Geometry

Nurit Zehavi and Giora Mann

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Abstract: *In a previous study that explored epistemological perspectives on solving problems with Computer Algebra Systems (CAS) we concluded that awareness of the special ways that the software utilizes symbols in algebraic manipulations and in implicit plotting should be encouraged (Zehavi, 2004). Such awareness is required for, and encouraged by treating geometry analytically with a symbolic-graphical system. In this paper we compare a traditional solution of a problem in analytic geometry with CAS-based solutions to the same problem. The discussion will focus on the role of reflective thinking, namely selection of techniques, monitoring of the solution process, insight, and conceptualization, play in the creation of instrumented techniques (Guin & Trouche, 1999). Teachers, who experienced learning activities from a resource e-book for teaching analytic geometry with CAS, contributed to the design of tasks and to the analysis of instrumented techniques.*

Introduction

Since 1996 a team at the Weizmann Institute of Science has been preparing CAS-based activities for junior high school, and for the senior high school. The activities complement the current syllabus aiming to broaden learning opportunities and to promote greater mathematical understanding. Research studies that accompany the development of the learning activities indicate that students' interaction with CAS and students' reflections are intertwined (Zehavi & Mann, 2003; Mann, Zehavi, & Halifa, 2003). We have recently developed a resource e-book for teaching Analytic Geometry, containing activities for students, and an extended teacher guide including annotated CAS files (we use *Derive*). Although symbolic-graphical technology is not allowed at this stage in the final exams, an increasing number of mathematics teachers incorporate this technology in their work. The activities were presented to in-service teachers in professional workshops as part of the formative development of the learning materials. The practicing of instrumented techniques led the teachers to extend the pedagogical scope of the activities. Here we discuss the epistemological value added to the pragmatic production of solutions by instrumented techniques, [see: Guin & Trouche (1999), Artigue (2002), and Lagrange (2005)]. We first analyze a traditional solution to the problem of finding the *director circle of an ellipse*. The analysis method we developed for this purpose links the cognitive and meta-cognitive levels, namely the execution of the solution and the reflective thinking. Then we analyze by the same method CAS-based solutions. Implications of the

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analysis to our understanding of the changes that computer algebra systems bring to mathematics education will appear in the concluding part.

The Analysis Method

The steps of the solution are analyzed in two levels: *execution* and *reflective thinking*. The basic components of the *execution* of problem solving in analytic geometry (or any other domain that requires modeling) are: constructing a mathematical *model* for the problem, *manipulations* within the model to obtain results, *interpretation* of the results in the contexts of the problem, and *representations* (graphical or symbolic) of the model or the manipulations or the interpretations. We use the term *reflective thinking* for the meta-cognitive level referring to four categories: selection of *techniques*, *monitoring* of the solution process, *insight* or ingenuity, and *conceptualization* (i.e. connecting concepts and meaning).

The reflective thinking components are inferred from the written 'execution' of the solution and from explanations given in textbooks. To make the reflective thinking more transparent we asked teachers and students to add annotations to their CAS worksheets and to discuss them verbally. The classification associated to solution steps, however, should be regarded as subjective.

A traditional solution

The problem is presented as a task: *"Find the locus of the points of intersection of perpendicular tangents to the ellipse defined by the equation* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$*".* This task appears in traditional textbooks and is regarded as quite sophisticated for high school students. Therefore, some textbooks provide a solution to the problem (For example, Barry, 1963). The steps of the traditional solution of this problem are described in the following (Chart 1).

Step 1

<u>Reflective thinking: selecting technique</u>

The equation of a tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $y = mx+\sqrt{a^2m^2+b^2}$ *or* $y = mx-\sqrt{a^2m^2+b^2}$.

A line parallel to the vertical axis is not considered in this equation.

Step 2

Execution: Modeling

A tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ passes through a point (p, q) if and only if $q = mp + \sqrt{a^2m^2+b^2}$ *or* $q = mp - \sqrt{a^2m^2+b^2}$

We look for values of *m* that satisfy the above condition.

Step 3

Reflective thinking: insight, selecting technique

In order to utilize Viète's formula the equation $q = mp \pm \sqrt{a^2m^2+b^2}$

should be "simplified" in a special way to get

a quadratic equation in the form $Am^2 + Bm + C = 0$.

Step 4

Execution > manipulations

…….. $(p^2-a^2)m^2 - 2pqm + q^2 - b^2 = 0$

Step 5

Reflective thinking: conceptualization

The product of the slopes of two orthogonal lines is -1.

Step 6

Execution: manipulations

Viète's formula states that $m_1 \cdot m_2 = \frac{q^2-b^2}{p^2-a^2}$. Thus we have $\frac{q^2-b^2}{p^2-a^2} = -1$.

Step 7

Execution: interpretation, representation

The standard form of a Cartesian equation for the locus of points whose coordinates (p, q) verify the equation $p^2+q^2=a^2+b^2$ is $x^2+y^2=a^2+b^2$.

Chart 1: Steps of a traditional solution

Only a few high school students can come up with such a solution that requires good mastering of the mathematical meaning of symbols and a global view of the

task. We dare to say that one should almost know the solution before actually working on it: the analysis indicates that conceptualization and insight are prior to the execution steps.

CAS-based solution

The task was presented to the teachers in a workshop. In order to get a visual product, the task involved a specific numerical example, $\frac{x^2}{9}+\frac{y^2}{4}=1$. In Chart 2 we present an example of a CAS-based solution using *Derive*'s notation.

Step 1

Execution: Modeling 1

The equation of a line (not parallel to the vertical axis) that passes through (p, q) is $y = mx - mp + q$. By substitution we get an equation for the *x* values of the intersection points of the ellipse and the line.

$$\frac{x^2}{9} + \frac{(m\cdot x - m\cdot p + q)^2}{4} = 1$$

Step 2

Reflective thinking: Selecting technique

$$x^2\cdot(9\cdot m^2 + 4) + 18\cdot m\cdot x\cdot(q - m\cdot p) + 9\cdot(m^2\cdot p^2 - 2\cdot m\cdot p\cdot q + q^2 - 4) = 0$$

Step 3

Execution: modeling 2

$$(18\cdot m\cdot(q - m\cdot p))^2 - 4\cdot(9\cdot m^2 + 4)\cdot 9\cdot(m^2\cdot p^2 - 2\cdot m\cdot p\cdot q + q^2 - 4) = 0$$

We look for values of *m* that satisfy the above condition, i.e. Discriminant = 0.

Step 4

Execution: manipulations

$$m = \frac{\sqrt{(4\cdot p^2 + 9\cdot(q^2 - 4))} - p\cdot q}{9 - p^2} \vee m = \frac{\sqrt{(4\cdot p^2 + 9\cdot(q^2 - 4))} + p\cdot q}{p^2 - 9}$$

Step 5

Reflective thinking: conceptualization

The product of the slopes of two orthogonal lines is -1, thus

$$\frac{\sqrt{(4\cdot p^2 + 9\cdot(q^2 - 4))} - p\cdot q}{9 - p^2}\cdot\frac{\sqrt{(4\cdot p^2 + 9\cdot(q^2 - 4))} + p\cdot q}{p^2 - 9} = -1$$

Step 6

Reflective thinking: monitoring

Plot the equation in Step 5

Where do the "holes" come from? (see later)

Is this a circle? Why?

Let' simplify the equation.

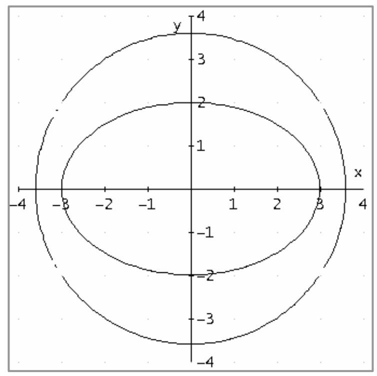


Figure1. Director circle with "holes"

Step 7

Execution: manipulations

Simplify the equation in Step 5,

$$\frac{q^2 - 4}{p^2 - 9} = -1$$

$$p^2 + q^2 = 13$$ and plot.

Step 8

Execution: interpretation, symbolic representation

The standard form of a Cartesian equation for the locus of points whose coordinates (p, q) verify the equation $p^2 + q^2 = 13$ is $x^2 + y^2 = 13$.

Chart 2: Steps of a CAS-based solution

In contrast to the traditional solution which began with prior reflection, the CAS solution started with writing a "simple" equation for finding the intersection points of a line with slope *m* that passes through a point (p, q) and the given ellipse. Selecting a familiar technique for simplifying the equation led to the well known model ($\Delta = 0$) and utilizing the symbolic mechanism of the software to obtain two algebraic solutions for *m*. Translating the necessary and sufficient condition (if and only if) for lines to be perpendicular into an equation (Step 5) gave a strange result that called for monitoring. In Step 6 the teachers used the software to plot the graph of this equation. Various reactions were heard: Where do the "holes" come from? Our error? Bug of the implicit plotting? Is this a circle? Why? Let's simplify the equation: $\frac{q^2-4}{p^2-9}=-1$.

Standard algebraic manipulations and interpretation yield the representation in the form of equation of the circle $x^2+y^2=13$. The circle and the given ellipse have the same center. In the general case, i.e. for an ellipse given by the canonical equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, the radius of circle which is obtained by a working session as above is equal to $\sqrt{a^2+b^2}$. This circle is called the *directo*r circle (or *orthoptic* circle, or *Monge* circle) of the ellipse given by the equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

The surprising holes around the four points (3, 2), (3, -2), (-3, -2), (-3, 2) are explained algebraically by the denominator in the equation; the graphical interpretation draws our attention to the exceptional tangents (to the ellipse) that are parallel to the x-y axes. The instrumented scheme that the teachers implemented has an epistemic value: The problem is that we work in a neighborhood of a singular point of the equation whose graph has been plotted (the singularity is caused by what we did at the beginning: we did not consider lines parallel to the y-axis). This is a general problem for computerized drawing of curves (see Dana-Picard, 2005)

After the surprising phenomenon of the holes has been understood, another question appeared: how can one see from Equation (1) that it would actually simplify to equation (2)? In the nominators of Equation (1) we can see the pattern (A – B)(A +B). At this stage the teachers became interested in investigating the expression under the square sign. Plotting the inequality $4p^2+9(q^2-4)\geq 0$ added more insight to the solution process: we see the outside of the given ellipse; since the expression under the square sign appear in the solution for the slope *m* of the tangent, the solution of the inequality shows, in fact, that it is impossible to draw a real tangent to the ellipse through a point within the circle.

The teachers suggested adding pragmatic value to the above exploration, namely, to produce pairs of perpendicular tangents to the ellipse given by the

equation $\frac{x^2}{9}+\frac{y^2}{4}=1$ (see Figure 2). Some of them claimed that this should be stated initially as the goal of the task, so that the efforts in identifying the geometric locus of points of intersection of such pairs of tangents would be the means to achieve the goal. Others argued against such a pragmatic goal and preferred to consider the animation of pairs of tangents as an implementation of the result. The instrumented technique needed for this task involves the use of a slider bar to view in a dynamic way pairs of tangents that intersect in a point T= $(p, \sqrt{13-p^2})$ on the director circle whose equation is $x^2+y^2=13$. We substitute $\sqrt{13-p^2}$ for *q* in one of the expression for *m* in Step 4, and write the equation of two perpendicular tangents through T.

$$m1 := \frac{\sqrt{(81-5\cdot p^2)} - p\cdot\sqrt{(13-p^2)}}{9-p^2}$$

$$\left[y = m1\cdot x + \sqrt{(13-p^2)} - m1\cdot p,\ y = -\frac{1}{m1}\cdot x + \sqrt{(13-p^2)} + \frac{1}{m1}\cdot p\right]$$

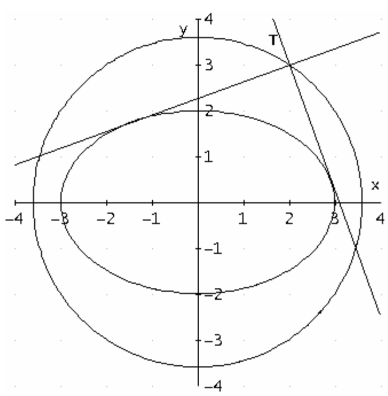


Figure 2. 'Animation' of perpendicular tangents

The teachers agreed that visualizing the tangents should be an integral part of the activity because it can provides feedback and control to student's actions. Not less important is the satisfaction feeling in obtaining a nice product.

Changes that computer algebra systems bring to mathematics education

Based on the example we described in this paper, and other similar examples we attempt to identify changes that CAS brings to the mathematical environment of teachers and students.

In a traditional solution one must have a full blown strategy from the beginning in order to solve the problem, and to master sophisticated methods of manipulations (e.g. Viète's formulas) to carry out the strategy. In a CAS solution one can start the solution process by using the symbolic power of the software to perform familiar manipulations and then obtain representations of the results. Having some result and being free from technical work one can gradually consolidate a solution strategy.

One implication of the above is that some topics of the core traditional curriculum may become obsolete. Viète's formulas and other algebraic ingenuities have been taught to facilitate manipulations by hand, but one can do without them when using software that was designed to perform the manipulations. These human culture developments should be appreciated and recognized, but not necessarily in the core mathematics curriculum. Instead we should develop strategies that develop awareness to pragmatic and epistemic values of instrumented techniques.

Our analysis indicates that in traditional solutions conceptualization and insight are prior to the execution steps, while in CAS solution the reflection steps (conceptualization, insight, monitoring, and selecting techniques) are inseparable from the execution steps.

A consequent implication is that advanced problems that have been traditional reserved for those few gifted with mathematical intuition, can now be accessed effectively by a greater population with appropriate instruction by the teachers.

The role of the teacher who teaches with modern technology is very complex, including aspects of the technology, of mathematics, and of didactics. Thus the structure of a computer based activity should initially be made clear to the teacher at a global level. To be able to guide effectively students in using the various instrumented techniques, teachers first need to review the relevant mathematical methods; they also need some experience and exposure to learning events that have the potential to intertwine execution and reflection. But most importantly, they should be partners in the task-design process. (This actually happened in one of our workshop that introduced the director circle of an ellipse.)

After finding the director circle of the ellipse the teacher usually explored loci of points of intersection of perpendicular tangents to an hyperbola and to a parabola, identifying the differences between the three cases. In one workshop some teachers were interested in finding the locus of the intersection point of tangent to an ellipse having an angle of 45^o between them. In the case of 90^o we had the simple equation $m_1 \cdot m_2 = -1$.

Here we have the equation $|\dfrac{m_1 - m_2}{1 + m_1 \cdot m_2}| = 1$.

Plotting this implicit equation for the ellipse we used before gives a graphical representation of the locus (Figure 3).

A more traditional symbolic representation can be obtained by algebraic manipulations.

