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Book X of The Elements: Ordering Irrationals

Jade Roskam

BOOK X OF *THE ELEMENTS*: ORDERING IRRATIONALS

Jade Roskam[1], *The University of Montana*

Abstract: Book X from The Elements contains more than three times the number of propositions in any of the other Books of Euclid. With length as a factor, anyone attempting to understand Euclidean geometry may be hoping for a manageable subject matter, something comparable to Book VII's investigation of number theory. They are instead faced with a dizzying array of new terminology aimed at the understanding of irrational magnitudes without a numerical analogue to aid understanding. The true beauty of Book X is seen in its systematic examination and labeling of irrational lines. This paper investigates the early theory of irrationals, the methodical presentation and interaction of these magnitudes presented in The Elements, and the application of Euclidean theory today.

Keywords: Book X; Euclid; Euclid's Elements; Geometry; History of mathematics; rationals and irrationals; Irrational numbers

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1. BACKGROUND

Book X of Euclid's *The Elements* is aimed at understanding rational and irrational lines using the ideas of commensurable and incommensurable lengths and squares. Unfortunately, a lack of documentation of the early study of incommensurables leads to speculation on its exact origin and discoverer. Wilbur R. Knorr in a 1998 article from The American Mathematical Monthly dates original knowledge, but not necessarily understanding, of irrational quantities to the Old Babylonian Dynasty Mesopotamians. The mathematical tables of these peoples, dating back to 1800-1500 BC, supposedly demonstrate knowledge of the fact that some values cannot be expressed as ratios of whole numbers. However, many sources disagree with Knorr's article and attribute original knowledge of irrational magnitudes to the school of Pythagoras around 430 BC (Fett, 2006; Greenburg, 2008; Robson, 2007; Posamentier, 2002). Given the most well-known accomplishment to come from the Pythagoreans, the Pythagorean theorem, it seems inevitable that this group of people would discover irrational values in the form of diagonals of right triangles. Take for example the length of the hypotenuse of an isosceles triangle with side lengths 1. This gives one of the most studied irrational quantities, $\sqrt{(2)}$. Prior to this inexorable discovery, the Pythagoreans viewed numbers as whole number ratios and therefore could not incorporate irrational quantities into their theory of numbers. Irrationals, considered to be an unfortunate discovery and the result of a cosmic error, were treated as mere magnitudes inexpressible in numerical form (Fett, 2006; Greenburg, 2008). These ideas were continued during the writing of The Elements, and would remain until the Islamic mathematician al-Karaji

[1] Jade.Roskam@umontana.edu

translated Euclidean terminology into irrational square roots of whole numbers approximately 13 centuries after Euclid wrote (Berggren, 2007).

The Pythagoreans attitude toward irrationals stunted any studying of the magnitudes beyond the incommensurability of a square's side to its diagonal. Fortunately, the superstition surrounding irrationals did not reach Plato's camp. Theodorus, a student of Plato, and one of Theodorus' own students, Theaetetus, took it upon themselves to study irrational magnitudes at length and put forth the first known theory of irrational lines (Knorr, 1975). Theodorus is cited as the first to produce varying classes of incommensurable lines through arithmetic methods argues Knorr (1975). However, Theodorus' discoveries were limited to specific cases, like lines cut in extreme-and-mean ratio, and he was unable to generalize his findings. It was his student, Theaetetus, who is generally considered as the first to put forth an organized, rigorous theory of irrationals, a work that started intuitively with his master but one that Theodorus ultimately could not prove (Knorr, 1975; 1983). The assembled findings of Theodorus and Theaetetus were published by Plato in a dialogue titled after the younger mathematician. Unfortunately much of Theaetetus has been lost over time and the little that is known about Theaetetus' early theory of irrationals comes from Eudemus, a student of Aristotle. Eudemus lived between the times of Plato and Euclid and is credited as having passed the early theory of irrational lines to Euclid's generation to be examined in full force in Book X of The Elements (Knorr, 1975; Euclid, 2006). If it was not for Plato's Theaetetus and the accounts from Eudemus, we may very well have attributed the entirety of the ideas of commensurable and incommensurable magnitudes to Euclid (Knorr, 1983).

Theaetetus is the one credited with having classified square roots as those commensurable in length versus those incommensurable (Knorr, 1983; Euclid, 2006). The three main classes of irrational magnitudes are the medial, binomial, and apotome. The medial line is defined as the side of a square whose area is equal to that of an irrational rectangle. The binomial and apotome oppose one another, as the binomial is formed by the addition of two lines commensurable in square only and the apotome is defined as the difference between two lines commensurable in square only. Each class of magnitude will be discussed in more detail later. Theaetetus is also said to have tied each class of magnitude with a unique mean: he medial is tied to the geometric mean, the binomial to the arithmetic, and the apotome to the harmonic mean (Euclid, 2006). However, these terms may just have been a replacement by Eudemus to tie irrational lines to Euclidean means, rather than the original correlations Theaetetus may have used (Knorr, 1983). The history behind the advancement of irrationality theory cannot exclude Euclid from its discussion. It was Euclid who generalized the idea of commensurable and incommensurable to squares, and also ordered the binomial and apotome irrational lines into six distinct classes each (Knorr, 1983). Most of the post-Euclidean advancement of the theory of irrational lines is found in propositions 111-114 of Book X which are generally considered to have been additions due to the lack of contiguity between these and the previous properties of irrationals presented. It is important to note that Book X details a theory of irrational *magnitudes* and not a theory of irrational *numbers* (Grattan-Guinness, 1996). Theaetetus' original theory of irrationals may have included numbers, but Euclidean theory deals solely with irrational lines and geometric lengths. The six classes of binomial and apotome are now more easily understood using algebra as the ordering of irrational magnitudes is explained through solutions of a general quadratic formula. The basis of this development is somewhat controversial. Knorr (1975) attributes some of the "geometric algebra" to Theodorus. Most sources believe this understanding of geometry through algebra originated in the 8th Century through the vast advances made by many Islamic mathematicians in the area of algebra (Gratten-Guinness, 1996; Berggren, 2007). Some now argue that much, if not most, of The Elements is actually algebra disguised as geometry (Gratten-Guinness, 1996). However, as will be discussed later, this idea is a hindrance to understanding Euclidean theory. While using the solutions to a general quadratic is a good way to help understand how each order of binomial and apotome is derived, it inherently ignores all irrationals that are not in the form of a square root and treats irrationals as values rather than magnitudes (Burnyeat, 1978).

