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ON THE USE OF REALISTIC FERMI PROBLEMS FOR INTRODUCING MATHEMATICAL MODELLING IN SCHOOL

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On the use of Realistic Fermi problems for introducing mathematical modelling in school

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Abstract: In this paper uses an analytical tool refereed to as the MAD (Modelling Activity Diagram) framework adapted from Schoenfeld's parsing protocol coding scheme to address the issues of how to introduce mathematical modelling to upper secondary students. The work of three groups of students engaged in solving so called realistic Fermi problems were analysed using this framework, and it was observed that the processes involved in a typical mathematical modelling cycle were richly represented in the groups' solving processes. The importance of the social interactions within the groups was noted, as well as the extensive use of extra-mathematical knowledge used by the students during the problem solving session.

Keywords: Fermi problems; Fermi estimates; Modelling cycles; Mathematical modelling; Modelling activity diagram;

1. INTRODUCTION

The study of mathematical modelling in mathematics education has been a steadily growing branch of research since at least the late 1960's (Blum, 1995). The arguments for including mathematical modelling in mathematics education have been collected under *the formative argument*; *critical competence*; *utility*; *picture of mathematics*; and *the promoting mathematics learning argument* (Blum & Niss, 1991; Niss, 1989), and mathematical modelling is nowadays explicitly included as part of the mathematics curricula in many countries all over the world.

Although mathematical modelling is getting more and more emphasised in governing curricula documents and despite extensive research efforts (e.g. ICTMA[2] publications), the adjustment and change in classroom practise on broad (national) scales are slow. Teacher-training courses, in-service courses for practicing teachers, textbooks and teaching materials, as well as ways of working with mathematical modelling in the classroom, need to be further developed and made accessible.

This paper addresses the issue of how to introduce mathematical modelling to upper secondary students in the very beginning of a modelling course or before engaging in a modelling project. In principle, such an introduction can be done using a direct or an indirect approach, where typically the direct approach is to present some sort of 'heuristics' to the students for how to model mathematically. This paper however, investigates the potential of an indirect approach, where groups of students are set to work on so called *Fermi problems*, which are open, non-

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[2] The International Community of Teachers of Modelling and Applications, http://www.ictma.net/

standard problems requiring the students to make assumptions about the problem situation and estimate relevant quantities before engaging in, often, simple calculations.

2. CONTEXT AND AIM OF THE STUDY

The study reported here is part of a bigger research project aiming to both get an overall picture of the past and present state and status of mathematical modelling in the Swedish upper secondary school, and to develop, design, implement and evaluate small teaching modules (sequences of lessons) on mathematical modelling in line with the Swedish national curriculum. In connection to the latter, the issue of how to introduce mathematical modelling in the beginning of these modules was first given priority. However, previous research and reports accounting for the designing and developing of courses and lessons on mathematical modelling (e.g. J. S. Berry, Burghes, Huntley, James, & Moscardini, 1987; Blum, Galbraith, Henn, & Niss, 2007, especially chapter 3.6) are sparse about how the actual introduction of mathematical modelling was implemented. One exception is Legé (2005), who contrasts a reductionist and a constructivist instructional approach to the introduction of mathematical modelling at high school level. Irrespective of some positive results, Legé's introductions of mathematical modelling are activities lasting over two weeks of time, which for my purpose is too long. To try to shred some light over if this could be done in a more time efficient way and to facilitate the design of the teaching modules, this pilot study was conducted.

The aim of the study reported on here was to investigate if Fermi problems could be used to introduce mathematical modelling at the Swedish upper secondary level. The question in focus can in general terms be formulated as: *What mathematical problem solving behaviour do groups of students display when engaged in solving Fermi problems?* This *preliminary* research question will be reformulated and specified in section 5 in terms of the theoretical framework that will be outlined.

The structure of the paper is as follows. First I briefly discuss perspectives on Fermi problems and mathematical modelling, before looking at previous research done in connection with this area. Then, I describe the methodology, the definition of *realistic* Fermi Problems, the concept of mathematical modelling sub-activities, and the developing of the MAD framework. The paper proceeds with accounting for the result and the analysis of the empirical study, before finally the discussion and conclusion are presented[3].

3 FERMI PROBLEMS

The term *Fermi problem* originates from the 1938 Italian Nobel Prize winner in physics Enrico Fermi (1901-1954), who was also a highly appreciated and popular teacher (Lan, 2002). He had a predilection for posing, as well as solving, problems like *How many railroad cars are there in US?* (Goldberger, 1999) or *How many piano tuners are there in the US?* (Efthimiou & Llewellyn, 2007), and by using a few reasonable assumptions and estimates, he gave astoundingly accurate and reasonable answers. Fermi was of the opinion that a good physicist as well as any thinking person could estimate any quantity quantitatively accurate just 'using one's head'; that is, just by reasoning, making some realistic and intelligent order of magnitude estimates and doing some simple calculations. Often, the questions came from everyday situations and phenomena he saw or experienced, as illustrated by Wattenberg (1988); "Upon

[3] This paper is an extend version of the paper by Berman Ärlebäck and Bergsten (in press)

seeing a dirty window, he [Fermi] asked us how thick can the dirt on a window pane get?" (p. 89). These types of problems are called *Fermi problems*, *back-of-envelope calculation problems* or *order of magnitude problems*. To my knowledge, Fermi himself did not define the characteristics of such problems explicitly and different authors in the literature emphasise different things.

3.1. Fermi Problem in physics education

Historically the ability to perform order of magnitude calculations was crucial for physicists before investing time and effort in engaging in a long and complicated calculation (Robinson, 2008). Nowadays, when today's extensive use of computers and computer packages easily and fast make these calculations for us, other arguments for including Fermi problems in physics, science and teacher education are used.

According to Chandler (1990), Fermi problems typically are intended to end up in estimates "to the nearest power of ten without using reference books or calculators" (p. 170). Carlson (1997), on the other hand, elaborates a bit more and describes the process and essence of solving Fermi problem as "the method of obtaining a quick approximation to a seemingly difficult mathematical process by using a series of 'educated guesses' and rounded calculations" (p. 308) and argues for their effective motivational potential in students. Following the same line of reasoning, Efthimiou and Llewellyn (2007, p. 254) characterize a Fermi problem as initially always seeming rather vague in its formulation, giving limited or no information on relevant facts or how to attack the problem. However, after a closer inspection and analysis, they undeniably allow an unfolding of the problem into simpler problems that eventually lead to a final answer to the original question. Another argument put forward by Efthimiou and Llewellyn (2007) is to use Fermi problems in general education and introductory science courses to foster students' critical thinking and reasoning.

Robinson (2008) puts forward a view in line with Carlson (1997), but is more specific when he writes that "[i]n order to solve a Fermi problem, one has to synthesize a physical model, examine the physical principles which are in operation, determine other constraints such as boundary conditions, decide how simple the model can be while still maintaining some realism, and only then apply some rough estimation to the problem." (p. 83). Drawing on this characterization, he argues that in the process of solving Fermi problems, the same set of skills is used which professional physicist use in their everyday work, but seldom are learned before the beginning graduate training. Thus, according to Robinson, the main argument for the use of Fermi problems in education is to introduce key skills and methodologies to students in an early stage of their schooling.

3.2. Fermi Problems in mathematics education

Turning to the field of mathematics education one can find both similarities and differences of how Fermi problems are characterized as compared to in physics education. Ross and Ross (1986) write that "[t]he essence of a Fermi problem is that a well-informed person can solve it (approximately) by a series of estimates" (p. 175), and that "[t]he distinguishing characteristic of a Fermi problem is a total reliance on information that is stored away in the head of the problem solver… Solving Fermi problems presents an artificial challenge" (p. 181). With a moderately free interpretation of the meaning of 'well-informed person' most authors agree with the first of these quotes, but concerning the latter there is diversity. Others, such as Peter-Koop (2004) and Sriraman and Lesh (2006), are of the opinion that the concept of Fermi problems is better and more useful if one allows the problems not to be purely intellectual in nature, but situated in the real world and in an everyday context.

Other characteristics ascribed to Fermi problems by some authors are their accessibility and/or self-differentiating nature, which means that the problem can be worked on and solved in different school grades as well as at different levels of complexity (Kittel & Marxer, 2005). Also, as expressed by Sowder (1992), there should not exist an exact answer: "[s]uch problems *must* be answered with an estimate, since the exact answer is not available" (p. 372, italics in original).

Some authors define the characteristics sequentially and more implicitly by describing the steps that are needed, or the understandings or insights that need to be achieved, to successfully come up with an answer. For example, Dirks and Edge (1983) list four "things typically required" when solving Fermi problems, namely "sufficient understanding of the problem to decide what data might be useful in solving it, insight to conceive of useful simplifying assumptions, an ability to estimate relevant physical quantities, and some specific scientific knowledge" (p. 602).

According to Ross and Ross (1986), the reason for teachers to use Fermi problems in teaching is twofold; first "to make an educational point: problem-solving ability is often limited *not* by incomplete information but the inability to use information that are already available" (p. 175, italics in original); and secondly, to give the students a more nuanced picture of mathematics, showing that doing mathematics is not always about getting exact answers through well-defined procedures. A more recent argument for the use of Fermi problem in mathematics education is the possibility to use them as a bridge between mathematics and other school subjects, engaging students in different interdisciplinary activities (Sriraman & Lesh, 2006).

