[The Mathematics Enthusiast](https://scholarworks.umt.edu/tme)

[Volume 6](https://scholarworks.umt.edu/tme/vol6) | [Number 3](https://scholarworks.umt.edu/tme/vol6/iss3) Article 13

7-2009

GRAPH ISOMORPHISMS AND MATRIX SIMILARITY: SWITCHING BETWEEN REPRESENTATIONS

Thierry Dana-Picard

Graph isomorphisms and matrix similarity: Switching between representations

Thierry Dana-Picard[1]
Jerusalem College of Technology, Israel

Abstract: A proof whether two graphs (possibly oriented graphs or multigraphs, etc.) are isomorphic or not can be derived by various methods. Some of them are reasonable for small numbers of vertices and/or edges, but not for larger numbers. Switching from iconic representation to a matrix representation transforms the problem of Graph Theory into a problem in Linear Algebra. The support provided by a Computer Algebra System is analyzed, in particular with regard to the building of new mathematical knowledge through a transition from graphical to algebraic representation. Moreover two important issues are discussed: a. the need for more than one representation; b. the direction of the switch between representations, which is non standard, from graphical to algebraic.

Keywords: Computer Algebra systems (CAS); Collegiate mathematics; Graph theory; Linear Algebra; Matrices; representations; isomorphisms;

I. Introduction.

Undergraduate mathematics is often taught as a collection of stand-alone courses, and students are not always aware of the bridges that exist between different areas of mathematics. Geometry and Linear Algebra are taught in separate courses (a nice exception is Dieudonné's book, 1969). Sometimes, Linear Algebra and Ordinary Differential Equations are taught together in one course, but usually not. Moreover, numerous topics relevant to applications of Analysis to Geometry disappeared from syllabi a long time ago. Thom (1962) expresses strongly his opposition to this trend.

In the present paper, we show and explore a bridge between two other mathematical fields, Graph Theory and Linear Algebra. Graph Theory is part of Discrete Mathematics, a branch of Mathematics which deals with objects that can be described by either finite or countable sets. In regular courses, Linear Algebra is presented over the real and the complex fields, in which cases it is understood as belonging to the continuous part of Mathematics, not to the discrete part. Linear Algebra over finite fields is taught in advanced courses, not aimed to every student. The discrete point of view provides numerous methods for proving theorems, different from the methods used in a continuous setting (see Grenier 2008). Switching from Graph Theory to Linear Algebra gives an opportunity to use other methods than the typical methods of Discrete Mathematics, i.e. exhaustion of cases (enumeration), induction, and so on. The technology is not responsible for the discovery of the bridge, but it helps to explore it, and then helps to study cases which would be unilluminating with only hand made computations.

[1] Email: dana@jct.ac.il

The Montana Mathematics Enthusiast, **ISSN 1551-3440, Vol. 6, no.3**, pp.477- 494

Moreover, we will show that the activities presented here lead to develop new mathematical knowledge simultaneously in two domains. The situation at the course starting point is as follows:

1. Matrix similarity is a standard topic in any course in Linear Algebra. But, as this topic appears at the end of the course, applications to other fields are rarely shown. This was the case for the students whose work is presented in this paper.
2. Algebraic graph theory is absent from numerous textbooks in Discrete Mathematics and from the syllabus of courses.

In one class, the teacher decided to have his students learn at least a few topics of algebraic graph theory, outsourcing to a Computer Algebra System (CAS) part of the operative knowledge. Later, the author had a discussion with a colleague teaching a parallel course. This colleague valued the introduction of these activities which enhance an important mathematical knowledge, but he said that doing the same thing with his own class was impossible, in particular because of the lack of CAS literacy of his students.

"Technology can be used to compute, ..., to reinforce, clarify, anticipate, or get acquainted with ideas, and to discover and investigate phenomena" (Selden, 2005). As showed by Dana-Picard (2005), the exploration of a cognitive neighborhood[2] for a given mathematical topic is mainly concerned by the last two components, discovery and investigation. How investigation can be fostered by switching between registers of representations has been studied by Duval (1999), Arcavi (2003), Presmeg (2006), Dana-Picard and Kidron (2008), etc... We elaborate on this issue in the last section. Not only the mathematical fields are different, but also the ways to use a CAS are different.

