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INTUITIONS OF "INFINITE NUMBERS": INFINITE MAGNITUDE VS. INFINITE REPRESENTATION

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Intuitions of "infinite numbers": Infinite magnitude vs. infinite representation

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Abstract. This study examines undergraduate students' emerging conceptions of infinity as manifested in their engagement with geometric tasks. Students' attempts to reduce the level of abstraction of infinity and properties of infinite quantities are described. Their arguments revealed they perceive infinity as an ongoing process, rather than a completed one, and fail to notice conflicting ideas. In particular, confusion between the infinite magnitude of points on a line segment and the infinite representation of real numbers was observed. Furthermore, students struggled to draw a connection between real numbers and their representation on a number line.

Keywords: Infinity; Infinite numbers; Intuition; Magnitudes; Real numbers; Representations;

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> *From time immemorial, the infinite has stirred men's emotions more than any other question. Hardly any other idea has stimulated the mind so fruitfully* (Hilbert, 1925, p.136).

Infinity has played an important role in the historical development of mathematics and mathematical thought. From as early as 450 BC, mathematicians and philosophers have been intrigued by the ethereal dance of infinity. Over the centuries, as an understanding of infinity developed and changed, mathematics too evolved, reflecting the community's emerging understanding of a concept so heavily shrouded in mystery. With time it eventually became clear that not one, but many, concepts of infinity have a place in mathematics. This paper is concerned with two types of infinity, and the interplay between them: *potential* infinity, that which is inexhaustible, and *actual* infinity, "the infinite present at a moment in time" (Dubinsky, Weller, McDonald, & Brown, 2005, p.341).

This study is part of broader investigations regarding university students' naïve and emerging conceptions of infinity and transfinite arithmetic as they attempt to coordinate intuition and reflection with formal instruction. In what follows, students' engagement with geometric representations of infinity are described and used as a lens to their understanding of infinity and arithmetic properties of 'infinite numbers'. In particular, students' conceptions as they attended to the number of points 'missing' from the shorter of two line segments are of interest. This paper also explores what sort of connection, if any, participants made between a geometric representation of infinity and a numeric one. These can be seen as the main contributions of this study, complementing and extending prior research, which focused on learners' conceptions regarding the comparison of infinite sets.

This story of 'infinite numbers' begins with an exposition of the related literature regarding students' conceptions of infinity, as well as the theoretical perspectives that guided this study. Following that, the design of the study is described, and key findings are presented and analysed. The paper concludes with a summary of the main findings and suggestions for future avenues of investigation.

1. BACKGROUND

Students' reasoning concerning cardinal infinity has been a popular focus of current research (see among others: Dreyfus & Tsamir 2004; Tsamir, 1999, 2001; Tsamir & Dreyfus, 2002; Weller, Brown, Dubinsky, McDonald, & Stenger, 2004). The body of literature ranges from expositions of learners' intuitive understanding of infinity (e.g. Fischbein, Tirosh, & Hess, 1979) to developing pedagogical tasks that will encourage a deliberate use of formal definitions (e.g. Tsamir & Tirosh, 1999). A prominent trend has been to examine learners' conceptions through a lens of set theory – that is, students are presented with numeric sets, such as {1, 2, 3, …} and {2, 4, 6, …}, and are asked to draw cardinality comparisons. Their conceptions are then analysed based on the techniques or principles they apply to the task.

In a study conducted by Tsamir and Tirosh (1999), they noticed that visual presentations of sets had an impact on high school students' intuitive responses. For instance, one task had students compare the cardinalities of the two sets {1, 2, 3, …} and {4, 8, 12, …}. When the sets were expressed numerically, many students relied on the inclusion or 'part-whole' method for comparison and concluded that the set of natural numbers was greater than the set of multiples of four. Tsamir and Tirosh (1999) created a follow up task that presented the corresponding sets geometrically in such a way as to emphasize their one-to-one correspondence. Students were asked to consider a set of line segments with increasing lengths – i.e. {1cm, 2cm, 3cm,…} – and then to imagine constructing squares in such a way that the segments were of the same lengths as the sides of the squares. Both the set of line segments and the set of squares were depicted pictorially with the lengths and perimeters written below each segment and square, respectively. Through this analogy students could attend to the natural correspondence between a side and a perimeter of a square, and as such, they were more likely to recognise the one-to-one correspondence between the sets {1, 2, 3, …} and {4, 8, 12, …}. Tsamir and Tirosh (1999) were able to make use of the tangible nature of a geometric figure in order to emphasise correspondences between numerical sets, and also to draw students' attention to the inconsistencies of comparing infinite sets with different methods.

