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FROM TRAPEZOIDS TO THE FUNDAMENTAL THEOREM OF CALCULUS

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From Trapezoids to the Fundamental Theorem of Calculus

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Abstract: The philosophy of Mathematics Education undergoes changes from the school to college level and students generally have a tough time coping with the transition. It is our endeavor to impress the importance of introducing college level topics at an early stage, so that students are not lost in the transition. Keeping this in mind, we suggest an early exposure to an important topic from Calculus; approximating the area of a planar region. Traditionally this topic is introduced using Riemann Sums but in this paper we try to follow a student's natural inclination in approximating areas and explain how this approach can be adopted at the middle school or high school level. It is our belief that using suitable technology like TI- 83/84 or Maple, this approach can be adapted to various other college level topics providing the student with a sound footing to cope with college level mathematics.

Keywords: Calculus; Collegiate math teaching; Fundamental theorem of Calculus; Riemann sums; Trapezoids; Teaching with technology

Introduction via rectangles

Every introductory calculus textbook, at some point or another, investigates the problem of finding the area of a region lying between two vertical lines, a curve, and the x-axis. An examination of

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several popular textbooks ([6];[7];[4]) follow similar approaches when introducing students to the solution of this problem.

- Remind students that for rectangles area = length × width
- Draw a picture showing an approximation using left-handed rectangles
- Draw a picture showing an approximation using right-handed rectangles
- Suggest that more rectangles result in a better approximation
- Introduce Riemann Sums
- Introduce limits and define the definite integral
- Demonstrate the need for the Trapezoidal Rule and Numerical Integration

This approach, with minor additions, subtractions, and other alterations serves as the standard approach to this topic. This approach, based on a student's ability to recognize the utility of rectangles in the approximation process, works superbly. However, it has been the authors' experience that not all students are inclined to use rectangles to arrive at their first approximation of the area. This paper will examine what can happen when students are allowed a different starting point. During this proposed mathematical journey students will discover a slope-area connection while discovering the need for mathematically rich concepts such as mathematical induction.

Introduction via trapezoids

Consider the following problem asked to a class of students studying calculus for the first time: Approximate the area between the x-axis, the curve $y = x^2$ and the lines $x = 0$ to $x = 3$. It has been the authors' experience that many student's first impulse is to use a triangle (see figure 1) to find the required approximation. This is not surprising since the hypotenuse of the triangle closely approximates the curve and students have known and used the formula for the area of a triangle is

A=*1/2bh,* where *b* is the length of the base of the triangle and *h* is the height of the triangle, for many years. Students rarely approximate this area with the rectangle of base 3 and height 9, after all it is obvious that the area of the rectangle is much larger that the requested area. This paper will investigate what can happen if students are allowed to explore their initial impulse rather than be immediately redirected to rectangles. Before students proceed they must agree that a fixed triangle can be moved through two-dimensional space without changing its area. After this agreement is reached students can be presented with the following question:

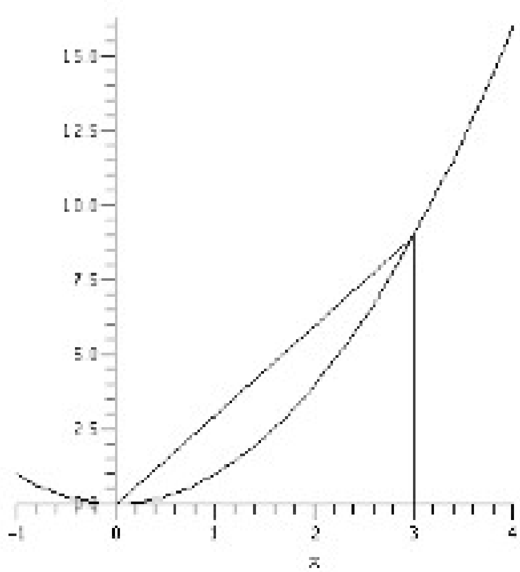
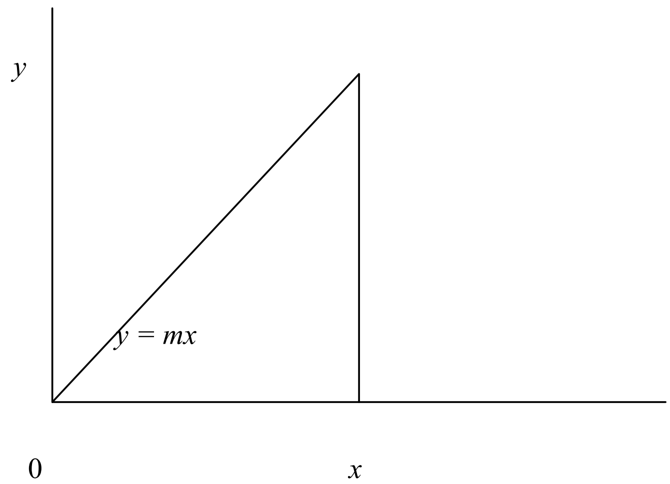


