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Two Applications of Art to Geometry

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Geometry and art exploit the same source of human pleasure: the exercise of our spatial intuition. It is not surprising, then, that interconnections between them abound. Applications of geometry to art, of which we shall indicate a few, go back at least to Alberti's *De Pictura* (1435). But although geometry started out, as it so often does, as a most courteous suitor in its relationship with art, it was soon to be affectionately rewarded. We shall study two of these rewards.

Geometry applied to art

Let us indicate briefly how geometry may be applied to art. A perspective painting distorts sizes and shapes. A building in the distance may be smaller than a man's head, the circular rim of a cup becomes an ellipse, etc. Lines, however, always remain lines. This simple fact is the key to drawing tiled floors (figure 1), as Alberti explained in *De Pictura*, because it guarantees that the diagonal of the first tile is also the diagonal of successive tiles. Furthermore, all lines parallel to the viewer's line of sight will meet at one point in the picture, namely the point perpendicularly in front of the viewer's eye (the so-called "centric point"). The horizon is the horizontal line through this point, because if the observer looks downwards from there, no matter how little, then the ray from his eye will hit the ground, whereas is he looks upwards it will not, so this is indeed the boundary between ground and sky (here we are assuming, of course, that the earth is flat). Thus, for example, placing the centric point close to the ground gives the viewer the impression that he is lying down. This trick is used to great effect by Mantegna in *St. James led to Execution* (figure 2). It also follows, in the words of Alberti, that the horizon is "a limit or boundary, which no quantity exceeds that is not higher than the eye of the spectator . . . This is why men depicted standing in the parallel [to the horizon] furthest away are a great deal smaller than those in the nearer ones—a phenomenon which is clearly demonstrated by nature herself, for in churches we see the heads of men walking about, moving at more or less the same height, while the feet of those further away may correspond to the knee-level of those in front." (*De Pictura*, Book I, §20, quoted from the Penguin edition, Alberti (1991, p. 58).) For more on the role of geometry in Renaissance art see, e.g., Kline (1985, ch. 10) and Ivins (1973).

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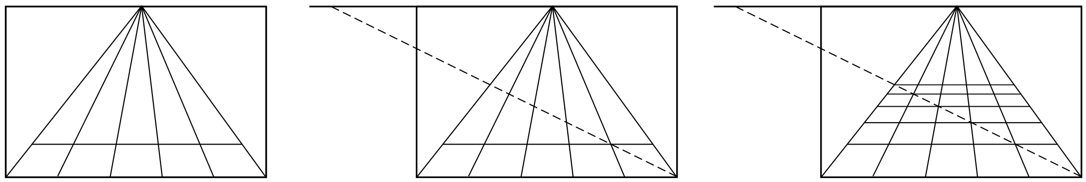


Figure 1: Drawing a tiled floor.

Figure 2: Mantegna's *St. James led to Execution*.

Newton's classification of cubic curves

Let us now turn to the applications of art to geometry. Our first example is Newton's classification of cubic curves. The classification of curves is the zoology of mathematics—indeed, Newton spoke of dividing curves into different "species." Art provides a picturesque criteria for whether two curves should be considered to be of the same species or not: two curves are of the same species if one is a projective view of the other, i.e., if when painting the picture of one curve you obtain the other. Newton (1695), §5, used this idea to classify cubics "by shadows," as he said, into the five equivalence classes illustrated in figure 3 (for more details see Newton (1981), vol. VII, pp. 410–433, Newton (1860), Ball (1890), Brieskorn and Knörrer (1986), Stillwell (2002)). We shall show where $y = x^3$ fits into this classification by showing that it is equivalent to $y^2 = x^3$ (the mirror image of the middle curve in figure 3). The classification of cubics is a natural setting for the use of projective ideas because cubics are the next step beyond conics, which are themselves too easy: projectively, they are all the same; any section of a double cone projected from the vertex of the cone (the eye point) onto a plane perpendicular to the axis (the canvas) comes out as a circle.