Compared to how Fermi problems are viewed in physics education, it is notable that both disciplines to some extent describe the same procedure about how to approach and solve such problems; *making simplifying assumptions*, *estimating*, and *doing rounded calculations* are important aspects of the problem solving process. In addition, the more recent references from both fields argue for the potential inherent in the problems for foster students' critical thinking. The biggest difference between mathematics and physics education is the view of why to the use Fermi problems. In the latter, Fermi problems *in themselves* are seen to illustrate and emphasise basic and fundamental principles of physics, whereas in mathematics education, at least in the early references, they are artificial in nature, used as *tools* for teaching and learning some mathematical content. However, this view expressed in some of the references seems to be changing (e.g. Sriraman & Lesh, 2006).

3.3. Some previous research involving Fermi problems in education

Focusing on mathematics education research, Fermi problems seem to be used fairly sparse and are in principle mentioned in two different contexts at the lower educational level. First and foremost, they are mentioned in connection with (measure) estimates (sometimes called numerosity problems); and secondly, they are mentioned in connection with modelling. Recently, Fermi problems have also been suggested and used in fostering students' critical thinking (Sriraman & Lesh, 2006; Sriraman & Knott, 2009). The focus in this paper is on the use of Fermi problems in connection with modelling. However, for a review of the research on estimates see Sowder (1992), and the more recent Hogan and Brezinski (2003).

Although there exist extensive literature about mathematical problem solving, only a handful of papers explicitly use Fermi problems. Furthermore, most of the articles are theoretical in the sense that they discuss 'what one can do with Fermi problems' (see the references in the section about Fermi problems in mathematics education above). However, Peter-Koop (2003; 2004; 2009) used Fermi problems in third and fourth grade to, among other things, investigate students' problem solving strategies. She concluded that (a) Fermi problems were solved in a sensible and meaningful way by the students, (b) the students developed new mathematical knowledge, and (c) solution processes "revealed multi-cyclic modelling processes in contrast to a single modelling cycle as suggested in the literature" (2004, p. 461). The impact and use of these results on a broader scale are however unclear and remains to be researched further.

Beerli (2003) reports on Swiss teaching materials for grades 7 to 9 which emphasize Fermi problems in a thematic way throughout; and, because of the way problem solving is connected to reality, opportunities are provided to use mathematization and modelling as means to develop mathematical knowledge and skills. Beerli also suggests that Fermi problems could effectively be used in assessment (pp. 90-91).

Schoenfeld (1985b, pp. 278-281) writes about a Fermi problem called *the cell problem* used for investigating cognitive issues connected to, and the developing of, methodologies for the study of students' problem solving processes. He describes how different constellations of students try to estimate how many cells there might be in an average-sized human body, to think about criteria of what might count as a reasonable upper/lower estimate, and to decide how much confidence they have in their estimates. The result of his analysis formed part of the foundation for the research approach he used in his book *Mathematical Problem Solving* (Schoenfeld, 1985b).

4. MATHEMATICAL MODELLING

In mathematics education one can find many different approaches to and perspectives on mathematical modelling in the research literature (Blum, Galbraith, Henn et al., 2007; Haines, Galbraith, Blum, & Khan, 2007). The variety of perspectives is illustrated by Sriraman and Kaiser (2006) in their report of an analysis of the papers presented in *Working Group 13: Applications and modelling* at the CERME4 conference[4] written by European scholars. They conclude "that there does not exist a homogenous understanding of modelling and its epistemological backgrounds within the international discussion on applications and modelling" (p. 45) and argue and call for a more precise clarification of the concepts involved in the different approaches to make communication and discussions more simple and fruitful.

However, Kaiser, Blomhøj and Sriraman (2006) are optimistic about the chances for such an understanding to develop and they argue that there already in certain respects exists "a global theory for teaching and learning mathematical modelling, in the sense of a system of connected viewpoints covering all didactical levels" (p. 82), but that this "theory of teaching and learning mathematical modelling is far from being complete" (p. 82). Hence, in the last years, efforts have been made to clarify and differentiate different approaches (Kaiser & Sriraman, 2006; Kaiser, Sriraman, Blomhøj, & García, 2007) and work in this direction continues (Blomhøj, 2008). According to Kaiser and Sriraman (2006), different perspectives on, and approaches to, mathematical modelling can be classification as *realistic or applied modelling*; *contextual modelling*; *educational modelling* (with either a *didactical* or *conceptual* focus); *socio-critical modelling*; *epistemological or theoretical modelling*, or *cognitive modelling*.

[4] The 4th conference organized by ERME, European society for Research in Mathematics Education, held in Sant Feliu de Guíxols, Spain, 17-21 February 2005.

Research on mathematical modelling in mathematics education, regardless which perspective on modelling is adapted, typically uses or develops some general description of the process of mathematical modelling (Kaiser et al., 2006). This general description is often given or summarised in a so called *modelling cycle*, which schematically and idealised illustrates how the modelling process connects the extra-mathematical world (domain) and the mathematical world (domain) (Blum, Galbraith, & Niss, 2007). Depending on the purpose and focus of the research these modelling cycles might look different and highlight different aspects of the modelling process (Haines & Crouch, in press; Jablonka, 1996).

In the Swedish context, where the present study is situated, the present mathematics curriculum and syllabus for the upper secondary level is founded in a reform from 1965. Since then, a number of reforms and revisions have been made with the affect that the emphasis on mathematical modelling in the written curricula documents governing the content in Swedish upper secondary mathematics courses has gradually been gaining more momentum (Ärlebäck, submitted). In the latest formulation from 2000, *using and working with mathematical models and modelling* is put forward as one of the four important aspects of the subject that, together with *problem solving*, *communication* and *the history of mathematical ideas*, should permeate all mathematics teaching (Skolverket, 2000). It is also stressed that "[a]n important part of solving problems is designing and using mathematical models" and that one of the goals to aim for is to "develop their [the students'] ability to design, fine-tune and use mathematical models, as well as critically assess the conditions, opportunities and limitations of different models" (Skolverket, 2000). However, a more detailed definition or description of what a mathematical model is or what it means to model mathematically, is not provided. Lingefjärd (2006) summarizes the development and situation as "it seems that the more mathematical modeling is pointed out as an important competence to obtain for each student in the Swedish school system, the vaguer the label becomes" (p. 96). The question that naturally arises is why this is the case and what can be done about it. Research in this area is lacking in Sweden, but challenges and barriers to overcome are likely to be similar to those reported by Burkhardt (2006). Some support for this assumption is found in research reports on the dominant role of the use of traditional textbooks in Swedish mathematics classrooms, greatly influencing class organisation as well as content (Skolverket, 2003).

The perspective on mathematical modelling I take in this paper is the view that mathematical modelling is a complex (iterative and/or cyclic) problem solving process, here illustrated in Figure 1. When analysing such complex problem solving process in more detail, one can do so using the notion of *competencies* (Blomhøj & Højgaard Jensen, 2007; Maaß, 2006), *modelling skills* (J. Berry, 2002), or dividing the modelling cycle into *sub-processes* or *sub-activities*. For example, Borromeo Ferri (2006) describes the modelling process in Figure 1 in terms of 6 *phases* (real situation, mental representation of the situation, real model, mathematical model, mathematical result, and real results) and *transitions* between these phases (understanding the task, simplifying/structuring the task, mathematizing, working mathematically, interpreting, and validating).

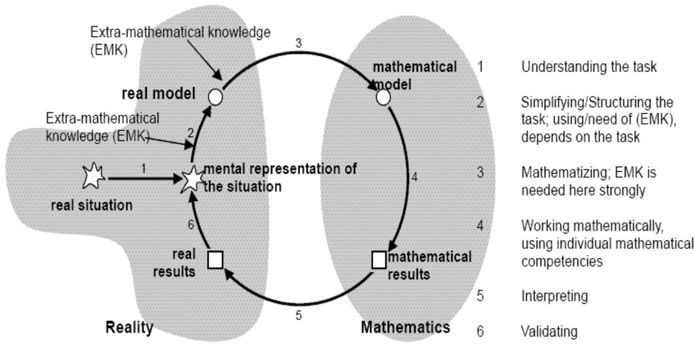

Figure 1: The modelling cycle by Blum and Leiβ (2007) as adapted and presented by Borromeo Ferri (2006, p. 92)

This choice of view on mathematical modelling is in line with how mathematical modelling is described in the Swedish upper secondary curriculum (see above) and other scholars have made analogous interpretations (e.g. Palm, Bergqvist, Eriksson, Hellström, & Häggström, 2004). In my interpretation of the classification of the approaches by Kaiser and Sriraman (2006) briefly mentioned above, this research is carried out in alignment with the educational perspective, where the goal to strive for is to learn students mathematical modelling. Some of the arguments of Kaiser, Blomhøj and Sriraman (2006) about different purposes for using such a view on the modelling process in research also motivate this choice of perspective; as the driving motivation is as *a didactical tool for planning* the very introduction of mathematical modelling in a modelling course or small modelling project, as *a point of reference to the particular curricular element* of modelling in governing curricula, and as the point of departure for developing *an analytical tool to retrospectively determine which part of the modelling cycle the students' have been working with*.