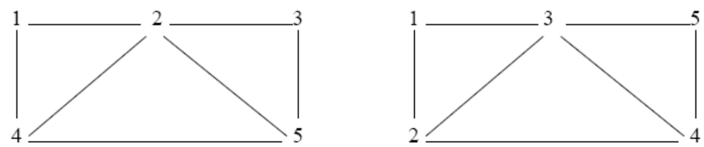
Various kinds of technological tools have been introduced into the mathematics classroom and into the researcher's lab, ranging from a graphical hand-held device to an interactive (non user-programmable) website and to a CAS. In particular, their graphical features are emphasized in order to provide visualizations, either fixed or animated, but all the other features, algebraic, numerical, etc., are important and they are used in classroom. In some CAS, algorithms specific to Graph Theory have been implemented, which enable the drawing of a picture of the graph from the abstract definition of the vertices and the edges. For the classroom activities the Derive software has been used. It has no implementation of specific features for Graph Theory and only the Linear Algebra algorithms were used. Note that the Linear Algebra packages of other CAS can assist this activity, sometimes with specific outputs. We address this issue in Section V.

II. The mathematical situation.

An important problem in computational complexity theory is determining whether, given two graphs G_1 and G_2, it is possible to re-label the vertices of one graph so

[2] Recall that a mathematical domain A is said to belong to a cognitive neighborhood of another mathematical domain B if theorems and/or methods from B can be applied to solve problems in A.

that it is identical to the other, or not. This re-labeling is called *a graph isomorphism* and we denote $G_1 \cong G_2$ .In simple words, two graphs are isomorphic if they can be represented with identical drawings. For example, see Figure 1: the permutation of vertices $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 5 & 2 & 4 \end{pmatrix}$ preserves the existence (resp. the non-existence) of an edge between vertices, whence ensures the fact that the two given iconic representations correspond to isomorphic graphs. A formal definition of a graph isomorphism can be found in Rosen's book (1999, p. 460).

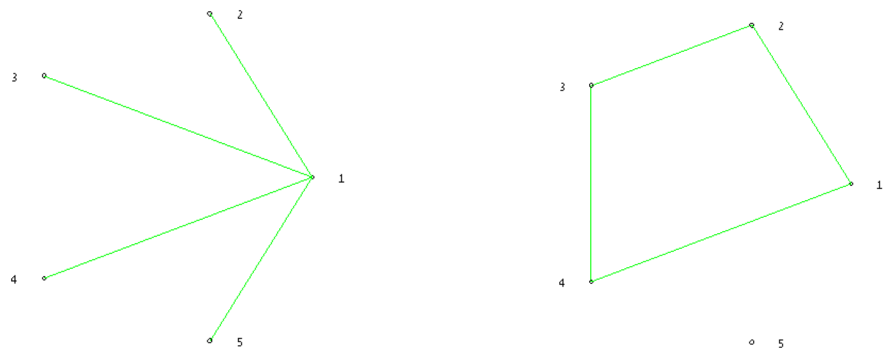


(a) First labeling **(b) Second labeling**

Figure 1: One picture, two sets of labels.

Adjacency matrices are used to describe graphs in a computational way. For a given graph, label the rows and the columns of a square matrix $A=(a_{ij})$ by the vertices of the graph. For a non-oriented graph, a_{ij} is the number of edges between vertices *i* and *j*. For an oriented graph a_{ij} is the number of arrows from vertex *i* to vertex *j*. Thus, the adjacency matrix of a non-oriented graph is symmetric and for an oriented graph the adjacency matrix can be either symmetric or non-symmetric. Relabeling the vertices of the graph changes the adjacency matrix in the same way reordering the vectors of a basis of a *n*-dimensional vector space changes the matrix of a linear operator: the original matrix *A* and the new one *B* are similar, i.e. there exists an invertible square matrix *P* of order *n* such that $B=P^{-1}AP$.