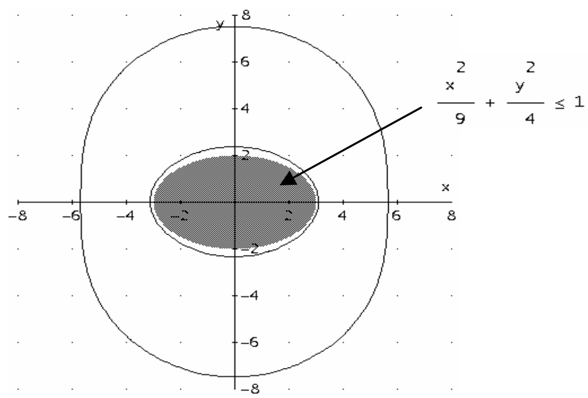
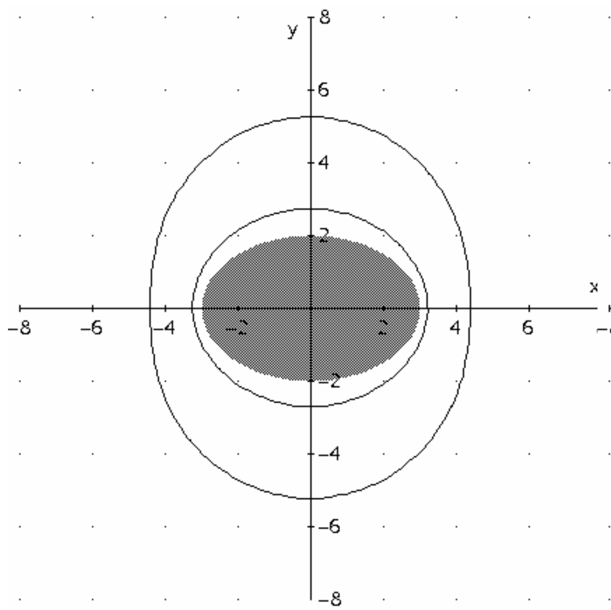


Figure 3. Seeing the ellipse in $45^{\circ}/135^{\circ}$

Now the questions come quick and fast: what about other angles (Figures 4, 5)? What about hyperbola, parabola?



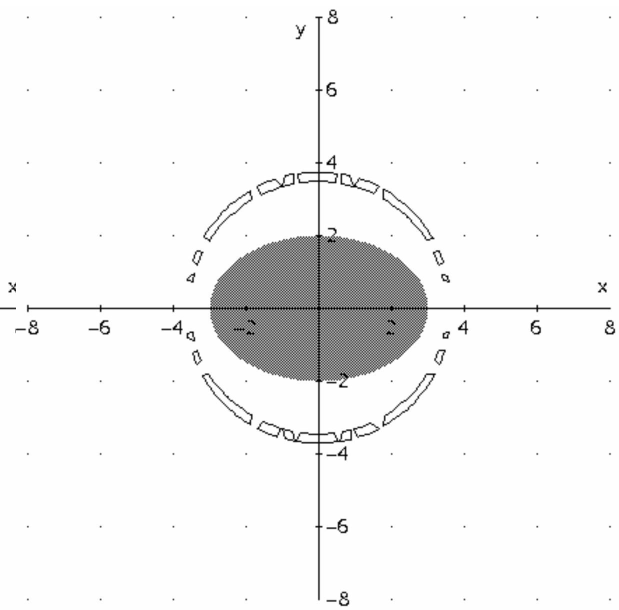


Figure 4. $|\dfrac{m_1 - m_2}{1 + m_1 \cdot m_2}| = 2$

Figure 5. $|\dfrac{m_1 - m_2}{1 + m_1 \cdot m_2}| = 20$

In this problem, as in the one we presented in detail, the implicit plotting plays an important role in making algebraic manipulation by the software and conceptual insight of the users inseparable. We invite the interested readers to explore the problem (with CAS, of course) and design a didactic sequence of tasks that suits their educational goals and the needs of their students.

Acknowledgement

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A Comparison of the Danish and the Virginia Secondary teacher Education System: Their values and Emphasis on Mathematics Content Knowledge

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Abstract: In this paper I will first examine an example of secondary mathematics teacher education in the USA, namely Virginia, then compare it with the secondary teacher education in Denmark. The purpose is both to investigate how much mathematics the students get in the respective systems and secondly to see what this type of teacher education communicates about the values emphasized in the various countries' education systems. I spent more time on explaining the Danish education system than that of the USA and the single states since it is assumed that the reader is familiar with these systems. One cannot necessarily deduce from number of courses how much mathematics the student actually "gets" since this depends on particular passing requirements as well as requirements of entry, the specific content of the courses both in terms of levels of difficulty and topics, etc. However, a comparison of course load indicates how much study of "mathematics" is perceived enough, or minimum, to teach secondary mathematics from the national or state political perspective (who might see a direct link between course load and knowledge).[1]

1. USA

The USA has on national level no direct legislation in education matters. It is up to each state to determine. However, there are some national legislations such as the No Child Left Behind Act of 2001 (NCLB)[2]. This federal legislation requires states to demonstrate progress from year to year in raising the percentage of students who are proficient in reading and mathematics and in narrowing the achievement gap. NCLB sets five performance goals for states:

[1] I wish to thank Gwen Lloyd, Associate Professor, Department of Mathematics, Virginia Tech for valuable comments to this paper. The errors remain my own.
[2] http://www.ed.gov/nclb/landing.jhtml?src=pb

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- All students will reach high standards, at a minimum attaining proficiency or better in reading/language arts and mathematics by 2013-2014.
- All limited English proficient students will become proficient in English and reach high academic standards, at a minimum attaining proficiency or better in reading/language arts and mathematics.
- All students will be taught by highly qualified teachers by 2005-2006.
- All students will learn in schools that are safe and drug free.
- All students will graduate from high school.

This legislation mainly sets goals for the teaching, it does not determine how teaching should take place, methods used, subjects to teach. In terms of teacher education there is on national level the National Council for Accreditation of Teacher Education (NCATE)[3], which is the professional accrediting organization for schools, colleges, and departments of education in the United States. It is a coalition of over 30 organizations representing teachers, teacher educators, policymakers, and the public. There is also the national Praxis I and II tests. The Praxis Series tests are currently required for teacher licensure in 39 states and U.S. jurisdictions. These tests are also used by several professional licensing agencies and by several hundred colleges and universities. Because The Praxis Series tests are used to license teachers in many states, teacher candidates can test in one state and submit their scores for licensure in any other Praxis user state.[4] Therefore to compare "USA's" teacher education system one must focus on the level of the states. In this paper I have decided to study Virginia.

1.1 Virginia

The Virginia Licensure requirements[5] for teaching mathematics grades 6-12 are that the student has completed a major in mathematics or 36 semester/credit hours of course work distributed in each of the following areas: Algebra (including linear and abstract algebra), Geometry (including Euclidean and non-Euclidean geometries), Analytic geometry, Probability and statistics, Discrete mathematics (including the study of mathematical properties of finite sets and systems and linear programming), Computer science (including computer programming), and Calculus (including multi-variable calculus). This should also include knowledge in the history of mathematics. Students should obtain passing scores of 147 on the Praxis II (mathematics content) test.

2. Denmark

The Danish school system is different from the US system. Formal schooling begins at the age of seven. Then follows 9 years of compulsory comprehensive

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[3] http://www.ncate.org
[4] http://www.ets.org/praxis/prxtest.html
[5] http://www.math.vt.edu/people/lloyd/math_licensure/VA_licensure.pdf

schooling. This is called the *Folkeskole* (People school). In the first grade the student is placed in a class according to age only. The students stay together through all 9 years. There is an optional one-year pre-school class from the age of six commonly called Grade 0. The class has it own room - the socalled "Classroom" and the class also has a "Class-teacher". The teachers have one big common room. Teacher preparation usually takes place at the teachers' private home. Hence the teachers are not at the school the whole day but often only around the times where they teach or if there is a meeting. There is an option of a grade 10 in the *Folkeskole*. This is usually chosen by students who are not sure what education they want.

The teacing during the first nine year covers the following subjects: Danish (all grades), English (3-9), Christian studies[6] (all level except the level where the confirmation preparation of the Evangelical Lutheran Danish National Church takes place), History (3-9), Social studies (8-9), PE and sports (all levels), Music (1-6), Art (1-5), Textile design, wood/metalwork and home economics (one or more levels within grades 4-7), Mathematics (all levels), Science/technology (1-6), Geography (7-8), Biology (7-9), Physics/chemistry (7-9), German (or sometimes French, non-compulsory, 7-9). Bilingual children (0-10) are being given instruction in Danish as a second language. The Minister of Education lays down provisions pertaining to the instruction in Danish as a second laguage for biligual children and to mother-tongue teaching of children from the European Union, the European Economic Area, the Faroe Islands, and Greenland. There is also a number of optional topics in grades 8-10.

The Danish Parliament makes the decisions governing the overall aims of the education, and the Minister of Education sets the targets for each subject. But the municipalities and schools decide how to reach these targets. The Ministry of Education publishes curriculum guidelines for the subjects, but these are seen as recommendations and are not mandatory for the municipalities. Schools are permitted to draw up their own curricula in accordance with the aims laid down by the Minister of Education. However, nearly all schools choose to confirm the centrally prepared guidelines as their binding curricula.[7] The *Folkeskole* is not an examination-oriented school and school failure is almost non-existing. The *Folkeskole* builds on the principle of differentiated teaching to sustain the

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[6] The Act (1 August 1994) states: "6. *(1)* The central knowledge area of the subject of Christian studies shall be the Evangelical Lutheran Christianity of the Danish National Church. At the oldest form levels, the instruction shall furthermore comprise foreign religions and other philosophies of life. *(2)* If requested, a child shall be exempted from participation in the instruction in the subject of Christian studies, when the person who has custody of the child submits a written declaration to the headteacher of the school to the effect that he/she will personally assume the responsibility of the child's religious instruction. ... If the child has reached the age of 15, exemption can only be granted with the child's own consent." 53(1) Upon negotiation between the municipal council and the ministers in the municipality, time shall be set aside for the preparation for confirmation. If agreement cannot be reached between the parties, the decision shall be taken by the municipal council upon consultation of the parish councils involved.

[7] http://www.uvm.dk

principle that all students should be given adequate challenges. At grades 8-10 the teaching may be organised in teams within the individual class and across classes and grades.

In grades 1-7, assessment is given either in writing or verbally in the form of meetings where student, parents, and Class-teacher take part. In grades 8-10 a grading system (13-point marking scale) is added and the student receives a written report at least twice a year for the leaving examination subjects. Examinations are offered at grades 9 and 10. There are national standard rules for all examinations. The papers for the written examinations are set and marked centrally. The other examination questions shall be drawn up by the teacher or by an external examiner according to a decision taken by the Minister. Examinations are not compulsory and each examination subject is assessed on its own merit; results cannot be summed up to give an average mark. The school shall issue a leaving certificate for students who leave school at the end of grade 7 or later.

88% attend municipality schools and 12% that attend various forms of private or “free schools” that are subject to various rules. The idea with the free-schools is that a number of parents or firms can get together and run a school with help from state funding (Selander, 2000, p. 65). The private schools receive a grant (“per student per year”) for their operational expenditures which in principle matches the public expenditures in the municipal schools - less the private school fees paid by the parents. This is to ensure that public expenditures for the private and municipal schools follow the same trend.

After the *Folkeskole* young people have five options: (1) The *Almene Gymnasium* (General Academic High Schools) is a three-year upper secondary education. (2) The *Højere Forberedelseseksamen (HF)* (Higher Preparatory Examination Course) is a two-year course that is meant for adults and for students who have completed the 10th grade of the *Folkeskole*. These two are preparatory for higher education. (3) The three-year Commercial High School (*Handelsskole/Handelsgymnasium*). (4) The three-year Technical High School (*Teknisk skole/Teknisk Gymnasium*). Both of these give access to higher education as well as prepare for professional activities in the private sector. (5) Vocational education and training courses (*Erhvervsuddannelserne*) with theoretical training (1/3) at technical schools and practical training as an apprentice (*lærling*) (2/3) at an enterprise. Teacher education for these 5 places is different. (1) and (2) are the same, and I will describe this more detailed below. Teacher education for the *Folkeskole* is also different. While the *Folkeskole* is administered on the level of municipality, the *Almene Gymnasium* and *HF* are administered at the level of county.

The subjects offered in the *Almene Gymnasium* and *HF* are the following: 1. Astronomy, Visual arts, Biology, Danish, Computer science, Design, Drama, Information technology, English, Business economics, Film and TV studies, Philosophy, French, Physics, Geography, Greek, History with civics, Physical

education and sport, Italian, Japanese, Chemistry, Latin, Mathematics, Music, Science, Classical studies, Psychology, Religious studies, Russian, Social studies, Spanish, Technology, German. The students cannot choose freely, but there are some bindings, which I will not go into detail with here as this is not the scope of the paper.

2.1 Teacher education for the *Folkeskole* (grades 1-9 (10))

There is a four-year unified training system for the whole compulsory nine-year schooling (including the optional grade 10). It takes place at one of the country's 18 *Seminarier* (College of Education). One study-year amounts to a full-time job. The school-year begins in the beginning of August and ends at the end of June. A study-year consists of teaching, lecturing, supervision, student teaching, independent work, and various study form such as group work, project work etc. Through the four years the student takes courses within the areas listed below. The number in brackets shows how much it weighs in relation to a one-year-study full time. The student is obliged to study four *Liniefag* “Line-subjects” which would be the subjects that the student will then mainly be teaching in the *Folkeskole*:

- Christian studies (0.2)
- The *Liniefag* Danish or Mathematics (0.7)
- 3 other *Liniefag* (3 x 0.55)
- Pedagogical subjects:
    - The school in the society (0.1)
    - Pedagogy (0.2)
    - Psychology (0.2)
    - General didactics (0.2)
- Teaching practics (0.6) (24 weeks at a school).

If a student chooses mathematics as the 0.7 *Liniefag*, this amounts to 1150 workings hours. At for instance Aalborg Seminarium, this in practice means 338 lesson-hours over a three year period. The rest of the time is spent on independent study and preparation for the exam.[8] The four *Liniefag* are being chosen among the 18 subjects that exist in the *Folkeskole*. One of the *Liniefag* must be either Danish or Mathematics, but the students can choose both subjects. The four *Liniefag* shall represent at least two of the following areas: The humanistic area, the natural science area, and the practical-musical area.

8 http://www.aalsem.dk/C1256D3B003DB5CC/0/B9FDD5096225DD02C1256EF200699407?OpenDocument

2.2 Teacher education for the *Almene Gymnasium* (grades 10-12)

One can usually not teach just mathematics in the *Almene Gymnasium*, one need to have *Pædagogikum* (pedagogical competence) and *Fagkompetence* (subject competence) in two topics. The formal route is first to study mathematics as well as another subject at a university. Most mathematics teachers in the *Almene Gymnasium* have either physics or chemistry as their second subject, but any combination is possible. One needs to study these two subjects to the level of a *Candidata/Candidatum* (female/male) degree. This degree can be translated into a US Master's degree. It consists of a *Hovedfag*, which directly translated means 'major subject', but not to be confused with what is called 'a major' when studying at a US university. In Denmark it is 3.5 years of full time study. The other part of the degree is a *Sidefag/Bifag*, which directly translated means 'minor subject', but not to be confused with what is called 'a minor' when studying at a US university. In Denmark it is 1.5 years of full time study, for instance the 3rd - 5th semester of the *Hovedfag* study. When the candidate studies mathematics, all courses from the first semester are mathematics courses. There are not any "general education courses" to give them an all-round knowledge. General education takes place at the *Almene Gymnasium*. At university one gets "specialized".

