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WHO CAN SOLVE 2x=1? AN ANALYSIS OF COGNITIVE LOAD RELATED TO LEARNING LINEAR EQUATION SOLVING

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Who Can Solve 2x=1? – An Analysis of Cognitive Load Related to Learning Linear Equation Solving

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Abstract: Using $2x=1$ as an example, we discuss the cognitive load related to learning linear equation solving. In the framework of the Cognitive Load Theory we consider especially the intrinsic cognitive load needed in arithmetical, geometrical and real analytical approach to linear equation solving. This will be done e.g. from the point of view of the conceptual and procedural knowledge of mathematics and the APOS Theory. Basing on our observations, in the end of the paper we design a setting for teaching linear equation solving.

Keywords: conceptual knowledge; cognitive load theory; linear equations; procedural knowledge

1. Introduction

Cognitive Load Theory, as Sweller (1988) defined it, proposes that optimum learning assumes conditions that are aligned with human cognitive architecture. While this architecture is not yet known precisely, there already exists consensus among cognition researchers that learning happens the easier the less short time working memory – the part of our mind that provides our consciousness and enables us to think, to solve problems, and to be creative etc. – is encumbered. The term cognitive load refers to the total amount of mental activity by which the working memory is oppressed at an instance in time. The most important factor that contributes to cognitive load is the number of knowledge elements that must be employed simultaneously. Basing on Miller (1956), Sweller suggests that most human beings can hardly deal with more than seven (plus minus two) elements in tandem. An immediate consequence of Cognitive Load Theory is that when we design instructional material or our action in mathematics class, we should try to minimize the working memory load by paying extra attention to choosing problem solving methods, how we represent background information, how we put forward exercises and so on.

This paper has got two purposes. We shall first study the cognitive related to a few approaches to solving linear equations. More precisely, we aim to clarify what kind of intrinsic cognitive load a learner encounters in arithmetical, geometrical and real analytic approaches to linear equations. This will be done e.g. by analyzing what conceptual and procedural knowledge (Hiebert & Lefevre, 1986; Haapasalo & Kadijevich, 2000; Star 2005) is required in these approaches. Further, we also refer to

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the APOS Theory (Asiala et al, 1997) when we consider the complexity of the learning processes related to these approaches.

The term intrinsic cognitive load refers to the load that is due to the content to be learned. The intrinsic cognitive load cannot be modified by instructional design but, of course, it must be acknowledged, for instance, in order to be able to customize the total cognitive load when designing teaching and instructional material etc. However, we shall also discuss the extraneous cognitive load, which is due to, for example, teacher's activity in the class. This will be done in the last section. For the more detailed description of the intrinsic and extranous cognitive load, we refer to Sweller (1988).

Another purpose of this paper is to give some aid in designing teaching linear equations. Modern technology makes possible to use illustrative methods also in teaching of arithmetic and algebra. Therefore geometrical aspect plays nowadays more essential role than in the past also on those fields of school mathematics where its potential has traditionally been seen very limited. Hence it is important to clarify, whether geometrical approach lightens – and if yes, then how – the cognitive load related to learning linear equation solving.

For the sake of perfection, we shall also shortly discuss the amount of the cognitive load that is related to mathematically complete understanding of linear equations of one real variable. We shall see, among other things, that solving $2x=1$ in ordered field with the least-upper-bound property requires much more than one might think at first glance. Of course, this real analytical approach cannot be taken into school as such but, in the last section, we shall ponder the pros and cons of all three approaches and then relying on our observations we shall design a more optimal approch for teaching linear equations both at school and in mathematics teacher training.

Naturally, linear equations have already appeared in several mathematics educational research. The most of these however seem to concentrate not on the challenge itself that lies in learning to solve linear equations but, if anything, on measuring the development of learners' arithmetical skills, or on the question how pupils learn to solve real life problems using linear equations, or they are some how related to the comprehension of the concept of equation, function etc. Nevertheless, some papers consider linear equations also from the perpective of cognitive scienses. For example, MacGregor and Stacey (1993) studied cognitive models underlying students' formulation of linear equations. Qin et al (2004) and Anderson (2005) focus merely on neuroscientific issues but are based on the data of a 6-day experiment in which children learned to solve linear equations and perfect their skills. Having browsed the ISI Web of Knowledge and Google Scholar, it seems that the present paper provides a new perspective on teaching linear equations.