We imagine ourselves standing on top of the flat part of $y = x^3$ and painting its image on a canvas standing perpendicular to the plane of the curve (figure 5a). I say that the painting comes out looking like figure 5b. First of all, the dashed line represents the horizon. Let us focus first

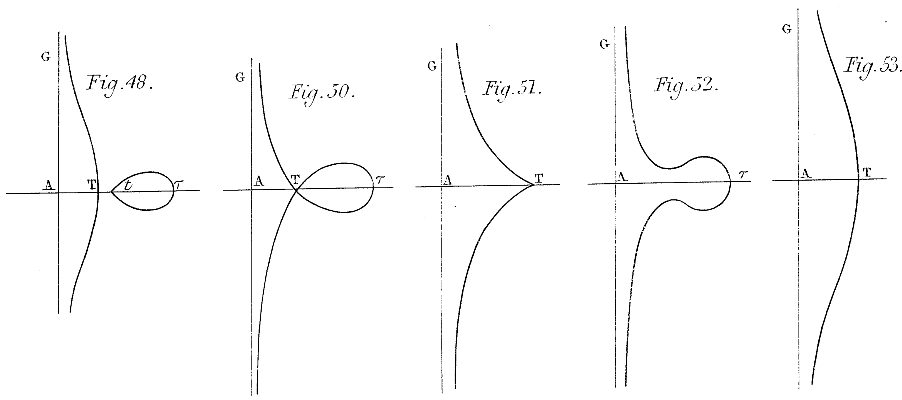


Figure 3: The five projective equivalence classes of cubic curves. (From Newton (1860).)

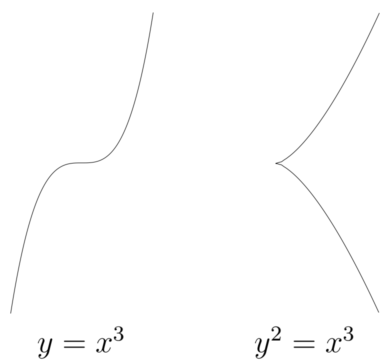
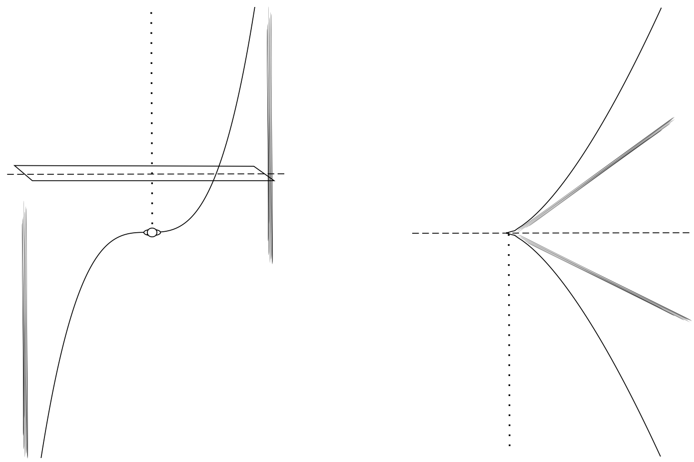


Figure 4: Two equivalent cubic curves.

on the part of figure 5b below the horizon, which is supposed to be the image of everything in front of us. Apparently, even though the curve $y = x^3$ goes of to our right, we will see it meeting the horizon straight ahead of us. We understand why by looking at the support lines drawn in the figures. The dotted line and the brush stroke line on our right are parallel so in the picture they should meet at the horizon (like railroad tracks, if you will). Since the curve $y = x^3$ essentially stays between these two lines (almost all of it, anyway), it must stay between them in the picture as well, so it is indeed forced to meet the horizon straight ahead of us. The part above the horizon is similar, but we must allow for a mathematical eye that can see through the neck, so to speak. To draw the image of any point in front of us we connect it to our eye with a line and mark where this line intersects the canvas. To draw the image of any point behind us we use the same procedure, ignoring the fact that the canvas is no longer between the eye and the point.



