The Mathematics Enthusiast

Volume 6 | Number 3 Article 5

7-2009

# MATHEMATICAL BEAUTY AND ITS CHARACTERISTICS- A STUDY ON THE STUDENT'S POINT OF VIEW

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# Mathematical Beauty and its Characteristics - A Study on the Students' Points of View

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Abstract*:* Based on the statement, that the experience of mathematical beauty has a positive influence on students' motivations and attitudes towards mathematics and its study, the focus of this paper is the aesthetic component of mathematics. First, the role of aesthetics for perception and education is addressed. The appreciation of the beauty of mathematics is one of the wellsprings of this subject, not only in research but also in school education. This should have implications for the teaching of mathematics. However the beauty making elements have not been very well analysed. In particular, it is not clear to what extent the criteria for aesthetics found in literature are in agreement with emotions of students. A study on this topic is presented below. It involves students out of grades 5 to 12, as well as university mathematics teacher students, and reveals similarities and differences between the views of students of different educational levels.

Keywords: Aesthetics; Affect; Attitudes; Beliefs; Emotions; Mathematical beauty;

## 1. THE ROLE OF AESTHETICS FOR PERCEPTION AND EDUCATION

In the literature there are many reports concerning the use of aesthetics as a guide when formulating a scientific theory, or selecting ideas for mathematical proofs (Brinkmann & Sriraman, 2009).

The first who introduced mathematical beauty as well as simplicity as criteria for a physical theory was Copernicus (Chandrasekhar 1973, p. 30). Since then, these criteria have continued to play an extremely important role in developing scientific theories (Chandrasekhar 1973, p. 30; 1979; 1987). This is especially so for truly, creative work that seems to be guided by aesthetic feeling rather than by any explicit intellectual process (Ghiselin 1952, p. 20). Dirac, for example, tells about Schrödinger and himself (Dirac 1977, p. 136):

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> It was a sort of act of faith with us that any questions which describe fundamental laws of nature must have great mathematical beauty in them. It was a very profitable religion to hold and can be considered as the basis of much of our success.

Van der Waerden (1953 ) reports that Poincaré and Hadamard pointed out the role of aesthetic feeling when choosing fruitful combinations in a mathematical solution process. More precisely, Poincaré asked how the unconscious should find out the right, that is fruitful, combinations among the many possible ones. He gave the answer: "by the sense of beauty, we prefer those combinations that we like" (Van der Waerden 1953, p. 129; see also Poincaré 1956, p. 2047-2048).

A similar statement is given by Hermann Weyl (Ebeling, Freund and Schweitzer 1998, p. 209):

> My work has always tried to unit the true with the beautiful and when I had to choose one or the other I usually chose the beautiful.

Thus theories, that have been described as extremely beautiful, as for example the general theory of relativity, have been compared to a work of art (Chandrasekhar 1987); Feyerabend (1984 ) even considers science as being a certain form of art.

Mathematics and mathematical thought are obviously directed towards beauty as one profound characteristic. Papert and Poincaré (Dreyfus and Eisenberg 1986, p. 2; Hofstadter 1979) even believe that aesthetics play the most central role in the process of mathematical thinking. The *appreciation of mathematical beauty by students* should thus be an integral component of mathematical education (Dreyfus and Eisenberg 1986). But Dreyfus and Eisenberg remarked in 1986, that developing an aesthetic appreciation for mathematics was not a major goal of school curricula (NCTM, 1980), and they suggested that "this is a tremendous mistake". However in the curricular guidelines of Northrhine-Westfalia, Germany (MSWWF 1999, p. 38), the development of students' appreciation of mathematical beauty is explicitly demanded in the context of the *fostering of long-life positive mathematical views*. The importance of this demand may be stressed by the following statement given by Davis and Hersh (Davis and Hersh 1981, p. 169):

> Blindness to the aesthetic element in mathematics is widespread and can account for a feeling that mathematics is dry as dust, as exciting as a telephone book, as remote as the laws of infangthief of fifteenth century Scotland. Contrariwise, appreciation of this element makes the subject live in a wonderful manner and burn as no other creation of the human mind seems to do.

