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AIMS AND SCOPE

The Montana Mathematics Enthusiast is an eclectic journal which focuses on mathematics content, mathematics education research, interdisciplinary issues and pedagogy. The articles appearing in the journal address issues related to mathematical thinking, teaching and learning at **all** levels. The secondary focus includes specific mathematics content and advances in that area, as well as broader political and social issues related to mathematics education. Journal articles cover a wide spectrum of topics such as mathematics content (including advanced mathematics), educational studies related to mathematics, and reports of innovative pedagogical practices with the hope of stimulating dialogue between pre-service and practicing teachers, university educators and mathematicians. The journal is also interested in research based articles as well as historical, philosophical and cross-cultural perspectives on mathematics content, its teaching and learning.

The journal is accessed from 94+ countries and its readers include students of mathematics, future and practicing teachers, university mathematicians, mathematics educators as well as those who pursue mathematics recreationally. The journal exists to create a forum for argumentative and critical positions on mathematics education, and especially welcomes articles which challenge commonly held assumptions about the nature and purpose of mathematics and mathematics education. Reactions or commentaries on previously published articles are welcomed. Manuscripts are to be submitted in electronic format to the editor preferably in APA style. The typical time period from submission to publication (including peer review) is 7-10 months. Please visit the journal website at <http://www.montanamath.org/TMME>

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Adult Students' Reasoning in Geometry: Teaching Mathematics through Collaborative Problem Solving in Teacher Education

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Abstract: This article reports research that is concerned with pre-service teachers² working collaboratively in a problem-solving context without teacher involvement. The aim is to focus on the students' heuristic strategies employed in the solution process while working on two problems in geometry. Two episodes from the dialogues in one group of students with limited mathematical backgrounds have been chosen to illustrate some mathematical movement throughout the group meetings, from working with the first problem to working with the second one. The findings reveal that three categories of strategies, visualising, monitoring, and questioning, play an important role in order to make progress with the problems. As a preparation for working on the two problems, metacognitive training in combination with cooperative learning was introduced to the students throughout a month. The study indicates that these critical components in the design of the instructional context stimulated the students with limited background in mathematics to improve their problem-solving skills. The analysis has particularly focused on the important role of the process writer that provokes the mathematical discussion by generating utterances categorised as looking-back questions. By recapitulating the solution process or the last idea introduced in the dialogue, the process writer stimulated the establishment of a common ground for the further discussion. The article also deals with issues of teacher involvement in students' mathematical discussions in collaborative working groups.

Keywords: geometry; teacher education; mathematical reasoning; heuristic strategies; collaborative problem solving; dialogical approach

1. Introduction

This study focuses on observation, analysis and interpretation of the mathematical discussion of one group of students working on two geometry problems in a collaborative problem-solving context without teacher intervention. From a socio-cognitive perspective, I am particularly interested to illustrate how the students express elements of mathematical reasoning in group

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² Pre-service teachers are students who intend to pursue a teaching career. I will refer to the pre-service teachers as 'students'.

dialogue when they are in the process of solving the problems. Goos et al. (2002) emphasise that research on small-group learning in mathematics has revealed few insights into how students think and learn while interacting with peers. These authors suggest that research should focus on the potential for small-group work in order to develop students' mathematical thinking and problem solving skills. In Bjuland (2004), I have illustrated how one group of students reflected on their collaborative small-group experience as learners of mathematics and on their future role as teachers of mathematics. The outcome of the analysis revealed that reflections on their own learning processes stimulated reflections on their preparation for the profession of teaching.

In this article, two episodes from the dialogues of one group of students are analysed in order to illustrate important aspects from their process as learners of mathematics (the same group as reported from in Bjuland, 2004). The students in this group have limited mathematical backgrounds. In the first episode, chosen from the student discussion of the first geometry problem, the students attribute meaning to the concept of distance from a point to a line. The second episode, taken from the second problem, illustrates how the students are able to find a solution to a complex geometry problem. Based on the analysis, it is not possible to conclude that the students have developed in their reasoning process. However, there has been some mathematical movement throughout the group meetings, from working with the first problem to working with the second one. This study indicates how it is possible within a particular instructional context for students in teacher education with limited background in mathematics to improve their problem-solving skills.

Based on three models of problem solving (Polya, 1945/1957; Mason et al., 1982; Mason and Davis, 1991; Borgersen, 1994), reasoning can be defined as five interrelated processes of mathematical thinking, categorised as *sense-making*, *conjecturing*, *convincing*, *reflecting*, and *generalising* (Bjuland, 2002). Instead of focusing on these overall processes of reasoning, this article focuses on how the students express their reasoning through their ways of employing heuristic strategies in the solution process. One central aim of this study is therefore to contribute to the understanding of how students are able to use constructive heuristic strategies in their solution process. For instance, by identifying the strategies of visualising, monitoring, and questioning, we gain insight into the students' ways of approaching and making sense of the problems given, and into their different attempts at finding possible solutions.

At a very general level, I am interested in what happens when you let a group of students in teacher education discuss mathematics in a collaborative problem-solving context without getting any input from their teacher. I am aware of the fact that collaborative work in a problem-solving context without teacher involvement is a radical position to take with respect to the teaching and learning of mathematics. In one respect it is natural to have a teacher visiting a group in order to stimulate the students' solution process. However, I want to suggest that students in teacher education can benefit from learning mathematics through collaborative problem solving without teacher involvement within a particular instructional context (see methodology section). This perspective is also exemplified by the study carried out by Borgersen (1994, 2004) with adult students working on problems in geometry in small groups. Throughout the analyses I am also concerned with the question: when is it appropriate for a teacher to become involved in the group discussion?

Based on these introductory considerations, the following research question has been formulated: Which heuristic strategies can be identified in the dialogues of a group of adult students when working on two geometry problems without teacher intervention? Based on this question it is natural to focus on whether these strategies stimulate mathematical progress in the students' solution process.

2. Theoretical background

Inspired by Barkatsas and Hunting (1996), the collaborative mathematical problem-solving process is defined as the cognitive, metacognitive, socio-cultural and affective process of figuring out how to solve a problem. My major concern for the analysis of the dialogues is the students' reasoning process, the cognitive and metacognitive component of the problem-solving process. In this article, elements of reasoning are specifically explored in the dialogues by the identification of the students' heuristic strategies employed in the solution process. More specifically, this study focuses on four groups of heuristic strategies categorised as *visualising*, *monitoring*, *questioning*, and *logical strategy*.

2.1 Heuristic strategies

In general, the concept of strategy is defined as 'a plan you use in order to achieve something' (Collins Cobuild Dictionary, 1993, p. 791). I am concerned with a particular branch of strategies well-known in problem-solving research based on the work of Polya (1945/1957). For the purpose of the analysis of the dialogues, the terms *problem-solving strategies* and *heuristic strategies* will be used interchangeably.

When students are translating a mathematical text into a visual representation by drawing an auxiliary figure or making a modification of a figure, they employ the strategy of visualising. Drawing a figure is widely accepted as a useful strategy in order to generate and manage the global picture of a problem situation (Duval, 1998; Mason and Davis, 1991; Polya, 1945/1957). According to Mason and Davis (op. cit), a figure provides structure, and encourages tying down thoughts and conjectures that buzz around in the pupils' minds. Duval (1998) claims that in a geometrical figure there are more constituent *gestalts* and more possible subconfigurations than the ones explicitly named in theorems.

In this study, the strategy of monitoring is related to the students' metacognitive activity throughout their problem solving, when they are concerned with monitoring their solution process and when they look back and consider a convincing argument. In the analysis, I am particularly concerned with the monitoring questions that stimulate the solution process. A monitoring question is identified both as a monitoring strategy and as a questioning strategy.

In the Socratic dialogues, one learns that it is more difficult to ask questions than to answer them. If a person is engaged in a dialogue only to prove himself right and not to gain insight, asking questions will seem to be easier than answering them (Gadamer, 1989). However, in order to be able to ask a question, one must wish to know, and that means knowing that one does not know. Asking mathematical questions is vital, both as part of the presentation of mathematics and in the context of problems for students (Mason, 2000). Mason claims that questions arise as pedagogic instruments in classrooms both for engaging students in and assessing students' grasp of ideas. It is the disturbance represented by the sudden shift of one's own attention that prompt a question

(Mason, op. cit.). When students work on problems in a small group without teacher intervention, they initiate the questions and ideas themselves. It is therefore interesting to categorise different types of questions (the strategy of questioning) that are constructive for the solution process.

In this study, the logical strategy is related to the students' attempts at building up a logical cause effect argument. In the analyses of the dialogues, I therefore focus on the students' use of *if-then structures* in the mathematical discussion.

2.2 A brief review of literature on problem solving

During the 1980s, research focused on case studies and interview research using thinking-aloud protocols to try to ascertain the distinctions in approaches to problem solving between successful and unsuccessful problem solvers, so-called experts and novices (Lester, 1994).

The findings of Schoenfeld (1985, 1992), based on empirical material of more than one hundred videotapes of college and high-school students, working on unfamiliar problems, indicate that typical students spend the full 20 minutes allocated for the problem session in unstructured exploration. Roughly 60% of the solution attempts have a solution-profile in which the students read the problem and quickly choose an approach to it, and pursue it in that direction without reconsidering or reversing it. Schoenfeld (1992) shows a time-line graph of a solution process for a typical student and of a mathematician respectively, attempting to solve a non-standard problem. While the typical student spent most of the time in unstructured exploration or moving quickly into implementation of the problem, the expert spent more than half of his allotted time trying to make sense of the problem. The mathematician did not move into implementation until she/he was sure she/he was working in the right direction.

Another study carried out by Goos and Galbraith (1996) confirms the fact that students do not spend much time on making sense of an unfamiliar problem. These authors focused on the nature and quality of the interactions between sixteen-year-old secondary school students working collaboratively on application problems. The structure of the students' problem-solving attempts showed an immediate jump into implementation after an initial quick reading and analysis of the problem.

Carlson and Bloom (2005) used a multidimensional problem-solving framework with four phases (orientation, planning, executing, and checking) in order to describe the problem-solving behaviours of 12 mathematicians as they worked on four mathematical problems. The effectiveness of these experts in making intelligent decisions, leading to mathematical solutions stemmed from their ability to draw on their various problem-solving attributes (conceptual knowledge, affect, heuristics, and monitoring) throughout the problem-solving process.

These studies from individual problem solving of mathematicians (Carlson and Bloom, op. cit), from non-experienced problem solvers (Schoenfeld, op. cit), and from interactions between students working collaboratively (Goos and Galbraith, op. cit), show that metacognitive awareness is an important element in order to succeed in solving a problem. In his musings about mathematical problem-solving research, Lester (1994) also confirms that metacognition was seen as the driving force in problem solving. He claims that research is only just beginning to

understand the degree to which metacognition influences problem-solving activity. However, Lester focuses on some results that have come to be generally accepted. One of these results shows that effective metacognitive activity during problem solving is quite difficult. It requires knowing not only what and when to monitor, but also how to monitor. The result shows that it is a difficult task to teach students how to monitor their behaviour.

During the 90s, research in mathematics education focused on peer interaction in small groups as an important issue (Cobb, 1995; Healy et al., 1995; Hoyles et al., 1991; Kieran and Dreyfus, 1998). Following Brodie (2000), I think that such a context could be crucial as an arena for learning since peer interaction is seen to provide support for the construction of mathematical meaning by students. It also allows more time for student talk and activity. According to Farr (1990), the dynamics of a three-person group changes dramatically compared to a two-person group, since it is possible to form coalitions in the former size of group, but not in the latter. As far as my five-person group is concerned, the dynamics of the group are quite complex since the perspective of every single student could be brought into the mathematical discussion.

Using Vygotskian terminology, Forman (1989) names three conditions needed for a Zone of Proximal Development, created by collaborating students, to be effective: Students must have mutual respect for each other's perspective on the task, there must be an equal distribution of knowledge, and there must be an equal distribution of power. According to Hiebert (1992), when students express themselves, they reveal different ways of thinking, ways that can be acquired by other members of the group. By expressing ideas and defending them in the face of others' questions, and by questioning others' ideas, the students are forced to deal with disagreements. I assert that this may stimulate the students to elaborate, clarify, and maybe reorganise their own thinking.

Goos et al. (2002) carried out a three year study concerning patterns of student-student social interaction that mediated metacognitive activity in senior secondary school mathematics classrooms. Analyses of dialogues of small group problem solving focused on how a collaborative zone of proximal development could be established through interaction between peers of comparable expertise. Unsuccessful problem solving was related to the students' poor metacognitive decisions during the problem-solving process and their lack of critically challenging each other's thinking. Successful outcomes were revealed if students challenged and rejected unhelpful ideas and actively stimulated constructive strategies.

It is necessary to ask, critically, whether it is sufficient to place students in collaborative groups in order to enhance mathematical reasoning. More specifically as Stacey (1992) puts the question: Are two heads better than one? In a study carried out by Stacey (op. cit) in which Year 9 students (average age 14 years) were given a written test of problem solving, groups of students did not acquire better results than individual student performance while solving the same problems. To investigate why this happened based on analyses from the dialogues of students solving problems in groups, Stacey (op. cit) observed that many ideas were brought into the discussions, but the students had difficulties in selecting those which would be effective. Constructive ideas were rejected in favor of simpler, but erroneous, ideas.

Other researchers have also put emphasis on the following question: Is learning mathematics through conversation as good as they say? (Sfard et al., 1998). A study carried out by Sfard and Kieran (2001), revealed that the students' communication was ineffective when two 13-year-old boys were learning algebra. Based on empirical material taken from a two-month-long series of group interactions, the findings indicated that the collaboration seemed unhelpful and it lacked the expected synergetic quality.

A study focused on the enhancement of mathematical reasoning in eighth-grade classrooms (384 students) by investigating the effects of four instructional methods on students' reasoning and metacognitive training (Kramarski and Mevarech, 2003). This study indicates that students need metacognitive training in combination with cooperative learning in order to enhance mathematical reasoning. These aspects are also critical elements in my design of the instructional context introduced in the methodology part.

3. METHOD

The data corpus of the study has been collected at a teacher-training college in Norway. In this particular year, 105 students attended the four-year teacher education programme in order to become teachers in primary (elementary) school or in lower secondary school. All the students had to participate in a problem-solving course in geometry in their first semester as a part of the mathematics programme. This course consisted of three parts: a first part of teaching over a month in September, a second part of small-group work without teacher involvement over three weeks in October, and a third part of teaching in which problems from the second part would be discussed and elaborated in some plenary lessons. During the research project I was a teacher in the first part of the problem-solving course and a researcher in the second one. In fact we were two teachers who carried out the teaching programme. During the first part it was important for us to reflect on the teaching at the end of each day. Another crucial aspect of these meetings was also to discuss and reflect on my two different roles during the project.

In the second part, I was only concerned with the observation of three groups of students, focusing on their problem-solving process with two geometry problems. The empirical material is based on the small-group work without teacher involvement from this period. The data comprises fieldnotes and audio transcripts of four group meetings (8 lessons) in each group, and 21 group reports from this collaborative small-group work. I chose not to use video in my data collection procedure since I believe that the pressure on the students working under video observation might influence the conversation more than audio recording.

3.1 Subjects

At the beginning of the semester, the students were divided into groups of five in alphabetical order by the administration at the college (21 groups). This means that I could not influence how the groups were arranged with regards to variables such as sex, mathematical attainment and so on. Three groups were randomly chosen for observation, and I am here concerned with one of these groups.

From table 1 below it can be seen that four of the students have only attended the compulsory course (1MA), and two of those students have low marks from this course. I do not know what the students have done in the period between upper secondary school (students graduate when

they are 19) and their beginning at the teacher-training college. However, from the group reflection at the end of the fourth meeting, I know that Liv has not done any mathematics for five years (see Bjuland, 1997, p. 194).

Name	Age	Gender	Mathematical background
Unn	20	F	1MA (H)
Mia	21	F	1MA (M)
Gry	21	F	1MA (L)
Liv	22	F	1MA (L)
Roy	24	M	1MA (M), 2MN (M), 3MN (M)

Table 1: Background knowledge of the subjects

1MA: Compulsory course in the first year in upper secondary school.

2MN, 3MN: Voluntary courses in the second and third year in upper secondary school, preparing for further studies in natural sciences.

(L), (M), (H): Represents categories of grading: (L): Low marks, (M): Middle-average marks, (H): High marks

From the background knowledge about the subjects, it is possible to argue that Roy could play a dominant role in the group discussion. He is the only male student, he is the oldest, and he has also the best mathematical background. However, from the group reflection at the fourth meeting, Roy says that he had limited background knowledge in geometry when he started at the teacher-training college. Later in the same group reflection (see Bjuland, 1997, p. 201), he makes clear that he could hardly remember how to construct a perpendicular. Based on the background knowledge about the subjects, I have chosen to categorise this as a group of students with limited mathematical background.

3.2 The instructional context

In Bjuland (1997) there is a detailed description of the 28 lessons of mathematics that were designed for the first part of the problem-solving course as a preparation for the collaborative small-group work in the second part. It is important to give a brief outline of this period since this provides the background for the analysis of the students' dialogues.

The aim of this teaching part was to focus on basic classical geometry, prepare students to work on problems in small groups, and stimulate students to experience mathematics as a process, as described by Borgersen (1994). We focused particularly on geometrical concepts that were relevant for the small-group work without teacher involvement in the second part, for instance concepts related to circles, similarity, cyclic quadrilaterals, and the relationship between the angle at the centre and the angle at the circumference (Thales' theorem).

Some advice introduced by Johnson and Johnson (1990) on how cooperative learning can be used in mathematics was also presented in order to focus on the effect of group dynamics. These authors suggest the following basic elements in their standard for cooperative learning: 1) *Positive interdependence* (group members should 'sink and swim together' to reach a common goal); 2) *Promotive interaction* (the participants assist, help, support and encourage each other's effort to achieve); 3) *Individual accountability* (students cannot 'hitchhike' on the work of

others, they are held responsible for their contribution to accomplishing the goal); 4) *Interpersonal and small group skills* (students should be taught in social skills which include leadership, decision-making, communication, trust-building and conflict-management skills); 5) *Group processing* (students reflect on their work and decide on ways to improve effectiveness).

We were particularly concerned with the fifth element since we used this in combination with metacognitive training. The students were asked to write about how the problem-solving process developed, how ideas or strategies emerged in the dialogue, and how every idea and suggestion was introduced or presented. These experiences were then brought into a plenary discussion afterwards. For instance, in the notes it was common that the students wrote something like: 'suddenly one of the group members came up with an idea and we solved the problem'. In the lectures, we stressed the importance of writing down explicitly which idea they had. We were aware of the fact that monitoring activity during problem solving is quite difficult (Lester, 1994). However by focusing on this process writing throughout one month, we aimed at giving the students the opportunity to focus on when to monitor and how to monitor during the problem-solving process.

The students were also introduced to two models that illustrate different stages in a problem-solving process. By introducing Polya's four-stage model (Polya 1945/1957), and an expansion of this model in seven main steps (Borgersen 1994), the students gained some insight into the various elements in a mathematical problem-solving process. An ongoing problem *Best place on Stadium*, formulated by Borgersen (1994) was used to illustrate the dynamic and cyclic stages of these models. The problem is adapted to a well-known everyday context for the students: 'As a student in Bergen, you would like to go to a football match in order to watch the local women's team Sandviken play against Trondheims-Ørn. If you have a ticket for the long side of the field, which place is the best for watching the goal scored by your home team?' (translation of the original Norwegian text, see Bjuland, 1997, p. 61).

In the plenary lectures we also presented dialogues from a mathematical classroom (Johnsen, 1996) similar to the classroom of Lampert (1990) in which the pupils learned mathematics through questions and answers in a conjecturing atmosphere. We analysed the nature of this classroom discourse and particularly focused on different types of questions emerging in the dialogue and how the teacher taught the mathematics through problem solving.

Throughout the group meetings of the second part, the students were expected to work on the problems without getting any help from their teacher. From an educational perspective, it can be argued that the students' collaborative problem solving of this part was not free from teacher intervention. The situation was within an instructional context and I, as an observer, was present throughout the group meetings. The students were also stimulated by certain objectives introduced on the sheet of paper with the mathematical problems, giving guidelines for a group report from the small-group work of the second part. This report was to consist mainly of three different elements: the solution of the problems, the process writing in which the students were to write down their ideas and strategies throughout the problem-solving process, and a reflection part in which the students were to reflect on their problem-solving process. Linked to this reflection, the report was also to include an evaluation of the totality of the small-group work, e.g. by answering questions like: What have we learned from this small-group work?

As far as the three project groups are concerned, my co-teacher was not involved in the students' mathematical discussion. He only visited the groups in order to give some general information or to check that the students were present since these meetings were a compulsory part of the course. The two geometry problems, given in the second part, would then be discussed and elaborated in some plenary lessons in the third part of the project.

3.3 Problem selection

The group members worked on the following problems:

Problem 1

- A. Choose a point P in the plane. Construct an equilateral triangle such that P is an interior point and such that the distance from P to the sides of the triangle is 3, 5 and 7 cm respectively.
- B. Choose an arbitrary equilateral triangle $\triangle ABC$. Let P be an interior point. Let d_a, d_b, d_c be the distances from P to the sides of the triangle (d_a is the distance from P to the side opposite of A , etc.)
 - a) Choose different positions for P and measure d_a, d_b, d_c each time. Make a table and look for patterns. Try to formulate a conjecture.
 - b) Try to prove the conjecture in a).
 - c) Try to generalise the problem above.

Problem 3

Given a right-angled triangle $\triangle ABC$ ($\angle B = 90^\circ$) and a semicircle Ω with centre O and diameter AQ , where Q is a point on AB . The points P ($P \neq A$) and R on Ω are given so that P is on AC and OR is perpendicular to AB .

- a) Find $\angle APR$ and $\angle QPC$.
- b) Prove that $\angle BQC = \angle BPC$.
- c) Prove that if B, P, R are collinear (are points on a line), then BC and BQ are of equal lengths.
- d) Formulate the converse of the theorem in c). Investigate whether this formulation is a theorem.

The experiences from observations of one student group working on these problems at another teaching-training college helped me to obtain a thorough understanding of the mathematics. For a group of students in their first semester at a teacher training college, I therefore believe that these problems are sufficiently difficult to ensure that the participants are dependent on one another in order to succeed in finding a solution. The students are challenged to experience the cyclic structure of the problem-solving process, as described by Borgersen (1994; 2004). These problems also have embedded in them some important geometrical concepts to be learned.

Problem 1A was meant to be an introductory problem, focusing on important constructions using a compass and straight edge. Even though this is a bit difficult introductory problem, it is possible to come up with a solution based on the lessons designed for the first part of the problem-solving course.

Problem 1B stimulates the students to experience the cyclic structure of the problem-solving process, from finding the distance sum based on drawings, measurements, and constructions of conjectures via attempts at proving the conjectures to generalisations and formulation of new

problems. Based on observation from students working on this problem, I suggest that most student groups of the problem-solving course would manage to solve problem 1Ba. Even though problem 1Bb and 1Bc are quite difficult, especially for some of these student groups, most of these groups are able to deal with a diversity of geometrical concepts. These discussions can then function as a starter for an elaborated dialogue in a plenary session in the third part.

Problem 3 recapitulates many geometrical concepts that were focused on in the teaching part. It could be argued that this problem does not offer the same rich setting for exploration as that of problem 1. Even though we have to take into consideration that these geometry problems are different in nature, I want to emphasise that problem 3 is not an easy problem for these students. First of all the students have to read and analyse carefully the introduction part. A crucial component of approaching and making sense of the text is to draw an auxiliary figure in order to visualise the problem. The text consists of mathematical concepts and symbols compressed within less than three lines. The students must interpret several pieces of information in order to make the transition from the presentation in the text into a visual representation. It is also possible to find two different solutions for angle $\angle APR$ (problem 3a). In order to solve problem 3b, the students must combine knowledge of cyclic quadrilaterals and apply Thales' theorem.

3.4 Unit of analysis

I have chosen the dialogical approach (Marková and Foppa, 1990; Linell, 1998) to the data analysis since this stance 'allows one to analyse the co-construction of formal language among participants in a defined situation' (Cestari, 1997, p. 41). More specifically, this approach permits me to identify interactional processes, which, in the analyses of these particular episodes, are the students' utterances expressing their heuristic strategies used in the solution process.

Following Wells (1999), I have divided the two episodes at three levels, indicating a gradually more detailed analysis. First, the episodes have been divided into thematic segments (sequences). Then each segment has been divided into exchanges. Finally, each exchange has been divided into utterances (moves), either initiating, responding or follow-up utterances. The exchange level constitutes the most appropriate unit for the analysis for me in order to capture the dynamic characteristics of the dialogues. This is quite in correspondence to the dialogical approach and Marková's (1990) three-step process as the unit of analysis. This means that in the sequence of conversation each utterance is interrelated to the previous utterance as well as to the subsequent one. Each utterance has to be interpreted according to the contexts in which it is expressed (Linell, 1998).

The students' utterances are presented in the left column, while the right column shows important heuristic strategies used in the students' solution process and other aspects from the analysis. Four categories of strategies are identified, given the following abbreviations in the right column: VS (Visualising strategy), MS (Monitoring strategy), QS (Questioning strategy), and LS (Logical strategy). The name in bold shows the student that initiates a strategy. This is exemplified in the following utterance from the introductory segment of the first episode analysed below:

223 **Gry** ...(6 sec.)... Should we write what we've done so far?... we have drawn... MS, QS (1. Monitoring question categorised as looking back. 2. Recapitulation of the solution process).

Gry's initiative consists of two monitoring strategies (MS), a *monitoring question* that elicits and triggers the mathematical activity of the group, and a *recapitulation* of the ongoing solution process. The monitoring questions are also included in another category QS.

4. Approaching and making sense of problem 1Ba

The students took about 70 minutes to come up with a solution on problem 1A, in addition, to spending 20 minutes discussing alternative ways of doing the problem. So, the students were concerned with problem 1A for the whole of the first meeting.

The aim of presenting the following episode is to identify crucial heuristic strategies used in the students' solution process of problem 1Ba. More specifically, the analysis of the students' discussion focuses on how the students attribute meaning to the concept of distance from a point to a line. The episode is taken from the second meeting and all five group members are present. The analysis of the episode is organised in four thematic segments.

The dialogue preceding this episode has shown that the students have read problem 1Ba, discussed the formulation and drawn a figure of an equilateral triangle. They are now ready to do the measurements in a). Gry, who was absent from the first meeting, has taken the responsibility of being the process writer. That means that she is concerned with writing down ideas and strategies that emerge in the problem-solving process.

4.1 Recapitulation and the role of the process-writer

223	Gry	...(6 sec.)... Should we write what we've done so far?... we have drawn...	MS, QS (1. Monitoring question categorised as looking back. 2. Recapitulation of the solution process).
224	Unn	We measure the distance from the line to the point...	Attuned response. The concept of distance is introduced.
225	Roy	Well each... each of us has placed the point P in different places... in order to get... well... shouldn't we make a	Linked to (223) and (224). Recapitulation – confirmative question focusing on the next activity: to make a table.
226	Unn	table?... or?...	
227	Roy	Yes...	Agreement.
228	Unn	Yes (low voice)	Agreement.
229	Roy	Then we place all the measurements... (in	Following up Roy (225).
230	Gry	a table) Yes... d_b is... d_a is... d_c is...(8 sec.)... Then we have... we have placed those points in different ways... and then we	Roy starts making a table - Silence MS, QS (1. Monitoring question categorised as looking back, asking for an explanation, 2.
231	Roy	have drawn... no... how should we say	Recapitulation, returning to the concept of distance).
232	Liv	it... those lines?...	
233	Roy	Well (simult.) We have measured... (simult.) Then we have d_a d_b d_c ... those are the distances from P to the sides of the triangle... we have to say (write) that it's the shortest distance to the sides... we've chosen...	Attunement. Attunement. MS (Recapitulation, summing up for process writing, giving an explanation, making explicit the distances).

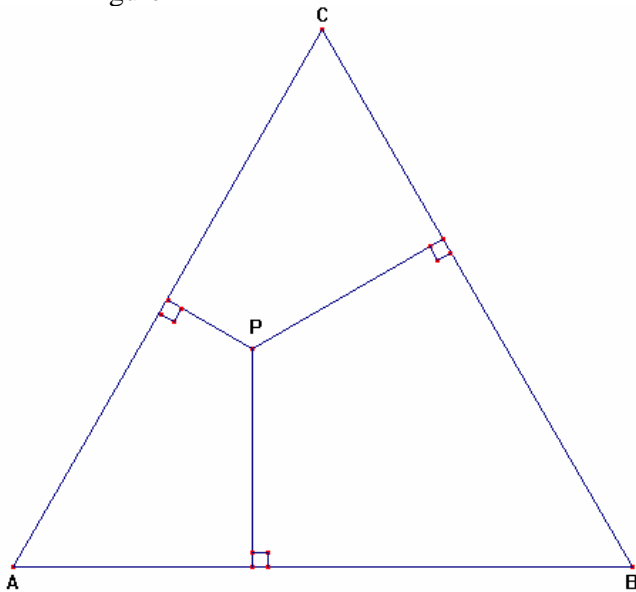
The dialogue illustrates two important monitoring strategies: Monitoring questions (also questioning strategy) and recapitulation of the solution process. It also shows the important role of Gry as the process writer since she elicits and triggers the mathematical activity in the group by initiating two crucial questions. Her *monitoring question*, categorised as a *looking-back question* (223), stimulates a *recapitulation* of the solution process, promoting the establishment of a common ground for the further discussion. The concept of distance from a point to a line has been introduced in the discourse. Gry's second *looking-back question* (230) challenges her colleagues to give an explanation how they have drawn the lines from P to the sides of the triangle. Both monitoring questions have been initiated after some silence in the group (223), (229). The students are attuned to Gry's initiatives. In his explanation, Roy makes it clear that they are concerned with the distances d_a , d_b , d_c , emphasising that they have chosen d_a , d_b , d_c to be 'the shortest distances' (233) from P to the sides of the triangle.

The monitoring strategies of posing looking back questions and recapitulation the ongoing solution process are related to the students' focus on the process writing. These strategies are also indicators of mutuality since the students, in this way, are establishing a common ground for the problem-solving activity.

4.2 Questioning generating a discussion about the concept of distance

The second segment is almost a continuation of the first one. Liv has been working on her figure, and she has constructed the perpendiculars from P to their intersections with the sides of the triangle (see figure 1 below). The dialogue below shows how some questions generate a mathematical discussion about the concept of distance among the students.

Figure 1

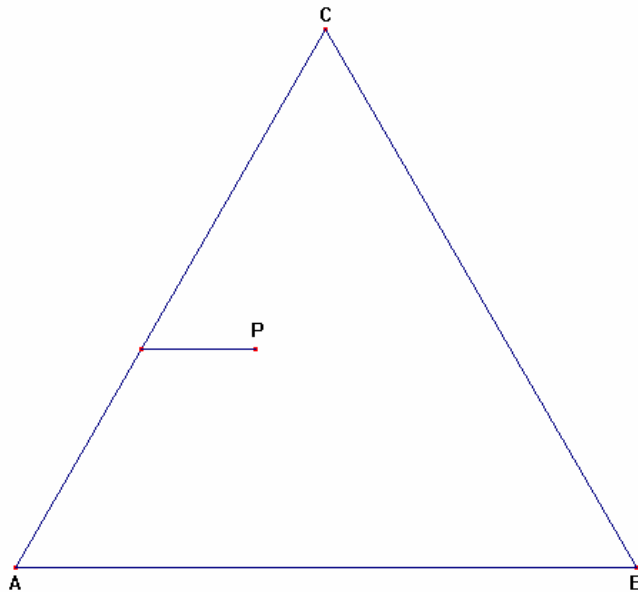


237	Unn	90 degrees?... is that relevant?...	MS, QS, VS (Monitoring questions related to figure 1, asking for an explanation n).
238	Roy	Well... in order to get the shortest... (inaudible, simult.)	Attuned response.
239	Liv	90 degrees is the shortest	Attuned response.
240	Unn	distance...(simult.) Yes but why couldn't we draw it just right out there?...	QS, VS (Why-question related to figure 2, introducing an alternative way of doing the measurements).
241			QS (Brief following up question, asking for clarification).
	Roy	Hmm?...	
242			QS (Why-question related to figure 1, challenging the '90 degrees' perspective).
	Unn	Why do you draw it up there?...	
243			
244	Liv	90 degrees is the shortest distance...	Repetition of the same explanation.
245	Roy	(simult.)...	Explanation is linked to the figure.
	Mia	90 degrees on that line there... (simult.)...	QS (Following up question, asking for further explanations).
246		Should it be 90 degrees?...	
	Liv		Repetition of the same explanation.
		Yes... 90 degrees is the shortest distance...	

The dialogue shows three open questions initiated by the same student, Unn. These questions stimulate a discussion about the concept of distance among the students. Unn has probably seen that Liv has been working on her figure. This seems to stimulate the *monitoring question* (237) that challenges the students to consider why they should use angles of 90 degrees in order to do the measurements (see figure 1). The *why question* (240) comes up with an alternative way of doing these measurements in which the distance should be measured along the line through *P* parallel to the base of the triangle (see figure 2 below). The third question (242) challenges the students to justify why 'the shortest distance' is the length of the perpendicular from *P* to its intersection with the side of the triangle. These questions are all related to the students' attempt at drawing a figure, the strategy of visualising. Mia enters the dialogue (245) with a *following up question* that challenges Roy and Liv to reconsider their argumentation. However, the response of the 90-degree angle has been constantly repeated (246).

The dialogue throughout this segment has shown that the three strategies *monitoring*, *questioning*, *visualising* (translating the problem into a visual representation) stimulate the students to become aware of two alternative ways of interpreting distance from a point to a line.

Figure 2



4.3 Elaboration on the two different perspectives

As a continuation of the dialogue, these two different perspectives on the idea of distance are discussed.

- | | | | |
|-----|------------|---|--|
| 256 | Unn | But I don't agree... I think that this is very illogical... because if I should have measured this... | LS (If-then structure, building up a logical cause effect argument). |
| 257 | Roy | It's nearly like this... (Roy refers to the perpendicular on his figure) | Focus on his figure. |
| 258 | Unn | Then I would only have laid the ruler there and said how far it is right out... | Continuation of (256), repetition of own perspective. |
| 259 | Liv | But that's not the shortest distance... | Attunement, challenging Unn (258). |
| 260 | Unn | I would only have done it like this... if it's 90 or 60 degrees... actually that doesn't make any difference to me... | LS (Begin the if-then structure, sticking to her own perspective). |
| 261 | Roy | But then... the distance differs then... | LS (Complete the if-then structure, giving the argument). |

The dialogue shows that the students are now ready to defend their own arguments and challenge the other students' points of view. By employing the logical strategy (LS), using an *if-then structure*, Unn repeats the perspective of measuring the distance along the line through *P* parallel to the base of the triangle (256), (258). From a mathematical point of view, this difficulty seems

to be quite stable since it is natural or logical to lay the ruler parallel to a horizontal base. The argument against this idea has been repeated, claiming that this is not 'the shortest distance' from P to the side of the triangle (259). Prior to this discussion, Liv has constantly repeated that they have to focus on the 90-degree angle.

The dialogue shows the great attunement among the students, particularly when Unn and Roy employ the logical strategy together. Unn introduces the *if-then structure* (260), while Roy is following up and giving the argument (261). By giving examples of 90-degree angles or 60-degree angles, indicating that it makes no difference how you measure, the students are introduced to the fact that the two different perspectives lead to different distances. The elaboration of the two different perspectives has brought a new element into the mathematical discussion since some of the students have seen that they do not get comparable results if they measure the lengths along different line segments from P to the sides of the triangle.

4.4 Justification leading to agreement

After having elaborated on the two different perspectives on the concept of distance from a point to a line, the dialogue continues with the important *why-question* below:

- | | | | |
|-----|------------|--|---|
| 269 | Liv | But why did you measure straight out like this?... | QS (Why-question, challenging Unn's perspective). |
| 270 | Unn | No... actually I don't know... that's a good question but it can... well... Hmm?... | Uncertainty. Difficulty to give an explanation. |
| 271 | Roy | No it was just accidental that I did it like this... but eeh... we have to find... shouldn't we decide to do this in the same way?... | QS (Brief following up question, asking for further explanations). |
| 272 | Unn | At any rate, we have to decide that the three angles are equal... if we should find any ratio between those... we can't have one angle of 30 degrees like this and then one of 60 and one of 90... that's... the three angles have to be equal if we're to find a pattern... | Questions (269), (271), leading to agreement. Unn's question triggers a confirmation. |
| 273 | Roy | | MS (Recapitulation, conclusion. Summary of the ongoing discussion).
LS (If-then structure, building up a logical cause effect argument). |

The *why-question* (269) challenges Unn to give an argument for her way of measuring the distances from P to the sides of the triangle. Unn has problems in giving reasons for her interpretation of distance, and she realises that Liv's question is good (270). In fact she is not that eager to defend her proposition. When Unn is invited to repeat her way of doing the measurements by Roy's *following up question* (271), she admits that her idea was just accidental (272). It seems as if Unn gradually realises that they have to decide to do the measurements in the same way in order to be able to compare their results.

Roy repeats the argumentation for doing the measurements in the same way in order to be able to find a pattern (273). He sums up the most important elements of the ongoing discussion and

draws a conclusion. The students have established agreement about the fact that they have to choose a particular and unique distance in order to find a pattern. The participants gradually come round to a single way of interpreting the concept of distance from a point to a line.

The students in this particular group do not have the experience and background in mathematics to go straight to the measurements of d_a , d_b and d_c respectively. However, by using the strategies of *monitoring* (recapitulation, monitoring questions), *questioning* (posing open questions, categorised as *monitoring*, *why* and *following up*) and the logical strategy (*if-then structure*), they make progress in their attempts at attributing meaning of the concept of distance from a point to a line.

4.5 Reconstruction leading to the conjecture $d_a + d_b + d_c = \text{constant}$

The students spent about 32 minutes on problem 1Ba. The analysis of the episode introduced above has been chosen from the first 10 minutes of this discussion. The reconstruction of the solution process for the final 22 minutes of the solution process is briefly summarised below, leading to the conjecture: $d_a + d_b + d_c = \text{constant}$.

Roy helps Mia to draw the perpendiculars on her figure (the strategy of visualising the problem, VS). She is still uncertain regarding the prior discussion about choosing the distance from P to a side of a triangle to be the length of the perpendicular segment from P to its intersection with that side for their measurements. Roy repeats the argument for measuring in the same way in order to be able to find a pattern. In his explanation, he also draws an arbitrary line from P to the side of the triangle in order to show Mia that this line segment is longer than the length of the perpendicular from P to its intersection with the side of the triangle. The strategy visualising (VS) is then used as a tool for his explanation.

Roy brings the idea into the discussion that the triangles should have equal sides if they should find a pattern. The students decide to place some more points in their own triangles, and Roy comes up with the following conjecture: $(d_a + d_b + d_c) / 3 = \text{constant}$. The students try out this conjecture, and they observe that they do not need to divide by 3.

The students spent about 25 minutes working on problem 1Bb without coming up with a convincing argument. In the discussion, they focused on cyclic quadrilaterals, the special quadrilateral kite, and the fact that the conjecture was based on equilateral triangles. They were also conjecturing about cyclic quadrilaterals being similar (Bjuland, 2002).

It might be asked why student teachers should work on this difficult problem without getting any help from their teacher? A supervising teacher could have guided the students in their “zone of proximal development” (Vygotsky, 1978). If there had been a teacher, when should he become involved in the group discussion? As far as problem 1Ba is concerned, a teacher could just have told them how distance is defined from a mathematical point of view.

However, a teacher often dominates the discussion since his voice represents the mathematical community. Based on the analysis of the group discussion from the solution process of problem 1Ba, there are reasons to believe that these students have established a constructive mathematical discussion. This is due to the fact that they got the opportunity to attribute meaning to the

concept of distance. This knowledge could then be affirmed in one of the plenary lessons in the third part of the project.