(a) (b)

Figure 5: Projective equivalence of $y = x^3$ and $y^2 = x^3$.

Desargues' theorem

We shall now see how Desargues' theorem emerges beautifully from natural ideas of perspective painting, namely the "visual ray construction" of 'sGravesande (1711) (see Andersen (2006) for a modern commentary). Desargues' theorem is one of the great results of projective geometry. Let us first look briefly at what it says and how we can think about it. The theorem says: if two triangles (ABC and $A'B'C'$) are in perspective (i.e., AA', BB', CC' all go through the same point, O) then the extensions of corresponding sides (AB and $A'B'$; BC and $B'C'$; AC and $A'C'$) meet on a line. Desargues' theorem is especially easy to think about in three dimensions, as indeed Desargues himself did (as conveyed to us by Bosse (1648); see Field and Gray (1987, chapter VIII)). Consider a triangular pyramid. Cut it with two planes to get two triangles. The three points of intersection of the extensions of corresponding sides will or course be on a line (the intersection of the two planes). By projecting the triangles onto one of the walls of the pyramid we get two plane triangles in perspective and the theorem holds for them also. So Desargues' theorem holds for any two triangles in perspective that can be obtained by projection from a triangular pyramid. We feel that any triangles in perspective can be obtained in this way so Desargues' theorem is proved. Now let us see what it has to do with art.

Visual ray construction of the image of a line. We shall draw the perspective image of a ground plane. To do this we rotate both the eye point and the ground plane into the picture plane: the ground plane is rotated down about its intersection with the picture plane (the "ground line") and the eye is rotated up about the horizon. Consider a line AB in the ground plane. The intersection of AB with the ground line is of course known. The image of AB intersects the horizon where the parallel to AB through the eye point meets the picture plane, and parallelity is clearly preserved by the turning-in process. So to construct the image of AB we turn it into the

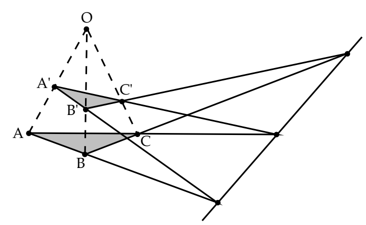


Figure 6: Desargues' theorem.

picture plane and mark its intersection with the ground line and then draw the parallel through the eye point and mark its intersection with the horizon; the image of AB is the line connecting these two points. □

Collinearity property of the visual ray construction. Draw the line connecting a turned-in point A and the turned-in eye point. The image of A is on this line because if we turn things back out the eye-point–to–horizon part of the line will be parallel to the A–to–ground line part of the line, so that the image part of this line is indeed the image of the A–to–ground-line line. □