Figure 1
y=x^2

Find the area of the triangle below:

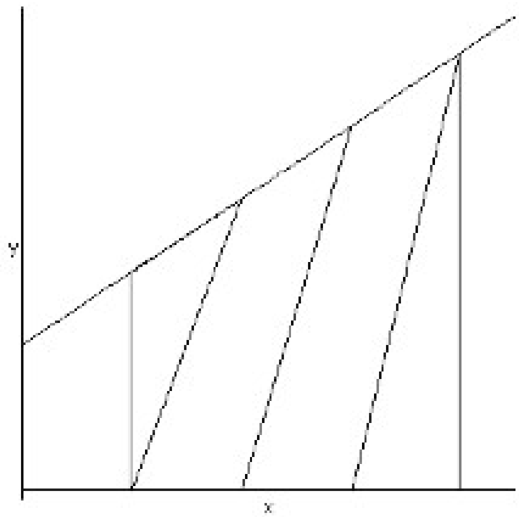
Example 1



Given this graph students will quickly state that *Area* = ½(x)(y) square units.

The authors suggest students be guided through the following derivation leading to an alternative form of this area formula. The first step in this alternative vision is to ask students to observe that the equation $y = mx$, defines the hypotenuse of the triangle in this position on the coordinate plane. Knowing this students see that these triangles have base = x and height = mx. Thus the area of such a triangle is $1/2\ (x)(mx) = 1/2mx^2$. This "new" formula suggests, that the area the area of a triangle can be thought of as dependent on the slope of the hypotenuse and the length of the base. Thus a link between slope and area has been established. At this point the line can be moved and the boundaries changed and the following graph (Figure 2) and question can be presented:

Figure 2
y= m*x + kT



Find the area of the shaded region bounded by the x-axis, the line $y = mx + k_T$ and the vertical lines $x = x_0$ and $x = x_1$.

Students will recognize that the shape created in figure 2 is a trapezoid. Some students will remember the formula for the area of a trapezoid to be $A = \frac{h(b_1 + b_2)}{2}$, where b_1 and b_2 are the lengths of the two parallel sides of the trapezoid and h is the height (distance separating the parallel sides) of the figure. Thus in the figure presented $b_1 = mx_1 + k_T$, $b_2 = mx_0 + k_T$. and $h = x_1 - x_0$. Thus :

$$
\begin{aligned}
Area &= \frac{(x_1 - x_0)[(mx_1 + k_T) + (mx_0 + k_T)]}{2} \\
&= \frac{x_1(mx_1 + k_T) + x_1(mx_0 + k_T) - x_0(mx_1 + k_T) - x_0(mx_0 + k_T)}{2} \\
&= \frac{mx_1^2 + x_1k_T + mx_0x_1 + x_1k_T - mx_0x_1 - x_0k_T - mx_0^2 - x_0k_T}{2} \\
&= \frac{mx_1^2 + 2x_1k_T - mx_0^2 - 2x_0k_T}{2} \\
&= \frac{m(x_1^2 - x_0^2) + 2k_T(x_1 - x_0)}{2} \\
&= \frac{m}{2}\left(x_1^2 - x_0^2\right) + k_T\left(x_1 - x_0\right) \qquad (1)
\end{aligned}
$$

Again students see a "new" formula in which the area of the figure is calculated using the slope of the line which forms one of its boundaries and the values of *x* that dictate its height and position on the x-axis.