5. From sense-making to convincing on problem 3b

The aim of presenting the second episode is to show how these students are able to succeed in finding a solution to problem 3b, which is quite a difficult problem for students with limited mathematical background. The analysis focuses on the students' main heuristic strategies in the solution process.

The students have started a 25-minutes discussion on problem 3 during the second meeting. They read the problem and discuss what is meant by the mathematical symbols ' Ω ' and ' \neq '. From this conversation, I have reconstructed how the students draw an auxiliary figure step by step in order to visualise the problem (see Bjuland, 2002). The students come up with two different figures:

Figure 3

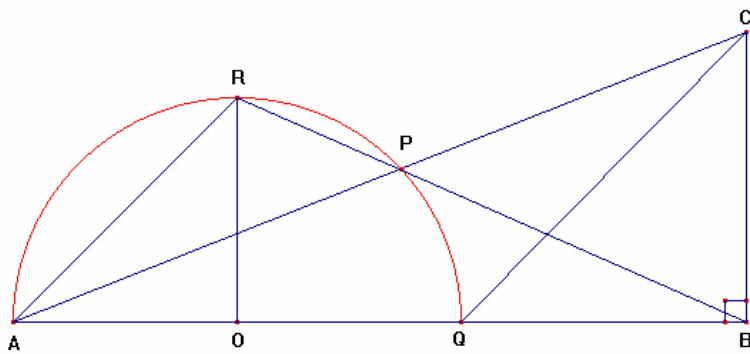
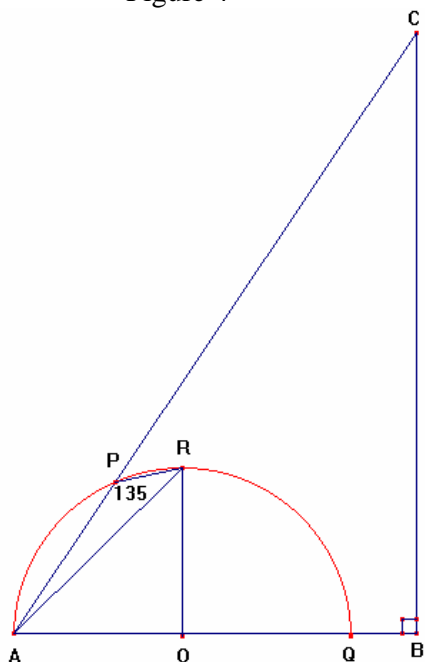


Figure 4



They agree on measuring angle $\angle APR$ by a protractor in order to get an idea of the size of the angle. The measurements from four of the students show that angle $\angle APR$ is 45 degrees. However, one of the students' measurement suggests that angle $\angle APR$ is 135 degrees. The students compare the two figures, but they do not find any mistake. They observe that point P is placed to the left of point R (figure 4). The student who comes up with the 135-degree angle of $\angle APR$ draws a new figure in which angle $\angle APR$ is 45 degrees. By doing this, the students avoid the fact that there are two solutions to the problem.

They find a solution for angle $\angle APR$ by introducing an argument that the angle $\angle AOR$ at the centre is double the angle $\angle APR$ at the circumference since they both subtend the same arc AR (see figure 3). In a similar way, they find the angle at the circumference $\angle QPA$ by applying Thales' theorem, observing that the angle $\angle AOQ$ at the centre is 180 degrees. After some efforts, they come up with a convincing argument that angle $\angle QPC$ is 90 degrees.

The students start working on problem 3b about 30 minutes into the third meeting. Figure 3 is the starting point for the mathematical discussion. The dialogue preceding this episode has shown that figures are drawn based on the information given in the problem, representing geometrical visualisations of the problem. Relevant subconfigurations from figure 3 have been found in order to make sense of the problem. The students have focused particularly on quadrilateral $QBCP$ on a separate figure. Two main questions and one idea have emerged in the discussion: Is $QBCP$ a cyclic quadrilateral? Is triangle $\triangle QPC$ similar to triangle $\triangle QBC$? The idea of reflecting triangle $\triangle QPC$ around the axis of reflection QC has also been discussed (Bjuland, 2002).

The episode is organised in three thematic segments. The analysis of the first segment illustrates that the monitoring strategy of looking back on the solution process, bringing formerly acquired ideas into the discussion, is crucial for the students' attempt at coming up with a reasonable solution.

5.1 Looking back on ideas generated in the solution process

927	Liv	We have APR (angle) which is 45 (degrees) there... then we have...	MS (Recapitulation of solution on problem $3a_1$). Continuation in (929).
928	Roy	That's why we could find...	Attunement. Continuation in (930).
929	Liv	Then there's 90 degrees there... then there's 180 degrees there... isn't there?...	Focus on the figure, probable $\angle QPA$ (90°), $\angle APC$ (180°), related to (930).
930	Roy	That's why we could find PQC ... QPC ... yes but eeh... we have that quadrilateral and we have some vertical angles... reflection... cyclic quadrilateral...	MS (Recapitulation of solution on problem $3a_2$ and different ideas previously discussed).
931	Gry	Yes but it doesn't meet... (the circle does not meet the vertices of the quadrilateral)	Following up Roy (930). The idea of a circle, circumscribing the quadrilateral QBCP.
932	Liv	Similarity then?...	QS (Brief following up question, is triangle $\triangle QPC$ similar to triangle $\triangle QBC$?).

The monitoring strategy of *recapitulation the ongoing solution process* is brought into the discussion by Liv's initiative (927). The use of the personal pronoun *we* (927), suggests that the students have developed a shared understanding of the solution for problem 3a. The students recapitulate previously acquired ideas and solutions in order to come up with a direction for the reasoning process (927) – (932). In one respect these utterances are elaborations on Liv's monitoring initiative (927). However, I have chosen to emphasise Roy's recapitulation (930) as a monitoring strategy since he focuses on specific ideas previous discussed in the solution process. Roy's initiative also stimulates Gry (931) to elaborate on one of these ideas since she focuses on the circle that circumscribes quadrilateral QBCP. The monitoring strategy of *recapitulation* helps the students to establish common ground, giving all of them the opportunity to participate in the mathematical discussion.

5.2 Focusing on the particular idea of cyclic quadrilaterals

933	Unn	Is it a cyclic quadrilateral?...	MS, QS (Monitoring question, asking for confirmation).
934	Roy	Yes... it's possible to find the centre... it's possible to find a centre...	Response. Related (932), a circle, circumscribing QBCP.
935	Mia	She got the circle around the whole... (Mia speaks about Liv's figure)...	MS (Monitoring other students' work, bringing the idea into the discussion).
936	Gry	Did she?...	MS (Monitoring other students' work, bringing the idea into the discussion).
937	Mia	Yes...	QS (following up question attuned to Mia).
938	Gry	I don't...	Response.
939	Liv	It's just... well it's just to bisect... then you get the centre... but it's not...	Response. Related to (931). VS (modify her figure by constructing the centre by bisecting the sides. Suggestion of rejecting this idea).

Based on all the ideas that have emerged in the discussion, a *monitoring question* encourages the students to focus on the particular idea whether quadrilateral *QBCP* is a cyclic quadrilateral or not (933). The dialogue illustrates that the students elaborate on this idea (934) – (939). Liv has, through all her drawings of quadrilateral *QBCP* (Bjuland, 2002), also modified her figure by drawing a circle that circumscribes the quadrilateral. However, she does not explain her work to the other students. Instead, it is Mia who brings this important element into the discussion (935), informing her colleagues about Liv's modified figure. This monitoring strategy provokes more attention (936) – (938), stimulating Liv to focus on her circle (939). She suggests how they could construct the centre of this particular circle in order to obtain a more accurate figure, indicating that she uses the strategy of modifying her visual representation. However, she also suggests that the construction of the circle is not the best idea in order to come up with a solution. Prior to this discussion, Liv has focused on the idea of similarity as a possible direction for the solution process.

In the continuation of the dialogue, Liv is still concerned with the idea of similarity. However, one of the students recapitulates the characteristics of a cyclic quadrilateral by searching for help from her textbook (953). This seems to help the students to conclude that *QBCP* is a cyclic quadrilateral (954) – (956).

953. Unn: A cyclic quadrilateral is a quadrilateral which can be circumscribed by a circle... in cyclic quadrilaterals opposite angles are supplementary angles... together they subtend the whole circumference... (Unn reads from her textbook)...

954. Liv: Mmm...

955. Mia: Yes but that shows that this is a cyclic quadrilateral...

956. Roy: Yes...

The textbook is here used as an important strategy in order to come up with a shared understanding about this particular concept. However, there is still a discussion about whether the opposite angles should be 90 degrees each. The dialogue in the next segment shows how this discussion develops among the participants.

5.3 Breakthrough: coming up with a convincing argument

962	Roy	Yes two of those are 90 and they're 180 together... but those others don't need to be... they are 180 together... but one of those can be 30 and one can be...	Characteristics of a cyclic quadrilateral.
963	Mia	One is... together... one is... yes one of those can be 60 and the other... can't it?... it just says 180 together...	Following up (962).
964	Unn	But... yes... but that we already knew...	
965	Mia	Yes... but she has at any rate... she has roughly done it to get the centre... but it	Confirmation of the characteristics. MS (Monitoring Liv's figure, informing about the circle, circumscribing QBCP).
966	Liv	Well... we can do it exactly then... then we get the centre... here is the centre... like this... do you think it has something to do with angles at the circumference?...	VS (Attuned to Mia (965), responding by making a proper construction of the circle.
967	Roy	Roy?... Hmm?...	Introducing the idea of angles at the circumference).
968	Liv	If we can find angles at the circumference	QS (Brief following up question, asking for a further explanation).
969	Mia	here...	Repetition of the idea of angles at the circumference.
970	Liv	In a circle... Yeeees!... look at this... there we are!... (she laughs and laughs)...	Difficult to interpret, maybe reading from her textbook. Breakthrough, find a solution.

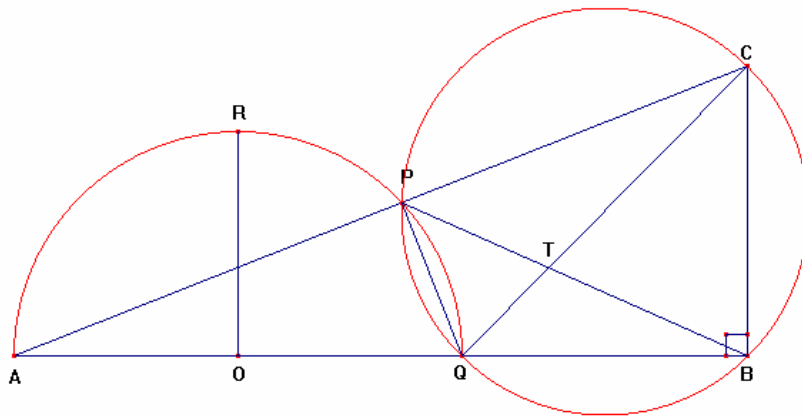
In his explanation, Roy (962) makes it clear that the particular quadrilateral *QBCP* consists of two opposite angles which are 90 degrees each. However, he goes on to emphasise that the other opposite angles do not need to be equal. By introducing examples in which one of the angles could be 30 degrees (962) or 60 degrees (963), both Roy and Mia stress that the sum of two opposite angles should be 180 degrees.

As shown in the analysis of the previous segment, Mia informed the other students about Liv's modified figure in which Liv has drawn a circle that circumscribed the cyclic quadrilateral *QBCP* (935). Sequentially linking it to the discussion above about the cyclic quadrilateral, Mia recapitulates this by reminding the students about this modified figure (965). The initiative of repeating this important step seems to trigger the breakthrough in the solution process (965) – (970). The strategy of monitoring another students' work provokes Liv to do the construction of the circle exactly (966). By focusing on the circle, Liv looks back on the idea of some angles at the circumference and brings this into the discussion by her request for agreement directed to Roy. His brief *following up question* (967) triggers a repetition of this idea. By focusing on some

angles at the circumference (968), Liv comes up with a solution to the problem (970). Her strong affective response suggests that she has observed that angle $\angle BQC$ and angle $\angle BPC$ are both angles at the circumference, subtending the same segment of the circle.

The students' breakthrough in the solution process has been brought about by four crucial steps, leading to the following figure:

Figure 5



The first step has been to recapitulate previously acquired ideas and solutions, conjecturing about cyclic quadrilaterals in particular. The second step has been to conclude that $QBCP$ is a cyclic quadrilateral. This conclusion has been triggered by a *monitoring question* and the students' initiative of recapitulating the characteristics of a cyclic quadrilateral by searching for help from a theoretical source. The third step in the solution process is to construct the circle that circumscribes quadrilateral $QBCP$ (see figure 5 above). This has been stimulated by the monitoring strategy from one of the participants in which the students are reminded of the modified figure where this circle has been approximately drawn around $QBCP$. Based on this, the construction of the circle has been done exactly. It has then been observed that angle $\angle BQC$ and angle $\angle BPC$ are both angles at the circumference, subtending the same arc. By Thales' theorem, these angles are then equal. This is the fourth step in the solution process, and the students have come up with a proper solution.

6. Discussion and conclusion

In order to respond to the research question, I focused my attention on the identification of heuristic strategies expressed in the mathematical discussion of a group of adult students working on two geometrical problems without teacher intervention. Related to this question, I also focused on the critical function of these strategies in order for the students to make mathematical progress in the solution process. Table 2 below summarises these findings.

Problem 1Ba	Problem 3b
Focus: The concept of distance from a point to a line	Focus: The concept of cyclic quadrilateral
<p>Heuristic strategies with critical mathematical function:</p> <ol style="list-style-type: none"> 1. Visualising strategy (VS). Transforming a written mathematical text into a visual representation: <ol style="list-style-type: none"> a) Draw a figure of an equilateral triangle. b) Place different positions of P as interior points of the triangle. c) Draw the distances d_a, d_b, d_c from P to the sides of the triangle. d) Use of figures as a support for their mathematical explanations. 2. Monitoring strategy (MS). Establishing common ground for the problem-solving activity: <ol style="list-style-type: none"> a) Recapitulation <ul style="list-style-type: none"> - Looking back on the solution process, returning to the concept of distance from a point to a line, - Conclusion, summing up the discussion for process writing. b) Monitoring questions <ul style="list-style-type: none"> - Looking back on the solution process, related to process writing 3. Questioning strategy (QS). Posing open questions, stimulating the mathematical discussion. <ol style="list-style-type: none"> a) Monitoring questions (see 2b). b) Why questions. c) Following up questions (asking for further clarifications and explanations) 4. Logical strategy (LS). Building up a logical cause effect argument <ol style="list-style-type: none"> a) If-then structure 	<p>Heuristic strategies with critical mathematical function:</p> <ol style="list-style-type: none"> 1. Visualising strategy (VS). Transforming a written mathematical text into a visual representation: <ol style="list-style-type: none"> a) Draw an auxiliary figure step by step in order to visualise the problem. b) Identify a subconfiguration from the initial figure, the quadrilateral $QBCP$. c) Draw $QBCP$ on a separate figure. d) Modify this figure by drawing a circle, circumscribing the quadrilateral. 2. Monitoring strategy (MS). Establishing common ground for the problem-solving activity: <ol style="list-style-type: none"> a) Recapitulation <ul style="list-style-type: none"> - Looking back on suggested ideas from the solution process, - Looking back on previously acquired solutions. b) Monitoring questions <ul style="list-style-type: none"> - Bringing specific ideas into the discussion. c) Monitoring other students' work <ul style="list-style-type: none"> - Bringing these ideas into the discussion 3. Questioning strategy (QS). Posing open questions, stimulating the mathematical discussion. <ol style="list-style-type: none"> a) Monitoring questions (see 2b). b) Following up questions (asking for further clarifications and explanations). 4. Using the textbook as a tool in order to discuss a particular mathematical concept.

Table 2: Critical heuristic strategies used in the solution process

The diversity of strategies indicates central elements of mathematical reasoning, corresponding to the second component of Schoenfeld's (1985, 1992) framework. I have particularly identified how the following categories of strategies have been constructive for the students' solution process: *visualising*, *monitoring*, *questioning*, and *logical strategy*. The analysis has also revealed that the students, almost simultaneously, use the strategies of modifying their figures, posing open questions, and monitoring their solution process. It seems as if these strategies are all related to the students' metacognitive activity. Lester (1994) confirms that metacognitive

awareness plays an important role in problem solving, but such an activity during problem solving is not common among novice problem solvers. The study carried out by Goos et al. (2002) also reported that unsuccessful problem solving was characterised by students' poor metacognitive decisions throughout the problem-solving process. It might be asked why the students in this particular group are able to monitor their solution process. What stimulates this monitoring activity?

The elements of metacognitive training in combination with cooperative learning were critical components in my design of the instructional context introduced in the methodology part. These aspects are important in order to benefit from collaborative problem solving. This is also emphasised in the study carried out by Kramarski and Mevarech (2003). As far as my particular group is concerned, the conjecturing atmosphere established in the discussion has showed that the communication is mutually supportive. Graumann (1995) emphasises the importance of establishing mutual relationships between participants in dialogues. The common ground among the students is one crucial condition for learning to take place.

The students are attuned to each other's perspectives. They show willingness to pose open questions, and they seek to explain and justify by collaborating with each other in order to come up with important ideas and possible solutions on the mathematical problems. In this respect the analysis of the dialogue is also an indicator of the students' equal status in the group even though one of the students seems to have more power based on the students' mathematical backgrounds and their experiences. In Stacey's study (1992) a lot of ideas were generated in the dialogues among the junior secondary students. However, these students did not seem to monitor or to carefully consider all the group initiatives. Motivation for learning and maturity among student teachers are of course aspects to take into consideration why these students have established a community that shares a commitment to caring, collaboration, and a dialogic mode of making meaning (Wells, 1999).

It is important for students in their teaching training programme to experience collaborative problem solving in small groups. Teaching mathematics through problem solving could stimulate students to develop a thorough understanding of mathematical concepts (Lester and Lambdin, 2004). The students in this particular group have without teacher involvement elaborated their understanding of the particular mathematical concepts: distance from a point to a line and cyclic quadrilaterals. The detailed analyses give us insight into how important the sense-making process is for the students' discussion on geometrical concepts and for the development of a mathematical solution (problem 3). The findings of Schoenfeld (1985, 1992) based on college and high-school students, working on unfamiliar problems, show that the students immediately jump into implementation after an initial quick reading and analysis of the problem. I am aware of the fact that it is difficult to make a comparison between Schoenfeld's novices and my adult students when it comes to their way of approaching and making sense of a given problem, due to different variables such as time and the nature of the problem. However, my analysis makes an important contribution to research since my findings reveal that it is possible for adult students with limited mathematical backgrounds to succeed in finding a solution to a complex geometry problem without teacher involvement.

Even though I take into consideration that problem 1 and problem 3b are different in nature, problem 3b is not an easy problem for these students. First of all they have to focus on a subconfiguration, a quadrilateral, in their figures. Then they have to find out that this is a cyclic quadrilateral. Thirdly, they have to know that it is possible to construct a circle that circumscribes the quadrilateral before using Thales' theorem and arguing for the fact that two angles are equal. The analysis has revealed that these students have identified and argued why the quadrilateral is cyclic. From a mathematical point of view, this is not obvious, and students need quite a lot of geometrical expertise to see this. We could ask why they manage to do this without teacher intervention? The teaching part in the first period of the project plays a crucial role. It is also possible that the monitoring awareness established in the group has helped them to carefully consider the ideas generated in the conversation before rejecting them. The analysis has particularly focused on the important role of the process writer that stimulates the mathematical discussion by generating utterances categorised as *looking-back questions*. She is concerned with recapitulating the solution process or the last idea introduced in the dialogue. This is promoting the establishment of a common ground for the further discussion. It seems as if the process writer really plays a critical role in the process of problem solving. This is an important result that I cannot find adequately addressed in earlier research. It could be tempting to ask what happened in the other 20 groups of students? Is this just a nice story from one group?

In Bjuland (2002), I have carefully analysed one of the other groups of students while working on the same geometry problems. The findings from this group are also promising. By using constructive heuristic strategies, particularly the strategy of posing monitoring questions, these students are also able to find a solution on problem 3b without any involvement from their teacher. In this group all students contribute with important monitoring strategies that stimulate the progress in the solution process. Since only three of the groups were randomly selected for observation, I have only the group reports from the other groups as empirical materials. It is therefore difficult to report on those groups as far as monitoring activity is concerned.

From a social scaffolding perspective (Wells, 1999; Wood, Bruner and Ross, 1976), I have already posed the following question: Could some open questions from a teacher have stimulated the students in their solution process on problem 1? It is also likely that a teacher could have guided the students in their discussions with stimulating questions in order to obtain an effective solution process. If that is the case, why should these students work on two geometry problems without getting any help from a teacher? In the group work, designed in the autumn of a later semester at this particular teacher-training college, other students worked on the same geometrical problems as reported from in my study. Being a teacher throughout these meetings, I had the opportunity to observe the students during their work and identify their difficulties with the problems. When great frustration appeared in a group, I observed that it was constructive for the solution process to stimulate the students' group dialogue by posing an open question linked to their discussion. Maybe such situations are the most suitable for teachers to become involved in the group discussion.

7. Final remarks

This study has focused on how elements of mathematical reasoning (the students' heuristic strategies) are expressed in dialogues. The corpus includes the students' utterances in interaction, socially contextualised in a problem-solving context. The analysis has been focused on student

conversation in one collaborative small group of students working on two geometry problems. The findings have revealed that monitoring activity, related to the use of the strategies of monitoring, questioning, and visualising, is crucial for mathematical progress in the solution process and for having a constructive discussion about mathematical concepts. As a pedagogical implication, this finding suggests that teacher education must stimulate metacognitive training in combination with cooperative learning among the students in order to develop problem-solving skills. More specifically, this also means that students in teacher education must be aware of the critical role of the process writer in the process of problem solving. I am fully aware that the corpus of this study is limited and therefore questionable as the groundwork for generalisation. Still, pre-service teacher educators and researchers can potentially use the findings of this study to help design and implement instruction that stimulate students to develop their problem-solving skills in collaborative small groups.

One possible direction for future research would be to focus more closely on observation, analysis and interpretation of the conversation of adult students working in groups on problems from other topics of mathematics. Is there anything about geometry, in general, or these tasks, in particular, that stimulate students to develop their reasoning or their problem-solving skills?

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References

- Barkatsas, A. N. & Hunting, R. (1996). A review of recent research on cognitive, metacognitive and affective aspects of problem solving. *Nordisk Matematikdidaktikk*, 4, (4), 7-30.
- Bjuland, R. (1997). Problemløsningsprosesser i geometri. Lærerstudenters samarbeid i smågrupper: En dialogisk tilnærming. Unpublished Master's degree thesis. Kristiansand: Agder University College.
- Bjuland, R. (2002). Problem solving in geometry. Reasoning processes of student teachers working in small groups: A dialogical approach. Doctoral dissertation. Bergen: University of Bergen.
- Bjuland, R. (2004). Student teachers' reflections on their learning process through collaborative problem solving in geometry. *Educational Studies in Mathematics*, 55, 199-225.
- Borgersen, H. E. (1994). Open ended problem solving in geometry. *Nordisk Matematikdidaktikk*, 2, (2), 6-35.
- Borgersen, H. E. (2004). Open ended problem solving in geometry revisited. *Nordisk Matematikdidaktikk*, 9, (3), 35-65.

Brodie, K. (2000). Teacher intervention in small-group work. *For the Learning of Mathematics*, 20, (1), 9-16.

Carlson, M. P. & Bloom, I. (2005). The cyclic nature of problem solving: An emergent multidimensional problem-solving framework. *Educational Studies in Mathematics*, 58, 45-75.

Cestari, M. L. (1997). Communication in mathematics classrooms. A dialogical approach. Doctoral dissertation. Oslo: University of Oslo.

Cobb, P. (1995). Mathematical learning and small-group interaction: Four case studies. In P. Cobb and H. Bauersfeld (Eds.), *The emergence of mathematical meaning: Interaction in classroom cultures* (pp. 25-129). Hillsdale, NJ: Lawrence Erlbaum.

Collins Cobuild Dictionary (1993). *Essential English dictionary*. Oslo: Kunnskapsforlaget.

Duval, R. (1998). Geometry from a cognitive point of view. In C. Mammana and V. Villani (Eds.), *Perspectives on the teaching of geometry for the 21st Century*, (pp. 29-37). Dordrecht: Kluwer.

Farr, R. (1990). The social psychology of the prefix 'inter': A prologue to the study of dialogue. In I. Marková & K. Foppa (Eds.), *The dynamics of dialogue* (pp. 25-44). Hemel Hempstead: Harvester Wheatsheaf.

Forman, E. (1989). The role of peer interaction in the social construction of mathematical knowledge. *International Journal of Educational Research*, 13, 55-70.

Gadamer, H. G. (1989). *Truth and method*, New York: Continuum.

Goos, M. & Galbraith, P. (1996). Do it this way! Metacognitive strategies in collaborative mathematical problem solving. *Educational Studies in Mathematics*, 30, 229-260.

Goos, M., Galbraith, P., & Renshaw, P. (2002). Socially mediated metacognition: Creating collaborative zones of proximal development in small group problem solving. *Educational Studies in Mathematics*, 49, 193-223.

Graumann, C. F. (1995). Commonality, mutuality, reciprocity: A conceptual introduction. In I. Marková, C. Graumann, & K. Foppa, (Eds.), *Mutualities in dialogue*, (pp.1-24). Cambridge: Cambridge University Press.

Healy, L., Pozzi, S., & Hoyles, C. (1995). Making sense of groups, computers and mathematics. *Cognition and Instruction*, 13, 505-523.

Hiebert, J. (1992). Reflection and communication: Cognitive considerations in school mathematics reform. *International Journal of Educational Research*, 17, 439-456.

Hoyles, C., Sutherland, R., & Healy, L. (1991). Children talking in computer environments: insights into the role of discussion in mathematics learning. In K. Durkin & B. Shire (Eds.), *Language in mathematical education: Research and practice* (pp. 162-175). Milton Keynes: Open University Press,

Johnsen, V. (1996). Hva er en vinkel? *Nordisk matematikdidaktikk*, 4, (1), 25-49.

Johnson, D. W. & Johnson, R. T. (1990). Using cooperative learning in mathematics. In N. Davidson (Ed.), *Cooperative learning in math. A handbook for teachers* (pp.103-125). Menlo Park, CA: Addison Wesley.

Kieran, C. & Dreyfus, T. (1998). Collaborative versus individual problem solving: Entering another's universe of thought. In A. Oliver & K. Newstead (Eds.), *Proceedings of 22nd PME Conference: Volume 3* (pp.112-119). Stellenbosch, South Africa.

Kramarski, B. & Mevarech, Z. R. (2003). Enhancing mathematical reasoning in the classroom: The effect of cooperative learning and metacognitive training. *American Educational Research Journal* 40 (1), 281-310.

Lampert, M. (1990). When the problem is not the question, and the solution is not the answer: Mathematical knowing and teaching. *American Educational Research Journal*. 27, 29-63.

Lester, F. K. (1994). Musings about mathematical problem-solving research 1970-1994. *Journal for Research in Mathematics Education*, 25, 660-675.

Lester, F. K. & Lambdin, D. V. (2004). Teaching mathematics through problem solving. In B. Clarke et al. (Eds.), *International perspectives on learning and teaching mathematics* (pp.189-203). NCM: Göteborg University.

Linell, P. (1998). *Approaching dialogue. Talk, interaction and contexts in dialogical perspectives*. Amsterdam: John Benjamins Publishing Company.

Marková, I. & Foppa, K (Eds.) (1990). *The dynamics of dialogue*. Hemel Hempstead: Harvester Wheatsheaf.

Marková, I. (1990). A three-step process as a unit of analysis in dialogue. In I. Marková & K. Foppa (Eds.), *The dynamics of dialogue* (pp.129-146). Hemel Hempstead: Harvester Wheatsheaf.

Mason, J., Burton, L., & Stacey, K. (1982). *Thinking mathematically*. London: Addison-Wesley.

Mason, J. & Davis, J (1991). *Fostering and sustaining mathematics thinking through problem solving*. Victoria: Deakin University Press.

Mason, J. (2000). Asking mathematical questions mathematically. *International Journal of Mathematical Education in Science and Technology*, 31, 97-111.

- Polya, G. (1957). *How to solve it*. Princeton, NJ: Princeton University Press. (Original work published 1945).
- Schoenfeld, A. H. (1985). *Mathematical problem solving*. Orlando, FL: Academic Press.
- Schoenfeld, A. H. (1992). Learning to think mathematically: Problem solving, metacognition and sense making in mathematics. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 334-370). New York, Macmillan.
- Sfard, A.; Neshet, P.; Streefland, L.; Cobb, P. & Mason, J. (1998). Learning mathematics through conversation: Is it as good as they say? *For the Learning of Mathematics* 18, (1), 41-51.
- Sfard, A. & Kieran, C. (2001). Cognition as communication: Rethinking learning-by-talking through multi-faceted analysis of students' mathematical interactions. *Mind, Culture, and Activity*, 8, (1), 42-76.
- Stacey, K. (1992). Mathematical problem solving in groups: Are two heads better than one? *Journal of Mathematical behavior*, 11, 261-275.
- Vygotsky, L. S. (1978). *Mind in society: The development of higher psychological processes*. Cambridge, Mass.: Harvard University Press.
- Wells, G. (1999). *Dialogic inquiry: Towards a sociocultural practice and theory of education*. Cambridge: Cambridge University Press.
- Wood, D. J., Bruner, J. S., & Ross, G. (1976). The role of tutoring in problem solving. *Journal of Child Psychology and Psychiatry*, 17, 89-100.

The Interplay of Processing Efficiency and Working Memory with the Development of Metacognitive Performance in Mathematics

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Abstract: The present study outlines a specific three level hierarchy of the cognitive system and especially the relations of specific cognitive and metacognitive processes in mathematics. The emphasis is on the impact of the development of processing efficiency and working memory ability on the development of metacognitive abilities and mathematical performance. We had used instruments measuring pupils' metacognitive ability, mathematical performance, working memory and processing efficiency. We administered them to 126 pupils (8-11 years old) three times, with breaks of 3-4 months between them. Results indicated that the development of each of the abilities was affected by the state of the others. Particularly, processing efficiency had a coordinator role on the growth of mathematical performance, while self-image, as a specific metacognitive ability, depended mainly on the previous working memory ability.

1. Introduction

There is an increasing consensus that intelligence is a hierarchical and multidimensional edifice that involves both general-purpose and specialized processes and abilities (Demetriou, Zhang, Spanoudis, Christou, Kyriakides & Platsidou, 2005). According to differential theory, individual differences in psychometric intelligence are associated with individual differences in processing efficiency and / or working memory (Engle, 2002; Jensen, 1998). According to developmental theory, developmental changes in thinking are associated with changes in processing speed or efficiency (Kail, 1991), central attentional energy or capacity (Pascual Leone, 2000), or working memory (Case, 1985; Demetriou, Efklides & Platsidou, 1993). In fact, recent research shows that processing efficiency is the developmental factor in regard to the development of working memory and reasoning whereas working memory is a factor of individual differences in regard to the development and functioning of reasoning. That is, changes in processing efficiency open possibilities for changes in working memory and thinking (Demetriou, 2004; Demetriou, Christou, Spanoudis, Platsidou, 2002). The research on the development of metacognitive abilities should be connected with the development of other cognitive abilities, such as speed of processing, control of processing, working memory, attention etc.

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We believe that mathematics does involve some special mechanisms of representation and mental processing which are appropriate for the representation and processing of quantitative relations. At the same time, we also believe that these mechanisms are constrained by the organization and the possibilities of the human brain. Thus, any research about the architecture and the development of mind in respect to mathematics will have to specify the domain-specific processes and functions that it involves, the general potentials and processes of the human mind that sustain and frame its functioning, and their dynamic relations in real time during problem solving.

The relationships among cognitive processes, such as control of processing, speed of processing and working memory with metacognitive processes, such as self-representation, self-evaluation and self-regulation, is one of the issues elaborated by Demetriou and his colleagues (Demetriou & Kazi, 2001; Demetriou et.al, 2002). The present study was motivated by the integrated model proposed by these authors, according to which any change in a particular system affects the functioning of the cognitive processes. For instance, practice in the application of arithmetic operations may make children more aware of their memory limitations. At the same time a change at the metacognitive system influences the functioning of the processing system.

There are important questions that are still debate in psychology and in mathematics education. The present study purports to contribute to the ongoing research on the impact of specific cognitive processes, on metacognitive abilities and on mathematical performance. The purpose of the present study was twofold: First to explore the impact of processing efficiency and working memory on metacognitive processes in respect to mathematics and secondly to explore if the above interrelations tend to change with development.

At the following sections we first present, the concept of metacognition and its relations with young pupils' mathematical performance. Then we connect the metacognitive abilities with the processing efficiency and working memory ability in order to realize its position at the whole cognitive system and its relation with the mathematical performance.

The concept of metacognition in mathematics education

In recent years metacognition has been receiving increased attention in cognitive psychology and mathematics education (Guterman, 2003; Pappas, Ginsburg & Jiang, 2003). The interest has focused on its role in human learning and performance. In modern psychological literature the term metacognition has been used to refer to two distinct areas of research: knowledge about cognition and regulation of cognition. The present study uses the term "metacognition" referring to the awareness and monitoring of one's own cognitive system and its functioning.

Metacognitive knowledge is "knowledge or beliefs about what factors or variables act and interact in what ways to affect the course and outcome of cognitive enterprises" (Flavell, 1999, p.4). The major categories of these factors or variables are person, task and strategy (Flavell, 1987). The "person" category encompasses everything that a person believes about the nature of him/herself and other people as cognitive processors. It refers to the kind of acquired knowledge and beliefs that concern what human beings are like as cognitive organisms. The "task" category concerns the information available to a person during a cognitive enterprise. Thinkers must recognize that different tasks entail different mental operations (Demetriou, 2000). The

“strategy” category includes a great deal of knowledge that can be acquired concerning what strategies are likely to be effective in achieving what goals and in what sort of cognitive undertakings.

Whereas Flavell uses taxonomy with the three categories (person, task, strategy) to define metacognitive knowledge, Brown (1987) has categorized metacognitive knowledge based on a person’s awareness of his/her metacognitive knowledge: declarative, procedural and conditional. Declarative knowledge is propositional knowledge which refers to “knowing what”, procedural knowledge refers to “knowing how” and conditional knowledge refers to “knowing why and when”. For example in mathematics education the knowledge that “I am not good in remembering mathematical rules” is a declarative knowledge, the knowledge that “I can remember information easier if I connect them with everyday regular experiences” is a procedural knowledge and knowing that “I reduce the big numbers of a problem in order to manipulate them better”, is a conditional knowledge.

The second dimension of metacognition, self-regulation refers to the processes that coordinate cognition. It is the ability to use metacognitive knowledge strategically to achieve cognitive goals; especially in cases that someone needs to overcome cognitive obstacles. It has become clear that one of the most important issues in self regulated learning is the students’ ability to select, combine and coordinate strategies in an affective way (Boekaerts, 1999). Successful learners are able to swiftly transfer the knowledge and strategies acquired in one situation to new situations, modifying and extending these strategies on the way. Self-regulated learners in school age are able to manage and monitor their own processes of knowledge and skill acquisition (DeCorte, Verschaffel & Op’t Eynde, 2000). Self-regulatory behavior in mathematics includes clarifying problem goals, understanding concepts, applying knowledge to each goal and monitoring progress toward a solution.

The relationship between cognitive and metacognitive processes is one of the issues elaborated by Demetriou and his colleagues (Demetriou & Kazi, 2001; Demetriou, et al., 2002). Actually the present study was motivated by the intergrated model proposed by Demetriou according to which, the mind includes three fundamental levels of organization. The structure of the mind, referring to the cognitive and metacognitive system is presented at the next section.

2. The Structure of the mind: processing efficiency, working memory and metacognition

The human mind can be described as a three level hierarchical system involving domain-general and domain-specific processes and functions (Demetriou & Kazi, 2001; Demetriou, 2004). Speed of processing, inhibition and control, and working memory are the basic dimensions that define the condition of this system.

According to this model, the mind includes two levels of knowing, one oriented to the environment and another oriented to the self. That is, the first level includes representational and understanding processes and functions that specialize in the representation and processing of information coming from the environment. The second level includes functions and processes

oriented to monitoring, representing, and regulating the environment-oriented functions and processes. Thus, the input to this level is information arising from the functioning of the environment-oriented systems under the current processing constraints (for example, sensations, feelings, and conceptions caused by mental activity). Optimum performance at any time depends on the interaction between the two levels, because efficient problem solving or decision making requires the application of environment oriented functions and processes, under the guidance of representations held about them at the level of self-oriented processes.

Two of the main cognitive processes that the present study investigates are information processing and working memory. The processing system is defined in terms of three main parameters: speed of processing, control of processing and working memory. The first parameter is the maximum speed at which a given mental act may be efficiently executed; it refers to the time needed by the system to record and give meaning to information and execute an operation. Control of processing determines the system's efficiency in selecting the appropriate mental action. The more demanding a task is, the more processing resources, monitoring, and regulation it requires. Finally, working memory refers to the quantity of processes, which enable a person to hold information until the current problem is solved (Demetriou & Kazi, 2001). A common measure of working memory is the maximum amount of information and mental acts that the mind can efficiently activate simultaneously.

The present study outlines this architecture with an emphasis on the interdependent relations of the system. This architecture has similarities and differences with models proposed by psychometric theories, which are presented below. The emphasis is on the impact of processing efficiency and working memory ability on the development of mathematical performance and metacognition.

2.1 The development of cognitive and metacognitive abilities

The neo-Piagetian perspective, as Demetriou's theory, explains the cognitive development in terms of information processing. The limits in working memory capacity impose constraints on cognitive processes, and vary with age. New Piagetian theorists consider the development of working memory to be a causal factor of cognitive growth across domains (Kemps, Rammelaere, & Desmet, 2000). The core of neo-Piagetian research is to explore whether the development increase in working memory can account for cognitive development at large.

There is evidence that processing speed changes uniformly with age, in an exponential fashion, across a wide variety of different types of information and task complexities. That is, change on speed of processing is fast at the beginning (i.e., from early to middle childhood) and it decelerates systematically (from early adolescence onwards) until it attains its maximum in early adulthood (Demetriou et al. 2002; Hale and Fry, 2000; Kail, 1991). This pattern of change reflects the fact, that, with age, the time taken by the brain to complete an operation becomes smaller due to improvements in the interconnectivity of the neural circuits in the brain and the improvements in the myelination of neuronal axons that insulate the communication between neurons. As a result, the representation and manipulation of information in the brain becomes faster and more efficient (Case, 1992; Thatcher, 1992).

Concerning the working memory there is general agreement that the capacity of all components of working memory (i.e., executive processes, phonological, and visual storage) do increase systematically with age. Additionally, there seems to be an inverse trade-off between the central executive and the storage buffers, so that the higher the involvement of executive processes the less is the manifest capacity of the modality-specific buffers. This is so because the executive operations themselves consume part of the available processing resources. However, with age, executive operations and information are chunked into integrated units. As a result, with development, the person can store increasingly more complex units of information (Case, 1985).

Concerning the metacognitive abilities Kail's research (1991) indicated that even preschoolers are capable of reflecting on their own prior knowledge. By the time young children begin to express and recognize them as enduring entities, they also begin to show major advances in their understanding of others (Rochat, 2003). Actually by 4-5 years, according to Schneider and Sodian (1998), children begin to be capable of holding multiple representations of themselves and others. By the age of about 4 years, children understand the relation between beliefs and knowing, while between the ages of 4-7 years children move to a more sophisticated understanding of the role of inferential processes in knowledge acquisition (Schneider & Sodian, 1998). However, it is important the investigation of the development of specific metacognitive dimensions, such as self-representation, in relation to cognitive abilities in childhood ages.