5. METHODOLOGY

Part of my research is done in collaboration with two teachers aiming to design a number of small *modelling teaching modules*[5] in line with the national curriculum for the Swedish upper secondary school. A natural question that arises in the early stages of the design process is how to introduce the topic of mathematical modelling in a gentle, efficient and interesting way. The paper entitled *Modeling conceptions revisited* by Sriraman and Lesh (2006), where they say in a paragraph about estimation and Fermi problems that "estimation activities can be used as a way to initiate mathematical modeling" (p. 248), caught my interest. However, not finding any reported empirical research on this specific matter, the decision was to design and conduct this small pilot study.

5.1. Developing an analytic tool

5.1.1 Schoenfeld's protocol coding scheme

For the present study I developed and used an adapted version of Schoenfeld's 'graphs of problem solving' (Schoenfeld, 1985b) to get a schematic picture of the problem solving process of students working on a Fermi problem. Originally, Schoenfeld's 'graphs of problem solving',

[5] A *modelling teaching module* is here a sequence of lessons with a planned focus on mathematical modelling.

adapted from Wood (1983, cited in Schoenfeld, 1985b), is part of "an analytic framework for the macroscopic analysis of problem-solving protocols, with emphasis on executive or control behaviour" (p. 271) in decision-making during problem-solving. His idea is to try to characterize the problem-solving process of an individual, or a group of individuals[6], by analysing transcriptions of verbal data generated during "out load problem-solving session" (p. 270). This he achieved through partitioning protocols[7], which is what Schoenfeld calls his verbal data, "into macroscopic chunks of consistent behaviour called *episodes*. An episode is a period of time during which an individual or a problem-solving group is engaged in one larger task … or a closely related body of tasks in the service of the same goal." (p. 292). Each episode was then characterized as either *reading*, *analysis*, *planning*, *implementation*, *exploration*, *verification*, or *transition*[8]. All the categories are briefly described together with so called *associated questions*, and "[a] full characterization of a protocol is obtained by parsing a protocol into episodes and providing answers to the associated questions." (p. 297).

In his original work, Schoenfeld claims that the reliability of the coding, done by three undergraduate students trained for the coding, is quite high (Schoenfeld, 1985b, p. 293), but no measure or methods of how this conclusion was drawn is presented. Scott (1994), attempting to replicate some of Schoenfeld's results, argues that "[t]here are clearly problems of interpretation in several of his [Schoenfeld's] behaviour categorisations" (p. 538), and through examples Scott illustrates that "fundamental ambiguity remains regarding the meaning of some of the parsing categories" (p. 527). However, Schoenfeld (1985b) ends the chapter on his framework for analysing the protocols in a humble way arguing that "it is best thought of as work in progress." (p. 314).

Although Scott (1994) is critical about the reliability of Schoenfeld's method, he expresses his conviction that "*concurrent verbalisation* with *no interviewer intervention* is least prone to the effects of the study environment, and to the incompleteness and inconsistency of some verbal data" (p. 537, italics in original). The debate on the suitability and usefulness of verbal data in research is reviewed and discussed in some length by Goos and Galbraith (1996). Drawing on the work by Nisbett and Wilson (1977), Ericsson and Simon (1980), Genest and Turk (1981) and Ginsburg, Kossan, Schwartz, and Swanson (1983), among others, they discuss the three different approaches of *talk/think aloud*, *concurrent probing* and *retrospective probin*g. Especially, they acknowledge differences in limitations in terms of *reactivity* (such as stress, task demands and influences from the direct environment), *incompleteness* (what is spoken out load is selective of what a subject thinks), *inconsistency* (observed behaviour do not correspond to what is verbalized), *idiosyncrasy* (the question of generalisability due to sensitivity for subjects individual differences), and *subjectivity* (the researches bias influences and interpretation).

Schoenfeld (1985a) also discusses the reliability and usefulness in mathematics education research of data in the form of "verbal reports (protocols) produced by individuals or groups" (p. 171), but at a more pragmatic level. He lists and discusses five 'variables' which might affect and reduce the limitations mentioned above when problems are solved 'in the load'. These five variables are *the number of persons being taped*, *the degree of intervention*, *the nature of instruction and intervention*, *the environment*, and *task variables* (pp. 174-176). These variables

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[6] In the case when the problem-solving process of a group is under consideration, the process refers to the process of the whole group as a collective, not being constituted of each individual problem-solving process.
[7] Or rather the transcription of the verbal protocols.
[8] Some episodes were characterized as both *planning* and *implementation* (Schoenfeld, 1985b, p. 296)

are not independent and can both reinforce or obstruct each other effects on the displayed behaviour. For example, an artificial environment (as a laboratory taped problem solving situation often is) making the students feel uneasy can be balanced with letting the students work in group constellations familiar to them. In a way, all five of Schoenfeld's variables deal with reactivity, and the 'values' of the variables will affect how comfortable the students are in a taped problem solving situation and the expectations and obligations felt by the students towards the task given them and toward the researcher. The variables controlling instruction and intervention are also connected to the limitations of incompleteness and inconsistency.

The idea to build on the work of Schoenfeld (1985b) is by no means new, and other researchers have used, modified and developed his ideas. For instance, in their study of two students' collaborative problem solving activity in an applied mathematics course, Goos and Galbraith (1996) used "a selective extension of Schoenfeld's episode analysis" (p. 241) combined with another framework. Stacy and Scott (2000) follows Schoenfeld's methodology "as closely as circumstances permitted" (p.123) when studying how and to what extent students use the problem solving strategy of *trying examples* in a problem solving situation. Exploring pairs of students' problem-solving process involving functions using a graphical calculator, Brown (2003) uses a modified framework for identifying and singling out interesting "defining moments" (p. 83) on a macroscopic level, before exploring these in greater detail on a microscopic scale in the search for possible explanation of the observed behaviour. In a closer look at decision making in group solutions' involving Bayes' formula in probability also Stillman (2005) makes use Schoenfeld's protocol parsing scheme. In contrast to Scott (1994), Goos and Galbraith (1996), Stacy and Scott (2000), and Stillman (2005), who use Schoenfeld's categories to code episodes as (part of) their framework, my adaptation is more in line with the approach taken by Brown (2003). Brown modified and extended the number of categories to better suit the characteristics of the problem solving situation she studied, as I did developing the *MAD framework* used in this study (see section 5.1.3.).

5.1.2. Realistic Fermi problems

Based on the earlier overview of the meaning and use of Fermi problems, I here describe my use of the concept by giving it the following definition. What I call *Realistic Fermi problems* are characterized by:

- their *accessibility*, meaning that they can be approached by all individual students or groups of students, and solved on both different educational levels and on different levels of complexity. A realistic Fermi problem does not necessarily demand any specific pre-mathematical knowledge;
- their clear real-world connection, to be *realistic*. As a consequence a Realistic Fermi problem is more than just an intellectual exercise, and I fully agree with Sriraman and Lesh (2006) when they argue that "Fermi problems which are directly related to the daily environment are more meaningful and offer more pedagogical possibilities" (p. 248);
- the *specifying and structuring of the relevant information and relationships* needed to tackle the problem. This characteristic prescribes the problem formulation to be open, not immediately associated with a know strategy or procedure to solve the problem, and hence urging the problem solvers to invoke prior constructs, conceptions, experiences, strategies and other cognitive skills in approaching the problem;

- the absence of numerical data, that is the *need to make reasonable estimates* of relevant quantities. An implication of this characteristic is that the context of the problem must be familiar, relevant and interesting for the subject(s) working in it;
- (in connection with the last two points above) their inner momentum to *promote discussion*, that as a group activity they invite to discussion on different matters such as what is relevant for the problem and how to estimate physical entities.

Using the nomenclature of Schoenfeld (1985a), the first four of these characteristics are all task variables that taken together define a type of problem quite different from the typical problems students normally encounter in their mathematics classes. In other words, one can expect that the students, at least initially, will behave a little lost, not knowing how to proceed. To some extend the last characteristic is also a task variable with the intention to, as a group activity, counteracting the problem solving process from stalling and the group to get stuck, which relates to the variable of number of students being taped.

The characteristics of Realistic Fermi problems were used for guidance when constructing the problems used in the study. From this point on, whenever the term (Realistic) Fermi problem is used in this paper, it refers to problems with these characteristics. It can be noted that there are some similarities to the six principles used in the *Models and Modeling perspectives* (Lesh & Doerr, 2003) for designing “thought revealing activities for research, assessment, and instruction” (Lesh, Hoover, Hole, Kelly, & Post, 2000, p. 595) called *modeling-eliciting activities*. Especially *the Reality Principle* bears resemblance to the realistic character of a Realistic Fermi problem; *the Self-evaluation Principle* is similar in the sense that a Realistic Fermi problem ‘invites’ the problem solver(s) to validate assumptions, estimates and calculations performed during the solving process; an analogue to *the Simplicity Principle* is the accessibility characteristic of a Realistic Fermi problem; and, as will be clear from the formulation of the Realistic Fermi problem used, also to *the Construct Document Principle*. The six principles for constructing modeling-eliciting activities of the Models and Modeling perspectives were not taken as point for departure for the construction of the Realistic Fermi problems used in this study, since these are embedding and situated in a multi-layer research framework with a broad focus facilitating a much wider research agenda.