Using Technology and Algebra

The following program can be used on the TI 83/84 to calculate areas using equation (1).

```
Input "slope",M
Input "Y Intercept",K
Input "Lower Limit",L
Input "Upper Limit",U
(M/2)*(U^2-L^2)+K*(U-L)→A
Disp A
```

The following program can be used on Maple 11 to calculate area using equation (1).

```
M: =; U: =; L: =; K: = ;    (Input values of M, U, L, K)
Area: = (M/2)*(U^2-L^2)+K*(U-L);
```

Area;

Evalf(%);

The area can be computed using the traditional trapezoidal rule as follows.

with (student);

M: =; U: =; L: =; K: =;   (Input values of M, U, L, K)

trapezoid (Mx+k, X=U..L, 1);

evalf(%);

What of students that do not remember the formula for the area of a trapezoid, are they left in the dark unable to create an argument that can lead to the insight above? No, they can approach this problem by dividing the figure into two more recognizable shapes, a triangle and a rectangle, calculating their areas and adding the results. Their work may look like what is found below.

$$Area = (mx_0 + k_T)(x_1 - x_0) + \tfrac{1}{2}[(mx_1 + k_T) - (mx_0 + k_T)](x_1 - x_0)$$

$$= (x_1 - x_0)\left[(mx_0 + k_T) + \left(\frac{mx_1 + k_T}{2}\right) - \left(\frac{mx_0 + k_T}{2}\right)\right]$$

$$= (x_1 - x_0)\left[\frac{mx_0 + k_T}{2} + \frac{mx_1 + k_T}{2}\right]$$

$$= \frac{x_1 - x_0}{2}[mx_0 + mx_1 + 2k_T]$$

$$= \frac{1}{2}[mx_1x_0 + mx_1^2 + 2k_Tx_1 - mx_0^2 - mx_0x_1 - 2k_Tx_0]$$

$$= \frac{1}{2}[mx_1^2 - mx_0^2 + 2k_Tx_1 - 2k_Tx_0]$$

$$= \frac{m}{2}[x_1^2 - x_0^2] + k_T[x_1 - x_0] \qquad (1)$$

This is the same result that was found when the direct rule for the area of a trapezoid was used.

It is the authors' belief that such an exposition serves three purposes:

First, students clearly see that in mathematics there can be different ways to view the same concept.

It illustrates the general slope-area connection central to the calculus.

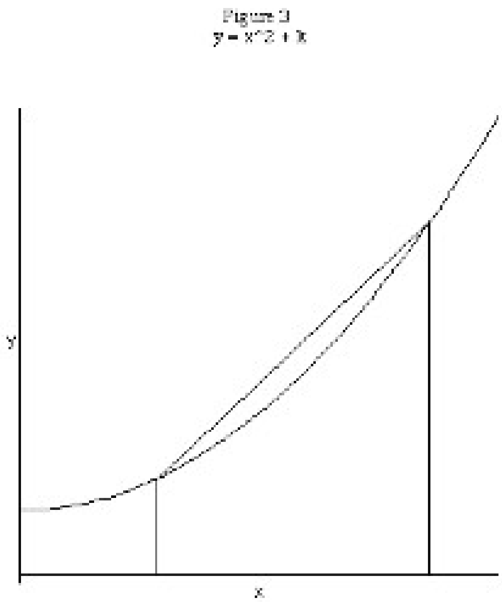
This introduction and some new notation:

Area under $y = mx + k$ between x_0 and x_1 is equal to

$$\frac{m}{2}[x_1^2 - x_0^2] + k[x_1 - x_0] = \frac{m}{2}\left[x^2\right]_{x_0}^{x_1} + k\left[x\right]_{x_0}^{x_1}$$

exposes students to an example of the Fundamental Theorem of Calculus which they will encounter in the near future.

Since not all graphed function result in lines, it is not inappropriate at this time to suggest students explore shapes with curved tops and learn the lessons they have to teach. A natural starting point for this journey is our starting point $y = x^2$ with a vertical shift $y = x^2 + k_p$. Since students probably have not learned a formula for the area under such a curve the first step in the discovery process is to find an approximation of the answer sought. Examination of the graph found below (figure 3) suggests that a trapezoidal approximation (using a single trapezoid) is a better first approximation than a rectangular approximation (using a single rectangle).