As we have already mentioned, the purpose of the present study was to investigate the interrelations among the cognitive processes of information processing and working memory with mathematical performance and the inner metacognitive process from a developmental perspective, depending on the model of Demetriou and his colleagues. The architecture of the mind postulated by this model bears similarities and differences to architecture postulated by others models, for example the hierarchical conception of the human intelligence, proposed by Custafsson (1988). Moreover, many of the functions proposed are common with the abilities described by psychometric theories (Case, Demetriou, Platsidou & Kazi, 2001). However, the present study aimed to go beyond general cognitive structure and include measures of both actual cognitive performance and metacognition, at the specific domain of mathematics and not generally at the whole performance. Even though, children's early understanding of themselves has been intensively investigated in the last decades, there is a lack of studies investigated at the same time cognitive abilities and metacognition, in respect to specific domains, such as mathematics. Particularly in mathematics education, research should be concentrated on the impact of cognitive factors on the development of the metacognition at the specific domain, and consequently on the respective performance. A reliable model depicting the development of

those cognitive and metacognitive abilities could be useful in two ways: On the theoretical level it will contribute to deeper understanding of this important interconnection and on the practical side it may be useful in developing teaching programs for the improvement of young pupils' metacognition in mathematics.

3. Method

In the present study we developed and used a self-reported inventory measuring metacognition and an inventory measuring mathematical performance. The exact procedure, which was used for the measurement of processing efficiency, working memory, mathematical performance and metacognition, is presented below.

3.1 Participants

Data were collected from 126 children, in grades three through five (about 8 to 11 years old), from six different urban elementary schools. Specifically, 37 (19 girls, 18 boys) were 3rd graders, 40 (18 girls, 22 boys) were 4th graders and 49 (24 girls, 25 boys) were 5th graders. The mean age of the overall sample was 9.5, with students ranging in age from 7.9 to 11.4 years, at the first time of testing. The mean age of the 3rd graders was 8.4, with students ranging in age from 7.9 to 8.8. The mean age of the 4th graders was 9.5 with students ranging in age from 8.9 to 9.9 and finally the mean age of the 5th graders was 10.7 years, with students ranging in age from 10 to 11.2.

3.2 Materials

Apart from self-reporting inventory for the measurement of metacognitive performance and mathematical performance, individual meetings were arranged with each one of the subjects for the measurement of processing efficiency and working memory.

3.3 The inventory for the measurement of the metacognitive performance

The inventory for the measurement of the metacognitive performance was comprised of 30 Likert type items, of five points (1=never, 2=seldom, 3=sometimes, 4=often, 5=always), reflecting pupils perceived behaviour during in-class problem solving. A specimen item is: "when I encounter a difficulty that confuses me in my attempt to solve a problem I try again". The responses provide an image of pupils' self-representation, which refers to how they perceive themselves in regard to a given mathematical problem. The 30 items are presented at the Appendix, as a part of a table presenting the factor loadings of them (Table 1). The reliability of the whole inventory was very high. Specifically, the Cronbach's α was 0.86.

3.4 Mathematical performance tasks

The individual's mathematical ability was measured through four numerical tasks, four analogical, four verbal and four matrices for the measurement of spatial ability taken from the Standard Progressive Matrices. All mathematical tasks were used in previous studies (Demetriou

et al., 2002). The reliability of the cognitive mathematical tasks was high. Specifically, the Cronbach's α was 0.87. All items in the mathematical performance inventory were scored on a pass-fail basis (0 and 1).

3.5 Stroop-like tasks

The pupils' information processing efficiency was measured using a series of stroop-like tasks devised to measure speed and control of processing, under three different symbol systems: numerical, verbal, and imaginal. To measure, for example, verbal speed of processing, participants were asked to read at the computer a number of words, denoting a colour written in the same ink-colour (for example the word green written in green) and they had to type the letter G at the keyboard, indicating the written word or the colour of the word. At the half of the items the instructions were to type G (for green), Y (for yellow) and R (for red) in order to indicate, as quickly as possible, the colour of the word and at the other half of the items the instruction were to type the respective buttons in order to indicate the written word. For verbal control of processing, the subjects were asked to recognize the ink colour of words denoting a colour different than the ink (for example the word green written in red). To measure the two dimensions of numerical processing, several number digits were composed of small digits. This task involved the numbers 4, 7 and 9. In the compatible condition the large digit was composed of the same digits, while in the incompatible condition the large digit was composed of one of the other digits. Pupils had to type at half of the items the small digit and at the rest the big digit, according to the given instructions. Reaction times to all three types of the compatible conditions (verbal, numerical, imaginal) were taken to indicate speed of processing, while reaction times to the incompatible conditions were considered indicative of the person's efficient control of processing. The tasks addressed to the imaginal system were similar to those used for the numerical system and comprised three geometrical figures: circle, triangle, square. The buttons on the keyboard, they had to use, were "S" for square, "C" for circle and "F" for triangle. Two examples of the given tasks are presented at Figure 1. The computer measured reactions times automatically.

The reliability of processing efficiency tasks was very high. Specifically, the Cronbach's α was 0.91.

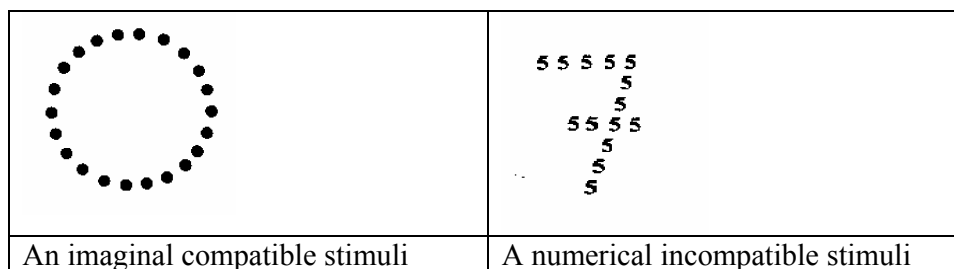


Figure 1: Examples of stroop like tasks

3.6 Working-memory tasks

To measure working memory, we asked pupils to recall a number of words, sets of numbers, and images. The verbal task, for example, combined six levels of difficulty, each of which was tested by two different trials. The difficulty level was defined in terms of the number of words in the task, which ranged from two to seven concrete nouns. The numerical tasks were structurally identical to the verbal task. Specifically in the easy trial, only decade numbers were involved, while in the difficult trial the two digits of the numbers were different. Both words and numbers presented to children as verbal stimuli. In the imaginal task, the stimuli were presented visually at the computer. The participants were shown a card on which a number (2-7) of geometrical figures were shown and they were asked to choose from four choices the card, which had the same figures, at the same relative position with the first one (Fig. 2).

The reliability of working memory tasks was very high. Specifically, the Cronbach' s α was 0.83.

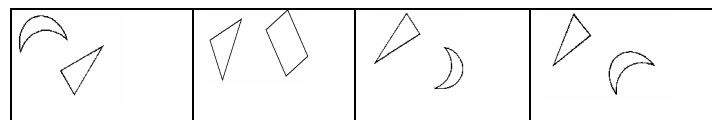


Figure 2: An example of working memory task

3.7 Procedure

To specify the nature of change in cognitive abilities in mathematics in relation to metacognition and the possible interrelations in the patterns of change in these aspects, a series of three repeated waves of measurements were taken, with a break of 3-4 months between successive measurements. The same materials were used at each wave of measurement. Each participant was tested individually on processing efficiency and working memory tasks. Testing took place in a quiet room, provided by the schools for the purpose of the experiment. The breaks of 3-4 months for the consecutive measurements were necessary in order to investigate the developmental changes of the specific variables. The investigation of the impact of development factors on specific abilities by repeated measurements with longer breaks was impossible, because of practical difficulties, which would not permit us to have the same sample of pupils.

3.8 The stoop-like tasks

The experimenter introduced the three tasks (numerical, verbal, imaginal) to the child, through the computer, one by one, using first several demonstration cards and then several practice cards to familiarize the child with the tasks, mainly in order to learn the buttons they had to use from the keyboard. For practical reasons, the presentation order of items within the symbol systems was the same across subjects.

3.9 The memory tasks

For verbal and numerical tasks, participants were instructed to recall the words or the numbers in the order of presentation as soon as the experimenter finished stating a series. The presentation order of difficulty levels was the same across participants, going from easy to difficult.

Administration of a task stopped if the participant failed to recall errorlessly the two trials involved in a level. In the spatial task, the participant was instructed to carefully choose the card with the figures in exactly the same position and orientation as the initial one.

3.10 The mathematical tasks and the inventory of metacognitive performance

The mathematical tasks and the inventory about metacognition were individually presented in a paper and pencil form and were individually administered. The experimenter explained each task and was available to answer questions as needed.

4. Results

4.1 The structure of the cognitive and metacognitive processes

The collected data of the inventory about metacognitive abilities were first subjected to exploratory factor analysis in order to examine whether the factors that guided the construction of the inventory were presented in the participants' responses. This analysis resulted in 10 factors with eigenvalues greater than 1, explaining 64.74% of the total variance (the factor loadings and the content of the items are shown in Table 1). After a content analysis of the ten factors, according to the results of the exploratory factor analysis, these factors were classified in the following four groups: "general self-image" (two factors), "strategies" (four factors), "motivation" (two factors), and "self-regulation" (two factors). The means of those four groups of factors were subsequently used in order to avoid a big number of variables at the structural equation modeling. It is important to note that reducing a large number of raw scores to a limited number of representative scores is an approach suggested by proponents of structural equation modeling (Gustafsson, 1988). The items that constructed the two factors about "self-image" referred to the beliefs and self-efficacy that pupils had about their abilities, in general, and while encountering specific situations, in particular. "Self-regulation" in mathematics, the other two factors of the analysis, included clarifying problem goals, understanding concepts, applying knowledge to each goal to develop a solution strategy and monitoring progress toward a solution. The factors of "strategies" consisted of items concerning the strategies pupils used in order to solve problems and in order to overcome cognitive obstacles. Finally, the factors of "motivation" consisted of pupils' beliefs about the impact of their effort and their will on their performance and the impact of their parents and teachers.

Thus the first and second order factor structure of the instruments was investigated to determine whether the general levels of the architecture of the mind, that is speed of processing, control of processing, working memory and self-awareness system, explain the variability in the different means scores. Confirmatory factor analysis model designed to test the multidimensionality of the materials were used in order to examine their construct validity. It was important to investigate the degree of the similarity of the models, which could be constructed for the repeated measurements. Structural equation modelling was used to test the hypothesis on the existence of seven first order factors, two second order factors and a third order factor, in all cases. The seven first order factors were processing efficiency, working memory, cognitive performance at mathematics, self-image, self-regulation, strategies and motivation. The a priori model hypothesized that the variables of all the measurements would be explained by those factors and each item would have a nonzero loading on the factor it was supposed to measure. Analysis was conducted using the EQS program (Bentler, 1995) and maximum likelihood estimation

procedures. Multiple criteria were used in the assessment of the model fit. The model was tested under the constraint that the error variances of some pair of scores associated with the same factor would have to be equal. This was an indication of the LMTEST, in order to arrive at an elaborated model in which the goodness of fit-index would be good in relation to typical standards (CFI>0.9, $\chi^2/df < 2$, RMSEA<0.05). This model was tested separately on the performance attained at each testing wave.

There were six measures representing the two dimensions (speed and control of processing) of processing efficiency tasks. That is, the three means representing performance on the three sets of stoop-like compatible tasks addressed to speed of processing and the three means representing performance on the three sets of stoop-like incompatible tasks addressed to control of processing through the verbal, the numerical and the imaginal symbol systems. Additionally there were six mean measures representing the easy and difficult tasks of numerical, verbal and imaginal working memory tasks. Finally there were four mean measures representing the performance of individuals on numerical, verbal and analogical mathematical tasks and matrices.

A series of models were tested separately for each one of the measurement waves. Specifically, a one-factor model was first tested. The first of the models that were tested, involved only ten first order uncorrelated factors. Kline (1998) argues, “even when the theory is precise about the number of factors of a model, the researcher should determine whether the fit of a simpler, one-factor model is comparable” (p.212). The fit of this model was very poor in all cases, as it was expected:

First testing: $X^2=1013.412$, $df=275$, $X^2/df=3.68$, $p<0.001$, CFI=0.015, RMSEA=0.160

Second testing: $X^2=690.571$, $df=251$, $X^2/df=2.75$, $p<0.001$, CFI=0.370, RMSEA=0.124

Third testing: $X^2=540.141$, $df=229$, $X^2/df=2.35$, $p<0.001$, CFI=0.477, RMSEA=0.107

The second model tested involved seven first order factors, that is processing efficiency, working memory, mathematical performance, self-image, self-regulation, strategies and motivation. Thus this model tests the assumption that each of the dimensions represented by the tasks is fully autonomous of each other. The fit of this model was also very poor in all cases:

First testing: $X^2=978.402$, $df=275$, $X^2/df=3.556$, $p<0.001$, CFI=0.115, RMSEA=0.171

Second testing: $X^2=710.603$, $df=251$, $X^2/df=2.831$, $p<0.001$, CFI=0.382, RMSEA=0.129

Third testing: $X^2=552.136$, $df=229$, $X^2/df=2.411$, $p<0.001$, CFI=0.493, RMSEA=0.118

The third model involved the three first order factors (processing efficiency, working memory and cognitive performance in mathematics) regressed on a second order factor and the four first order factors (self-image, self-regulation, strategies and motivation) regressed on a different second order factor. This model was found to fit well.

First testing: $X^2=295$, $df=268$, $X^2/df=1.103$, $p<0.001$, CFI=0.904, RMSEA=0.025

Second testing: $X^2=299.481$, $df=244$, $X^2/df=1.227$, $p<0.001$, CFI=0.927, RMSEA=0.030

Third testing: $X^2=257.255$, $df=222$, $X^2/df=1.158$, $p<0.001$, CFI=0.898, RMSEA=0.032

Finally the organization of cognitive and metacognitive processes and abilities were tested using the model shown in Figure 3. It was a three level model, which was consistent with the theory. It involved three types of factors. The seven first order factors were: processing efficiency,

working memory, cognition, general self-image, self-regulation, strategies and motivation. Those factors were regressed on two second order factors: the general cognition and the general self-representation. Those second order factors of cognitive and metacognitive processes were regressed on a third order factor that concerned the cognitive and metacognitive abilities in mathematics.

First testing: $X^2=279.949$, $df=262$, $X^2/df=1.068$, $p=0.213$, $CFI=0.974$, $RMSEA=0.026$

Second testing: $X^2=265.413$, $df=236$, $X^2/df=1.124$, $p=0.091$, $CFI=0.956$, $RMSEA=0.034$

Third testing: $X^2=224.374$, $df=195$, $X^2/df=1.150$, $p=0.073$, $CFI=0.952$, $RMSEA=0.036$

The parameter estimates of this final model for the three waves are shown in Figure 3. The fit of the model was very good and the values of the estimates were high in all cases. It is clear therefore that the three-level architecture accurately captures the data.

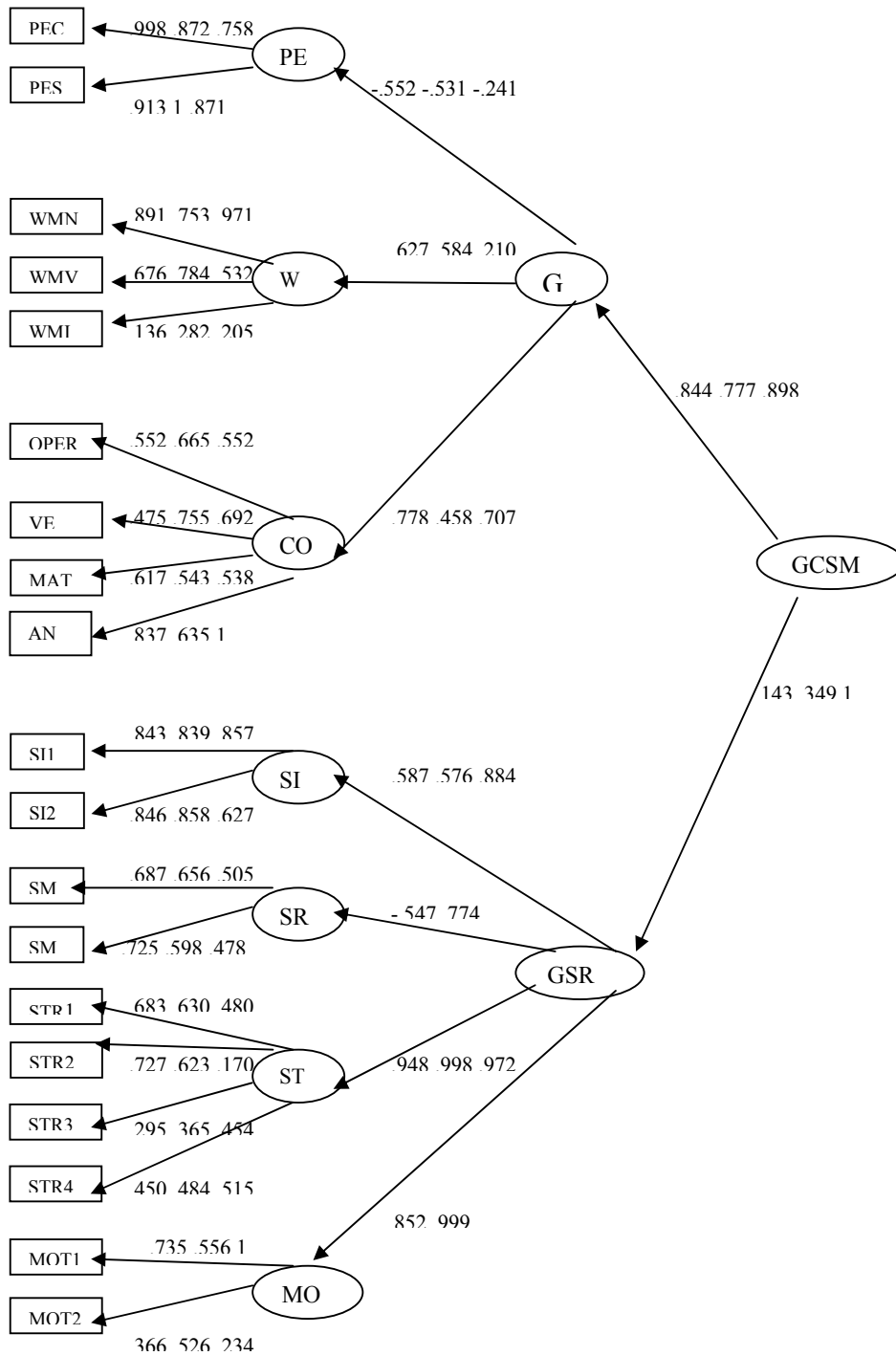


Figure 3: The third level model of metacognitive and cognitive abilities in mathematics in three testing waves²

² PE = Processing Efficiency, PEC= Control of Processing, PES= Speed of Processing, WM = Working Memory, WMN= Numerical, WMV= Verbal, WMI= Imaginal, CO= Cognitive performance in mathematics, OPER= Operations, VE= Verbal, MAT= Matrices, AN= Analogies, SI = Self Image, SR = Self-regulation, STR= Strategies,

4.2 The development of cognitive and metacognitive abilities

In order to specify the dynamic relations between mathematical performance and metacognition with processing efficiency and working memory, during the period of the study, dynamic modelling was used. The dynamic model explored possible relations among cognitive variables (processing efficiency, working memory, and cognitive performance in mathematics) and metacognitive variables (self-image and self-regulation) across the three waves of measurement. The variables of strategies and motivation excluded from the last analysis in order to avoid testing a complicated model with too many variables and consequently many limitations with the statistical analysis. We believed that self-image and self-regulation had a stronger relationship with the general self-representation than the use of strategies and motivation. Self-image about personal strengths and limitations, in comparison to the abilities of others, is a part of the general self-representation. While self-regulation is one of the two basic dimensions of metacognitive ability and it is too important in order to overcome obstacles encountering while solving a mathematical problem.

The dynamic model explored relations among cognitive and metacognitive variables across the three waves of measurement. The main hypothesis was that all the variables at the second measurement were affected by the respective variables at the first measurement and the variables at the third measurement were affected by the respective variables at the first and the second measurement. Furthermore, the second hypothesis was that significant relations would connect the different cognitive and metacognitive variables at each wave of the measurements.

The initial fit of the model tested, without any correlations among the five variables (processing efficiency, working memory, cognitive performance in mathematics, self-image, self-regulation) in each wave of measurement, was very poor ($X^2=999.359$, $df=410$, $X^2/df=2.42$, $p<0.001$, $CFI=0.581$, $RMSEA=0.114$). It improved, however, dramatically after the above two hypotheses were tested ($X^2=482.319$, $df=376$, $X^2/df=1.28$, $p=0.001$, $CFI=0.924$, $RMSEA=0.051$), indicating the impact of the first measurement on the respective abilities at the second and the third measurements and the connection of the different cognitive and metacognitive abilities at each wave of the measurements. After a few error variances were allowed to correlate, according to the indications of the LMTEST, the fit of the model was excellent ($X^2=434.964$, $df=373$, $X^2/df=1.16$, $p=0.01$, $CFI=0.956$, $RMSEA=0.039$). The parameter estimates of this model are shown in Figure 4.

The results of the above dynamic model underline the predominance of the processing efficiency and the working memory for the structure of the cognitive mathematical performance and

MOT= Motivation, GSR= General Self Representation, GC = General Cognitive Abilities, GCSM= General Cognitive and Self-representation Abilities in Mathematics

The three numbers indicated the loadings of the variables and factors at the three consecutive measurements, respectively.

Detailed tables with Means, Standard Deviations and Correlations between the Variables Used in Structural Modeling can be obtained from the author.

individuals' metacognitive performance. The cognitive performance in mathematics at the third measurement (COG3) was affected significantly by the initial condition of the processing (PE1) efficiency (0.397) and the condition of working memory (WM2) at the second measurement (0.495). It was important that the effect of the initial mathematical (COG1) performance (.226) was lower than the effect of the initial processing efficiency (PE1) and the previous working memory ability (WM2). Individuals' differences on the two cognitive processes remained the same after one year and specify the differences at the cognitive performance and the self-image. In particular, these results indicated that individuals with high processing efficiency and high working memory ability had high self-regulation ability and a positive self-image. It is very important that there were no statistically significant correlations between self-image and self-regulation with the cognitive performance in mathematics, at the first and second measurement. This indicated that the individuals' self-image, except the third measurement, did not depend on their mathematical performance. This was a non-expectable result that underlined the important role of working memory on the development of the cognitive system and the impact of the most recent experiences on the structure of the self-image.

The model parameters (Figure 4) show that there was a general pattern of individuals' differences at the first measurement that persisted at the second and the third measurement in the case of working memory. This is evidenced from the continuing significant loadings of each variable at different measurements. Specifically, the loading of the working memory ability at the first measurement (WM1) on the working memory ability at the second measurement (WM2) was 0.814. Similarly, the loading of the working memory ability at the first measurement, and the loading of the same variable at the second measurement on the working memory ability at the third measurement (WM3) were 0.767, and 0.349, respectively. The behaviour was not the same in the case of the other variables. The difference remained at the second measurement, but changed at the third one.

A notable finding from the specific dynamic model was the predominant role played by the processing efficiency, affecting significantly all the others cognitive and metacognitive variables at the first measurement. The statistically significant loading of processing efficiency (PE1) on working memory (WM1) was -0.337, on mathematical performance (COG1) was -0.206, on self-image (SI1) was -0.198 and on self-regulation (SR1) was -0.262. At the same time, the predominant role of processing efficiency on the whole system was underlined by the result that the loading of processing efficiency at the first measurement on cognitive mathematical performance at the third measurement was significant (-0.397). The loading of working memory at the second measurement (WM2) on the mathematical performance at the third measurement (COG3) was significant as well (0.226). Consequently the mathematical performance depended on the previous processing efficiency and the working memory.

The performance of self-image at the third measurement (SI3) was affected by the initial condition (WM1) of the working memory (0.239) and the mathematical performance (COG3) at the third measurement (0.530). This is an important indication of the factors that affect individuals' self-image in mathematics. Actually the impact of the mathematical performance at the same measurement was expectable, because of the recent experiences. Nevertheless the impact of the initial condition of the working memory ability indicated the predominant role of cognitive processes and abilities on the self-image.

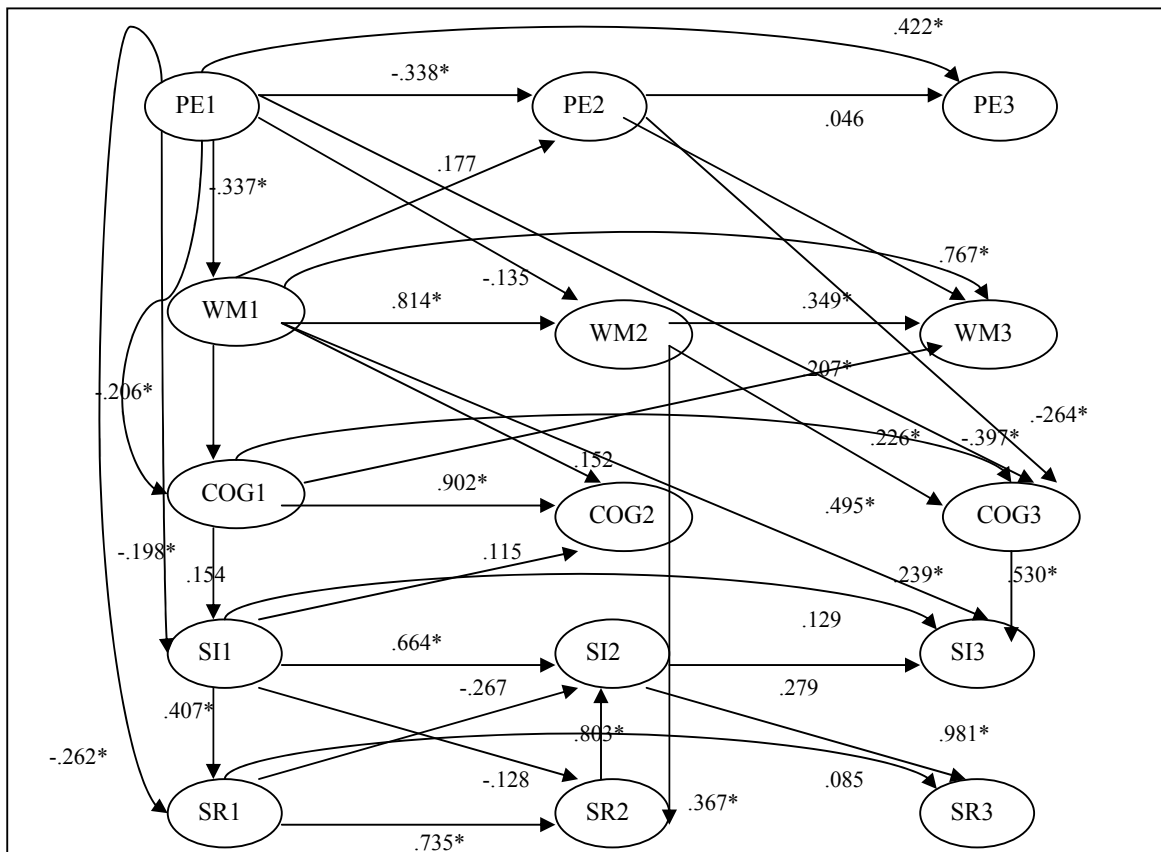


Figure 4: The dynamic model of cognitive and metacognitive abilities through the three measurements³

We next used growth modeling to explore the nature of change in cognitive and metacognitive abilities in mathematics, and the possible interrelations in the patterns of change in these variables. This model was estimated with the MPLUS packet (Muthen & Muthen, 2001). The basic latent growth model was composed of two latent factors: The first one represented the initial status – the intercept, and the second one was the latent growth rate – the slope, and was defined by fixing it to 0, 1, and 2. Figure 5 illustrates the general model that was tested. The following twelve manifest variables were used in this model: three for processing efficiency, three for working memory, three for cognitive memory, three for cognitive ability in mathematics and three for self-image. Each variable was a composite measure of the performance attained on the tasks at each of the three testing waves. Thus, processing efficiency

³ PE= Processing Efficiency, WM= Working Memory, COG= Cognitive Abilities in Mathematics, SI= Self Image, SR= Self-regulation

* Significance at the .05 level

1= first measurement 2= second measurement 3 =third measurement

was the mean of performance attained on the processing speed and control of processing tasks; working memory was the mean of performance attained on numerical, verbal and imaginal memory tasks; cognitive ability in mathematics attained on the performance on numerical, verbal, analogical mathematics tasks and matrices. Self-image was the mean of the two factors, which were the results of the exploratory factor analysis for self-image.

At the initial run of the model, the slope variable was fixed to have a relation of 0, 1, and 2 with all manifest variables at the first, second and third testing waves, respectively. This constraint expresses the modelling assumption that change is a linear function of time. The slope of the processing efficiency was the only fixed variable; it was assigned the relation of -2, -1, 0, because of the predictable reduction of reaction time. For the best fitting of the model the intercepts and the slopes were changed, as shown in Figure 5, in order to justify the differences of means, without changing the linearity of the model (e.g., the slope variable for the cognitive performance was fixed to have a relation of 0, 0.38, 0.90). The overall model fit statistics were $\chi^2=64.60$, $df= 41$, $\chi^2/df=1.575$, $p=0.001$, $CFI=0.934$, $RMSEA=0.06$, suggesting an excellent fit of the model to the data. The parameter estimates of the model are presented in Figure 5.

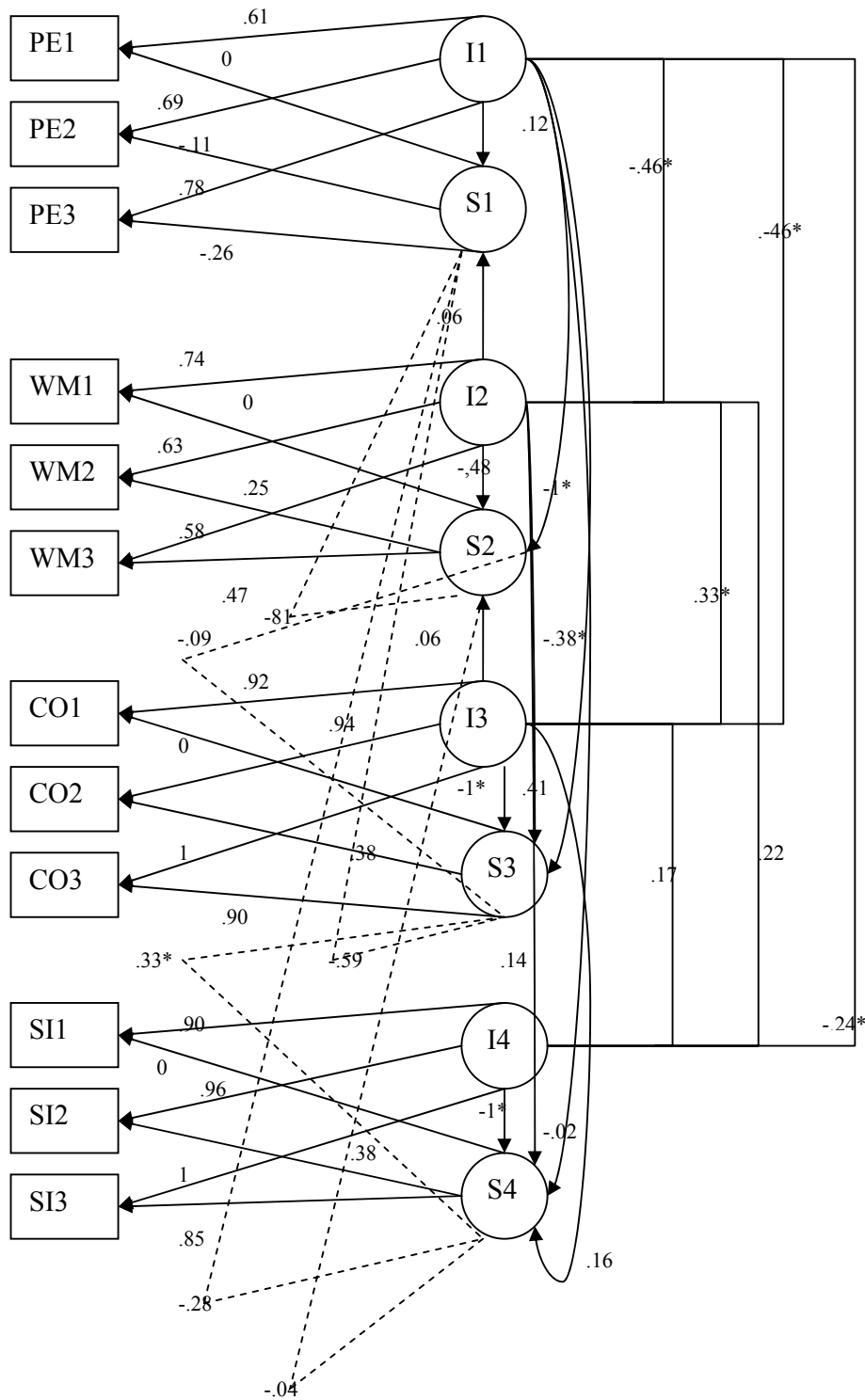


Figure 5: The best fitting growth model for processing efficiency, working memory, cognitive mathematical performance and self-image across the three testing waves⁴.

⁴ PE = Processing Efficiency, WM = Working Memory, CO = Cognitive performance in mathematics, SI = Self Image, I = Intercept, S = Slope

The correlation among the intercepts and between the intercept and the slope were very important. The regression of the processing-efficiency intercept on the working memory intercept and the regression of the working memory intercept on the cognitive abilities intercept were not significant. The correlations between the intercept of processing efficiency with the slope of working memory and the intercept of processing efficiency with the slope of cognitive ability were significant. Both of them were negative (-0.46, -0.38, respectively) meaning that individuals who had higher mean value of the initial scores on the time of processing efficiency (meaning slower speed) had a weaker rate of increase. Quite notable is the finding that the initial condition of self-image depends on the corresponding processing efficiency (-0.24), while its growth depends on the growth of mathematical performance (0.33). The positive correlation among the growth of the two variables indicates that individuals who improved on mathematical performance had improved on self-image as well.

The pattern of findings presented, suggest several important conclusions. First, that there are significant individual differences in the pupils' attainment in cognitive and metacognitive variables. The significant correlations among the intercepts indicate that there are strong interrelations among the initial conditions of the three cognitive aspects of the mind and the metacognitive aspect. However, growth over time affects the development of each function of the mind differently. The fact that the slope of cognitive ability and working memory depended on the initial condition of the processing efficiency indicated that processing efficiency had a coordinator role on the development of other cognitive and metacognitive processes.

The existence of significant intercept correlations among different abilities (processing efficiency with working memory and processing efficiency with cognitive performance in mathematics) suggest that growth in each of the abilities was affected by the state of the others, especially the state of processing efficiency at a given point of time. On the other hand, the lack of intercept – slope relations between self-image and cognitive abilities suggests that growth of self-image was not affected by the state of the processing efficiency or working memory at a given point in time. The relation of the slope of self-image with the slope of cognitive abilities indicates that the advancement on self-image depended on the advancement of mathematical performance.

5. Discussion

The findings of this study lead to some potentially important conclusions about the development of cognitive and metacognitive processes. Although complicated figures are presented at the above section, few are the results that should be underlined regarding the interrelations metacognition, cognitive processes and mathematical performance. Firstly, there was stability on the models which were constructed for the repeated measurements, indicating the stability of the structure of the specific variables the study investigated. Secondly, results indicated that the development of each of the cognitive abilities and dimensions of metacognition was affected by the state of the others. Particularly, processing efficiency had a basic impact on the growth of mathematical performance and on working memory. The mathematical performance depended on the previous working memory ability, as well. Finally, the self-image, as a significant part of

self-representation, depended mainly on the previous working memory ability and on the recently mathematical performance.

The human mind is much more complex than simply cognitive abilities and processes and their presentations (Demetriou & Kazi, 2001). Metacognition is constrained by the processing potentials of the mind. The existence of significant correlations among different cognitive abilities, especially between processing efficiency with working memory and cognitive performance in mathematics suggest that growth in each of the abilities is affected by the state of the other variables, especially the state of processing efficiency at a given point of time. From the analysis of the dynamic model, it is quite clear that the processing efficiency has a coordinator role on the cognitive system and the individual's metacognitive performance, even from the first measurement. This result was observable from the fact that processing efficiency is strongly associated with all the other factors, and actually affects them significantly.

The lack of relations between self-image and cognitive abilities suggests that growth of metacognitive performance is not directly affected by the state of the processing efficiency or working memory, at a given point of time. It is mainly affected by the initial condition of those abilities. Individuals' self-image depended mainly on previous working memory ability and partially on the recent mathematical performance. It is very important the effect of mathematical performance on the self-image at the final measurement. It seems that mathematical performance is the only cognitive ability, for which individuals have direct consequences which are expressed by remarks, awards and most often rewards by significant others i.e., teachers and parents. This is in line with Reder (1996) conclusions that consciousness is a necessary condition of a more precise self-representation.

Demetriou et al. (2002) suggest that both the working memory and the processing efficiency are associated with individual development differences on thinking. A change at the metacognitive system influences the functioning of the cognitive system and vice-versa. The results of the present study indicated that changes on thinking and metacognitive performance might be associated with processing efficiency and working memory, even at the years of the primary education, at the specific domain of mathematics. Additionally the present study found that change on processing efficiency or working memory may be necessary, but not sufficient for changes on functions residing at other levels of the mental architecture.

The present study has provided evidence about the relations and interconnections among cognitive and metacognitive processes with respect to mathematical performance. It is too important the predominant role of processing efficiency and working memory ability on mathematical performance and metacognition. Further investigation could lead to intervention programs for the improvement of self-representation in mathematics. Future studies could investigate whether changes on cognitive performance, especially on cognitive processes, such as processing efficiency and working memory capacity, tend to follow changes on metacognitive knowledge, self-evaluation, self-regulation and self-representation. In the area of mathematics, a number of important questions remain unanswered about metacognition. Are people aware of cognitive processes when they do mathematics, even in early childhood? Are they accurate in their self-representations of strengths and weaknesses in mathematics? Much more research is needed to study the different aspects of metacognition in a more systematic, detailed way. It should continue on the possible developmental changes in the interrelations among specific cognitive processes and metacognitive processes.

References

- Bentler, P.M. (1995). *EQS: Structural equations program manual*. California: Multivariate Software Inc.
- Boekaerts, M. (1999). Self-regulated learning: where we are today. *International Journal of Educational Research*, 31, 445-457.
- Brown, A. (1987). Metacognition, executive control, self-regulation and other more mysterious mechanisms. In F.Weinert & R. Kluwe (Eds), *Metacognition, Motivation and Understanding* (65-115). London: LEA.
- Case, R. (1985). *Intellectual development. Birth to adulthood*. New York: Academic Press.
- Case, R. (1992). *The mind's staircase: Exploring the conceptual underpinnings of children's thought and knowledge*. Hillsdale, NJ: Erlbaum.
- Case, R., Demetriou, A., Platsidou, M., & Kazi, S. (2001). Integrating concepts and tests of intelligence from the differential and developmental traditions. *Intelligence*, 29 (4), 307-336.
- De Corte, E., Verschaffel, L., & Op't Eynde, P. (2000). Self-regulation, A characteristic and a goal of mathematics education. In M. Boekaerts, P. Pintrich, & M. Zeider (Eds.), *Handbook of Self-regulation* (687-726). USA: Academic press.
- Demetriou, A., (2000). Organization and development of self-understanding and self regulation: Toward a general theory. In M. Boekaerts, P. R. Pintrich, & M. Zeidner (Eds.), *Handbook of self-regulation* (pp. 209-251). Academic Press.
- Demetriou, A. (2004). Mind Intelligence and Development: A cognitive, differential, and developmental theory of intelligence. In A. Demetriou, & A. Raftopoulos (Eds.), *Emergence and Transformation in the Mind: Modeling and Measuring Cognitive Change* (21-73). Cambridge University Press.
- Demetriou, A., Efklides, A., & Platsidou, M. (1993) The architecture and dynamics of developing mind: Experiential structuralism as a frame for unifying cognitive developmental theories. *Monographs of the Society for Research in Child Development*, 58 (5-6, Serial No. 234).
- Demetriou, A., & Kazi, S. (2001). *Unity and Modulatory in the Mind and the Self*. London: Routledge.
- Demetriou, A., Christou, C., Spanoudis, G., & Platsidou, M. (2002). The development of mental processing: efficiency, working memory and thinking. *Monographs of the Society for Research in Child Development*, 67 (1).
- Demetriou, A., Zhang Xiang, K., Spanoudis, G., Christou C., Kyriakides, L., & Platsidou, M. (2005). The architecture, dynamics and development of mental processing: Greek, Chinese or Universal?. *Intelligence*, 33, 109-141.
- Engle, R. W. (2002). Working memory capacity as executive attention. *Current Directions in Psychological Science*, 11, 19-23.
- Flavell, J. (1987). Speculations about the nature and development of metacognition. In F. Weinert & R. Kluwe (Eds.), *Metacognition, Motivation and Understanding* (21-29). London: LEA.
- Flavell, J. (1999). Cognitive development: children's knowledge about the mind. *Annual review of psychology* (on line). Available http://www.findarticles.com/cf_dls/m0961/1999_Annual/54442292/p/article.html.
- Gustafsson , J. E. (1988). The development of mental abilities. In R. J. Sternberg (Ed.), *Advances in the psychology of intelligence* (Vol. 4, pp.35-71). Hillsdale, NJ: Erlbaum.