Inconsistencies in middle school students' intuitions about infinity were documented by Fischbein et al. (1979), who interpreted students' intuitions as they addressed issues such as the divisibility of line segments of different lengths, or the number of points on geometric figures of different dimension. The divisibility task consisted of comparing the number of times two line segments could be halved. The majority of students reasoned that although both line segments could be halved infinitely, the process would finish sooner on the shorter segment. Similarly, when comparing the set of points on a line segment with the set of points on a square, the common response alluded to infinities of different 'size'. Students appealed to 'part-whole' arguments, and reasoned that as the line segment was included as part of the square, the two sets must have different cardinalities, though both were infinite. These responses were in contrast to other observations of Fischbein et al. (1979), which suggested infinity was conceived of as a single, endless entity. Fischbein et al. concluded that the intuition of infinity is very labile and "sensitive to the conceptual and figural context of the problem" (1979, p.31).

The belief that there is only a single, endless infinite surfaced as a persuasive intuition of middle school students when they addressed set comparison tasks in a similar study by Fischbein et al. (1981). As part of the study, participants were asked to compare the cardinality of the set of natural numbers with the cardinality of the set of real numbers represented as a number line. The typical response that "there is an infinity of points on the line, and there is an infinity of natural numbers" (Fischbein et al., 1981, p.506), and so the two sets must be equinumerous is incorrect when judged by mathematical convention. Students' responses indicated that infinity was conceived of mainly as potential, that is, as an inexhaustible process. The association of infinity with inexhaustibility has also surfaced in undergraduate university students' views regarding limits in calculus (Sierpinska, 1987; Schwarzenberger & Tall, 1978; Williams, 1991). Fischbein suggested that such an association is "the essential reason for which, intuitively, there is only one kind, one level of infinity. An infinity which is equivalent with inexhaustible cannot be surpassed by a richer infinity" (2001, p.324).

2. THEORETICAL FRAMEWORK

Three inter-related frameworks are used in this study to interpret students' intuitions of infinity as well as their ideas after instruction: reducing abstraction (Hazzan, 1999), APOS: Action, Process, Object, Schema (Dubinsky & McDonald, 2001), and 'measuring infinity' (Tall, 1980).

In Hazzan's (1999) perspective, reducing the level of abstraction of a mathematical entity occurs as a learner attempts to understand unfamiliar and abstract concepts. Hazzan (1999) described several ways students make sense of new concepts by reducing levels of abstraction. For instance, Hazzan noted "students' tendency to work with canonical procedures in problem solving situations" (1999, p.80). That is, by basing arguments on familiar mathematical entities to cope with unfamiliar concepts, students lower the level of abstraction of those concepts. In the context of infinity, one such example is students' use of familiar (finite) measuring properties to interpret infinite quantities of measurable entities, such as the quantity of points on a line segment. This example of reducing the level of abstraction of infinity relates to Tall's (1980) notion of 'measuring infinity.'

Tall (1980) suggested intuitions of infinity can develop by extrapolating measuring, rather than cardinal, properties of numbers. Many of our everyday experiences with measurement and comparison associate 'longer' with 'more.' For example, a longer inseam on a pair of pants corresponds to more material. Likewise, a longer distance to travel corresponds to more steps one must walk. Tall (1980) proposed extrapolating this notion can lead to an intuition of infinities of 'different sizes.' A measuring intuition of infinity coincides with the notion that although any line segment has infinitely many points, the longer of two line segments will have a 'larger' infinite number of points. Tall (1980) called this notion 'measuring infinity' and suggested it is a reasonable and natural interpretation of infinite quantities, especially when dealing with measurable entities such as line segments. I would like to suggest that the intuition of 'measuring infinity' might develop as a consequence of learners' attempts to lower the level of abstraction of comparing the infinite cardinalities of points on line segments of different lengths.

Reducing the level of abstraction is further proposed by Hazzan (1999) to reflect a process conception of an entity. Process and object conceptions of mathematical entities are described in another of the theoretical frameworks to which I refer: that of the APOS (Action, Process, Object, Schema) theory (Dubinsky & McDonald, 2001). Dubinsky, Weller, McDonald, and Brown (2005) proposed an APOS analysis of two conceptions of infinity: actual and potential. The distinction between *potential* infinity, which can be thought of as endless, and *actual* infinity, a completed entity that encompasses what was potential, was first made by Aristotle. He, like many after him, denied the existence of actual infinity (Moore, 1995). The idea that infinitely many objects could be gathered together and thought of as a totality, was, and continues to be, very difficult. A more natural conception of infinity is that of potential, or dynamic, infinity (Fischbein, 2001). Fischbein considered dynamic infinity as "processes, which are, at every moment, finite, but continue endlessly" (2001, p.310).

Dubinsky et al. (2005) suggested that an understanding of potential infinity corresponds to a process conception in APOS terminology. That is, infinity is imagined as performing an endless action, although without having to execute each and every step. Conversely, an understanding of actual infinity develops when one is able to consider the process as a totality, i.e., when one can encapsulate it into an object. To connect this perspective to the infinite number of points on a line segment, a conception of potential infinity would correspond to, say, an action of marking or 'creating' points on a segment that is imagined to continue indefinitely. While actual infinity is illustrated by the idea that the infinite number of points exists as a completed entity, without needing to be marked.