After the end of the 5-year *Candidata/Candidatum* degree, the teacher candidate can apply for 2-year "Education positions" advertised at the *Alment Gymnasium*. Sometimes these are also adverticed by the County Council. If accepted the student enters into a two-year program which includes a 6-month (one semester) *sidefagssupplering* (Minor Subject Supplementary) of 700 working hours to give teachers extra subject area knowledge. The two-year education is then divided into Subject Competence and and Pedagogical Competence. The pedagogical competence is divided into a Practical *Pædagogikum* and a Theoretical *Pædagogikum*. It is the principal who has the overall responsibility and who appoints a main course responsible (usually one of the teachers at the school). The course responsible plans and coordinates the candidate's education. The course responsible supervises the candidate during the Practical Pædagogikum and determines in collaboaration with a person appointed by the Ministry of Education if the candidate has passed.

2.2.1 Subject Competence

The students who aspire to become teachers study mathematics alongside any other mathematics student. Some universities have previously had education classes, but these things changed alongside the recent reforms (2004) of the licensure. The principal assigns *Faglig Kompetence* on the basis of the candidate showing mastery of the central area of the subjects and its terminology and methods, have knowledge of use of ICT, is able to gain new knowledge when the subjects develops, and has sufficient knowledge of the Danish language.

2.2.2 Practical *Pædagogikum*

During the first year of Practical *Pædagogikum*, the candidate teaches a ”practice class” under supervision by the subjects' teachers. The planning of the teaching takes place in collaboration with the teachers. Parallel to this is a discussion of the students' preconditions, lesson planning, questioning techniques, working methods as well as assessment and evaluation. The total number of teaching and observation lessons are 180 hours corresponding to 480 working hours which includes preparation and work after class. The candidate must also participate in general pedagogical tasks at the school. During the second year the candidate teaches more independently in his or her own classes while periodically receiving supervision. At the same time the candidate must write a final project.

2.2.3 Theoretical *Pædagogikum*

Theoretical Pædagogikum consists of general pedagoy and subject pedagogy. The amount of teaching and preparation for candidates with two subjects is around 530 hours. The teaching is partly distance education and weekend courses. Subject pedagogy consists of three parts: one course in subject didactics in each of the candidate's subjects (around 60 hours per subject), course in subject didactics in subjects related to the candidate's subjects (around 60 hours), and a course in use of ICT in the subjects (around 50 hours). General pedagogy consists of three parts: One course in general pedagogy (around 170 hours) which consists of lesson planning, communication, evaluation, different teaching strategies, teacher collaboration, learning processes, theories of learning, teacher roles. Another course is in organisation culture and school development (around 30 hours) and the third course is in general *Gymnasium* relevant topics (around 30 hours) such as knowledge of the school's computer system, use of ICT as communication between teachers and teachers and students, and it use in the school administration. There is also a Common Course in Subject pedagogy and General pedagogy (around 50 hours). The goal is to make the candidate able to reflect about the subjects and the relation between them in terms of the Almene *Gymnasium* in general, as well as in relation to value and ideas of education and the development of a general and broad competency for the students.

2.3 Values and the principle of general education

2.3.1 Denmark

As stated in Dahl and Stedøy (2004), Denmark has, in line with the rest of the Nordic countries (Iceland, Finland, Norway, and Sweden) the same educational objectives which are equal access to (lifelong) learning, teaching democracy, independence, equality, and the development of critical awareness in students. The focus is broad and comprehensive as opposed to elitist (Andersen, 1999, p. 27). A central goal in Swedish education policy is that students must learn more

than mere knowledge and therefore the teaching of respect for human values is equally important (Swedish Ministry of Education, 2000b). It is necessary to develop a "democratic mentality" in the students (Swedish Ministry of Education, 2000a, pp. 6-9). The purpose of the pre-school as well as the whole compulsory education is to develop the child's ability to function and act socially responsibly, to make sure that solidarity and tolerance are learnt at an early stage, and to counteract traditional sex roles (Swedish Ministry of Education, 2000a, pp. 113-114). One can see a similar focus in Norway in the curriculum of 1997 for grades 1-10. Here it is written in the preface that the general education shall built on basic Christian and humanistic values. It shall promote equality between the sexes and solidarity among different groups in the society (Norwegian Ministry of Education, 1997, pp. 17-18). Also the Danish law for the *Folkeskole* from 1975 reflects this, as it is written that a task for this school is to prepare the students to participation and decision-making in a democratic society and to share the responsibility for solving common tasks (Selander, 2000, p. 70). Therefore the school's education and daily life must build on freedom of spirit and democracy. The present Act of the Danish *Folkeskole* states the following:[9]

> (1) The *Folkeskole* shall – in cooperation with the parents – further the students' acquisition of knowledge, skills, working methods and ways of expressing themselves and thus contribute to the all-round personal development of the individual student.
> (2) The *Folkeskole* shall endeavor to create such opportunities for experience, industry and absorption that the students develop awareness, imagination and an urge to learn, so that they acquire confidence in their own possibilities and a background for forming independent judgments and for taking personal action.
> (3) The *Folkeskole* shall familiarize the students with Danish culture and contribute to their understanding of other cultures and of man's interaction with nature. The school shall prepare the students for active participation, joint responsibility, rights and duties in a society based on freedom and democracy. The teaching of the school and its daily life must therefore build on intellectual freedom, equality and democracy.

The Swedish Education Act states that all children and youths shall have equal access to education, regardless of gender or social or economic factors. This right of education also extends to adults. The education shall "provide the students with knowledge and, in co-operation with the homes, promote their harmonious development into responsible human beings and members of the community" (Skolverket, 2003).

The Nordic countries have therefore the same educational objectives in common, which are equal access to (lifelong) learning, teaching democracy, independence, equality, and the development of critical awareness in students. The focus is broad and comprehensive as opposed to elitism (Andersen, 1999, p. 27). The 'Nordic dimension in education' as discussed by Dahl (2003), is therefore that the teaching of democratic values is as important as the teaching of knowledge. The focus is on a "school for all", adult (lifelong) education, equality, democracy, and a high number of people receiving further education. The systems are decentralised school system with possibilities for choice. The whole school structure is organised in a single track.

[9] http://www.uvm.dk

2.3.2 Virginia

Virginia implemented state-mandated Standards of Learning (SOL) and associated tests (Virginia, 1995). The purpose of the SOL's is:

> The intent of the Virginia Board of Education and the Superintendent of Public Instruction to establish high academic standards for our young people and greater accountability for our public schools throughout the Commonwealth. The Board and the Superintendent concur that Virginia's academic standards need to be measurable in order that parents and taxpayers may see how their students and schools are performing against these high academic standards. While not compromising the rigor which will demand higher performance, we also believe that Virginia's standards must address the educational expectations for ALL Virginia students.
>
> …
>
> The Board and Superintendent believe that high academic standards are the beginning of a multi-year journey to improve educational achievement. For these new standards to make a real difference, we will need to develop accountability measures and consequences for students and teachers, invest in new teaching materials, provide extensive professional development, expand the use of technology, involve parents in the education of their children at the school level, and expect our students to work harder, including doing more work at home.

More specifically the Board and the Superintendent have identified various areas that are critical to the discussion relating to academic standards and accountability. These areas are centrered on that accountability as best being addressed at the school building level. The local school boards must hold individual teachers accountable for their performance and the achievement of their students. The preferred method of school improvement is to reward schools for achieving the targeted improvement in student performance. It is furthermore the state's responsibility is to set expectations for <u>what</u> students should know at key points but it is the responsibility of the local school boards and schools is to determine <u>how</u> the students reach these expectations. There also need to be consequences for students. Hence, in the Virginia Board of Directors resolution from 1995, there is no reference to "democratic education" or development of critical awarenesses of the students. In terms of the topic mathematics, The Board, stated that "Students today require stronger mathematical knowledge and skills to pursue higher education, to compete in a technologically oriented workforce, and to be informed citizens. Students must gain an understanding of fundamental ideas in arithmetic, measurement, geometry, probability, data analysis and statistics, and algebra and functions, and develop proficiency in mathematical skills." What comes closest to the "Nordic model" here is the remark about being "informed citizens". But being an "informed citizens" is not the same as preparing a student for active participation in the democracy nor something that contribute to an all-round personal development.

However, when one looks at the SOL[10] of history and the social sciences, one does see more of the things mentioned in the Nordic countries. For instance the overgoal is stated as follows:

> The study of history and the social sciences is vital in a democratic society. All students need to know and understand our national heritage in order to become informed participants in shaping our nation's future. The History and Social Science Standards of Learning were developed with the assistance of educators, parents, business leaders, and others with an interest in public education.
> The History and Social Science Standards of Learning are designed to
> - develop the knowledge and skills of history, geography, civics, and economics that enable students to place the people, ideas, and events that have shaped our state and our nation in perspective;
> - enable students to understand the basic values, principles, and operation of American constitutional democracy;
> - prepare students for informed and responsible citizenship;
> - develop students' skills in debate, discussion, and writing; and
> - provide students with a framework for continuing education in history and the social sciences.

Particularly point 2 and 3 shows that it seems, that the Virginia SOL does emphasise teaching of democratic values. However, the main difference between Denmark and Virginia still seems to be if this is part of the overall goal of the education system, or something that is “reserved” for particular topics. This is not to downplay the quality of the teaching in history and social science, but of the whole Virginia system seems different than that of the Nordic countries where the education in democracy is clearly stated as part of the overall goal, alongside the teaching of knowledge whereas the overall goal of the Virginia Board of Educators seems to be on accountability and tests. And as stated above, formal examinations do not exist in the Danish school system until grade 8.

3. Discussion

How many mathematics hours?

What does “full time” means? One year of “full time study” in Denmark is equivalent to 60 ECTS (European Credit and Transfer System) of study. This is independent on whether the student is a graduate or undergraduate. In fact there is not this distinction in the Danish system. One is instead an x-semester student, a *Hovedfags*-student, *Sidefags*-student, or when one writes one’s Master’s thesis (*Speciale* = “Specialization thesis”) within one’s *Hovedfag*, one is a *Hovedfags*-student. Depending on where one looks for a “translation” from the European system into the system in the USA, 1 ECTS credits represent the value of 1/2 US credit[11] or 1 ECTS credit could be considered equivalent to 2/3 US semester

[10] http://www.knowledge.state.va.us/main/sol/solview.cfm?curriculum_abb=HSS
[11] http://www.goglobal.ch/incoming/pages/ects.html

credit hours.[12] "Full time" in Denmark is also defined as the student on average spending 37 hours per week "studying" i.e. attending lectures, preparing for classes etc. At Aalborg University, Department of Mathematics, 1 ECTS is defined as 5 "4-hour lessons".[13] "Full-time" is also what as good as all students would do, since this is the only way to receive the State Education Grant.[14] These grants are giving freely to all students on state approved programs and most often these programs are "packages" i.e. that if one decides to study mathematics, then each semester has a number of predetermined courses – so a course in one sense is a x-semester course. At some point during the study, one can choose between a number of "directions" that each has its own "package" of courses that one follows.

In Virginia, at for instance Virginia Tech[15], a full-time undergraduate student takes 12 or more credit hours, a graduate student 9 credit hours or more. During the summer terms the same numbers are 5 credit hours for the undergraduates and 3 credit hours for the graduates. However, during Fall and Spring semester, an undergraduate student can take up to 19 credit hours[16], a graduates 18 credit hours[17], before it is considered "overload" and the student needs special permission.

For the academic year 2004-2005, at Virginia Tech, the fall semester began 23 August and ended 16 December, while the spring semester began 17 January and ends 11 May. The two summer terms are 23 May – 2 July and 5 July – 13 August. In Denmark, the fall semester begins the first week of September and ends the last week of January, Spring semester begins the first week of February and ends the last week of June. January and June are usually examination months. Holidays are at Christmas, Easter, Pentecost, and Assension Day as well as July and September. This means that the Virginia Fall semester + Spring semester + "half" the summer term and the Danish Fall semester + Spring semester are approximately of equal length.

What is clear from this "comparision" is that nothing is "clear" and that a "fair comparison" of what is "full time" becomes very difficult. For the sake of convenience and as a "compromise" between the different interpretations, I will use 15 USA credit hours as being equivalent to the European 30 ECTS. When a future Virginia secondary mathematics teacher have studied mathematics 36 credit, or semester, hours, most would have done this as part of their undergraduate degree. This means that the minimum state requirement for a secondary mathematics teacher in Virginia amounts to 36/30 = 1.2 years "full time" study of mathematics.

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[12] http://www.ncsu.edu/studyabroad/staff/equiv/ects.pdf
[13] http://www.math.aau.dk/index.html
[14] http://www.su.dk/
[15] http://www.registrar.vt.edu/registration/enrollstatus.html
[16] http://www.clahs.vt.edu/UAAO/pol-procedure%20page.htm
[17] http://www.registrar.vt.edu/registration/minmaxenroll.html

The Danish *Alment Gymnasium* mathematics teacher has studied mathematics full time at least 2 years (if they have mathematics as *Sidefag* and then have taken the half-year *Sidefagssupplering*), and many would have studies mathematics full time for 3.5 years (if they have mathematics as *Hovedfag*). The Danish *Folkeskole* teacher has studies mathematics either 0.55 or 0.70 years.

Country/State	**Denmark**		**Virginia**
Grade-level	1-9 (10)	10-12	6-12
Years of mathematics	0.55 – 0.70	2 – 3.5	1.2 (1.8)

This means that the Danish *Gymnasium* teacher has taken between 1.7 and 2.9 times more mathematics courses than the general Virginia teacher. However, the Virginia teacher (grades 6-9) has between 1.4 and 1.8 times more mathematics courses than the Danish *Folkeskole* teacher. Furthermore, in terms of years of study, in Denmark, the *Almene Gymnasium* teacher with teacher qualification has done around 7 years of study, while a *Folkeskole* teacher has done 4 years of study. Secondary mathematics teacher in Virginia have, if they have a Master’s degree, have done 5 years of study.

This is when one looks at the state requirements in both Virginia and Denmark. In practice in Virginia, the universities require more of their students. For instance at Virginia Tech, students in the 5-year program for secondary mathematics teachers must take 30 credit hours of mathematics as undergraduates, two mathematics electives, 3 credit hours in computer science, 3 in probability and statistics, and 10 credit hours of mathematics courses designed for teachers. This adds up to 46 hours of mathematics and mathematics-related coursework as undergraduates. As graduates, they must take 2 graduate mathematics electives and one more mathematics course designed for teachers. This adds up to 9 credit hours. Altogether a student with a graduate degree from the Virginia Tech Secondary Mathematics Education Teacher Licensure Program, Master of Arts in Education, would have taken around 55 credit hours of mathematics.[18] This means that at Virginia Tech, the students take courses amounting to 55/30 = 1.8 years of full time study, hence the 1.8 in the table above. The reason for this is that the state requires “minimum” while the universities often require a higher standard.

Regardless of this, it seems in general that the Danish *Almene Gymnasium* teacher is better prepared in terms of mathematics content knowledge (if one can deduce directly from number of course taken to amount of knowledge) than the Virginia secondary teacher, while the Virginia teacher is better prepared than the Danish *Folkeskole* teacher.

[18] The 55 credits hours comes if one includes the computer science classes as well as the mathematics courses designed for teachers where in some of the courses the mathematics “level” is grade 6-12 but the students “relearn” it while at the same time learn how to teach this level of mathematics.

What about the content knowledge and the more “soft values”?

It seems that Denmark values education in democracy as much as the teaching of knowledge whereas teaching a democratic awareness does not seem to be emphased a lot in the Virginia system - where instead content knowledge and tests are emphasised. It might therefore seem perculiar that student teachers in the US system have less mathematics than in Denmark. Another peculiarity is that the Danish systems emphasises the teaching of general values. This can both be seen in the teacher education for the *Folkeskole* - for instance in the mandatory Christian subject, as well as some of the courses in the theoretical *pædagogikum*.