One easily thinks that, for example, $2x=1$ is so simple equation that finding its solution hardly encumbers our cognition. On second thought, this is not the whole truth. There are several contexts in which this equation bears remarkably different content, e.g. mathematical models of rational numbers and real numbers differ from each others fundamentally, and on some more complicated occasions even the perception of the meaning of the symbols "2" and "1" may be an untrivial task. Indeed, the expression $ax=b,\ a\neq 0,$ is reasonble in some contexts even thought symbols a, b and x were not numbers, vectors or any other numeric variables. Nevertheless, we shall confine ourselves to dealing only with rational or real numbers.

Using $2x = 1$ as an example we now study linear equation solving and what kind of cognitive load is related to solving this equation with profound understanding in arithmetical, geometrical and real analytical (i.e., in the contexts of the ordered field that has the least-upper-bound property) approaches.

What constitutes a single knowledge element or cognitive load unit? It depends on both the learner's familiarity and expertise on the subject to be studied and the content itself. According to the APOS Theory, an expert can handle several concepts, procedures etc. as a single schema whereas a novice may already be confused about the details related to a single concept. Therefore we consider only the relative intrinsic cognitive load of different approaches and do not give any quantitative measure of the load for each approach. That would require a large empirical data because the cognition research has already revealed that human brains can digest illustrated data easier than data given in form of lists, tables etc. In a theoretical paper like this one, it is not possible to realize a reliable quantitative comparision of the total cognitive load that an individual learner actually experiences in geometrical and other approaches and thus we only can reveal and discuss the mathematical details that constitute the intrinsic cognitive load.

2. Arithmetical approach

Lithner (2003) has noticed that even at university students most often base their reasoning and problem solving strategies on the identification of similarities. Since linear equations are easily identifiable, it is also very probable that most mathematics teachers in their teaching – and along them their pupils, too – strongly aim at constructing *one* general algorithm for linear equation solving. Such an arithmetical algorithm apparently presumes that $ax = b, a \neq 0,$ is solved by applying the equivalence

$$ax = b \Leftrightarrow x = \frac{b}{a}.$$

Applying this division-based rule is eventually a routine procedure and, therefore, the intrinsic cognitive load required to produce a correct solution for $2x = 1$ and other such linear equations may seem to be quite limited. However, from the point of view of conceptual knowledge, linear equations are not only related to division but also to multiplication and rational numbers. It is well-known that these concepts are not at all trivial for most pupils at school. Hence it is not so surprising to notice that, e.g., only 45 percent of the eight-graders who took part in TIMMS 2003 gained full credits in "If $4(x+5) = 80$, then $x =$" (Gonzales et al, 2004).

Moreover, many pupils, and even some university students, find it difficult to perceive that the division is actually carried out by the multiplication by the inverse of a:

$$ax = b \Leftrightarrow a^{-1}ax = a^{-1}b \Leftrightarrow 1x = a^{-1}b \Leftrightarrow x = \frac{1}{a} \cdot b = \frac{b}{a}. \qquad (1)$$

This is quite natural, since the chain of equivalences in (1) consists of several arithmetical operations and equalities and there are at least two ways to denote the inverse. Hence the number of knowledge elements that all must be pieced together in order to fully understand the operational equivalence between division and multiplication by an inverse is significant.

Looking at the conceptual knowledge related to solving both $2x = 1$ and $ax = b$ deeper, a natural question arises: Do we have to understand what rational numbers really are in order to be able to comprehend the division-based solving procedure of linear equations or is it vice versa: we learn the concept of rational number through solving linear equations? According to Haapasalo & Kadijevich (2000) both orders appear. The comprehension of the concepts of division and rational numbers cannot thus be separated from the deeper appreciation of the solving algorithm of linear equations.

