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MATHEMATICAL CURIOSITIES ABOUT DIVISION OF INTEGERS

Jérôme Proulx

Mary Beisiegel

Mathematical curiosities about division of integers

Jérôme Proulx[1]
Université du Québec à Montréal

&

Mary Beisiegel,
University of Alberta[2]

As mathematics educators, our focus of attention is mainly placed on the learning and teaching of mathematics. But, as we study phenomena of mathematical learning and teaching, we often come across intriguing *mathematical* phenomena that capture our interest. We find ourselves often bouncing mathematical ideas back and forth, not just looking for (new/better) ways of teaching or presenting a mathematical concept, but also of uncovering and discovering potential understandings of the concept. These mathematical issues we encounter represent for us a significant aspect of our work, and are also very stimulating. One of these issues arose for us as we were tackling issues of division of numbers and of conventions relating to the remainder; issues that are, mathematically speaking, as we hope to communicate, very interesting and thought provoking. Thus, we explore four different avenues/curiosities about division, where operations with positive *and* negative numbers are considered, as well as the meaning one can draw out of these operations.

Curiosity 1: Division, integers and conventions

Let's take a very simple division, like $18 \div 4$. One answer to this operation is "4 remainder 2." That said, what about $3r6$, $2r10$, $5r^{-}2$? The usual answer when dividing numbers requires one to ask how many times does 4 go into 18, and then describe what is leftover as the remainder after having taken out all the 4's you can from 18. Thus, in this case, $18 \div 4 = 4r2$. However, one could argue that all four answers given above are equivalent and make sense mathematically. Indeed, they are all mathematically correct and represent an understanding that division represents a partitioning of a number (the dividend) into equally sized parts (the divisor), where in some answers the dividend has not been fully partitioned. For $3r6$, the number 18 has had three groups of size four taken out with six parts remaining, which can be represented by $18 = 3 \cdot 4 + 6$. This is correct, but the idea of taking out as many 4's as possible is not yet complete.

With this in mind, as Brown (1981) explains, the division algorithm respects some conventions, given by its definition, since the remainder (r) is defined as, and needs to be, between zero and the divisor (i.e., $0 \leq remainder <$ divisor). Thus, 18 ÷ 4 gives $4r2$ and not $3r6$, even though both are conceptually acceptable. With infinitely many possibilities in any division

[1]E-mail : proulx.jerome@uqam.ca
[2] The order of author names in this paper was decided on a coin flip.

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problem, and in order to use the procedure appropriately, some conventions have to be respected. The issue of convention and definition play a significant role in the answer. Therefore, the last three answers ($3r6$, $2r10$, $5r^{-}2$) would be ruled out because of the mathematical convention, just as $16 \div 4 = 3r4$ would be ruled out even if it is conceptually adequate.[3]

With this rule or convention established for the remainder, what would now happen if negative numbers are used? If we attempt to calculate $^{-}18 \div 4$ by using the above convention, we obtain $^{-}18 \div 4 = ^{-}5r2$ (and not $^{-}18 \div 4 = ^{-}4r^{-}2$). This seems counterintuitive in comparison to our calculations for the previous example 18 ÷ 4. As we could not go "over" or beyond 18 when calculating $18 \div 4$ (e.g., with $18 \div 4 = 5r^{-}2$), in the case of $^{-}18 \div 4 = ^{-}5r2$ we do. Again the mathematical convention guides the way in which division and its algorithm are to be used. In that sense, one needs to know the adequate mathematical convention in order to obtain a mathematically acceptable answer, even if alternatives are conceptually meaningful.