Dubinsky et al. proposed encapsulation occurs once one is able to think of infinite quantities "as objects to which actions and processes (e.g., arithmetic operations, comparison of sets) could be applied" (2005, p.346). They also suggested that encapsulation of infinity entails "a radical shift in the nature of one's conceptualisation" (2005, p.347). In terms of APOS theory, Hazzan argued that a "process conception of a mathematical concept can be interpreted as on a lower level of abstraction than its conception as an object" (1999, p.79).

APOS theory and the idea of 'measuring infinity' are used in my study to interpret undergraduate students' emergent conceptions as they attempt to reduce the level of abstraction of infinity. Specifically, the questions addressed in this paper are: 1. What connections do students make between geometric and numeric representations of infinity, i.e. between points on a line and real numbers? 2. What can be learned about students' conceptions of infinity as they address properties of transfinite arithmetic?

3. SETTING AND METHODOLOGY

The participants of this study were 24 undergraduate university students in an interdisciplinary design and technology program, who had no mathematical background beyond high school. They were enrolled in the course "Foundations of Academic Numeracy", which was designed to develop quantitative and analytic reasoning. One of the objectives of the course was to provide an opportunity for students to engage in critical analysis and reflection regarding some of the fundamental ideas in mathematics. The topic of infinity was included as one of these fundamental ideas.

Data collection relied on two main sources: (i) individual written responses to "reflection activities", and (ii) follow up interviews with two of the participants. The "reflection activities" were essentially a series of written questionnaires administered over several weeks. The rationale behind these reflections was to elicit students' naïve conceptions and then to encourage them to reconsider, develop, and critique the underlying ideas through subsequent questioning. Tasks were formulated based on students' previous responses and common themes that emerged from the class. It was important, both for research and instructional purposes, that students' responses were not affected by seemingly correct solutions or the desire to appease their instructor. In order to avoid swaying students' responses, very little instruction was provided initially, and it was made clear that there was no one 'right' answer being sought. The activities reflected this in their design by, for example, recalling students' previous responses and presented them with a slight twist, so as to encourage them to challenge the issues they had unearthed. Other questions presented students with a dubious argument that claimed to be from one of their peers, in order to provoke a critique of the ideas involved. The basis for both styles of question was to avoid presenting an authoritative position. Students addressed each issue based on its appeal to their own emerging ideas.

At the end of the course, an instructional discussion on cardinality and infinite sets occurred. The discussion included comparing cardinalities of countable and uncountable infinite sets through one-to-one correspondences, or the idea of 'coupling'. Some of the specific conceptions that arose in students' reflections were also addressed. In the subsequent months, follow up interviews were conducted with two students, Lily and Jack. The interviews further explored their naïve and emerging conceptions of infinity.

The study began with two preliminary questionnaires, which included items a) and b) below. These tasks set the stage for exploring students' connection between numeric and geometric representations of infinity.

a) How many fractions can you find between the numbers $\frac{1}{19}$ and $\frac{1}{17}$? How do you know?

b) How many points are there on a line segment? How do you know?

Later questionnaires focused on the sets of points on line segments of varying lengths, and were intended to investigate ideas regarding 'infinite numbers' as well as 'infinite number properties.' Due to the contingent nature of the activities, details concerning the specific questions are developed in the following section.

The primary focus of this paper is on students' responses to two questionnaires in particular. The first (Q1, section 4.2) confronted students with an idiosyncrasy of infinite quantities and asked for an explanation. Of particular interest was the response of one participant, Lily. Her attempt to formulate an argument that was consistent with her experiences and intuitions prompted a follow up to Q1. In this follow-up (Q2, section 4.3), students were asked to respond to Lily's argument as well as to a variation of it.

4. RESULTS AND DISCUSSION

4.1 Infinite values, finite points

From the early stages of the study, a clear disconnect in students' conceptions of points on a real number line and numbers was observed. Typical arguments to item a), which concerned the number of fractions between $\frac{1}{19}$ and $\frac{1}{17}$, are exemplified by the following two responses:

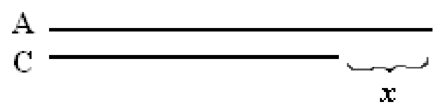
> "Infinite. Because there are endless numbers that can be put into the numerator or the denominator and still making sure the fraction is larger than $\frac{1}{19}$ and smaller than $\frac{1}{17}$";
>
> and
>
> "You can find an infinite amount of fractions in between $\frac{1}{17}$ and $\frac{1}{19}$ because you can continue to add digits after the decimal point forever (e.g. $\frac{1}{18}, \frac{1.3}{18}, \frac{1.3625}{18}$, etc.) making the fractions a little bigger or smaller."