- Guterman, E. (2003). Integrating written metacognitive awareness guidance as a 'psychological tool' to improve student performance. *Learning and Instruction, 13*, 633-651.
- Hale, S., & Fry, A. F. (2000). Relationships among processing speed, working memory, and fluid intelligence in children, *Biological Psychology, 54*, 1-34
- Jensen, A. R. (1998). *The G factor: The science of mental ability*. New York: Praeger.
- Kail, R. (1991). *The development of memory in children*. New York: W.H. Freeman.
- Kemps, E., Rammelaere, S., & Desmet, T. (2000). The development of working memory: exploring the complementarity of two models. *Journal of Experimental Child Psychology, 77*, 89-109.
- Kline, R.B. (1998). *Principles and practice of structural equation modelling*. New York: The Guilford Press.
- Muthen, L.K., & Muthen, B.O.(2001). *Mplus: The comprehensive modelling program for applied researchers. User's guide*. Los Angeles: Muthen & Muthen.
- Pappas, S., Ginsburg, H., & Jiang, M. (2003). SES differences in young children's metacognition in the context of mathematical problem solving, *Cognitive Development, 18*, 431-450.
- Pascual-Leone, J. (2000). Reflections on working memory: Are the two models complementary? *Journal of Experimental Child Psychology, 77*, 138-154.
- Reder, L. (1996). Different research programs on metacognition: Are the boundaries imaginary? *Learning and Individual Differences, 8* (4), 383-390.
- Rochat, P. (2003). Five levels of self-awareness as they unfold early in life. *Consciousness and Cognition, 12*, 717-731.
- Schneider, W., & Sodian, B. (1998). Memory strategy development: Lessons from longitudinal research. *Developmental Review, 17*, 442-461.
- Thatcher, R. W. (1992). Cyclical cortical reorganizations during early childhood. *Brain and Cognition, 20*, 24-54.

Questionnaire items	Factors									
	SI1	SI2	SR1	SR2	STR1	STR2	STR3	STR4	MOT1	
I examine my own performance while I am studying a new subject.	.759									
When I read a problem I know whether I can solve it.	.599									
After I finish my work I know how well I performed on it.	.696									
I know ways to remember knowledge I have learned in Mathematics.		.823								
I understand a problem better if I write down its data.		.640								
When I cannot solve a problem, I know the factors of the difficulty.		.605								
I know how well I have understood a subject I have studied.		.539								
I define specific goals before my attempt to learn something.			.736							
After I finish my work I wonder whether there was an easier way to do it.			.680							
When I encounter a difficulty on problem solving I reread the problem.			.639							
When I encounter a difficulty that confuses me in my attempt to solve a problem I try to resolve it.			.758							
I can learn more about a subject on which I have previous knowledge.				.650						
I can learn more about a subject on which I have a special interest.				.763						
When I am solving a problem I wonder whether I answer its major question.				.580						
I try to use ways of studying that had been proved to be successful.					.549					
For the better understanding of a subject I use my own examples.					.782					
In order to solve a problem I try to remember the solution of similar problems.					.557					
I understand something better if I use pictures or diagrams.						.648				
I concentrate my attention on the data of a problem.						.634				
When I try to solve a problem I pose questions to myself in order to concentrate my attention on it.						.553				
After I finish my work I wonder whether I have learned new important things.							.754			
After I finish my work I repeat the most important points in order to be sure I have learned them.							.633			
Before I present the final solution of a problem I try to find some other solutions as well.							.658			
When I do not understand something I ask for the help of others.								.671		
While I am solving a problem I try to realize which its aspects that I cannot understand are.								.730		
When I encounter a difficulty in problem solving I am looking for teacher's help.								.505		
My performance depends on my will.										.778
My performance depends on my effort.										.599
My teacher believes that I must be a good student in mathematics.										
My parents believe that I must be a good student in mathematics.										

Table 1: Varimax Rotated Factor Matrix

Students' Conceptions of Limits: High Achievers versus Low Achievers

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Abstract: Learning an advanced mathematical concept, limits of functions in this case, is not a linear development equal for all learners. Intentions and abilities influence students' learning paths and results. Students' learning developments of limits were studied in terms of concept images (Tall & Vinner, 1981) in the sense that their actions, such as problem solving and reasoning, were considered traces of their mental representations of concepts. High achievers' developments were compared to low achievers' developments to for the duration of a semester to reveal differences and similarities.

1. Introduction

Students learning limits of functions perceive and treat limits differently. Embracing limits of functions demands certain abstraction skills from the students. There are several cognitively challenging issues to deal with, such as understanding the quantifiers' roles in the formal definition or linking formally expressed theory to everyday problem solving. Students accept different levels of understanding as they have different priorities and abilities. Each student has his or her own conceptual development during a course and the question is; how do high achieving students' conceptual developments differ from low achieving students' developments?

A study on students' conceptual development of limits of functions was conducted at a Swedish university (Juter, 2006a) with the purpose to describe students' developments as they learned limits in a basic calculus course. The results imply differences in high achieving and low achieving students' work with limits, but also a lack of differences at some points as will be discussed further on in this article.

2. A model of concept representations

Tall (2004) has introduced three worlds of mathematics to distinguish different modes of mathematical thinking, with the purpose to "gain an overview of the full range of mathematical cognitive development" (Tall, 2004, p. 287). The theory of the three worlds emphasizes the construction of mental representations of concepts and has emerged from several theories on concept development, such as Sfard's (1991) work on encapsulation of processes to objects and Piaget's abstraction theories (Tall, 2004). The three worlds are

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somewhat hierarchical in the sense that there is a development from just perceiving a concept through actions to formal comprehension of the concept. The first world is called the embodied world and here individuals use their physical perceptions of the real world to perform mental experiments to build mental conceptions of mathematical concepts. The mental experiments can be children's categorisations of real-world objects, such as an odd number of items or, later, students' explorations of intuitive perceptions of limits of functions. The second world is called the proceptual world. Here individuals start with procedural actions on mental conceptions from the first world, as counting, which by using symbols become encapsulated as concepts. The symbols represent both processes and concepts, for example counting and number or addition and sum. The symbols, together with the processes and the concepts, are called procepts (Gray & Tall, 1994) and are used dually as processes and concepts depending on the context. The third world is called the formal world and here properties are expressed with formal definitions as axioms. There is a change from the second world with connections between objects and processes to the formal world with axiomatic theories comprising formal proofs and deductions. Individuals go between the worlds as their needs and experiences change and mental representations of concepts are formed and altered.

Not all mathematical concepts can be regarded as an object and a process, e.g. a circle or an equivalence class that are both pure objects, though in limits this duality is very obvious. Limits can be handled through an explorative approach with tables of function values and graphs from the beginning and later as symbolically expressed entities. Learning limits of functions demands leaping between operational and static perceptions (Cottrill et al., 1996). There is a challenge in understanding the termination of an infinite procedure as a finite

object, such as $\lim_{x \rightarrow \infty} \frac{\sin\left(\frac{1}{x}\right)}{\frac{1}{x}} = 1$. It is important to reach all significant stages and be able to

change between the different stages. Only then can an individual fully understand the concept if understanding of a mathematical concept is defined as Hiebert and Carpenter did (1992), i.e. to be something an individual has achieved when he or she can handle the concept as part of a mental network. The more connections between the mental representations, the better the individual understands the concept (Dreyfus, 1991; Hiebert & Carpenter, 1992).

In an attempt to create a model for concept development, I have used theories about concept images (Tall & Vinner, 1981; Vinner, 1991) as a complement to the theory of the three worlds. A concept image for a concept is an individual's total cognitive representation for that concept. The concept image comprises all representations from experiences linked to the concept, of which there may be several sets of representations constructed in different contexts that possibly merge as the individual becomes more mathematically mature. Multiple representations of the same concept can co-exist if the individual is unaware of the fact that they represent the same concept. Possible inconsistencies may remain unnoticed if the inconsistent parts are not evoked simultaneously. Concept images are created as individuals go through the developments represented by the three worlds. The model in Figure 1 shows how part of a concept image can be structured as I consider it. The three types of symbols used each represent a concept at the stage of one of Tall's three worlds, as described in the figure. The concepts can be, for instance, geometric series, derivatives of polynomials, definitions of derivatives and limits of functions, theorems, proofs, and

examples of topics of related concepts. More links and more representations of concepts exist around the formal world representations of concepts. There are also parts that are not very well connected to other parts. This situation can occur when individuals use rote learning as they try to cope with mathematics. Students who are unable to encapsulate processes as objects or take the step from procepts to a strictly formalistic exposition can use rote learning as a substitute.

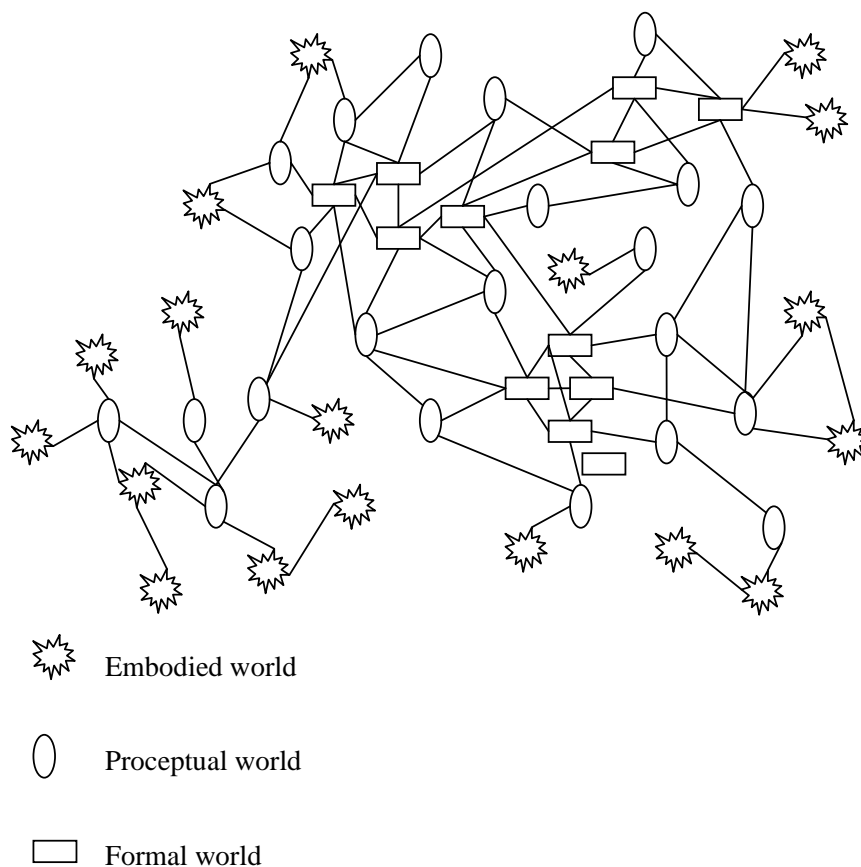


Figure 1. Model of part of a concept image at one time. Each node is a representation of a concept at one of the different stages of Tall's three worlds.

Mental representations can be depicted in terms of *topic areas* as a complement to the levels of abstraction shown in Figure 1. A topic area refers to the areas of mathematics with components of a certain topic, e.g. 'functions' or 'limits'. The components are such nodes as those in Figure 1. The sizes of topic areas vary according to what context they appear in, for instance large areas such as 'functions', or smaller areas such as 'polynomial functions'. The classification in topic areas means sub-topic areas at several levels. A component in one topic area can in itself be a topic area. Weierstrass's limit definition belongs to the topic area of 'limit', as do 'limits of rational functions' and the symbols used to express limits. The symbols also belong to the topic areas 'derivatives' and 'continuity'. Topic areas overlap this way as illustrated by the simplistic model in Figure 2.

If a concept is represented in more than one topic area in a concept image and the topic areas the representations belong to are disjoint, then inconsistencies may occur in the way aforementioned. Inconsistencies can appear within a topic area as well, but they are easier to detect due to the relatedness of the topic. The development of concept images never ends and the mental representations generate a dynamical system linked together at various levels.

An example of a topic area, marked by a wider contour line in Figure 2 represents the topic area 'limits'. It comprises a marked oval component representing the limit definition, which is also part of the topic areas 'derivatives', TA2, and 'continuity', TA3. The black rectangular component represents the definition of derivatives. The figure only shows some nodes in each topic area to describe the structures of the complicated relations. There are, in most real cases, more nodes linked in more intricate constellations.

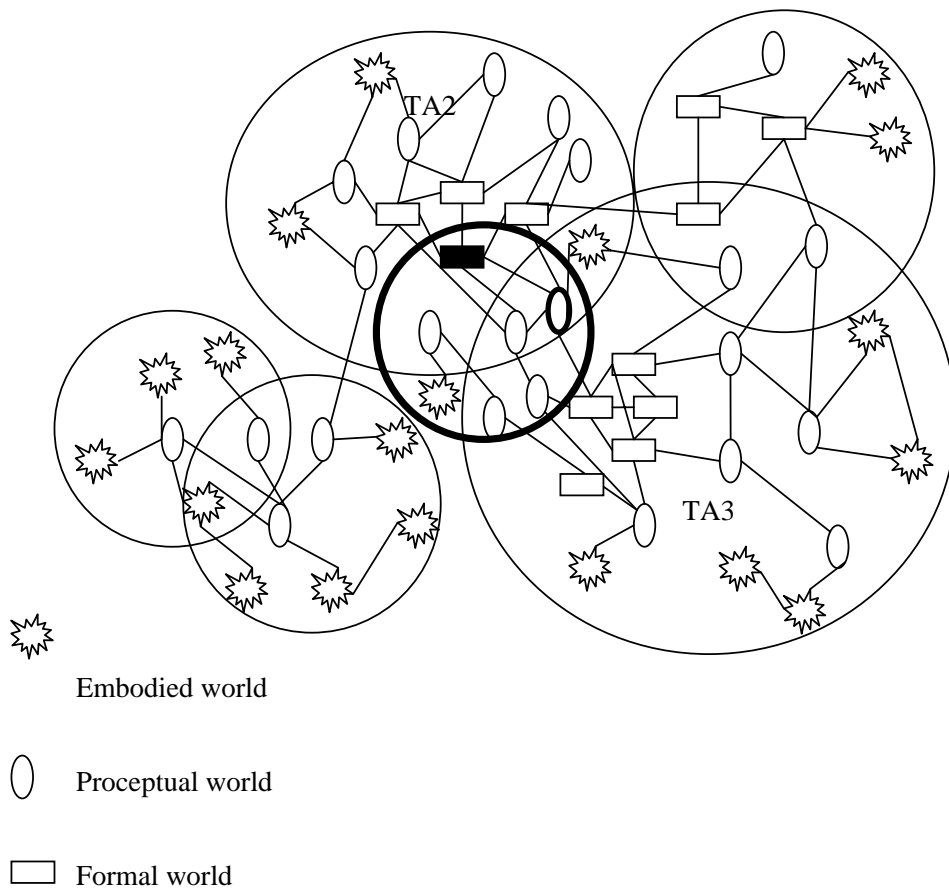


Figure 2. Topic areas and components with links in a model of a concept image. The marked part is the topic area 'limits'. TA2 and TA3 represent the topic areas 'derivatives' and 'continuity' respectively.

Concept images change on account of outer and inner stimuli, such as discussions, thoughts and problem solving, and a model such as the one in Figure 2 is hence in constant change. It is nevertheless a tool suitable for describing students' concept developments of limits of functions.

3. The empirical study

This section describes the sample of students studied and the course they were enrolled in, followed by an outline of methods and instruments used.

3.1 The students and the course

There were 112 students participating in the study, of these, 33 were female. The students were aged 19 and up. They were enrolled in a first level university course in mathematics that was divided into two sub-courses. Both of them dealt with calculus and algebra and were given over 20 weeks full time (10 weeks for each course). The students had two lectures (the whole group with one lecturer) and two sessions for task solving (in sub-groups of 30 students with a teacher in each sub-group) three days per week. Each lecture and session lasted 45 minutes. Thus the total teaching time for each course was 90 hours.

The notion of limits of functions was presented in the first course before derivatives. The lectures and sessions dealing with limits are outlined here to describe the students' first encounter with limits of functions at university level. On the first lecture on limits, the lecturer followed the textbook presenting formal definitions and theorems on indefinite and definite limits of functions and limits of monotonic functions as x tends to infinity (for functions depending on x). The textbook has an intuitive approach in the initial pages of the book, but the exposition becomes strictly formal after that. On the following task solving session, the students in the group were reluctant to go up to the black board to solve tasks, and the teacher ended up solving seven of ten tasks for that session. Students tried to solve three of the seven tasks before the teacher solved them. The students said that absolute values confused them and that was also one of the problems with the tasks.

The second lecture dealt with standard limit values and some proofs were rapidly presented (some comments on the speed were whispered among the students). The number e was introduced and so were ε - δ definitions as x tends to a number. Continuity was then presented with some following theorems. Parts of the proofs were omitted. The lecturer kept on following the textbook to help the students follow his reasoning. The second task solving session was very similar to the first where the teacher solved most of the tasks. The triangle inequality was discussed. A question of whether a function can have several limits was posed and answered.

In the third lecture the lecturer continued to prove theorems from last lecture. Trigonometric formulas were repeated from lectures before limits were taught. Some theorems were made plausible through pictures. Derivatives were introduced. The lecturer said that derivatives and integrals were what the course is all about. On the following task solving session, there were

many questions from the students about theorems and definitions and how to use the theories they had met. Continuity and monotonic functions were particularly discussed. The rest of the session was about derivatives.

The following lectures and sessions dealt with derivatives and integrals. Limits were taught again in the second course in different settings such as integrals and series. The first course had a written exam and the second had a written exam followed by an oral one. The marks awarded were *IG* for not passing, *G* for passing and *VG* for passing with a good margin.

3.2 Methods

Different methods were used to collect different types of data, such as students' solutions to limit tasks and responses to attitudinal queries. The sets of data were collected at different stages in the students' developments. The instruments used were designed to take those differences into account. The limit tasks were of increasing difficulty and the attitudinal part was mainly in the beginning of the semester. The students were confronted with tasks at five times during the semester, called *stage A* to *stage E*.

The students got a questionnaire at stage A in the beginning of the semester. It contained easy tasks about limits and some attitudinal queries. The scope of these and subsequent tasks is described in the instruments section. The students were also asked about the situations in which they had met the concept before they started their university studies. The attitudinal data are not presented in this article.

After limits had been taught in the first course, as described in the former section, the students received a second questionnaire at stage B, with more limit tasks at different levels of difficulty. The aim was for the students to reveal their habits of calculating, their abilities to explain what they did, and their attitudes in some areas. The students were asked if they were willing to participate in two individual interviews later that semester. Thirty-eight students agreed to do so; of these, 18 students were selected for two individual interviews each. The selection was done with respect to the students' responses to the questionnaires so that the sample would as much as possible resemble the whole group. The gender composition of the whole group was also considered in the choices.

The first session of interviews was held at stage C in the beginning of the second course. Each interview was about 45 minutes long. The students were asked about definitions of limits, both the formal one from their textbook and their individual ways to define a limit of a function. They also solved limit tasks of various types with the purpose to reveal their perceptions of limits and commented on their own solutions from the questionnaires to clarify their written responses where it was needed.

The students received a third questionnaire at the end of the semester, at stage D. It contained just one task. Two fictional students' discussion about a problem was described. One reasoned incorrectly and the other one objected and proposed an argument to the objection. The students in the study were asked to decide who was correct and why.

A second interview was carried through at stage E after the exams. Each interview lasted for about 20 minutes. Of the 18 students, 15 were interviewed at this point. The remaining three students were unable to participate for various reasons. The students commented on the last questionnaire and, linked to that, the definition was scrutinized again. The quantifiers *for every* and *there exists* in the ε - δ definition were discussed thoroughly. Of the 15 interviewed

students, three low achievers and three high achievers were selected for comparisons as indicated in Table 1.

Table 1: Students' marks

Name	Mark
Martin	G1/G2
Tommy	G1/G2
Anna	G1/G2
Julia	VG1/VG1
Dennis	VG1/VG1
Emma	VG1/VG1

Field notes were taken during the students' task solving sessions and at the lectures when limits were treated to give a sense of how the concept was presented to the students and how the students responded to it. Tasks and results from other parts of the study are described in more detail in other articles (Juter, 2005a-2006c).

3.3 Instruments

The students solved some easy tasks about limits of functions at stage A, such as the following example:

Example 1: $f(x) = \frac{x^2}{x^2 + 1}$. What happens with $f(x)$ if x tends to infinity?

The tasks did not mention limits per se, but were designed as a means to explore if the students could investigate functions with respect to limits.

At stage B the tasks were more demanding. Some of the tasks were influenced by Szydlik (2000) and Tall and Vinner (1981). Three tasks had the following structure:

Example 2: a) Decide the limit: $\lim_{x \rightarrow \infty} \frac{x^3 - 2}{x^3 + 1}$.

b) Explanation.

c) Can the function $f(x) = \frac{x^3 - 2}{x^3 + 1}$ attain the limit value in 2a?

d) Why?

Example 2 is what I regard to be a routine task. There were also non-routine tasks. A solution to a task was presented to the students. It could be incomplete or wrong and the students were to make it complete and correct. There were two such tasks. The students were also asked to formulate a definition of a limit, not necessarily the one in their textbooks.

At stage C, which was the first set of interviews, the students were asked to comment on statements very similar to those used by Williams (1991) in a study about students' models of limits. The statements the students commented on are the following (translation from Swedish):

1. A limit value describes how a function moves as x tends to a certain point.
2. A limit value is a number or a point beyond which a function can not attain values.
3. A limit value is a number which y -values of a function can get arbitrarily close to through restrictions on the x -values.
4. A limit value is a number or a point which the function approaches but never reaches.
5. A limit value is an approximation, which can be as accurate as desired.
6. A limit value is decided by inserting numbers closer and closer to a given number until the limit value is reached.

The reason for having these statements was to get to know the students' perceptions about the ability of functions to attain limit values and other characteristics of limits. The students were given the statements to have something to compare with their own thoughts. There were other tasks designed to make the students consider the formal definition to clarify what it really says, and tasks about attainability, for example:

Example 3: Is it the same thing to say "For every $\delta > 0$ there exists an $\varepsilon > 0$ such that $|f(x) - A| < \varepsilon$ for every x in the domain with $0 < |x - a| < \delta$ " as "For every $\varepsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - A| < \varepsilon$ for every x in the domain with $0 < |x - a| < \delta$ "? What is the difference if any?

As indicated before, at stage D the students got a task with a description of two students arguing over a solution to a task (translation from Swedish):

Example 4: Two students discuss a problem. They do not agree on the solution. The problem discussed is about the following limit: $\lim_{x \rightarrow 0} \frac{\cos(x^{-2})}{10000}$.

The student S1 claims that the limit exists and is zero with the following explanation:

I use the definition for limits and write $|f(x) - A|$ where A is the limit. I try $A = 0$ since I think that is the limit. I get: $\left| \frac{\cos(x^{-2})}{10000} - 0 \right| \leq \frac{1}{10000} < \frac{1}{9999}$. This is true for all x except $x = 0$, but

that can never be the case since we then would have zero in the denominator in x^{-2} , so it is true for all x in the domain of $f(x) = \frac{\cos(x^{-2})}{10000}$. This means that if we chose $\varepsilon = \frac{1}{9999}$ for all possible δ with $0 < |x - 0| < \delta$, then the definition is met and the limit is zero.

The student S2 does not agree and claims that if one, for example, chose $\varepsilon = \frac{1}{100000}$ then one can not find a $\delta > 0$ with $|f(x) - A| < \varepsilon$ where $0 < |x - 0| < \delta$ for all x in the domain, that is to say all $x \neq 0$. Therefore, the student S2 claims that $\lim_{x \rightarrow 0} \frac{\cos(x^{-2})}{10000}$ has no limit according to the definition of limits.

At stage E, the second set of interviews, the students' written responses to the task at stage D were discussed. Example 3 was also brought up again in connection with the task at stage D.

4. Typical patterns for the six students' developments of limits

The students' responses to tasks and questions in the questionnaires and interviews have been analysed and categorised. Table 2 shows the typical developments of the students in the categories *High achievers* and *Low achievers* respectively. A developmental portrait was done for each of the 15 interviewed students (published in Juter (2006b)) from which the combined descriptions were drawn.

Table 2: Typical student developments in the two categories through the semester

Stage	High achievers	Low achievers
A	Links limits to prior studies. Solves easy tasks well.	Links limits to prior studies and other topics. Solves easy tasks well.
B	Limits are attainable in problem solving. Solves tasks and explains well. Problems to state a limit definition.	Limits are attainable in problem solving. Solves routine-tasks and explains with some flaws, other tasks problematic. Cannot state a limit definition.
C	Limits are attainable in problem solving and in theory. Prefers statement 3. Problems to state the definition. Can identify the definition with uncertainty about the quantifiers' meanings. Solves tasks fairly well.	Limits are attainable in problem solving but not in theory. Prefers statements 1 and 4. Cannot state the definition. Problems to identify the definition, quantifiers not understood. Problems to solve tasks. Not an actual limit if attainable by the function.
D	Identifies the error.	Problems to identify the error.
E	Can identify the definition. Can explain the quantifiers' meanings.	Problems to identify the definition. Cannot explain the quantifiers' meanings.

There are obvious similarities between the two categories. The quantifiers in the definition caused confusion for all students. There was an opinion among some students that ϵ and δ in Example 3 at stage C come in pairs and can therefore be placed either way in the example. Mostly high achieving students showed traces of this conception, which can be explained by the fact that low achieving students had not integrated the theory well enough in their concept images to even identify the definition next to a wrong one. The high achieving students did not have this misunderstanding at stage E as they were able to explain the meaning of the quantifiers. The low achieving students did not understand the quantifiers meaning in the definition for the duration of the course.

The students' problems to connect theory to problem solving became particularly apparent from their difficulties to determine whether limits are attainable for functions or not. Many students interpreted the strict inequalities in the formal definition to say that limits are not attainable. Examples where limits were attainable did not change the low achieving students' beliefs about the definitions' meaning. Some students became frustrated when they saw examples of attainable limits and were asked questions about the definition because they were unable to create a coherent picture of the situation. The students' concept images were divided in disjoint topic areas; one for limits in theory and one for limits in problem solving. High achievers were able to link the two topic areas at the end of the course. Such linking often requires hard work which sometimes involves substantial changes in the students' concept images and the prospect of that makes them disregard inconsistencies and simply see parts that do not cohere with the rest of the concept image as minor exceptions.

Students with positive attitudes to mathematics in general were better limit problem solvers. Most of the high achieving students thought that they had control over the concept of limits, but many of the low achieving students also claimed to have control even if that was not the case. An unjustifiably strong self confidence can prevent students from further work on erroneous or incomplete parts of their concept images.

The students' responses to Example 2 from the instruments section revealed part of the low achieving students' confusion:

4.1 Low achievers:

Martin

a) $\rightarrow 1$ b) $\frac{1 - \frac{2}{x^3}}{1 + \frac{1}{x^3}} \rightarrow 1$ as $x \rightarrow \infty$. c) Yes. d) –

Tommy

a) 1 b) For large x , -1 and $+2$ can be neglected, denominator and numerator are identical and thereby attains one.

c) No. d) Trivial, $x^3 - 2 \neq x^3 + 1$, for the function to attain one requires $x^3 - 2 = x^3 + 1$, logically impossible.

Anna

a) $\rightarrow 1$ b) $\frac{x^3 \left(1 - \frac{2}{x^3}\right)}{x^3 \left(1 + \frac{1}{x^3}\right)} = \frac{1}{1} \rightarrow 1$. [Arrows showing that $\frac{2}{x^3}$ and $\frac{1}{x^3}$ tend to zero]

c) No. d) At really large x the value tends to 1 but there will still be a difference between denominator and numerator.

4.2 High achievers:

Julia

a) 1 b) $\frac{x^3 - 2}{x^3 + 1} = \frac{x^3 \left(1 - \frac{2}{x^3}\right)}{x^3 \left(1 + \frac{1}{x^3}\right)}$ i.e. when $x \rightarrow \infty$, -2 and $+1$ can be neglected since

they are so much smaller than x .

c) No. d) You cannot insert $x = \infty$.

Dennis

a) 1 b) When x becomes very large, -2 and 1 make a smaller difference for the value.

c) No. d) There are different numbers in the denominator and numerator, i.e. they can in fact not be equal and the fraction can therefore not be one.

Emma

a) 1 b) When x is very large, the numerator and denominator are in what matters

equal. We also have $\frac{x^3 - 2}{x^3 + 1} = \frac{x^3}{x^3} \cdot \frac{1 - \frac{2}{x^3}}{1 + \frac{1}{x^3}}$ where we can see that for large x the expression is

approximately equal to one.

c) No. d) $x^3 - 2 < x^3 + 1 \quad \forall x > 0$. The numerator and denominator are therefore never equal and the fraction is always smaller than one.

High achievers did not use arrows in part a, but the low achievers did. This is one example of students' confusion of limits with function values. The students were uncertain of what happens at the critical point. Anna, in particular, revealed this type of uncertainty as she mixed up her answer in part b. Tommy was also not sure what he believed to be true as he stated two opposite opinions in b and c. He displayed traces of a concept image with multiple incoherent representations. The high achievers did not show this type of confusion, but they

were not always clear in their explanations either. Julia, for example, had a vague response to part d where it is impossible to determine whether she reasons correctly or not.

Learning limits requires skills from many mathematical areas. Students need to be able to understand formal expositions, perform algebraic manipulations, understand the meanings of quantifiers and absolute values, which students found problematic, and link theory to their every day problem solving. They also need to find inspiration and reasons to go through the hard work to make the knowledge meaningful in their concept images. High achievers have richer concept images enabling them to create many high quality links and therefore the concept image becomes useful in a variety of situations, which gives the students a broader and clearer view of the topic at hand.

5. Concluding remarks

As could be expected, high achieving students' abstraction abilities were more developed than other students'. The former group was to a much higher degree than the latter able to link theory to problem solving and explain the meaning of, for example, the limit definition. The students were studied during a semester and for that time there were similarities of the high achieving students' developments with the historical development of limits that the other students did not reveal. The similarities were mainly linked to abstraction and formality as the students started with an operational approach with a focus on problem solving rather than theory and then gradually understood the links between theory and problem solving.

There were no clear patterns of students' mental representations of limits as exact values or approximations, limits as objects or processes, and limits as attainable or unattainable for functions. Of the 15 students interviewed, only two showed a coherent trace of their concept images. Both students were high achievers. The lack of patterns in all students' concept images, particularly in the high achievers', points to the complex nature of limits and the challenge to teach and learn limits.

References

- Cottrill, J., Dubinsky, E., Nichols, D., Schwingendorf, K., & Vidakovic, D. (1996). Understanding the limit concept: Beginning with a coordinated process scheme. *Journal of Mathematical Behaviour* 15, 167-192.
- Dreyfus, T. (1991). Advanced mathematical thinking processes. In D. Tall (Ed.), *Advanced mathematical thinking* (pp. 25-41). Dordrecht: Kluwer Academic Publishers.
- Gray, E., & Tall, D. (1994). Duality, ambiguity and flexibility: A procedural view of simple arithmetic. *The Journal for Research in Mathematics Education* 26(2), 115-141.
- Hiebert, J., & Carpenter, T. P. (1992). Learning and teaching with understanding. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 65-97). New York: Macmillan.
- Juter, K. (2005a). Limits of functions – how do students handle them? *Pythagoras* 61, 11-20.
- Juter, K. (2005b). Students' attitudes to mathematics and performance in limits of functions. *Mathematics Education Research Journal* 17(2), 91-110.

- Juter, K. (2005c). Limits of functions: Traces of students' concept images. *Nordic Studies in Mathematics Education* 10(3-4), 65-82.
- Juter, K. (2006a). Limits of functions – University students' concept development. PhD thesis. Sweden: Luleå University of Technology.
- Juter, K. (2006b). Limits of functions as they developed through time and as students learn them today, to appear in *Mathematical Thinking and Learning*.
- Juter, K. (2006c). Limits of functions - students solving tasks. *Australian Senior Mathematics Journal* 20(1), 15-30.
- Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics* 22, 1-36.
- Szydlik, J. (2000). Mathematical beliefs and conceptual understanding of the limit of a function. *Journal for Research in Mathematics Education* 31(3), 258-276.
- Tall, D. (2004). Thinking through three worlds of mathematics. In M. Johnsen Høines, & A. B. Fuglestad (Eds.), *Proceedings of the 28:th conference of the International Group for the Psychology of Mathematics Education* (pp. 281-288). Norway: Bergen University College.
- Tall, D. & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics*, 12, 151-169.
- Vinner, S. (1991). The role of definitions in the teaching and learning of mathematics. In D. Tall (Ed.), *Advanced mathematical thinking* (pp. 65-81). Dordrecht: Kluwer Academic Publishers.
- Williams, S. (1991). Models of limit held by college calculus students. *Journal for Research in Mathematics Education* 22(3), 219-236.

Today's Mathematics Students

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1. Today's Mathematics Students

A common mistake that undergraduate mathematics professors make when teaching is to assume that students are younger versions of themselves. Since many mathematics professors are above average in intelligence and were quite good students, the assumption that students are just like themselves can cause pedagogical difficulties (Krantz, 1993). To teach effectively, it is important to understand students. Yet, understanding today's students is literally like bridging a generation gap (Hawk, 2005).

Studying generation gaps is, perhaps, unfamiliar territory to mathematics educators. In mathematics, one counterexample proves a theorem false, but in making generalizations about generations of mathematics students, one does not examine individual cases. In fact, any individual may have none of the described characteristics (and yet be a member of a given generation) and most individuals will not have all the characteristics. In fact, it is possible that variation within a generation is greater than variation between generations. Yet, sociologists do attempt to describe generations by providing a variety of information, which could prove useful to educators.

Such information might lead educators to understand their students' culture and, thus, the influences on their students, even if such influence is manifested to a more or less degree in individual students. Sociologists attempt to describe the motivations, interests, personalities, and other traits of a generation by describing the so-called "personality of the cohort." For example, Neil Howe and William Strauss (authors of *Millennials Rising: The Next Great Generation*, 2000) define generations as "a cohort group whose length approximates the span or a phase of life and whose boundaries are fixed by peer personality" (Howe & Strauss, 1991, p. 60). Further, the peer personality is defined as a "generational persona recognized and determined by (1) common age location; (2) common beliefs and behavior; and (3) perceived membership in a common generation" (Howe & Strauss, 1991, p. 64). It is possible, then, to distinguish one generation of students from another.

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2. A New Generation

The generation of people born between 1980 and 2000 are called the "Millennial" generation and many of the earlier born are in college now. In fact, the term Millennial originated because the earlier born entered college in 2000 and the last born will enter college in 2020 (Franklin, 2005). Sociologists claim that Millennials are different from Baby Boomers (born between 1946 and 1964) and Generation X-ers (born in the 60s and 70s). Millennials are confident, hopeful, goal- and achievement-oriented, civic-minded, impatient, and inclusive (Raines, 2000). While Generation X-ers are certainly used to television and probably personal computers, the Millennials in college now came of age while immersed in the Internet and email. Of course, the Baby Boomers were not brought up on the computer at all. Consider the following dates.

- The World Wide Web was invented in 1989-1991.
- The DVD was invented in 1995 (the VCR was invented in 1971).
- The Apple Macintosh was invented in 1984.
- The Windows program was invented in 1985.
- The TI graphing calculator was introduced in 1990 (the very first graphing calculators were invented around 1985).
- Ebay began in 1995, as did Amazon.
- What professors consider new, today's college students have lived with for a majority of their lives.

3. Mathematics Background

Perhaps even more important than understanding the cultural differences between today's college professors and today's college students is understanding the mathematics background of today's college students. Based on the fact that more students take remedial mathematics courses than ever before, students are entering college with different mathematics, such as less algebra (Levine & Cureton, 1998). According to a national survey in 2000, approximately 75% of mathematics teachers agree with the principles of the National Council of Teachers and Mathematics (NCTM) and implement them to at least a moderate degree (Horizon Research, 2002). It is important to understand, then, what the principles of the NCTM are. Roughly, the NCTM calls for less attention to procedural understanding and by-hand-symbol manipulation, and more attention to problem solving, conceptual understanding, and technology use.

Yet, based on standardized test results it appears that mathematical skills are either the same or improved. The national average ACT math score has been about the same from 1995 through 2005 (available from ACT, <http://www.act.org>). The national average SAT score is up 14 points since 1995 (available from SAT, <http://www.sat.org>). NAEP scores are up since 1995 (available from the National Center for Education Statistics, <http://nces.ed.gov/nationsreportcard/>).

How can we reconcile decreasing college mathematics placement scores and increasing standardized mathematics test scores? One possible answer is that these two sets of instruments measure different skills. Certainly increasing scores on standardized tests is encouraging. However, college mathematics placement tests consist of numerous algebra manipulation problems. These type of problems are virtually absent from standardized tests such as the ACT and SAT.

In sum, incoming students have fewer algebra skills than previous generations of students. Of course, one might argue that more students enter college than in previous generations. Some studies suggest that incoming freshmen have more general problem-solving skills, and increased knowledge of statistics and data analysis (Senk & Thompson, 2003). Students are savvier with all kinds of technology than previous generations were (Noeth & Volkov, 2004). There is even some evidence that students have better attitudes towards mathematics than previous generations did (Senk & Thompson, 2003).

4. The College Experience

Once in college, today's students have a different experience than most college professors had when they were students. For example, in 1973, only 36% of all full-time college students were employed, but by 1995, the percent was at 69 (Hansen, 1998). In 1999, the percent of full-time college students who were also employed was an astonishing 80% (Oblinger, 2003). Bluntly, parents no longer pay for college (Levin & Cureton, 1998).

While living on campus is on the decline, drug and alcohol use are on the rise among college students, as is binge drinking and casual sex (Levin & Cureton, 1998). In addition, today's college students have the most severe psychological problems of any generation of college students (Kitzrow, 2003). Obesity, asthma, and attention-deficit disorder are also on the rise (Howe & Strauss, 2000).

During class, students have shorter attention spans than they once did, and they often try to multitask, even during lecture. Today's students spend time (when they could be studying) surfing the net. Obviously, this was not an issue for yesterday's students. My own university (a medium-sized Midwest university) conducted a survey via email in 2005 of all 642 students with a major in science, mathematics, or engineering, and 189 replied (a response rate of approximately 29%). When asked what gets in the way of their academic performance (students could check off as many items as they desired from a list of items, including "other"), 32% checked the answer "surfing the web," which was the third most popular response. The first most popular response was procrastination (65%) and the second most popular response was "social activity" (36%). However, only 16% said "partying." By the way, "loneliness" was also rather high at 29%. "Playing computer games" came in at 28%.