Using adjacency matrices, we translate a problem in Graph Theory into a problem in Linear Algebra. The second one is not easier than the first one. To determine whether two given square matrices of the same order are similar is easy when both are diagonalizable. If they have the same eigenvalues, with the same respective multiplicities, then they have the same diagonalization, up to a re-ordering of the chosen eigenvectors. The set of eigenvalues (each one is written a number of times equal to its multiplicity; for example we write {1,1,2} if 1 is a double eigenvalue and 2 a simple eigenvalue). Suppose that diagonalizations of the matrices A_1 and A_2 exist and are given by $D = P_1^{-1}A_1P_1$ and $D = P_2^{-1}A_2P_2$, for appropriate invertible matrices P_1 and P_2 , then $A_2 = (P_1P_2)^{-1}A_1(P_1P_2)$ i.e. A_1 and A_2 are similar. If the matrices are not diagonalizable, similarity is harder to check. Of course, if one matrix is diagonalizable and the other is not, they are non similar. Note that the theorem sustaining the classroom activities is a "if ... then ..." theorem, not a "if and only if" theorem. If the graphs G_1 and G_2 are isomorphic, then their adjacency matrices have the same eigenvalues, but the converse is not true (see Cvetkovič et al. 1995, pages 61 sq.). The smallest known pair of non isomorphic graphs with the same spectrum is given by Skiena (1990, page 85); see Figure 2. Both graphs have two simple eigenvalues 0 and -2, and a triple eigenvalue equal 0, and their adjacency matrices are similar. The non existence of an isomorphism can be found at first glance: the graph in (a) is connected and the graph (b) is not.



(a) **(b)**

Figure 2: Non isomorphic graphs with the same spectrum.

The volume of the computations increases very fast with the number of vertices of the graphs. Here a Computer Algebra System reveals useful for technical assistance on computing. But not only for this assistance. Outsourcing of the computations to the CAS and careful observation of the output may yield a better understanding of the mathematical situation and enhance understanding of older knowledge. "Technology can be used to compute, to reinforce, clarify, anticipate, or get acquainted with ideas, and to discover and investigate phenomena" (Selden, 2005).

III. The study frame.

The Jerusalem College of Technology (JCT) is an Engineering School for High-Tech and Orot College is a Teacher Training College. In both institutions students learn a one-year course in Linear Algebra and an introduction to Graph Theory is given as part of a subsequent course in Discrete Mathematics. Matrix similarity belongs to the Linear Algebra syllabus. For various reasons, this topic has been taught at the very end of the course and quite no application to other fields of mathematics has been shown, beyond the fact that a basis change transforms the matrix of a linear transformation into a similar matrix.

Isomorphisms of graphs are an important topic in the syllabus. Conversations with colleagues teaching parallel courses revealed that students learn generally existence theorems related to degrees of vertices. Several textbooks do not mention more than this and the exercises are based either on the definition only or on such theorems about degrees of vertices (or in-degree and out-degree for directed graphs). Students are often reluctant to use adjacency matrices beyond writing the adjacency matrix of a given graph, or conversely drawing a picture of a graph whose adjacency matrix is given. "The computations are heavy", they say (for example, recall that paths of a given length n are counted using the n^{th} power of the adjacency matrix). Therefore a Computer Algebra System (namely Derive) has been used in the classroom activities, in particular the Linear Algebra algorithms. Note that the Linear Algebra packages of other CAS can assist this activity, sometimes with specific outputs. More than two thirds of the students in the class had a good CAS literacy, as a result of two previous courses strongly based on CAS use. Other students had an opportunity to improve their knowledge and their know-how with regards to software.

The central topic of the activities is new for the students. It has to be introduced and developed explicitly by the teacher, using *strategic scaffolding*, one of the scaffolding categories detailed by Hobsbaum et al (1996); Anghileri (2006, p. 36) elaborates on this issue. The main characteristics are:

- A measured amount of teacher support;
- A careful selection of the tasks and of their difficulty level;
- Students' ability to build a mathematical meaning from the given tasks;
- Explicit strategies.

The extended literature about scaffolding emphasizes the fact that scaffolding is relevant for one student-one teacher situations. Here the teacher had to provide such a scaffolding separately to every student, but also a form of "global" scaffolding to the class as a whole.

Within the global frame of the group, each student can have his/her own learning process. Therefore consolidation of knowledge (Dreyfus and Tsamir, 2004) has to be observed on an individual basis. As the work described in this paper is based on classroom activities, i.e. not in an individual frame, we will elaborate only briefly on the consolidation issue, in the last section.