To be exact, solving $2x = 1$ using division-based algorithm does not necessarily require complete understanding of rational numbers and their arithmetics because in this case the division needs be applied only on integers. Since calculating the ratio of two non-integer rationals is eventually multiplication of a rational number by an inverse of a rational number, i.e.,

$$\frac{p}{q} \div \frac{r}{s} = \frac{p}{q} \cdot \left(\frac{r}{s}\right)^{-1} = \frac{p}{q} \cdot \frac{1}{\left(\frac{r}{s}\right)} = \frac{p}{q} \cdot \frac{s}{r},$$

the cognitive load related to conceptual understanding of the division-based solving algorithm is in the case of $2x = 1$ considerably lower than in the general case. More precisely, in this case, a learner can produce the correct answer $x = \frac{1}{2}$ with reasonable conceptual understanding if he or she does not know the arithmetics of non-integer rationals but only perceives that $\frac{1}{2}$, and more generally any rational number, is a ratio of two integers.

Of course, it is possible to solve $2x = 1$ also without using division but by simply observing that $2 \cdot \frac{1}{2} = 1$ or $\frac{1}{2} + \frac{1}{2} = 1$. These approaches are clearly less burdening in the sense of intrinsic conceptual cognitive load than the division-based one above but, on the other hand, they rely on intuitive knowing or guessing the correct answer and then representing the left-hand side of the original equation as a suitable product or sum and thus are not as general as the one based on division.

All in all, there are several acceptable arithmetical methods that may provide the correct solution for $2x = 1$ and similar linear equations. What can we say about the eventual cognitive load related to this approach?

According to the APOS Theory, on the higher level a learner is, the more and more versatilely he or she exploits automated and routine procedures. For a learner at the level of Scheme (S) or Object (O), the division-based algorithm may constitute only a one single knowledge element and for a learner on the level of Action (A) or Process

(P) already the number of details in (1) may exceed the capacity of his or her perceptive skills (for the definition of the APOS levels, see Asiala et al, 1997). And as the TIMMS 2003 results show, learners with same educational background can be in very different stage of their learning process. This complicates even further giving any quantitative estimate of the cognitive load.

On the other hand, it is very plausible that linear equations are in most cases introduced at school in such a way which we classify belonging to the arithmetical approach in this paper. Therefore we think that, instead of giving any numeric estimate of the cognitive load, it is more reasonable to compare the load of the other two approaches to the one of the arithmetical approach and then design, if possible, an optimal approach piggybacking onto pros of each three approaches.

We conclude this section by observing that all procedures considered above share at least one fundamental problem: they do not explicitly say why there are no other solutions but $x = \frac{1}{2}$ for $2x = 1$.

3. Geometrical approach

Presumably only few mathematics teachers have applied, at least until the existence of modern computers and mathematical softwares, illustrations as a principal tool for finding the solution for linear equations but maybe a little more often they have used images for convincing their pupils of the fact that there are no other solutions. On the other hand, the more central role computing machinery takes in mathematics education, the more central geometrical approach also in solving equations may become.

Before discussing the details, it is worth to consider shortly what solving equations in geometrical context really means. In geometry we first and foremost deal with geometrical objects. Straight lines, curves etc. are geometrical objects; equations, expressions etc. are primarily not. Lines and curves intersect, coincide and so on; equations and expressions have roots, factorize and so on. In other words, we ask different questions about geometrical objects and non-geometrical objects. Analytical geometry is the field of mathematics that relates these different kind of worlds to each others and hence it is possible to solve arithmetical problems also geometrically. For example, in the xy-plane solving $2x = 1$ is reasonable and it means finding the x-parameters of the intersection points of the curves $y = 2x$ and $y = 1$. In Euclidean or other non-analytic geometry, we could speak only of the intersection points of curves without any chance to join this action to arithmetical concepts.