2. Euclid on Irrationals

At the start of Book X Euclid (2006) provides definitions for commensurability and rationality. For commensurability Euclid states that "magnitudes are said to be commensurable which are measured by the same measure, and those incommensurable which cannot have any common measure" and that "Straight lines are commensurable in square when the squares on them are measured by the same area, and incommensurable in square when the squares on them cannot possibly have any area as a common measure" (p. 693). Euclid (2006) then moves to rationality which he defines as:

Let then the assigned straight line be called rational, and those straight lines which are commensurable with it, whether in length and in square, or in square only, rational, but those that are incommensurable with it irrational….And then let the square on the assigned straight line be called rational, and those areas which are commensurable with it rational, but those which are incommensurable with it irrational, and the straight lines which produce them irrational (p. 693).

To simplify, given a rational length (or number), all lengths (numbers) that have common measure with the rational and/or with the square of the rational are also rational. Those lengths that do not have a common measure with the given line are irrational. Squaring a rational length produces a rational area, and those areas that are commensurable with the rational area are rational and those incommensurable with the rational area are irrational. If an area is irrational, the length that was squared to create the irrational area is also irrational.

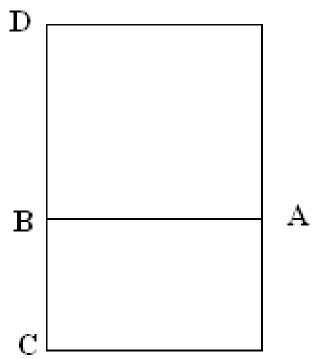
In total, there are 13 distinct irrational straight lines. In addition to the medial, Euclid sets up six orders of binomials and six orders of apotomes. The Elements also defines a subgroup of irrational lines that can be constructed from the thirteen distinct irrationals which include first and second order bimedial lines, first and second order apotome of a medial line, major, and minor, the first four of which will be discussed briefly.

According to Euclid (2006), a medial is formed when a rectangle contained by two rational straight lines commensurable in square only is irrational and the side of the square equal to it is irrational. The side of the square is called the medial (X. 21) [1].

Book X. Proposition 21

In the diagram below, lines *AB*, *BC* are assumed to be rational lengths that are commensurable in square only. That is to say, the square on *AB* and the square on *BC* have the ratio of a whole number to a whole number, but lengths *AB*, *BC* do not have a common measure. Now construct the square *AD* such that *AB* = *BD*.

[1]Note that for ease, I will denote propositions from The Elements by (Book. Proposition Number). For instance, Proposition 47 from Book I will be cited as (I. 47).



Then the square *AD* is rational since $AD = AB^2$ and *AB* is rational. We know that *AB* and *BC* are incommensurable in length, which implies that *BD*, *BC* are also incommensurable in length. Note that

$$\frac{BD}{BC} = \frac{BD * AB}{BC * AB} = \frac{AD}{AC}$$

Since *BD*, *BC* are incommensurable, this implies that *AD*, *AC* are also incommensurable. But we know that *AD* is rational, so *AC* must be irrational. Since *AC* is an irrational area, a square with equal area will also be irrational and, by definition, will have a side of irrational length. This irrational side length is known as a medial.

Binomials on the other hand are formed when two rational straight lines commensurable in square only are added together, making the whole irrational. The following is adapted from The Elements (X. 36):

Let lines *x*, *y* be rational and commensurable in square only, meaning that nothing measures both *x* and *y*, but x^2 and y^2 have a common measure. It is proposed that their sum, $x + y$, will be irrational and, as per The Elements, called a binomial.

(*i*) Since *x* is commensurable in square only with *y*, *x* and *y* are incommensurable in length. Therefore, since

$$\frac{x}{y} = \frac{x^2}{x * y}$$

(*ii*) It follows that x^2 and $x*y$ are also incommensurable.
But since x^2 and y^2 have a common measure, *a*, then

$$n * a = x^2$$
$$m * a = y^2$$

where m, n are integers.
By substitution,

$$x^2 + y^2 = n * a + m * a = a * (n + m)$$

(iii) So a measures x^2 and a measures $(x^2 + y^2)$, which implies that x^2 and $x^2 + y^2$ are commensurable.
(iv) It is obvious that x*y is commensurable with $2*(x*y)$.
(v) Since (iii) x^2 and $x^2 + y^2$ are commensurable, (iv) $x*y$ and $2*(x*y)$ are commensurable, but (ii) x^2 and $x*y$ are incommensurable, it follows that $x^2 + y^2$ and $2*(x*y)$ are incommensurable. From this, we see that
$(x^2 + y^2 + 2*(x*y))$ must be incommensurable with $(x^2 + y^2)$. Rearranging the first term, $(x + y)^2$ and $(x^2 + y^2)$ are incommensurable. Since x, y are rational, then x^2, y^2 are also rational and it follows that $x^2 + y^2$ is rational. This implies that $(x + y)^2$ is irrational, and therefore $(x + y)$ is irrational.