5. Implications

If we are convinced that current students have a significantly different background, both culturally and educationally, and have a different college experience than current professors, one is left wondering what should be done with this knowledge. Some researchers suggest that current students need certain pedagogy in place in order to learn (Carlson, 2005; Frand, 2000; Levine & Cureton, 1998; Oblinger, 2003). Current students need to:

- be engaged in the learning process and not sit passively taking notes (Carlson, 2005).
- have a say in what happens in the classroom (Frand, 2000).
- be treated like the customer (Oblinger, 2003).
- In addition, current students cannot tolerate delays of any kind and they want to be doing rather than knowing (Frand, 2000).

In practical terms, this could take the form of structuring one's classes so that students work in groups, learning by trial and error (or one might say "discovery learning"). Technology, such as graphing calculators, would be a vital part of one's courses. And, the professor's role would be one of facilitator and giver of feedback.

These are, of course, radical changes in the normal lecture format of college mathematics classrooms. One must consider, however, if the radical changes in college students do not, in fact, demand such radical changes. Yet, one also has to consider if all of this is more theory than reality. For example, if we asked Millennial students who are undergraduate mathematics majors how they want their mathematics courses taught and what they think of technology, what would they say?

6. A Survey

In 2006, an email survey was conducted at my own university. Recall that it is a university in the Midwest. It is the second largest university in the state, with a reputation as a solid university. The university is both a teaching and research university, granting bachelor and master degrees in most disciplines, but no doctorates are granted. The Mathematics Department is a large service department, as engineering degrees, along with science degrees, are extremely popular at the university. Also, at any given time, there are approximately 100 students majoring in mathematics. The Mathematics Department made a decision in 2000 to attempt to ease the transition of incoming mathematics students from high school to college mathematics courses. One idea the department formed was to have cooperative learning in small groups in some (but not all) of the mathematics classes.

To form the sample for the email survey, the researcher began with a list of all mathematics majors (again, approximately 100), and eliminated those students who were born earlier than the definition of Millennial allows. The researcher then eliminated students who had not experienced at least one mathematics course taught in a traditional lecture format and one mathematics course that included small groups. Although this does not eliminate all possible bias (e.g., perhaps the professors who teach one type of class are somehow better than the professors who teach another type of class), it at least makes it possible to state that the students in the survey had experienced both types of learning. This left 63 students in the sample. The survey consisted of two open-ended questions:

1. In your mathematics classes, do you prefer listening to a lecture, working in small groups, or something else (if something else, what)?
2. What is your opinion about the use of calculators in mathematics classes?

Twenty-eight students replied (a response rate of approximately 44%). Twenty-five (or 89%) said that they definitely preferred lectures, with the remaining three (11%) students saying they wanted a combination of lecture and small groups. Nobody preferred small groups to lectures and nobody suggested another alternative. However, all the students put qualifications on how the lectures should be. Consider the following three quotes:

I personally enjoy a lecture as long as the instructor uses good in-class examples on how to solve the problems that we are expected to know how to solve on homework and tests.

I prefer lecture, if it is well prepared and there is time for questions.

Personally, I just like lectures for math but one thing that professors should do more of is examples

So the Millennial students at one university (or at least those who felt strongly enough about it to answer the email) like lectures. However, the lectures that they like are a tad bit different from the traditional definition of lecture. The following statement may be the best summary: "I prefer listening to a professor lecture with some class involvement in the lecture" [italics added]. It is not acceptable to students that a professor simply stands in front of a class and talks. The professor should be aware of the needs of the class. The lecture format is fine with students, but the professor should include examples and take questions. The professor should be aware of the class enough that he or she adjusts the pace of the lecture accordingly. The professor can still take the role of a leader, but she or he must try to help the students during the lecture with what the students will actually be doing. However, many of the Millennial mathematics majors do not want to "use up" classtime working with classmates in small groups.

Only one student mentioned that the professor might consider using technology, such as PowerPoint. Still, even that student said it should be "every so often" as a supplement to the chalk/chalkboard approach, just for "variety". One student mentioned that professors should photocopy their handwritten notes for students to use during class (not a very technology savvy approach at all!). This last suggestion was for pacing reasons, as professors were accused by most students as going too fast during lecture. Consider this quote, "I think professors forget that we haven't been doing this for 10+ years and we need time to catch on."

The students who preferred lectures viewed small group discussions as either a waste of time (because mathematics is best learned individually) or something that one could easily facilitate on one's own. And even students who preferred a mixture of lecture and small group discussion thought that the small group discussions needed to be carefully monitored by the professors. I don't like to spend long periods of time in class working with the group because most students procrastinate and waste the time. For in-class groups I think it's better to do only simple problems that can be completed in less than 3 minutes. [This student preferred both lecture and small groups.]

It's easy to set up small groups to study for classes, and I think that helps students clarify any material they may be confused with. [This student preferred only lecture during class time.] I like to work in small groups when doing homework. It's kind of like getting the best of both worlds. [This student preferred only lecture during class time.]

The calculator question was asked to help ascertain students' perspective on technology use. The answers showed more variation than the answers to the lecture question. In the Mathematics Department, some courses are taught with graphing calculators required, and some courses are taught with graphing calculators banned. However, a previous survey of all the majors revealed to us that almost all students (99%) own a graphing calculator and made considerable use of it in high school. Returning to our current survey, 16 (or 57%) students supported calculator use; eight (29%) supported no calculator use; and the remaining four (14%) felt neutral about the issue. Those that supported calculator use tended to support the use of a calculator in all things, at all times. The justification for this philosophy was that it is a technological world and school prepares one for the world. Consider this representative quote:

I believe that school should be viewed as preparation for the "real world" and in the "real world" your boss wouldn't ask you to solve a problem without a calculator!

However, there was also the group of students who did not like calculators.

I hate being dependent on a calculator.

Calculators are rather pointless, since you need to show all of your work.

There are so many different things that a calculator can do which takes away what the student should be able to do (i.e., derivatives, integration, standard deviation, combination, permutation).

I'm against calculators, as any good math student should be.

My opinion on calculator use is that it is really abused in the classroom.

I have found them nearly useless.

One interesting response discussed that if "professors really got to know every student they would be able to build a level of trust and respect that would prevent the students from using functions that are not allowed." This student is saying that there may be functions on the calculator that the professor does not want students to use. Regardless, that professor should have such a close relationship with his or her students, that if he or she says, "don't touch that button", it is guaranteed that students won't. That is an amazing level of closeness, much beyond what is the norm in the past. Calculators are viewed by some of the Millennial students as simply a part of life and therefore should be a part of the classroom. However, this is not the case with all the Millennial students. Thus, it is not the case that all Millennial students need technology use in order to see the worth of a course of study.

The results of this small survey support the idea Millennial students desire their mathematics professors be aware of them and their needs. In return, the Millennial students offer their respect and attentiveness (to a well-prepared lecture, for example). However, the absolute need to work in groups or be submerged in technology was not supported in this survey.

7. Closing Statement

Theorists suggest that the Millennials are very different students from previous students. Implications are that undergraduate mathematics professors ought to teach mathematics in a different format than for previous students. A survey at one Midwest institution revealed that students do not dislike the lecture format, per se, nor are all of the Millennials convinced that calculators are a must. Rather, these students want personal attention during the lecture. A mathematics professor, when lecturing, should try to use lots of examples, pace the lecture, have a relationship of trust with students, allow calculators to the degree that makes sense for a particular class (and even perhaps negotiate this with students), and, in general, should be checking with students on how it is all going. It is not less of a relationship that students want with professors, but more. Rather than exclusive use of small group discussion, these students want an "interactive lecture." Perhaps it is appropriate to end this paper with two sentences used earlier: These are, of course, radical changes in the normal lecture format of college mathematics classrooms. One must consider, however, if the radical changes in college students do not, in fact, demand such radical changes.

References

- Carlson, S. (2005). The net generation goes to college. *The Chronicle of Higher Education*, 52(7), A34-A36.
- Frand, J. L. (2000, September/October). The information-age mindset: Changes in students and implications for higher education. *EDUCAUSEreview*, 15-24.
- Franklin, R. (April, 2005). *Who will our students be in a postmodern, post-denominational, and materialistic age?* Paper presented at the conference Future of Baptist Higher Education, George W. Truett Theological Seminary, Baylor University, Waco, TX.
- Hansen, E. (1998). Essential demographics of today's college students. *AAHE Bulletin*, 51(3), 3-5.
- Hawk, J. R. (2005). Reconnecting with our postmodern students. *Thought & Action*, XXI, 37-46.
- Horizon Research. (2002). *Status of mathematics high school teaching*. Author: Chapel Hill, NC.
- Howe, N., & Strauss, W. (2000) *Millennials rising: The next great generation*. New York: Vintage Books.
- Howe, N., & Strauss, W. (1991). *Generations: The history of America's future*. New York: Morrow.
- Kitzrow, M. A. (2003). The mental health needs of today's college students: Challenges and recommendations. *NASPA Journal*, 41(1), 167-181.
- Krantz, S. G. (1993). *How to teach mathematics*. Washington DC: American Mathematical Society.
- Levine, A., & Cureton, J. S. (1998, March/April). What we know about today's college students. *About Campus*, 4-9.
- Noeth, R. J., & Volkov, B. B. (2004). *Evaluating the effectiveness of technology in our schools*. Iowa City: ACT.
- Oblinger, D. (2003, July/August). Boomers, Gen-xers, & Millennials: Understanding the new students. *EDUCAUSEreview*, 37-47.
- Raines, C. (2002). Managing Millennial. *Generations at Work*. [on-line] <http://www.generationsatwork.com/articles/millennials.htm>
- Senk, S. L., & Thompson, D. R. (Eds.). (2003). *Standards-based school mathematics curricula*. Mahwah, NJ: Erlbaum.

The Need for an Inclusive Framework for Students' Thinking in School Geometry

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Abstract: This study is the outcome of a research that investigated how students who were assigned varying levels of geometric thinking attempted problems requiring some amount of algebraic thinking in geometry. The study reports that students' thinking in geometry also requires facility with algebra and as such there is a need for a framework that provides a more inclusive view of what constitutes geometric thinking in school mathematics.

Key words: geometric thinking, van Hiele framework, school geometry

1. Introduction

The problem solving movement of the 1980s witnessed a burgeoning interest in the processes of learning that was enhanced further by the rise of constructivism within the field of mathematics education. The emphasis was not only on general mathematical thinking but also on thinking in specific content domains. Subsequently, several studies focused on geometric thinking (Burger & Shaughnessy, 1986; Devilliers & Njisane, 1987; Senk, 1989), on probabilistic thinking (Jones, Langrall, Thornton, & Mogill, 1999), on algebraic thinking (Herbert & Brown, 1999) and on statistical thinking (Mooney, 2002; Groth, 2003). In many such studies, the idea was to identify a framework that could aid in assessing a student's level of thinking in that particular content domain so as to facilitate instruction. One of the earliest known frameworks for thinking in a content domain within mathematics education is in geometry, proposed by the van Hieles in the 1950s (husband and wife) in the Netherlands. The primacy of the van Hiele framework attests to the very special status of geometry in mathematics as an essential component of school mathematics curricula all over the world.

2. Van Hiele Theory

The van Hiele theory has since been extensively used in studies to conceptualize students' thinking in geometry at various levels. The van Hieles proposed five hierarchical levels that describe growth in student thinking in geometry. These levels are not necessarily age-bound as in

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the Piagetian cognitive developmental theory. The van Hiele initially used Level 0 to be the lowest level and Level 4 to be the highest of the five levels (Crowley, 1987), however some authors use a numbering from 1 to 5 (Pegg & Davey, 1998). The levels are consecutively: Recognition, Analysis, Informal Deduction, Formal Deduction, and Rigour. Van Hiele (1986) mentioned that tracing the levels of thinking in geometry is not a simple affair, for the levels are not situated in the subject matter but in the thinking of man. Although van Hiele claimed that the roots of his theory are found in the theories of Piaget, progression from one level to the next is not the result of maturation or natural development. All depends on the quality of the experience that one is exposed to.

One of the earliest studies using the van Hiele framework in the United States was carried out by Usiskin (1982) at the University of Chicago. Since, a significant amount of research in school geometry has focused on the van Hiele levels of thinking (e.g., Burger, & Shaughnessy, 1986; Senk, 1989; Guitiérrez, Jaime, & Fortuny, 1991). An increased focus on the van Hiele theory has led researchers to capitalize on the strengths of the theory and also to highlight some shortcomings. The van Hieles developed their theory in the late 1950s, at a time when school geometry was primarily Euclidean in nature. However, the nature of school geometry has undergone major changes since the time when the van Hiele framework was developed.

3. The Nature of School Geometry

The reform of the 1960s in mathematics education brought major changes in the school geometry content. New approaches to geometry such as coordinate, transformational, and vector approaches were emphasized in the school curriculum. Although, the reform movement met with several obstacles, it was nevertheless significant in establishing a prominent place for algebraic approaches to the teaching and learning of geometry in school mathematics. There is a greater emphasis now in the geometry curriculum on writing algebraic expressions, substitution into an expression, setting up and solving equations; all of which require an understanding of the notion of variable and unknown. Whereas geometry has a separate subject status in the high school curriculum in several countries, such as the United States, it is integrated in an inclusive mathematics curriculum in many other countries. High school geometry builds on elementary school geometry which traditionally has emphasized measurement and the informal development of the basic concepts required in geometry at the high school level. The topics on measurements of perimeter, of area, and of volume which are revisited in the high school curriculum provide excellent opportunities for further applications of algebraic concepts in geometry.

On the other hand, Clements and Battista (1992) have claimed that school geometry refers almost universally to Euclidean geometry, even though there are numerous approaches to the study of a particular topic. While this may be true at lower levels, there are very strong connections between algebra and geometry at higher levels. Also, algebra and geometry have strong historical links. The use of literal symbols in the form of variables, constants, parameters and so on abounds in algebra. Symbols abound in school geometry as well. Students work with variables and unknowns when generalizing results or solving problems such as finding unknown sides or angle measures. The idea of a variable is also used in geometry using a variable point as in problems involving loci. Other simple uses of algebra in geometry as far as symbols are concerned involve labelling points or vertices, sides, and angles of figures. Some other connections between algebra and geometry in the high school curriculum arise in problem solving and modelling, and in the various modes of representations – graphical, algebraic, and

numeric. The symbolic representations pose problems for the students. Duval (2002) has claimed that there is no direct access to mathematical objects other than through their representations, and thus we can only work on and from semiotic representations, because they provide a means of processing. In geometry, this implies working in different registers (natural language, symbolic, and figurative) and moving in between registers. Algebra offers geometry a powerful form of symbolic representation.

Many of the concepts in geometry have their counterpart in algebra. For example, a point in geometry corresponds to an ordered pair (x, y) of numbers in algebra, a line corresponds to a set of ordered pairs satisfying an equation of the form $ax + by = c$ ($a, b, c \in R$), the intersection of two lines to the set of ordered pairs that satisfy the corresponding equations, and a transformation corresponds to a function in algebra (National Council of Teachers of Mathematics [NCTM], 1989). Algebraic results can be achieved geometrically and geometrical results can be demonstrated using algebra. For example, Pythagorean theorem for a right triangle having sides of lengths a, b , and c , can be represented algebraically using the formula $a^2 + b^2 = c^2$.

4. Geometric Thinking

As any form of mathematical thinking, geometric thinking is quite difficult to conceptualize. It is definitely a form of mathematical thinking within a specific content domain.

It would be simpler to consider what students are expected to be able do in geometry and accordingly model and understand their thinking. For example, the Standards (NCTM, 2000) highlighted the following aspects of school geometry for grades 9 – 12: analyze characteristics and properties of two- and three-dimensional geometric shapes and develop mathematical arguments about geometric relationships; specify locations and describe spatial relationships using coordinate geometry and other representational systems; apply transformations and use symmetry to analyze mathematical situations; and use visualization, spatial reasoning, and geometric modeling to solve problems (p. 41). The NCTM Standards clearly emphasize the link between algebra and geometry when it mentions describing spatial relationships using coordinate geometry.

In addition, geometric thinking is inherent in the types of skills we want to nurture in students. Hoffer (1981) has proposed a set of five categories of basic skills relevant to school students: (1) visual skills - recognition, observation of properties, interpreting maps, imaging, recognition from different angles; (2) verbal skills - correct use of terminology and accurate communication in describing spatial concepts and relationships; (3) drawing skills- communicating through drawing, ability to represent geometric shapes in 2-D and 3-D, to make scale diagrams, sketch isometric figures; (4) logical skills - classification, recognition of essential properties as criteria, discerning patterns, formulating and testing hypothesis, making inferences, using counter-examples; and (5) applied skills - real-life applications using geometric results learnt and real uses of geometry e.g. for designing packages. Although Hoffer seems to focus on Euclidean geometry, it is difficult to imagine how accurate communication in describing spatial concepts and relationships can be done without some form of algebraic support. Most high schools in the United States follow the Algebra I – Geometry – Algebra II sequence, in the first three years of high school. There is a clear emphasis on the importance of algebra for the study of geometry. So, how do students think in school geometry when solving problems? Furthermore, another

question to ask is: Does the van Hiele theory still hold in a context where school geometry has changed considerably to accommodate various algebraic approaches?

This paper is the outcome of a study on students' use of algebraic thinking in geometry at high school level. The research did not specifically focus on levels of geometric thinking but focused on how students used the following three forms of algebraic thinking in high school geometry: symbols and algebraic manipulation, different forms of representation, and generalization. The emphasis is on issues about what constitutes geometric thinking and the need to conceptualize geometric thinking within a broader framework that is inclusive of algebraic thinking in school geometry.

5. Methodology

This qualitative study took place over a three-month period during the first semester of the academic year in two large Midwestern rural high schools in the United States. One geometry class (post-Algebra I) was selected from each high school (school X and school Y). Class A from school X had 21 students and class B from school Y had 18 students. Two tests were administered to the students from these two classes: an algebra test (constructed by the researcher) and a van Hiele test (developed by Usiskin at the University of Chicago, 1982). The algebra test was finalized based on comments from the two classroom teachers and three other experts in the field. The van Hiele test from the Usiskin study has met with criticism (see Crowley, 1990; Wilson, 1990). However, it was deemed relevant for the particular purpose of selecting the focus students in this study and so was not modified. Based on their performances on the two tests, three students were selected from each of the two classes: Anton, Beth, and Mary from class A in school X and Kelly, Phil and Ashley from class B in school Y. It is to be noted that Anton was the only student in the sample with the highest assigned van Hiele level whereas Kelly was the student with the highest algebra test score (27 out of 30). Mary had the lowest algebra test score (5 out of 30) whereas none of the students were assigned a van Hiele level 0.

Table 1
Focus Students

School	Students	Algebra Score (out of 30)	Van Hiele Level (scale 0 – 4)
	Anton	23	4
School X	Mary	5	1
	Beth	13	1
	Kelly	27	3
School Y	Ashley	17	1
	Phil	24	1

The six focus students were interviewed four times for about 40 minutes each time. During these interviews the focus students were asked to solve sets of problems which involved the use of algebra in geometry. The problems were finalized based on the schools' mathematics programs with the help of three experts in the field. These problems included the use of variables and unknowns, the writing and solution of simple linear equations, the writing and solution of linear simultaneous equations in two unknowns, the substitution of values in expressions, and the recall and use of formulae within geometry. Besides, the two classrooms were observed for about three months and 12 lessons from each class were videotaped. Artifacts, such as tests, quizzes, and homework of the focus students were also collected. The two teachers from these two classrooms were interviewed twice for about 30 minutes each time.

6. Discussion

It is not possible to describe in detail how each of the focus students attempted the set of tasks in geometry, as the research had a different focus and was not specially looking at levels of thinking in geometry. So, in what follows, a general description is given about how these students with varying levels of performance on the algebra and van Hiele tests solved the problems involving algebraic thinking in geometry.

Three aspects of algebraic thinking were investigated: the use of symbols and algebraic relations, the use of representations, and the use of generalizations within geometrical contexts. Four of the focus students were assigned van Hiele Level 1 (scale 0-4): Mary, Beth, Phil, and Ashley. In comparison to each other, these students showed different dispositions towards the use of algebraic thinking in geometry. Mary and Beth had difficulties in the use of each of the three aspects of algebraic thinking enumerated above. They encountered difficulties when working with variables and unknowns, and writing and solving equations. They also found the use of different representations difficult. They could work with some linear geometric patterns with some help but not with non-linear ones. However, they could remember some of the formulae, like the one for finding distance between two points in the coordinate plane. Mary, who had the lowest score on the algebra test (5 out of 30), struggled in most of the problems that she was asked to solve. She had a dislike for mathematics because she did not like the many formulae she had to remember. In general, Mary and Beth had a more instrumental approach to the use of algebraic thinking in geometry.

On the other hand, Phil and Ashley (with algebra test scores of 24 and 17 out of 30 respectively) were better than Mary and Beth at using the different forms of algebraic thinking. They could confidently work with variables and unknowns. It was interesting to note that Ashley did quite well in most of the interview problems that she tackled. Phil and Ashley had a more positive approach to problem solving and a quest for finding the solution, compared to Mary and Beth. However, both Phil and Ashley were not good at remembering formulae. In this episode with the interviewer (R), Ashley is solving the problem: *If an isosceles triangle has two sides of lengths 10 cm and 4 cm, what will be its perimeter?*

Ashley: So...it will be 10, ... [*she mumbles and draws a triangle showing the sides as 10, 10, 4*]

- R: So what would be the perimeter in that case?
 Ashley: Just 24.
 R: Ok. What could it be otherwise?
 Ashley: 18
 R: Can you show it on a diagram?
 Ashley: Hmm...[*she draws some diagrams*]
 R: So now, are both of these possible, only one of them is possible, what do you think?
 Ashley: Hmm...this one couldn't really be possible [*referring to the 4, 4, 10 triangle*] because it is 10. If you add the two sides it does not give 10.
 R: In other words the answer will be ..what?
 Ashley: 24.
 R: Alright...and suppose I give you a triangle having sides a , b , and c ...[*I draw a triangle ABC with sides a, b, c*] what condition must be satisfied by a , b , and c ?

 R: So if I have the sides of lengths a , b , and c what condition should be satisfied then?
 Ashley: So $a + b \geq c$.
 R: Can you write it down?
 A: [*she writes $a + b \geq c$*] ...and the others...[$b + c \geq a$, $c + a \geq b$, *she writes these statements without any prompts*]
 R: When you have an 'equal to' sign what will happen?
 Ashley: It will be equilateral?...I don't know..
 R: Hmm...suppose I had like 10, 5, and 5 what would happen in that case? [*I use a line segment of length 10 and illustrate the condition*]
 Ashley: Oh.. that wouldn't be a triangle.. it would be just a line.
 R: Ok...

Although, Ashley had a fairly low algebra score, she was quick to generalize to a triangle with sides of lengths a , b , and c . In comparison, Anton who had the highest assigned van Hiele level and a fairly high algebra score had major difficulties with this problem.

Anton (Level 4) and Kelly (Level 3) were the two focus students with the highest van Hiele levels and fairly high algebra scores (23 and 27 out of 30 respectively). In general they were much better in using algebraic thinking in geometry than the other focus students, except possibly Ashley. They could work with variables, unknowns, and equations with greater confidence. But they did have a few difficulties as well. It is worth noting that Anton and Kelly were not very good at remembering formulae. They seemed to be generally more relational or conceptual in their use of algebraic thinking. In this episode with the interviewer (R), Kelly is asked to solve the problem: *The supplement of an angle is four times its complement. What is the angle?*

- R: So if you don't know the angle how do you start the problem?
 Kelly: With x ?
 R: Ok...let's try so if x is the angle. First, what would be its complement?

- Kelly: ...is it $90-x$?...I don't know...
- R: Hmm... like for 40 it was 50 isn't it? For x it will be...*[she writes $90-x$]*. Ok. And what will be the supplement?
- Kelly: $180-x$?
- R: Ok. So one of them is four times the other which one is four times the other?
- Kelly: Four times its complement is the supplement.
- R: Ok, alright. Can you write an equation from there?
[she writes $180 - x = 4(90-x)$] Can you solve it? *[she eventually solves the problem but has some difficulties with the algebra.]*

Kelly initially had a difficulty understanding the geometrical concepts supplement and complement of an angle. She had no difficulty in setting up the equation correctly. However, she did seem to have a few difficulties in reaching the solution. On the other hand, Anton did not do well in this problem. He proceeded to work on the problem as he described it below:

- Anton: Let us see, supplement of an angle. Let's just say S for the supplement and minus A for angle equals four times C , the complement. *[he writes $S-A = 4C$]*.. hmmm so $S = 180$... no wait we are talking about the complement. I keep thinking that the total...hmmm but $A + S = 180$ and $A + C = 90$ so $4C$ would equal $S-A$. So $4C + A =$ supplement *[he writes $4C + A = S$]*. So if you take this ...

Anton was completely lost with his use of variables and did not recover from his initial slip to set up the right equation to get a solution. He knew what the complement and the supplement of angle stood for. Thus Anton, who had the highest van Hiele score and understood the geometrical concepts, had difficulties with the algebra in this question.

The results of this study show that algebraic thinking has strong connections to thinking in geometry. There is a significant amount of algebra in the geometry curriculum at high school level. Hence, students studying geometry need to be well prepared in algebra. The use of tests such as the van Hiele test targets students' thinking in an exclusively Euclidean context and tend to give a limited view of students' thinking in school geometry, which incorporates a significant amount of algebra. It is important to have tests for assessing students' thinking in geometry that would include the use of algebraic thinking as well. Phil and Ashley were both assigned van Hiele level 1, but they were both quite versatile in their use of algebra in problems requiring algebraic thinking in geometry. On the other hand, Anton was assigned the highest van Hiele level of four, and he had a few difficulties working on the selected problems. Hence, the van Hiele levels of thinking in geometry should be interpreted with greater care.

Most researchers agree that there is a certain hierarchic development of cognition as far as geometry is concerned and the van Hiele levels provide a valuable framework for studying geometric thinking. Even studies in non-Western contexts have provided support for the van Hiele theory. For example, De Villiers and Njisane (1987), working with high school students in South Africa, investigated how eight different Geometric Thinking Categories (GTCs) corresponded with the van Hiele model. The categories were: recognition and representation of figure types; visual recognition of properties; use and understanding of terminology; verbal description of a figure (or its recognition for a verbal description); one step deduction; longer

deduction; hierarchical classification; and reading and interpreting given definitions. They found that *hierarchical classification* was the most difficult GTC for pupils. Roughly they concluded that the first two GTCs corresponded to the first van Hiele level, the next two to the second van Hiele level and the next two to the third van Hiele level.

However, research has also shown that there exist some concerns regarding this theory. For example, Pandiscio and Orton (1998) have claimed that one of the weaknesses of the van Hiele theory is that it appears to lack generality and thus each strategy may need to be revised for different content domains. On the other hand, Senk (1989) claimed that van Hiele did not acknowledge the existence of a “nonlevel”; instead she asserted that all students entered geometry at ground level, which is Level 0 (scale 0 – 4), with the ability to identify common geometric features by sight.

Another issue that some researchers (Gutiérrez, Jaime, & Fortuny, 1991) have brought forth is that a student can possibly develop two consecutive van Hiele levels of reasoning at the same time. They found that, depending on the complexity of the problem, students used several levels of reasoning. However, they claimed that this was not to be interpreted as a rejection of the hierarchical structure of the van Hiele theory, but rather that the theory should be adapted to the complexity of human reasoning processes. People do not behave in a simple, linear manner, which the assignment of one single level would lead us to expect.

Regarding proof in geometry, research has shown that high school students' achievement in writing geometric proofs is positively related to the van Hiele levels of geometric thought and to achievement on standard nonproof geometry content (see Senk, 1989). According to the van Hieles, students below Level 2 (scale 0-4) should not be able to do proofs at all other than memorization; students at Level 2 might be able to do short proofs based on empirically derived premises; but only students at Levels 3 or 4 should be expected to write formal proofs consistently (Levels are based on a scale of 0 to 4). Senk (1989) claimed that this is only partially supported by her research. Students at Level 2 or higher substantially outperformed students at lower levels; however when concurrent knowledge of standard nonproof content was controlled, students at Level 3 or 4 did not score consistently higher than those at Level 2.

The van Hieles developed their levels of thinking in geometry while the secondary geometrical content was mostly Euclidean, but the advent of the mathematics reform in the 1960s changed the face of geometry. Different types of geometries were introduced such as coordinate, vector, and transformational. This in turn introduced a significant amount of algebraic manipulation in high school geometry. Do the van Hiele levels of thinking still hold in a geometrical context where algebra plays a significant role?

With a view to give an alternative framework for studying geometric thinking, researchers have proposed combining the van Hiele theory with other popular theories. For example, Olive (1991) analyzed the Logo work of 30 ninth-grade students from three different theoretical perspectives: the van Hiele levels of thinking, the SOLO taxonomy (Structure of the Observed Learning Outcomes from Biggs, & Collis, 1982), and Skemp's (1987) model of mathematical understanding. Pandiscio and Orton (1998), in their theoretical paper, argued for a synthesis of van Hiele's and Piaget's perspectives. Pegg and Davey (1998) have also argued for a synthesis of

the van Hiele and SOLO models for research in geometry. Some studies have looked at other cognitive aspects of learning geometry. For example, Chinnapan (1998) examined the nature of prior mathematical knowledge that facilitates the construction of useful problem representations in geometry. Lawson and Chinnapan (2000), for instance, explored the relationship between problem-solving performance and the organization of students' knowledge. They reported findings on the extent to which content and connectedness indicators differentiated between high- and low-achieving groups of students undertaking geometrical tasks.

7. Conclusion

There is agreement among researchers about a hierarchic development of geometrical thinking. The van Hiele theory provides a strong framework. However, research has also shown that this theory has some limitations. Amongst others, there have been concerns about the distinctness of the levels of thinking and the possibility of different levels in different topic areas. The present study adds another dimension to this issue, namely that the levels of thinking in geometry cannot ignore the significant connection between algebra and geometry.

The argument that students who were assigned varying van Hiele levels performed differently on the problems in geometry that required algebraic thinking, rests on the validity of the van Hiele test and a broader definition of school geometry. The instrument from the Usiskin (1982) uses only multiple choice items. This instrument is known to have been criticized by several researchers, such as Crowley (1990) and Wilson (1990). However, this instrument provided base-line data for selecting the focus students. In future studies, a more conclusive test that can combine written tests and interviews, may possibly provide better information.

Besides, the connections between algebra and geometry, levels of thinking in school geometry may also be influenced by technology. Various types of Dynamic Geometry Software (DGS) are now available for students to explore geometrical concepts. In addition, earlier work with LOGO has shown that students with LOGO experience gained more than control students in geometry (Scally, 1987). As such there are various avenues to explore when considering geometric thinking. There is still a strong interest in how students think in geometry. It is essential that researchers come up with models or frameworks that address some of the shortcomings of the van Hiele theory or possibly come up with a totally new framework. This study simply reports on the need for such a framework.

References

- Biggs, J., & Collis, K. (1982). *Evaluating the quality of learning: The SOLO taxonomy*. New York: Academic Press.
- Burger, W. F., & Shaughnessy, M. (1986). Characterizing the van Hiele levels of development in geometry. *Journal for Research in Mathematics Education*, 17(1), 31-48.
- Chinnapan, M. (1998). The assessing of geometry schemas by high school students. *Mathematics Education Research Journal*, 10(2), 27-45.

- Clements, D. H., & Battista, M. T. (1992). Geometry and spatial reasoning. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 420-464). New York: Macmillan.
- Crowley, M. L. (1987). The van Hiele Model of development of geometric thought. In M. M. Lindquist, & A. P. Shulte (Eds.), *Learning and teaching geometry, K-12, 1987 Yearbook* (pp. 1-16). Reston, VA: National Council of Teachers of Mathematics.
- Crowley, M. L. (1990). Criterion-referenced reliability indices associated with the van Hiele Geometry Test. *Journal for Research in Mathematics Education*, 21(3), 238-241.
- Gutiérrez, A., Jaime, A., & Fortuny, J. M. (1991). An alternative paradigm to assess the acquisition of van Hiele levels. *Journal for Research in Mathematics Education*, 22(3), 237-251.
- De Villiers, M. D., & Njisane, R. M. (1987). The development of geometric thinking among high school pupils in Kwazulhu. In J. C. Bergeron, N. Herscovics, & C. Kieran (Eds.), *Proceedings of the 11th International Conference for the Psychology of Mathematics Education* (Vol. 3, pp. 117-123). Montreal, Canada: Université de Montréal.
- Duval, R. (2002). Representation, vision, and visualization: Cognitive functions in mathematical Thinking. Basic Issues for Learning. In F. Hitt (Ed.), *Representations and mathematics visualization* (pp. 311-336). Mexico: PME-NA-Cinvestav-IPN.
- Groth, R. E. (2003). High school students' levels of thinking in regard to statistical study design. *Mathematics Education Research Journal*, 15(3), 252-269.
- Herbert, K., & Brown, R. (1999). Patterns as tools for algebraic thinking. In B. Moses (Ed.), *Algebraic thinking: Grades K - 12* (pp. 123-128). Reston, VA: National Council of Teachers of Mathematics.
- Hoffer, A. (1981). Geometry is more than proof. *Mathematics Teacher*, 74, 11-18.
- Hoffer, A. (1983). Van Hiele-based research. In R. Lesh, & M. Landau (Eds.), *Acquisition of mathematics concepts and processes* (pp. 205-227). New York: Academic Press.
- Jones, G. A., Langrall, W. C., Thornton, C. A., & Mogill, T. (1999). Students' Probabilistic Thinking in Instruction. *Journal for Research in Mathematics Education*, 30(5), 487-519.
- Lawson, M. J., & Chinnappan, M. (2000). Knowledge connectedness in geometry problem solving. *Journal for Research in Mathematics Education*, 31(1), 26-43.
- Mooney, E. S. (2002). A framework for characterizing middle school students' statistical thinking. *Mathematical Thinking and Learning*, 4(1), 23-63.
- National Council of Teachers of Mathematics. (1989). *Curriculum and evaluation standards for school mathematics*. Reston, VA: Author.
- National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: Author.
- Olive, J. (1991). Logo programming and geometric understanding: An In-depth study. *Journal for Research in Mathematics Education*, 22(2), 90-111.
- Pandiscio, E., & Orton, R. E. (1998). Geometry and metacognition: An analysis of Piaget's and van Hiele's perspectives. *Focus on Learning Problems in Mathematics*, 20(2&3), 78-87.
- Pegg, J., & Davey, G. (1998). Interpreting student understanding of geometry: A synthesis of two models. In R. Lehrer & D. Chazan (Eds.), *Designing learning environments for developing understanding of geometry and space* (pp. 109-135). Mahwah, NJ: Lawrence Erlbaum Associates.

- Scally, S. P. (1987). The effects of learning Logo on ninth grade students' understanding of geometric relations. In J. C. Bergeron, N. Herscovics, & C. Kieran (Eds.), *Proceedings of the 11th International Conference for the Psychology of Mathematics Education* (Vol. 2, pp. 46-52). Montreal, Canada: Université de Montréal.
- Senk, S. L. (1989). Van Hiele levels and achievement in writing geometry proofs. *Journal for Research in Mathematics Education*, 20(3), 309-321.
- Skemp, R. R. (1987). *The psychology of learning mathematics*. Hillsdale, NJ: Lawrence Erlbaum Associates.
- Usiskin, Z. (1982). *Van Hiele levels and achievement in secondary school geometry* (Final report of the Cognitive Development and Achievement in Secondary School Project). Chicago, IL; University of Chicago, Department of Education. (ERIC Document Reproduction Service No. ED 220 288).
- Van Hiele, P. M. (1986). *Structure and insight*. Orlando, FL: Academic Press.
- Wilson, M. (1990). Measuring a van Hiele geometry sequence: A reanalysis. *Journal for Research in Mathematics Education*, 21(3), 230-237.

Numerical Methods with MS Excel

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Abstract: In this note we show how MS Excel can be used to to perform numerical Integration, specifically Trapezoidal Rule and Simson's rule. Futhermore, we illustrate how to generate Lagranges Interpolation polynomial.

1. Introduction

MS Excel is the most commonly used spreadsheet, and has now grown into powerful software that can be used virtually by all branches of science and engineering. The availability of the program in almost all PCs makes its usage appears to be at the increase. The program has been use in teaching and solving many mathematical problems in many different ways. And this usage ranges from lower level to advanced level courses. The nature of the program makes numerical methods much easier to be implemented. However, since in Excel the users assume more responsibility in designing the application, and are in full control of the implementation, the program requires certain degree of creativity.

In this paper, we intend to illustrate how teachers and students can use Excel to implement three well-known numerical methods: Simpson's Rule, Trapezoidal Rules and Lagrange's interpolation. It should be noted that using Excel in addressing the first two integration techniques is not new (see references), however, our approach is much simpler, more direct, and do not require any *macros* for execution. In addition, our worksheet needs no modifications ones is developed, except for entering the interval of integration and the number of divisions (≤ 100). This gives us motivation to share our experience with larger mathematics education community.

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The motivation for Lagrange's interpolation stems from a question asked to Dr. Math in MathForum.org (<http://mathforum.org/library/drmath/view/63984.html>). The questioner wanted to know among other things how to use Excel to work out Lagrange's interpolation. Although the two other questions were answered sufficiently, the Excel implementation was skipped by Dr. Math. So in this note, we plan to answer that question as well as share our experience with larger mathematics community.

2. Numerical Integration with Excel

In this section we illustrate how Excel worksheets can be used to implement the trapezoidal rule and the Simpson's rule for numerical integration. The problem is to find a numerical approximation for the integral

$$I = \int_a^b f(x)dx .$$

2.1 The trapezoidal rule

The trapezoidal rule works by approximating the function $f(x)$ by a piecewise linear function and evaluate the integral of each piece. If the interval $[a, b]$ is divided up into n equal subinterval, each of width $h = \frac{b-a}{n}$, then the approximate integral is

$$I \approx \frac{h}{2} \sum_{i=1}^n f(x_{i-1}) + f(x_i), \text{ where, } x_i = a + ih, \text{ and } i = 0, 1, \dots, n$$

We illustrate with the following example

$$\int_0^1 (14x^6 + 7)dx.$$

The method is explained as follows. The endpoints (initial and terminal) of the interval, and the number of divisions are entered in the cells A2, B2, C2, respectively. The value of h is calculated in the cell D2 by entering the formula $= (B2-A2)/C2$.

To generate the x_i s we take the following steps:

1. In cell E2 enter the formula $=A2$. This copies the value of $a = x_0$ into E2. The following figure shows the upper part of the Excel worksheet implementation of the method.

	A	B	C	D	E	F	G	H
1	a	b	n	h	x_i	$f(x_i)$	A_i	
2	0	1	50	0.02	0	7		
3					0.02	7	0.14	
4					0.04	7	0.14	
5					0.06	7	0.14	
6					0.08	7	0.14	
7					0.1	7	0.14	
8					0.12	7	0.14	
9					0.14	7	0.14	
10					0.16	7	0.14	
11					0.18	7	0.14	
12					0.2	7.001	0.14	
13					0.22	7.002	0.14	
14					0.24	7.003	0.14	
15					0.26	7.004	0.14	
16					0.28	7.007	0.14	

2. The next values are generated with the formula $=IF(E2>= \$B\$2, \$B\$2, E2+ \$D\$2)$ in E3. This formula adds h to the previous value until we reach the value of b . Afterwards it keeps entering the value of b . This mechanism is used to enable changing the value of n to get more control on the accuracy of the solution as explained in Step 5.
3. Copy this formula to the next 100 cells or so below E4. The below figure shows a part of the sheet further down, where you can see the value of b being repeated.
4. The function f is entered in the column labeled $f(x_i)$ by entering the formula $=7+14 * E2^6$ in the cell F2 and copying it along the corresponding cells for the x_i s.
5. We then form the elements of the summation in the trapezoidal rule by entering the formula $= (E3-E2)/2*(F2+F3)$ in cell G3 and copying it along the corresponding cells for the x_i s. Observe that, instead of using the value of h generated in cell D2, we used the equivalent difference $E3-E2$. This has two advantages:
 - a) The formula produces zeros when we go past the right endpoint b . In this way, the final sum of these numbers is not affected by the repetition of b .
 - b) It allows the use of the trapezoidal rule with non-uniform divisions of the interval $[a, b]$.
6. The last step is to add the terms in column G to get the approximation of the integral. Select the range of cells that contains the summation terms and then click the sum button (\sum) on the toolbar. The result is shown below.