5.1.3 The MAD framework

Comparing the phases and transitions in the modelling process described according to Borromeo Ferri (2006), and the character of a Realistic Fermi problem as presented above, one can see that there are obvious similarities. One might therefore try to describe and analyse the process of solving Fermi problems using this framework as it is. However, since the *estimating* of different sorts of quantities when solving Fermi problems is essential, and this is usually not a typical feature of a mathematical modelling problem, it seems that this activity also needs to be incorporated into the framework to give a more nuanced picture of the problem solving process of a realistic Fermi problem.

To adopt Schoenfeld’s (1985b) categories to the present study, I started from the view of modelling presented above and included the central estimation feature of Realistic Fermi problems to identify the following six *modelling sub-activities* to be used as codes for the activities the students engage in when solving a Femi problem:

Reading: this involves the reading of the task and getting an initial understanding of the task
Making model: simplifying and structuring the task and mathematizing

Estimating: making estimates of a quantitative nature
Calculating: doing maths, for example performing calculations and rewriting equations, drawing pictures or diagrams
Validating: interpreting, verifying and validating results, calculations and the model itself
Writing: summarizing the findings and results in a report, writing up the solution

Here, the activity of *reading* is similar to Borromeo Ferri's 'understanding the task'; *making model* incorporates parts of both 'simplifying/structuring the task' and 'mathematizing'; *calculating* is the same as 'working mathematically'; and *validating* is both 'interpreting' and 'validating'. The reason for these fusions is that it is often hard to separate 'simplifying/structuring the task' from 'mathematizing' and vice versa, and that to some extent 'interpreting' and 'validating' are intertwined. The sub-activity of *estimating* is implicit in Borromeo Ferri's modelling cycle, in my understanding found both as a component of 'simplifying/structuring the task' when constructing a 'real model', and as a component of 'mathematizing' in the transition from a 'real model' towards a 'mathematical model'.

A graphical representation of the problem solving process, analogue to the graphs Schoenfeld (1985b; 1992) describes, using these categories is called a *modelling activity diagram*, and is used as an analytical tool in this study to capture the macroscopic behaviour displayed by the students engaged in the activity of solving Fermi problems[9]. Examples of how such diagrams can look like are shown in Figures 2, 3 and 4 below.

5.1.4. Research question

Using the developed vocabulary from the previous sections it is now possible to rephrase the preliminary research question presented before in more specific terms:

What mathematical modelling sub-activities do groups of student display when the engage in solving Realistic Fermi problems?

[9] I will refer to this framework as the *MAD framework*.

6. METHOD

Using the characteristics of Realistic Fermi problems, two such tasks were constructed for this pilot study. This paper reports from students' work on one of these problems (the problem the students started with), *The Empire State Building Problem*. In this problem students were asked to come up with an answer for the time it takes to go from the street level to the top observatory floor in the Empire State Building using the elevator and stairs respectively. The very formulation of the problem given to the students was the following:

> There is an information desk on the street level in the Empire State Building. The two most frequently asked questions to the staff are:
> - *How long does the tourist elevator take to the top floor observatory?*
> - *If one instead decides to walk the stairs, how long does this take?*
>
> Your task is to write a short letter answering these questions, including the assumptions on which you base your reasoning, to the staff at the information desk.

An a priori analysis was made of the two questions of the problem to identify what the students reasonably *must* estimate and model to be able to solve these problems. In addition to this scrutinizing of the problems, possible extension and more elaborated features to incorporate in the situation were also identified. As a result of this analysis, the information that students reasonably must use in the case of the Empire State Building problem are the height of the Empire State Building, the speed of the elevator and the speed when walking the stairs. As for more elaborate extensions to include in the model, we have the elevator queuing time and the capacity of the elevator. The time for getting in and out of the elevator might also be considered. In the problem on walking the stairs on the other hand one could start thinking of how to model the endurance and one's fitness.

Seven students volunteered, all from a class enrolled in a university preparatory year taking the upper secondary courses in mathematics taught by the author, and divided themselves into three groups, A, B and C. In group A these were the three male students Axel, Anders, and Axel; group B was constituted by one female, Birgitta, and one male student, Björn; and group C's members were the two male students Christer and Claes. All the names used are pseudonyms. The group constellations are by no means random; in class they normally sit together helping each other out on the problems to be covered in a specific lesson. After a short introduction, dealing with ethical issues of the study and urging the students to do their best and to think aloud, the groups were placed in different rooms equipped with videotape recorders and were set to work. The two problems were distributed one at a time, and the groups worked on each problem for as long as they wanted.

The work of all three groups on the problems captured on the videotapes was transcribed using a modified and simplified version of the TalkBank conversational analysis codes[10] as a guide for the transcription. The students' written short answers were also collected.

[10] www.talkbank.org. In this paper '(.)' means a short pause, and '((*text*))' is a comment added by the researcher to clarify the context or meaning.

The transcriptions were coded by the categories of the six modelling sub-activities described above. The categorization was done both on the level of *utterances*, here taken to be "stretch of continuous talk by one person, regardless of length and structure" (Linell, 1998, p. 160), and on the level of *dialogues* constituted by a sequence of utterances made by the group members taking turns making utterances. The question asked to, and guiding the categorization of, each utterance and dialogue was 'What sub-activity is the utterance/dialogue indicating that the student/group is engaged in?'. This process was repeated to refine the coding and test the reliability of the process, and the procedure was validated by looking at the video-recordings as well as the written short answers from the three groups. Examples of the coding are given in next section. The final result of this analysis was graphed in a *modelling activity diagram* for each group, showing the time spent and the moves between the different modelling sub-activities during the work with the problem. It was decided to graph the sub-activates in the modelling activity diagram using time intervals of 15 seconds to make the description as clear as possible.

7. EMPIRICAL RESULTS AND ANALYSIS

When the groups were given the Empire State Building Problem, the students' first reactions in all three groups involved surprise and frustration of not knowing the data needed to solve the problem:

"It's just to estimate everything!" (Christer, group C)

"It's just to make something up!" (Axel, group A)

"That's a bad question since we don't get to know how high it ((*the Empire State Building*)) is!" (Björn, group B)

However, after this initial shock, the groups were very active and spent approximately 30 minutes engaged in solving the two parts of the problem. The work of the groups naturally divides into three main phases. After the initial reading of the problem formulation, these are the two phases dealing with the solving of the two parts of the problem (phase one dealing with the elevator and phase two with the stairs), and a third writing phase where they compose the letter asked for in the task. All groups spent about one minute reading the problem. How the groups distributed their time on the three main phases is summarised in Table 1.

Table 1. Summary of how the groups used their time on the three main phases

Group	Time spent on first part	Time spent on second part	Time spent on writing	Total time spent on task
A	5 minutes	18 minutes	6 minutes	30 minutes
B	9 minutes	10 minutes	11 minutes	31 minutes
C	7.5 minutes	16.5 minutes	7.5 minutes	32.5 minutes

When solving the first part of the *Empire State Building Problem* regarding the use of the elevator, none of the groups elaborated their solution to incorporate the extensions of the problem suggested from the a priori analysis presented above or any other elaborations. All groups simply calculated that time equals the height divided by the average speed, a model taken for granted. However, in dealing with the second part of the problem, they developed quite different ways to approach how to model the endurance of the stair climbers, and also spent time discussing how the stairs actually where constructed (if it was a spiral staircase, if there were landings on each floor, and so on and so fourth). Group A used the same model as in the problem with the elevator (using an estimated average speed), and group C a variant of the same approach estimating the average time needed to walk the stairs of one floor and adding some additional time for resting before walking the next one. Group B developed a more advanced model where the time taken for a given floor depended on how high up in the building the floor is situated. Some of the key estimations done by the groups, and their answers to the problem, can be found in Table 2 below[11]. It could be noticed that the mathematical demands were kept at a very elementary level throughout the problem solving activity in all groups.

Table 2. Some of the groups' estimated quantities and their answers to the problems (see the Appendix)

Group	Estimated height (m)	Estimated elevator speed (m/s)	Answer, time for using the elevator	Answer, time for using the stairs
A	300	5	60 s	40 min
B	175	3,6	48,6 s	16 min 15 s
C	350	2	2 min 55 s	1 h 55 min

The *modelling activity diagram* for the solving process for group A, B and C is shown in Figure 2, 3 and 4, respectively. On the vertical axis the modelling sub-activities ***R****eading,* ***M****aking model,* ***E****stimating,* ***V****alidating,* ***C****alculating* and ***W****riting* are displayed. Time in minutes, starting from the point when the students got the problem formulation and to the time they handed in the written solution (the letter), is displayed on the horizontal axis.