But, again, what would happen for 18 ÷ $^{-}4$? If we attempt to follow the convention, we need to have the remainder lying between 0 and the divisor. Hence $0 \leq \textit{remainder} < {}^{-}4$, which is mathematically impossible. So, analyzing two potential answers, we obtain $18 \div {}^{-}4 = {}^{-}4r2$ or $18 \div {}^{-}4 = {}^{-}5r^{-}2$. In both answers the remainders 2 or $^{-}2$ are bigger than $^{-}4$. The only answer that could satisfy the requirement that the remainder be smaller than $^{-}4$ would be $18 \div {}^{-}4 = {}^{-}6r^{-}6$, an answer that clearly goes "over" 18 and that appears conceptually acceptable, but would still be inadequate because $^{-}6$ is smaller than 0, the lower bound for the remainder.

Through browsing and searching different definitions for the remainder, one way that we have found to step away from this inconsistency for various cases of signed numbers is to redefine the remainder in terms of the divisor's absolute value: $0 \leq \textit{remainder} < |\text{divisor}|$.[4] In this case, $18 \div {}^{-}4 = {}^{-}4r2$ where the remainder is both bigger than 0 and smaller than $|{}^{-}4|$. But this step, as often happens in mathematics (Hersh, 1987; Lakatos, 1976), requires reworking the definition; in this case for what a reminder is. Notice also in this case that the answer does not require us to go "over" 18 as was done for $^{-}18 \div 4$. That said, what about $^{-}18 \div {}^{-}4$? With the new definition of the remainder, we obtain $^{-}18 \div {}^{-}4 = 5r2$, which goes "over" $^{-}18$. We therefore obtain two cases where the product of the quotient and divisor go "over" or beyond the dividend, and two cases where the product stays "under" or below the dividend.

On an interesting note, one could argue that each time we claimed to go "over" $^{-}18$ in the divisions, we in fact obtained a number that was "under" $^{-}18$ (by attaining a smaller number than it). For example, $^{-}18 \div 4 = {}^{-}5r2$ resulted in $^{-}20 + 2$, and $^{-}18 \div {}^{-}4 = 5r2$ resulted also in $^{-}20 + 2$. We explore this issue in the next sections, as we attempt to understand what these computations mean conceptually and how we can contextualize them.

This sort of interplay of convention and concepts is often hidden within the procedures we use, or even is taken for granted as part of the conceptual understanding of it. In this case, we are able to see the mathematical richness in digging deeper to understand the role that the conventions and algorithms are playing in the answers we obtain, both in regard to the concept

[3] Brown (1981) explores the meaning of these various possibilities after one child, Sharon, offered him a similar answer to a division procedure.

[4] Another option could have been to remove the lower bound, as Brown (1981) does. But, in this case with integers, we wanted to explore the décision of taking the divisor's absolute value as a lower bound.

itself and the conventional way of reporting it.[5] This raises, we think, the interest in pulling these mathematical notions apart and exploring them in depth, and represents, as Brown (*ibid.*) suggests, "one way in which we can relate elementary and advanced knowledge of the discipline so that both perspectives are enriched rather than destroyed by the linkages" (p. 13).

Curiosity 2: Conceptualizations of division

Several metaphors or conceptualizations exist for dividing numbers. Here we work with three of these ideas in the context of the examples presented above. In particular, we look at division as a measurement concept and as a partitioning concept. Then, as division is the inverse of multiplication, we explore the connections between multiplication as repeated addition and division as repeated subtraction and how these conceptualizations can become quite difficult to make sense of when working with integers.

One of the first conceptualizations of division is that of a measurement problem, where for the problem 18 ÷ 4 we can think of asking 'how many groups of size four can be found in 18 things?' Here we are trying to find the *number of groups* when we already know the size of each group. In the second conceptualization, partitioning, we can think of 18 ÷ 4 as asking 'if four people were to share 18 things equally, how many things would each person get?' Here we are trying to find the *size of each group* when we know how many groups we have. The results will be the same for each conceptualization, but represent something different depending on how one approaches the problem (see, e.g., Hart, 1981; Simon, 1993). However, both of these conceptualizations can be difficult to make sense of as they sometimes break down when working with integers while at the same time continuing to adhere to the convention for the remainder.