For the CAS assisted part of the work, we refer to Fischer's (1991) didactical principle of *outsourcing operative knowledge and operative skills*. Peschek and Schneider (2001) regard operative knowledge as a means to generate new mathematical knowledge (see also Peschek 2005). In fact they distinguish three *fields of competence*: basic knowledge, operative knowledge and skills, and reflection. In the following activities, the needed basic knowledge is matrix similarity, acquired at the end of the Linear Algebra course. Because of a lack of time in this course, the topic has been shown but not applied in concrete situations. Students have now an opportunity to manipulate this knowledge in an applied situation. The operative skills are outsourced to the CAS. Most students (but not all of them) had already good operative skills for matrix computations using the CAS, including computation of eigenvalues and eigenvectors. During the sessions, they could improve these skills and discover new commands of the CAS. Moreover, new mathematical knowledge has been constructed, new CAS literacy being part of it.

IV. Classroom activities with CAS.

We present here classroom activities which took place with a group of 15 students (about 20 years old). Their course in Graph Theory comes one semester after the course in Linear Algebra. This enables them to use eigenvalues and diagonalization of matrices in a situation very different to what has been met either in Linear Algebra or in other courses with some geometric flavor. The students were already used to switch from iconic representation to algebraic representation and from algebraic representation to iconic representation.

1. First activity.

Consider the graph with two different vertex labeling given in Figure 1. The respective adjacency matrices are

$$A_1 = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{pmatrix} \text{ and } A_2 = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}.$$

Operative knowledge: using Derive's command **eigenvalues**, the students found that both matrices have the same five distinct real eigenvalues. Thus, the matrices A_1 and A_2 are diagonalizable, and for suitable eigenvector orderings, both matrices have the same diagonalization. It follows that the matrices are similar, whence $G_1 \cong G_2$.

Reflective thinking:

Mina: This is not new; we knew already that the graphs are isomorphic!
Vered: So what?
Mina: Why did we do all this work?
Vered: We are now convinced that our way of working is right. Not?

Silence for a while. The second student sees that something still "disturbs" the first student. So she adds:

Vered: We see always a trivial example when learning something new. So we are really sure that the theorem is right. I will do the same thing when I"ll teach.

This remark was important for the teacher. It shows that Vered is aware not only of the new mathematical knowledge she is currently leaning, but also of the structure of the educative sequence.

2. Second activity.

We consider the two graphs shown in Rosen's book (1999, p. 461, example 10); see Figure 3. The vertices of the graph G_1 will be denoted by u_k and the vertices of G_2 by v_k, $k = 1,...,8$.

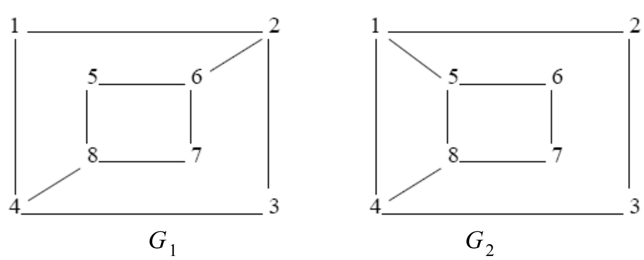


Figure 3: Two non isomorphic graphs

Reflective thinking:

Teacher: Let us check whether these graphs are isomorphic or not.
Vered: Easy! We check the degrees of the vertices.

Short silence, everybody computes.

Vered: These are the same degrees.
Teacher: So, what is your conclusion?

Leah: The graphs are isomorphic.
Short silence.
Hadas: Maybe not.
Vered: Why not?
Hadas: The degrees are not at the same place in the two graphs.
A couple of students, together: She is right!

The teacher asks for a clearer explanation of what happens. One student explains that the degree 3 vertices compose a connected subgraph in G_1, but not in G_2. This convinces the class that the two graphs are not isomorphic, but more than 10 students demand what they call "a stronger algebraic proof".

Operative knowledge:

Vered: Let's use matrices as we did before!
Teacher: Good idea, do it. Please write down the adjacency matrices.

The adjacency matrices of the two given graphs are

$$A_1 = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \end{pmatrix} \text{ and } A_2 = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \end{pmatrix}$$

With Derive's command **eigenvalues**, the students determine the eigenvalues of A_1.

The output is: 0, -1, 1, $\mathbf{\frac{1}{2}+\frac{\sqrt{17}}{2}}$, $\mathbf{\frac{1}{2}-\frac{\sqrt{17}}{2}}$, $\mathbf{-\frac{1}{2}+\frac{\sqrt{17}}{2}}$ and $\mathbf{-\frac{1}{2}-\frac{\sqrt{17}}{2}}$.