There are two common threads in these responses. One is the idea of potential infinity. The notions of "endless numbers" or adding "digits after the decimal point forever" imply infinity is conceived of as a process. The idea of changing the numerator or denominator corresponds to an action that is imagined to continue "forever", and is consistent with Fischbein et al.'s (1981) suggestion that infinity is intuitively thought of as inexhaustible. The second common thread in these and similar responses relates to students' conceptions of number. Both of these arguments describe processes being carried out with fractions. That is, students were attending to the rational numbers within the interval, but failed to address the irrational numbers. This might a consequence of the task itself, as the endpoints of the interval were rational numbers rather than irrational ones. However it may be more likely due to students' familiarity and comfort with rational numbers over irrational ones.

In response to item b), regarding the number of points on a line segment, the majority of participants (17 out of 24) indicated that points were either the places that a line segment begins and ends, or else they were markers that partition a line segment into equal units. These responses were surprising in light of students' responses to item a), and their ideas regarding the infinite number of 'values' on any line segment. Students' arguments supporting an infinite number of 'values' on a line segment were similar in nature to their arguments regarding item a) above. They described processes of finding "as many values as we want", however they distinguished between the finite number of points that existed on a line segment and the infinite number of points that could be "given a value" or labelled. As before, these arguments indicate a process conception of infinity. Further, the idea of 'finding values', or 'creating points' by assigning them values, may be interpreted as an attempt to reduce the level of abstraction of an infinite yet bounded quantity.

Students' distinction between point and value prompted a class discussion regarding the geometry of points and lines to establish a shared understanding (to use the term loosely) of the infinite magnitude of points (rather than 'values') on a line segment. The questionnaire following this discussion related to the number of points on line segments of different lengths, and prompted students to reflect on the number of points 'missing' from the shorter of the two segments. The following specific question was posed:

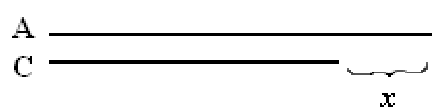
> Consider line segments A and C again. Suppose that the length of A is equal to the length of C + x, where x is some number greater than zero, as depicted below. What can you say about the number of points on the portion of A whose length is x?



In order to investigate both students' rationale when comparing the number of points on line segments of different lengths, and students' intuitions regarding subtracting infinite quantities, Q_1 presented their conclusions with a slight twist.

4.2 Subtracting infinity

> **Q_1.** On a previous question, you reasoned that two line segments A and C both have infinitely many points.



> Suppose that the length of A is equal to the length of C + x, where x is some number greater than zero. You also previously suggested that the segment with length x has infinitely many points. That is, the ∞ points on A minus the ∞ points on C leaves an ∞ number of points on the segment with length x. Put another way,
>
> $$\infty - \infty = \infty.$$
>
> Do you agree with this statement? Please explain.

Participants' responses to Q_1 revealed inconsistencies in students' conceptions, as well as a strong intuitive resistance to the idea of subtracting infinite quantities. Jack, for instance, experienced a conflict as a conception of infinity emerged that contrasted his intuition.

Previously, Jack had described infinity as a "hypothetical number" that is "the biggest number you can get", and for which "you'd have to count your whole life and you still would never get there." Intuitively, Jack seemed to conceive of infinity as an unattainable extension of 'very big'. His comment that counting your whole life "still would never get [you] there" typifies a process conception of infinity. However, this fundamental notion of infinity was challenged by the visual representation of the two line segments. In response to Q_1 Jack wrote:

> What I'm thinking is that if you got infinite points on A and if you got infinite on C, well, you're seeing that they're not equal. So how can you say that infinite points are equal? Like, visually, you're seeing that A is bigger, so therefore the infinite number has to be bigger on A than the infinite number on C. But then again, infinite is the largest you can get, so that's kind of confusing.

Jack observed that the two line segments are not equal in length, and thus concluded that the two could not have an equal amount of infinite points despite his insistence that infinity is "the largest you can get." The conflict in Jack's conceptions might be attributed to an attempt to extrapolate everyday experiences with finite measurements, where length and quantity are often directly proportional. Using familiar experiences to make sense of novel situations is considered by Hazzan (1999) as an attempt to reduce the level of abstraction of the new concept. In the case of infinity, extrapolating experiences with measurement can be deemed as a conception of 'measuring infinity'. Jack's conception of 'measuring infinity' is at odds with his intuition of a single, never-ending infinity, and his recognition of this created a cognitive conflict that he was unable to resolve.

The notion of 'measuring infinity' surfaced in several students' responses to Q_1, however most students neglected the inconsistency between it and their intuition of potential infinity. For instance, Rosemary rationalized the expression "$\infty - \infty = \infty$" by arguing that while any line segment will have infinitely many points, a longer segment would have a larger infinite number of points. She also claimed that subtracting an infinite quantity from another (albeit "larger") infinite quantity would leave "a lot of points… extending into infinity" and "it will take forever" to count them. The inconsistency between a process conception of infinity, as exhibited by Rosemary's description of "extending into infinity" and taking "forever", and her measuring conception of a "larger" infinity went unnoticed.