Euclid defines an apotome in proposition 73 of Book X as the remainder of two rational straight lines, the less subtracted from the greater, which are commensurable in square only. It is, in essence, the counterpart of the binomial. Euclid's proof that the apotome is irrational follows the same logical steps as those used to prove the irrationality of the binomial. We start will the same basic assumption, that lines x, y are rational and commensurable in square only. It is proposed that the apotome, $x – y$, is irrational. Steps (i) through (v) are identical to the proof of Proposition 36. For the apotome, note that

$$x^2 + y^2 = 2 * x * y + (x - y)^2$$

Since (v) $x^2 + y^2$ and $2*(x*y)$ are incommensurable, it follows that $x^2 + y^2$ and $(x - y)^2$ are also incommensurable.
But since x, y are rational by construction, $x^2 + y^2$ must be rational. This implies that $(x – y)^2$ is irrational, from which it follows that $x – y$ is irrational. Thus we have proven that if a rational straight line is subtracted from a rational straight line, and the two are commensurable in square only, the remainder will be irrational.

It was stated earlier that Theaetetus tied the three known types of irrationals at the time to unique means: the medial with the geometric, the binomial with the arithmetic, and the apotome with the harmonic. The first two of these pairings follow somewhat simply. The medial is tied to the geometric mean, which can be found using the following general formula.

$$G(\mathrm{x}_1, \mathrm{x}_2, \ldots, \mathrm{x}_\mathrm{n}) = \sqrt[n]{(\mathrm{x}_1 * \mathrm{x}_2 * \ldots * \mathrm{x}_\mathrm{n})}$$

A medial is defined as the length of the side of a square whose area is equal to that an irrational rectangle formed by two rational lines commensurable in square only. Using our above diagram, the square on the medial, we will call it MN for simplicity, is equal to the area of rectangle AC. Algebraically,

$$MN^2 = AC = AB * BC$$
$$MN = \sqrt{AB * BC}$$

Medials can therefore be represented symbolically as the geometric mean

$$\sqrt{x * y}$$

for two given rational magnitudes *x* and *y* commensurable in square only.
As stated earlier, the binomial is defined as the sum of two rational straight lines commensurable in square only and is closely related to the arithmetic mean, of which the following is the general formula.

$$A(\mathrm{x}_1, \mathrm{x}_2, \ldots, \mathrm{x}_\mathrm{n}) = \frac{1}{n} * (\mathrm{x}_1 + \mathrm{x}_2 + \cdots + \mathrm{x}_\mathrm{n})$$

It is obvious how the representation of the binomial ($x + y$) is closely linked to this formula. However, the coupling of the apotome and the harmonic mean is more complex. To explain, the harmonic mean of two numbers, *x* and *y*, is

$$\frac{2 * x * y}{x + y}$$

If you consider the propositions X.112-4, you can see that if a rational area has a binomial for one of its sides, the other side will be an apotome commensurable with the binomial and of corresponding order. Using our knowledge of the general form of an apotome and a binomial, we can see that this area would be

$$(x + y) * (x - y) = x^2 - y^2$$

With $x*y$ representing a medial area and the above equation for the given rational area, we see that

$$\frac{2 * x * y}{x^2 - y^2} * (x - y)$$

with (x – y) representing the basic form of an apotome. Again, this seems like a stretch given the ease with which the binomial and medial are tied to their respective means. It should be noted that this relationship between the apotome and the harmonic mean is explained in the commentary by Woepcke in an Arabic translation of Book X of The Elements (Euclid, 2006). Whether this was Theaetetus' original reasoning for pairing the apotome and harmonic mean is unknown. Again, these algebraic explanations are not the original work of Euclid, but theories imposed upon his work by later mathematicians. This is important to note because Euclidean theory pertained solely to irrational magnitudes and not to irrational numbers. Since most of Theaetetus' originally theory is lost, it cannot be determined conclusively whether the Platonic mathematician described the above relationships. The ties between three types of Euclidean quantities and three Aristotelian quadrivium is seen elsewhere in The Elements. According to Ivor Gratten-Guinness' 1996 article, the three types of quantities Euclid addresses, number, magnitude, and ratio, correlate to arithmetic, geometry and harmonics, respectively. These relationships certainly follow more readily than the irrational magnitudes to the corresponding means, and the latter associations may have been formed in response to the former.

3. Ordering Irrationals

A class is defined as a set of objects connected in the mind due to similar features and common properties (Forder, 1927). All magnitudes in the class of binomials are of the form $(x + y)$ where x, y are lines commensurable in square only. All binomials share common features, which will be discussed later. The same is true of apotomes. All are of the form $(x - y)$ where x, y are lines commensurable in square only, and they share common properties. These represent two of the three classes proposed in Theaetetus' early theory of irrationals. Within each of these classes, Euclid defines six orders, or sub-classes, of each. Each member of a sub-class contains all the properties common to the class as a whole, but has different properties from members of other sub-classes (Forder, 1927). Theaetetus is credited with ideas of the medial, binomial, and apotome, but he makes no reference to the six orders of binomials and apotomes listed in The Elements. Therefore it was up to Euclid's discretion on how to best order the sub-classes. The difficulty in Euclidean theory of irrationals lies in the overlap of properties between sub-classes. As will be discussed in detail, the six orders of each class are paired into three groups, with one from each pair representing commensurability and one from each pair representing incommensurability. The struggle arises from what is most important in the class, the commensurability or the pairing with another sub-class. Despite the algebraic understanding of Euclid's irrationals making the pairing of sub-classes easier to follow, Euclid chose to first break each class of irrational line into commensurable versus incommensurable, and then pair the members in each.