Reflection:

Teacher: Any comments?
Short silence.
Shira: There are not enough.
Teacher: Not enough what?
Shira: Not enough eigenvalues. There are only 7.
Teacher: What did you expect?
Myriam: Eight.
Teacher: So, what happened?
Short silence.
Vered: There must be one double.
Teacher: Why?
Vered: The matrix is symmetric, it must have a diagonalization.
Teacher: Very nice. How can we know who is the double eigenvalue?
Short silence.
Yael: (with a short hesitation) how can we compute the characteristic polynomial?
Myriam: It's a determinant, there must be a command.
Teacher: Right. Who knows?

Operative knowledge:

A couple of students answer that the command is **charpoly.** The teacher recalls the syntax. With this command, the result is

$$P(\lambda)=\lambda^{8}-\mathbf{10}\lambda^{6}+\mathbf{25}\lambda^{4}-\mathbf{16}\lambda^{2}=\lambda^{2}\left(\lambda^{6}-\mathbf{10}\lambda^{4}+\mathbf{25}\lambda^{2}-\mathbf{16}\right)$$

Vered: Here it is; it's 0.
Teacher: Vered, what is 0?
Vered: The double eigenvalue.
Myriam: How nice!!
Tehila: OK, but what do we do now?
Vered: The same thing with the other matrix.

The computation for A_2 is performed the same way, using Derive. The eigenvalues are

$$\frac{1}{4}\left(3+\sqrt{5}+\sqrt{22-2\sqrt{5}}\right),\frac{1}{4}\left(3+\sqrt{5}-\sqrt{22-2\sqrt{5}}\right),\frac{1}{4}\left(3-\sqrt{5}+\sqrt{22-2\sqrt{5}}\right),$$
$$\frac{1}{4}\left(3-\sqrt{5}-\sqrt{22-2\sqrt{5}}\right),-\frac{1}{4}\left(3+\sqrt{5}+\sqrt{22-2\sqrt{5}}\right),-\frac{1}{4}\left(3+\sqrt{5}-\sqrt{22-2\sqrt{5}}\right),$$
$$-\frac{1}{4}\left(3-\sqrt{5}+\sqrt{22-2\sqrt{5}}\right),-\frac{1}{4}\left(3-\sqrt{5}-\sqrt{22-2\sqrt{5}}\right).$$

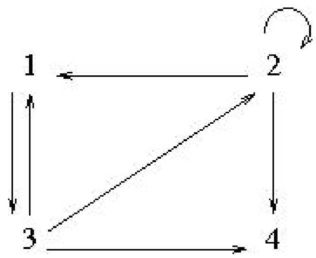
Reflection:
Teacher: What do you see?
Shira and Vered: (at the same time) they are different.
Teacher: Different from what?
Shira: From A_1 .
Vered: We did it! The matrices are not similar.

3. Third activity.

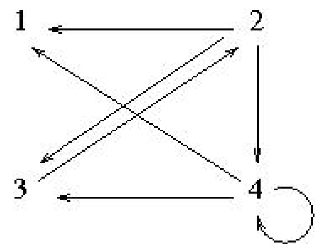
After the second activity, students received homework assignments (check whether couple of pairs of graphs are isomorphic or not). The next meeting took place one week later, with a third classroom activity. The task was to show that two given graphs with the same number of vertices and the same number of edges are non isomorphic. The teacher could let the students work on their own, and almost no intervention was necessary.

The next step in the same meeting consisted in turning students' attention towards similar situations, either with oriented graphs or with multigraphs. Precise definitions are given by Rosen (1999). One example is shown by Figure 4. The class dealt with the new situation using the same algebraic and technological tools, but in a different algebraic situation. The same CAS commands were used as in the previous sessions.

First example. The graphs in Figure 4 are given and the students are asked to check whether they are isomorphic or not.



The graph G_1.



The graph G_2.

Figure 4 : Two oriented graphs - first example .

The graphs are oriented graphs, and their respective adjacency matrices are not symmetric. We have:

$$A_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \end{pmatrix}$$

A few students note that the theorem on the diagonalizability of symmetric matrices cannot apply, and do not know how to proceed. A couple of students propose immediately to use the CAS. They determine the eigenvalues of A_1 and the eigenvalues of A_2: for matrices, the eigenvalues are $0, \frac{1+\sqrt{5}}{2}$ and $\frac{1-\sqrt{5}}{2}$.