This is not to say that one system is “bad” the other one is “good”. But in seeing the differences - and perhaps the internal inconsistencies in them – one might learn more about one’s “own” system, how it can be improved, by seeing how other’s have chosen to do it. Perhaps the Danish *Folkeskole* teacher education system needs to require more mathematical content knowledge, perhaps the Virginia state requirements needs to be higher in terms of content knowledge, perhaps the Danish Folkeskole teachers need more than 0.2 years of Christian studies, since “soft values” obviously are important, perhaps the Virginia school system need more emphasis on bringing up good citizens, etc. etc. The reader can make up his or her own mind.

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Algebraic Insight Underpins the Use of CAS for Modeling

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Abstract: Computer Algebra Systems (CAS) performs algorithmic processes quickly and correctly. Concern is commonly expressed that students using CAS will merely be pushing buttons but this paper indicates that, while CAS may assist students, this facility impacts on only one section of the mathematical modeling process: CAS may be used to help find mathematical solutions to mathematically formulated problems. Controlling and monitoring the use of CAS to perform the necessary routine processes requires the mathematical thinking referred to as algebraic insight. This paper sets out a framework of the aspects, and elements of algebraic insight and illustrates the importance of students developing each of the two key aspects: algebraic expectation and ability to link representations. This framework may be used for both planning teaching and monitoring students' progress.

CAS Support Mathematical Analysis in the Modeling Process

Mathematical analysis tools are now not only increasingly powerful but affordable and available. In particular, Computer Algebra Systems (CAS), available for PC's and hand held calculators, offer students support to allow them to work successfully through more complicated or time consuming mathematical manipulations and calculations. Heid (2003) describes clearly three key ways in which CAS can function as a cognitive technology:

- Students can use CAS for the repeated execution of routine symbolic procedures in rapid succession, without diminished accuracy and increased fatigue usually associated with the repetitive execution of by-hand routines…
- Students can assign rote symbolic tasks to the CAS so that they can concentrate on making 'executive' decisions…
- Students can use the CAS to apply routine symbolic algorithms to complicated algebraic expressions, without the confusion students sometimes experience when trying to apply a routine procedure to a complicated expression. (pp. 34-35).

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This capacity for CAS to be used by students to share cognitive load has obvious advantages for mathematical modeling. A CAS allows the user to work in numeric, graphic or symbolic modes and to move between these with mathematical precision and relative ease. For example, in modeling real world situations in order to solve estimation or optimization problems, it is common to begin by collecting and entering numeric data into a software package. CAS allows us to use the graphic mode to examine any pattern in this data; make use of the CAS's statistical capabilities to perform an appropriate regression on the data; and store the result in the graphic function editor ready for graphing or transfer to the symbolic mode. The model which has been created can be examined and refined for the particular case then the impact of changing the various parameters may be explored until a general model is developed or that notion discarded.

Monitoring CAS Work Requires Algebraic Insight

It must be clear though that CAS does not reduce the need for students to develop their skills in mathematical thinking. Figure 1, below, illustrates the typical process for mathematical modeling. Starting with a real world situation (top left) which must be formulated as a mathematical problem, the mathematician typically collects numeric data or moves immediately to a symbolic representation of the situation (top right). Using symbolic, graphic, numeric or geometric methods the mathematician works on the abstract version of the problem in order to progress towards some particular or general solution. Once a mathematical solution has been developed (bottom right) this abstract solution must be interpreted in terms of the real world (bottom left) and checked for applicability in the situation where this process began. If the solution is not adequate then the process must be repeated. This diagram highlights the fact that, currently, technologies like CAS only impact on one section of the modeling cycle, that is, the process of moving from the mathematically formulated problem to a symbolically formulated particular or general solution.

CAS assists with routines but does not take over the role of mathematical thinking. This is illustrated by Pierce and Stacey, (2001a) who report the following extract from a group interview conducted with first year undergraduate mathematics students working with CAS available for all aspects of learning and assessment:

> **Interviewer:** One of the other things that people argue about is whether or not people are really doing mathematics when working with a computer-algebra system. Are you doing it or is the machine doing it? Who's doing the maths?
>
> **Student A:** I reckon that we are actually doing it. The computer only spits out an answer to what you type into it
>
> **Student B:** It's just like with a calculator…it's just going a bit further, we're not just doing multiplication and division quickly, we're doing simple differentiations and stuff quickly.
>
> **Student C:** Also, you still have to interpret the answer or for that matter interpret the question so you can convert it into what the

computer wants …you're still doing a lot of mathematics. (pp. 153-154).

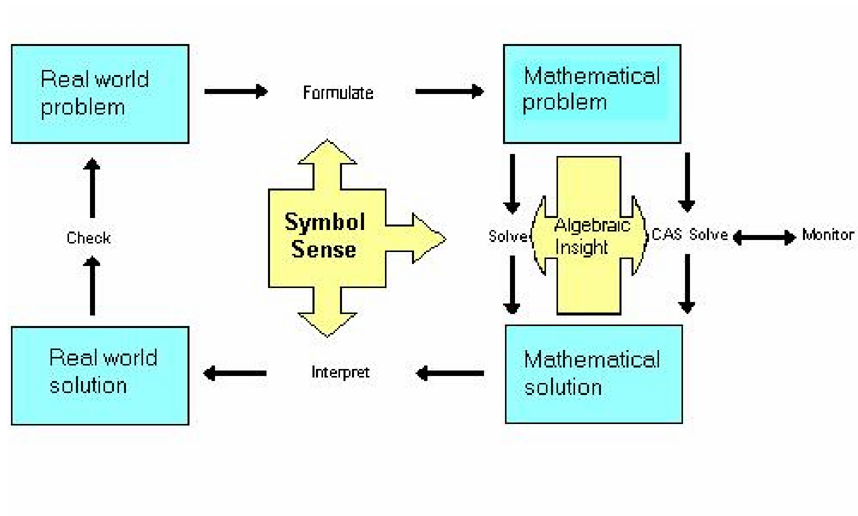

Figure 1. A model of problem solving showing the places of symbol sense and algebraic insight (Pierce & Stacey, 2002)

The processes of formulating and solving the mathematical problem then interpreting the solution *all* require what Fey (1990) and Arcavi (1994) call *symbol sense*. As Fey (1990) pointed out:

> Even if machines take over the bulk of computation, it remains important for users of those machines to plan correct operations and to interpret results intelligently. Planning calculations requires sound understanding of the meaning of operations – of the characteristics of actions that corresponds to various arithmetic operations. Interpretation of results requires judgement about the likelihood that the machine output is correct or that an error may have been made in data entry, choice of operations, or machine performance. (p.79)

Symbol sense is a broad concept encompassing a feel for the power of symbols; an ability to use symbols to express relationships; a sense of when to use symbols and when to use another approach; a sense for which symbolic manipulations will aid progress towards solution of a problem; an ability to recognise equivalent symbolic expressions; an ability to interpret the meaning of symbols in a given context and much more. In this paper we concentrate on the part of symbol sense required to monitor progress towards the solution of a mathematically formulated problem. This is the phase of the modelling process where a CAS may be able to perform the algorithmic tasks involved accurately and quickly. However, in order to direct and monitor this work the user needs the part of symbol sense we call *algebraic insight.*

Technology to date does not impact on the processes of formulation and interpretation; it does however offer alternative methods to progress between the mathematically formulated problem and a mathematical solution. Methods which were, in the past, considered too time consuming or tedious are now accessible. For mathematics teachers and students, limited by the constraints of class timetables and a crowded curriculum, CAS can offer the possibility of tackling interesting real problems which could not previously have been tackled in the time available. The support of CAS to correctly execute the algorithmic routines and manipulation required in a solution process may allow students to test their conjectures and develop their higher level mathematical thinking instead of setting their focus at the micro level of the steps involved in these routines. However, studying the value of the output from such a process of shared cognition will be dependent on correct input and the execution of appropriate commands.

Checking that mathematical expressions have been correctly entered in to CAS and that the output at each stage makes sense certainly requires *symbol sense*. As stated above, to draw specific attention to this part of symbol sense we refer to it as *algebraic insight*. Its place in the broader scheme of thinking required to work within and between the three mathematical representations typically afforded by CAS is illustrated in Figure 2 and the key aspects, elements and some common instances of this concept are outlined in Figure 3.

Figure 2 indicates that *algebraic insight* has two key aspects: first the thinking which allows us to monitor working within the symbolic mode of operating, that is *algebraic expectation;* and second the *ability to link representations*, in this case to link the symbolic with graphical or numeric representations. These two elements of algebraic insight will be discussed and illustrated in the following section.

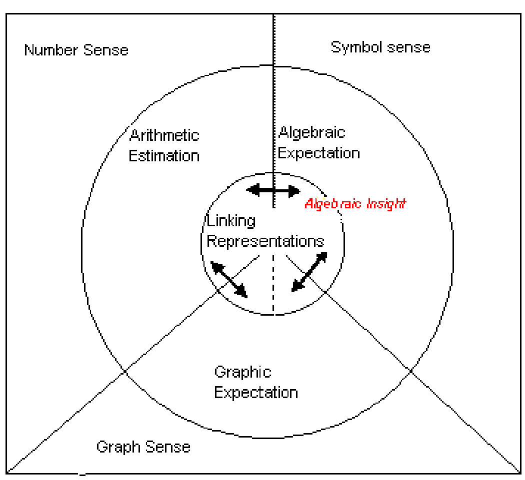

Figure 2. The place of algebraic insight and its components within the senses needed when working with CAS. (Pierce and Stacey, 2004)

Algebraic Insight

The framework set out in Figure 3, is designed to encourage reflection on the skills of algebraic insight and to serve as a basis for teachers in planning and assessing. The framework divides the first aspect of algebraic insight, algebraic expectation, into three elements relating to conventions and basic properties, structure and key features. The second aspect, ability to link representations, has elements which link the symbolic to graphic and numeric representations. The framework is not proposed as a catalogue of specific, itemized skills: the common instances chosen are merely illustrative and will, in practice, be age and stage appropriate.

The divisions within the framework are neither mutually exclusive nor exhaustive. Whilst these features would be desirable, the author does not believe they are fully attainable. The framework was developed in response to the literature and the author's experience of teaching with CAS. It is an attempt to analyze what it is that 'expert' mathematicians do when they look at a result to an algebraic problem and say 'there is a mistake here' or 'that looks all right'.

This is the thinking used in, what the problem solving literature, for example Schoenfeld (1985), calls 'monitoring' or 'control'. Examples of the application of the thinking summarized in the framework are described below.

Aspects	**Elements**	**Common Instances**
1. Algebraic Expectation	1.1 Recognition of conventions and basic properties	1.1.1 Know meaning of symbols
		1.1.2 Know order of operations
		1.1.3 Know properties of operations
	1.2 Identification of structure	1.2.1 Identify objects
		1.2.2 Identify strategic groups of components
		1.2.3 Recognise simple factors
	1.3 Identification of key features	1.3.1 Identify form
		1.3.2 Identify dominant term
		1.3.3 Link form with solution type
2. Ability to Link representations	2.1 Linking of symbolic and graphic representations	2.1.1 Link form with shape
		2.1.2 Link key features with likely position
		2.1.3 Link key features with

	intercepts and asymptotes
2.2 Linking of symbolic and numeric representations	2.2.1 Link number patterns or type with form
	2.2.2 Link key features with suitable increment for table
	2.2.3 Link key features with critical intervals of table

Figure 3 A. Framework for algebraic insight (Pierce and Stacey, 2001b)

Algebraic Expectation

The term Algebraic Expectation is used to name the thinking process which takes place when an experienced mathematician considers the nature of the result they expect to obtain as the outcome of some algebraic process. First, *recognition of conventions and basic properties* of mathematics is a skill based on both knowledge and understanding of the meaning of symbols. At a basic level much of this knowledge will transfer from experience with numbers and arithmetic processes. In addition, to make mathematical meaning explicit our symbols must be arranged in a conventional manner, for example the meaning of , '$2\ dx\ x\ \sin \int$ ' is quite unclear. In this case several alternatives such as $2\int \sin x dx$, or $\int \sin(2x) dx$ are possible and the correct sequence of symbols will rely on the user understanding both the context of the problem and role of each symbol, especially '2' in each of these expressions. Recognition of conventions and basic properties is demonstrated, for example, in three common instances: when students know the meaning of symbols; the appropriate order of operations; and the basic properties of operations.

The second element of *algebraic expectation* involves *identifying structure*. Consider, for example, $\dfrac{a(x+1)^5 + b(x+1)^2}{(x+1)}$. The vinculum indicates the first level of structure in this expression. The numerator can be seen as a strategic group of components consisting of two terms, while the denominator may be viewed as a single object. Considered at another level, $(x+1)$ can be identified as an object which is common to each of three terms which make up this expression. Common instances of identification of structure occur when students identify objects, strategic groups of components or simple factors.

Finally *identifying key features* forms the third element of *algebraic expectation*. Mathematical expressions can be scanned for key features: features that identify the form of the expression indicating whether it is, for example, trigonometric, exponential, or polynomial. Key features also provide information by which expectations may be formed. For functions, for example, these features may lead to expected number of solutions, solution type, number of maxima and minima, and domain and range.

Algebraic expectation may be thought of as a parallel to the arithmetic skill of estimation. One of the most common examples of the need for algebraic expectation is seen when a mathematician looks at two expressions and decides, without doing any explicit calculations or manipulations, whether they are likely to be equivalent. This skill is particularly important for those working with CAS: checking the correct entry of mathematical expressions and matching CAS outputs with conventional by-hand presentation of various mathematical expressions.

The three elements of algebraic expectation may be thought of as three different lights illuminating the attributes of a mathematical expression and hence providing possible clues to inform our algebraic expectation. Students should be encouraged to consider any mathematical expression in the light of each of these three elements as part of their routine in making judgments about how best to progress the solution of a problem or in monitoring their working by-hand or by CAS.

Consider a rule to describe the surface area of a cylinder of given volume V:

$$A = 2\pi r^2 + \frac{2v}{r}.$$

Encouraging a*lgebraic expectation* means asking questions related to each of the elements outlined above. Initially we as teachers need to guide this process until it becomes a habit in our students' mathematical thinking. A 'checklist' of fundamental questions would include "What do each of the letters in this expression represent?" "What is the structure of this expression? Are there any simple factors? What are the key features that you notice and what do they tell us about the function and its possible solutions?

In the example given above:

Recognition of conventions and basic properties could involve: identifying *r, A* and *v* as variables; knowing the convention that the Greek letter π is used to represent a special irrational number; knowing the conventions of implicit multiplication and index notation so that evaluation of $2\pi r^2$ requires $2 \times \pi \times r \times r$; knowing the convention for order of operations so that the multiplication and division precede the addition of the two terms.

Identification of structure means recognising that the two terms on the right hand side may be seen as two processes which could be treated as objects; there is a simple common factor of 2 on the right hand side; and the value of A depends on the value of *r*.

Identification of key features means recognising that the expression in r consists of the sum of a quadratic and a reciprocal function; the dominant term will be the term with r^2; key features such as the squared term mean that the equation may have none, one or two solutions; division by r means that there will be a restriction on the domain since $r \neq 0$.