In addition to the positive influence on students' attitudes towards mathematics, the experience of mathematical beauty would surely have as well *a positive influence on students' motivations for the study of mathematics*. Of course, this statement can only be confirmed on the basis of a classroom teaching that emphasizes students' aesthetic feelings.

## 2. CRITERIA OF AESTHETICS

If we want students to experience mathematical beauty, we first have to bring out the characteristics of mathematical aesthetics. What does it mean, for example, that a theorem, a proof, a problem, a solution of a problem (the process leading up to a solution, as well as the finished solution), a geometric figure, or a geometric construction is beautiful?

Although assessments about beauty are very personal, there is a far-reaching agreement among scholars as to what arguments are beautiful (Dirac 1977). Thus it makes a sense to search for factors contributing to aesthetic appeal. Before starting on this journey, Hofstadter (1979, p. 555) sounds a note of warning when suggesting, that it is impossible to define the aesthetics of a mathematical argument or structure in an inclusive or exclusive way:

> There exists no set of rules which delineates what it is that makes a peace beautiful, nor could there ever exist such a set of rules.

However we can find in the literature several indications of criteria determining the aesthetic rating.

The Pythagoreans took the view that beauty grows out of the mathematical *structure*, found in the mathematical *relationships* that bring together what are initially quite independent parts in such a way to form a unitary whole (Heisenberg 1985). Chandrasekhar (1979) names as aesthetic criteria for theories their display of "*a proper conformity of the parts to one another and to the whole*" while still showing "some *strangeness* in their proportion". Weyl (1952, p. 11) states that beauty is closely connected with *symmetry,* and Stewart (1998, p. 91) points out that *imperfect symmetry* is often even more beautiful than exact mathematical symmetry, as our mind loves surprise. Davis and Hersh (1981, p. 172) take the view that:

> A sense of strong personal aesthetic delight derives from the phenomenon that can be termed *order out of chaos*.

And they add:

> To some extent the whole object of mathematics is to create order where previously chaos seemed to reign, to extract structure and invariance from the midst of disarray and turmoil.

Whitcombe (1988 ) lists as aesthetic elements a number of vague concepts as: *structure, form, relations, visualisation, economy, simplicity, elegance, order*. Dreyfus and Eisenberg (1986) state, according to a study they carried out, that *simplicity, conciseness* and *clarity* of an argument are the principle factors that contribute to the aesthetic value of mathematical thought. Further relevant aspects they name are: *structure, power, cleverness* and *surprise*. Cuoco, Goldenberg and Mark (1995, p. 183) take the view that:

> The beauty of mathematics lies largely in the *interrelatedness of its ideas*. ... If students can make these connections, will they also see beauty in mathematics? We think so...

Ebeling, Freund and Schweitzer (1998, p. 230) point out, that the beautiful is as a rule connected with *complexity*; complexity is necessary, even though not sufficient, for aesthetics.

Complexity and simplicity are both named as principal factors for aesthetics: how do these notions fit together? If simplicity is named, it is mainly the simplicity of a solution of a complex problem, the simplicity of a proof to a theorem describing complex relationships, or the simplicity of representations of complex structures. It looks as if simplicity has to be combined in this way with complexity, in order to bring out aesthetic feelings (Brinkmann 2000).

The criteria for aesthetics might give us an idea of how to choose mathematical objects for presentation in classroom, if we want to bring out aesthetic feelings in the students. However, we have to consider that the criteria for aesthetics noted above, have in the main been developed by mathematicians and scientists. Sinclair (2004) suggests, according to some prevalent experience of teachers, that there are stimuli that commonly trigger also students' aesthetic response. But, it is not at all clear whether the criteria brought out above will point to worthwhile classroom activities which in turn will give rise to the looked for emotions of students.

Furthermore, the quoted criteria for aesthetics are given by qualitative characteristics[2], and hence by their nature they are fuzzy quantities. Thus aesthetic considerations will depend on individual judgements. Accordingly another point of interest will be to find out whether in mathematic classes the aesthetic sensation of students can be expected to be relatively homogenous.