Of the various responses to Q_1, Lily's was unique. In her response, she disagreed with the possibility that $\infty - \infty = \infty$. She wrote:

> I disagree with this statement. For example, π is an infinite (on going) number. If we subtract $\pi - \pi$ the answer is 0, NOT ∞. But, if there is a restriction that says we can't subtract by the same number it could still be an infinite number, but just a smaller value. For example, $\pi - 2\pi = -\pi$, is still an infinite number, only negative.

Lily appeared to conceive of infinity as potential – her use of the qualifier "on going" to describe her notion of an "infinite number" corresponds to a process conception of infinity. However, the on-going process in Lily's conception is applied, not to the magnitude of her "infinite number", but to its infinite decimal representation. Lily's objection to Q_1 seems to stem from confusion between an infinite magnitude, such as the number of points on a line segment, and the infinite number of digits in the decimal representation of π. Her use of π to justify claims about infinite magnitudes is indication of a disconnect between points on a line and real numbers. Further, not only did Lily overlook the particular value of π itself, but she also failed to distinguish the differences between acting on one specific element as opposed to infinitely many. Lily reasoned that since π is an "infinite (on going) number" and $\pi - \pi = 0$, then the difference $\infty - \infty$ must also be 0. Lily's generalization of properties of π to draw conclusions about the entire set of points can be interpreted as an attempt to reduce the level of abstraction of dealing with an infinite number of elements. The use of one number to explain properties of infinitely many coincides with Hazzan's (1999) observation that students will try to reduce the level of abstraction of a set by operating on one of its elements rather than all of them.

Another interesting aspect of Lily's response was her use of "restrictions." She proposed that the difference of two 'infinite numbers' might be another 'infinite number' if there are appropriate restrictions placed on the quantities. By restricting the 'values of infinity' she reasoned that it is possible to attain "an infinite number, it [will] just be a smaller value." Appending "restrictions" allowed Lily to conceive of 'infinite numbers' with different sizes, despite the conflict with her description of infinity as "on going". The notion of infinities with 'different values' is consistent with an intuition of measuring infinity (Tall, 1980), and serves as an example of reducing the level of abstraction. According to Hazzan, this can be seen as a case of using familiar procedures to cope with novel and abstract concepts: Lily applies the familiar procedure of subtracting real numbers to cope with the concept of subtracting transfinite ones.

4.3 'Infinite numbers'

Lily's confusion between an infinite number of elements and an infinite number of digits in one particular element emphasised the disconnect between numeric and geometric representations of infinity that appeared in the early stages of the study. The question of whether other students shared Lily's ideas regarding the magnitude of a number with infinite decimal expansion naturally arose. Thus, a follow up questionnaire (Q_2) recalled Q_1, presented Lily's argument verbatim, as well as a similar one, and asked students to elaborate on whether or not they agreed with the arguments.

Q_2. Recall [Q1 as quoted above].

Student X: [Lily's response as quoted above]

Student Y: I disagree with this statement. You can subtract two infinite numbers and NOT end up with ∞. For example, $\frac{1}{3}$ is an infinite number, but $\frac{1}{3} - \frac{1}{3} = 0$, NOT ∞. Also, $\frac{4}{6}$ and $\frac{1}{6}$ are both infinite (on going) numbers, but if we subtract $\frac{4}{6} - \frac{1}{6} = \frac{3}{6} = ½ = 0.5$, which is not an infinite number. But sometimes it's possible to subtract two infinite numbers and get an infinite number. For example, $\frac{1}{3} - \frac{1}{6} = \frac{1}{6}$, which is infinite and smaller than $\frac{1}{3}$. So, sometimes ∞ - ∞ = ∞, but usually not.

Most participants (22 out of 24) agreed with at least one of the arguments in Q_2, which came as a surprise in light of the common description of infinity as the "largest you can get". The confusion between infinite magnitude and infinite decimal representation revealed two distinct interpretations of 'infinite numbers'. For the students who agreed with both arguments, confusion between magnitude and representation was broad: they ignored the finite magnitude of both rational and irrational numbers. For instance, Jim wrote:

$\frac{4}{6}$ and $\frac{1}{6}$ are both infinite (on going) numbers but when subtracting them your result is $\frac{1}{2}$ which is not infinite. This proves that an infinite number subtracting by another infinite number is not always another infinite number. As a result the statement ∞ - ∞ = ∞ is not true because sometimes the result is infinite but a different value and other times the result is not infinite.