<u>Myriam:</u> There are only three.
<u>Shira:</u> Yes, one is double.
<u>Myriam:</u> (Asks the teacher) We look for eigenvectors?
<u>Vered:</u> Yes, with the computer.

Most of the students determine the eigenvectors with the CAS and conclude that both matrices are diagonalizable, with the same diagonalization, whence the graphs are isomorphic.

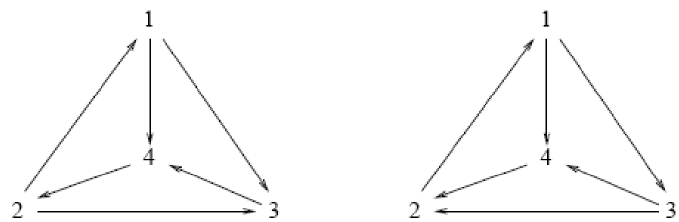
At this point, something interesting happens.

<u>Yael:</u> I computed the characteristic polynomial of the matrices. It is the same. So they are surely similar.

At the same time, one student says ``yes!!", and another one says ``No! you don't know!". A discussion follows, recalling that having the same characteristic polynomial is a necessary condition for matrices to be similar, not a sufficient condition. The student who said ``no", named Rachel, explains that they must look for eigenvectors.

Rachel: There exists a basis of eigenvectors for each matrix, therefore they are similar (she means ``the graphs are similar'').

Second example. Now the students are given the graphs displayed in Figure 5.



The graph G_1. **The graph G_2.**

Figure 5: Two oriented graphs - second example .

Their respective adjacency matrices are

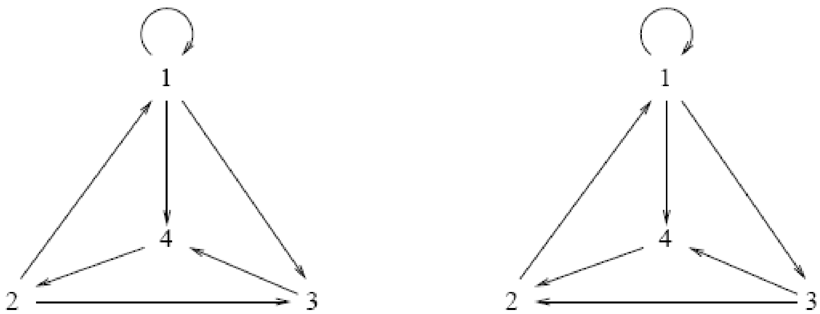
$$A_1 = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} \text{ and } A_2 = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Here all the students follow Yael's way and compute the characteristic polynomials. In both cases they obtain $P(\lambda) = \lambda^4 - 2\lambda - 1$. Using once again the software, they determine the eigenvalues. Most of them appear with very complicated expressions (complex numbers whose real part and imaginary part are given by non rational expressions).

Vered: It's ugly!
Teacher: Why?
Vered: Impossible to understand.
Teacher: Why?
Rachel: Complex numbers.
Teacher: Is this a problem, from an algebraic point of view?
Vered: But they are four.
Teacher: So, what is your conclusion?
Vered: We did not learn matrices with complexes, but ... (She waits a few seconds) this means that the matrices are diagonalizable?

Finally, the teacher has to explain that here the algebraic properties (for determinants, characteristic polynomial of a matrix, etc.) are the same over the reals and over the complex numbers. The class concludes that the two matrices are similar, whence the two graphs are isomorphic.

Third example. Finally, the teacher modifies slightly the graphs and gives to the students the graphs displayed in Figure 6



The graph G_1. **The graph G_2.**

Figure 6: Two oriented graphs - third example .

The respective adjacency matrices are

$$A_1 = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} \text{ and } A_2 = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

All the students but two compute immediately the characteristic polynomials. They are respectively $P_1(\lambda) = \lambda^4 - \lambda^3 - 2\lambda$ and $P_2(\lambda) = \lambda^4 - \lambda^3 - 2\lambda - 1$. The conclusion is shouted by three students at the same time: ``they are not similar!"