## 3. STUDENTS' JUDGEMENTS ON MATHEMATICAL BEAUTY – A STUDY

In order to gain more insight into the aesthetic feelings of students, a study was carried out by the author in Germany.[3]

### 3.1 Design of the study

The participants of the study were on the one hand 168 students attending two gymnasiums. They were in grades 5 to 8 (96 students) and grades 11 and 12 (72 students). On the other hand 85 university mathematics teacher students were included in the study.

The students were asked to work on the questionnaire given in Figure 1, which had been developed by the author. (The consecutive number in the left column in Figure 1 has been added here with regard to the evaluation of the study data.)

[2] Birkhoff (1956) made an attempt to quantifying aesthetics in a general way, but his proposal seems not to be very convincing.
[3] First results have been published in Brinkmann 2004a, 2004b and 2006.

*Figure 1: Questionnaire*

**What is a beautiful mathematical problem?**

1. Write down one ore several mathematical problems, which appeal to you, respectively write down their contents.
2. What do you think is a “beautiful” mathematical problem?
3. Which are, in your view, characteristics of a beautiful mathematical problem?

| | | |
|---|---|---|
| 1 | ☐ | The problem is tricky. |
| 2 | ☐ | The problem has a simple solution. |
| 3 | ☐ | The problem is complicated. |
| 4 | ☐ | The problem looks complicated but it has a simple solution. |
| 5 | ☐ | The problem is simple. |
| 6 | ☐ | The problem has an elegant solution. |
| 7 | ☐ | The problem is complex. |
| 8 | ☐ | The problem and its solution are easily to be understood. |
| 9 | ☐ | The problem is unfamiliar for me. |
| 10 | ☐ | The topic of the problem is interesting. |
| 11 | ☐ | The problem has a surprising solution. |
| 12 | ☐ | The solution of the problem is obvious. |
| 13 | ☐ | The nature of the problem is familiar to me. |
| 14 | ☐ | The nature of the problem is new to me. |
| 15 | ☐ | The solution of the problem can easily be guessed. |
| 16 | ☐ | The problem looks simple but it has a complicated solution. |
| 17 | ☐ | There are considered regular patterns or structures. |
| 18 | ☐ | The solution of the problem is complicated. |
| 19 | ☐ | The problem is a puzzle. |
| 20 | ☐ | The problem requires several solution steps. |
| 21 | ☐ | The problem is about symmetric figures. |
| 22 | ☐ | It is possible to solve the problem by logical considerations, without calculating. |
| 23 | ☐ | The solution of the problem shows unexpected regularities. |
| 24 | ☐ | The problem refers to realistic applications. |
| 25 | ☐ | The problem has got more than one possible solution. |
| 26 | ☐ | The problem respectively its solution are clearly structured. |
| 27 | ☐ | The facts are presented clearly. |
| 28 | ☐ | The solution of the problem is significant for further applications. |
| 29 | ☐ | The problem requires a complex intellectual examination. |
| 30 | ☐ | … |

While the first two items of the questionnaire are of an open format, the third item uses a closed format to refer to a number of specific characteristics of mathematical problems that might be related to aesthetical feelings of students. In this item, beauty-making elements referred to in the literature are used:

- A number of statements refer to different nuances of simplicity (statements 2, 5, 8, 12, 13, 15), or complexity (statements 1, 3, 7, 9, 18, 19, 29) of a mathematical problem or its solution, or to combinations of both (statements 4, 16).
- The statements 26 and 27 refer to clarity and structure.
- The statement 6 refers to the characteristic of elegance.
- The statements 11 and 23 refer to the characteristic of surprise.
- The statements 17, 21 and 23 refer to symmetry and regularities.
- The statement 28 refers to the feature of power.

Based on experiences of the author as a teacher, as well as on advice given by colleagues, further possibly beauty making characteristics of a mathematical problem were included: the aspect of interest (statement 10), the feature of novelty / not novelty (statements 14, 13), the reference to applications (statements 24, 28), the open ended problem feature (statement 25).