Case 1: 18 ÷ 4: In this case, where both the dividend and the divisor are positive, both conceptualizations are simple to apply. For the measurement concept, as was mentioned, we can ask 'if we have 18 things and 4 things are given out at once, how many people/groups will be given four things?' Here the answer 4*r*2 tells us that four people will receive four things and we will have two things left over. For the partitioning concept, we ask 'if we have 18 things to be given equally to four people, how many things will each person have?' The answer 4*r*2 denotes that each person will be given four things and we will have two things left over. In both of these conceptualizations, going "over" the quantity of 18, as was done previously to satisfy conventions, does not make sense because we cannot give out more things than we have.

Case 2: $^{-}18 \div 4$: In this case, the conceptualizations of division become problematic. In particular, from the measurement perspective, we ask the question 'if we have $^{-}18$ things and four things are given out at once, how many people/groups will be given four things?' It is difficult to imagine having $^{-}18$ things. And, more importantly, when we complete the problem with the result $^{-}18 \div 4 = {}^{-}5r2$, the $^{-}5$ represents the number of groups that have each been given four things. But how can we have $^{-}5$ groups? This is hard to imagine, and it has haunted mathematicians for years in the historical developments of negative numbers!

Under the partitioning concept, we can make a bit more sense of this problem. From this approach, we ask 'if we have $^{-}18$ things to be given equally to four people, how many things will each person have?' We can alter this question slightly to be a financial question, as we often do

[5] Something reminiscent of Davis (1973) and Brown (1981) exploration of non-standard ways of children for subtracting and dividing numbers, as well as Kieren's (1999, 2004) "missing fraction mysteries" task where children had to find fractions between ¼ and ¾ and began writing fractions like 5.9/6, 5.99/6, 5.999/6, etc.

to ease one's understanding. We ask 'if four people owe \$18 and they are to split the debt evenly, how much money will they each owe?' This makes sense on a conceptual level, where debt is represented by a negative value. However, when we look at the answer $^{-}5r2$, again $^{-}5$ could be said to be problematic since the debt is overpaid, though it helps to make sense division-wise, and it acts as a viable option to cover the entire expense! That said, for the next two cases, things become even trickier.

Case 3: $18 \div {}^{-}4$: What are we to do in this case? Under the idea of division as measurement, the question becomes 'how many groups of size $^{-}4$ can we take out of 18 things' and within the idea of division as a partitioning, we ask 'how will negative four people share 18 things?' Taking negative things out of positive ones does not make sense, nor does the idea that we have negative people. In the case of a positive dividend and negative divisor, the conceptualizations we are working with here do not help to make sense of the calculations and are unreasonable.

Case 4: $^{-}18 \div {}^{-}4$: The measurement conceptualization is interesting with the example $^{-}18 \div {}^{-}4$. The question we ask here is 'how many groups of size $^{-}4$ can be found in $^{-}18$ things?' Here we find that $^{-}18$ can be divided up into groups of $^{-}4$. When we do this, we see that we have four groups of $^{-}4$. If we stop there, however, we have $^{-}2$ remaining. And, as we mentioned previously, while this is mathematically correct, it is not appropriate in regard to the convention for the remainder, giving $4r^{-}2$. So, we have to take out another group of $^{-}4$, so this leaves us with five groups of $^{-}4$ and a surplus remainder of 2 (i.e., $5r2$). Unfortunately here, however, going "over" makes the conceptualization a bit hazy, whereas $4r^{-}2$ makes more sense.

The partitioning concept in this case is a bit more difficult to see. We ask 'if we have $^{-}18$ things to be given to negative four people, how many will each person receive?' Again, the idea of having negative people or a negative entity that is supposed to receive something is hard to imagine. The partitioning concept appears limited in helping to make sense of this case.

Turning our attention to the connection between multiplication and division, we obtain additional, yet different ways of making sense of division. Multiplication is often presented as repeated addition. If we have $3 \cdot 4$, we can write this as $4 + 4 + 4$. Since multiplication and division are closely connected as inverse operations of each other, if multiplication is repeated addition then division can be seen as repeated subtraction. This way of seeing division, fruitful in cases when the dividend and the divisor are of same sign, becomes quite complicated in the other cases presented above.