7.1. Group work

7.1.1. Group A

The work of group A is driven forward by Alfred and Axel, who talk approximately twice as much as Anders[12], and it is their initiatives and ideas that form the strategies the group chooses to pursue. On the few occasions where Anders comes with a suggestion or a comment outside the black box in which Alfred and Axel are working for the moment, his ideas are given very little attention and eventually fade away. Alfred and Axel are both high achieving students in class, while Anders is just above average. The group is focused in the sense that all three students seem to be engaged in the same sub-activity most of the time. During the session a legible difference in attitude toward the problem and what the solution should look like and contain becomes apparent between Alfred and Axel. Axel takes the task to get as good estimates as possible very seriously. He is the one that initiates the group in the sub-activity of *validating* most of the times and makes it his priority to make the group's solution as realistic as possible. Alfred, on the other hand, is not interested in getting good estimates of the needed quantities, but rather more focused

[11] From http://www.esbnyc.com/ one can read that the 86th Floor Observatory is situated 320 meters above street level and that for the 102th Floor Tower the figure is 373 meters. There are in all 73 elevators in the building operating at speeds between 3 m/s and 5 m/s, and that it is possible to ride from the lobby to the 86th floor in less than a minute (20090403 one can find several video-clips on http://www.youtube.com documenting this ride). The number of stairs are 1576 and each step is approximately 19 cm in height. Normally, walking the stairs is not permitted, but there is an annual stair climbing race, "the Empire State Building Annual Run Up", and the record for running the stairs from the lobby to the 86th floor is 9 minutes and 37 seconds

[12] The number of utterances coded uttered by Anders is 113, by Axel 209, and by Alex 272.

on the principle arguments behind how to achieve the answer. In Alfred's opinion, if the actual height of the Empire State Building is 200 meters or 300 meters is not important; what matters is how to use the estimated height to come up with an answer. This difference occasionally creates tensions in the group, but Axel is the most persistent one, and usually persuades Alfred (and Anders) to reconsider their sometimes naïve assumptions and estimates.

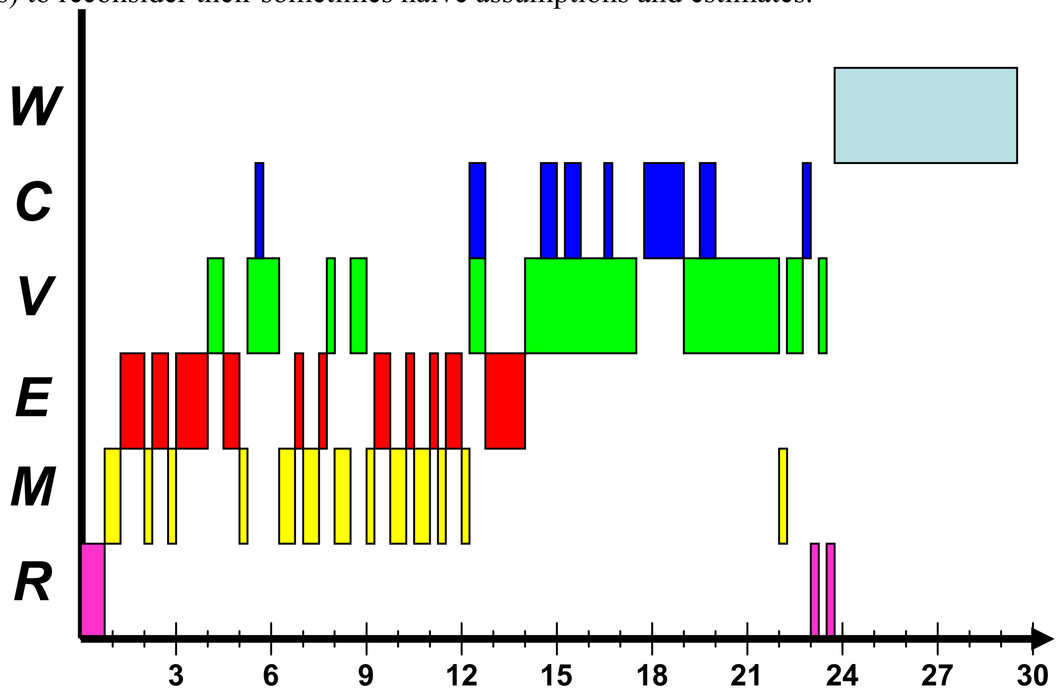


Figure 2. The modelling activity diagram for the Empire State Building problem (group A)

Figure 2 displays the modelling activity diagram of the work of group A on the Empire State Building problem. After a short initial *reading* phase, about 5 minutes are spent on the first part of the problem consisting of the dialectic interplay between *making a model* and *estimating* (with some minor elements of *validating* and *calculating*). When the group engages in solving the second part of the problem they first continue in this dialectic manner for another 8 minutes, followed by approximately 10 minutes composed of *validating* with a parallel element of *calculating.* The problem solving session ends with a 7 minutes phase of *writing*. In the writing phase all three members of the group are involved in dictating the letter with Alfred functioning as the secretary. The letter can be found in the appendix.

Alfred, Anders and Axel use the same strategy to solve both parts of the problem, namely to calculate the time using an estimated height and an average speed of the elevator. However, in the second part, they first use this model to get time 33 minutes for climbing the stairs, but adjust this calculation to 40 minutes since their model of how the stairs look includes landings on every floor.

7.1.2. Group B

Birgitta and Björn are two of the two most ambitious students in the class and they are used to be able to solve most of the textbook problems they encounter in class without too much effort. During this non-standard problem solving session they have difficulty in deciding whether or not an estimate is good enough and utterances expressing this insecureness are common (this is exemplified in the two dialogue extracts from the group found later in this paper). At one point in the beginning of their problem solving session, when I entered the room to check on the recording device, Birgitta and Björn quite obstinately wanted me to confirm whether their estimation of the height of the Empire State Building was correct, but naturally I neither confirmed nor rejected their estimate.

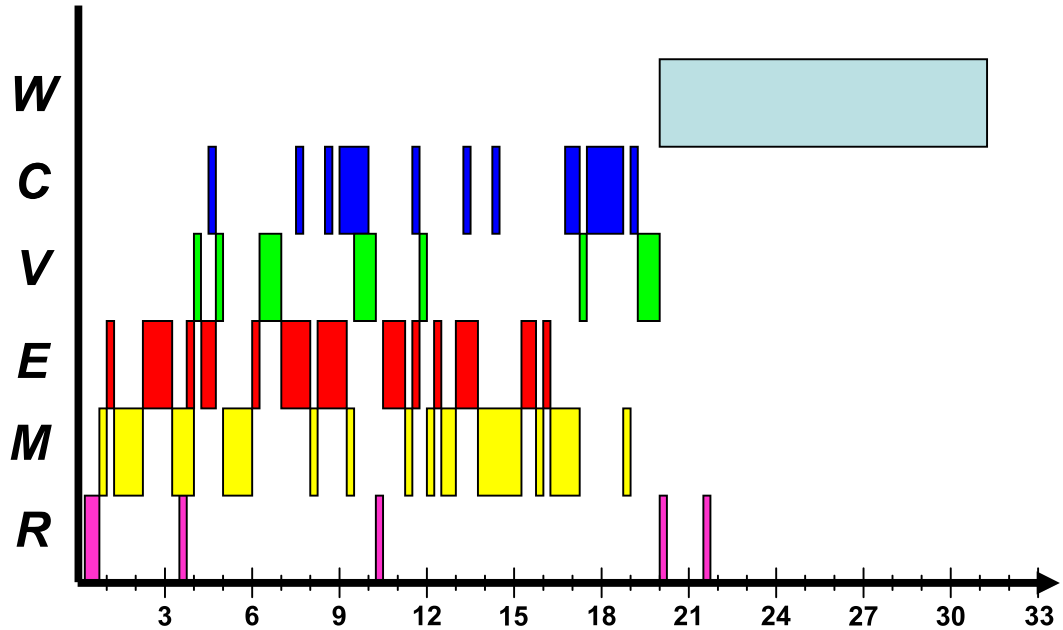


Figure 3. The modelling activity diagram for the Empire State Building problem (group B)

Group B starts with a short initial phase of *reading* (see figure 3) before Britta and Björn spend approximately 9 minutes on the first part of the problem, partly struggling to understand and make sense of what to do and how to do it. About the same amount of time, 10 minutes, is then devoted to the second part of the problem, followed by 11 minutes of *writing*.

The model activity diagram for group B looks a bit different from group A. Their problem solving process is not as concord as the process group A displayed, and Björn and Britta's modelling activity diagram appears to sprawl more than the diagram of group A. After about the first three minutes the problem solving process displayed in figure 3 jumps between *making a model*, *estimating*, *validating* and *calculating* (occasionally returning to the problem formulation for some re-*reading*) in a seemingly random but uniformed distributed manner until the phase of *writing* begins.

To answer the second part of the problem, group B develop a model of how physical tiring it is to climb the stairs in a high building. In their model they group their estimated number of floors in Empire State Building, 50, into groups of 5. They then estimate that it will take 15 seconds to walk from floor 1 to floor 5, and their model predicts that the next 5 floors will take the same amount of time that climbing the last 5 floors did plus additional as many seconds longer as the number of floors you already climbed. Hence, to climb from floor 6 to floor 10 will take 15 + 5 = 20 seconds, from floor 11 to floor 15 will take 20 + 10 = 30 seconds, and so on (see the Appendix, group B's letter for details). As a consequence of this model, the time it takes to climb the last 5 floors, from floor 46 to floor 50, is 240 seconds, meaning that climbing one floor in average takes almost a minute (58 s.). However, this is something neither noted or reflected on by Björn and Birgitta.

Of the three groups, group B is the one that spent the longest time on *writing*. One reason for this is that Birgitta starts writing the letter but has trouble in formulating their reasoning in the second part of the problem so Björn has to take over. In addition, their solution to the second problem is the most sophisticated and complicated of the groups, which contribute to the larger amount of time needed to go through and account for all the details in the writing process. Bigitta's and Björn's letter can be found in the appendix.

During the problem solving process Birgitta's beliefs about what constitutes an answer to a (school) mathematics problem surface as the excerpt below illustrates.