In his response, Jim readily accepted the arguments of students X and Y, neglecting the differences between a particular (finite) value and an infinite quantity. Jim used the infinity symbol to represent numbers of different magnitudes, and as such, exemplified students' notions that infinity has no 'specific value'. The dynamic nature of this conception can be interpreted as an attempt to reduce the level of abstraction of an entity that is beyond the realm of his imagination. Jim's attempt to extrapolate his experiences with finite quantities, and also to use them explicitly (though perhaps unknowingly) to justify his notions of infinity, is further indication of an attempt to reduce the level of abstraction of the expression '∞ - ∞'.

Other students held a slightly different conception of 'infinite number' – they recognized rational numbers as finite quantities and associated them with points on a number line, but did not make the same association with irrational numbers, mistaking them with infinite quantities. This interpretation was exemplified in Rosemary's response to Q_2. When addressing student X, Rosemary remarked:

> $\pi - \pi = 0$ that is correct because one is taking away the same amount of points from what they initially began with will give 0, but in the line segment question, the amount of points in x (which is ∞ amount) is much less than the amount of points in A and C. Which because of this, I agree with Student X's second statement of how there should be restrictions. In this case, points in x are less than points in A or C.

As in Q_1, Rosemary's response is consistent with the idea of 'measuring infinity', using Lily's notion of 'restrictions' to accommodate the possibility that a longer segment will have a greater number of points. Further, Rosemary identified with Lily's argument regarding $\pi - \pi$, and alludes to the possibility of a line segment having π-many points. Her remark that $\pi - \pi = 0$ is correct because "one is taking away the same amount of points from what the initially began with" illustrates participants' general confusion regarding the magnitude of irrational numbers.

Additional evidence of Rosemary's attempts to reduce the level of abstraction of subtracting transfinite numbers is seen in her response to student Y:

> Student Y states: $\frac{1}{3} - \frac{1}{6} = \frac{1}{6}$ (which is an ∞ number) but $\frac{4}{6} - \frac{1}{6} = \frac{3}{6}$ (which is only 0.5 and not an ∞ number). Well, when we represent these numbers on a number line *[drew two line segments, one from 0 to $\frac{1}{6}$ and one from 0 to $\frac{1}{2}$, and labelled the segments A*

> *and B, respectively]* then won't both line segments have ∞ points? (But of course segment B will have more than segment A)

Once again, Rosemary appealed to her intuition of 'measuring infinity' as she related student Y's numeric example to its geometric representation. In contrast to her use of π, Rosemary distinguished rational numbers from infinite quantities. Although she stated that $\frac{1}{6}$ was an "infinite number," she observed its specific value on the number line. Similarly, she remarked that though $\frac{1}{2}$ was not infinite itself (it "is only 0.5"), when represented on a number line she acknowledged there were still infinitely many points between 0 and $\frac{1}{2}$. This distinct handling of rational and irrational numbers suggests a misconception about real numbers: whereas rational numbers were associated with points, irrational numbers were not. Nevertheless, Rosemary seemed to use the words "infinite number", both to represent a number with infinitely many (nonzero) digits in a decimal representation, as well as to represent the infinite quantity of points on a line segment. It would be interesting to see if Rosemary's measuring conception would be so persuasive had she not applied the same terminology to two different notions.

4.4 After instruction: Lily and Jack

At the end of the course, the class was instructed on equivalences of infinite sets, as well as the distinction between an infinite decimal expansion and an infinite quantity. Specifics of the instruction are detailed below. In the months following the end of the course, follow up interviews were conducted with two students: Lily and Jack.

The interview with Lily took place roughly six months after instruction regarding the distinction between infinite magnitude and infinite representation, and included a discussion on the finite value of π. The interview with Lily focused on her conception of π as an 'infinite number', and since it was the number of decimal digits that gave π it's infinite quality, Lily was asked to speculate on the number of decimal digits of a rational scalar of π. She reasoned, "if we times [π] by 3 it'll just be a bigger number, with more digits." As with the line segments, Lily expressed ideas consistent with 'measuring infinity': she associated "bigger" with "more," believing that 3π would be infinite but a "bigger infinite" than π.

Lily's perception of the "infinite size" of π persisted despite instruction and also in conflict with her ideas regarding 3.14 as an approximation of π. She claimed that 3π was "3 times a number that's really big." To determine the magnitude of 3π, Lily used the familiar number 3.14, yet she was surprised to calculate that triple this number was only about 9: "let's say π is 3.14, then times 3 is going to be big. Well, not big, but (pause) well, kind of triple?" Notwithstanding Lily's attempts to reduce the level of abstraction of π by working with 3.14, it seemed difficult for her to accept π as a small number. When asked about the possibility of measuring a length of π cm, she claimed that one would need "a really big ruler" with huge spaces between each whole number to accommodate all of π's decimal digits. She argued that since π's expansion was infinite and never-ending, then any segment of length π would have to be "really long, until, if possible, there's an end to it." Lily seemed to ignore the actual magnitude of each of π's decimal digits, which, together with her process conception of a never-ending infinite, might have contributed to her notion of π as very large, despite the relatively small magnitude of 3.14.