Euclid defines each of the six orders of binomial and apotome in Definitions II and III, respectively, of Book X and the introduction to Book X provides an algebraic understanding of how each type is derived. To clarify the definitions given by Euclid, we will represent a binomial using the general form $(x + y)$ with x being the greater segment and y being the lesser segment and an apotome using the general form $(x - y)$ with x being the whole and y being the annex (or what is subtracted from the whole).

Consider the general quadratic formula

$$x^2 \pm 2 * a * x * p \pm b * p = 0$$

where p is a rational straight line and a, b are coefficients. Only positive roots of this equation will be considered as x must be a straight line. Those roots include

$$x_1 = p * (a + \sqrt{a^2 - b})$$
$$x_1^* = p * (a - \sqrt{a^2 - b})$$
$$x_2 = p * (\sqrt{a^2 + b} + a)$$
$$x_2^* = p * (\sqrt{a^2 + b} - a)$$

First, consider the expressions for x_1 and x_1*. Suppose *a, b* do not contain any surds. That is to say, they are either integers or of the form $\frac{m}{n}$, where *m, n* are integers. If this is the case, either

$(i)\ b = \frac{m^2}{n^2} * a^2$

Or

$(ii)\ b \neq \frac{m^2}{n^2} * a^2$

If (*i*), then x_1 is a first binomial and x_1* is a first apotome. Euclid defines the first order in Definitions II, for the binomial, and III, for the apotome, in Book X as:

Given a straight line and a binomial/apotome…the square on the greater term/whole [*x*] is greater than the square on the lesser/annex [*y*] by the square on a straight line *commensurable* (emphasis added) in length with the greater/whole…the greater term/whole commensurable in length with the rational straight line set out then the entire segment is known as a first binomial/first apotome (p. 784, 860). This wordy definition is translated by Charles Hutton in his 1795 two volume edition of *A Mathematical and Philosophical Dictionary* to more comprehensible terminology: the larger term, *x*, is commensurable with a rational and is thus a rational itself and $x^2 - y^2 = z^2$, where *z* is commensurable in length with *x*, so *z* must also be rational. Using this new definition and numerical examples provided by Islamic mathematician al-Karaji in the 10th Century, we can understand better what Euclid was representing geometrically (Berggren, 2007). For instance, 3+√5 would be considered a first binomial and 3-√5 would be considered a first apotome. The greater term (3) is rational and

$$3^2 - (\sqrt{5})^2 = 9 - 5 = 4 = 2^2$$

If (*ii*), then x_1 is a fourth binomial and x_1* is a fourth apotome.

The fourth order of binomial and apotome are defined as opposing the first order. Euclid's definition for the fourth order of each class of irrational states:

If the square on the greater term/whole [*x*] be greater than the square on the lesser/annex [*y*] by the square on the straight line *incommensurable* (emphasis added) in length with the greater/whole, then if the greater term/whole be commensurable in length with the rational straight line set out then the entire segment is called a fourth binomial/fourth apotome (p. 784, 860). Much like the first binomial and first apotome, the greater term, *x*, will be rational. However, unlike the first order, the square root of the difference of the squares of the two terms, *z*, will be incommensurable with *x*, meaning that *z* will not have a rational ratio to *x*. Take, for example 4-√3. The greater term (4) is a rational number, and

$$\frac{\sqrt{4^2-(\sqrt{3})^2}}{4} = \frac{\sqrt{16-3}}{4} = \frac{\sqrt{13}}{4}$$

which is not a rational ratio.

Now look at the possibilities for the x_2, x_2* expressions. If we stick to our supposition that *a, b* do not contain surds, then either

$(i)\ b = \frac{m^2}{n^2-m^2} * a^2$

Or

$(ii)\ b \neq \frac{m^2}{n^2-m^2} * a^2$

If (*i*), then x_2 is a second binomial and x_2* is a second apotome.
Like the first order, the second order of binomials/apotomes has the square root of the difference of squares of the two terms (*z*) commensurable with the greater term/whole (*x*). The difference between the first and second order is that the lesser term/annex (*y*) is the segment that is commensurable with the rational straight line set out. This indicates that the lesser term, *y*, is rational, and that the ratio of the square root of the difference of the squares of the two terms, *z*, and the greater term, *x*, is a rational ratio expressible in whole numbers. Again, a wordy definition easily explained with actual values, like $\sqrt{18} \pm 4$. The lesser term (4) is a rational number and

$$\frac{\sqrt{\left(\sqrt{18}\right)^2 - 4^2}}{\sqrt{18}} = \frac{\sqrt{18-16}}{\sqrt{18}} = \frac{\sqrt{2}}{\sqrt{18}} = \frac{\sqrt{1}}{\sqrt{9}} = \frac{1}{3}$$

which is a rational ratio expressed in whole numbers.

If (*ii*), then x_2 is a fifth binomial and x_2* is a fifth apotome.
The fifth order of binomial and apotome is a combination of the second and third order definitions. Like the second order, the lesser of the two terms (*y*) is commensurable with the rational straight line set out. However, the square root of the difference between the two terms (*z*) is incommensurable in length with the greater term/whole (*x*). This means that again the lesser term is rational and that the ratio of the square root of the difference of the squares of two terms, *z*, and the greater term, *x*, is not a rational ratio. For instance, $\sqrt{6} \pm 2$. 2 is a rational number and

$$\frac{\sqrt{\left(\sqrt{6}\right)^2 - 2^2}}{\sqrt{6}} = \frac{\sqrt{6-4}}{\sqrt{6}} = \frac{\sqrt{2}}{\sqrt{6}} = \frac{\sqrt{1}}{\sqrt{3}} = \frac{1}{\sqrt{3}}$$

which is not a rational ratio.