Thus, after the equation of the line containing the points $\left(x_0, x_0^2 + k_p\right)$ and $\left(x_1, x_1^2 + k_P\right)$is found formula (1), derived above, can be used to find a *"*good*"* approximation of the desired area under the parabola. The slope, *m*, of the line is:

$$m = \frac{(x_1^2 + k_P) - (x_o^2 + k_P)}{x_1 - x_0}$$
$$= \frac{x_1^2 - x_0^2}{x_1 - x_0}$$
$$= x_1 + x_0.$$

The y-intercept, k_T, of the line is found as follows:

$$x_0^2 + k_P = (x_1 + x_0)(x_0) + k_T$$
$$x_0^2 + k_P = x_1 x_0 + x_0^2 + k_T$$
$$k_P - x_1 x_0 = k_T$$

Thus, the equation of the non-perpendicular line connecting the two parallel sides of the trapezoid is $y = (x_1 + x_0)x + (k_P - x_1x_0)$ and the approximate area of this figure is:

$$\frac{x_1 + x_0}{2}\left[x_1^2 - x_0^2\right] + \left[(k_P - x_1x_0)(x_1 - x_0)\right]. \quad (2)$$

From this we see that:

$$Area \approx \frac{x_1 + x_0}{2}\left[x_1^2 - x_0^2\right] + \left[(k_P - x_1x_0)(x_1 - x_0)\right]$$
$$(x_1 - x_0)\left[\left(\frac{(x_1 + x_0)^2}{2}\right) + (k_P - x_1x_0)\right]$$
$$\left(\frac{x_1 - x_0}{2}\right)\left[\left((x_1 + x_0)^2\right) + 2(k_P - x_1x_0)\right]$$
$$\left(\frac{x_1 - x_0}{2}\right)\left[x_1^2 + 2x_1x_0 + x_0^2 + 2k_P - 2x_1x_0\right]$$
$$\left(\frac{x_1 - x_0}{2}\right)\left[x_1^2 + x_0^2 + 2k_P\right]$$
$$\left(\frac{x_1 - x_0}{2}\right)\left[(x_1^2 + k_p) + (x_0^2 + k_P)\right]$$
$$\left(\frac{x_1 - x_0}{2}\right)(y_1 + y_0)$$

which is the traditional trapezoidal rule for one subdivision. One can now observe that

$$\left(\frac{x_1 - x_0}{2}\right)(y_1 + y_0) = \left(\frac{h(b_1 + b_0)}{2}\right).$$

which demonstrates that the trapezoidal rule for one subdivision is simply a functional form of the elementary school formula for the area of a trapezoid. Once again emphasizing to students that the same mathematical concept can be viewed in various ways.

Those wishing to program equation (2) into the TI 83/84 may use the following:

Input “Constant”,K

Input “Lower Limit”,L

Input "Upper Limit",U

((U+L)/2)*(U^2-L^2)*((K-U*L)*(U_L))→A

Display A

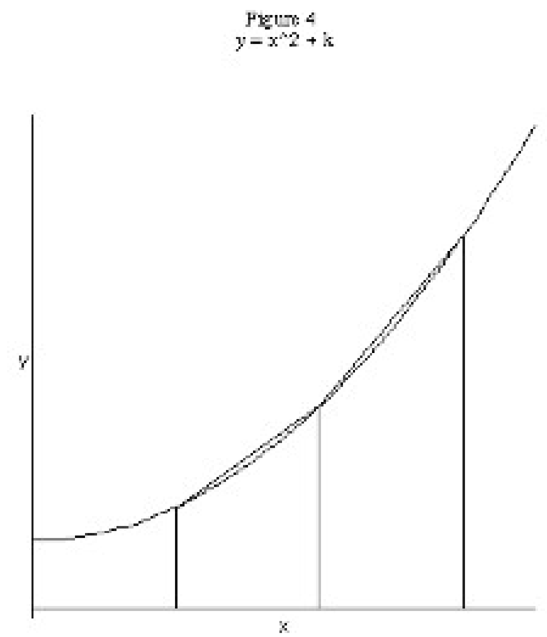
Also a student interested in programming in Maple 11 may use the following:

$$T := ((U+L)/2)*(U^2 - L^2)) + ((K - U*L)*(U - L));$$
$$K :=; U :=; L :=;$$
$$evalf(T);$$

The area may be also computed by using the traditional trapezoidal rule by using following code:

$$with(student);$$
$$K :=; U :=; L :=;$$
$$trapezoid(x^2 + K, x = L..U, 1);$$
$$evalf(\%);$$

It quickly becomes obvious to students that breaking the region up into two trapezoids, as seen in figure 4 below, results in an even better approximation of the area desired.



$$Area \approx \left\{ \frac{x_1 + x_0}{2} \left[x_1^2 - x_0^2 \right] + \left[(k_p - x_1 x_0)(x_1 - x_0) \right] \right\} + \left\{ \frac{x_2 + x_1}{2} \left[x_2^2 - x_1^2 \right] + \left[(k_p - x_2 x_1)(x_2 - x_1) \right] \right\}$$