The struggle to accommodate conflicting ideas, such as Lily faced with her conceptions of π, also surfaced in the interview with Jack. In his written responses, Jack had struggled with the conflict between his competing conceptions of potential and measuring infinity. Following instruction, Jack continued to express inconsistent notions of infinity as he attempted to reconcile his naïve understanding with a normative one. The interview with Jack, which took place two months after the end of the course, began by recalling class instruction on the correspondence between points on line segments of different lengths.

The instructional class discussion included the following well-known geometric construction of a bijection between two line segments AB and CD. The construction begins by connecting the endpoints of AB and CD with line segments that extended past the endpoints of CD to meet at a point labelled p, as depicted in Figure 1. An arbitrary point, *w*, can be labelled on AB and connected to the point *p* by a line segment. The connecting segment will intersect CD at a point *r*, as depicted in Figure 2. With this construction, it is possible to pair up each point on AB with exactly one point on CD. Conversely, a ray from *p* to any point on CD can be extended to meet a point on AB in a unique way. In this manner, every point on CD is paired with exactly one point on AB. Thus a one-to-one correspondence is constructed between the set of points on AB and the set of points on CD. Most students easily followed the construction, though there was significant resistance to the idea that the longer line segment would not have more points.

Figure 1:

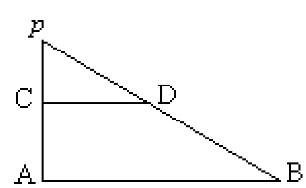
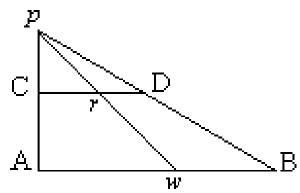
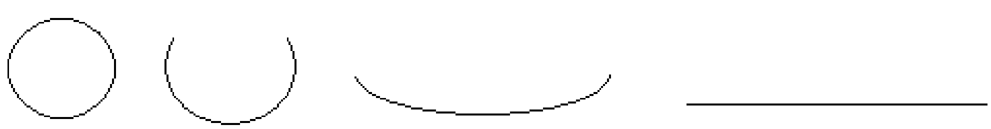


Figure 2:



Jack had no trouble recreating the above argument. However, he insisted, “that A [AB] is bigger, so therefore the infinite number has to be bigger on A [AB] than the infinite number on C [CD].” Jack’s conception of measuring infinity was very compelling, and he continued to struggle with the conflict between it and his intuition that infinity “is the largest you can get” and is “never-ending.” In an attempt to challenge his measuring intuition, Jack was asked to consider the number of points on two circles of different circumference. He claimed there were an infinite number of points because “drawing a line from the centre to the side [drew the radius of the circle], you can draw infinite of them.” Furthermore, he noted that the circles would have the same number of points because “you’re not caring about the length of the radius, which makes your circle bigger or smaller. You’re caring about the 360 degrees,” that is, the number of radii, which is the same in both circles. We then proceeded to ‘cut open’ and ‘flatten’ each circle, such as in Figure 3.

Figure 3:



Jack judged that even though the shape of the circles was now different, the number of points had not changed[2]. Jack reasoned that the two flattened circles would still have an equinumerous set of points because "you still have that imaginary [centre] point, and all the [radii] connecting to it." This construction is essentially the same as the triangle argument above: the number of rays from *p* that intersect with the longer line segment is the same as the number that intersect with the shorter line segment. The visual representation had a significant effect on Jack's perceptions. Comparing and equating the number of radii of two circles was canonical, even when they were flattened. However, Jack noted "if you go back to this [lines AB and CD], still, if you look at it this way it still doesn't make sense. The circle way kind of does. Well, not kind of, it actually does." Eventually, Jack accepted that two line segments of different lengths could have the same quantity of points, stating it was "hard to believe, but it makes sense."

5. CONCLUDING REMARKS

This paper examines undergraduate students' emerging conceptions of infinity during their efforts to coordinate intuition with conventional mathematical properties. As students grappled with properties of actual infinity, they unearthed features that were at odds with their personal experiences – participants were challenged by competing and inconsistent notions of infinity as endless or as a large number whose size was relative. In resonance with earlier work (e.g. Fischbein et al., 1979), students often remained unaware of these inconsistencies. Further, students' responses support the argument that infinity is conceived of intuitively as an

[2] Topologically, the line segment and circle do differ: an open line segment is isomorphic to $S^1 \setminus \{N\}$, for some point N. However, since the goal was to compare two circles in their 'new form' and not to compare the line segment with the circle, this fact was not addressed at that moment in the conversation.

inexhaustible process, rather than a completed object, in APOS terminology. However, it is notable that the conception of 'measuring infinity' which emerged in students' attempts to reduce the level of abstraction of comparing geometric infinite sets was a persuasive factor in students' reasoning, and at times overshadowed the association of infinite with endless.