To obtain the final two orders of binomial and apotome, we must consider the case where

$$a = \sqrt{\frac{m}{n}}$$

where *m, n* are integers. To abbreviate, let $\lambda = \frac{m}{n}$. Therefore

$$x_1 = p * (\sqrt{\lambda} + \sqrt{\lambda - b})$$
$$x_1^* = p * (\sqrt{\lambda} - \sqrt{\lambda - b})$$
$$x_2 = p * (\sqrt{\lambda + b} + \sqrt{\lambda})$$
$$x_2^* = p * (\sqrt{\lambda + b} - \sqrt{\lambda})$$

"If $\sqrt{\lambda - b}$ in x_1, x_1* is not surd but of the form $(\frac{m}{n})$, and if $\sqrt{\lambda + b}$ in x_2, x_2* is not surd but of the form $(\frac{m}{n})$, the roots are comprised among the forms already shown" (X. Introduction). To explain, in our original equations for x_1, x_1*, x_2, and x_2*, *a* was assumed to be rational (containing no surds) and $\sqrt{a^2 \pm b}$ would then be irrational. In our new equations, we define *a* as being irrational. The above quote states that if $\sqrt{\lambda \pm b}$ is rational (containing no surds) then we well again obtain the 1st, 2nd, 4th, and 5th order binomials or apotomes. The original x_1, x_1* and the new x_2, x_2* are taking a rational magnitude plus (binomial) or minus (apotome) an irrational to obtain the 1st and 4th orders. The original x_2, x_2* and newly formed x_1, x_1* start with an irrational magnitude and add (binomial) or subtract (apotome) a rational magnitude, forming the 2nd and 5th orders of binomial and apotome. Therefore, the only case that needs to be investigated is the case where an irrational magnitude is added or subtracted from another irrational magnitude.

If $\sqrt{\lambda - b}$ in x_1, x_1* is surd, then either

(*i*) $b = \frac{m^2}{n^2} * \lambda$

Or

(*ii*) $b \neq \frac{m^2}{n^2} * \lambda$

If (*i*), then x_1 is a third binomial and x_1* is a third apotome.

In the case of the third order of each type of irrational, we again have a connection to the language describing the first order. The square on the greater term/whole (*x*) is greater than the square on the lesser term/annex (*y*) by the square on a straight line commensurable with the greater/whole. However, in this order neither of the terms, *x* or *y*, are commensurable with the rational straight line set out. In terms of real numbers, both *x* and *y* must be irrational and ratio of the square root of the difference of the squares of the terms, *z*, and the greater term, *x*, is a rational ratio expressible in whole numbers. To explain by example, look at the third order binomial (√24+√18) or the third order apotome (√24-√18). Both terms are irrational and

$$\frac{\sqrt{(\sqrt{24})^2 - (\sqrt{18})^2}}{\sqrt{24}} = \frac{\sqrt{24 - 18}}{\sqrt{24}} = \frac{\sqrt{6}}{\sqrt{24}} = \frac{\sqrt{1}}{\sqrt{4}} = \frac{1}{2}$$

with $\frac{1}{2}$ being a rational ratio expressible in whole numbers.

If (*ii*), then x_1 is a sixth binomial and x_1* is a sixth apotome.

Much like the third, in a sixth order of binomial and apotome neither the lesser term (y) nor the greater term (x) are commensurable in length with the rational straight line set out, but the square on the greater/whole is greater than the square on the lesser/annex by the square on a straight line incommensurable in length with the greater term/whole. This translates to the sixth apotome having the form of two irrational terms with the ratio of the square root of the difference of the squares of the two terms, z, to the greater term, x, being an irrational ratio. Look at the sixth order binomial/apotome $\sqrt{6}\pm\sqrt{2}$. Both terms are irrational and

$$\frac{\sqrt{\left(\sqrt{6}\right)^2-\left(\sqrt{2}\right)^2}}{\sqrt{6}}=\frac{\sqrt{6-2}}{\sqrt{6}}=\frac{\sqrt{4}}{\sqrt{6}}=\frac{2}{\sqrt{6}}$$

which is not a rational ratio.

The attached table summarizes the six orders of binomial and apotome.

Euclid also designates two orders of bimedial lines and two orders of apotome of medial straight lines. Bimedial lines are the sum of two medial lines which are commensurable in square only. Proposition 37 demonstrates how to construct a first bimedial line (two medial lines commensurable in square only and containing a rational rectangle can be added together). Constructing a second bimedial line is discussed in proposition 38, where by all the same conditions as the first bimedial apply, accept that the two medial lines form a medial rectangle instead of a rational rectangle. An apotome of a medial is defined as the difference between two medial lines, the lesser of which is commensurable with the whole in square only. If a rational rectangle is contained with the square of the whole, then the remainder is a first apotome of a medial straight line (X. 74). If a medial rectangle is contained with the square of the whole, the remainder is known as a second apotome of a medial straight line (X. 75). An obvious connection can be drawn between bimedial lines and apotome of medial lines. All four types are constructed by manipulating two medial lines, with the first orders of each referring to a contained rational rectangle and the second orders of each having a medial rectangle contained by the two medial lines. The name apotome of a medial is fitting in an obvious way: the line is formed by the difference of two medials ($x - y$). What is confusing is the naming of the bimedial. With the connection between the bimedial and apotome of a medial mentioned above and the definition of the bimedial as the sum of two medial lines ($x + y$), it is interesting that Euclid did not use the more obvious title of binomial of a medial line. It is possible that the original terminology was binomial of a medial line and through translation was shortened to bimedial, but this is mere speculation.