For example, $18 \div 4$ can be solved in the following way $18 - 4 = 14$; $14 - 4 = 10$; $10 - 4 = 6$; $6 - 4 = 2$ at which we stop because we can not take out another 4 (because it would lead to $^{-}2$). Our result is then $4r2$. A similar thing happens when we have $^{-}18 \div {}^{-}4$. Here we have $^{-}18 - ({}^{-}4) = {}^{-}14$; $^{-}14 - ({}^{-}4) = {}^{-}10$; $^{-}10 - ({}^{-}4) = {}^{-}6$; $^{-}6 - ({}^{-}4) = {}^{-}2$; and finally $^{-}2 - ({}^{-}4) = +2$ (if we accept going "over" $^{-}18$).

However, when the dividend and divisor are of opposite signs, this conceptualization of division becomes problematic. Let's look at these two possibilities. Under the idea of repeated subtraction $^{-}18 \div 4$ becomes $^{-}18 - 4 = {}^{-}22$; $^{-}22 - 4 = {}^{-}26$; $^{-}26 - 4 = {}^{-}30$; and so on. The result becomes more negative and we can 'pull out' infinitely many 4's from the number $^{-}18$ without ever closing in on an answer. $18 \div {}^{-}4$ appears similarly troublesome. $18 - ({}^{-}4) = 22$; $22 - ({}^{-}4) = 26$; $26 - ({}^{-}4) = 30$; and so on. In this case, our answer becomes more positive as we 'take out' $^{-}4$'s from 18 and again do not succeed on closing in an answer. Additionally, aside from the computations here, this appears difficult to conceptualize as how is one to remove negative quantities from a positive quantity, and vice-versa?

Thus, what the three different conceptualizations offer us is the fact that each of them can help to some extent in making more sense of the operations of division of integers, but they are also limited. There does not seem to exist a definitive conceptualization working for all of these examples as all have their limits and need to be reflected upon – and we believe it is in the thinking through that they become mathematically interesting and significant. That said, other aspects to look into concern the play with numbers, independently of any context or constraints. We look at this through the use of simple calculators in the next section.

Curiosity 3: Division and calculators

As we saw above, a purely conceptual approach made some aspects of division difficult to make sense of for different cases of integers. This raises the issue of exploring the numbers themselves and ideas of dividing integers and the remainder with a calculator. Again, in this calculator-context, we look at some of the previously explored outcomes and how these sometimes connect and sometimes don't connect to results obtained through using a calculator. For example, in the case of $18 \div 4$ the calculator produces the result 4.5, which makes sense for the remainder. One can look at it in the following way: $18 \div 4 = 4.5 = 4 + 0.5 = 4 + \frac{2}{4}$, where the 2 of $\frac{2}{4}$ was our remainder from the previous exploration of this case. Thus, the decimal number result coincides with the remainder result.

In the case of $^{-}18 \div 4$, the calculator produces the result $^{-}4.5$. Compare this to our previous result that followed the convention for the remainder where we had $^{-}18 \div 4 = {}^{-}5r2$. Clearly the algorithm for division in the calculator is not following the convention for the remainder as the decimal portion of the result represents $^{-}0.5$, which is $^{-}\frac{2}{4}$, where $^{-}2$ is the remainder. Thus, a question arises 'are these the same result numerically?' Our previous answer of $^{-}5r2$ can be re-written as $^{-}5 + \frac{2}{4} = {}^{-}5 + 0.5 = {}^{-}4.5$. So, these different approaches yield the same answer numerically, yet they go about finding and representing the solution differently.