Continuing this line of reasoning they see that three is better than two and one hundred would be better that three etc. A TI 83/84 program for this alternative formula for this approximation using n subintervals is found below.

```
Input"Constant", K
Input "Number of Subintervals", N
Input "Lower Limit", L
Input "Upper Limit", U
(U-L)/N→W
0→S
For (I,1,N,1)
```

L+W→U

((U+L)/2)*(U^2-L^2) + ((K-U*L)*(U-L))→T

S+T→S

U→L

END

Disp S

Again a Maple program for the above is found below:

N: =; U: =; L(0):=; W:=$\frac{U-L(0)}{N}$;R=0;

For j from 0 by 1 to n-1 do

U(j)=L(j)+W;

$$A(j) := R + \left(\left(\frac{U(j)+L(j)}{2}\right) * \left(U(j)^2 - L(j)^2\right) + \left(K - U(j) * L(j)\right) * \left(U(j) - L(j)\right)\right);$$

$R = A(j)$;

od;

(Readers wishing to use MS Excel to execute the trapezoidal rule are referred to [3].)

Mathematical Induction

Let us look at the number of subdivisions used to approximate the area under a curve. For n=1, we have the area of a trapezoid. For n=2, with subdivisions, x_0, x_1, x_2 we have the area approximated by the sum of two trapezoids i.e

$$Area \approx \frac{x_2 - x_0}{2\times 2}(y_0 + y_1) + \frac{x_2 - x_0}{2\times 2}(y_1 + y_2) = \frac{x_2 - x_0}{2\times 2}(y_0 + 2y_1 + y_2).$$

The authors suggest that the students be asked to verify that for *n*=3, you get $Area \approx \frac{x_3 - x_0}{2 \times 3}(y_0 + 2y_1 + 2y_2 + y_3)$. Now, for the problem with 100 sub-divisions the student has most likely made an educated guess and written down the formula $Area \approx \frac{x_{100} - x_0}{2 \times 100}(y_0 + 2y_1 + 2y_2 + 2y_3 + ... + 2y_{99} + y_{100})$. At this point one can conjecture that the general formula based on the number of subdivisions, say *n*, it will be $Area \approx \frac{x_n - x_0}{2 \times n}(y_0 + 2y_1 + 2y_2 + 2y_3 + ... + 2y_{n-1} + y_n)$.

Although it seems to be true from the pattern, it is just a guess and should not be considered a rule at this stage. In order to elevate this conjecture to the status of a rule we have to provide a convincing argument. The convincing mathematical argument is called a proof. There should be no loopholes in the proof. There are different methods of proof. One of them is called the method of induction. The logic behind this method is follows.

Suppose: P(n) represents some sort of argument involving natural numbers. Example: P(n) can represent the statement: The area under a curve y=f(x) using n subdivisions is approximately $\frac{x_n - x_0}{2 \times n}(y_0 + 2y_1 + 2y_2 + 2y_3 + ... + 2y_{n-1} + y_n)$. We can use an inductive argument as follows: P(1) is verified to be true: i.e. the result is true for n=1. In our situation P(1) would represent the statement, Area ≈ $\left(\frac{x_1 - x_0}{2}\right)(y_1 + y_0)$ which is true. Now, suppose the result is true for some natural number *k* i.e. P(*k*) is true. This statement is known as the induction hypothesis. The induction hypothesis in this problem is the assumption that the area under a curve y = f(x) with k subdivisions on the interval [a, b] is approximately $\frac{b-a}{2 \times k}(y_0 + 2y_1 + 2y_2 + 2y_3 + ... + 2y_{k-1} + y_k)$. We will use this to prove that the result is true for next integer n = k + 1 or P(k + 1). At that point we would have clearly shown that if the result is true for any given integer, then it is also true for the next integer.

1. The result is clearly true for n=1 since it is the well-known formula for the area of a trapezoid.
2. Assume the induction hypothesis: with *k* subdivisions, each of length $l = \frac{b-a}{k}$, the area is approximately $\frac{l}{2}(y_0 + 2y_1 + 2y_2 + 2y_3 + \ldots + 2y_{k-1} + y_k)$.
3. Let us work out the last step: Show that P(k+1) is true. We need to somehow use the induction hypothesis to prove P(k + 1). This can be done easily. For k + 1 subdivisions, each of length $l = \frac{b-a}{k+1}$, we can sum up the area of the trapezoids corresponding to the first k subdivisions and then to this add the area corresponding to the subdivision $[x_k, x_{k+1}]$. By the induction hypothesis, the Area corresponding to the first k subdivisions is approximated by $\frac{l}{2}(y_0 + 2y_1 + 2y_2 + 2y_3 + \ldots + 2y_{k-1} + y_k)$. The area corresponding to the subdivision $[x_k, x_{k+1}]$ is $\frac{x_{k+1} - x_k}{2}(y_k + y_{k+1})$. But $x_{k+1} - x_k = l$. So the area with *n* =*k* + 1 subdivisions is approximately