4. Properties and Interactions

One of the most fascinating things about the three main types of irrational lines is studying the ways that they interact with each other. One example of this is the algebraic representation of binomials and apotomes. Binomials can be understood as a process of addition represented by $(x + y)$. The opposite is true of the apotome which is represented as $(x - y)$. These in turn have a product of $(x^2 - y^2)$. It is obvious that there are numerous relations that these lines hold with each other, and yet for all of their similarities, each of the categories of irrational lines are mutually exclusive. Euclid goes so far as to say, "The apotome and the irrational straight lines following it are neither the same with the medial straight line nor with one another" (X. 111). However, these lines are not just mutually exclusive categories, but are also unique in their division into parts. Proposition 42 demonstrates that if AB is a binomial, then there is only one point C between A and B such that AC, BC are rational and commensurable in square only. This proves that for a given binomial, there is only one way to separate its length into greater and lesser segments. The same is proven of a first bimedial (X. 43) and a second bimedial (X. 44). Likewise, from a given apotome, only one length can be subtracted such that both segments are rational and commensurable in square only (X. 79). Again, Euclid goes on to prove in propositions 80 and 81 the uniqueness of first and second apotome of a medial lines.

We must first look at the major properties of medials, binomials, and apotomes before we can delve into the interactions between these lines.

Common to all of the types of irrational lines is that fact that lines commensurable with the given length are of the same type and order where applicable (X. 23(medial), 66(binomial), 67(bimedial), 103(apotome), 104(apotome of a medial)). Unique to medials are the ideas that rectangles contained by medial lines commensurable in length is medial (X. 24), that rectangles contained by medial lines commensurable in square only are either rational or medial areas (X. 25), and that the difference between two medial areas will never be a rational area (X. 26). In maybe the most important proposition of book X, Euclid proves that an infinite number of unique irrational lines arise from a medial line (115). Interestingly, he chose to make this the last proposition in the book, possibly with the hopes that future students would use this property of medials to further investigate the theory of irrationals, possibly coming up with new unknown forms of irrational lines or a new classification system. There are a few propositions that deal with only binomials or apotomes, but these are usually taken in sets with the ensuing propositions using a bimedial or apotome of a medial, and thus will be discussed later. However, propositions 48-53 do deal strictly with binomials, in that each describes how to find a binomial of particular order. Propositions 85-90 perform the same action for orders of apotome.

Why Euclid chose to classify apotomes and binomials such that the first and fourth, second and fifth, and third and sixth orders were paired is not explained. We can note that the first three orders deal with commensurable lengths between the differences in squares explained above and the greater segments while the last three orders have greater terms being incommensurable with the difference in squares. We can also note each of the pairings are based on which term (greater, lesser, or neither) are rational. From this, it is plausible to assume Euclid's ordering is first based on the commensurability of given aspects of the line, and second based on which part of the given line is rational, leading to the two fold classification system seen today. Whether this was Euclid's reasoning or not, it does appear that he was not particularly concerned with functional order throughout the elements. For example, the first time a reader is introduced to a line cut in extreme-and-mean ratio is in the beginning of Book II. Yet it is not until Book X that the properties of such a line (with greater length is an apotome and lesser length a first apotome) are explained and not until Book XIII that this type of line is applied, which will be discussed in more detail later. It is also interesting to note Euclid devotes books VII, VIII, and IX to investigating numbers and number theory, but certain properties of numbers appear in many of the other ten books, including addressing ratios of numbers in Book V well before putting forth a theory of numbers (Grattan-Guinness, 1996).

Despite what may or may not be a flawed ordering system, the vast majority of Book X is devoted to exploring the interactions between the classes of irrational lines. It should be noted that for each property of binomial lines, the same property is proven just thirty-seven short propositions later for apotome lines. Starting with propositions 54 and 91, Euclid proves that if a rectangle is formed by a rational line and a first order binomial or apotome, the "side" or diagonal of that rectangle will be a binomial or apotome. As I mentioned before, the propositions describing the interactions of irrational lines often come in sets. Just as 54 and 91 prove the above statements, propositions 55, 56 and 92, 93 prove a similar situation occurs with bimedials and apotome of medial lines. An area formed by a rational and a second order binomial has for its side a first order bimedial (X. 55). For a rational and a third order binomial, the second bimedial is the diagonal (X. 56). Switching "apotome" for binomial and "apotome of a medial" for bimedial, we have the statements of propositions 92 and 93. We learn that if a rectangle is formed with rational length and area equal to a binomial squared, the width of the rectangle will be a first order binomial in proposition 60. The likewise is true of apotomes (X. 97). Propositions 61-62 and 98-99 are devoted to proving a similar statement: that if a straight line *AB* is a first bimedial or apotome of a medial (or second order for proposition 99), and a rational straight line *CD* is the side of rectangle *CE* such that the area of *CE* is equal to the square on *AB*, then the other side of rectangle *CE*, side *CF*, is a second binomial or apotome (third order for 99). Finally, as stated previously, propositions 112-113 prove that if a rational area has a binomial for one of its sides, the other side will be an apotome commensurable with the binomial and of corresponding order, with 114 proving that if a binomial and apotome that are commensurable and of the same order form the length and width of a rectangle, the diagonal will be rational.