Similar to the example above, for $18 \div {}^{-}4$ a calculator gives $^{-}4.5$ as a result. Applying the convention for the remainder to this problem yields $^{-}4r2$. This is quite interesting in that while the calculator's results are identical for both $18 \div {}^{-}4$ and $^{-}18 \div 4$, the algorithm for division does not give the same result for them. Again, looking at the result $^{-}4r2$ one can wonder if the result is the same as $^{-}4.5$. Note here that $^{-}4r2$ can be written as $^{-}4 + \frac{2}{^{-}4}$, where 2 is the remainder and $^{-}4$ is the divisor. This then becomes $^{-}4 + ({}^{-}0.5) = {}^{-}4.5$, which is indeed the same as our calculator's calculation.

For the final case of $^{-}18 \div {}^{-}4$, the calculator offers 4.5 as a result. Our previous work with this problem and the remainder gave us the result $5r2$. These results don't appear to be the same. But if we look at $5r2$, this can be written as $5 + \frac{2}{^{-}4} = 5 + ({}^{-}0.5) = 4.5$, leading to the same numerical value in the end, but coming from different answers. These issues for calculators and of considering numbers only for themselves have in fact led us to consider issues about the long division algorithm, how it can function in these cases and how one can make sense of it in relation with integers. We explore this as our next and last curiosity.

Curiosity 4: Long division algorithm

This final curiosity about the long division algorithm ties back to the conventions about the remainder, and has an obvious connection to the above explorations on calculators. The long division algorithm is peculiar in the sense that there are not necessarily conventions attached to it, but rather there are specific steps that one needs to follow to obtain the answer. For example, with two positive numbers the steps to solve 18 ÷ 4 with long division looks like the following. Step 1 (**Figure 1a**): How many times does 4 go into 18? 4 times. Then, we multiply 4 by 4 and obtain 16. 18 minus 16 gives 2. There are two options here for Step 2: one is to obtain the answer in terms of remainder, which gives 4*r*2, where the remainder 2 leads to $\frac{2}{4}$ or ½ (**Figure 1b**). The second choice is to opt for a decimal representation, leading one to place a decimal point after the 4 and add a zero after the two (**Figure 1c**).

<table>
<tr>
<td>4
18)4
– 16

2

Figure 1a</td>
<td>4*r*2 **or** 4 $\frac{2}{4}$
18)4
– 16

2

Figure 1b</td>
<td>4.5
18)4
– 16

20
– 20

0

Figure 1c</td>
</tr>
</table>

<table>
<tr>
<td>4
18)4
– 16

2

Figure 1a</td>
<td>4*r*2 **or** 4 $\frac{2}{4}$
18)4
– 16

2

Figure 1b</td>
<td>4.5
18)4
– 16

20
– 20

0

Figure 1c</td>
</tr>
</table>

Then, the question is 'how many times does 4 go into 20?' 5 times. 5 multiplied by 4 equals 20, 20 minus 20 gives 0. We could spend time explaining the ins and outs of this procedure, but because we want to underline other dimensions for the division, we assume the reader is aware of these rationales and the reasons why it functions. Obviously, the aspects we want to emphasize concern the play with integers and how it causes us to take a step back and question the steps we are taking and the coherence of these steps. We approach this in a similar fashion as with the conventions in the first section.