$$\frac{l}{2}(y_0 + 2y_1 + 2y_2 + 2y_3 + \ldots + 2y_{k-1} + y_k) + \frac{l}{2}(y_k + y_{k+1})$$

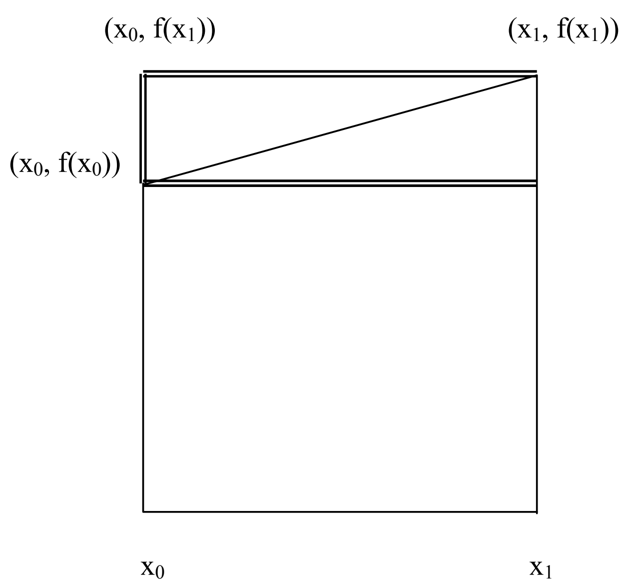
$$= \frac{l}{2}(y_0 + 2y_1 + 2y_2 + 2y_3 + \ldots + 2y_{k-1} + 2y_k + y_{k+1})$$

It should be clear to students that the formula is independent of the equation of the curve and so that the above discussion will work for any continuous curve.

The Trapezoidal Rule

It can now be pointed out that the area of a trapezoid is simply the average of the areas of two rectangles as demonstrated on figure 5 below.

Figure 5



$$\text{Area of Trapezoid} = \frac{(x_1 - x_0)(f(x_1) + f(x_0))}{2}$$

$$= \frac{(x_1 - x_0) f(x_1) + (x_1 - x_0) f(x_0)}{2}$$

$$= \frac{\text{Area of rectangle with side } (x_1 - x_0) \text{ and side f}(x_1) + \text{Area of rectangle with side } (x_1 - x_0) \text{ and side f}(x_0)}{2}$$

At this point students have been led to the idea of left and right, or upper and lower, rectangular sums which then leads to the concept of the Riemann Sum and finally the definition of the definite integral. At this time teachers can introduce any student interested in the historical development of mathematics to the ancient Greek method of area by exhaustion used by, among others, Archimedes (interested readers are referred to [6];[1];[2];[5]).

Conclusion

The journey suggested above can be summarized as follows:

- Allow students to use a triangle to approximate a desired area
- Allow students to discover the connection among triangles, slope, and area
- Have students use a trapezoid to approximate a desired area
- Allow students to discover the connection among trapezoids, slope, and area
- Suggest that more trapezoids result in a better approximation
- Deduce the trapezoidal rule and validate using mathematical induction
- Show that the area of a trapezoid is the mean of the areas of two rectangles
- Draw a picture showing an approximation using left-handed rectangles
- Draw a picture showing an approximation using right-handed rectangles
- Suggest that more rectangles result in a better approximation
- Introduce Riemann Sums
- Introduce limits and define the definite integral.

This discovery path is interesting in several respects. It begins with a problem involving the use of low degree polynomials functions, things students are familiar with and have mastery over. It then allows them the freedom to follow a path of discovery suggested by their intuition. This path leads them to some interesting discoveries: there is a connection between area and slope, for special cases the "slope formula" can take different and interesting forms, there is a trapezoid rectangle connection, examination of simple problems can lead to generalizations from which the need for mathematical induction arises naturally. Simply stated this demonstration follows a natural progression that prepares students for both the Fundamental Theorem of calculus and future work in numerical analysis while introducing them to the need for proof.

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