Birgitta:	But if you think about it (.) we are supposed to give them an answer
Björn:	Mm
Birgitta:	But since we don't have the values we can't. We have to do like you said before ((*to give and estimate*))
Björn:	Mm
Birgitta:	We have to sort of (.) but how do we do that without having any values?
Björn:	Mm (.) we can estimate
Birgitta:	Yes but then it'll be (.) estimations
Björn:	Mm
Birgitta:	So that's not an answer
Björn:	Mm
Birgitta:	Is it (.) or?
Björn:	Hm ((*indicating that perhaps it is*))

Another aspect of the same belief is the fact that they give an answer to the first part of the problem with a very high accuracy, 48.6 s, despite the rough estimates done for the calculation producing the answer.

7.1.3 Group C

Christer and Claes in group C are two lively, open, very talkative and mischievous students, which is reflected in their work on the problem. Being full of fun they now and then take more or less serious and reasonable assumption under consideration for inclusion in their solution. An example is when they work on how long it would take to climb the stairs and discuss the need for some extra time for stops and argue with the accompanying mother-in-law every now and then. Concerning their abilities in mathematics, Claes is an average student and Christer just below average. During the problem solving activity Christer shows a focus on getting an answer, whereas Claes wants to try different approaches and see if the answers these provide agree. Claes is the enthusiastic one of the two who brings many of the ideas to the table, while Christer is the one questioning their work initiating validation processes. Among the three groups, group C seems to be the one enjoying the problem solving activity the most.

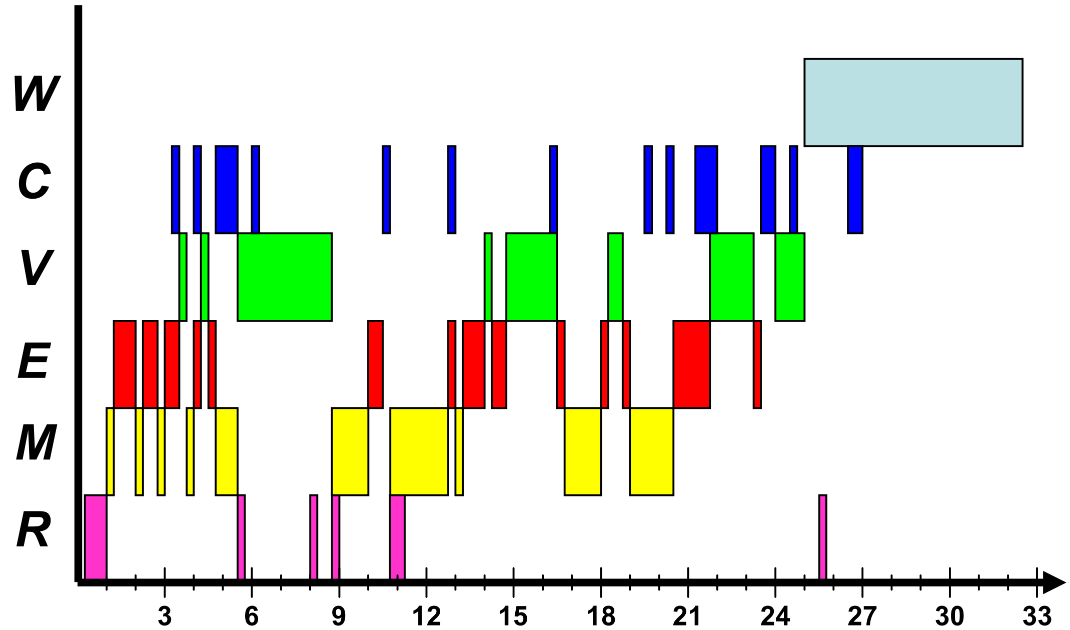


Figure 4. The modelling activity diagram for the Empire State Building problem (group C)

In Figure 4 the modelling activity diagram of group C starts with a short initial *reading* phase, followed by about 7.5 minutes spent on the first part of the problem and then approximately 16.5 minutes on the second part. The *writing* phase that ends the problem solving session is about 7.5 minutes. The modelling activity diagram of group C bears resemblance to the diagrams of both the other groups; the sub-activities Christer and Claes engage in when they struggle with the first part of the problem are similar to the one displayed by group A. However, when solving the second part of the problem, there are more similarities with the way group B engage in the sub-activities. It is Christer who under silence does all the *writing* while Claes devotes himself to drawing pictures of skyscrapers; the letter can be found in the appendix.

7.2. Examples of the coded categories

The following section will provide examples of, and comments on, how the data was coded using the categories of the MAD framework.

7.2.1. Making model

A typical segment of the group work, which was categorized as *making model* deals with negotiating and agreeing on how to structure the problem and which assumptions or idealizations to make. This is briefly illustrated in the following excerpt which takes place around 3 minutes into the discussion of group A. The group just estimated the height of the Empire State Building to be 200 meters and now turned to properties of the elevator:

Alfred: But in the case of a tourist elevator (.) shall we then assume that it

just goes from the ground to the absolute top ((*floor*))?
Anders: Yes
Axel: Without stopping
Anders: Yes

Another example of a small dialogue coded as *making model* is the following when Christer and Claes decide on with method to use to calculate an estimate for the time it takes to use the stairs. They have just spent quite some time discussing how to take into account that it is physically tiring to walk the stairs in terms of resting a given amount of time after a given interval of floors. So far, their idea has been to let the amount of time spent resting depend on how high up in the building you are, but they had not come to an agreement on the details:

Christer: Ok, shall we try to use some sort of average in this case to (.) 'case it's not interesting to calculate (.) I mean it's (.) 'case it'll (.) it'll (.) the velocity will decrease the higher up one gets
Claes: Mm, but (.) but
Christer: Let's settle for using an average all the time
Claes: ((*nodding*)) Then, then we can (.) Put it like this, we can make an estimate (.) let's say that you start to rest on every 10th floor and rest for a minute. Otherwise, if you're a fatso you'll never make it

7.2.2. Estimating

Estimation segments are often initiated with a direct question as for example "Yes, but how high is the poor building?", "How fast can an elevator run?" or "How many floors are there?". The discussions that follow such a question are all aimed to produce an estimate of some quantity, a number:

Alfred: Shall we estimate a speed by which an elevator can travel upwards? A typical elevator in a building of a regular height of say 200 meters (.) going all the way up (.) in meters per seconds
Axel: Yes, say that (.) a typical elevator might travel by the speed of 3-4, 3 meters per second
Alfred: Shall we say 5? It's a quite fast elevator
Axel: Mm, yes, let's settle for 5 ((*m/s*)).
Alfred: 5 ((*m/s*))?
Anders: Hm ((*nodding his head*))

The following excerpt comes from when Björn and Birgitta try to get an estimate of the height of the Empire State Building. They have just come to terms with the fact that they have to make some estimates to be able to solve the problem. This example also to some extent illustrates the insecureness that characterizes Birgitta and Björn's way of working:

Birgitta: Ok, sure. Let's start making up some values then
Björn: Mm
Birgitta: Mm (.) eh (.) Empire State (.) State Building (.) well, how high was the World Trade Centre for example?
Björn: How high was what?

Birgitta: World Trade Centre
Björn: What? ((*inaudible*))
Birgitta: I stink at this (.) estimating stuff
Björn: Yeah, me too ((*inaudible*))
Birgitta: ((*laughter*)) But
Björn: But (.) I don't know (.) it must (.) take 50 floors (.) it can't be more than say (.) sort of 100 meter say (.) more than 300 meter sort of (.) no (.) 150 ((*inaudible*)) (.) Say that a floor is like 3 meters
Birgitta: Yes
Björn: Or if you think about (.) they might be higher that that ((*inaudible*))

Before Björn and Birgitta use the estimated number of floors and how high each floor is to calculate the height of the building in a validation process of the suggested height of 150 meters by Björn above, the dialogue when Birgitta expresses her beliefs about what should constitute a (mathematical) answer to the problem takes place (see the section on group B above). Eventually, they agree to estimate the height of the Empire State Building to 175 meters.

7.2.3. Validating

Questions also often start up segments of *validation* ("Shall we really say 100 floors?"), as do statements of doubt ("It feels like that's too fast (.) for that building"). In these segments previous assumptions, estimates, calculations and results are looked at critically and are either made manifest or rejected in favour of better versions, as in the following example where the calculated elevator time evokes questioning of the previous estimated height of the *Empire State Building* (of 200 meters):

Axel: It feels like that's too fast (.) for that building
Anders: Hm (.) it does
Axel: I think it is (.) and now I'm stretching it (.) I think it is 300 ((*meters*)) (.) I think it is 280 meters, that I was off with a hundred ((*meters*)) before
Alfred: We can say that it ((*the Empire State Building*)) is 200 meters, that's fine
Axel: Yes ((*10 seconds of quiet mumbling, Axel is working on his calculator*)) 60 seconds, one minute to go up there. Is that reasonable?
Anders: Yes
Alfred: If we assume, how high
Axel: Yes, 'cause that's almost probable
Alfred: Yes, 300 meters (.) 300 meters
Axel: One minute then. One minute can be really long
Alfred: Yes, especially in an elevator

Although Axel's second statement expresses a re-estimation of the height of the Empire State Building, and indeed that he afterwards is doing some calculations, this excerpt is coded as the group being engaged in the activity of *validating*. This extract exemplifies the type of considerations that must be dealt with when using the codes of the MAD framework; it codes the activity on the group level and not on the individual level. Nevertheless, Axels' contribution in this excerpt (estimating) has a big influence on the groups' final result.