This study sheds new light on students' emerging conceptions of infinity as manifested in their engagement with geometric tasks. Geometric representations provided a useful analogy for demonstrating qualities of transfinite arithmetic, and as such, confronted students with the property that transfinite subtraction is undefined. It has been shown that many students are tempted to treat infinity as simply a very big number (e.g. Sierpinska, 1987), however students' conceptions regarding arithmetic with transfinite numbers is lacking in mathematics education literature. This study offers a first glimpse at learners' attempts to reduce the level of abstraction of transfinite subtraction. The issue of learners' conceptions regarding transfinite arithmetic is of interest in my ongoing investigations.

Students' attempts to cope with the expression "$\infty - \infty$" revealed significant misconceptions regarding the size of real numbers. Their confusion between the infinite magnitude of points on a line segment and the infinite decimal representation of both rational and irrational numbers created an obstacle to a conventional understanding of mathematical infinity, and demonstrated a shortcoming in their understanding of number and place value. Furthermore, students' failure to identify specific numbers as points on a number line highlighted a disconnect between their conceptions regarding numeric and geometric representations of infinity. The use of finite quantities to explain phenomena of transfinite ones misguided students' intuitions and, ultimately, their understanding. Students' various attempts to reduce the level of abstraction of infinitely many points on a line segment by considering properties of a single point revealed an intuition of infinity that may be at odds with future instruction on limits and set theory.

This study opens the door for further investigation regarding some issues that may be taken for granted, such as the relationship between magnitude and representation, and the connection between points on a line and numbers. Future research will attend to the persuasive factors that can influence change in learners' emerging conceptions, as well as to the different conceptual challenges learners face when addressing properties of 'infinite numbers' and transfinite arithmetic.

5. REFERENCES

Dreyfus, T. & Tsamir, P. (2004). Ben's consolidation of knowledge structures about infinite sets. *Journal of Mathematical Behavior*, 23, 271–300.

Dubinsky, E. & McDonald, M. A. (2001). APOS: A constructivist theory of learning in undergraduate mathematics education research. In Derek Holton, et al. (Eds.), *The Teaching and Learning of Mathematics at University Level: An ICMI Study*, (pp. 273–280). Netherlands: Kluwer Academic Publishers.

Dubinsky, E., Weller, K., McDonald, M. A., & Brown, A. (2005). Some historical issues and paradoxes regarding the concept of infinity: an APOS-based analysis: Part 1. *Educational Studies in Mathematics*, 58, 335–359.

Fischbein, E. (2001). Tacit models of infinity. *Educational Studies in Mathematics*, 48, 309–329.

Fischbein, E., Tirosh, D., & Hess, P. (1979). The intuition of infinity. *Educational Studies in Mathematics*, 10, 3–40.

Fischbein, E., Tirosh, D., & Melamed, U. (1981). Is it possible to measure the intuitive acceptance of a mathematical statement? *Educational Studies in Mathematics*, 12, 491-512.

Hazzan, O. (1999). Reducing Abstraction Level When Learning Abstract Algebra Concepts. *Educational Studies in Mathematics*, 40, 71-90.

Hilbert, D. (1925). On the infinite. In P. Benacerraf & H. Putnam (Eds.), *Philosophy of Mathematics: Selected Readings* (pp. 134 – 151). New Jersey: Prentice-Hall, Inc.

Moore, A. W. (1995). A brief history of infinity. Scientific American, 272 (4), 112 – 116.

Sierpinska, A. (1987). Humanities students and epistemological obstacles related to limits. *Educational Studies in Mathematics*, 18, 371–397.

Tall, D. (2001). Natural and formal infinities. *Educational Studies in Mathematics*, 48, 199–238.

Tall, D. (1980). The notion of infinite measuring numbers and its relevance in the intuition of infinity. *Educational Studies in Mathematics*, 11, 271–284.

Tirosh, D. & Tsamir, P. (1996). The role of representations in students' intuitive thinking about infinity. *Journal of Mathematical Education in Science and Technology*, 27, 33–40.

Tsamir, P. (2001). When "the same" is not perceived as such: the case of infinite sets. *Educational Studies in Mathematics*, 48, 289-307.

Tsamir, P. (1999). The transition from the comparison of finite sets to the comparison of infinite sets: teaching prospective teachers. *Educational Studies in Mathematics*, 38, 209–234.

Tsamir, P., & Dreyfus, T. (2002). Comparing infinite sets — a process of abstraction: The case of Ben. *Journal of Mathematical Behavior*, 21, 1–23.

Tsamir, P. & Tirosh, D. (1999). Consistency and representations: The case of actual infinity. *Journal for Research in Mathematics Education*, 30, 213–219.

Weller, K., Brown, A., Dubinsky, E., McDonald, M., & Stenger, C. (2004). Intimations of infinity. *Notices of the AMS*, 51(7), 740 – 750.