Mathematically most natural and the only reasonable setting to study the solution of $2x = 1$ in an illustrative way thus is the xy-coordinate plane. By presenting the both sides of the equation as straight lines and then studying the set of points where these lines intersect we find the complete solution of the equation. The illustration is given in Figure 1.

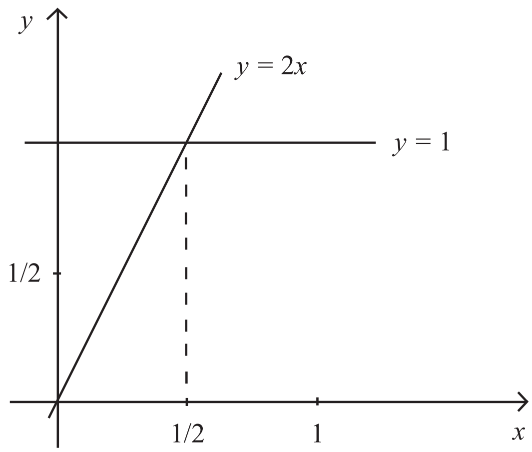

Figure 1. The illustration of $2x = 1$.

In order to be able to solve $2x = 1$ completely in this setting, a learner should at the minimum know that two non-parallel straight lines always intersect exactly at one point. Again, at first glance, this may seem to be a piece of cake but does a learner really know that? Is it only an intuitive conclusion justified by a prompted observation from elementary Euclidean geometry or can it been explained in any other way than by solving linear equations? Being punctilious, it seems that prerequisities to use this approach are more challenging than the problem itself to be solved or we must fool ourselves and accept at least one of the fundamental and non-trivial features of the machinery for granted. After all, mathematical reasoning should be beyond everyday facts!

Let us now consider in more detail the load on working memory needed in understanding the relationship between the illustration in Figure 1 and the solution of $2x = 1$. First a learner must transform a single equation $2x = 1$ into a pair of equations

$$\begin{cases} y = 1, \\ y = 2x, \end{cases}$$

then construct the graphs of these equations, find the intersection point of the lines in the plane, identify the value of the parameter x of this point, and then finally go these steps backwards in order to be able to interpret this value as the only solution of the original equation. The number of operations and processes to be controlled simultaneously in the working memomy seems to exceed the magical seven easily if a learner has not yet gained, with respect to the APOS Theory, O- or S-level capacity in using the coordinate system.

It is worth observing that a learner must go through all of the above steps also in that case if computers are applied. The most remarkable difference is that computers can provide ready-made operations for some of the subroutines, e.g. for finding the x-parameter of the intersection point. In other words, computers can only lighten the arithmetical load but not provide an escape from understanding the relationship of the original problem and the illustration which constitutes the core of the cognitive load of the whole manoeuvre.

It seems that also the necessary conceptual knowledge in this manoeuvre readily exceeds the knowledge needed in the arithmetical approach; for example, can we assume the facility to create or read graphs of straight lines in the coordinate system without good knowledge of arithmetics of (at least) rational numbers? Already finding the correct slope requires good understanding of proportions.

On the other hand, human brains can receive and manipulate data better in an illustrated than in a pure arithmetical form. Most propably, human brains can group larger data as a single schema or an information element for working memory if the data is given figuratively. Hence, let us look at Figure 1 once again. If it were, say, a Java applet based dynamic figure such that using it a learner easily perceived how the straight lines and the expressions $2x$ and 1 are related to each others, and the figure automatically produced the cutted line and the value for the x-coordinate of the intersection point, this setting could provide all tools for controlling the geometrical solving of $2x = 1$ as a single schema. From this point of view, at least procedurally the geometrical approach is not more burdening than the arithmetical approach.

Using the similar thinking as above, one may conclude that illustrations *always* makes mathematics easier. Counterexamples do however exist, as the following one related to elementary algebra verifies.

Even at college and university one can meet every now and then student who claim that $\frac{x}{2} + \frac{x}{3} = \frac{2x}{5}$. Having asked other students how this student could be corrected, a common answer has been that teacher should equip the example with an image like the one in Figure 2.