Item 1 and 2 of the questionnaire provide qualitative responses that help us

- to interpret the answers given to item 3 (e. g. to become an idea of that what is denoted as a simple problem),
- to find further beauty making characteristics of problems, that are not yet considered in item 3.

In the primarily developed questionnaire, the statements 20 and 22 were not yet enclosed. They have been added later, according to first study results, as these criteria have been named by some students under item 2 (see Brinkmann 2004a). Hence, 36 of the students out of grades 5 to 8 and 72 students out of grades 11 and 12 (i.e. the participants of the first studies) had not been working on the completed questionnaire.

The students were instructed to tick as many statements in item 3 they felt were correct. Of course it might be that the statements marked by a student do not have equal weighting. However the focus of the study was to explore beauty-making elements, without emphasizing individual rankings, hence this matter was not a real issue.

Out of the students in grades 7 and 8, 36 had participated at the “Math Kangaroo” competition shortly before they completed this questionnaire, and thus could name problems out of this competition when working on task 1.

The responses given by the 11$^{th}$ and 12$^{th}$ graders were differentiated according to the students’ mathematical achievement in school lessons (belonging to the best third of the class/course or not).

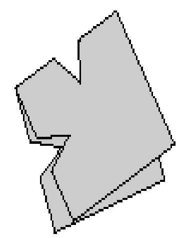
## 3.2 Results

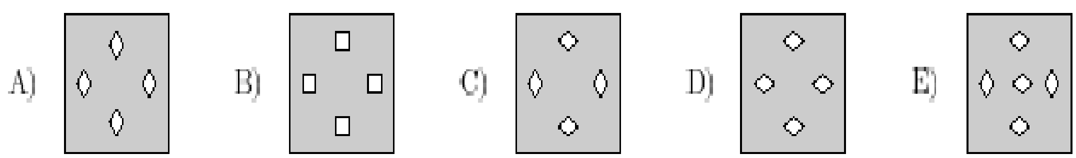
*Students in grades 5 to 8:*

About two third of the 5$^{th}$ to 8$^{th}$ graders named *puzzles* as beautiful mathematical problems, but most of these students added that the puzzles should not be too difficult. One third of the students specified problems, that require drawing activities, as beautiful, especially geometry problems of this kind.

As an example, the following problem out of the Math Kangaroo competition 2003 was named by about half of the students that had participated before at this competition (see above):

A square piece of paper is folded twice and cut in the way you can see in the picture. How will the piece of paper look after unfolding?

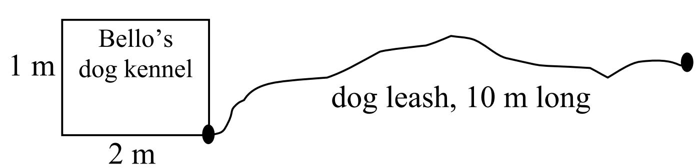





Further beautiful problems given by the students were mostly modelling problems, where mathematics is applied in real world contexts, e. g.:

- The problem of the garden cottage: The site plan is given on a scale of 1:500, and the allowance that the distance to the site borders has to be at least 2.5m and the distance to a street at least 5m. Find out the area where the cottage is permitted to be built.

- The dog kennel problem: Draw the area where Bello can move.




- Sparkling mineral water automat: Is it more favourable to make sparkling mineral water with a home automat, or to buy ready-made sparkling mineral water?

- The helicopter problem: There is given the map of an area where 2 helicopters have to be positioned, and also the maximal possible velocity of the helicopters. Where should the helicopters been positioned, such that every place of the area can be reached within 10 minutes?

Each of these problems had been named by some 15% to 30% of the students, which knew them.

Regarding *item 2*, the majority of the students answered that a problem and its solution must be simple to be beautiful. As examples of this point they gave among others 1x1=1, and a problem without fractions or percents. Many students made their statement of simplicity more precise by adding that a problem is *not* beautiful if they cannot solve it by themselves, or if it is so complicated that they have no idea what to do. But a simple problem is also *not* beautiful if they had to work on this problem many times, or if it is boring, and too simple. One student wrote in response to item 2 that a very difficult but solvable problem is beautiful.