Looking at $^{-}18 \div 4$ and using the same steps as above, we obtain the following. Step 1: How many times does 4 go into $^{-}18$? Right here, at Step 1, we have also two options. One option is to follow the same reasoning as for the convention and opt for $^{-}5$ and the other is to opt for $^{-}4$. Let's have a look at the former (**Figure 2a**). $^{-}5$ times 4 gives $^{-}20$. $^{-}18$ minus $^{-}20$ gives $^{+}2$. At this stage, again, there are two options: stopping with the remainder or continuing on with decimals. If we stop with the remainder (**Figure 2b**), it gives $^{-}5r2$ where the remainder 2 leads to $2/4$. But, then the question becomes 'is it $^{+}2/4$ or $^{-}2/4$?' Taking $^{+}2/4$ seems counter-intuitive, as the quotient and its value created with the remainder would not be of the same sign ($^{-}5$ and $^{+}2/4$). However, taking $^{-}2/4$ would mean $^{-}(5\,2/4)$ and this is clearly wrong. The same thing happens if we opt for decimals (**Figure 2c**), as it gives $^{-}5$ and ".5". Is the ".5" positive or negative? In other words, is it "$^{-}5$ + $^{+}.5$" giving $^{-}4.5$ or is it "$^{-}5$ + $^{-}.5$" giving $^{-}5.5$? The former, $^{-}4.5$, is definitely the answer, which means that the various quotient values need to be computed (added) in order to find the final answer. Therefore, the decimal point "$^{+}.5$" does not belong to the quotient $^{-}5$, but stands on its own and has its own sign (in this case, positive). This obviously does not happen when it is only positive numbers, as all quotients have the same sign – it also illustrates the mathematical richness underlying these operations as we address later. Therefore, the answers to $^{-}18 \div 4$ are "$^{-}5$ + $^{+}.5$" or "$^{-}5$ + $^{+}2/4$", leading to $^{-}4.5$ or $^{-}(4\,2/4)$.

$^{-}5$ $^{-}18\ \overline{)4}$ $-\ ^{-}20$ ------- 2	$^{-}5r2 = {}^{-}5\ ^{-}2/4$ **or** $^{-}5\ ^{+}2/4$ $^{-}18\ \overline{)4}$ **??** $-\ ^{-}20$ ------- 2	$^{-}5.5$ **or** "$^{-}5 + .5$" $^{-}18\ \overline{)4}$ **??** $-\ ^{-}20$ ------- 20
Figure 2a	**Figure 2b**	**Figure 2c**

This leads us to the second route in solving this problem, which is taking $^{-}4$ for the quotient (**Figure 3a**). $^{-}4$ times 4 gives $^{-}16$. $^{-}18$ minus $^{-}16$ gives $^{-}2$. Again there are two options at this point: stopping with the remainder or continuing with decimals. The remainder option gives $^{-}4r^{-}2$ which means $^{-}4$ and $^{-}2/4$ (**Figure 3b)**. Here, because there is a sign attached to it, we know directly that both parts of the quotient obtained are the same sign and can be added together, giving $^{-}(4\,2/4)$, the same answer we had above. For the decimals (**Figure 3c**), the question becomes 'how many times does 4 go into $^{-}20$?' giving $^{-}5$ as an answer. Here again, both parts of the quotient obtained are of the same sign, making it easy to see how they add and leading to the answer "$^{-}4 + {}^{-}.5$" or $^{-}4.5$. However, it appears quite unfamiliar to see a sign before the tenths place after the decimal point. Also, some could raise the issue, with reason, that we have not respected the procedure, since $^{-}16$ is bigger than $^{-}18$ and therefore we would have taken too many 4's from $^{-}18$; the impact of which is that we obtain $^{-}2$ as the result of $^{-}18 - {}^{-}16$, something that should not happen as one is not supposed to obtain a negative number at this stage since it indicates to the solver that the number taken is too big. This is a very interesting argument because it requires that one rethink what it means "to take all there is to be taken from the dividend." In this case, again, what appears important is the understanding and the mathematical rationale one develops, and not the steps one follows.

$^{-}4$ $^{-}18\ \overline{)4}$ $-\ ^{-}16$ ------- $^{-}2$ **Figure 3a**	$^{-}4r^{-}2$; $^{-}4\ ^{-}2/4$; $^{-}(4\,2/4)$ $^{-}18\ \overline{)4}$ $-\ ^{-}16$ ------- $^{-}2$ **Figure 3b**	$^{-}4.5$ $^{-}18\ \overline{)4}$ $-\ ^{-}16$ ------- $^{-}20$ **Figure 3c**