In this section we have briefly outlined the elements of *algebraic expectation* and considered an illustration applying this thinking to a practical example. This analysis of the symbolic expression does not provide a solution for a problem but alerts the student to the attributes of the expression which may provide important insights for the process of monitoring the solution for a particular problem.
Further *algebraic insight* may be gained by linking the symbolic representation with graphic or numeric representation. In the example above, linking the symbolic form of the quadratic and reciprocal function to a parabola and hyperbola then visually adding the ordinates to gain an approximate image of the sum of these terms will give a visual impression of possible values for A. CAS can assist a student in examining how A varies with r and explore the effect of setting different values of the parameter V.
Next we will focus on the second aspect of *algebraic insight*: *ability to link representations.*

Ability to Link Representations

The process of progressing from working with a single data set to developing a general model will commonly start with collection of data and examination of this data set. A student with algebraic insight will be looking for patterns in the data which will be indicative of the form of a suitable symbolic model. For example, if for equally spaced values of the independent variable there is a very rapid increase in the size of the dependent variable this is likely to indicate exponential growth while a recurring pattern of values will indicate that a trigonometric function may provide the basic form of a suitable model. If the raw data has no obvious pattern then examination of first or second difference or ratios may quickly demonstrate whether the data is best modeled by a linear, quadratic or cubic polynomial or if an exponential function is the more appropriate choice. However, students commonly find using tables of values to identify patterns, and therefore algebraic form, quite difficult.

They commonly find the visual representation provided by a graph of the data more helpful. Ability to link symbolic and graphic representations and ability to link symbolic and numeric representation form the two elements of the second aspect of algebraic insight. We will now consider an example showing some ways in which algebraic insight may support the modeling process. Links to the algebraic insight framework, Figure 3, are included in parentheses.

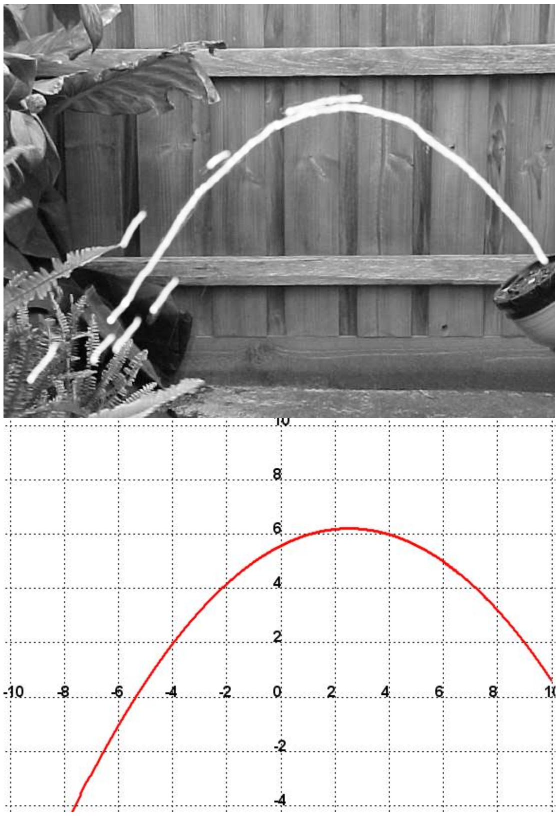

Figure 4. Garden Hose Spray and Graphic Representation

Algebraic Insight Supporting the Modeling Process

Consider the task of creating a mathematical model for the curve formed by a spray of a garden hose. First, working from a photo of a garden spray the student could aim to find a rule for a function whose graph would match this particular spray. In this case algebraic insight will be shown by the student who looks at the image formed by the spray from a garden hose, as shown in Figure 4, and recognizes that this is likely to be best modeled by a quadratic function (2.1). Further, key features such a the critical values of maximum, minimum or intercepts may be identified from a graph and in turn linked to values of various parameters of a function (2.1). A student who knew that a quadratic may be described by several equivalent expressions and that in this case the form $f(x) = a(x-h)^2 + k$ would prove easiest for finding a symbolic expression to describe the path of the water demonstrates a deeper level of algebraic insight (1.2, 1.3, 2.1). Algebraic insight allows the student to make such links between the numeric or graphic representation and their symbolic equivalent.

Recognising that the function rule which describes this graph, $f(x) = -0.1(x - 2.5)^2 + 6.2$, will be equivalent to an algebraic expression which will also be a polynomial of degree 2, with a co-efficient of -0.l, a term in *x,* and a constant term with a value between 5 and 6 requires algebraic insight (2.1). Once a symbolic representation of the particular set of data has been achieved then the consequences of changing various parameters may be explored in a systematic manner (1.1, 1.3). Students may be encouraged to make conjectures and discover "what happens if….". This may be done as an abstract exercise without regard to the initial context but equally results obtained this way may also be interpreted in terms of the real life scenario and checked for reasonableness. In this way a student may move from the particular rule which matched this hose spray to a general rule which may be adapted, according to guidelines, to fit other sprays.

CAS Support Learning Algebra through Strategic Exploration

Developing students' algebraic expectation is important if they are to harness the power of CAS to support their working for iterative, complex or other time consuming manipulations where working by hand would take much longer or be open to simple errors. Students require a basic level of such understanding in order to even enter expressions correctly into a CAS (1.1), in particular to identify structure (1.2) and hence make appropriate use of parentheses. Once some very basic facility with the CAS is established it is also possible to use CAS to assist in the further development of students' algebraic expectation. For example, recognition of familiar patterns and relationships is the key to progressing work with symbols. This includes such strategies as identifying common factors, difference of two squares, perfect squares; coming to understand = as indicating the equality of the expressions linked by this symbol ; and later rules for derivatives and anti-derivatives. CAS may be used to explore strategic sets of examples which will give the student exposure to many correct simplifications, for example. Our experience is that as students start to see a pattern they may make conjectures which they test with CAS then progress to finding that working in their own head can be more efficient than using the CAS. At the same time, knowing that the support of CAS is available increases students' confidence to progress in mathematics.

Conclusions

CAS may be used to support and extend students' work in mathematics and it may also be used as a pedagogical tool. CAS may be used effectively to support students' work in mathematical modeling. The use of CAS does not preclude the need for mathematical thinking, it in fact highlights the need for symbol sense and in particular the two aspects of algebraic insight, namely algebraic expectation and ability to link representations. Mathematics teaching has, out of necessity, focused a great deal of time and attention on algorithmic routines. Since CAS does these effectively, attention may now be directed towards deliberately teaching these skills of algebraic insight.

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Von Neumann and Computers

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The title of this paper should not be Von Neumann and computers, but Von Neumann and the Von Neumann machine. Von Neumann may be famous for many things but humility was not one of them. Yet no one had anything bad to say about 'good time' Johnny Von Neumann; he just was too likeable. He gave massive parties and loved women, fast cars, jokes, noise, Mexican food, fine wine, and, most of all, mathematics. 'Unbelievable', said one of Von Neumann's old friends, 'He knew how to have a good time. His parties were once if not twice a week at 26 Wetcott Road. Waiters came around with drinks all night long. Dancing and loud laughter. With Von Neumann at the centre of it all he was a fantastically witty man.'

Johnny von Neumann (1903 – 1957)

In a way Von Neumann could afford to party, since he had been born lucky - lucky enough to have a great mind, that did not forget. He had a true photographic memory and never forgot a thing. The story goes that you could ask him to quote from anything he had ever read, and the only question he would ask you was, 'When do you want me to stop?' In addition to having a great memory, he was also very fast. Start asking him a question and, before you finished, he would be answering, and suggesting interesting follow-ups that you should consider. It's no wonder, with a super mind like his, that he is credited with inventing the present-day computer.

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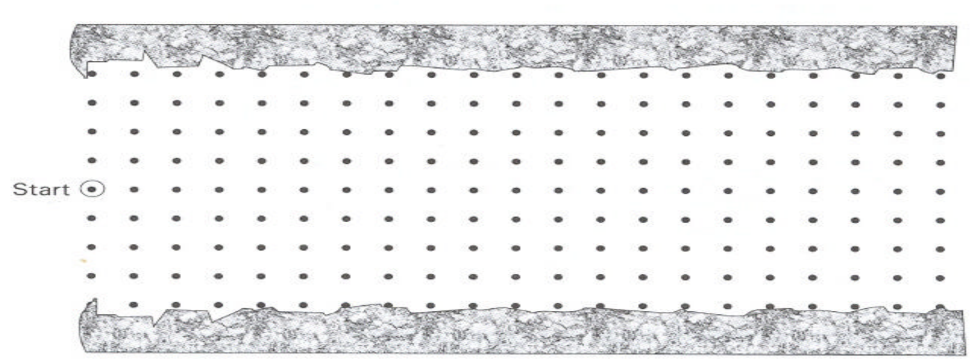
When one of the first computers was to be tested, Von Neumann was on hand to help. The test for the computer was to work out powers of 2, and to find the first number to have 7 as its fourth digit from the right. The computer and Von Neumann started at the same time, and you guessed it, Johnny finished first!
As well as working on the development of the computer, Von Neumann also worked on the atom bomb and created a branch of mathematics called game theory. His work in these areas argued strategies for the Cold War and inspired the movies *Or Strangelove* and *War Games.*

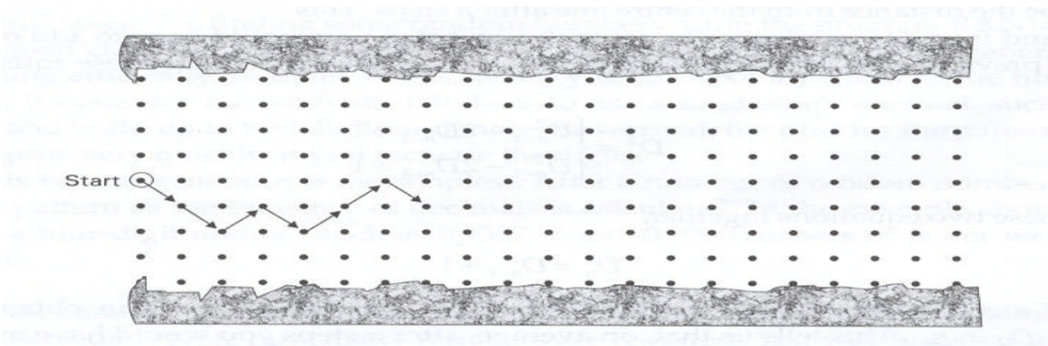
When Von Neumann finished building his computer he had to find a use for it. In his eyes the only useful thing to do was mathematics so he and fellow mathematicans Fermi and Ulam invented a simulation method that they called the Monte Carlo method. This method simulates random events using the computer. In Buffon's needle problem the JAVA applet uses Von Neumann's simulation method to validate the correct value of pi. To simulate the unpredictable event of throwing a needle, the computer has to use something called a pseudo-random number generator. The pattern of numbers generated are deterministic, yet of sufficient complexity to cause the outcome to appear unpredictable (random). In the following experiments we will look in more detail at random number generators.

Experiment Time

There is a problem in mathematics called the random walk. You start with a player at some point. He tosses a coin, and how it falls decides his direction of motion. This randomly makes the choice of which way to go.
To set the scene for students, imagine that you are on a narrow mountain ridge. It is windy and the rain is battering down on you. This makes it difficult to see where you are going. Every time you move forward you get blown randomly to the left or right. What are your chances of making it across the ridge?



Start at the base camp dot, and throw a coin. If it is heads move to the right and if it is tails move to the left. For example, if you throw HHTHTTH then it would look like this:



Keep throwing the coin until you make it to the other side or fall off the edge.

What is the chance that you will:

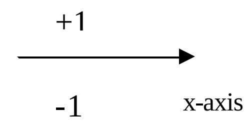
- fall off the cliff
- make it to the other side
- fall off before the tenth move
- fall off before the fifteenth move
- fall off the right or left side of the ridge?

Before you start this experiment using paper, it is a good idea to let the students try to walk the ridge for 'real' in the classroom. With a pre-defined ridge marked on the floor, you can flip the coin to say move left or right. This works well as a starter to get them thinking about how wide the ridge needs to be if they are to make it to the other side. After this introduction get students to perform this experiment on paper at least five times, then collect the data on the board. Be as detailed in this collection phase as the class's ability allows. For example, you may ask if anyone went over the edge in the first 5 moves, or between 6 and 10 moves, or 11 and 15 moves, and so on. Once you have collected the results the class can then talk about the probability of crossing the ridge.
Here are some extension ideas for this task:

Instead of using a coin use the random number button (RAN#) on your calculator, moving to the right if the number generated is in the range 0-0.5, and to the left for higher numbers. What happens if the wind blows harder from one side? Does this make it more difficult to cross the ridge? Simulate this by moving to the right for random numbers 0-0.3, and to the left otherwise, or something similar.
Include the chance that you will be blown two dots to the right or left.

Using mathematics to predict this random event

One question you can ask is how far on average the ridge walker will move away from the centre line after the start. Let the centre line be the *x-axis*. If you move to the left this is +1, and if you move to the right this is -1. Let *On* be the distance from the centre line after *n* steps. This can be found from *On-I'* and since to get to the next step you would have to add or subtract one to the previous step, $D_n = D_{n-1} + 1 \text{or} D_n + D_{n-1} - 1$ If you then square these equations you obtain



$$D_n^2 = \left\{ \begin{array}{l} D_{n-1}^2 + 2D_{n-1} + 1 \\ D_{n-1}^2 - 2D_{n-1} + 1 \end{array} \right.$$

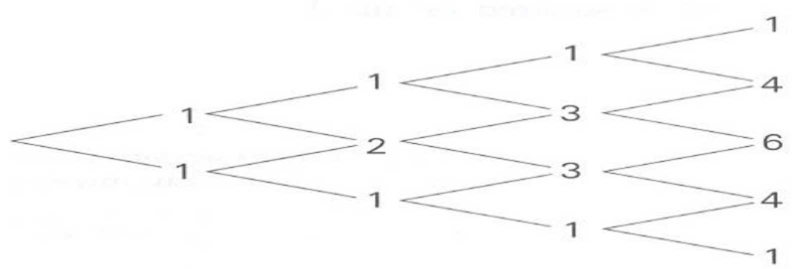
Adding these two equations together,

$$D_n^2 = D_{n-1}^2 + 1 \qquad [1]$$

After one step, $D_1^2 = 1$. Using this, and equation 1 repeatedly, we can obtain $D_n^2 = n$. Therefore $D_n = \sqrt{n}$. This tells us that, on average, after *n* steps you would have moved $\sqrt{n}$ away from the centre line.

Calculating the Probabilities

To calculate the probabilities involved in the random walk, you can use Pascal's triangle to get the chances of following different routes in a random path, for at each step you are making a decision to go left or right. The numbers in the triangle indicate the number of routes from the start position to that point. To find the next line in Pascal's triangle add together the two roots to that point.



Extension Ideas

The 147 random number generator works in the following way. First it selects a decimal between 0 and 1, which it then multiples by 147. Then it takes the fractional part of this result and multiplies it by 10. The integer it produces is the random number.
For example, if the decimal between 0 and 1 that is selected is 0.1357, we have

0.1357 x 147 = 19.9479
The fractional part is 0.9479
10 x 0.9497 = 9.479
So the random number is 9.

If you want a larger random number using this method, just multiply by 100 or 1000 in the final stage. Try finding some random numbers using the program RAN147.