In *item 3,* about two third of the students ticked *interesting topic* (statement 10), *puzzles* (statement 19), *simple solution / simple problem* (statements 2 and 5). Statement 22 (*a problem solvable by logical considerations, without calculating*), that had been added in the questionnaire after the first studies, turned out to be relevant: it has been ticked by 58% of the students that had this choice. As well the second later added statement 20 (several solution steps required) is of importance (27%). Further statements to matter are statement 4 (*a problem that looks complicated but has a simple solution*), statement 8 (*the problem and its solution are easily to be understood*) and statement 12 (*obvious solution*).

Just the reference to realistic applications plays a subordinate role, although one might expect this when regarding the examples given to item 1. Rather the interesting topic is decisive.

Table 1 shows a more detailed summary of the outcomes for item 3.

*Table 1: Outcomes in item 3 of the questionnaire, grades 5 to 8*[4]

| Nr. | Statement | % |
|---|---|---|
| 10 | The topic of the problem is interesting. | 67 |
| 2 | The problem has a simple solution. | 64 |
| 19 | The problem is a puzzle. | 61 |
| 5 | The problem is simple. | 60 |
| 22 | It is possible to solve the problem by logical considerations, without calculating. | 58 |
| 4 | The problem looks complicated but it has a simple solution. | 47 |
| 8 | The problem and its solution are easily to be understood. | 44 |
| 12 | The solution of the problem is obvious. | 44 |
| 11 | The problem has a surprising solution. | 30 |
| 1 | The problem is tricky. | 29 |
| 21 | The problem is about symmetric figures. | 28 |
| 20 | The problem requires several solution steps. | 27 |
| 25 | The problem has got more than one possible solution. | 27 |

The number of statements ticked by the students for item 3 differed mostly from 3 to 7. The most preferred combination was statement 2 (simple solution), statement 5 (simple problem), and statement 12 (obvious solution).

*Students in grades 11 and 12:*

When working on *item 1* of the questionnaire most of the students stated that they never really had come across a beautiful mathematical problem, but that it would be nice to do so. As "beautiful" examples out of the known problems, they mainly named very simple arithmetical problems (such as $1 \cdot a = a$), binomial formulas, and problems that can be solved by simple algorithms (e.g. systems of linear equations). Only in isolated cases (3 higher-achieving students) were some comparatively complex problems quoted (e.g. derivation of Pythagoras' theorem, calculation of volumes with integrals).

The answers to *task 2* were quite varied. For the *lower-achieving students* a beautiful problem is mostly a problem that can be solved easily by well-known formulas or a well-known algorithm, or a problem where one sees what is to do immediately. But for these students the solution should not be too simple because that would make it boring, and hence it should consist of several (simple) steps. As well the solution should not be too obvious; it is better to have to first

[4] There are listed only those statements ticked by at least 25% of the students. The percentages to number 20 and 22 refer only to the 60 students that had the possibility to tick the respective statements.

think a little about the problem. Contrarily to the younger students, a connection to the real world is also important, as is the feeling that the problem could be useful for life.

For the *higher-achieving students* a beautiful problem must be a problem that he or she can solve, and it must be presented in a clear way. But these students did not stress the need for the problem to be solvable by using well-known formulas and algorithms. On the contrary, the aesthetic appeal seemed to be greater if one has to think about the problem in an unconventional way, if connections within mathematics have to be seen, if more than one well-known formula has to be used, and the successful combination of these formulas has to be found out by oneself.