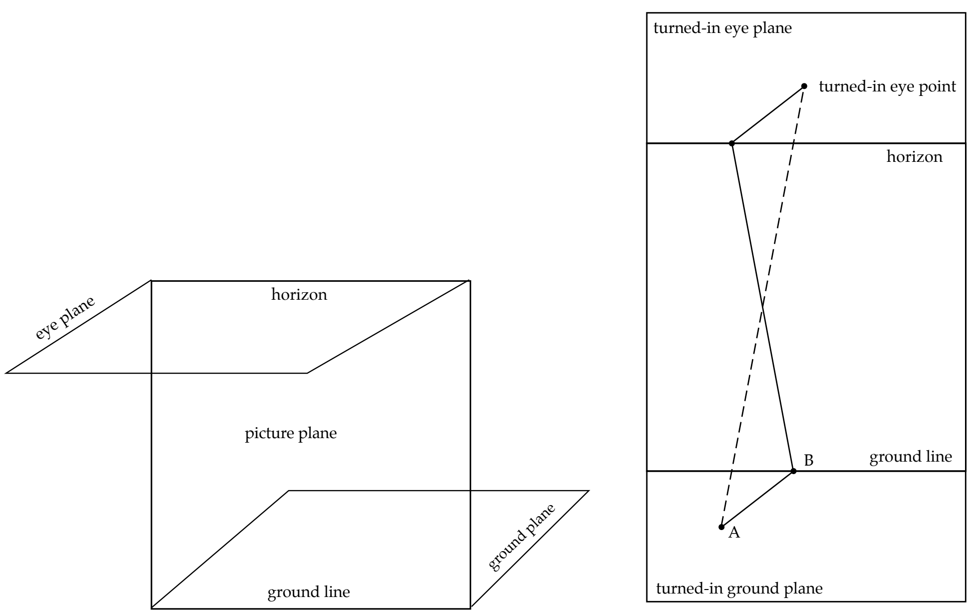


Figure 7: The visual ray construction.

Desargues' theorem by the visual ray construction. Construct the perspective image $A'B'C'$ of a triangle ABC. By the image-of-a-line construction, intersections of extensions of corresponding sides are all on a line, namely the ground line, and by the collinearity property $A'B'C'$ and ABC are in perspective from the eye point, so we have Desargues' theorem. □

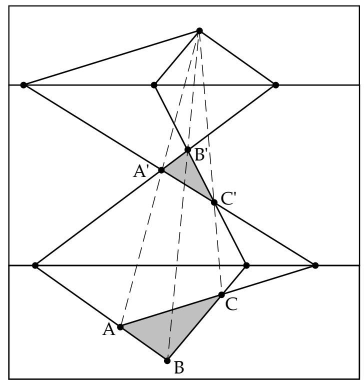


Figure 8: Desargues' theorem by the visual ray construction.

A more conventional proof of Desargues' theorem would be to use projective simplification, following Poncelet (1822, cf. §168). This proof is less directly influenced by art, but nevertheless the basic idea comes from our intuition with paintings, namely the idea of the horizon—"the line at infinity." In real life the horizon is intangible, but in a painting it is just a line like any other. And in real life parallel lines never meet, but in the painting they meet at the horizon, at a point like any other. Thus art suggests an alternative to Euclidean geometry where the line at infinity is just as real as any other line and where there is no such things as lines that never meet. Now let us use these ideas to prove Desargues' theorem. AB and $A'B'$ will meet somewhere, and BC and $B'C'$ will meet somewhere; grab the line determined by these two points and put it at the line at infinity, which is, as we said, a line like any other. This means that, in our picture (figure 9), AB will be parallel to $A'B'$, and BC will be parallel to $B'C'$. We need to show that AC and $A'C'$ meet at the same line, i.e., at the line at infinity, i.e., that AC and $A'C'$ are also parallel. Recall that AC is parallel to $A'C'$ if and only if $OA/AA' = OC/CC'$. Using this result on the two pairs of lines that we already know are parallel gives $OA/AA' = OB/BB'$ and $OB/BB' = OC/CC'$. So $OA/AA' = OC/CC'$ and thus AC and $A'C'$ are parallel.

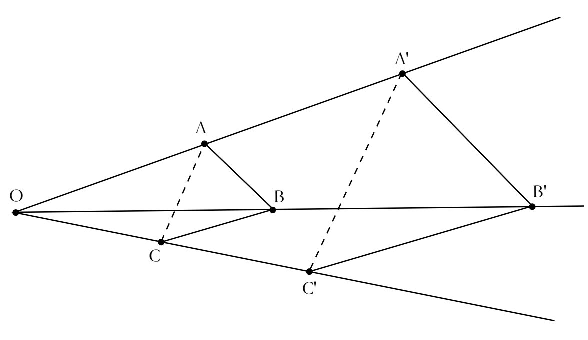


Figure 9: The simplified Desargues' configuration.

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