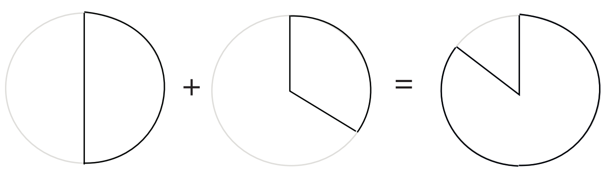

Figure 2. The illustration of $\frac{x}{2} + \frac{x}{3} = \frac{5x}{6}$.

Now, what is the point in this image? A half and a third of a disk is not equal to $\frac{2}{5}$ of the disk but $\frac{5}{6}$ of the disk. But do we really think that understanding *this* is problematic to our student? Obviously not but more propably he or she does not sense any meaning for $\frac{x}{2}$ and $\frac{x}{3}$ and hence cannot apply proper arithmetical rules for them. For the same reason the cognitive load that student must take over in order to be able to understand the correspondence between the image and polynomial expressions is greater compared to the aid that the image can provide.

All in all, the cognitive load related to linear equation solving in the geometrical approach depends remarkably both on the learner's capacity to use the coordinate system and also on the computing tools that are available. For a learner at A- or P-level in using the coordinate representations this approach is propably more burdening than the arithmetical approach and for an advanced user, the load is quite the same as in the arithmetical case.

Nevertheless, geometrical approach provides a somewhat sufficient explanation for the uniqueness of the solution, at least in the context of school mathematics; the complete explanation would take good knowledge of the algebraic structure called group, which already belongs to university mathematics and to the real analytical approach in this paper.

4. Real analytical approach

Now we study $2x = 1$ in the context of real numbers as they are ultimately defined in real analysis, i.e., in the context of an ordered field that has the least-upper-bound property (e.g. Rudin, 1976, 8). To solve $2x = 1$ means then finding the sequence of necessary axioms to establish the chain of equivalences (or implications) between the equation $2x = 1$ and the solution. This is quite typical conventional problem in academic mathematics; it is to be solved using so-called means-ends analysis (see e.g. Larkin et al., 1980) whose principal idea is reducing differences between the current problem state and the goal state. Although this strategy is forceful in obtaining answers, unfortunately, it unavoidably induces high levels of cognitive load. This is because the strategy requires attention to be directed simultaneously to the current state, the goal state, differences between them, procedures to reduce those differences and any possible subgoals that may lead to solution. (Sweller, 1988).

As a matter of fact, $2x = 1$ must be read so that it is the abbreviation for $x \oplus x = 1$ since it is not stated in the axioms that the natural numbers belonged to such an algebraic structure we are dealing with. This implies, for example, that we can solve the original equation by multiplying the equation by the inverse of 2 only if we are able to show that the natural number 2 belongs to the algebraic structure. Following this method – and there hardly are any other available – we soon run into a surprising challenge: there exist examples of fields, e.g. $\{0, 1\}$ equipped with the usual (mod 2) –arithmetic, where $x \oplus x = 1$ does not have any solution! Hence we deduce that in the field of real numbers, in addition to the axioms related to addition and multiplication, we need at least the axioms of order – in other words, the properties of inequalities! – in order to be able to solve this seemingly simple equation $2x = 1$. The same holds again for the general case, too. Ultimately, as anyone familiar with axioms of real numbers can withness, it takes several hours of lectures to provide all necessary details and hence in most mathematics teacher training programs students never see them.

As one could assume beforehand, in this setting both the conceptual and procedural knowledge required are of much greater dimension than in arithmetical or geometrical approaches. But this is the only approach that provides mathematically complete answer to $2x = 1$. It is also self-evident that one cannot use this approach at school. A classical dilemma follows: the more advanced mathematical education we give to mathematics teacher students the less they benefit from it in the pedagogical sense.