	A	B	C	D	E	F	G	H	I
96					1	21	0		
97					1	21	0		
98					1	21	0		
99					1	21	0		
100					1	21	0		
101					1	21	0		
102					1	21	0		
103							9.003		

Note: (a) If you now change the number of divisions n to 100, the new, more accurate approximation will appear in the same cell (G103).

(b) To change the interval of integration all you need to do is to change the values a, b in cells A2, B2.

(c) To change the integrated function enter the new formula in cell F2 and copy it to cell F103.

2.2 Simpson's rule

Simpson's rule finds an approximation of the value of the integral I by approximating the integrand with a piecewise polynomial of degree 2 and then evaluate the integral over each piece. Simpson's formula is

$$I \approx \frac{h}{3} \sum_{i=1}^{n-1} [f(x_{i-1}) + 4f(x_i) + f(x_{i+1}))],$$

Where the interval $[a, b]$ is divided into n intervals, each of length $h = \frac{b-a}{n}$. The Excel implementation of Simpson's rule is very much similar to that of the trapezoidal rule, except for some details. The following figure shows the upper part of the worksheet implementation.

	A	B	C	D	E	F	G	H	I
1	a	b	n	h	xi	f(xi)	Si		
2	0	1	40	0.03	0	7			
3					0.03	7	0.35		
4					0.05	7			
5					0.08	7	0.35		
6					0.1	7			
7					0.13	7	0.35		
8					0.15	7			
9					0.18	7	0.35		
10					0.2	7			
11					0.23	7	0.35		
12					0.25	7			
13					0.28	7.01	0.35		
14					0.3	7.01			
15					0.33	7.02	0.35		
16					0.35	7.03			

As you can see from this figure the table is exactly the same as that for the trapezoidal rule except for the last column. The last column is generated as follows. In the cell G3 enter the formula

$$=(E4-E2)/6*(F2+4*F3+F4).$$

Since Simpson's formula spans two subintervals for each entry in the summation, the copying of the formula is done as follows. Select the range of two cells G2, G3 (note that G2 is actually empty). Using the formula copying technique, drag the two cells down to cell G103. The result is that the formula is copied to every other cell. One more thing to notice here is that, in the above formula, the value of h is replaced by the difference over 3 cells divided by 2. This way the same skipping is achieved and no problem arises as a result of repeating the values of b . When you select the range G2:G103 and click the sum button, you will see the result 9.000003644, which is more accurate than the result of the trapezoidal rule as the theory predicts.

3. Lagrange Interpolation

The idea of interpolation as succinctly elaborated in the site (<http://mathforum.org/library/drmath/view/63984.html>) is to find a function of a specified form which passes through a given list of points. Lagrange interpolation uses polynomials. For instance, it is well known that given two points (x_1, y_1) and (x_2, y_2) , there is exactly one line (degree one polynomial) that can be generated from these two points. It is also true that given three distinct points, there are two possibilities, either the three points lie in a line, or else there is exactly one quadratic polynomial that the three points pass through. If you have four points, there will be a cubic (at most) polynomial through them; if you have 10 points, there will be a polynomial of degree at most 9 through them. In general, given a list of n distinct points $\{(x_1, y_1), (x_2, y_2), (x_3, y_3), \dots, (x_n, y_n)\}$, there will be a polynomial of degree at most

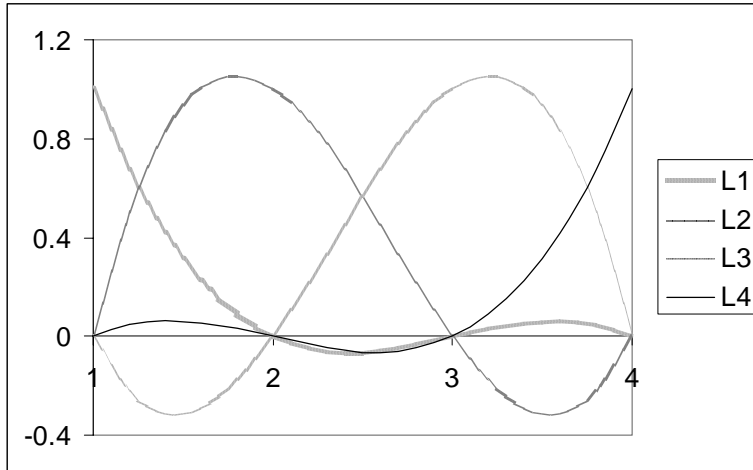
($n-1$) passing through the given points. In this section, we illustrate how one can use Excel to generate this polynomial. The steps are as follows:

1. Mark a range (say R(2,3):R(2,n+2)) as $x_1, x_2, x_3, \dots, x_n$ (this is a row of n cells starting at C2).
2. Mark a column (say A4:An+3 (R(4,1):R(n+3,1))) as $x_1, x_2, x_3, \dots, x_n$.
3. In the range R(3,3):R(3,n+2) enter the values of the interpolation points x_k from the interval $[a, b]$.
4. In the column B4:Bn+3, that is (R(4,2):R(n+3,2)) enter the values of the interpolation points x_k from the interval $[a, b]$.
5. In the cell C4 (i.e., R(4,3)) enter the formula = IF(C\$3-\$B4<>0;C\$3-\$B4;1). Notice that row 3 and column B are fixed in this formula.
6. Copy the formula through the range R(4,3):R(n+3,n+2). This should produce the differences $(x_i - x_j)$ for all i, j but replacing the difference $x_j - x_j$ by one instead of its true value 0. The net effect is to cancel the calculation of $x_j - x_j$.

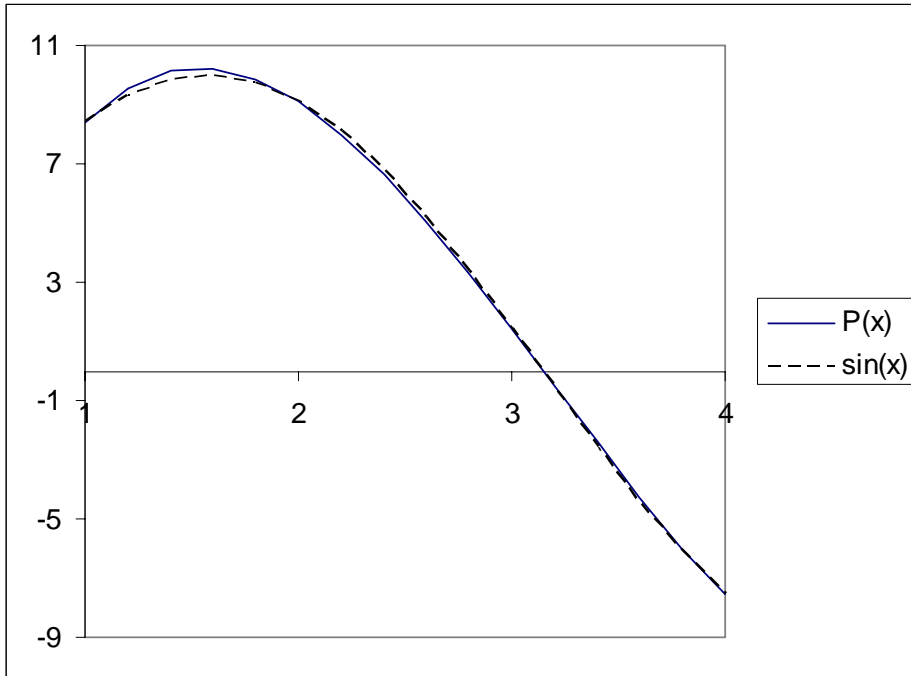
Interpolation points		x1	x2	x3	x4	products (x-xj) ->
		1	2	3	4	
x1	1	1	1	2	3	
x2	2	-1	1	1	2	
x3	3	-2	-1	1	1	
x4	4	-3	-2	-1	1	
Prod(xj-xi) ->		-6	2	-2	6	
x values	1	0	-1	-2	-3	0
	1.2	0.2	-0.8	-1.8	-2.8	-0.8064
	1.4	0.4	-0.6	-1.6	-2.6	-0.9984
	1.6	0.6	-0.4	-1.4	-2.4	-0.8064
	1.8	0.8	-0.2	-1.2	-2.2	-0.4224
	2	1	0	-1	-2	0
	2.2	1.2	0.2	-0.8	-1.8	0.3456
	2.4	1.4	0.4	-0.6	-1.6	0.5376
	2.6	1.6	0.6	-0.4	-1.4	0.5376
	2.8	1.8	0.8	-0.2	-1.2	0.3456
	3	2	1	0	-1	0
	3.2	2.2	1.2	0.2	-0.8	-0.4224
	3.4	2.4	1.4	0.4	-0.6	-0.8064
	3.6	2.6	1.6	0.6	-0.4	-0.9984
	3.8	2.8	1.8	0.8	-0.2	-0.8064
	4	3	2	1	0	0

7. In the cell R(n+4,3) enter the formula = Product(R(4,3):R(n+3,3)). The selection of the argument of the product sheet function can be done by dragging with the mouse. Copy the formula in R(n+4,3) through the range R(n+4,3): R(n+4,n+2) by dragging with the mouse. This gives the products that go in the denominators of the Lagrange basis functions.
8. Generate the set of x values that you want to compute at in the range R(n+5,2):R(n+m+5,2) from the interval $[a, b]$ by any method of sequence generation.
9. In the cell R(n+5,3) enter the formula =R(n+5,\$2)-R(\$3,3). Drag this formula through the range R(n+5,3):R(n+m+5,n+2). this generates a table of the values of $(x - x_i)$, $1 \leq i \leq n$.
10. In the cell R(n+5,n+3) enter the formula = Product(R(n+5,3):R(n+5,n+2). Drag this formula through the range R(n+5,n+3):R(n+m+5,n+3).
11. Now we are ready to generate the values of the Lagrange basis functions at the points x chosen in step 7 . Label the cells (should be blank) R(n+4,n+4):R(n+4,2n+3) as L_1, L_2, \dots, L_n .

L1	L2	L3	L4	f(xi)	P(x)	sin(x)
1	0	0	0	8.41471	8.41471	8.41471
0.672	0.504	-0.224	0.048	9.092974	9.55817	9.320391
0.416	0.832	-0.312	0.064	1.4112	10.14123	9.854497
0.224	1.008	-0.288	0.056	-7.56802	10.22038	9.995736
0.088	1.056	-0.176	0.032		9.852127	9.738476
0	1	0	0		9.092974	9.092974
-0.048	0.864	0.216	-0.032		7.99942	8.084964
-0.064	0.672	0.448	-0.056		6.627964	6.754632
-0.056	0.448	0.672	-0.064		5.035109	5.155014
-0.032	0.216	0.864	-0.048		3.277354	3.349882
0	0	1	0		1.4112	1.4112
0.032	-0.176	1.056	0.088		-0.50685	-0.58374
0.056	-0.288	1.008	0.224		-2.4203	-2.55541
0.064	-0.312	0.832	0.416		-4.27265	-4.4252
0.048	-0.224	0.504	0.672		-6.00739	-6.11858
0	0	0	1		-7.56802	-7.56802



12. In the cell $R(n+5,n+4)$ enter the formula $\text{If}(R(n+5,3)=0;1;R(n+5,\$n+3)/R(n+5,3)/R(\$n+4,3))$. Drag this formula through the range cell $R(n+5,n+4):R(n+m+5,2n+m+4)$. This generates the values of the Lagrange basis functions at the m chosen points x .
13. The final stage is to generate the Lagrange polynomial. Label the cells $R(n+4,2n+4)$ and $R(n+4,2n+5)$ as $f(x_i)$ and $P(x)$ respectively.
14. In the range $R(n+5,2n+4):R(2n+4:2n+4)$ enter the values of $f(x_i)$, $i = 1, 2, \dots, n$.
15. Select the range of cells (empty by now) $R(n+5,2n+5):R(n+m+5,2n+5)$ and enter the formula $=\text{MMULT}(R(n+5,n+4):R(n+m+5,2n+m+4); R(n+5,2n+4):R(2n+4:2n+4))$. This produces the values of $P(x)$ at the m points x .
16. The values of the original function can be produced next and then $f(x_i)$ and $P(x)$ can be plotted against x on the same graph.



References

<http://oregonstate.edu/instruct/ch490/lessons/lesson11.htm>

<http://oregonstate.edu/~haggertr/487/integrate.htm>

<http://www.math.villanova.edu/connections/Excel/>

<http://mathforum.org/library/drmath/view/63984.html>

Can our Learners Model in Mathematics?

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Abstract: Mathematical modeling of real world conditions should be part of mathematics classroom activities. In this paper I argue that when real world problems are taught at schools learners are not able to cope on their own, without the assistance of their educator. There is very little or no emphasis placed on this aspect of mathematics at schools, although it is just beginning to make an appearance in our new Outcomes Based Curriculum. I also discuss an experiment conducted with Grade 10 learners (15 year old) and their responses to real world problems and the conditions that need to be considered. There is ample evidence that a lot of work on mathematical modeling of real world problems has been done elsewhere in the world, but not much has been done in South Africa. This experiment was fully conducted using Sketchpad as a mediating tool. This in itself was a difficult task because our learners have not been really exposed to dynamic geometry environments.

1. The Teaching Experiment

This paper reports on a modeling experiment that was conducted, to teach concepts such as perpendicular bisectors and concurrency, with grade 10 learners, using a pseudo real world problem. The learners selected for this research were from a school in Chatsworth in KwaZulu-Natal, South Africa. These learners came from lower to middle income families and were all average to below average performers in their school mathematics examinations and tests. In this particular school, very little or no modeling was done with them, although certain word problems may have, to a small extent, required some modeling activity. The learners themselves were selected randomly from a computer studies class by their computer studies educator. The problem given to them was contextualized within the South African rural background. The question was:

In a developing country like South Africa, there are many remote villages where people do not have access to safe, clean water and are dependent on nearby streams or rivers for their water supply. With the recent outbreak of cholera in these areas, untreated water from these streams and rivers has become dangerous for human consumption. Suppose you were asked to determine the site for a water reservoir and purification plant so that it would be the same distance away from four remote villages. Where would you recommend the building of this plant?

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2. The learners' ability to create and use mathematical models to solve the specific real world problem

Rather than immediately starting with *Sketchpad* (a dynamic geometry software), the researcher first asked the learners to attempt to find a solution on their own, using any previous knowledge. All learners "guessed" a solution somewhere in the "middle" of the quadrilateral, but none could find a precise solution. More importantly, few of them even thought of or tried to test their solution. This extract from the interview with Pravanie (see below) was typical of several interviews. The learners seemed to feel that this type of question was not within their ability to solve it and in some cases the learners explicitly said that this was because this type of question had never been taught or asked of them. This is a very significant point. Learners seemed to feel that they could not solve problems not seen before. Moreover it seemed that a real-world context such as this was rather novel to them.

- RESEARCHER** Where do you think that we should build the reservoir?
PRAVANIE I don't know ... all we are only given is this diagram ...
RESEARCHER Do you think that you will be able to find the most suitable point?
 You can use any method you know, to do so.
PRAVANIE I don't know sir ... this is too difficult ... please don't interview me
 (pleading).
RESEARCHER Are you saying that you cannot find any way of solving this
 problem?
PRAVANIE I can't ... I'm not so good in maths ... maybe at the centre here
 (pointing to the middle of the quadrilateral).
RESEARCHER Will you be able to justify your answer? Will you be able to tell me
 why?
PRAVANIE (silence)..... not really ...
RESEARCHER Don't you even want to try?
PRAVANIE I don't know what to do ...

From the above it is clear that Pravanie was very uncomfortable being faced with an unknown problem not seen before, and that she seemed to have been so intimidated that she did not even want to continue with the interview. She seems a typical product of the traditional approach where learners acquire a "learned helplessness" that is, an unwillingness to attempt problems on their own.

Christina's approach though was different (see below). Although she could not determine a precise way of finding a satisfactory solution, she was initially prepared to use pencil and ruler. After she discovered that using the pencil and ruler only gave an approximate solution she simply gave up.

- RESEARCHER** Where do you think that we should build the reservoir?
CHRISTINA Can I use my ruler and pencil?
RESEARCHER You can use any method you know, to do so.
CHRISTINA (after a while) I thought it will be here ... (pointing to the middle)
 ...(measuring the distances) ... but when I measure the distances,
 it's not the same.
RESEARCHER Are you saying that you cannot find any way of solving this
 problem?

CHRISTINA I think it is in the middle ... maybe at the centre here (pointing to the middle of the quadrilateral but not continuing further).

When asked to explain why she thought the solution was somewhere in the middle, she emphatically refused to provide an explanation or to continue looking for a better solution.

RESEARCHER Will you be able to justify your answer? Will you be able to tell me why?

CHRISTINA No! (emphatic)

RESEARCHER Don't you want to try?

CHRISTINA No!

Faeza displayed good self-confidence initially (see below). Her comments such as : “*It's easy to understand*” and “*Ya ... can I measure with my ruler?*” showed that she felt that she was capable of solving the problem. Although she was sure that she could prove that her guess (somewhere in the middle of the quadrilateral) was correct by measuring, she suddenly lost some of her confidence when she discovered that her guess was not correct. Yet again, just like the others, she could not think of another method. It is interesting to note that the learners became irritated when prompted by the researcher to try and find another method. Faeza's teacher dependence acquired from traditional teaching is also clearly highlighted by her question “... *what method should I use?*”

RESEARCHER Faeza, you've read the question, do you understand it?

FAEEZA It's easy to understand.

RESEARCHER Where do you think that we should build the reservoir?

FAEEZA About here (pointing to the centre of the quadrilateral).

RESEARCHER Can you prove to me that this is the correct point?

FAEEZA Ya ... can I measure with my ruler?

RESEARCHER Yes you may.

FAEEZA (After a while) I don't think I'm measuring correctly ...am I? ...

RESEARCHER Do you want to perhaps try another method?

FAEEZA I don't know ... what method should I use?

RESEARCHER Do you know of any method?

FAEEZA No!

RESEARCHER Are you sure?

FAEEZA I really don't know (irritably).

Similarly, Nigel, Roxanne and Schofield had no idea about what they should be doing.

RESEARCHER Where do you think that we should build the reservoir?

NIGEL There (pointing to the middle of the quadrilateral).

RESEARCHER Can you prove that your answer is correct?

NIGEL No, I can't.

RESEARCHER Are you saying that you cannot find **any** way of proving this?

NIGEL I can't.

RESEARCHER Don't you want to just try?

NIGEL I wouldn't know were to start ... I can't.

In all cases the learners could not think of any method (not even successive trial and error) that they could use to verify their guesses. The fossilized teacher-dependence of the learners is aptly summarized by Schofield's comment that: "*We didn't do this in class before ... I can't do it!*" Clearly all these learners seemed to have been only accustomed to teaching-learning situations in which the teacher would always first present a new method or technique and all that is required of them is to practice it. It did not appear at all as if these learners had ever been exposed to a problem-centred (or modeling) approach before, namely, where they are expected to regularly tackle problems not seen before.

Roxanne's interview.

- RESEARCHER** Are you saying that there is no way to do it or is it that you don't know how to do it?
- ROXANNE** I can't. There might be a way to do it but I can't.
- RESEARCHER** Wouldn't you want to try?
- ROXANNE** I don't know what to do!

Schofield's interview.

- SCHOFIELD** I ... I don't know ...
- RESEARCHER** Do you think that there is a way of showing why your answer is correct?
- SCHOFIELD** I don't think so.
- RESEARCHER** Don't you want to try?
- SCHOFIELD** We didn't do this in class before ... I can't do it!

3. Learners' conjectures and their justifications

It is perhaps not surprising to note that all learners conjectured that the most suitable point was somewhere in the 'centre', presumably relying on their visual intuition to locate an approximate point equidistant to the vertices.

- FAEEZA** Ya. It should be situated towards the middle.
- CHRISTINA** In the centre
- NIGEL** Towards the middle.
- ROXANNE** In the middle.

The inaccuracy of their visual perception created surprise when the learners later discovered that their conjectures were not correct. This surprise in turn created some level of curiosity.

Despite their difficulty or reluctance to use successive guessing and testing with ruler or compass, they nonetheless were able to realize in the *Sketchpad* environment that the distances should be measured from each vertex to the constructed point, and that by dragging the point around one could change the distances until they were the same.

- RESEARCHER** In the centre would mean about there in the middle? Is that correct? (Christina nods her head). Now how will we be able to determine whether that is the correct point?
- CHRISTINA** We measure from that point to that (pointing to the vertices of the quadrilateral).
- RESEARCHER** (after Christina measures the distances) What do you observe?
- CHRISTINA** The distances are different...
- RESEARCHER** How then can we find a suitable point?
- CHRISTINA** Move the point around.
- RESEARCHER** Move the point around. (After awhile) Is it easy to find this point?
- CHRISTINA** No.
- RESEARCHER** So what happens if we move the point around?
- CHRISTINA** The distances will change.

Faeza, who showed much confidence during the interview, was not intimidated by the fact that she was in a new environment (doing mathematics using the computer was new to them). Her reaction to questions and the tone of her replies conveyed the impression that she was quite comfortable with using *Sketchpad*. Other learners displayed similar behaviour, although they hesitated at some stages with their responses.

- RESEARCHER** Now, Faeza, draw a point somewhere in this quadrilateral ... this is where you suggested the building should be. How will we know what the distances are from that point to the various villages?
- FAEEZA** Obviously you need to measure it.
- RESEARCHER** Will you measure it then? (after a while) Ok...there are all the distances from the point that you chose to the different villages. So what do we do now?
- FAEEZA** Drag the point around until we get those distances equal.

Schofield also showed that he was comfortable with using *Sketchpad*.

- SCHOFIELD** It is obvious that it must be in the middle. We can show this by measuring the distances from the village to the point that is chosen.
- RESEARCHER** Do just that then...(after a while) ... what do you observe?
- SCHOFIELD** (silence)... the distances are different...
- RESEARCHER** What should we do then to get the distances to be equal?
- SCHOFIELD** Move this point around?
- RESEARCHER** Go ahead and do that.
- SCHOFIELD** (after a long while)... this must be the point.

Roxanne was the only one who initially stated that the ideal point should be at the centre and thereafter asked for the point to be shifted to the side. Although she had not found the accurate point, her moving of the point towards the right was getting closer to the correct position.

- RESEARCHER** So do you think that it should be in the middle here? (pointing)
- ROXANNE** Ehhh...maybe more towards the side here...(indicating a shift to the right)
- RESEARCHER** All right, then how do you think we should go about checking whether it is correct?

- ROXANNE** By measuring the distances.
RESEARCHER Quickly do that ... (after a while) okay. These are all the measurements. But the distances are not equal. How can we get them to be equal?
ROXANNE You can take the point around and try to find the spot for which the distances will be equal.

4. The recognition of real world conditions when modeling

At this point the child was told that real world situations are extremely complex and they usually must be simplified before mathematics can be applied to it. The learners were then asked to give some of the assumptions that they thought may have been made in order to simplify the problem that may not be true in real life. The responses received here showed that given an opportunity learners would be able to reflect on real life conditions as compared to traditional classroom situations where these assumptions are usually not discussed.

The learners being interviewed were able to recognize factors that could have affected the position they chose. At the expense of belaboring this point it may be essential to list the responses of all the learners just so that a clear impression can be obtained with regard to the way the learners construct their reality.

- PRAVANIE** Apart from the fact that there might be a valley over there, there could be a mountain, there could be a building already constructed. There could be endangered species.
RESEARCHER ... endangered species of what?
PRAVANIE Any plants or animals and stuff like that ... a nature reserve.

Rivers and mountains were common responses but the endangered species of plants and animals was an interesting response. It must be remembered that these learners had not seen this question before so their responses were spontaneous.

Christina's responses were similar to earlier responses but Faeza's and Roxanne's responses about the chief's house or the chief's kraal being situated there was highly realistic.

- CHRISTINA** There might be... a hard rock.
RESEARCHER Yes....
CHRISTINA Other buildings
RESEARCHER What kind of buildings do you think?
CHRISTINA Police station or something....
RESEARCHER Any other reasons?
CHRISTINA If there's like a stream or river you won't be able to build

FAEEZA Maybe there is building
- RESEARCHER** What kind of building do you think?
FAEEZA Could be a school, I don't know.... Could be a shopping complex.
RESEARCHER Remember that these are remote villages
- FAEEZA** Might be the chief's house.....
RESEARCHER Yes...

- FAEEZA** There might be a mountain there ... anything, like a big rock. The cost factor must be too great to get rid of the rocks. It won't be cheap to build there.
- ROXANNE** Maybe there is a mine or a building.
RESEARCHER Remember that these villages are remote villages. What kind of building might be there?
- ROXANNE** There might be kraals there or the chief's house...
RESEARCHER Do you think that there might be other reasons?
ROXANNE The place might be a mountain, with hard rocks...

Nigel's responses below about a mine being at the exact spot or hazardous nuclear waste being stored there, were responses that were not expected and it showed that given the opportunity, learners could be very creative. This suggests the viability of a more concerted effort be made to encourage learners to think creatively about the real-world and its relationships with mathematics in the mathematics class.

- NIGEL** There might be a mine
- RESEARCHER** Yes, any other reasons?
- NIGEL** If the area is very rocky or has mountains.....maybe they are storing hazardous nuclear waste nearby.....

Perhaps a response more suited to a country like South Africa was that of Schofield. He spoke of financial constraints and the social problems by referring to the fact that the people themselves might be unhappy with the location. This showed that learners themselves could be politically mature enough to consider a wide range of issues.

- SCHOFIELD** There might be buildings like a school
RESEARCHER Remember that these are remote villages.
SCHOFIELD ... there may be a river close to the point... if there are no roads then it is going to cost more money to first build roads...the people in the villages may say that they don't want the reservoir at that point...what about very hard rocks...

A matter of concern regarding the learners' responses regarding the unsuitability of the chosen position was the fact that none of the learners focused on the issues more specifically related to mathematics. With the exception of Pravanie none of the other learners stated that it was assumed that the land was flat and though a chosen position may be mathematically ideal, it may not be correct if one considered the possibility of hills and valleys. None of the learners realized that the relative sizes of villages might have some influence on the chosen position. For example, if one village was substantially larger than the others it may make practical sense to put the reservoir closer to it. It is also implicitly assumed that the positions of the reservoir and the villages can be represented by points, that is, their sizes are insignificant compared to the distances in question. If not, this raises several questions. Were the distances being calculated related to the centres of the villages or to the outer boundaries of the villages? Would this therefore not affect the position chosen?

These are critical questions that learners need to focus on when working with real world problems. It seems that their lack of experience in working with real world problems played a role.

The learners' responses to the real-life problems that could be experienced indicated a reasonable level of understanding. In fact, three of the learners directly indicated that mathematics alone cannot always be used for solving real life problems, that is, problems can be experienced.

NIGEL That sometimes in real-life we may not be able to use exact mathematics to solve problems.

ROXANNE We are saying that we can work out a place on the computer or by calculating it but it does not mean that it will be the right place.

SCHOFIELD Maths might be one thing but reality is another ... sometimes we can't use maths on its own.

The other learners could see that a mathematically determined point may be unsuitable in the real world. This does not in any way indicate that mathematics is not effective but it does show that they had some understanding that with real life problems other factors must be considered.

PRAVANIE Very often we think a certain place will work but when we go there we notice that there is a problem.

CHRISTINA Sometimes finding the point might not work because ... because the place might not be suitable.

FAEEZA Because we are trying to show that we can choose a point but that point is not always good... maybe we should only choose the point after we see the place.

5. Conclusions

Given the results of this experiment it was obvious that learners had not previously been exposed to real world problems which they had to solve using modeling strategies. In fact some learners clearly felt threatened and inadequately prepared to solve such a problem. One learner went as far as asking the researcher not to interview her because the work seemed too difficult. Another learner indicated that this type of problem was too difficult whilst another felt that he did not do this type of problem in class. Evidently learners seem to be good imitators of their educators in class. If the educator does a particular type of example then the learners are able to copy those strategies. Unfortunately it currently seems that few educators are engaging learners in mathematical modeling of problems. As a result learners may successfully answer ordinary mathematics questions, but may encounter difficulties when facing questions of the real world.

None of the learners interviewed in this research knew or could devise a precise method of finding a solution to the problem. All some of them could do was to attempt a trial and error

approach and measure the distances using a ruler. Although this leads to an approximate solution, it is rather time consuming (and often not accurate). It however generally shows that learners can understand and cope with real world, at least at the trial and error level. This finding therefore shows that teaching via modeling is possible as any real world problem can at least be approached in a trial and error fashion. Indeed, not all real world problems can be solved in a precise way. In many cases, the best solution can only be obtained by trial and error methods.

There is little doubt that much work still has to be done in encouraging the development of mathematical modeling skills in learners. Furthermore, this development must start at an early age. It is often difficult to start encouraging learners to use different modeling strategies at the end of their schooling careers if they were not exposed to such methods already. The Grade 10 learners interviewed in this experiment showed considerable unease initially because of the lack of knowledge of strategies to work with this problem. It is this unease that transmitted itself to the researcher as mild irritation. Their emphatic “*I don’t know what to do*” or “*I really don’t know*” was said in a tone that could not be captured on the transcript. They were visibly frustrated that the researcher was attempting to coax a solution out of them. This is possibly the result of inadequate preparation for not only tackling real world problems, but also on having to rely on their own ingenuity to invent an appropriate strategy.

The researcher is convinced that exposure to various modeling strategies and dynamic computer software would create a conducive environment for the solution of real world problems and would instill in learners greater confidence when working with different problems. It certainly would prevent learners from becoming overwhelmed by the seemingly insurmountable nature of the problem (“*I don’t know sir.... This is too difficult... please don’t interview me* (pleading)”). The fact that the learners were able to eventually, through a guided interview, arrive at correct solutions to problems 1, 2 and 3, indicates that with the correct guidance, available strategies and confidence, they could become more successful at solving such problems.

In conclusion of this section it must also be stated that learners showed immense awareness of real world conditions. They were able to recognize why mathematics alone cannot be used in real life, but has to be interpreted and adapted taking local conditions into consideration when considering the suitability of the solutions. It is the opinion of the researcher that this aspect of learning and teaching is often neglected and it might be a useful way of involving learners in a problem whilst at the same time it offers them a look at real world conditions – aspects of which they will encounter as engineers, land surveyors, business people and so on.

Learners obtain a vast amount of experience as they interact with the world and environment around them. When providing reasons for why a particular position was not suitable learners gave reasons that were interesting and realistic such as the position may be the actual homestead of the Chief of the tribe living there. For political and tribal reasons one cannot simply move a Chief out to build a reservoir. Educators ought to start utilizing these experiences to direct the learners’ thinking when solving real world problems to finding real world solutions. For example, a problem involving bags of cement may result in a solution of 3,5 bags (50 kilograms each) of cement. Learners are already aware that cement cannot be purchased in smaller quantities – they are only sold in 50-kilogram units. Thus the solution must be 4 bags of cement.

So we know now that children do realize that there are real world conditions. The question that arises is, if and when they do recognize real world conditions, how do they cope with them mathematically? This is an interesting research topic which the researcher has just touched upon with these few problems.

References

Mudaly, V. (2004). *The Role of Sketchpad as a Modeling tool in Secondary Schools*. University of KwaZulu-Natal. Durban.

Learning Mathematics with Understanding: A Critical Consideration of the Learning Principle in the *Principles and Standards for School Mathematics*¹

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Abstract: Learning with understanding has increasingly received attention from educators and psychologists, and has progressively been elevated to one of the most important goals for all students in all subjects. However, the realization of this goal has been problematic, especially in the domain of mathematics. To this might have contributed the fact that, although the vision of students learning mathematics with understanding has often appeared in curriculum frameworks, this vision has tended to be poorly described, thereby offering limited support to curriculum development and policy. The Learning Principle in the Principles and Standards for School Mathematics, an influential mathematics curriculum framework in the United States, seems to make an effort to break this tradition by offering a research-based description of what is involved for students to learn mathematics with understanding. In this article, we examine the extent to which the Learning Principle meets this goal in light of seminal scholarly work on learning mathematics with understanding. By solidifying some key ideas set forth in the Learning Principle and by identifying ideas for further consideration, the article contributes to the development of better descriptions in curriculum frameworks of issues related to promoting meaningful learning in school.

1. Introduction

How is it that there are so many minds that are incapable of understanding mathematics? Is there not something paradoxical in this? Here is a science which appeals only to the fundamental principles of logic, to the principle of contradiction, for instance, to what forms, so to speak, the skeleton of our understanding, to what we could not be deprived of without ceasing to think, and yet there are people who find it obscure, and actually they are the majority. (Poincaré, 1914, pp. 117-118)

Henri Poincaré's statement captures eloquently both the inextricable relation between mathematics and understanding, and the difficulty that learning mathematics with understanding

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entails. While learning mathematics with understanding has increasingly received attention from mathematics educators and psychologists and has progressively been elevated to one of the most important goals of the mathematical education of all students, the realization of this goal has long been problematic. Many factors might account for this, such as teachers' knowledge and pedagogy, the curriculum, etc. In this article, we consider one of those factors, namely, the curriculum, focusing on one curriculum framework's description of issues related to promoting meaningful learning in school. This focus is important because, although the vision of students learning mathematics with understanding has often appeared in curriculum frameworks, this vision has tended to be poorly described, thereby offering limited support to curriculum developers.

The *Principles and Standards for School Mathematics*, a mathematics curriculum framework recently released by the National Council of Teachers of Mathematics (NCTM, 2000) in the United States (US), seems to make an effort to break this tradition. The *Standards* document offers, in a section called the Learning Principle (NCTM, 2000, pp. 20-21), a research-based description of what is involved for students to learn mathematics with understanding. This article has been motivated by the increased value assigned nowadays to learning mathematics with understanding as a principal instructional goal for all students and by the high potential of the *Standards* to influence curriculum development.

Our primary goal in this article is to consider critically the research-based vision about meaningful learning in school mathematics that is elaborated in the Learning Principle (LP). We pursue this goal by discussing both strengths and weaknesses of the LP in light of scholarly work that could be considered seminal with regard to the theme of learning mathematics with understanding. Although our article is about a US mathematics curriculum framework, the discussion we conduct can be of interest to a broader audience. There are two primary reasons for this. First, the case of the US can be seen as indicative of the current trend in many countries to emphasize meaningful learning in school curricula across all subject areas (especially in mathematics and science). Second, the *Standards* have influenced the authors of curriculum frameworks in many countries.

The article is structured into two sections. In the first section, we provide evidence in favor of some key ideas advanced by the LP. Our discussion 'unpacks' these ideas, elaborating how they find support from existing research (part of which is not referenced in the LP). In the second section, we discuss some points that, although important and warranted by research, are insufficiently addressed in the LP. The issues raised in this section represent recommendations for how the LP could be enhanced. By solidifying some key ideas set forth in the LP and by identifying ideas for further consideration, the article contributes to the development of better descriptions in curriculum frameworks of issues related to promoting meaningful learning in school. A word of caution here is that our examination of the LP focuses on what is said or being referenced *in* the LP. One could argue that our analysis should have considered the entire *Standards* document in which the LP is embedded. We decided not to do that because one of the most interesting aspects of the LP is its seeming effort to describe – in a short and self-contained text – the essence of the *Standards'* vision with regard to students' learning mathematics with understanding.

The article as a whole can also be viewed as a survey of literature on learning mathematics with understanding, offering an interpretation and synthesis of some important points on this topic where there is consensus. Our focus on issues of consensus is deliberate, as we believe that a curriculum framework should primarily be judged based on its potential to communicate to curriculum developers well-established points that can serve as guiding principles in their efforts to design effective curricula.

2. Evidence in Favor of Key Ideas Set Forth by the Learning Principle

The LP supports the claim that learning with understanding is both *essential* and *possible* in school mathematics. The argument in favor of meaningful learning in school mathematics was made and supported experimentally as early as the 1930s (Brownell, 1935, 1940, 1947), and has been elaborated since then by many proponents of learning with understanding (e.g., Skemp, 1976). It has also been corroborated by the results of many recent studies of varying instructional and theoretical approaches. These studies: (1) collectively emphasize the importance of having meaning related to learning activities of students of varied ages, backgrounds, and abilities (Cobb et al., 1991; Fennema & Romberg, 1999; Hiebert & Wearne, 1993; Silver & Stein, 1996; Zohar & Dori, 2003), and (2) reveal the need for more instructional attention to sense-making as part of school mathematics instruction (e.g., Schoenfeld, 1988; Silver et al., 1993). In support of the LP, this mounting body of research suggests that *all* students can understand and apply important mathematical concepts. Also, this scholarly work emphasizes the merits of students developing conceptual understanding, and stresses the importance of the powerful connections established between procedures and concepts when one practices this kind of learning.

An important point set forth by the LP is that memorization of facts or procedures without understanding often results in fragile learning. This remark corresponds to research which has shown that mastery of facts and rote performance of procedures are not sufficient in thinking mathematically (Schoenfeld, 1988), getting the right answers does not necessarily imply mathematical proficiency (Erlwanger, 1973), and learning computational formulas is a poor substitute for developing understanding of the underlying concepts (Pollatsek et al., 1981). What is perhaps more important is that the LP goes a step further to note that conceptual understanding is only one out of at least three major components of proficiency, the other two being factual knowledge and procedural facility, and that the alliance of the three makes them usable in powerful ways. This claim finds support from research that has demonstrated the compatibility and close interrelation between factual and procedural competence, and learning with understanding (Bransford et al., 2000; Hiebert & Carpenter, 1992; Silver, 1987). Silver (1987), for example, emphasizes that pure forms of either conceptual or procedural knowledge are seldom exhibited, if ever, and that “it is the *relationship* between the knowledge types that gives one’s knowledge the power of application in a wide variety of settings” (p. 183; emphasis added).

Related to the above is the LP’s emphasis on the relationship between students developing mathematical understanding on the one hand and making connections among mathematical ideas and procedures on the other. Hiebert and Carpenter’s (1992) definition of mathematical understanding in terms of the way knowledge is structured illuminates this relationship:

A mathematical idea or procedure or fact is understood if it is part of an internal network. More specifically, the mathematics is understood if its mental representation is part of a network of representations. The degree of understanding is determined by the number and the strength of the connections. A mathematical idea, procedure, or fact is understood thoroughly if it is linked to existing networks with stronger or more numerous connections. (p. 67)

Well-connected and conceptually grounded ideas enable their holders to both remember them and see them as part of a larger whole within which each part shares reciprocal relationships with other parts (Resnick & Ford, 1981; Romberg & Kaput, 1999; Schoenfeld, 1988, 1992). In addition, ideas with these characteristics are fluently accessed for use in new situations (Skemp, 1976) and empower their holders with the ability of *transfer* – that is, the ability to use what they have learned in new and unfamiliar problems, and to learn related information more quickly (Bransford et al., 2000; Carpenter & Lehrer, 1999; Hiebert & Carpenter, 1992; Resnick & Ford, 1981; Schoenfeld, 1988). In sum, learning mathematics with understanding involves *making connections* among ideas; these connections are considered to facilitate the transfer of prior knowledge to novel situations. Transfer is essential because most new problems require solution via previously learned strategies; it would be impossible for one to become mathematically competent if each problem required a separate strategy.