For *task 3* (see Table 2) there were no great differences between high-achievers and low-achievers with regards the frequency of selection of the statements. The most important characteristic of a beautiful problem for these students is an *interesting topic* (81%), followed by the *reference to realistic applications* (65%). Also important are a *simple problem solution* (60%), a *familiar problem nature* (60%, although more emphasized by low-achievers), and a *clear presentation* of the facts (58%). The characteristics of a *simple problem*, a *complicated problem with simple solution*, and a problem with a *clear structure*, were each marked by about 40% of the students (the second one more emphasized by high-achievers). Three fourth of the high-achievers ticked problems with more than one possible solution. In contrast to the younger students, only one third of the students (about half of the high-achievers) marked puzzles and tricky problems as beautiful. About half of the high-achievers ticked problems with a surprising solution, and problems that require a complex intellectual examination.

*Table 2: Outcomes in item 3 of the questionnaire, grades 11 and 12*[5]

| Nr. | Statement | % all | % of low | % of high |
|---|---|---|---|---|
| 10 | The topic of the problem is interesting. | 81 | 79 | 83 |
| 24 | The problem refers to realistic applications. | 65 | 67 | 63 |
| 2 | The problem has a simple solution. | 60 | 65 | 50 |
| 13 | The nature of the problem is familiar to me. | 60 | 71 | 38 |
| 27 | The facts are presented clearly. | 58 | 56 | 63 |
| 5 | The problem is simple. | 43 | 46 | 38 |
| 4 | The problem looks complicated but it has a simple solution. | 42 | 31 | 63 |
| 26 | The problem respectively its solution are clearly structured. | 39 | 42 | 33 |
| 25 | The problem has got more than one possible solution. | 36 | 17 | 75 |
| 1 | The problem is tricky. | 31 | 21 | 50 |
| 19 | The problem is a puzzle. | 28 | 19 | 46 |
| 28 | The solution of the problem is significant for further applications. | 26 | 27 | 25 |
| 11 | The problem has a surprising solution. | 22 | 10 | 46 |
| 29 | The problem requires a complex intellectual examination. | 22 | 8 | 50 |
| 15 | The solution of the problem can easily be guessed. | 22 | 23 | 21 |

The number of statements ticked by an older student for item 3 was generally greater compared to the number selected by a younger student: 22% ticked 2-5 statements, 64% ticked 6-10 statements and the remaining 14% ticked 11-14 statements. Every listed statement was ticked at least once. In spite of this, there are visible favourites named by the majority of the students.

*University mathematics teacher students:*

The university mathematics teacher students named under *item 1* mainly examples of

- problems that have the character of a puzzle (e. g. : "How many squares can you draw in a chessboard?"),
- problems with astonishing or unexpected solutions (e. g.: "You have a rope that is 1m longer than the equator. Imagine, you put this rope around the equator and stretch it concentrically. What distance would the rope than have from the earth surface?" Solution: $1/(2\pi)$ m, these are round 16 cm – a surprising solution, as it is contradictory to human intuition.),
- problems where mathematics is applied in real life situations, as well as using geometry, algebra or combinatorics,
- problems requiring logical thinking,

[5] There are listed only those statements ticked by more than 20% of the students.

- problems requiring drafts or drawings.

As for *item 2*, the students characterized puzzles, tricky problems, problems that prompt thinking as beautiful. Further they mainly named as characteristics of a beautiful problem the reference to realistic applications, clearness, the possibility for creative solutions, the existence of more than one solution, complex solutions composed of several steps.

For *item 3* the mathematics teacher students indicated as most important features of a beautiful mathematical problem, similar as the elder school students, an *interesting topic,* the *reference to realistic applications* and a *clear presentation* of a problem. Further *tricky problems, puzzles* and problems with a *surprising solution* are also identified by these students as beautiful. In contrast to the younger school students, the feature of having *more than one possible solution* is relevant for these students; but simplicity is not a beauty making element, though it is the combination of a *complicated look but having a simple solution*. Similar as for scientists, *elegance* is felt as a weighty feature. More detailed information is provided by table 3.