5. A cognitive load generated effected approach to teaching linear equation solving

As a summary of the previous sections, we can say that the cognitive load related to learning linear equation solving is quite the same in the arithmetical and geometrical approaches and remarkably heavier in the real analytical approach. Taking into account also the discussion in the beginning of Section 3, one is easily led to think that the most suitable educational arrangement is such that pupils are first put to solve linear equations in the arithmetical context and then they proceed to studying the graphs of linear functions in analytical geometry, and then finally, those few who wish to be real mathematicians, study axioms of real numbers at university.

On the other hand, the arithmetical approach has least tools for motivation of the uniqueness of the solution and the geometrical approach provides at least a plausible solution to that. Moreover, the analysis in the previous sections merely deals with the intrinsic cognitive load and the total cognitive load that a learner experiences is remarkably affected also by the extraneous cognitive load, which is due to e.g. how the instructional materials is used to present information in actual teaching. Clever instructional solutions may smooth the peaks of the intrinsic load in minimizing the total load.

So, could we enhance learning linear equation solving by modifying the traditional practice? Especially, if we evaluate the capacity to study problems in whole higher than the capacity to produce single solutions quickly, the uniqueness of the solution of linear equation should be emphasized right from the beginning. Representing this point of view, we now present the keynotes of an approach to teaching linear equations in which we try to apply as many cognitive load generated effects, i.e., instructional techniques that have been developed in Cognitive Load Theory to facilitate learning, as possible. In Table 1 the most typical effects are listed and compared to standard practice by Cooper (1998). The term 'goal free effect' refers to generating goal free problems which is just the opposite to generating problems that require the means-ends analysis. This effect should automatically induce forwards working solution paths and thus impose low leves of cognitive load (Cooper, 1998 and the references therein). **See Table 1 in Appendix**

In our view, an ideal setting for learning general linear equation (i.e. $ax+b=cx+d$) solving is a dynamic two-part figure which combines the arithmetical and geometrical approaches so that

1. In the arithmetic window, as the equation to be solved have been entered, the left-hand side of the equation of is displayed, say, in blue color and the right-hand side in red color. The original equation and the current equivalent equation on which a learner performs arithmetical operations are both shown;

2. The figure automatically generates in the graphics window (the xy-coordinate plane) the graphs of $y=ax+b$ and $y=cx+d$ with the corresponding colors displaying also the equations of these straight lines;

3. In the arithmetics window, a learner can choose and perform any arithmetical operation, e.g. “Divide by 5”, and the figure performs the corresponding operation for both straight lines and their formulas in the graphics window;

4. A learner is auditorily guided to manipulate (using arithmetical operations) the original equation first into form $ex = f$ and then finally to divide this by e so that it becomes $x = \frac{f}{e}$;

5. Especially in the last stages of the process, a learner is encouraged to pay attention to the positions of the straight lines and notice that one of the lines is horizontal and the other one goes through the origin;

6. When the solution is found, i.e., when the current equivalent equation takes the form $x = x_0$, the figure automatically generates an extra vertical line through $x = x_0$, the line through the intersection point of the blue and red lines, marking the solution. The figure also displays this value numerically.

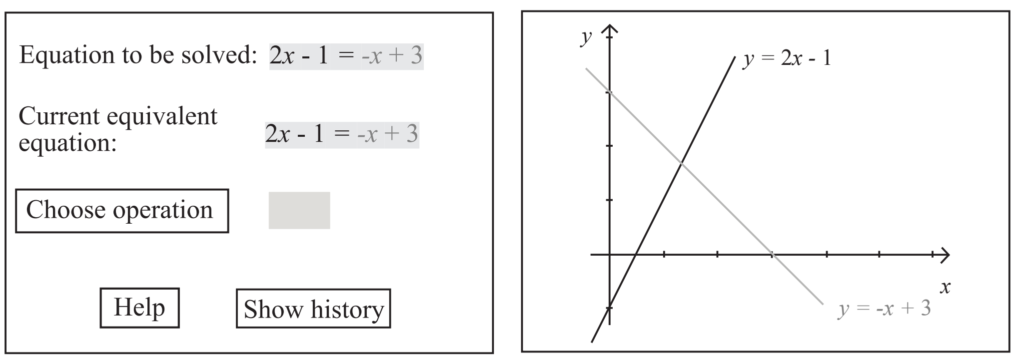

Figure 3. An exemplar view of a dynamic figure for learning linear equation solving.