A good challenge for the students is to discover when the random generator is not working efficiently. In other words, when you can spot a pattern in the numbers. To make it easier for the students, tell them to use a single-digit decimal, such as 0.7, at first, and build up to two digits and more. Be warned, the time for the pattern to repeat will grow very quickly as you increase the digits.

This type of generator is the simplest. After a number of random numbers you will see a pattern as the sequence of decimals is calculated. In the example above, starting with a four-digit number such as 0.1357, after 10000 numbers or fewer we will get a repeat.

Add the random numbers generated from your calculator's random button until their total exceeds 1. Note how many numbers are required. You will find that the average number of random numbers required is e = 2.7182818 . . . Changing the total to 3, so that the average becomes 8, makes a good challenge for students.

A similar idea is to find the average random number between 0 and 1. This is a problem that the students can work on using their calculators by finding the average of blocks of ten numbers. The answer comes to

$$\int_0^1 x\,dx = \frac{1}{2}$$

In other words, this is the area under the curve y=x from x = 0 to 1. What about the average of random numbers squared, or cubed. . .?

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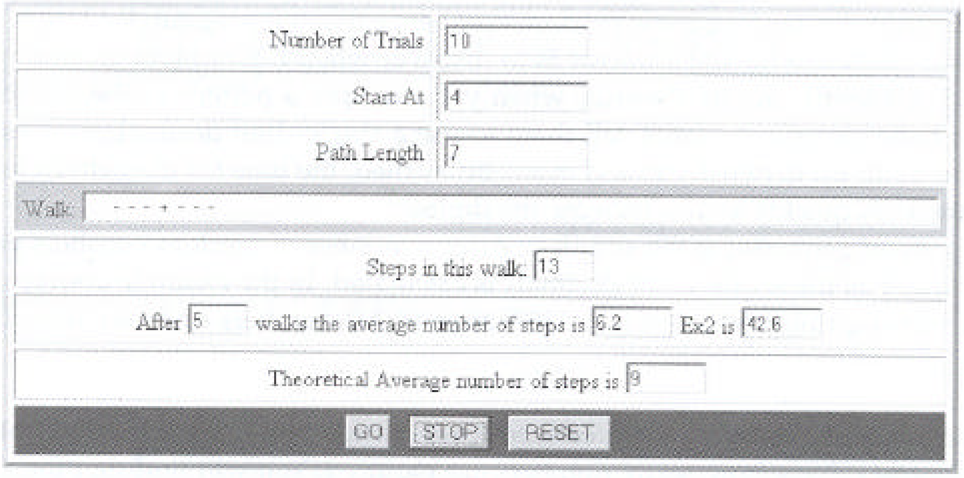
For a number of random walk movies, see
http:/ /cic.nist.gov /lipman/ sciviz/ random

See also: www.math.uah.edu/Stat/walk/http:/ /math.furman.edu/ -dcs/java/rw
http:/ /polymer.bu.edu/java/ www.random.org/ essay.html.

For a computer simulation of a random walk along a line see:
www.math.sc.edu/ -sumner /Random Walkhtml. Here is a screen shot from it.


Number of Trials 10
Start At 4
Path Length 7
Walk: - - - + - - -
Steps in this walk: 13
After 5 walks the average number of steps is 6.2 Ex2 is 42.6
Theoretical Average number of steps is 9
GO
STOP
RESET

Problem Solving and Problem Posing in a Dynamic Geometry Environment

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Abstract: In this study, we considered dynamic geometry software (DGS) as the tool that mediates students' strategies in solving and posing problems. The purpose of the present study was twofold. First, to understand the way in which students can solve problems in the setting of a dynamic geometry environment, and second, to investigate how DGS provides opportunities for posing new problems. Two mathematical problems were presented to six pre-service teachers with prior experience in dynamic geometry. Each student participated in two interview sessions which were video recorded. The results of the study showed that DGS, as a mediation tool, encouraged students to use in problem solving and posing the processes of modeling, conjecturing, experimenting and generalizing. Furthermore, we found that DGS can play a significant role in engendering problem solving and posing by bringing about surprise and cognitive conflict as students use the dragging and measuring facilities of the software.

1. Introduction

In an attempt to inform the development of better pedagogical models, this paper reports some of the findings from a study of the integration of dynamic geometry software (DGS) in mathematics classrooms. One of the distinguishing features of DGS is the facility to construct geometrical objects and specify relationships between them. Within the computer environment the geometrical objects created on the screen can be manipulated, moved and reshaped interactively with the use of the mouse. The tools, definitions, exploration techniques, and visual representations associated with dynamic geometry contribute to a learning environment fundamentally removed from its straightedge-and-compass counterpart (Laborde, 1998).

The focus of this paper is on students' problem solving and posing processes in the learning environment of dynamic geometry when they work on problem solving and posing. The paper also examines how the increasing use of DGS may give rise to new problems and new ways of introducing problem solving in a variety of contexts.

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The study is based on the theoretical premise that computers are being introduced in education not only because they do a *better* job but because but they do the job *differently* (Aviram, 1999; 2001). There are good reasons to support that DGS has the potential to help students improve their abilities to solve a variety of mathematical problems in novel ways and provide a powerful means of posing new problems by applying different heuristic approaches (Gomes & Vergnaud, 2004).

The first part of the paper presents the theoretical background of the study with special reference to computers as mediation tools, and to the learning mediated through employing DGS. The second part of the study provides support to the theoretical part by indicating that students in the era of computers can do mathematics differently (Aviram, 2001) by solving and extending problems in ways they could not do with paper and pencil, and by exploring and investigating different possible answers to a problem.

2. Theoretical Background

2.1 DGS as a mediation tool

In this study, we investigated pre-service teachers' abilities to construct geometrical objects and solve and pose problems in a computer-based environment, which served as a mediation tool (artifact). The types of artifacts are closely related to the knowledge that students construct (Artique, 2002), and are central to the processes by which students mathematize their activities. In addition, artifacts support students' mathematical development by anticipating how students might act with particular tools, and what they might learn as they do so (Cobb, 1997). Jones (1997) asserted that artifacts stand between the learners and the knowledge that students are intended to learn. This assumes that learning within a DGS environment involves the interaction between students and the software, as they submit their previous knowledge to revision, modification, completion or rejection in the process of acquiring new knowledge (Jones, 2000). This interaction is more clearly explained as the interaction between two systems (Brousseau, 1997; Jones, 1997). The first system refers to students who attempt to solve or pose a problem, and the second system refers to the environment, which offers opportunities to students to act and react. The environment also includes the tools that mediate students' actions and exists between the students and the world of mathematics (Artigue, 2002), and, most importantly, transforms the students' activities upon the world.

Gomes and Vergnaud (2004) considered DGS as an integral part of the didactical environment, performing a specific mediation of knowledge. In problem solving and posing, DGS makes possible for students to generate and use specific strategies (Hölzl, 1996). Hölzl, for example, identified two components of the epistemology of DGS: first the nature of the interface, and second the

consequences on students' conceptualizations. Specifically, the structure of a particular interface is a key determinant of the characteristics of the knowledge evolved using it. Problem solving and posing, using DGS, involves the direct and indirect effects of the software's interface on students' procedures and understandings. In addition, DGS's interface provides students with the opportunity to use visual reasoning in mathematics and helps them, through the dragging facilities, to generalize problems and relationships (Sinclair, 2004).

The main issue, however, is whether the involvement of students in such learning settings may result in understandings that could not be achieved through traditional instruction (Artique, 2002), and whether DGS is actually used and transformed by students in visually confirming and negating conjectures and in developing a new perspective on solving and posing original problems (Meira, 1998; Sinclair, 2004).

2.2 Problem solving and posing

Problem posing, problem solving, and conjecturing are three important mathematical activities (National Council of Teachers of Mathematics (NCTM), 2000). In geometry, these activities involve some tasks that technology performs efficiently and well, such as computing and graphing. In this study, two problems were assigned to students, which show how the computing and graphing capabilities of DGS can be used in making conjectures, in problem solving and problem posing within geometry tasks. The importance and relevance of these mathematical activities is supported by "The Principles and Standards for School Mathematics Document" (NCTM, 2000). For example, this document states that instructional programs should provide opportunities for all students to "use visualization, spatial reasoning, and geometric modeling to solve problems" (p. 308). The document also calls for students to "formulate interesting problems based on a wide variety of situations, both within and outside mathematics" (p. 258). In addition, the document recommends that students should make and investigate mathematical conjectures and learn how to generalize and extend problems by posing follow-up questions. Moreover, Silver and Cai (1996) refer explicitly to problem posing by arguing that students should have some experience in recognizing and formulating their own problems. Most authors agree that problem posing is a term used to mean both the generation of new problems and the reformulation of given problems (Cai & Hwang, 2002; English, 2003; Silver & Cai, 1996).

3. The Present Study

In this study, based on the theoretical dimensions discussed in the previous section, we considered DGS as the tool that mediates students' strategies in solving and posing problems. We perceived the construct of mediation in two different ways. First, in a cognitive framework, we addressed the students' attempts that arose from the exploration of the possible extensions and solutions

in the assigned problems, and second, in a social framework, we discussed the results or the solutions through the evidence provided by the software.

3.1 The Purpose of the Study

The purpose of the present study was twofold. First, to understand the way in which students can solve non-routine problems in the setting of DGS, and second, to investigate how DGS provides opportunities to students for posing new problems. Both purposes are explored under the assumption that DGS constitutes the artifact that helps us understand how computers can be used in education in doing mathematics in a different way. Thus, the research questions concern firstly the exploration of DGS as a tool that fosters the development of problem solving processes, and secondly the investigation of how DGS mediates the generation of new problems. More specifically, the questions of the present study are:

(a) In what ways does the DGS mediates students' problem solving processes in geometry problems?
(b) In what way the DGS environment provides opportunities for students to pose and solve their own geometry problems?

In order to meet the purposes of the study, two mathematical problems or situations were presented to students. These problems illustrate how students may be engaged in conjecturing, problem solving, and problem posing with the aid of DGS. The problems also illustrate the power of such environments to engage students at various levels of mathematical sophistication.

4. Method

4.1 The Problems of the Study

In order to answer the research questions formulated above, the following two problems, as were slightly adapted from Contreras (2003), were assigned to the students:

Problem 1. The authorities of four towns are planning to build an airport that will serve the needs of their citizens. Identify the optimal place for the location of the airport so that the needs of the four towns are served in a fair way.

Problem 2. What is the figure formed by the angle bisectors of the interior angles of a parallelogram?

The first problem is open-ended and purposefully not well defined. Thus, students had to provide a context for the problem in order to clarify the situation in which they would work. The second problem is a pure geometrical situation, which allows students to explore a seemingly standard problem, but in the solving process they may encounter surprising results. Both problems provided the opportunity to students to generate new problems by altering the situations or extending and reformulating the given problems in different ways (English, 1997, 2003; Silver & Cai, 1996).

4.2 The Participants

The participants of the study were six pre-service teachers with prior experience in dynamic geometry. All students attended a course on the integration of computers in elementary education. The course focused on mathematical applications in the teaching and learning of mathematics in grades 1-6, with special emphasis on the integration of dynamic geometry. Therefore, students had a basic understanding of Geometer's Sketchpad's drawings, menus, and construction features.

4.3 Data Collection and Procedures

The interviewees participated on a voluntary basis. Six students were interviewed while they were working on two non-routine problems. Three of the students worked on the airport problem and three on the bisectors problem. Each student participated in two interview sessions, which corresponded to the two aspects of the mediation construct, namely, the cognitive and social aspects. During the first session, students were asked to solve the problems. During the second session, we worked with the students and discussed not only their solutions but also possible ways of extending, posing and solving new problems.

The interviews were conducted in the mathematics laboratory, which was equipped with computers loaded with the Greek version of the Geometer's Sketchpad. A video recording of the sessions (as opposed to audio) was decided as the means of recording the interviews since we wished to capture not only the discussions but also the actions occurring on the computer screen as interviewees talked about their work. The setting was informal with students being able to analyze and build geometric constructions that they thought would help them solve the problems without any time constraints being set. The data was collected during unstructured interviews. One of the most important benefits of the unstructured interview approach has been described by Cobb (1986) as the process of "negotiating meaning". It gives the opportunity to the researchers to ask the subjects to clarify or explain their activities or comments.

Analysis of the data followed interpretative techniques (Miles & Huberman, 1994). Video records helped us identify the unique ways the software facilitated the students to solve the problems as well as the sequence of the cognitive processes and strategies used during the solution of the problems. The interviews helped to identify the ways in which students used the software in order to pose new problems and how these new problems were related to the original problems. Detailed analysis of all the data was then used to develop categories of problem posing and solving processes that could be checked against participants' own accounts of their work.

4. Results

4.1 The Airport Problem

The airport problem created a lot of discussion between the researchers and the individual students. The discussion revolved around (a) the meaning of the needs of the four towns and how one could interpret them, and (b) the meaning of the word “fair”, which produced some disagreements concerning the population of each town. Following the discussions about the context and the meaning of the words involved in the problem, some of the students decided to consider the concept of “fairness” as “equidistance”.

When the researcher asked students to solve the problem using the Sketchpad software, the three students who worked on that problem, modelled it, assuming that they should take into consideration the distance of the four towns from the airport. The students were not used to working with such investigations, since geometry textbooks dissuade students from making conjectures based on the limited evidence provided by a single shape. As a result, the students, trying out to find a solution to the airport problem, built “prototype conjectures” (see Hanna, 2000), which were based on common geometrical shapes such as rectangles, parallelograms or squares. Specifically, two of the students investigated the problem by assuming that the four towns were the vertices of a rectangle or a square. The following extract shows their attempts in finding a reasonable solution to the problem:

Student A: I assumed that the four towns are on the vertices of rectangle. Then I hypothesized that the best location for the airport should be in the centre of the rectangle.
Researchers: What do you mean by the “centre” of the rectangle?
Student A: Probably, this is the intersection of diagonals.
Researcher: Ok. How can you check your hypothesis?
Student A: (He points to the diagram on the computer screen). *I defined a point inside the rectangle and constructed the segments from the vertices to this point. I moved the point inside the rectangle* (See Figure 1). *In the meantime, I measured the length of each segment.*

In this case, student A found that the optimal location of the airport was the intersection of the diagonals of the shape, by dragging a point into the shape. He actually based his conjectures on measurements showing that each town is equidistant from the point of the intersection of the diagonals (see Figure 1).

At this point, when students were asked to generalize their findings to include all quadrilaterals, two of them intuitively answered that the point of the intersection of the diagonals should be the best location for the airport. However, they could not provide a reasonable justification, although they empirically tried other points inside an arbitrary quadrilateral. The dragging facilities of the software helped them to work inductively, i.e., from two or three examples they generalized to all

quadrilaterals. Of course, they recognized the need for a formal proof in order to convince themselves and others about the truth of their generalization.

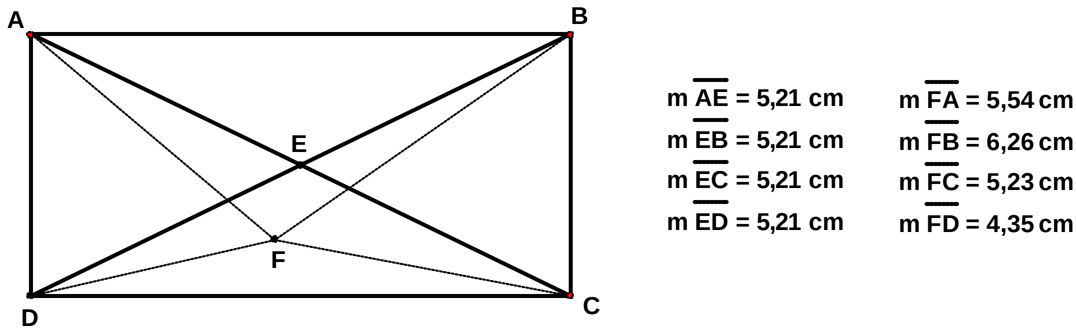


Fig. 1. Students' answers based on "prototype conjectures".