7.2.4. Calculating, Reading and Writing

Calculating is an activity normally preformed by one of the group members in the background of some other activity, so here the video recording is crucial for the coding. Occasionally the whole group focus on the actual calculation, but regardless of how it is obtained and by whom, the result of a calculation is important for how the solving process evolves.

The sub-activities *reading* and *writing* are rather self-explaining, but it is notable that when the group came to writing down their answers (in the form of a letter) they just reproduced and retold what they said before without any reflections or critical scrutiny.

7.3. The use of extra-mathematical knowledge

It could also be observed in the data that the students frequently used their personal extra-mathematical knowledge and experiences from outside schools in the solving process. It seems that they did this in at least three different ways: in a *creative* way to construct a model or to make an estimate, in the process of *validating* a result or an estimate, and finally in a *social* way as a narrative anecdote.

In the following excerpt Axel shares a personal experience from an amusement park in a creative way in hope to easier get an estimate of the elevator speed. However, in this specific case his reasoning makes him question the estimated height of the Empire State Building (200 m) which the group had agreed on two minutes earlier. Hence he is also using this piece of extra-mathematical knowledge to (involuntary) initiate a validating process:

Axel: Hm (.) Did anyone ride the FREE FALL[13]
Anders: ((*in unison with Alfred*)) Nope.
Axel: I was thinking that since (.) hm, it is 90 meters ((*high*)), how long does it take to get up there? (.) I think it takes 15, 20-25 seconds (.) and that's 90 meters.
Alfred: Yes.
Axel: It ((*the Empire State Building*)) must be higher than 200 meters.

As it turned out, all seven participants had a common friend living at the top apartment in a four storage apartment building. All groups used their knowledge and experiences about this apartment whereabouts as point of departure for modelling and estimating the elevator speed, the height and number of the steps of the stairs, time for walking the stairs. Other extra-mathematical knowledge invoked included experience from working as a postman, visiting a high tower in Malaysia, the whereabouts of other friends'/relatives' apartments, different elevator-experiences, climbing up in a radio tower on a Jacobs' ladder, and mounting climbing, just to mention a few.

7.4. Similarities and differences between the groups

From the constructed modelling activity diagrams (figures 2, 3 and 4) one can observe that the students engage in all of the predefined different sub-activities, that they do spend a considerable amount of time in each sub-activity, and that they go back and forth between the different types

[13] *FREE FALL* is an attraction in the amusement park *Gröna Lund* in Stockholm, Sweden.

of activities numerous times. In other words, the processes involved in the mathematical modelling cycle pictured in Figure 1 are richly represented in the groups' problem solving processes.

Looking at the problem solving process of the three groups one can observe both similarities and differences. To get an overview of the amount of time categorized as spent on the different sub-activities for each group and phase of the problem solving process Table 3 was complied. From this table it is clear the group A spent by far the least time engaged in *Making models* compared to the other two groups. One reason for this might be that group A quickly decided to base their work on the model that time equals height divided by average speed, and did not have to engage in discussing more advanced models. Group B had initial problems in coming to terms with how to attack the problem, which explains why they devote twice as much time making model during the first part of the problem than the other groups. In the second part of the problem, on the other hand, group C spend the most time *Making models* since they started to think in the same line as group B, but gave up this idea and ended up in just a slightly different model than the one adapted by group A.

Table 3. Summary of the amount of time categorized as the different sub-activities for each group and part of the problem solving process, respectively

	First part			Second part			Writing			Total		
	A	B	C	A	B	C	A	B	C	A	B	C
R	0:45	0:45	1:15	0:30	0:15	0:45	0:00	0:30	0:15	1:15	1:30	2:15
M	1:15	3:30	1:45	3:30	4:00	6:15	0:00	0:00	0:00	4:45	7:30	8:00
E	2:45	4:15	2:15	3:15	2:30	4:15	0:00	0:00	0:00	6:00	6:45	6:30
V	1:30	2:00	3:45	8:30	1:15	5:00	0:00	0:00	0:00	10:00	3:15	8:45
C	0:15	1:45	1:30	3:45	2:45	2:45	0:00	0:00	0:30	4:00	4:30	4:45
W	0:00	0:00	0:00	0:00	0:00	0:00	5:45	11:15	7:30	5:45	11:00	7:30
										31:45	34:30	37:45

On the whole the groups spent approximately the same amount of time *Estimating* and *Calculating*. However, group B is the only group who spend more time *Estimating* on the first part of the problem relative to the second part. One explanation of this behaviour might be the insecureness expressed by the group, due to not being able to relate the Empire State Problem to the types of problem they normally encounter in their mathematics classrooms. This insecureness makes the group seriously doubt their capabilities and even hard to engage in making assumptions and estimates at all. When solving the second part, on the other hand, they more or less focus on the procedure of how to come up with an answer and *Estimating* becomes secondary.

When it comes to *Validating*, group B engage by far the least in this sub-activity of the three groups. This is due to the little time group B spent validating on the second part of the problem, and the group seems to be happy when they have calculated an answer and do not express any inclination to question their numbers coming from a calculation. To a certain extent this behaviour is not surprising knowing some of the beliefs expressed by Birgitta about (school) mathematics.

In *Writing* the letter asked for in the task, the groups used between approximately six to eleven minutes; group A used the least amount of time and group B the most time. Why group B spent so much time writing in comparison to the other two groups has already been discussed (see 7.1.2.), and looking at the letter produced by group A and C (see the Appendix) one can observe that much more effort and detail is put into the letter by group C, giving a possible explanation for group C using more time than group A in writing the letter.

Finally, taking a look at how much time was spent totally on the different sub-activities, one can see that group A spends the least time, group B somewhat more time, and group C the most time. This is in line with the observed behaviours of the groups; group A works in a cohesive way where the members of the group dynamically follow each other if one starts to engage in another sub-activity. This makes group A's work very focused and thus, relative to the other groups, not so many sub-activities are being engaged upon simultaneously. The other two groups exhibit less of this behaviour. In the case of group B their issue of not having a clear way to approach the problem makes their behaviour more searching and ambivalent in trying to cope with the situation. The behaviour of group C, on the other hand, is more or less the opposite to the one found in group A, and is explained by that fact that group C is engaged in multiple sub-activities four times as much as group A.

8. SUMMARY AND DISCUSSION

The modelling activity diagram is a different way to describe students' modelling processes than has been done in many empirical studies. Borromeo Ferri (2007a; 2007b) pictured what she called "individual modelling routes" of her students by drawing arrows in the modelling cycle shown in Figure 1. In the context of picturing the processes engaged in during solving modelling tasks, the modelling activity diagram can be used to visualize a group or an individual student's modelling sub-activities in a more linear way along a timeline. Thus, it provides a simple dynamical picture of the activities involved. From the results presented above, the modelling activity diagram shows that Fermi problems might serve well as a means to introduce mathematical modelling at this school level: All modelling sub-activities are richly represented and contributed in a dialectic progression towards a solution to the task.

It is clear from the three modelling activity diagrams produced, that the view presented on modelling (see Figure 1) as a cyclic process is highly idealised, artificial and simplified. This way of conceiving mathematical modelling was useful for the developing of the MAD framework, but real authentic modelling processes are better described as haphazard jumps between different stages and activities, as is also noted by Haines and Crouch (in press).

Borromeo Ferri (2006; 2007a; 2007b) also notes that students use extra-mathematical knowledge in the modelling process when deriving the mathematical model and validating results. In the latter case, she differentiates between "intuitive" and "knowledge-based validation" (2006, p. 93) and notes that students mostly only make what she calls "inner-mathematical validation", and that validating for students means "calculating". However, the data in this study provides numerous examples where personal extra-mathematical knowledge is used by the students in the validation of both models and estimates as well as in the validation of calculations.

One of the reasons for using Realistic Fermi problems was to urge the groups into discussions about the problem setting and how to approach the problem. In my opinion this worked out nicely, but the study also suggests that to some extent such problems take the focus away from the mathematics, which I believe students experience hard to discuss. It also makes the problem available in an indirect way through the discussions about how to structure the problem and what (and how) to estimate. In this respect the realistic feature of the problem is crucial. Although the mathematics was kept at a very elementary level, one could have tried to deepen it by explicitly asking for, say, an equation relating height and time spent in the elevator or in the stairs.

Looking at the three letters produced by the groups (see the Appendix), and taking into account how much time the they spent on composing them, it is astonishing how little information they contain about the groups' activities during the 30-minutes long problem solving session. This brings to the fore the issue of how to assess modelling work and the development of modelling competencies, an active area of research where some results and methods are emerging, but which in my opinion still needs to be further researched.

In the data material one can note that the group dynamics are essential for the evolution of and activation of the different sub-activities during the problem solving process. It is the discussions and interactions in the groups, when different beliefs and opinions are confronted, that drive and shape the modelling process. Group behaviour is strongly influenced by individual preferences and group composition, making it one of the most important task variables to consider. Group B is an example of a constellation which was not optimized, whereas group C displayed a better blend of personalities, in the sense that members of group C complemented each other, bringing different attitudes and perspectives to the collaboration. The members of group B on the other hand, were in a sense too alike, with the result that the group got stuck within their expectations and way of thinking. Indeed, this social dimension of the problem solving process is something I feel the framework and methodological choices need to take into account more seriously in the future.