Clearly, this setting exploits the split attention and the modality effetcs. Also the redundancy effect is made good use of although two equation are shown at every turn. If the original equation is not shown, a learner may have a greater cognitive load in remembering the original task and in checking whether he or she got the right answer. And while a single arithmetical operation performed by a learner induces several changes in both arithmetic and graphics windows, it is necessary to display all these expressions in order to indicate the correspondence between the aritmetical and geometrical viewpoints. It is a little more difficult to say whether the straight lines corresponding to the original equation should be displayd throughout. On the other hand, it was logical and informative, on the other hand, it may be redundant. A possible solution is that these lines are displayed shadowedly in background after the first non-trivial arithmetical operation is performed or a learned is encouraged to use Show history –function so that an extra attention is paid to the position of the straight lines.

How well the goal free effect and the worked example effect are made use of depends merely on the expertise of instructor. A pro of this setting is that all arithmetical operations induce a single simple geometrical action. In other words, geometrically multiplication and division are not more complicated processes than addition or subtraction. Therefore it possible to head to solving general linear equations right after having worked a few examples of type $x+a=b$ and $ax=b$. This fact, in our view, is perhaps the most significant advantage of this setting compared with the traditional practices in which several weeks may be spent on solving only $x+a=b$ and $ax=b$. Anyway, this kind of dynamics should easily allow using goal free problems and studying versatile worked examples also collaboratively.

A few critical questions may also arise: for example, should we allow a learner also to move straight lines in the graphics window and let the figure automatically perform the corresponding arithmetical operations in the arithmetics window? Or should the figure somehow underline the coordinate values of the intersection point of $y=ax+b$ and $y=cx+d$ from the beginning? The answer to the first question is: No. Although freedom to move these lines may help a learner to understand the correspondence between the sides of the original equation and the lines, it easily leads to misconceptions and diversion, e.g. if a learner translates the lines in the graphics window so that the intesection point of lines remains fixed, a learner may think that he or she is still solving an equation equivalent to the original one. The latter question may also be answered negative while it is not so obvious. Namely, in this process the x-coordinate of the intersection point remains, of course, fixed. Hence, there is no urgent educational need to emphasize this value until the geometric solution is in its most visible form especially if this multiplied the cognitive load in perceiving the actions in the graphics window. On the other hand, seeing the coordinates of the intersection point at every turn would be of some relevance. Thus the best solution might be such that a user could choose whether the coordinates are displayed or hidden.

The setting described above also facilitates so-called trialogical approach to learning which is related to innovative knowledge communites and especially to the knowledge-creation metaphor of learning. The term "trialogical" refers to the fact that in this approach the emphasis is not only on individuals or on community but also on the way people collaboratively develop mediating artifacts. (Paavola & Hakkarainen, 2005).

More precisely, if the dynamic figure is equipped with saving function, a learner can always trace back with his or her instructor or other learners the steps that he or she has performed. Moreover, since there are only a limited number of possible operations that lead to correct solution, it is possible to program the figure to interactively help a learner to perform necessary steps correctly. It is important to notice that although learners may adopt using the means-ends analysis in linear equation solving, it is also possible to program the help function of the figure so that the goal free effect and thus more communicative learning is applied.

Finally, are there any elements in the real analytic approach that could be utilized in this approach, too? Perhaps, there is. First, the help function can be programmed so that in the arithmetics window it actively motivates a learner to pay attention to that subtraction and division are, respectively, addition of opposite number and multiplication by inverse. Moreover, if the figure allows a learner to enter also combinations of linear expressions to both sides of the equation to be solved, also the need to operate properly with the distributive laws can be discussed within the figure's interactive interface. Second, the need to solve also the existence of the solution can be discussed easily in this framework if the figure is also programmed to generate equations to be solved in varied domains.