*Table 3: Outcomes in item 3 of the questionnaire, mathematics teacher students*[6]

| Nr. | Statement | % |
|---|---|---|
| 24 | The problem refers to realistic applications. | 78 |
| 10 | The topic of the problem is interesting. | 76 |
| 1 | The problem is tricky. | 67 |
| 27 | The facts are presented clearly. | 58 |
| 6 | The problem has an elegant solution. | 54 |
| 4 | The problem looks complicated but it has a simple solution. | 49 |
| 25 | The problem has got more than one possible solution. | 41 |
| 11 | The problem has a surprising solution. | 32 |
| 19 | The problem is a puzzle. | 31 |
| 22 | It is possible to solve the problem by logical considerations, without calculating. | 29 |
| 20 | The problem requires several solution steps. | 28 |
| 14 | The nature of the problem is new to me. | 25 |
| 23 | The solution of the problem shows unexpected regularities | 21 |

The majority of the mathematics teacher students ticked 6-10 of the listed statements.

[6] There are listed only those statements ticked by at least 20% of the students.

## 4. LIMITATIONS

It goes without saying, that the results of the study are dependent on the learning experiences of the students involved: The students' judgements can only refer on that what the students got to know. Further, the teacher's personality, especially the enthusiasm with which a teacher deals with a certain mathematical problem, might have an influence on the emotions of learners and thus also on there judgements about aesthetics.

Item 1 and 2 of the questionnaire provide qualitative statements, which may help us

- to interpret the answers given in item 3 (e.g. to get an idea of what a student means when ticking "a simple problem"),
- but also to get new ideas of what it is that makes a problem beautiful for a student.

Clearly a characteristic expressed for items 1 or 2 does not necessarily occur in the list of item 3. In this case a quantitative statement giving the weight of such a characteristic is not possible at this stage, nor was it an aim of this project. For example, one such example is the statement that a problem requires several solution steps, expressed in one of the first studies to the topic and added later to the questionnaire (see above), or the statement given by mathematics teacher students that a problem should give possibility for creative solutions. In order to explore the relevance of such an argument, further studies would have to be carried out on the pre-condition that the participating students knew already respective problems.

## 5. DISCUSSION

In face of existing limitations, the study allows some core statements. The aesthetic feelings of school students towards a mathematical problem seem to be strongly connected with interest, with the problem having realistic applications (especially for elder students), and also giving students *feelings of security and success*: It seems to be a necessary condition that one has got to have the feeling that you could succeed in solving a certain mathematical problem, if it is going to be perceived as beautiful. In this respect it would appear that beautiful problems have to be simple enough for the group of students under observation.

But a beautiful problem is not just a simple problem. On the contrary, it has to have a certain degree of complexity: it is more beautiful if one has to think about the problem, for example if the problem is a puzzle (especially for younger students); if the solution consists of several steps; and if one has to combine a number of formulas to get a solution. The different answers given by lower-achievers compared to those given by the higher-achievers on this point leads to the conclusion that the permissible degree of complexity for a beautiful problem depends on the mathematical ability of each individual. [7]

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[7] The feedback of three teachers, who repeated the study in their school classes/courses confirm the results reported in this paper regarding school students.

With regard to university mathematics teacher students, feelings of security and success in the solving process of a mathematical problem seem not to be a pre-condition for aesthetic feelings to arise. These students could also get pleasure from a problem which they probably could not solve by themselves. Viewing the fact, that these students will be in future mathematics teachers, an important outcome of the study are the similarities to assert when comparing the judgements of these students with those of school students expressed in item 3. For both, the notions of an interesting topic and of a reference to realistic applications, are relevant, and a great deal of every student group consider tricky problems and puzzles as beautiful. However, it is not at all clear if the idea of that what an "interesting topic" is coincides – as for this point further studies are needed.

If we want in schools to bring mathematical beauty within students' experiences, we need to use a different style of mathematical problem. We have to consider the interests of the students and choose, if possible, problems referring to realistic applications. When doing so, we have to be very aware of the abilities of our students, in order to present to them challenging problems that they can solve.

However, in my opinion it is desirable that students' aesthetic feelings are not only restricted to those problems they feel they can solve by themselves. Thus, as a further conclusion of the study, we should create phases in classrooms, which have an atmosphere that is not predominated by the demand of success. On the contrary, these phases should give the students time for leisure, time and freedom to just enjoy mathematics.

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