5. Modern Implications

Euclid's' dialogue on irrational lines is not restricted to Book X. Indeed he puts forth an important application of the apotome in Book XIII. Proposition 6 states that if a line is cut in extreme-and-mean ratio (first introduced in II. 11), then the greater segment will be an apotome and the lesser segment a first apotome. This one proposition has enormous implications for the theory of irrational magnitudes. The golden ratio, one of the most applicable and well-studied areas of math, is created by a line cut in extreme-and mean ratio. This is an important topic to understand due to the vast number of properties held by objects that contain this ratio. One example is the logarithmic spiral which is formed through the construction of both golden rectangles, whose sides, when taken in proportion, equal the golden ratio, and the golden triangle, whose angles are 72°, 72°, 36°. Logarithmic spirals are seen throughout nature. Ram horns, elephant tusks, nautilus shells, pine cones, sun flowers and many other living things grow in accordance with the golden ratio (Fett, 2006). This proportion is said to be the most aesthetically pleasing to look at, which is why many great paintings and sculptures contain the golden ratio. The Parthenon in Athens, which not only houses sculptures containing the golden ratio but in fact can be inscribed in a golden rectangle, and five of Leonardo da Vinci's works, including two of his most famous "Madonna on the Rocks" and "Mona Lisa", are also said to contain the golden ratio (Fett, 2006). Many plastic surgeons still use the golden ratio several times over to construct what is believed to be a universal standard of beauty (Fett, 2006).

Each of the five Platonic solids, the only existing solids to have identical and equilateral faces and convex vertices, incorporates the golden ratio in its construction. The tetrahedron, octahedron and icosahedron are based on equilateral triangles, while the cube and dodecahedron are based on the square and pentagon. These shapes are discussed in detail in Book XIII of The Elements after the introduction of the line cut in extreme-and-mean ratio. Of particular interest are the dodecahedron and the icosahedron. Exodus of Cnidus, who lived after Theaetetus, is credited with having first discovered the irrationality of a line divided in extreme and mean ratio after working with the problem of inscribing a regular pentagon in a given circle (Knorr, 1983). The pentagon is actually formed by three golden triangles, and the ratio of the shorter side to the longer is equal to the golden ratio. This implies the construction of the icosahedron is dependent upon the golden ratio. The golden ratio is also present in the calculation of surface area and volume of the dodecahedron as well as the volume of the icosahedron (Fett, 2006). Since Theaetetus is credited with first discovering the icosahedron, Book XIII, along with Book X, is firmly based in the Athenian's work. In fact, M. F. Burnyeat quotes B. L. Van der Walden in his 1978 journal article as saying "The author of Book XIII knew the results of Book X, but…moreover, the theory of Book X was developed with a view to its applications in Book XIII. This makes inevitable the conclusion that the two books are due to the same author…Theaetetus".

The golden ratio is also seen in the comparison of sequential values in the also well-studied Fibonacci sequence. The Fibonacci numbers are defined by the recursive formula

$$f_{n+1} = f_n + f_{n-1}$$

for $f_n > 2$ with $f_1 = 1$ and $f_2 = 1$ (Fett, 2006; Posamentier, 2002; Rosen, 2005). When comparing the n^{th} Fibonacci number with the $(n\text{-}1)^{th}$, the ratio will approach the golden ratio as n increases. Not surprisingly given its relationship with the golden ratio, the Fibonacci sequence is often found in the growth of natural objects. For example, the number of spirals in plants that grow in a phyllotaxis pattern will always be a Fibonacci number (Rosen, 2005). Another sequence related to the both the Fibonacci numbers and the golden ratio is the Lucas numbers. These are defined using the same recursive formula as the Fibonacci numbers and still begins the sequence with 1, but $\ell_2 = 3$ instead of 1 (Posamentier, 2002; Rosen, 2005). Interestingly enough, the same relationship exists between Lucas numbers and the golden ratio as the Fibonacci numbers and the golden ratio (Posamentier, 2002).

6. Conclusions

It is unfortunate that so little is known about the early theory of irrationals or who advanced our understanding to what it is today. What is also regrettable is the lack of progress we have made since the days of Euclid. On the positive, we can be thankful for the meticulous systematic presentation of irrational magnitudes and their properties and interactions demonstrated in Book

X of The Elements. While this chapter of Euclidean geometry has not been developed to the degree that most of his work has, the multitude of Book X leaves us with plenty of information on irrationals.

We know that there are 13 types of irrational lines, with each category being mutually exclusive. We also know that for every irrational line, there is only one way to divide the line to meet the criteria of its category and order, proving the uniqueness of each. Maybe most importantly, we know from proposition 115 that there are infinitely many irrational lines. Euclid provides us with the application of this theory in Book XIII, showing how the pentagon and icosahedron utilize irrational magnitudes in the construction of each. This application of the extreme-and-mean ratio has led to significant discoveries in the area of aesthetics, art, and music through the apotome known as the golden ratio.

Perhaps this information was enough to satisfy mathematicians throughout history. Or possibly our Euclidean understanding of irrationals is complete. We highly doubt the latter, but believe so much time is spent simply trying to understand the already burdensome theory of irrational lines that little is left for the advancement of the theory. Many mathematicians have devoted time to aiding future students in understanding the classification of commensurable and incommensurable magnitudes. Hopefully this will eventually lead to a more readily comprehensible theory, a base step from which a more innovative, improved theory of irrational lines can be developed.