Student C perceived the problem in a quite different way. She conjectured that the four points representing the towns should be points on a circle. The following extract shows how DGS helped her to understand that her reasoning was true only under very specific circumstances.

Student C: The best location for the airport should be the centre of the circle, since the centre is equidistant from any point on the circumference of the circle. (She showed her work by drawing a circle and constructing four points on it, as shown in Figure 2a).

Researcher: Drag one of the vertices of your figure.

Student C: Oh! The centre of the circle (see Figure 2b) *is not always the best solution. In this case* (she points to Figure 2b) *there should be another point.*

Researcher: Where should that be?

Student C: Probably it is inside the quadrilateral. … A point inside the quadrilateral could serve the towns in a more appropriate or economically better way.

By constructing an arbitrary point inside the polygon (see point K in Figure 2c) and measuring the distances of that point from the four towns, she identified that her initial rule, i.e., the best location is the centre of the circle did not work. She then constructed a quadrilateral hypothesizing that the towns should be the vertices of that quadrilateral and with trial and error she tried to find the solution to the problem.

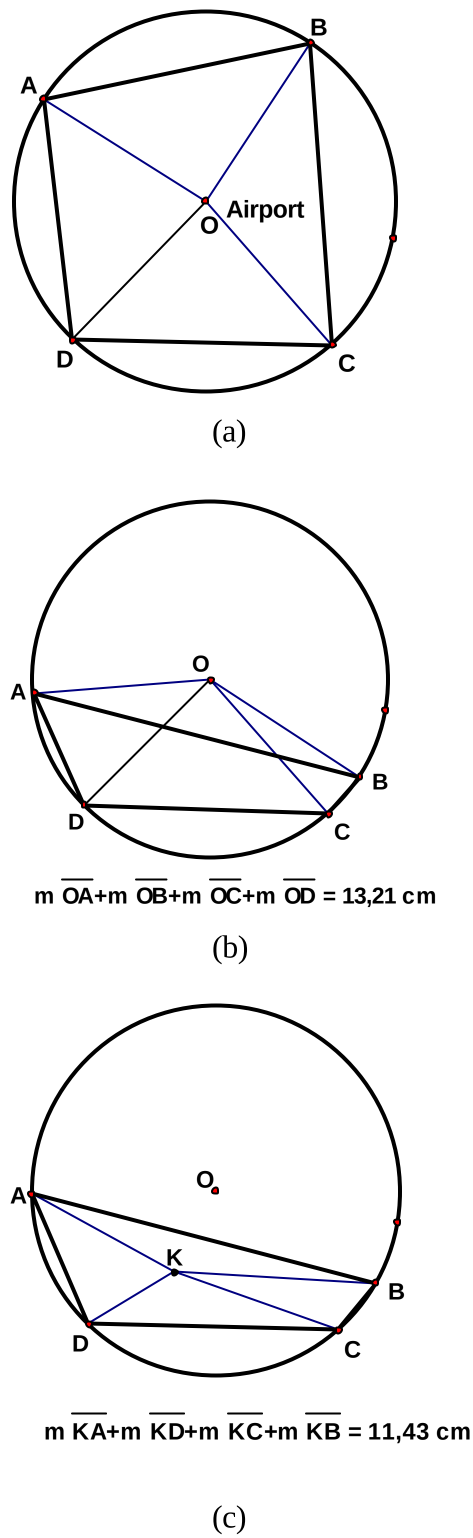
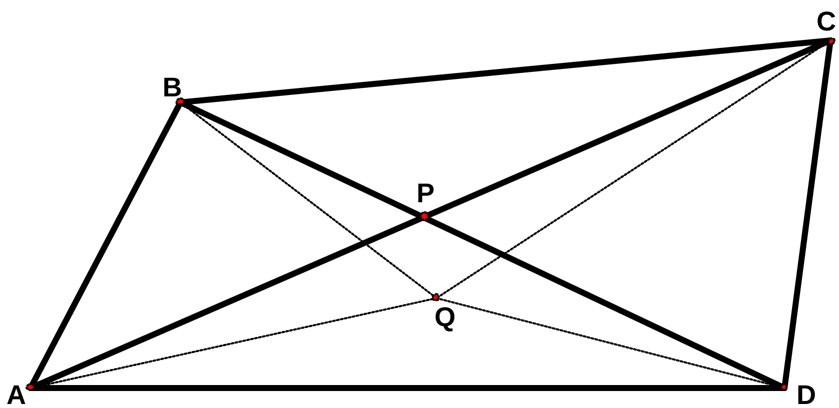


Fig. 2. Student's C conjectures and processes for finding the solution of problem1

The students who individually worked on the airport problem could not consider all the possibilities and were not able to generalize their solutions. The day following the experiment, a meeting took place with the researchers and the students in order to present their solutions and find ways to extend the problem. This meeting lasted for an hour, and we realized that students, with the help of the software, had the enthusiasm to work further on the problem. We started the discussion with the work of student C. We prompted them to find the sum of the distances (a) from the centre (see Figure 2b), and (b) from a point inside the polygon (see Figure 2c). They realized that a point inside the quadrilateral would be a better location than the centre of the circle. This gave the opportunity to students construct the two diagonals of the quadrilateral and their point of intersection and label it P. In an attempt to add experimental evidence to support the conjecture, they moved point Q around other possible locations and observed that the point P seemed to be the optimal point (See Figure 3). They realized that the point of diagonals intersection P is the point for which the sum of its distances from each of the four points (i.e. the cities) is the smallest possible. The next step was to provide a mathematical proof of the conjecture so that it could become a mathematical theorem for the students (proofs are beyond the scope of the current paper).



$$m\ \overline{PA}+m\ \overline{PB}+m\ \overline{PC}+m\ \overline{PD} = 24{,}11\ cm$$

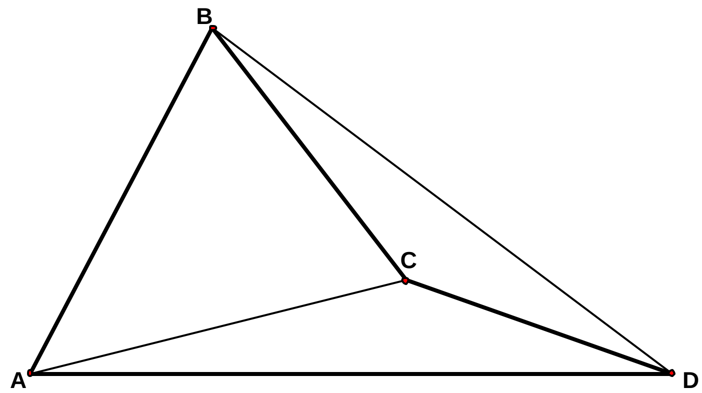
$$m\ \overline{QA}+m\ \overline{QB}+m\ \overline{QC}+m\ \overline{QD} = 24{,}55\ cm$$

Fig. 3. P is the point where the sum of its distances to each of the four cities is the smallest possible sum.

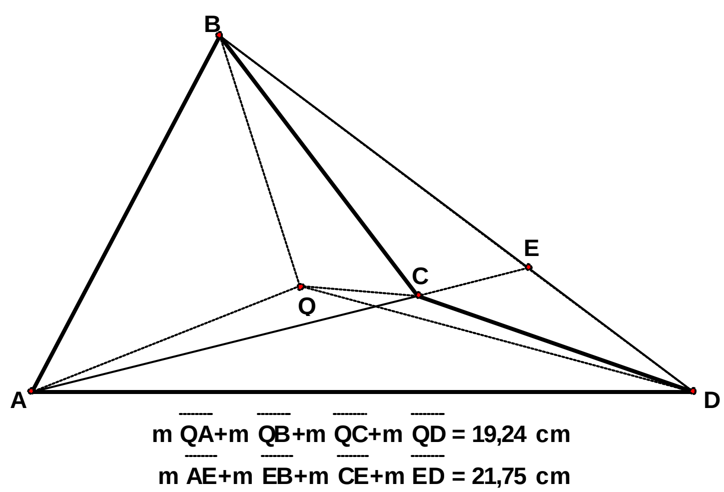
The discussion up to this point seemed to satisfy the students who concluded that the intersection of the diagonals would be the optimal point and the optimal solution of the problem. However, the prompt of the researcher led to further investigations of the problem by considering general or special cases and posing other follow-up problems as shown in the following extracts.

Researcher: Draw a quadrilateral and drag it in such a way as to transform it to a non –convex quadrilateral (see Figure 4a, which illustrates the drawing of a non-convex quadrilateral ABCD as constructed by student C). *What do you observe?*
Student C: The diagonals AC and BD do not intersect "insight" the figure.
Researcher: Does it mean that your previous conclusion is not correct?
Student C: I don't know.
Student A: We can find their point of intersection by extending AC. This may be the optimal point (point E in Figure 4b).
(At this point of the discussion students used the dragging and measure facilities of DGS to examine their hypothesis).
Student B: No, the point of diagonals' intersection is not the correct answer. (Student B points to her diagram). *I constructed a point into the figure and I measured the distances from it.*
Researcher: What are you looking for?
Student B: The best location. I mean, I am trying to find the point from which the sum of the distances is the smallest one.
Researcher: How can you find it?
Student B: (She explains her reasoning using the diagram on the screen). *I moved this point and I found that the total distance from any point inside the figure is always smaller than the distance from any point outside the figure.* (The student showed her work to the group).
Researchers: Ok. What is your answer to the problem?
Student A: We have to try by dragging the point. (All students worked by dragging a point inside the figure*).*
Student B: It seems that as the point reaches C, the total distance gets smaller.
Student A: Yes, it should be vertex C. Is it correct?

The environment of DGS provided students the opportunity to investigate the location of the new optimal point. Again, by moving point Q, they obtained the tentative location of the new optimal point as shown in Figure 4b. The above extract also addressed the conflicts that usually arise from the exploration of the possible extensions in the assigned problems. Surprisingly, students discovered that the optimal point coincided with point C. In other words, the optimal point seems to be the vertex of the reflex interior angle. Students concluded that the location of the optimal point depends on the type of quadrilateral (convex or non-convex). It should be noted that the dragging capabilities of DGS allowed them to discover that the optimal point of a quadrilateral is not always the point of intersection of the diagonals because such point does not always exist. Finally, students extended the previous conjecture or problem not only to non-convex quadrilaterals but also to the case where three or more points are collinear or to the cases where there are fewer or more points.



(a)



(b)

Fig. 4. The extension of problem 1 to a non-convex quadrilateral.

To answer the first question of the study, i.e., to find out the solution strategies employed by students in the DGS environment, we need first to summarize and interpret the students' work during the solution of the airport problem. In fact, all students used the strategies of modeling, conjecturing, experimenting, and generalizing. Specifically, all students first modelled the problem by representing it in different ways (the cities as points on the circumference of a circle, as vertices of quadrilaterals, etc). Second, they hypothesized the solution of the problem based on how they perceived and modelled the problem and tried to verify their conjectures by dragging and measuring. Finally, they tried to

generalize their solutions in their attempts to provide a solution to the problem at hand. However, students seemed to over-generalize their solutions based on certain cases and failed to extend the problem to all possible situations. The latter was achieved during the discussion among the students and the researcher, where the emphasis was to extend the problem and help students to pose new problems.

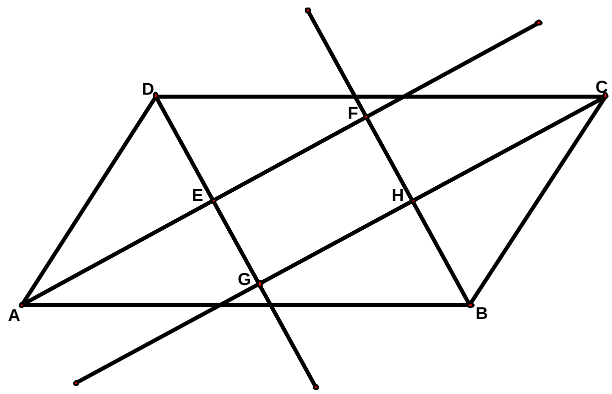
4.2 The Problem with the Bisectors of the interior angles of a parallelogram

To answer the second question of the study (i.e., to find ways in which DGS provides opportunities for students to pose and solve their own geometry problems), students were asked to solve the bisectors problem, which is a common geometrical problem found in most geometry textbooks. The software helped students to construct the parallelogram as well as the bisectors of the interior angles. Figure 5a shows students' construction of the angle bisectors of the interior angles of a parallelogram, and the following extracts show the way in which the DGS helped them to pose and solve new problems.

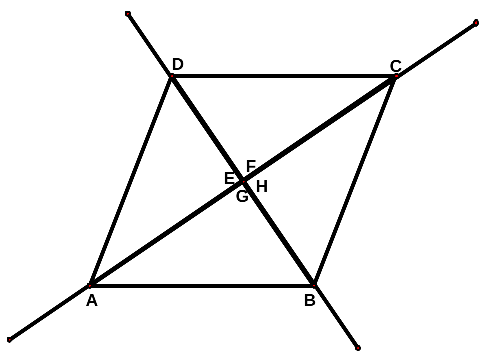
Student D: The figure formed by the bisectors of the interior angles seems to be a rectangle. (He dragged one of its vertices, and verified his answer).
Researcher: Is that always true? (Students tried to transform the parallelogram to other shapes such as rectangles, squares and rhombuses).
Student E: This is not always true. If you drag the parallelogram until it becomes a rhombus, the interior figure disappears. (See Figure 5b)
Student D: It didn't disappear. It became a point.
Student F: … If the original parallelogram becomes a rectangle, the figure is a square. (She verified it by dragging one of the vertices of the original rectangle). (See Figure 6).
Researcher: Try to extend the problem to other quadrilaterals.
Student D: This is an isosceles trapezium.
Student F: The angle bisectors of the interior angles of an isosceles trapezoid form a kite with two right angles. (See Figure 7).
Researcher: What about the figure formed in a non-isosceles trapezium?
Student D: It may be a kite. (Student D constructed the appropriate shape). *Yes, it's a kite without two right angles.*

The above constructions could certainly be reached without the computer, and the students could also prove the conjecture. However, without the use of the dynamic software students would not be able to add experimental evidence to their conjectures as they did by dragging any of the flexible points of the parallelogram and notice, as previously conjectured, that the figure might be a rectangle. An important finding lies on the fact that by dragging one of the flexible points of the parallelogram until it becomes a rhombus, the students observed that the figure no longer formed a rectangle but a point (See Figure 5b). This revealed to them that their first conjecture does not always hold and led them to consider a point as a degenerated rectangle! Again, it was evident that the

dragging capabilities of DGS allowed individuals to consider extreme cases of a geometric configuration, cases that textbook authors fail to consider.



(a)



(b)

Fig. 5. The figure formed by the bisectors of a parallelogram and a rhombus.