Hiebert and Carpenter (1992) emphasize that, “[o]ne observation that assumes near axiomatic status in cognitive science is that students’ prior knowledge influences what they learn and how they perform” (p. 80). The LP makes a strong point about the power of using children’s experience and prior knowledge in learning mathematics with understanding. Research suggests that students bring to school a considerable amount of knowledge and experience, and that students construct meaning for a new idea by relating it to ideas that they already know or have experience with (Bransford et al. 2000; Gagnon & Collay, 2001). In the particular domain of mathematics, research shows that children begin to construct mathematical relations long before they come to school. These early forms of knowledge can serve as the basis for developing several components of the formal elementary mathematics curriculum and for further expanding children’s understanding of mathematics (Carpenter et al., 1981, 1996; Carpenter & Lehrer, 1999; Hiebert & Carpenter, 1992; Schoenfeld, 1992). For example, the work of Carpenter and his colleagues demonstrates that, by the time children begin school, they already have a considerable knowledge store relevant to arithmetic. Although children may lack the symbolic representations of addition and subtraction, they have experiences of adding and subtracting numbers of items in their everyday play, and they can solve a wide range of addition and subtraction problems. If children’s knowledge is tapped and built on as teachers attempt to teach them the formal operations of addition and subtraction, it is likely that children will acquire a coherent and thorough understanding of these processes. Mack’s (1990) study with eight sixth-graders offers a different example. It shows that students at this level of schooling can construct meaningful algorithms in learning fractions, given that instruction builds appropriately upon their informal/prior knowledge on this domain. One other important point made by the LP is that learning mathematics with understanding can be promoted through students’ engagement in *problem solving* activities. Several researchers emphasize that curriculum activities that engage students in problem solving reflect an emphasis on learning mathematics with understanding (Fennema et al., 1999; Romberg & Kaput, 1999; Schoenfeld, 1992). For example, Fennema et al. (1999) note:

Because the goal of mathematics education should be the development of understanding by all students, the majority of the curriculum should be composed of tasks that provide students with problem *situations*. Two reasons support this claim. The first is that mathematics that is worth learning is most closely represented in problem solving tasks. The second is that students are more apt to engage in the mental activities required to develop understanding when they are confronted with mathematics embedded in problem situations. (p. 187)

From an epistemological point of view, problems are the source of the meaning of mathematical knowledge. As Vergnaud (1982) remarks, “[n]ot only in its practical aspects, but also in its theoretical aspects, knowledge emerges from problems to be solved and situations to be mastered” (p. 31). Intellectual productions become knowledge only if they prove to be efficient and reliable in solving problems that have been identified as being important practically (they need to be solved frequently) or theoretically (their solution allows a new understanding of the related conceptual domain). Inextricably related to engaging in problem solving is getting involved in activities related to *mathematical reasoning* and *proof*: exploring patterns; making, testing, and evaluating conjectures; and developing mathematically sound arguments for or against mathematical statements. Several scholars have elaborated on the connection between learning mathematics with understanding and reasoning and proof. Ball and Bass (2003) emphasize that “mathematical reasoning is inseparable from knowing mathematics with understanding” (p. 42). In the same spirit, Hanna and Jahnke (1996) note that “[p]roof in its full range of manifestations is ... an essential tool for promoting mathematical understanding in the classroom” (p. 877).

A final point we wish to highlight is that the LP considers both the *cognitive* and *social* aspects of learning. The consideration of both psychological and sociological conceptions of learning agrees with the current trend of integrating these two perspectives (see, e.g., Cobb & Bauersfeld, 1995; Yackel & Cobb, 1996). With regard to the psychological approach to learning, the LP acknowledges the constructivist idea that understanding is a continuing activity of individuals organizing their own knowledge structures, a dynamic process rather than an acquisition of categories of knowing (Confrey, 1994; Gagnon & Collay, 2001; Piaget, 1948/1973; Pirie & Kieren, 1994). The LP also notes that learning with understanding supports the creation of autonomous learners – that is, learners who “can take control of their learning by defining their goals and monitoring their progress” (NCTM, 2000, p. 21). The notion of autonomy goes back at least to the work of Piaget (1948/1973) who proposed that the main goal of education should be the cultivation of learners’ autonomy. With regard to the sociological perspective to learning, the LP (NCTM, 2000) advances the idea that “[l]earning with understanding can be further enhanced by classroom interactions, as students propose mathematical ideas and conjectures, [and] learn to evaluate their own thinking and that of others,” and notes that “[c]lassroom discourse and social interaction can be used to promote the recognition and connection among ideas and the reorganization of knowledge” (p. 21). Examples of this kind of classroom environments can be found in the several research reports (e.g., Ball & Bass, 2003; Lampert, 1986, 1990; Yackel & Cobb, 1996).

3. Recommendations for Enhancing the Learning Principle

As the discussion in the previous section suggests, the LP summarizes well some key ideas regarding learning with understanding in the context of school mathematics. However, there are some other important ideas that, although warranted by research, are insufficiently addressed in

the LP. In this section, we discuss four such ideas and provide recommendations about how these issues could be addressed in future versions of this or other curriculum frameworks.

Our first point is that the LP does not discuss the important idea that what we consider a desirable learning outcome (e.g., development of understanding or acquisition of procedural fluency) determines the relative worth of instructional methods. The way in which Brownell (1935, 1947) and Skemp (1976) – two proponents of teaching and learning for understanding – approach this issue can serve as a useful model for the LP.

Although both scholars favor meaningful learning and teaching of arithmetic and believe that thorough understanding of computational procedures cannot be achieved without a solid conceptual basis, neither of them rejects non-meaningful ways of teaching and learning arithmetic. Brownell (1935) notes that “*drill* is recommended when ideas and processes, already understood, are to be practiced to increase proficiency, to be fixed for retention, or to be rehabilitated after disuse” (p. 19; emphasis added). Skemp (1976) mentions three advantages of teaching *instrumental mathematics*, that is, rules without reasons: (1) Within its own context, instrumental mathematics is usually easier to understand; (2) The rewards are more immediate and apparent; and (3) Because less knowledge is involved, instrumental thinking can often help one achieve the right answer more quickly and reliably than relational thinking. Furthermore, both Brownell and Skemp advance the argument that, depending on what learning outcomes are valued, different methods should be employed; accordingly, there is no absolute instructional method. Brownell supports this argument by using his experimental findings regarding the learning of the subtraction algorithm for ‘borrowing’ from tens. On the one hand, he found that the method of ‘decomposition’ was more effective than ‘equal additions’ when the desired learning outcomes were the development of students’ understanding and the enhancement of their ability to transfer their knowledge. On the other hand, he found evidence that the equal additions method was superior to decomposition when both were taught mechanically. Skemp (1976) expresses a similar idea when he notes, for example, that, “[i]f students are being taught instrumentally, then a ‘traditional’ syllabus will probably benefit them more” (p. 156). The foregoing remarks indicate that the two researchers recognize that a fundamental question in teaching is what are the desired learning outcomes. Given the interdependence between desirable learning outcomes and appropriate instructional methods, teaching for understanding might not always be the most appropriate instructional method.

The results of Hiebert and Wearne’s (1993) investigation about the relationships between teaching and learning place value and multi-digit addition and subtraction in six second-grade classrooms support a similar idea to the one advanced in the previous paragraph. Hiebert and Wearne found that students in classrooms that emphasized the construction of relationships between place value and computation strategies received fewer problems than their more traditionally taught peers, spent more time with each problem, were asked more questions to describe and explain alternative strategies, and showed higher levels of performance by the end of the year on most types of activities. These findings initially appeared to be at odds with some conclusions derived by earlier work (e.g., Good et al., 1978; Leinhardt, 1986). This is because earlier work suggested that teachers who stimulate high rates of student achievement, with regard to basic skills at the primary grades, have the following characteristics: they teach quickly paced lessons, they ask more recall than process/explanation questions, and they present more problems

per lesson than do novices. This ‘discrepancy’ could be attributed to the relationship between desirable learning outcomes and instructional procedures. The characteristics of ‘effective teachers’ derived from studies prior to Hiebert and Wearne (1993) were probably limited to traditionally taught classrooms (see Brophy & Good, 1986). As Hiebert and Wearne (1993) note, “these characteristics may relate to higher achievement if compared with other classrooms using a *similar* (but not as effectively implemented) instructional approach” (p. 422; emphasis added).

We turn now to our second point, which relates to the issue of knowledge transfer from one situation to another. While the LP captures well the idea that conceptually grounded knowledge is more likely to be transferred to new problem situations, it does not consider situations where transfer does *not* happen. One way to address this issue would be to draw on the extensive body of research that considers the situated character of learning (Boaler, 1998; Carraher et al., 1985, 1987; Lave et al., 1984). The theory of *situated cognition* explains why transfer does not happen in terms of the idea that learning is linked to the situation or context in which it takes place. This theory accounts, for example, for cases where adults do not use their school-learned arithmetic in grocery shopping; adults often do not recognize mathematically similar situations, thus choosing procedures depending more on the context rather than on the mathematical aspects of the tasks (Lave et al., 1984). Situated cognition also accounts for cases where children are more successful in solving arithmetic problems in word context than when solving equivalent but purely symbolic problems (Carraher et al., 1987), or for ones where children demonstrate superior performance when solving problems in the market as compared to the school-like setting (Carraher et al., 1985).

Related to the above is our third point. The LP does not discuss the possibility of students’ prior knowledge and experience becoming a burden in their future learning of mathematics, thereby appearing to suggest that prior knowledge and experience always facilitate subsequent learning. However, as Bransford et al. (2000) point out, “[p]revious knowledge can *help* or *hinder* the understanding of new information” (p. 78; emphasis added). One way to account for this issue would be for the LP to caution the readers that prior experience, even when correct in the context it was generated, might not necessarily be readily applicable in new contexts. Several studies help exemplify this point. Bell et al. (1981) showed that children have difficulties with verbal problems about decimal numbers because of beliefs they acquired from their previous engagement in other mathematical domains and that are resistant to change. Two such beliefs, probably originating from students’ experience with whole numbers, are ‘multiplication makes bigger’ and ‘the smaller number must always be divided into the larger.’ Fischbein et al. (1985) propose that there are certain types of intuitive models linked with arithmetic operations and used in the initial instruction that “become so deeply rooted in the learner’s mind that they continue to exert an unconscious control over mental behaviour even after the learner has acquired formal mathematical notions that are solid and correct” (p. 16; in original the whole segment is emphasized). The repeated addition and partitive models are two such examples for multiplication and division, respectively. Instructors often choose these models “as initial didactical devices because they correspond best to the mental requirements of elementary school children at the concrete operational period and because they provide the most natural way of understanding the new concept” (Fischbein et al., 1985, p. 15). While these models are not wrong, they are incomplete and do not capture all the different meanings of multiplication and division. Along similar lines, Resnick et al. (1989), based on an analysis of children’s errors as

they learn decimals, conclude that “errors are a natural concomitant of students’ attempts to integrate new material that they are taught with already established knowledge” (p. 8).

We get now to our last point. Although the LP considers mathematical learning both at the level of the individual and the social group, it seems to overlook the important role of the *cultural* context in which learning takes place. According to learning theories in the domain of cultural discourses, “learning and knowing, whether focusing on the level of the individual or the social group, can only be understood when considered in the broader cultural context” (Davis et al., 2000, p. 69). Jerome Bruner (1996), a leader in the field of *cultural psychology*, argues that one “cannot understand mental activity unless [one] takes into account the cultural setting and its resources, the very things that give mind its shape and scope” (pp. x-xi). The crucial role of culture in shaping an individual’s understanding suggests that learning can take different forms for students with different cultural backgrounds. In turn, this emphasizes the importance of ‘culturally relevant pedagogy’ (Ladson-Billings, 1994) to cultivate learning with understanding in classroom environments with diverse student populations.

4. Conclusion

While learning mathematics with understanding is an important instructional goal for all students, forms of classroom mathematics practice that foster meaningful learning seem to deviate from the norm, at least in US mathematics instruction (Hiebert et al., 2003; Manaster, 1998). This state of affairs is in part due to the challenges that arise from trying to make learning with understanding a consistent part of all students’ everyday mathematical experiences.

One promising way to gain leverage on helping students learn mathematics with understanding is to equip teachers with curriculum materials (student textbooks and teacher editions) that provide them with the necessary guidance. This argument finds support in the large body of research that suggests that the mathematical activity that takes place in classrooms, including teachers’ decisions about what mathematics tasks to implement and how, are mediated through the curriculum materials they use (Beaton et al., 1996; Burstein, 1993; Nathan et al., 2002; Porter, 1989; Remillard, 2000; Romberg, 1992; Schmidt et al., 1997; Stein et al., 1996; Zaslavsky, 2005). But the design of curriculum materials that can be used by teachers to engage their students in meaningful learning is a complex endeavor and so the guidance that curriculum frameworks (such as the NCTM *Standards*) can offer to curriculum developers on this issue is crucial.

Some questions that curriculum frameworks need to address in regard to integrating understanding into a coherent conception of mathematics learning in school curriculum materials are the following: What might be the relationship among factual knowledge, procedural facility, and understanding in mathematical learning? What might be the role of problem solving and reasoning and proof in learning mathematics with understanding? What might be the influence of students’ prior knowledge and experiences in learning with understanding, and how might these be addressed or used effectively by instruction? What might account for knowledge transfer in some situations and what might inhibit this process in others? What might be the role of the broader cultural context in which students’ learning unfolds, and how might this relate to individual and social factors? What might be the relationship between learning outcomes and

instructional methods, and what might this relationship imply for learning mathematics with understanding?

The LP makes a serious attempt to position itself on many of the above issues, thereby guiding in significant ways the efforts of curriculum developers who are committed to improving the quality of students' learning of mathematics. By recognizing both the importance and the complexity of the goal to describe issues of promoting meaningful learning in school, we used seminal scholarly work in this area to examine the extent to which the LP meets this goal. Our examination revealed that the LP substantially captures some key points elaborated in the literature related to learning with understanding but insufficiently addresses some other important points. Our discussion in this article of the points that are insufficiently addressed in the LP is not meant to devalue the importance and potential contributions of the LP to curriculum development; rather it is meant to suggest ways in which the LP could be further improved.

References

Ball, D. L., & Bass, H. (2003). Making mathematics reasonable in school. In J. Kilpatrick, W. G. Martin, & D. Schifter (Eds.), *A research companion to Principles and Standards for School Mathematics* (pp. 27-44). Reston, VA: National Council of Teachers of Mathematics.

Beaton, A. E., Mullis, I. V. S., Martin, M. O., Gonzalez, E. J., Kelly, D. L., & Smith, T. A. (1996). *Mathematics achievement in the middle school years: IEA's Third International Mathematics and Science Study (TIMSS)*. Chestnut Hill, MA: Boston College.

Bell, A., Swan, W., & Taylor, G. (1981). Choice of operation in verbal problems with decimal numbers. *Educational Studies in Mathematics*, 12, 399-420.

Boaler, J. (1988). Open and closed mathematics: Student experiences and understandings. *Journal for Research in Mathematics Education*, 29(1), 41-62.

Bransford, J., Brown, A., & Cocking, R. (2000). *How people learn: Brain, mind, experience, and school*. Washington, DC: National Academy Press.

Brophy, J. E., & Good, T. L. (1986). Teacher behavior and student achievement. In M. C. Wittrock (Ed.), *Handbook of research on teaching* (3rd ed., pp. 328-375). New York: Macmillan.

Brownell, W. A. (1935). Psychological considerations in the learning and teaching of arithmetic. In W. D. Reeve (Ed.), *The teaching of arithmetic. 10th Yearbook of the National Council of Teachers of Mathematics* (pp. 1-33). New York: Teachers College Press.

Brownell, W. A. (1940). Borrowing in subtraction. *Journal of Educational Research*, 33, 415-424.

Brownell, W. A. (1947). An experiment on borrowing in third-grade arithmetic. *Journal of Educational Research*, 41(3), 161-171.

Bruner, J. (1996). *The culture of education*. Cambridge, Massachusetts: Harvard University Press.

Burstein, L. (Ed.). (1993). *The IEA study of mathematics III: Student growth and classroom processes*. Oxford: Pergamon Press.

Carpenter, T. P., Fennema, E., & Franke, M. L. (1996). Cognitively Guided Instruction: A knowledge base for reform in primary mathematics instruction. *The Elementary School Journal*, 97(1), 3-20.

Carpenter, T. P., Hiebert, J., & Moser, J. M. (1981). Problem structure and first-grade children's initial solution processes for simple addition and subtraction problems. *Journal for Research in Mathematics Education*, 12, 27-39.

Carpenter, T., & Lehrer, R. (1999). Teaching and learning mathematics with understanding. In E. Fennema, & T. Romberg (Eds.), *Mathematics classrooms that promote understanding* (pp. 19-32). Mahwah, NJ: Lawrence Erlbaum Associates.

Carraher, T. N., Carraher, D. W., & Schliemann, A. D. (1985). Mathematics in the streets and in schools. *British Journal of Developmental Psychology*, 3, 21-29.

Carraher, T. N., Carraher, D. W., & Schliemann, A. D. (1987). Written and oral mathematics. *Journal for Research in Mathematics Education*, 18, 83-97.

Cobb, P., & Bauersfeld, H. (Eds.). (1995). *The emergence of mathematical meaning: Interaction in classroom cultures*. Hillsdale, NJ: Lawrence Erlbaum Associates.

Cobb, P., Wood, T., Yackel, E., Nicholls, J., Wheatley, G., Trigatti, B., & Perlwitz, M. (1991). Assessment of a problem-centered second-grade mathematics project. *Journal for Research in Mathematics Education*, 22, 3-29.

Confrey, J. (1994). A theory of intellectual development, Part I. *For the Learning of Mathematics*, 14(3), 2-8.

Davis, B., Sumara, D., & Luce-Kapler, R. (2000). *Learning minds: Learning and teaching in a complex world*. Mahwah, NJ: Lawrence Erlbaum Associates.

Erlwanger, S. H. (1973). Benny's conception of rules and answers in IPI Mathematics. *Journal of Children's Mathematical Behavior*, 1(2), 7-26.

Fennema, E., & Romberg, T. (Eds.). (1999). *Mathematics classrooms that promote understanding*. Mahwah, NJ: Lawrence Erlbaum Associates.

Fennema, E., Sowder, J., & Carpenter, T. (1999). Creating classrooms that promote understanding. In E. Fennema, & T. Romberg (Eds.), *Mathematics classrooms that promote understanding* (pp. 185-199). Mahwah, NJ: Lawrence Erlbaum Associates.

Fischbein, E., Deri, M., Nello, M. S., & Marino, M. S. (1985). The role of implicit models in solving verbal problems in multiplication and division. *Journal for Research in Mathematics Education*, 16, 3-17.

Gagnon, G., & Collay, M. (2001). *Designing for learning: Six elements in constructivist classrooms*. Thousand Oaks, CA: Corwin Press.

Good, T., Grouws, D. A., & Beckerman, T. (1978). Curriculum pacing: Some empirical data in mathematics. *Journal of Curriculum Studies*, 10, 75-81.

Hanna, G., & Jahnke, H. N. (1996). Proof and proving. In A. Bishop, K. Clements, C. Keitel, J. Kilpatrick, & C. Laborde (Eds.), *International handbook of mathematics education* (pp. 877-908). Dordrecht: Kluwer Academic Publishers.

Hiebert, J., & Carpenter, T. P. (1992). Learning and teaching with understanding. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 65-97). New York: Mcmillan.

Hiebert, J., Gallimore, R., Garnier, H., Givvin, K. B., Hollingsworth, H., Jacobs, J., Chui, A. M., Wearne, D., Smith, M., Kersting, N., Manaster, A., Tseng, E., Etterbeek, W., Manaster, C., Gonzales, P., & Stigler, J. (2003). *Teaching mathematics in seven countries: Results from the TIMSS 1999 Video Study*. NCEES 2003-013. Washington, DC: U.S. Department of Education, Institute of Education Sciences.

Hiebert, J., & Wearne, D. (1993). Instructional tasks, classroom discourse, and students' learning in second-grade arithmetic. *American Educational Research Journal*, 30, 393-425.

- Ladson-Billings, G. (1994). *The dreamkeepers: Successful teachers for African American children*. San Francisco: Jossey-Bass.
- Lampert, M. (1986). Knowing, doing, and teaching multiplication. *Cognition and Instruction*, 3(4), 305-342.
- Lampert, M. (1990). When the problem is not the question and the solution is not the answer: Mathematical knowing and teaching. *American Educational Research Journal*, 27, 29-63.
- Lave, J., Murtaugh, M., & de la Rocha, O. (1984). The dialectic of arithmetic in grocery shopping. In B. Rogoff, & J. Lave (Eds.), *Everyday cognition* (pp. 67-94). Cambridge, MA: Harvard University Press.
- Leinhardt, G. (1986). Expertise in math teaching. *Educational Leadership*, 43(7), 28-33.
- Mack, N. K. (1990). Learning fractions with understanding: Building on informal knowledge. *Journal for Research in Mathematics Education*, 21, 16-32.
- Manaster, A. B. (1998). Some characteristics of eighth grade mathematics classes in the TIMSS videotape study. *American Mathematical Monthly*, 105(9), 793-805.
- Nathan, M. J., Long, S. D., & Alibali, M. W. (2002). The symbol precedence view of mathematical development: a corpus analysis of the rhetorical structure of textbooks. *Discourse Processes*, 33(1), 1-21.
- National Council of Teachers of Mathematics. (2000). *Principles and Standards for School Mathematics*. Reston, VA: National Council of Teachers of Mathematics.
- Piaget, J. (1948/1973). *To understand is to invent*. New York: Grossman.
- Pirie, S., & Kieren, T. (1994). Growth in mathematical understanding: How can we characterize it and how can we represent it? *Educational Studies in Mathematics*, 26, 165-190.
- Poincaré, H. (1914). Mathematical definitions and education. In, *Science and method* (trans. F. Maitland) (pp. 117-142). New York: Thomas Nelson and Sons.
- Pollatsek, A., Lima, S., & Well, A. D. (1981). Concept of computation: Students' understanding of the mean. *Educational Studies in Mathematics*, 12, 191-204.
- Porter, A. (1989). A curriculum out of balance: The case of elementary school mathematics. *Educational Researcher*, 18(5), 9-15.
- Remillard, J. T. (2000). Can curriculum materials support teachers' learning? Two fourth-grade teachers' use of a new mathematics text. *Elementary School Journal*, 100, 331-350.
- Resnick, L. B., & Ford, W. W. (1981). *The psychology of mathematics for instruction*. Hillsdale, NJ: Lawrence Erlbaum Associates.
- Resnick, L. B., Nesher, P., Leonard, F., Magone, M., Omanson, S., & Peled, I. (1989). Conceptual bases of arithmetic errors: The case of decimal fractions. *Journal for Research in Mathematics Education*, 20(1), 8-27.
- Romberg, T. A. (1992). Problematic features of the school mathematics curriculum. In P. W. Jackson (Ed.), *Handbook of research on curriculum* (pp. 749-788). New York: Macmillan.
- Romberg, T., & Kaput, J. (1999). Mathematics worth teaching, mathematics worth understanding. In E. Fennema, & T. Romberg (Eds.), *Mathematics classrooms that promote understanding* (pp. 3-18). Mahwah, NJ: Lawrence Erlbaum Associates.
- Schmidt, W. H., McKnight, C. C., & Raizen, S. A. (1997). *A splintered vision: An investigation of U.S. science and mathematics education*. The Netherlands: Kluwer Academic Publishers.
- Schoenfeld, A. H. (1988). When good teaching leads to bad results: The disasters of well taught mathematics classes. *Educational Psychologist*, 23(2), 145-166.

Schoenfeld, A. H. (1992). Learning to think mathematically: Problem solving, metacognition, and sense making in mathematics. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 334-370). New York: Macmillan.

Silver, E. A. (1987). Using conceptual and procedural knowledge: A focus on relationships. In J. Hiebert (Ed.), *Conceptual and procedural knowledge: The case of mathematics* (pp. 181-198). Hillsdale, NJ: Lawrence Erlbaum Associates.

Silver, E. A., & Stein, M. K. (1996). The QUASAR Project: The 'revolution of the possible' in mathematics instructional reform in urban middle schools. *Urban Education, 30*(4), 476-521.

Silver, E. A., Shapiro, L. J., & Deutsch, A. (1993). Sense-making and the solution of division problems involving remainders: An examination of middle school students' solution processes and their interpretations of solutions. *Journal for Research in Mathematics Education, 24*, 117-135.

Skemp, R. (1976). Relational understanding and instrumental understanding. *Mathematics Teaching, 77*(December), 20-26.

Stein, M. K., Grover, B., & Henningsen, M. (1996). Building student capacity for mathematical thinking and reasoning: An analysis of mathematical tasks used in reform classrooms. *American Educational Research Journal, 33*, 455-488.

Vergnaud, G. (1982). Cognitive and developmental psychology and research in mathematics education: Some theoretical and methodological issues. *For the Learning of Mathematics, 3*(2), 31-41.

Yackel, E., & Cobb, P. (1996). Sociomathematical norms, argumentation, and autonomy in mathematics. *Journal for Research in Mathematics Education, 27*(4), 458-477.

Zaslavsky, O. (2005). Seizing the opportunity to create uncertainty in learning mathematics. *Educational Studies in Mathematics, 60*, 297-321.

Zohar, A., & Dori, Y. J. (2003). Higher order thinking skills and low-achieving students: Are they mutually exclusive? *The Journal of the Learning Sciences, 12*(2), 145-181.

Erratum: An Improvement on the article *Taxi Cab Geometry: History and Applications*, TMME, Vol2, no.1, p. 38 – 64

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Abstract: As a high school student from Germany I did a Mathematics research paper entitled Taxicab Geometry: Fundamentals and Applications. During my research I found the article Taxicab Geometry: History and Applications by Chip Reinhardt which was published in the edition Vol.2, no. 1 (p. 38 – 64) of this journal. The article was very helpful for my coursework and I'd like to compliment the author and everyone who is involved in the journal on their work. In my coursework I created an application example similar to the one Chip Reinhardt used in his article. I solved my example in the same way Mr. Reinhardt solved his, but I made some improvements. In his article there were some errors in calculations which I revised. In the following article I'd like to suggest a rectified solution to Chip Reinhardt's example.

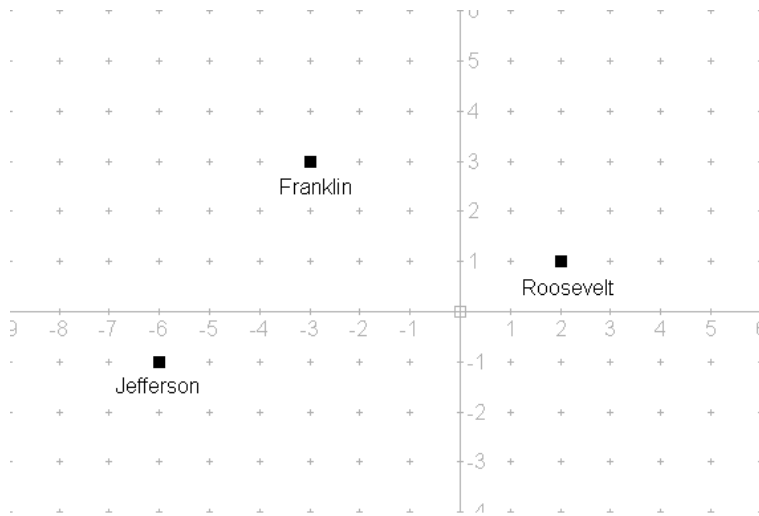
1. Creation of school districts

In New York school district boundaries should be created such that every student is going to the closest school.

The application of Taxicab geometry to solve this problem is a logical choice as the students have to stick to the streets on their way to school.

There are three schools in between which the school boundaries need to be found: Franklin at $(-3 / 3)$, Jefferson at $(-6 / 1)$ and Roosevelt at $(2 / 1)$. In a co-ordinate system the schools are located as following:

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To solve the whole problem we need to break the solution down into different sections.

Section 1

In the first section you focus on the boundary between Franklin and Jefferson school. To find the boundary line between these two schools you need to find the points which are equal distance from Franklin and Jefferson school. Expressed in an equation, which will be named a_1 in the following calculation, you need:

$$d(\text{Jefferson}) = d(\text{Franklin})$$

$$d[(-6, -1), (x, y)] = d[(-3, 3), (x, y)]$$

$$|-6 - x| + |-1 - y| = |-3 - x| + |y - 3|$$

$$|x + 6| + |y + 1| = |x + 3| + |y - 3|$$

To solve this equation for x and y we look at nine different cases as the solution with the absolute values gets difficult. The different cases are:

	$-1 \leq y \leq 3$	$-1 > y$	$3 < y$
$-6 \leq x \leq -3$	Case I	Case IV	Case VII
$-6 > x$	Case II	Case V	Case IIX
$-3 < x$	Case III	Case VI	Case IX

Chip Reinhardt continued like this :

We look at these cases because the absolute values will change the solutions.

Case I: $-6 \leq x \leq -3$ and $-1 \leq y \leq 3$

Since $|x+6| \geq 0$, when $-6 \leq x \leq -3$, $|x+6| = x+6$

Since $|y+1| \geq 0$ when $-1 \leq y \leq 3$, $|y+1| = y+1$

Since $|x+3| \leq 0$ when $-6 \leq x \leq -3$, $|x+3| = -x+3$

Since $|y-3| \leq 0$ when $-1 \leq y \leq 3$, $|y-3| = -y+3$

The underlined, red part is mathematically **incorrect** because $|x+3|$ and $|y-3|$ is always positive or equal to 0 so it is always ≥ 0 and

Improvement:

To avoid this mistake I chose to write the simplifications/calculations for each case like this:

Case I: Under the terms $-6 \leq x \leq -3$ and $-1 \leq y \leq 3$ you can simplify the starting equation a_1 as following:

Because $x+6 \geq 0$, when $-6 \leq x \leq -3$, you can replace $|x+6|$ by $x+6$.

Because $y+1 \geq 0$, when $-1 \leq y \leq 3$, you can replace $|y+1|$ by $y+1$

Because $x+3 \leq 0$, when $-6 \leq x \leq -3$, you can replace $|x+3|$ by $-x+3$

Because $y-3 \leq 0$, when $-1 \leq y \leq 3$, you can replace $|y-3|$ by $-y+3$

<u>x</u>	<u>y</u>
-6	5/2
-5	3/2
-4	1/2
-3	-1/2

If these simplifications are substituted into equation a_1 you can observe:

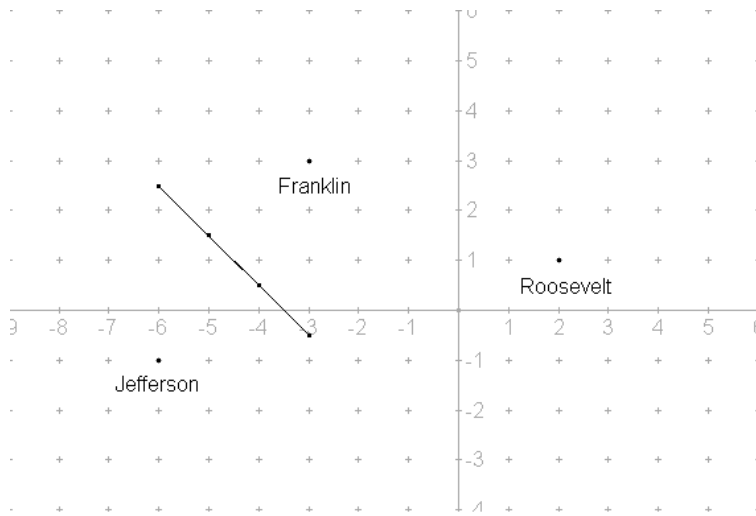
$$x+6+y+1 = -x-3-y+3$$

$$x+y+7 = -x-y \quad | -x+y/2$$

$$y = -x-7/2$$

To be able to draw the boundaries easier in at the end of the calculation you put your values into a table.

Drawn into the co-ordinate system the boundary between Franklin and Jefferson in the region where $-6 \leq x \leq -3$ and $-1 \leq y \leq 3$ looks like this:



Case II: If $-6 > x$ and $-1 \leq y \leq 3$ you can simplify the equation a_1 from the beginning as following:

Because $x+6 < 0$, when $-6 > x$, you can replace $|x+6|$ by $-x-6$.

Because $y+1 \geq 0$, when $-1 \leq y \leq 3$, you can replace $|y+1|$ by $y+1$

Because $x+3 < 0$, when $-6 > x$, you can replace $|x+3|$ by $-x-3$

Because $y-3 \leq 0$, when $-1 \leq y \leq 3$, you can replace $|y-3|$ by $-y+3$

Substituted into a_1 you can observe:

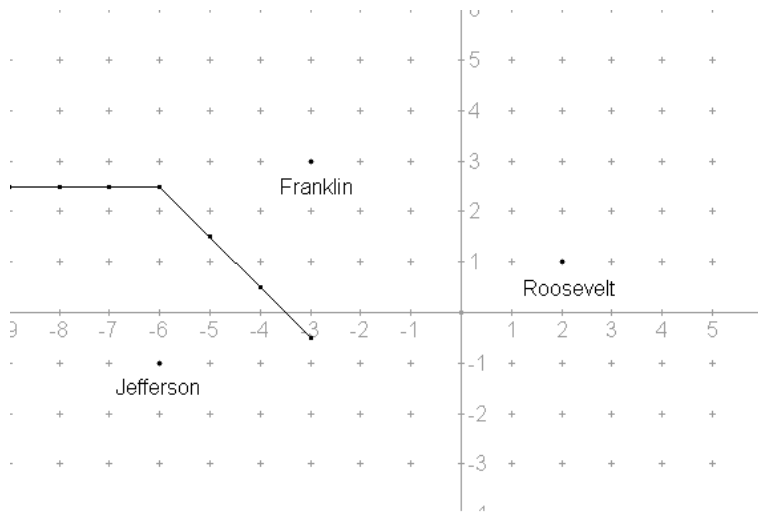
$$-x-6+y+1=-x-3-y+3$$

$$-x+y-5=-x-y$$

$$y=5/2$$

x	y
-7	5/2
-8	5/2
-9	5/2
...	5/2

In the following picture you can see the boundary between Franklin and Jefferson in the region where $-6 > x$ and $-1 \leq y \leq 3$.



Case III: Under the conditions $x > -3$ and $-1 \leq y \leq 3$ you can simplify the starting equation a_1 like this:

Because $x + 6 > 0$, when $-3 < x$, you can replace $|x + 6|$ by $x + 6$.

Because $y + 1 \geq 0$, when $-1 \leq y \leq 3$, you can replace $|y + 1|$ by $y + 1$

Because $x + 3 > 0$, when $-3 < x$, you can replace $|x + 3|$ by $x + 3$

Because $y - 3 \leq 0$, when $-1 \leq y \leq 3$, you can replace $|y - 3|$ by $-y + 3$

If you substitute these simplifications into the starting equation a_1 you can observe:

$$x + 6 + y + 1 = x + 3 - y + 3$$

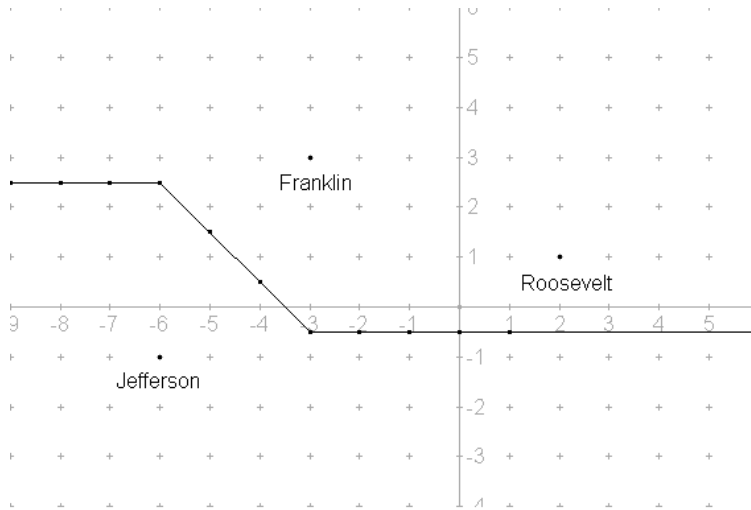
$$x + y + 7 = x - y + 6$$

$$2y = -1$$

$$y = -1/2$$

<u>x</u>	<u>y</u>
-3	-1/2
-2	-1/2
-1	-1/2
...	-1/2

Now you can see the whole boundary line, including the region $x > -3$ and $-1 \leq y \leq 3$, between Jefferson and Franklin at which the points and lines are equal distance from both schools:
 $d(\text{Jefferson}) = d(\text{Franklin})$



In case IV – IX no solutions exist. You don't necessarily need to do the algebra because its obvious if you look at the graph above that no other solutions exist. In the other quadrants one school will always be closer to the points/lines than the other.

Section 2

In section two you look at the boundary between Franklin and Roosevelt school. At first you need to look at the points which are equal distance from both schools to create a boundary. Expressed in an equation (a_2) you need:

$$d(\text{Franklin}) = d(\text{Roosevelt})$$

$$d[(-3,3), (x, y)] = d[(2,1), (x, y)]$$

$$|x+3| + |y-3| = |x-2| + |y-1|$$

To solve this equation for x and y we divide the calculation into different cases again:

	$1 \leq y \leq 3$	$1 > y$	$3 < y$
$-3 \leq x \leq 2$	Case I	Case II	Case III

$2 < x$	Case IV	Case V	Case VI
$-3 > x$	Case VII	Case IIX	Case IX

Case I: If these terms apply: $-3 \leq x \leq 2$ and $1 \leq y \leq 3$, you can simply a_2 as following:

Because $x+3 \geq 0$, when $-3 \leq x \leq 2$, you can replace $|x+3|$ by $x+3$.

Because $y-3 \leq 0$, when $1 \leq y \leq 3$, you can replace $|y-3|$ by $-y+3$

Because $x-2 \leq 0$, when $-3 \leq x \leq 2$, you can replace $|x-2|$ by $-x+2$

Because $y-1 \geq 0$, when $1 \leq y \leq 3$, you can replace $|y-1|$ by $y-1$

Substituted into a_2 you can simplify:

$$x+3-y+3=-x+2+y-1$$

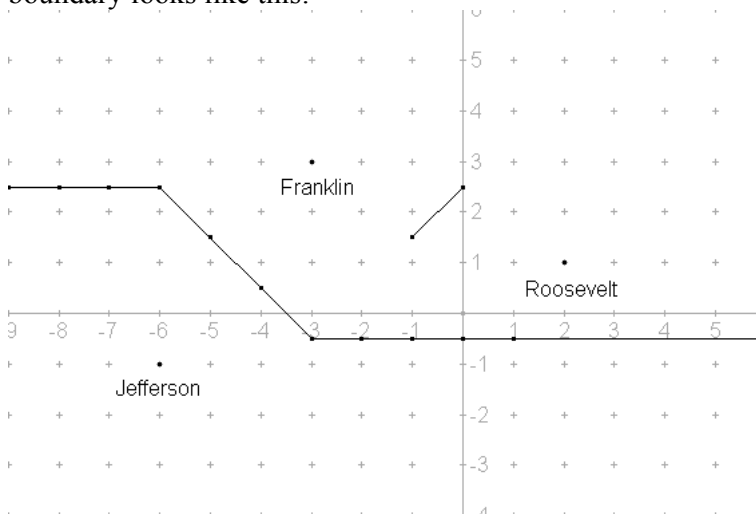
$$x-y+6=-x+y+1$$

$$-2y=-2x-5$$

$$y=x+5/2$$

\underline{x}	\underline{y}
0	5/2
-1	3/2

In the table you only need to plug the values which are in the region. If $x = -2$; $x = 1$; $x = 2$, the condition $1 \leq y \leq 3$ doesn't apply and you don't need to plug these values. Plotted the boundary looks like this:



Case II: If $-3 \leq x \leq 2$ and $1 > y$ you can simplify the equation a_2 as following:

Because $x+3 \geq 0$, when $-3 \leq x \leq 2$, you can replace $|x+3|$ by $x+3$.