N. Sirotic and R. Zazkis (2005) conducted a research project to find out how much we retain of the Euclidean theory of irrational lengths. They asked a group of college students studying to be secondary teachers if it was possible to locate $\sqrt{5}$ on a number line. The results were somewhat frightening. Less than 20% of the participants used a geometric construction to find $\sqrt{5}$ on the number line, most of those having used the Pythagorean Theorem, approximately 65% used some sort of decimal approximation in varying degrees of exactness, and an abysmal 15% either did not answer, or worse, argued it was not possible to find exactly where $\sqrt{5}$ falls on a number line. Most of those who argued it was impossible reasoned that since $\sqrt{5}$ was irrational, the decimal approximation in infinite and non-repeating and that was why it cannot be accurately positioned. However, these same participants believed a repeating infinite decimal, like $\frac{1}{3}$, could be placed in its exact position, but could not explain why whether the decimal repeated or not made a difference. This means that 80% of those future secondary teachers could not think past our understanding of decimal approximations to use a well-known and highly practiced idea (the Pythagorean Theorem) to find where $\sqrt{5}$ falls on a number line. To these people, it seems the number line is really a *rational* number line and irrational numbers cannot be placed exactly since "because [the decimal] never ends we can never know the exact value" (Sirotic, 2005). This is an unfortunate side effect of Theodorus' and al'Karaji's work to aid students in understanding irrationals. The original understanding of irrational lines using geometry is lost to the more easily comprehensible geometric algebra presented in almost all current editions of The Elements. While the use of algebra is integral to helping students understand this dense topic, a return to irrational magnitudes' geometric roots appears to be just as important for students to gain a true understanding of Book X.

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References

Burnyeat, M. F. (1978) The Philosophical Sense of Theaetetus' Mathematics. ISIS: The Official Journal of the History of Science Society. 69(4), 489-513

Euclid (2006) The Elements. (Sir T. L. Heath, Trans.) New York: Barnes & Noble Publishing, Inc. (Original work written around 300 BC).

Fett, B. (2006) An In-Depth Investigation of the Divine Ratio. The Montana Mathematics Enthusiast. 3(2), 157-175.

Forder, H. G. (1927) The Foundations of Euclidean Geometry. London: Cambridge University Press.

Grattan-Guinness, I. (1996) Numbers, Magnitudes, Ratios, and Proportions in Euclid's *Elements*: How Did He Handle Them? Historia Mathematica. 23, 355-375.

Greenburg, M. J. (2008) Euclidean and Non-Euclidean Geometries (4th ed.). New York: W. H. Freeman and Company.

Hutton, C. (1795) A Mathematical and Philosophical Dictionary. London

Imhausen, A., Robson, E., Dauben, J., Plofker, K., and Berggren, J. L. (2007) The Mathematics of Egypt, Mesopotamia, China, India, and Islam: A Sourcebook. (V. Katz, eds.) Princeton: Princeton University Press.

Knorr, W. R. (1975) The Evolution of the Euclidean Elements: A Study of the Theory f Incommensurable Magnitudes and Its Significance for Early Greek Geometry. Dordrecht, Holland: D. Reidal Publishing Company.

Knorr, W. (1983) Euclidean Theory of Irrational Lines. Bulletin of the American Mathematical Society. 9(1), 41-69.

Knorr, W. (1998) 'Rational Diameters' and the Discovery of Incommensurability. The American Mathematical Monthly. 105(5), 421-429

Kyburg, H. Jr. (1997) Quantities, Magnitudes, and Numbers. Philosophy of Science. 64, 377-410.

Posamentier, A. S. (2002) Advanced Euclidean Geometry. New York: Key College Publishing.

Rosen, K. H. (2005) Elementary Number Theory and Its Applications (5th ed.). Boston: Pearson Addison Wesley.

Sirotic, N. and Zazkis, R. (2007). Irrational numbers on the number line – where are they? International Journal of Mathematical Education in Science and Technology. 38(4), 477-488.

	Definition of Binomial (Apotome)		**Algebraic Intreprepation of the Roots of General Quadratic Formula**
Order	**Square on the greater segment (whole) greater than the square on the lesser segment (annex) by the square on a straight line ________ in length with the greater term (whole)**	**_________ be commensurable in length with the rational straight line set out Rational Term**	$x_1 = p\cdot\left(a + \sqrt{a^2 - b}\right)$ $x_1{*} = p\cdot\left(a - \sqrt{a^2 - b}\right)$ $x_2 = p\cdot\left(\sqrt{a^2 + b} + a\right)$ $x_2{*} = p\cdot\left(\sqrt{a^2 + b} - a\right)$
First	commensurable	greater segment (whole) greater term is rational	If *a, b* do not contain surds and $b = \frac{m^2}{n^2}\cdot a^2$ x_1 is a binomial and x_1^* is an apotome
Second	commensurable	lesser term (annex) lesser term is rational	If *a, b* do not contain surds and $b = \frac{m^2}{n^2 - m^2}\cdot a^2$ x_2 is a binomial and x_2^* is an apotome
Third	commensurable	neither segment (neither) neither term is rational	If $\sqrt{(a^2 - b)}$ is surd, $a = \sqrt{\frac{m}{n}}$ $b = \frac{m^2}{n^2}\cdot\lambda$ x_1 is a binomial and x_1^* is an apotome
Fourth	incommensurable	greater segment (whole) greater term is rational	If *a, b* do not contain surds and $b \neq \frac{m^2}{n^2}\cdot a^2$ x_1 is a binomial and x_1^* is an apotome
Fifth	incommensurable	lesser term (annex) lesser term is rational	If *a, b* do not contain surds and $b \neq \frac{m^2}{n^2 - m^2}\cdot a^2$ x_2 is a binomial and x_2^* is an apotome
Sixth	incommensurable	neither segment (neither) neither term is rational	If $\sqrt{(a^2 - b)}$ is surd, $a = \sqrt{\frac{m}{n}}$ $b \neq \frac{m^2}{n^2}\cdot\lambda$ x_1 is a binomial and x_1^* is an apotome