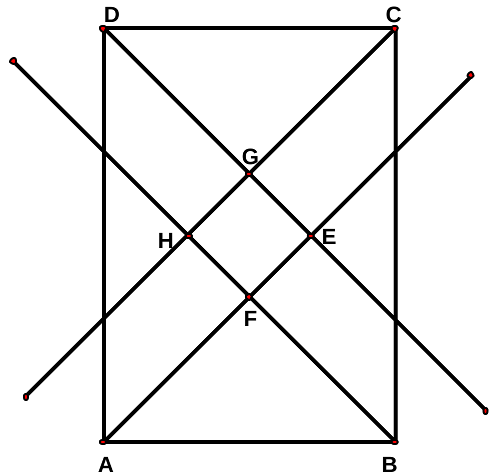


Fig. 6. The figure formed by the bisectors of a rectangle.

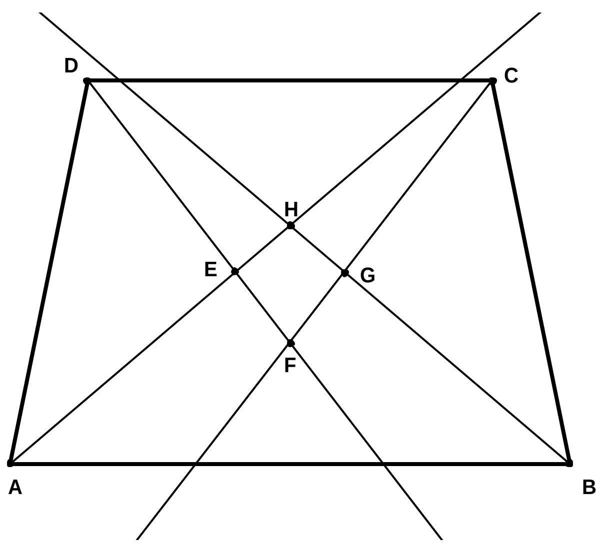


Fig. 7. The figure formed by the bisectors of an isosceles trapezium.

These activities led students to engage in problem posing by experimenting, generalizing, specializing, and extending the problem through the modification of the conditions of the given problem. A special case of the problem was to start with a rectangle instead of a parallelogram as done by student F. Figure 6 suggests that the figure formed by the angle bisectors of the interior angles of a rectangle is possibly a square. Students also considered general cases. They conjectured from Figure 7 and by dragging one of the flexible points that the

figure formed by the angle bisectors of the interior angles of an isosceles trapezoid is a kite with two right angles.

Another interesting extension to this problem was posed during the discussion, when one of the students suggested that it would be interesting to find out the figure, which can be formed by joining the mid-points of the figures constructed by the bisectors (Figure 8). This, of course, led to different conjectures based on the solution of the original bisectors problem. For example, one student conjectured that the new shape would be similar to the shape formed by the bisectors, while others generalized the theorem of the mid-points of a quadrilateral, predicting that in all cases, with the exception of squares and rhombuses, the shape would be a parallelogram. Figure 9 shows most of the constructions proposed by students in extending the original problem.

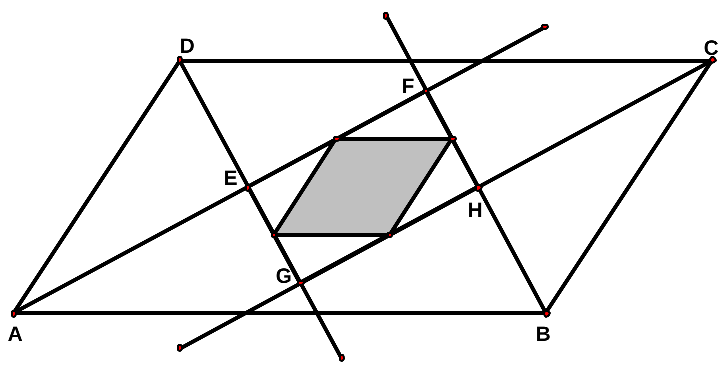


Fig. 8. The figure formed by the mid points of the segments defined by the bisectors of the parallelogram.

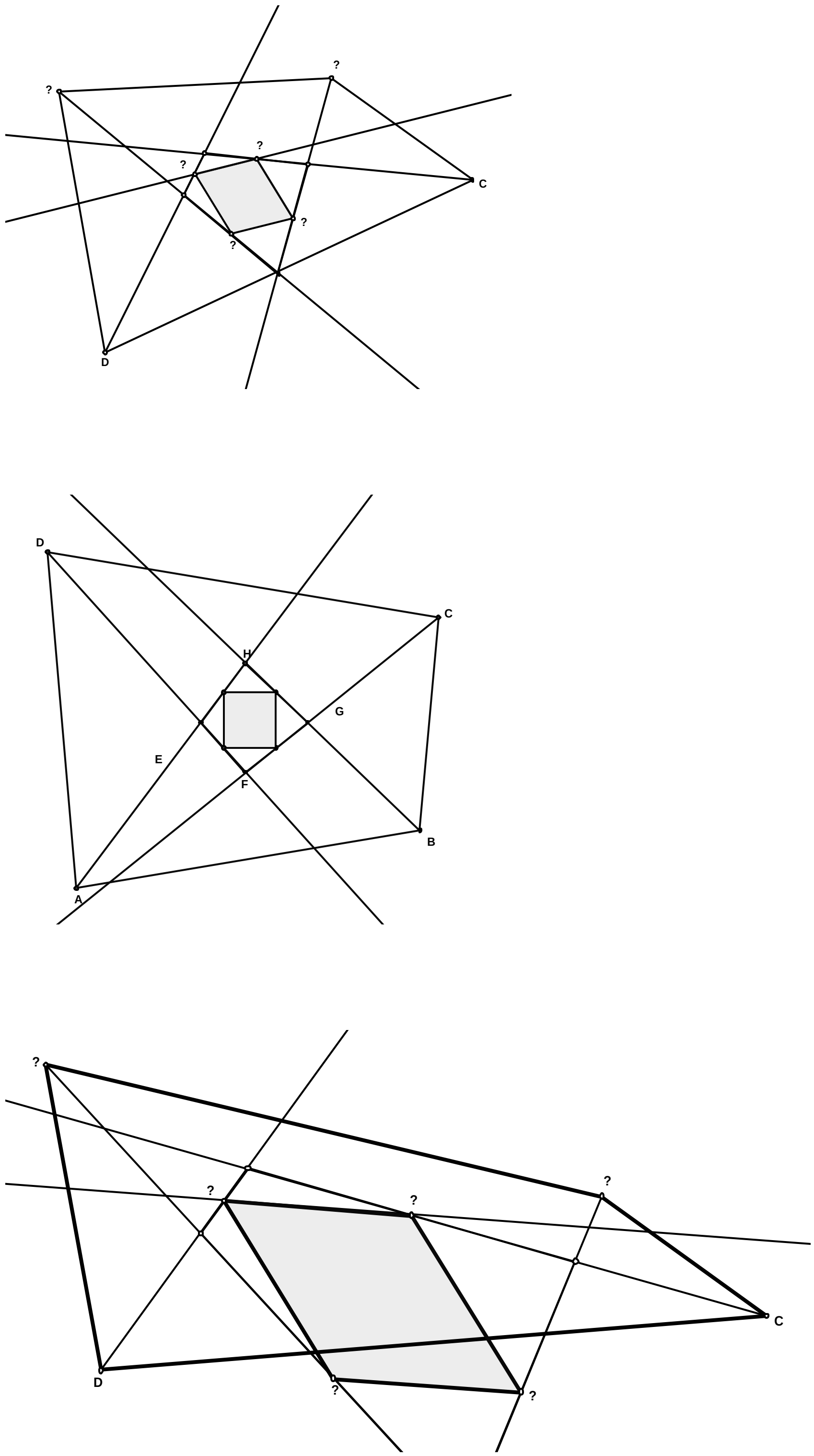


Fig. 9. The figure formed by the mid points of the segments defined by the bisectors of the quadrilateral.

5. Conclusions

The growing and nearly universal availability of technological tools facilitates teachers in teaching and improving the mathematical experiences of students. This paper focused on the use of a DGS in problem solving, inquiry, and exploration in mathematics. We provided some ideas on how students can use the tools of DGS to solve and pose mathematical problems. The paper also addressed the ways in which DGS may be associated with new problems that do not usually appear in the traditional geometry textbooks, and new ways of introducing problem solving and posing in a variety of contexts. Two examples were provided to show how the computing, graphing, and dragging capabilities of dynamic geometry software can enable students to explore and make mathematical conjectures, solve problems, and pose related problems.

In the two examples, DGS acted as a mediation tool (Artigue, 2000; Jones, 2000) in the implementation of an inquiry approach to teaching and learning mathematics as recommended in current mathematics education documents (NCTM, 2000). Specifically, this study showed that DGS can play a significant role in engendering problem solving and posing. First, the new information students obtained through dragging and measuring helped them understand the problems, and added challenge to the exploration of the possible answers to a problem. It was shown that dragging is an important tool for problem solving and posing, and measuring is an important tool for checking the correctness of students' conjectures.

Second, DGS as a mediation tool, encouraged students to use in problem solving and posing the processes of modeling, conjecturing, experimenting and generalizing. Through modeling, students constructed accurate images of the problems, which helped them to visually explore the problems and reflect on them. The meaning students extracted from the constructed images enabled them to explore at a perceptual level and to make conjectures about the possible solutions to the problems. Through experimentation, students also visually confirmed or negated their conjectures, and thus proceeded to suggest possible solutions or extensions to the assigned problems. The results of this study also show that in the DGS environment the problem solving processes involve the generation of new problems, supporting the relationship between problem solving strategy use and the tendency to pose extension problems (Cai & Hwang, 2002).

Third, DGS provided a context in which we can do mathematics in a different way (Aviram, 2001). The data of the study showed that DGS environment can bring about cognitive conflict and/or surprise, as it appeared mainly in the airport problem with the vertex quadrilateral. Since a particular paper and pencil figure usually displays a general case, it is difficult for students to appreciate the significance of special cases. However, students using the DGS are very likely to drag a figure past a special case, and thus more likely to stop on a special case and be faced with the consequences.

Finally, on a practical level, the present study of DGS learning can benefit teachers, and curriculum developers. Teachers faced with limited time and crowded computer labs may use research results to identify fruitful ideas in the language and construction actions of their students. In addition, curriculum developers may find inspiration for new activities aimed at the needs of dynamic geometry learners.

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Meet the Authors

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CALL FOR PAPERS

The Montana Mathematics Enthusiast is the official journal of the Montana Council of Teachers of Mathematics (MCTM). This biannual e-journal provides its readers with a lively blend of mathematics content, education theory and practice. The journal addresses the role of teaching and learning at all levels. Journal issues typically focus on both mathematics content and innovative pedagogical practices with the hope of stimulating dialogue between pre-service and practicing teachers and university educators. The journal strives to introduce research based as well as historical and cross-cultural perspectives to mathematics content, its teaching and learning. We welcome submissions from pre-service and practicing teachers, administrators, math education researchers, mathematicians and university educators. Submissions are externally reviewed in a timely manner. The journal also welcomes commentaries to previously published articles.

Please send an electronic version of your manuscript in APA style to sriramanb@mso.umt.edu

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TMME WEBSITE CIRCULATION STATISTICS

This page gives the statistics of journal circulation for 7 days in Sept.2004 immediately after the release of vol1,no2. The numbers since then have been consistently higher.

General Summary

(Figures in parentheses refer to the 7 days to 24-Sep-2004 00:58).
Successful requests: 10,681 (3,277)
Average successful requests per day: 382 (468)
Successful requests for pages: 3,738 (947)
Average successful requests for pages per day: 133 (135)
Distinct files requested: 297 (210)
Distinct hosts served: 882 (225)
Data transferred: 215.756 Mbytes (47.495 Mbytes)
Average data transferred per day: 7.717 Mbytes (6.785 Mbytes)

Domain Report

Listing domains, sorted by the amount of traffic.

reqs: %bytes: domain
----: ------: ------
2061: 29.85%: [unresolved numerical addresses]
3561: 24.13%: .net (Network)
2725: 21.25%: .com (Commercial)
1100: 10.65%: .edu (USA Educational)
550: 2.94%: .us (United States)
85: 2.39%: .au (Australia)
70: 1.43%: .br (Brazil)
90: 1.33%: .de (Germany)
30: 0.84%: .nl (Netherlands)
18: 0.63%: .cz (Czech Republic)
35: 0.61%: .nz (New Zealand)
113: 0.54%: .org (Non-Profit Making Organisations)
5: 0.43%: .gov (USA Government)
14: 0.38%: .th (Thailand)
54: 0.35%: .il (Israel)
20: 0.35%: .mx (Mexico)
22: 0.26%: .cy (Cyprus)
25: 0.21%: .tw (Taiwan)

6: 0.16%: .be (Belgium)
5: 0.16%: .uk (United Kingdom)
16: 0.16%: .my (Malaysia)
6: 0.15%: .mil (USA Military)
2: 0.14%: [unknown domain]
3: 0.12%: .fr (France)
5: 0.08%: .it (Italy)
6: 0.08%: .sg (Singapore)
14: 0.06%: .jp (Japan)
2: 0.06%: .co (Colombia)
2: 0.05%: .tv (Tuvalu)
4: 0.05%: .za (South Africa)
2: 0.04%: .pt (Portugal)
13: 0.04%: .no (Norway)
4: 0.02%: .ca (Canada)
4: 0.02%: .sc (Seychelles)
2: 0.01%: .ph (Philippines)
2: : .pl (Poland)
1: : .lv (Latvia)
1: : [domain not given]
1: : .kr (South Korea)
2: : .sa (Saudi Arabia)

Organisation Report

Listing the first 20 organisations by the number of requests, sorted by the number of requests.

reqs: %bytes: organisation
----: ------: ------------
2062: 29.85%: [unresolved numerical addresses]
1410: 3.22%: direcpc.com
1104: 4.57%: bresnan.net
805: 3.27%: qwest.net
547: 2.42%: umontana.edu
515: 3.29%: cutthroatcom.net
299: 1.45%: vnet-inc.com
197: 0.55%: montana.edu
151: 1.96%: comcast.net
151: 3.38%: googlebot.com
145: 0.78%: k12.mt.us
138: 0.94%: infoave.net
137: 3.16%: inktomisearch.com
122: 0.36%: bps.k12.mt.us
108: 1.33%: aol.com

90: 0.52%: adelphia.net
75: 1.16%: fiz-karlsruhe.de
74: 0.77%: teoma.com
73: 1.38%: dir.com
69: 0.51%: centurytel.net
2409: 35.13%: [not listed: 250 organisations]

Directory Report

Listing directories with at least 0.01% of the traffic, sorted by the amount of traffic.

reqs: %bytes: directory
----: ------: ---------
2493: 41.59%: /TMME/
4475: 22.51%: [root directory]
167: 16.42%: /newsletters/
308: 5.03%: /contest/
587: 4.43%: /msipics/
336: 1.25%: /cometinfo_files/
77: 1.22%: /mathprojects/
103: 1.04%: /classroomideas/
78: 0.59%: /SSheets/
418: 0.41%: /meahelena/
75: 0.24%: /graphfun/
41: 0.23%: /boardmin/
86: 0.13%: /nwconf04_files/
6: 0.12%: /wshop/
28: 0.08%: /mtinstitute_files/
37: 0.05%: /stars12040205_files/
13: 0.01%: [not listed: 4 directories]