An issue arises again if we opt for a division that takes one additional step, for example in the case of $^{-}19 \div 4$. We won't go into all the details but we look into the subtleties that could happen if one decides to go "over" $^{-}19$. Thus, as in **Figure 4a**, our answer to the question 'how many times does 4 go into $^{-}19$?' is $^{-}4$ times, which leads to $^{-}19$ minus $^{-}16$ giving $^{-}3$. Here, one can take directly the remainder and obtain $^{-}4r^{-}3$ and then $^{-}(4\tfrac{3}{4})$, albeit of course the convention for the remainder is not respected. A curious aspect, however, resides in the decimal answers (**Figure 4b to 4e**). Here, after having positioned the decimal point and added the 0 to $^{-}3$ (giving $^{-}30$), one still has to consider two decisions: continuing to go "over" the number and then choosing $^{-}8$ to give $^{-}32$, or staying "under" and going with $^{-}7$ to give $^{-}28$. Of course, one could continue with steps similar to those previously taken with the quotient of $^{-}4$, that is, to not go "over." But, as we have seen, what appears most important is the meaning one gives to each step rather than the taking of these steps. In the case of going "over" (**Figure 4b**), we obtain $^{-}4$ and $^{-}.8$, and with $^{+}2$ as a resultant of the operation. In the other case (**Figure 4c**), we obtain $^{-}4$ and $^{-}.7$, with $^{-}2$ as the resultant. The next step is interesting but tricky, since in the case of **Figure 4b** the question is 'how many times does 4 go into 20?', and in the **Figure 4c** the question is 'how many times does 4 goes into $^{-}20$?' Thus, in the latter case, as is reported in **Figure 4e**, we obtain $^{-}.05$ as an answer, leading to $^{-}4$ with $^{-}.7$ and $^{-}.05$ giving $^{-}4.75$ as the result for the division; all values obtained to form the final quotient being of the same sign. But, in the former case, as is reported in **Figure 4d**, we obtain $^{+}.05$ as the second decimal answer. This leads to the following sequence to obtain the resulting answer to the division: $^{-}4 + {}^{-}.8 + {}^{+}.05 = {}^{-}4.8 + {}^{+}.05 = {}^{-}4.75$. Both cases offer the same resulting value, albeit in different formats, but also require a different way of processing them as it could be easy to end up with $^{-}4.85$ for the answer in the case of **Figure 4d**. These represent insightful subtleties inherent to these operations that require one to pay important attention to the meaning of each step and calculation.

```
        ¯4
 ¯19   )4
– ¯16
-------
   ¯3
```

Figure 4a

```
       ¯4 ¯.8
 ¯19   )4
– ¯16
-------
  ¯30
– ¯32
--------
    +2
```

Figure 4b

```
       ¯4 ¯.7
 ¯19   )4
– ¯16
-------
  ¯30
– ¯28
--------
    ¯2
```

Figure 4c

```
       ¯4 ¯.8 +.05
 ¯19   )4
– ¯16
-------
  ¯30
– ¯32
--------
   +20
```

Figure 4d

```
       ¯4 ¯.7 ¯.05
 ¯19   )4
– ¯16
-------
  ¯30
– ¯28
--------
   ¯20
```

Figure 4e

What also appears fascinating and that emerges from issues of long division, as well as the play with calculators in the previous section, is the fact that the remainder is not considered alone in the production of an additional quotient, but gets assigned a "negative sign" when combined with the divisor. This leads to the realization that the sign of the numerical value produced by the combination of the remainder and the divisor needs to be reflected upon and is often taken for granted as giving a positive result. In these cases, as we have seen, the remainder is always connected to a divisor and the value of that additional part of the quotient takes a sign in relation to both. This is reminiscent of work done on comparison of fractions where a fraction can only be compared and understood in regard to its referent. Hart's (1981) study is famous for having asked students a question of the type: *If Mary spends ½ of her amount and Johnny spends the ¼ of it, who spent the most?* (see p. 72), leading students to consider that ½ and ¼ are in relation to something (½ of a small amount can be smaller than the ¼ of a large amount). Thus, as well, in the case of the remainder and divisor, the value produced that completes the division quotient is always in relation not only to the divisor and the remainder but also to the sign of both of these.