References

Anderson, J. R. (2005). Human symbol manipulation within an integrated cognitive architecture. *Cognitive Science*, 29 (3), 313-341.

Asiala, M., Cottrill, J., Dubinsky, E. & Schwingendorf, K. (1997). The development of students' graphical understanding of the derivative. *Journal of mathematical behavior*, 16 (4), 399-431.

Cooper, G. (1998). Research into Cognitive Load Theory and Instructional Design at UNSW. http://education.arts.unsw.edu.au/staff/sweller/clt/.

Gonzales, P., Guzmán, J. C., Partelow, L., Pahlke, E., Jocelyn, L., Kastberg, D. and Williams, T. (2004). *Highlights From the Trends in International Mathematics and Science Study (TIMSS) 2003 (NCES 2005–005)*. U.S. Department of Education, National Center for Education Statistics. Washington, DC: U.S. Government Printing Office.

Haapasalo, L. & Kadijevich, Dj. (2000). Two types of mathematical knowledge and their relation. *Journal für Mathematik-Didaktik*, 21 (2), 139-157.

Hiebert, J. & Lefevre, P. (1986). Conceptual and procedural knowledge in mathematics: An introductory analysis. In J. Hiebert (Ed.), *Conceptual and procedural knowledge: the case of mathematics* (pp. 1-27). Hillsdale (NJ): Lawrence Erbaum.

Larkin, H., McDermott, J., Simon, D., & Simon, H. (1980). Models of competence in solving physics problems. *Cognitive Science*, 11, 65-99.

Lithner, J. (2003). Student's mathematical reasoning in university textbook exercises. *Educational Studies in Mathematics*, 52, 29-55.

MacGregor, M. & Stacey, K. (1993). Cognitive models underlying students' formulation of simple linear equations. *Journal for Research in Mathematics Education*, 24 (3), 217-232.

Miller, G. A. (1956). The magical number seven, plus or minus two: Some limits on our capacity for processing information. *Psychological Review*, 63, 81-97.

Paavola, S. & Hakkarainen, K. (2005). The Knowledge Creation Metaphor - An Emergent Epistemological Approach to Learning. *Science & Education* 14, 535-557.

Qin, Y. L., Carter, C. S., Silk, E. M., Stenger, V. A., Fissell, K. Goode, A. & Anderson, J. R. (2004). The change of the brain activation patterns as children learn algebra equation solving. *Proceedings of the National Academy of Sciences of the United States of America*, 101 (15), 5686-5691.

Rudin, W. (1976). *Principles of Mathematical Analysis*, 3rd Edition. Singapore: McGraw-Hill.

Star, J.R. (2005). Reconceptualizing procedural knowledge. *Journal for Research in Mathematics Education*, 36 (5), 404-411.

Sweller, J. (1988). Cognitive load during problem solving: Effects on learning. *Cognitive Science* 12 (1), 257-285.

Appendix

Table 1. Cognitive load generated effects (Cooper,1998)

Standard Practice	Cognitive load generated effect
1. Use conventional problems which specify the goal so that students "know what they have to find"	The goal free effect Use goal free problems
2. Students need to solve many problems to learn because "practice makes perfect"	The worked example effect Students learn by studying worked examples. Problem solving is used to test if learning has bee effective
3. Instructional materials which require both textual and graphical sources of instruction should be presented in a "neat and tidy" fashion where the text and graphics are located separately	The split attention effect Instructional materials which require both textual and graphical sources of instruction should integrate the text into the graphic in such a way that the relationships between textual components and graphical components are clearly indicated
4. The same information should be presented in several different ways at the same time	The redundancy effect Simultaneous presentations of similar (redundant) content must be avoided
5. Similar to-be-learned information should be presented using an identical media format to ensure consistency in the instructional presentation	The modality effect Mix media, so that some to-be-learned information is presented visually, while the remainder is presented auditorily