Schoenfeld reminds us that "any framework for gathering and analyzing verbal data will illuminate certain aspects of cognitive processes and obscure others" (1985a, p. 174). In line with Schoenfeld (1985a), I wanted to minimize my *degree of intervention* once the groups were set to work, and after giving the students the initial instructions I only briefly visited them to check on the recording devices. On some occasion I got a question from the students (see the section on group B above), but otherwise the *nature of instruction and intervention* was limited to the initial instruction and later the problem formulation, in order not to affect and interfere in the problem solving activity more than absolutely necessary. The environment in which the groups where situated was naturally superficial. Although the departmental group rooms where nicely furnished, the students had never been there before, and in addition there was a camera directed at them. They were also asked to 'think aloud' and to work on a non-standard problem. The question now is if the subjects felt so uncomfortable that it induced an atypical and pathological behaviour in the students' problem solving process? I would argue that most circumstances above had little or no effect on the students' behaviour. For one thing, they had all volunteered to participate, and more over, they all knew me very well, infusing some sort of confidence in them. In addition, working in groups of two or three students quickly seemed to make them forget the camera and the direct surroundings. However, all these *task variables* are hard to keep track of and to try to realise their consequences for the displayed behaviour.

For the validity of the study using adult students in a special university preparatory course may be questioned. However, the mathematical pre-knowledge was the same as for adolescent students in school taking the same courses, and the increased extra-mathematical knowledge that came into play when working on the problems studied were not of a kind that could not have been experienced also by the latter group. That the participants chose the groups themselves can be considered to be an advantage for this pilot study, facilitating openness in the discussions. An alternative method of using a grounded theory approach could also have been applied to the data. However, the results of a study can only be interpreted within the research framework chosen, which in this case was linked to the mathematical modelling perspective where the pre-defined categories used make it easier to relate to and locate the results in previous research.

9. CONCLUSIONS AND FUTURE RESEARCH

Returning to the research question, one can conclude that all the modelling sub-activities proposed by the framework (*reading, making model, estimating, calculating, validating* and *writing*) are richly and dynamically represented when the students get engaged in solving Realistic Fermi problems. Thus this study shows that small group work on Realistic Fermi problems may provide a good and potentially fruitful opportunity to introduce mathematical modelling at upper secondary school level.

This research may be continued in a number of ways. First, the tool *modelling activity diagram* as an instrument of analysis has a potential to be developed further in different directions, depending on what it will be used for. One idea is to incorporate the group dynamics into the diagram by indicating in a sub-activity segment how much each group member contributes to the discussion. One could also try to modify the framework to be more general; *reading* in this study is just reading, but in a more general setting *reading* could stand for the gathering of any external information. In some situations it may be needed to try to split the sub-activity *making model* into the two sub-activities *structuring* and *mathematizing*.

Second, in a setting of teaching mathematical modelling, a joint follow-up session with the three groups in close connection to the problem solving session could serve well as a meta-cognitive activity, discussing the problem and their way of solving it, in order to highlight the processes involved in mathematical modelling. In future research the outcome of such an intervention might be fruitful to investigate.

Third, the data also pose interesting questions about how the students validate their results, models and estimates – why do they choose to do this the way they do? A (calculated) result depends on the model developed, the estimates done and the performed calculation. So, in validating a result it is desirable that all these three “influences” are looked upon critically (see Figure 5). Since all of these three types of validation are present in the data it would be interesting to investigate this phenomenon in more detail.

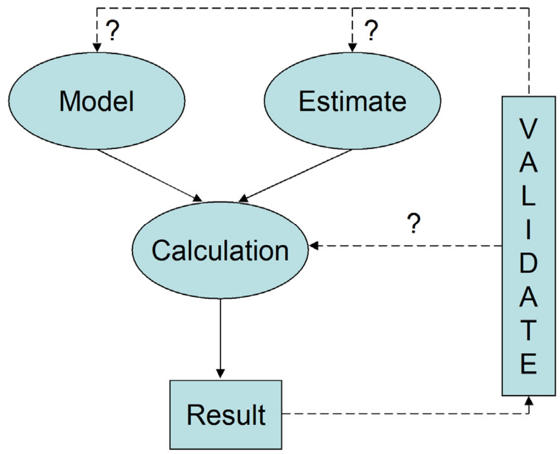


Figure 5. Possible validation routes

One of the most evident results produced by the MAD framework, illustrated in figures 2, 3, and 4 respectively, is the non-cyclic nature of the modelling process. Although the idealised view of mathematical modelling as described in terms of a modelling cycle has been much employed in mathematics education research, the discrepancy with what actually happens when students engage in modelling activities is palpable. My opinion is that this 'inconsistency' is something that researchers ought to take more seriously to refine current theories and methods to be able to better validate our research findings. Reference to elaborated epistemological analyses of mathematical modelling and authentic mathematical models in relation to education (e.g. Jablonka, 1996) is needed as well as discussions from the learner's perspective. Using a competence approach (Blomhøj & Højgaard Jensen, 2007; Maaß, 2006) might provide one alternative to the commonly used modelling cycle, but surely there must be other routes as well to investigate?

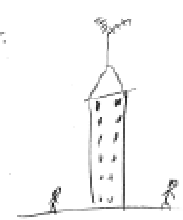
APPENDIX, Letter produced by group A

Goddag!

Om man skulle vilja ta hissen upp för denna 300 m höga byggnad, så tar det ca 60 sekunder, då farten på vår hiss är 5 m/s

I fall man istället vill ta trapporna upp för denna 100 våningar höga byggnad, med dess ungefär 3000 trappsteg, tar det istället runt 40 minuter för dig med hyfsad kondition och raskt steg.

I genomsnitt kommer då en våning ta runt 24 sekunder.



Good morning/afternoon/evening!

If one would like to take the elevator up in this 300 meter high building, it will take approximately 60 seconds, since the speed of our elevator is 5 m/s.

In the case one instead would like to walk up the stairs in this building with its 100 floors, and its' roughly 3000 steps, this will instead take about 40 minutes if you are in moderately good physical shape and walk with a swift pace.
On average one floor then takes about 24 seconds.

APPENDIX, Letter produced by group B

- Vi antar att byggnaden är 175 m hög dvs 50 våningar · 3,5 m
  —"— hissen rör sig med hastigheten 3,6 m/s

$$\frac{175 \text{ m}}{3{,}6 \text{ m/s}} = 48{,}6 \text{ s}$$

- Vi antar att en människa rör sig med hastigheten 1,5 m/s i början av trapporna. Då borde det rimligtvis ta 3 s att gå upp 1 våning (= 3,5 m) eftersom man blir trött. Vi antar att man i snitt per 5-våningar tröttnar med lika många sekunder per fem våningar som våningar som man har passerat. dvs att när man har passerat 10 våningar så tar nästa 5 10 sekunder längre
  totalt: 15+20+30+45+65+90+120+155+195+240 = 975 s = 16 minuter 15 sekunder

- We assume that the building is 175 meters high, that is 50 floors x 3.5 m.
  We assume that the elevator travels with a speed of 3.6 m/s

$$\frac{175 \text{ m}}{3.6 \text{ m/s}} = 48.6 \text{ s}$$

- We assume that a person moves with the speed of 1.5 m/s in the beginning of the staircase. Then, it should reasonably take 3 s to walk one floor (= 3.5 m) as you become tired. Vi assume that you in average per every 5 floors get tired with as many seconds per every 5 floors as floors you already walked, that is when you walked 10 floors the next 5 floors will take 10 more seconds.
  Totally: 15+20+45+65+90+120+155+195+240=975 s = 16 minutes 15 seconds

Appendix, Letter produced by group C


Uppskattade värden:
h = 350 m
v = 2 m/s
t = 350m / 2m/s = 175 s = 2 min 55 s

h = 350m
v = 2m/s

Antal våningar (3m/vån) = 350m / 3m ≈ 116
Lobby lite högre i tak → 115 vån.

Att gå en våning tar uppskattningsvis (för lagom otränad person)
45s + 15s vila = 1min.
(Vi räknar med att man vilar ca 45s var 3e vån)

Att gå hela vägen tar ~115 minuter
≈ 1h 55min
(Fika får man göra när man kommit upp!)

Hissen ska färdas 350m med medelhastigheten, v = 2m/s.
s = v/t ⇒ t = v/s
350m / 2m/s = 175s ≈ 2min 55s
Det tar nästan 3minuter att åka hela vägen upp.

Estimated values:
h = 350 m
v = 2 m/s
t = = 175 s = 2 min. 55 s.

Number of floors (3m/floor) $= \frac{350m}{3m} \approx 116$

The ceiling is a bite higher in the lobby →115 floors

To climb one floor takes approximately (if you are a moderate unfit person)
45s + 15 s rest = 1 minute
(Vi assume that one rests for roughly 45s on every third floor)

To walk the whole way takes ~115 minutes ≈1h55min.
(Have some coffee can you get when you arrive at the top!)

The elevator travels 350 m with an average speed of v=2m/s.

$$v = \frac{s}{t} \Rightarrow t = \frac{s}{v}$$

$$\frac{350m}{2m/s} \approx 175s \approx 2\,min\,55s$$

It will take almost 3 minutes to ride all the way up.

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