At this point, we have looked at two possibilities: $18 \div 4$ and $^{-}18 \div 4$. What happens with $18 \div {}^{-}4$ and $^{-}18 \div {}^{-}4$? Similar issues appear to pop-up as the play with the remainder requires that the solver pay attention to the signs attached to them, as well as consciously making the decisions to opt for going "over" or staying "under" for the first quotient when beginning the division. As a way of pushing your thoughts and developing your own ways of making sense of these, we do not explore these options here and leave them for you to try them out with both $18 \div {}^{-}4$ and $^{-}18 \div {}^{-}4$. As Descartes was famous for doing and announcing in his writing, we leave you the joy of working through these illuminating ideas on your own…

Concluding remarks

This paper raises an intriguing phenomenon that is present within other mathematical topics – that depending on the aspect we pay attention to (convention, conceptualizations, calculator, long division), the orientations taken sometimes make sense and sometimes do not. A fascinating aspect here is that, for the case of dividing integers, there does not appear to be a pattern present in the difficulties: each orientation helps to make sense of different type of division or hinders it (e.g., the conceptualization of measurement helped to make sense of $^{-}18 \div {}^{-}4$ but partitioning did not, whereas it was the opposite for $^{-}18 \div 4$; other simplification and difficulties emerged for long-division or conventions). What this means is (1) each operation can be clarified by some orientations but blurred by others. It does not appear that one sort of division was easier to make sense of through all the means and conceptualizations explored (except, of course, cases of positive divided by positive). And, (2) it illustrates all the attention one needs to pay to, and the mathematical richness one can draw from, these operations and ways of approaching them. These mathematical explorations of division with integers cannot be taken care of in a machine-like manner without deep mathematical thinking; they require important mathematical investments in the ideas by the solver. These are, therefore, rich mathematical contexts and situations to probe into.

All this makes us rethink issues of understanding of division, as often one will offer bigger and bigger numbers to verify one's understanding, assuming that if a person is able to operate on big numbers, then that person surely understands or even has demonstrated understanding of the concept at hand. We have offered here a different view in our explorations: that of staying with small numbers if one wishes to, but of digging into the concept itself through analysing its functioning and the meaning of the answers one obtains with integers.

These issues raise for us the significance of working on the exploration of mathematical concepts as a genuine activity of mathematics educators. Albeit this is not research *per se* in its traditional sense, yet these explorations have something to offer to our understanding of the very concepts that we work on with students in classrooms. We see it important to delve deeply into mathematical concepts and ideas, to understand the concepts, to make sense of what is happening, to gain a stronger footing in our own understanding of seemingly simple ideas. These sorts of mathematical developments of school mathematics appear here as initiatives driven to enhance our understandings of mathematics, a clear intention of all work being done in mathematics education.

References

Brown, S. I. (1981). Sharon's 'Kye'. *Mathematics Teaching*, *94*, 11-17.

Davis, R. B. (1973). The misuse of educational objectives. *Educational technology*, *13*(11), 34-36.

Hart, K. M. (Eds.) (1981). *Children's understanding of mathematics: 11-16*. London, UK: John Murray.

Hersh, R. (1997). *What is mathematics, really?* New York: Oxford University Press.

Kieren, T. (1999). Language use in embodied action and interaction in knowing fractions. *Proceedings of the XXIst Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (vol.1, 109-127). Cuernavaca, Morelos, Mexico.

Kieren, T. (2004). *Conversations with the cognitive sciences : Some notes on transformations used in mathematics education*. Paper presented at ICME-10, Copenhague, Denmark.

Lakatos, I. (1976). *Proofs and refutations: The logic of mathematical discovery*. New York: Cambridge University Press.

Simon, M. A. (1993). Prospective elementary teachers' knowledge of division. *Journal for Research in Mathematics Education*, *24*(3), 233-254.