Because $y-3 < 0$, when $1 > y$, you can replace $|y-3|$ by $-y+3$

Because $x-2 \leq 0$, when $-3 \leq x \leq 2$, you can replace $|x-2|$ by $-x+2$

Because $y - 1 < 0$, when $1 > y$, you can replace $|y - 1|$ by $-y + 1$

Substituted into a_2 you can observe:

$$x + 3 - y + 3 = -x + 2 - y + 1$$

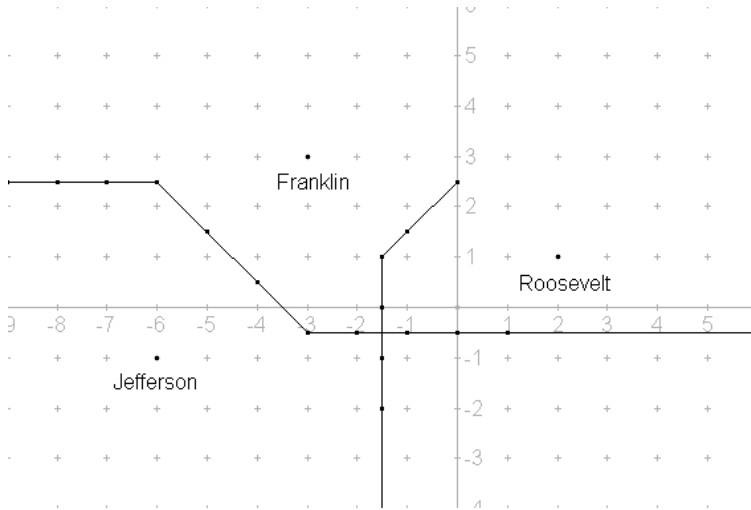
$$x - y + 6 = -x - y + 3$$

$$2x = -3$$

$$x = -3/2$$

\underline{x}	\underline{y}
-3/2	1
-3/2	0
-3/2	-1
-3/2	-2

In the following image you can see the plotted boundary in the region $-3 \leq x \leq 2$ and $1 > y$:



Case III: Now we have a look at the region where $-3 \leq x \leq 2$ and $3 < y$. Under these terms following simplifications for a_2 can be made:

Because $x + 3 \geq 0$, when $-3 \leq x \leq 2$, you can replace $|x + 3|$ by $x + 3$.

Because $y - 3 > 0$, when $3 < y$, you can replace $|y - 3|$ by $y - 3$

Because $x - 2 \leq 0$, when $-3 \leq x \leq 2$, you can replace $|x - 2|$ by $-x + 2$

Because $y - 1 < 0$, when $3 < y$, you can replace $|y - 1|$ by $y - 1$

If these simplifications are put into the starting equation a_2 you can observe:

$$x + 3 + y - 3 = -x + 2 + y - 1$$

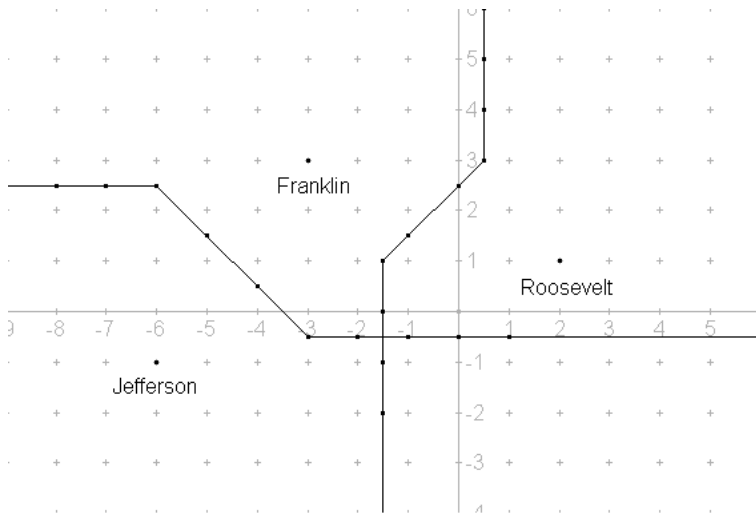
$$x + y = -x + y + 1$$

$$2x = 1$$

$$x = 1/2$$

\underline{x}	\underline{y}
1/2	3
1/2	4
1/2	5

The complete solution for section two was added to the solution of section 1 and can be seen below. The boundary was created such that $d(\text{Franklin}) = d(\text{Roosevelt})$:



In case IV – IX of this section no solutions exist either. You don't necessarily need to do the algebra because its obvious if you look at the graph above that no other solutions exist. In the other quadrants one school will always be closer to the points than the other.

Section 3

In section three you have a closer look at the boundary between Jefferson and Roosevelt school. Again you need the points where $d(\text{Jefferson}) = d(\text{Roosevelt})$, meaning the points which are equal distance from both schools, to create the boundary. The equation a_3 to find these points is:

$$d(\text{Jefferson}) = d(\text{Roosevelt})$$

$$d[(-6, -1), (x, y)] = d[(2, 1), (x, y)]$$

$$|x + 6| + |y + 1| = |x - 2| + |y - 1|$$

To solve this equation easier for x and y we divide the calculation into different cases again:

	$-1 \leq y \leq 1$	$-1 > y$	$1 < y$
$-6 \leq x \leq 2$	Case I	Case II	Case III
$2 < x$	Case IV	Case V	Case VI

$-6 > x$	Case VII	Case IIX	Case IX
----------	----------	----------	---------

Case I: Under the terms $-6 \leq x \leq 2$ and $-1 \leq y \leq 1$ you can simplify the starting equation a_3 as following:

Because $x+6 \geq 0$, when $-6 \leq x \leq 2$, you can replace $|x+6|$ by $x+6$.

Because $y+1 \geq 0$, when $-1 \leq y \leq 1$, you can replace $|y+1|$ by $y+1$

Because $x-2 \leq 0$, when $-6 \leq x \leq 2$, you can replace $|x-2|$ by $-x+2$

Because $y-1 \leq 0$, when $-1 \leq y \leq 1$, you can replace $|y-1|$ by $-y+1$

The simplified starting equation a_3 can be solved as following:

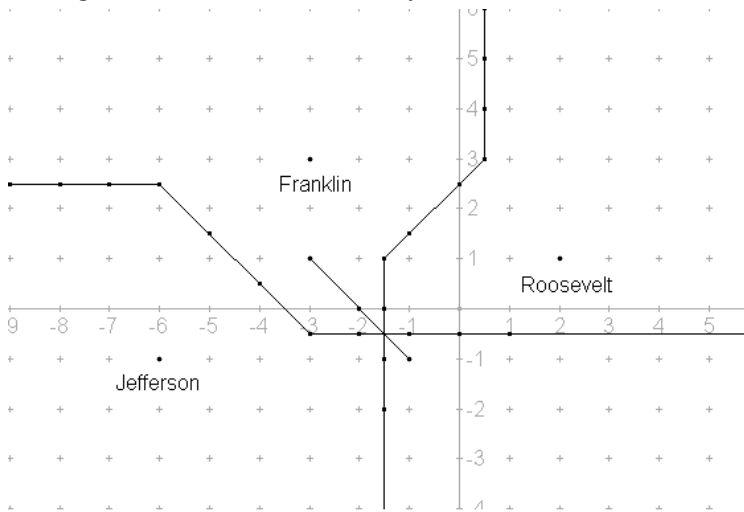
$$x+6+y+1=-x+2-y+1$$

$$x+y=-x-y-4$$

$$y=-x-2$$

\underline{x}	\underline{y}
-1	-1
-2	0
-3	1

In the picture below you can see the plotted boundary between Jefferson and Roosevelt school in the region $-6 \leq x \leq 2$ and $-1 \leq y \leq 1$:



Case II: Under these terms: $-6 \leq x \leq 2$ and $-1 > y$, you can simply a_3 as following:

Because $x+6 \geq 0$, when $-6 \leq x \leq 2$, you can replace $|x+6|$ by $x+6$.

Because $y+1 > 0$, when $-1 > y$, you can replace $|y+1|$ by $y+1$.

Because $x-2 \leq 0$, when $-6 \leq x \leq 2$, you can replace $|x-2|$ by $-x+2$.

Because $y-1 < 0$, when $-1 > y$, you can replace $|y-1|$ by $-y+1$.

Substituted into a_3 you can observe:

$$x + 6 - y - 1 = -x + 2 - y + 1$$

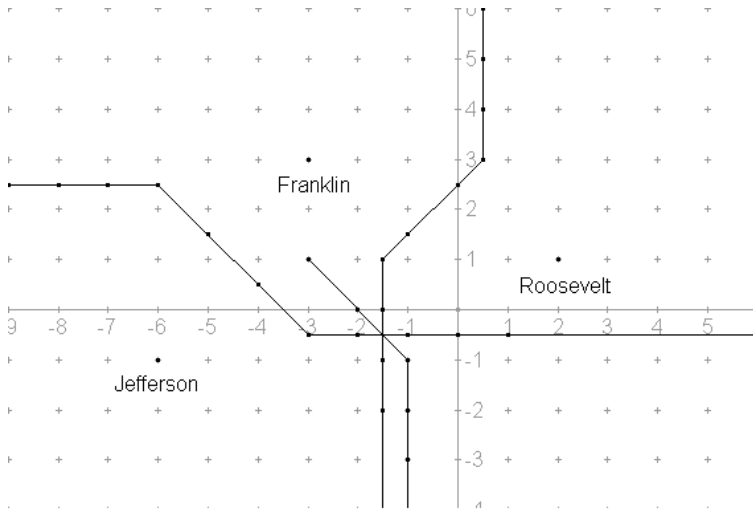
$$x - y + 5 = -x - y + 3$$

$$2x = -2$$

$$x = -1$$

x	y
-1	-1
-1	-2
-1	-3
-1	...

Below you can see the plotted boundary:



Case III: If $-6 \leq x \leq 2$ and $1 < y$ you can simplify the equation a_3 from the beginning as following:

Because $x + 6 \geq 0$, when $-6 \leq x \leq 2$, you can replace $|x + 6|$ by $x + 6$.

Because $y + 1 > 0$, when $1 < y$, you can replace $|y + 1|$ by $y + 1$.

Because $x - 2 \leq 0$, when $-6 \leq x \leq 2$, you can replace $|x - 2|$ by $-x + 2$.

Because $y - 1 < 0$, when $1 < y$, you can replace $|y - 1|$ by $y - 1$.

If these simplifications are substituted into a_3 you can solve the equation as following:

$$x + 6 + y + 1 = -x + 2 + y - 1$$

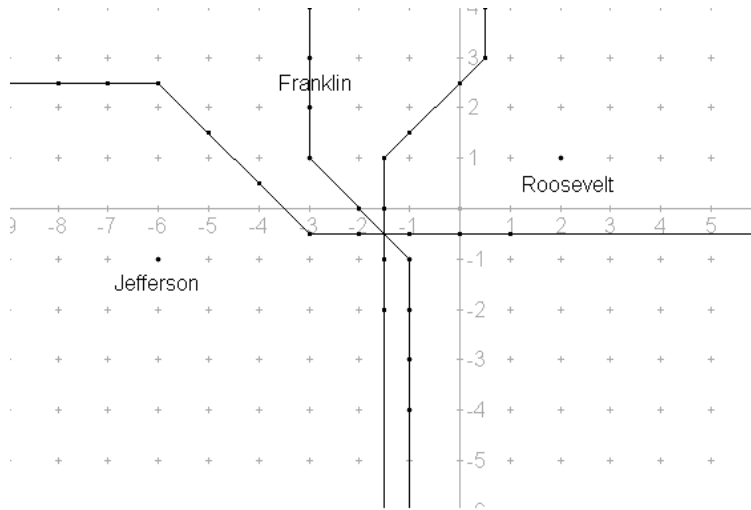
$$x + y + 7 = -x + y + 1$$

$$2x = -6$$

$$x = -3$$

x	y
-3	2
-3	3
-3	4
-3	...

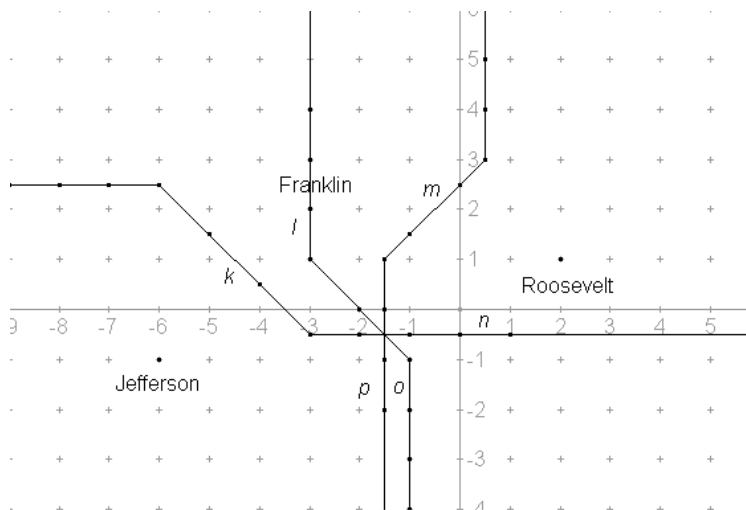
The last boundary which needed to be created was added to the other boundaries. The image which shows each calculated boundary looks like this:



In case IV – IX of this section no solutions exist either. Again you don't necessarily need to do the algebra because its obvious if you look at the graph above that no other solutions exist. In the other quadrants one school will always be closer to the points than the other

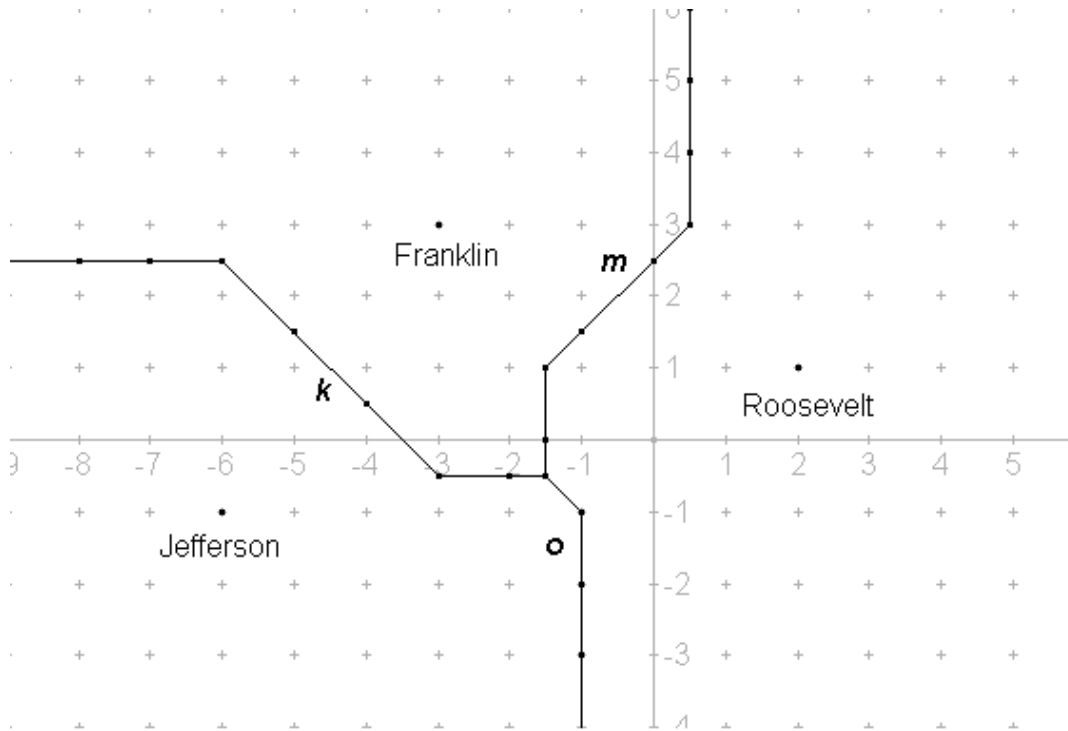
Section 4

In section 4 we need to look at the information we got from the previous sections and use some basic logic to interpret it. In the last image above you can see all boundaries between the three schools. At first we look at the point where all boundaries intersect which is in this example $(-1/2 / -1/2)$. From this point we will label every boundary for ease of explanation and look at each in particular:



At first we'll look at boundary *k*. Boundary *k* is a *correct* boundary because there is no other boundary which would divide Franklin and Jefferson school. On the contrary, boundary *l* was intended to divide Jefferson and Roosevelt but is a *wrong* boundary because the points on the line are closer to Franklin than to any other school. Since we now removed *l*, *m* must stay to

maintain a boundary between Franklin and Roosevelt. Boundary n is not necessary and can be removed because it is part of the boundary which was created to divide Jefferson and Franklin. In addition, every points on this boundary are closest to Roosevelt. The same is true for boundary p . The points on p are closest to Jefferson and p is part of the boundary which was created to divide Franklin and Roosevelt, so p can be removed. To maintain a boundary between Jefferson and Roosevelt boundary o has to stay. After we have removed some of the boundaries we have now our final solution. The final solution to a school district such that every pupil attends the school they live closest to can be seen in the picture below:



Lagrange: A Well-Behaved Function

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Abstract

This paper outlines the biography and achievements of Joseph Louis Lagrange (1736–1813) and includes a detailed explanation, with examples, of the Lagrange Multiplier method for optimizing multivariate functions subject to constraint. The Lagrange Multiplier is widely used in chemistry, physics, and economics, in particular. The paper considers the origin of economics' use of the multiplier and provides a concrete example of how it is used in microeconomic theory. While the focus is on the multiplier's application to microeconomics, the intended audience includes all teachers and students who encounter any of Lagrange's contributions. Since Lagrange's contributions to mathematics are numerous, so too are those who might benefit from learning more about the man and his time.

keywords: Lagrange, Lagrange Multiplier, Economics

1 Introduction

Joseph Louis Lagrange (1736–1813) lived during one of the most exciting periods in human history. Newton had published the *Principia Mathematica* less than 50 years, and died less than 10 years, before Lagrange's birth. The *Principia* (1687) not only provided a mathematical framework for analyzing physical phenomena, but it also signaled that the natural world functions according to a set of immutable laws. Insofar as Newton's audience adopted this message, the *Principia* set into motion a widely accepted belief that humans need only learn these laws in order to fully understand the natural world. "The eighteenth century," therefore, "was an age in which the power of the human mind seemed unlimited" (Grabier 1990). This positivist world view was adopted and expounded by mathematicians and philosophers such as Votaire, Kant, Euler, Laplace, Goethe, and others. Eighteenth century mathematicians, "almost without thinking", manipulated the calculus that Newton and Leibniz introduced and pushed it to its limits "without careful attention to convergence of series, without knowing under what conditions one might change the order of taking limits, [without well-founded knowledge] of integration..." (*ibid.*). Grabier (1990) further notes that

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physicists had been so successful with application of mathematics in explaining the natural world that they did not stop to show *why* mathematics was appropriate for describing the physical universe with a rigor that did not appeal to intuition.

Lagrange, no doubt, shared the optimism about mathematics and human progress that characterized his time. He chose the role, however, of the skeptic. His primary objective in mathematics—and life, it appears—was not to add to the list of new applications of calculus, but rather to revisit its foundations and offer a more rigorous explanation of how and why calculus works (Grabier (1990); Sarton (1944)). Before Lagrange, no formal definition of limits existed. Instead, the idea was explained by the *ad hoc* concepts of infinity and infinitesimals rather than by algebraic expressions of real numbers. Lagrange believed that conceptual or intuitive explanations had no place in a truly rigorous subject, and so Lagrange endeavored to reduce the foundations of calculus to algebra, which had already been established as a dependable representation of the real world.¹

The purpose of this paper is to give an introduction to Lagrange’s life, work, and influence on contemporary explanations and applications of calculus. Those who teach Lagrange’s methods in calculus and applied sciences will find this a useful supplement to enrich lectures or to assign to students. Those in the economics profession, in particular, will benefit from this paper since every intermediate microeconomics student must be intimately familiar with one of Lagrange’s techniques, the Lagrange multiplier. Few economists, however, learn about the multiplier’s origins. In the spirit of its creator, this paper seeks to transform merely a useful tool into an rigorously understood science. The discussion, therefore, will proceed as follows: Section 2 will present Lagrange’s biography, using the various cities in which he lived as a geographical framework for the phases of his intellectual growth. Section 3 will describe some of the contributions Lagrange made to mathematics and other disciplines, and it will give a specific example of how theoretical microeconomists regularly utilize his work. Section 4 will conclude.

2 Biography

2.1 Piedmont (1736–1755)

Lagrange was born in Piedmont, northern Italy, to where his great-grandfather had immigrated from Touraine, France in order to direct the Treasury of Construction and Fortifications under Charles Emanuel II Duke of Savoy.² His family was apparently wealthy, although much of that wealth was lost to failed financial speculation during his boyhood.

¹Geometry had long been accepted as an appropriate tool for describing and analyzing natural phenomena. Descartes’ work demonstrated the relationship between algebra and geometry and, by extension, the relationship between algebra and the natural world. Lagrange recognized, then, that by linking calculus to algebra, he would authenticate its use in describing the physical world.

²While his great-grandfather and name came from France, Lagrange can hardly be classified as French, himself. His grandfather and father both married Italian women, and Lagrange was raised in Italy. Nevertheless, we cannot so easily classify Lagrange as an Italian, either. Early in his life, he relocated to Berlin and then later moved to Paris. Additionally, Lagrange composed practically all his scholarship in French.

During his early childhood, Lagrange's interests were more classical than scientific (Simons (1972)), but he became enthusiastic about mathematics after reading a paper by Edmund Halley on the use of algebra in optics (1693). Sarton (1944) writes that Lagrange thereafter mastered "...with incredible speed the works of Newton, Leibniz, Euler, the Bernouli, [and] d'Alembert." By the time he was 18 years old, Lagrange began independent research into outstanding mathematical problems and established correspondence with Euler and Fagnano. Lagrange's first paper, which initially appeared as a letter to Fagnano from July 1754, drew an analogy between the binomial theorem and the series of derivatives of the product of functions. He also sent his findings to Euler, who was especially impressed with the young Lagrange, whose discovery of "a new series for the differentials and integrals of every grade corresponding to the Newtonian series for powers and roots" was spoiled only by the chance fact that Leibniz happened to have published the same discovery 50 years earlier. Lagrange's fear of being labeled a plagiarist fueled further efforts to make scientific contributions of real value. He began to work on finding solutions for the maxima and minima of the tautochrone, a curve on which a weighted particle will always arrive at a fixed point at the same time, regardless of its initial position (Chiang (1984); Koetsier (1986)). Lagrange's talent was apparent. The following year (1755), The Royal School of Mathematics and Artillery invited the him to teach in Turin.

2.2 Turin (1755–1766)

Lagrange continued to nurture his relationships with Euler, Fagnano, and d'Alembert during his tenure at the Royal School. During the first year there, Lagrange made up for the embarrassment of his first paper by helping Euler both solve some of the isoperimetric problems that had been plaguing him and also develop the calculus of variations. Euler, like Lagrange after him, was concerned with explaining calculus in algebraic and analytical terms: that is, without using intuition or the metaphor of geometry. In his 1744 classic, *A Method for Finding Curved Lines Enjoying Properties of Maxima or Minima*, Euler lamented that the solution to one of the problems considered in the book not purely analytical, writing "...we have no method that is independent of the geometrical solution..." (Koetsier (1986)). Lagrange soon thereafter presented Euler, who was 30 years older than Lagrange, with a purely analytic solution. Euler was so pleased that he wrote to his junior colleague, "...I am not able to admire you enough..." (Grabier (1990); Sarton (1944)).

Around 1759, Lagrange collaborated with the chemist, Saluzzo de Monesioglio, and the anatomist, Gian Francesco Cigna, to found the Turin Academy of Sciences. Academies provided an environment in which well-funded research could proceed unimpeded by the distraction of teaching. Accordingly, the Turin Academy quickly began to publish journals, the first volume of which contained 3 papers by Lagrange (Sarton (1944)). These papers featured his research on the nature and propagation of sound, the movement of the moon, and on the satellites of Jupiter. Meanwhile, he published a paper in the *Mémoires de l'Académie de Berlin* on an analytical expression of tautochronous curves. These papers earned Lagrange both prizes from the Parisian Academy as well as recognition throughout Europe. By 1766, Lagrange was ready to leave Turin, and his friend, Euler, played a major

role in ushering Lagrange to the next phase in his life.

2.3 Berlin (1766–1787)

Leibniz founded the Prussian Academy in 1700; and under the management of Frederick II (Frederick the Great), it became one of the premier academies in Europe, housing people like Euler, d’Alembert, Lambert, Kant, and Diderot. Euler left his position as Director of the Mathematical Section at the Prussian Academy for St. Petersburg in early 1766, and Lagrange assumed that post by the end of the year with the recommendations Euler (who was leaving) and d’Alembert (who refused the first invitation). The King famously wrote to d’Alembert after attaining Lagrange that he was happy to have “...replaced a one-eyed geometer with a two-eyed one...”³ (Sarton (1944); Koetsier (1986); Grabier (1990)).

Lagrange’s was most productive in Berlin. He worked mostly on mathematical and mechanical problems, and his writings were frequently published by the Academies of Berlin, Turin, and Paris. During the same time, Lagrange wrote a 315 page supplement to the second volume of the French translation of Euler’s *Elémens d’algèbre*. Sarton (1944) describes Lagrange’s time in Berlin as “...the richest years of his life...”.

Despite the admiration that Lagrange’s work afforded him, his peers and mathematical historians characterize him as exceedingly modest. Whereas Lambert famously enumerates the leading mathematicians, “Euler and d’Alembert are first and equal, Lagrange is third, and I am fourth, and there is no need of going any further as I cannot think of anybody else worth quoting after these”, Lagrange was much more humble. His first and almost ritualistic response to any question was always “*Je ne sais pas*”⁴. Lagrange was diffident when presenting new ideas to friends like d’Alembert. Sarton (1944) quotes Lagrange as having bemoaned the fact that he could not present d’Alembert with something better than the *Mécanique Analytique* to win the latter’s favor. Sarton (1944) points out that Lagrange’s diffidence went beyond “...the conventional exaggeration which eighteenth-century courtesy required.”

During his 21 years in Berlin, Lagrange developed the reputation of being somewhat 1-dimensional. That is, he seems to have been primarily—even solely—concerned with conducting mathematical research. When d’Alembert wrote to congratulate Lagrange on his first marriage and the happiness he was certain to derive from it, Lagrange responded that he had no taste for marriage, and would have avoided it altogether if not for his need of a nurse and housekeeper to provide him with the quiet conveniences necessary for a life of research (Sarton (1944)). Lagrange had no passion for musical performances, except that they provided a nice environment in which he could ponder mathematical problems without interruption. His daily diet and routine were unconventional and strict, and each day included at least 8 hours of uninterrupted and isolated study. Lagrange did not have many close acquaintances.

³Euler and Frederick did not have particular fondness for one another, and the King is here poking fun at Euler’s loss of sight that was just beginning. He was completely blind at the time of his death.

⁴“I do not know.”

Lagrange's sacrifice of companionship and his focus paid off. While in Berlin, he composed most of the *Mécanique Analytique*. His masterpiece, however, was published after Lagrange arrived in Paris. When Frederick the Great, to whom Lagrange was *persona grata*, died, the latter knew his position at the Prussian Academy would soon either end or become remarkably less agreeable under the new direction of Frederick William II. The French government offered to install Lagrange at the Paris Academy, and he quickly accepted. He left Berlin on 18 May, 1787 (Sarton (1944)).

2.4 Paris (1787–1813)

In Paris, Lagrange completed, edited, and published his magnum opus, the *Mécanique Analytique* (1788). Many of Lagrange's time and afterward welcomed the work with enthusiasm. Ernst Mach called it "...a stupendous contribution to the economy of thought..." (Grabier (1990)), Vittorio Fossombroni called it "immortal" (Sarton (1944)), and William Rowan Hamilton described the *Mécanique Analytique* as "a kind of scientific poem" (Simons (1972)). In the book, Lagrange distilled classical mechanics to the calculus of variations, which he then further reduced to pure algebraic analysis. Koetsier (1986) quotes Lagrange as writing in an advertisement for the book: "There are no figures at all in this work. The methods that I demonstrate require neither constructions nor geometrical or mechanical reasoning, but only algebraic operations subjected to a regular and uniform procedure. Those who love analysis will be pleased to see that mechanics is becoming a new branch of it and they will be grateful to me for having thus extended its domain."

After the publication of the *Mécanique Analytique*, Sarton (1944) writes, Lagrange was mentally exhausted and his contributions lessened. His inactivity could also be due, in part, to his being distracted by his work at the newly formed Commission of Weights and Measures, and also to the stress caused by the French Revolution, which began 1 year later. He had been invited to France by its pre-Revolutionary government, had stayed at the Louvre, and could easily have been marked as an enemy to the revolution during the Terror (1793–1794). Most of his colleagues were removed from their positions on the Commission, and Lavoisier was put to death. Lagrange, however, was retained, probably due to his characteristic modesty and aversion to conflict.⁵ During his position on the Commission, Lagrange promoted the official adoption by the French government of the metric system, including the use of decimals.

After the Revolution, he fell into the favor of Napoleon Bonaparte, who enjoyed sharing geometrical puzzles with Lagrange and Laplace. Napoleon's goodwill afforded Lagrange more security and comfort than upheaval of the Revolution. Lagrange spent the first part of the 19th century—and the final part of his life—teaching and revising many of his earlier works. During his last conversation with his colleagues, Lagrange said, "Oh! death is not to be feared and when it comes without pain it is a last 'function' which is neither distressing nor disagreeable...I have had a long career; I have obtained some fame in mathematics. I have never hated anybody, I have done no harm, and there must be an end..." (Sarton (1944); Koetsier (1986)). Lagrange died 10 April, 1813. He lived 77 years.

⁵"I don't know why they kept me", Lagrange said (Sarton (1944)).

3 Lagrange's Mathematical Contributions

Many of the practices of students in modern calculus courses derive from the work of Lagrange. Most generally, Lagrange, with other mathematicians, endeavored to strengthen the foundation of calculus, which had so far rested on geometrical and intuitive arguments. As mentioned above, proving the fundamentals of calculus in algebraic terms lent a more detailed and rigorous understanding of the system, established calculus' place in the canon of accepted branches of mathematics, and paved the way for advances in the science.

Lagrange's contributions to how we understand the limit of a function is an example of how he worked to introduce rigor into calculus. During Lagrange's time, mathematicians understood what we now call the derivative to be the slope of the a function when the change in the input (x) is *infinitesimally* small. *Infinitesimals* make sense to modern calculus students who can compare the concept to the formal definition of the derivative as the limit of the difference quotient as the denominator approaches 0. For Lagrange, however, there was no formal definition to which he could relate the concept of *infinitesimals*. There was no definition of continuity or convergence of sequences. In 1774, Lagrange and his colleagues at the Prussian Academy hosted a contest for advancements in the foundations of calculus (Grabier (1983)). And while Lagrange, himself, never developed satisfactory explanation for the foundations of calculus, he played a major role in creating an atmosphere in which such an explanation was necessary.

His work with inequalities, also, helped Cauchy develop a formal definition of the limit, the modern $\delta - \epsilon$ proof of the derivative, and the definition of the derivative as the limit of the difference quotient (Grabier (1983)). These definitions (and the proofs that accompany them) were, of course, exactly the kind of advancement for which Lagrange was looking. They translate nearly 200 years of intuitive understanding of calculus into a form with the certainty of algebra.

Some of Lagrange's more superficial—but no less important—contributions include the prime notation, whereby for a function, $f(x)$, its derivative is denoted $f'(x)$. Lagrange was the first to use the term, *dérivée*, from which we get the term 'derivative'. He also developed the use of ∂ to indicate a partial derivative in a multivariate function. And, for that matter, he developed the multiplier method for calculating optimizations of one multivariate function that is constrained by another.

3.1 The Lagrangian Multiplier

During his exchange of letters with Euler, Lagrange became inspired by the former's 1744 work, *A Method for Finding Curved Lines Enjoying Properties of Maxima or Minima*. In it, Euler worked to find the extrema of functions that are defined on a set of functions rather than a set of numbers on points (Koetsier (1986)). His solutions to these problems were, however, not purely analytical; and it was Lagrange who provided the following strictly analytical method for solving constrained optimizations.

3.1.1 The General Case

Suppose we wanted to find the optimization of some function $z = f(x, y)$, where the variables x and y are not independent. Instead, they are dependent on the side condition, $g(x, y) = 0$, which we call the constraint. That is, we want to:

$$\max_{x,y} z = f(x, y) \quad \text{subject to} \quad g(x, y) = 0$$

The usual way to solve this problem is to arbitrarily assume 1 of the variables is independent and the other dependent. If choose x to be the independent variable, then we can compute $\frac{dy}{dx}$ from the constraint function:

$$\frac{\partial g}{\partial x} + \frac{\partial g}{\partial y} \frac{dy}{dx} = 0 \tag{1}$$

which can be rewritten as:

$$\frac{dy}{dx} = -\frac{\partial g / \partial x}{\partial g / \partial y} \tag{2}$$

Now, we can do the same for the function, finding $\frac{dz}{dx}$ and setting it equal to 0 (by definition of the maximum or minimum of the function), such that:

$$\frac{dz}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \frac{dy}{dx} = 0 \tag{3}$$

If we plug equation (2) into (3), we get:

$$\frac{dz}{dx} = \frac{\partial f}{\partial x} - \frac{\partial f}{\partial y} \frac{\partial g / \partial x}{\partial g / \partial y} = 0 \tag{4}$$

If we solve (4) and the constraint function simultaneously, we get the required points (x, y) to optimize z . One problem with this method, however, is that x and y occur symmetrically, but are treated unsymmetrically (Simons (1972)).

Lagrange's method, which Simons (1972) calls more elegant, begins by forming a new function:

$$F(x, y, \lambda) = f(x, y) + \lambda g(x, y) \tag{5}$$

Since the constraint is now built in to the new function, F , we can treat it like an unconstrained optimization problem (Chiang (1984)). So, we take the partial derivatives of F with respect to x, y and λ and setting them equal to 0:

$$\begin{aligned} \frac{\partial F}{\partial x} &= \frac{\partial f}{\partial x} + \lambda \frac{\partial g}{\partial x} = 0 \\ \frac{\partial F}{\partial y} &= \frac{\partial f}{\partial y} + \lambda \frac{\partial g}{\partial y} = 0 \\ \frac{\partial F}{\partial \lambda} &= g(x, y) = 0 \end{aligned} \tag{6}$$

If we solve $\frac{\partial F}{\partial x}$ and $\frac{\partial F}{\partial y}$ for λ and set them equal to each other, we get:

$$-\frac{\partial f/\partial x}{\partial g/\partial x} = -\frac{\partial f/\partial y}{\partial g/\partial y} \quad (7)$$

Which can be written, along with $\frac{\partial F}{\partial \lambda}$, to give:

$$\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y} \frac{\partial g/\partial x}{\partial g/\partial y} = 0 \quad \text{and} \quad g(x, y) = 0 \quad (8)$$

In this way, Lagrange reached the same conclusion as (4) without introducing the arbitrary assignment of an independent variable or disturbing the symmetry of the problem. Introducing the new variable, λ , which falls out of the problem, is a small expense. The parameter, λ , is called the *Lagrange multiplier*; and economists love it.

3.1.2 The Lagrange Multiplier in Economic Theory

All economic inquiries, at their most basic levels, ask how decision makers can achieve the greatest benefit, given their situation. The benefit can come from a wide range of possibilities, including time spent with family, the number of trees planted in one's yard, or the number of deep dish meat lover's pizzas one eats. But when decision makers make their decisions, they do not do so based only on the amount of the benefit that would satisfy them, but also on the amount of that benefit they can afford. That is, with no constraints, a father might want to spend all of his time with his family; but at the very least there is the time constraint whereby he needs to devote at least a few hours each day to sleep. A homeowner may want to plant hundreds of trees in her yard, but she may only have enough land to support 50 trees. And that dieter may want to eat 3 of those pizzas before lunch is over, but she is constrained by a diet that only affords her 900 calories a day. In short, economists are concerned with maximizing benefit (commonly referred to as "utility") subject to the constraint of limited resources. Lagrange's method provides a simple way of representing these problems. It was first used by Westergaard in 1876, over 100 years after Lagrange developed it (Davidson (1986)).

3.1.3 The Lagrange Multiplier in Economic Theory: an Example

Suppose there were some decision maker, Kelda, who enjoys cooking with xanthan gum and yams; and the pleasure (utility) she derives from cooking with these ingredients is characterized by the function $U = x^{.5} + y^{.5}$, where x represents the amount of xanthan gum she cooks with, and y represents the number of yams she uses. Suppose that Kelda spends all of her income, I , on these two ingredients. Then, the problem of maximizing Kelda's pleasure is given by:

$$\max_{x,y} U = x^{.5} + y^{.5} \quad \text{subject to} \quad P_x x + P_y y = I$$

Where P_x and P_y are the prices of xanthan gum and yams, respectively. We can build a new function, F , to incorporate this information about Kelda's pleasure, such that:

$$F = x^{.5} + y^{.5} + \lambda(I - P_x x - P_y y) \quad (9)$$

Now, we maximize F by taking the partial derivatives with respect to x, y , and λ , so:

$$\begin{aligned} \frac{\partial F}{\partial x} &= .5x^{-.5} - \lambda P_x = 0 \\ \frac{\partial F}{\partial y} &= .5y^{-.5} - \lambda P_y = 0 \\ \frac{\partial F}{\partial \lambda} &= I - P_x x - P_y y = 0 \end{aligned} \quad (10)$$

If we eliminate λ from $\frac{\partial F}{\partial x}$ and $\frac{\partial F}{\partial y}$ and solve for x , we get:

$$x = \frac{P_y y}{P_x} \quad (11)$$

If we plug this into the constraint, then we get the optimal number of yams:

$$y^* = \frac{I}{2P_y} \quad (12)$$

and the optimal amount of xanthan gum:

$$x^* = \frac{I}{2P_x} \quad (13)$$

With each of these last 2 equations, we have calculated a mathematical expression of what we intuitively know: namely, that if income stays the same but the price of a good increases, we will buy less of that good, holding all else constant. We feel good for having done such a thing and are grateful to Lagrange for having provided us with a way to do it.

4 Conclusion

One of the greatest pleasures of studying the history of mathematics comes from discovering the strange habits mathematicians had. The Pythagoreans did not eat beans. Descartes never got out of bed before 11 o'clock in the morning (until, that is, he quickly died after being forced to begin his day at 6 AM to tutor queen Christina). And Lagrange was a shy man who kept a very regular schedule, avoided all conflict, answered each question, "Je ne sais pas", and seemed to think only of math. What is so wonderful about learning these traits is that it makes living people of these text book figures whom we would otherwise know only through the mathematical theorems, coordinate systems, and notation styles named after them. That is, learning about the lives of mathematicians allows teachers and students of mathematics to carry their education beyond lifeless memorization and

mechanical manipulation to something wholly more fulfilling. Knowing Lagrange's humanity makes his mathematical achievements somehow more extraordinary and less-easily taken for granted. During his tireless devotion to a goal he never saw achieved, Lagrange—almost as an aside—provided generations with the tools necessary to achieve that goal and many others.

References

- Chiang, Alpha C.**, *Fundamental Methods of Mathematical Economics*, 3 ed., McGraw-Hill International Editions, Economics Series, 1984.
- Davidson, Thorkild**, "Westergaard, Edgeworth and the Use of Lagrange Multipliers in Economics," *The Economic Journal*, September 1986, 96 (383), 808–811.
- Grabier, Judith V.**, "Who Gave You the Epsilon? Cauchy and the Origins of Rigorous Calculus," *The American Mathematical Monthly*, 1983, 90 (3), 185–194.
- , *The Calculus as Algebra: J. L. Lagrange, 1736–1813* Harvard Dissertations in the History of Science, Garland Publishing, Inc., 1990.
- Koetsier, T.**, "Joseph Louis Lagrange (1736–1813) : his life, his work and his personality," *Nieuw Archief Voor Wiskunde*, 1986, 4 (3), 191–205.
- Sarton, G.**, "Lagrange's Personality (1736–1813)," *Proceedings of the American Philosophical Society*, 1944, 88, 457–496.
- Simons, George F.**, *Differential Equations: With Applications and Historical Notes*, McGraw-Hill Book Company,