Teacher: Who are not similar?
Yael: The matrices.
Teacher: Why?
Naomi: (speaking for the first time) The polynomials are different, so the eigenvalues are different.
Teacher: All of them?
Naomi: No. At least one. And here we have 0 for A_1 and not for A_2.
Teacher: Remind us what the question was?
Yael: If the graphs are isomorphic.
Vered: OK, the graphs are not isomorphic.

3. Brief description of further activities.

Further work and activities have been done with the same class. After the second activity, students received homework assignments. The next meeting took place one week later. A central task was to show that two given graphs with the same number of vertices and the same number of edges are non isomorphic. For this, almost no teacher intervention was necessary. Another task was devoted to understanding the non-reversibility of the theorem described at the end of the first section (see Cvetkovič et al. 1995, pages 61 sq.); its description does not fit in this paper.

Part of the third meeting was devoted first to oriented graphs because they may provide non-symmetric adjacency matrices, i.e. matrices which can be non diagonalizable. Then the students had to study a pair of non isomorphic graphs having the same set of eigenvalues. The goal of this last example was to convince the students that the whole study relied on a one-way theorem: if two matrices are similar, then they have the same set of eigenvalues, but the converse is not true.

After the last meeting, the teacher had an informal discussion with the students. He asked for remarks about the CAS assisted work. Here are a few excerpts from the discussion between students (the first one was not previously quoted):
Student A: I'm sure that I would not have worked out all these examples by hand.
Student B: And so? You would not have learnt this?
Student A: No, I would have waited to see the answer from somebody else.
Student B: And so you would not have learnt the topic!
Student A: (hesitating) Maybe you are right, …, not so well.

V. Discussion.

1. The classroom activities.

In the first activity, the teacher chose an example where the isomorphism between the graphs is trivial. The graphical display itself proposes an invertible mapping between the sets of vertices. This enabled the students to discover how to work, in a situation where they have control on the results. Vered expressed this clearly. During this first activity, the students gained conviction that the working pattern is suitable. Therefore they were more independent from the teacher during the second activity. He helped somehow with passing from one step of reflective thinking to the next one, or with providing some new operative knowledge, such as an appropriate command of the CAS. The teacher's support was gradually faded; it was limited to questions. Reflection and interpretation were made by the students.

Teacher's support has been gradually removed during the third activity. At the end all the students but two were totally independent of teacher assistance. This has been checked with an assignment which included the study of one pair of non oriented multigraphs and of one pair of oriented multigraphs. Finally, the educative segment has been spread over a little more than two weeks, and gradually developed, meeting Anghileri's request (2006). The strategy has been made clear already from start:

- A progressive choice of examples: non oriented graphs, in order to have benefit of the theorem on the diagonalizability of symmetric matrices, then non oriented multigraphs and oriented graphs for which the theorem does not apply.
- Translation into notions from Linear Algebra and use of a CAS.

2. Switching between representations.

The original definition of a graph as a pair of sets (V,E), where $V=\{v_1,v_2,\dots,v_n\}$ is the set of vertices and $E=\{e_1,e_2,\dots,e_p\}$ is the set of edges, contains in itself a first kind of representation. Let us call this an *enumerative representation*. For small n and small p, it is possible to prove that two given graphs are isomorphic by *construction* of a specific isomorphism. Such a proof by construction becomes quickly unilluminating when the number of vertices and/or the number of edges increases. In a situation where the graphs are not isomorphic, besides the "boring" aspect of enumeration, there is a need to prove that all the cases have been considered (proof by exhaustion of cases). This is a formal proof, using combinatorial formulas, i.e. the point of view has been partly switched towards Combinatorics. Of course switching from the *enumerative representation* (V, E) to an *iconic representation* helps, but increasing *n* and/or *p* has a similar effect in the new setting as in the old one.

Considering only *iconic representations* of graphs does not yield enough insight into the concept of an isomorphism of graphs beyond simple examples, as suggested by Leah's reaction, and more by Student A in the last discussion. The matrix representation and its companion algebraic tools provide a possibility to have a more profound insight. "In some cases the current representations may prove an obstacle to the full development of a concept" (Ferrari 2003). Out of the record, students claimed that iconic representation is more readable for them, but others said that they felt more comfortable with matrix representation, as "they can do computations". A foreign colleague of the author said (free translation): "I observe everyday researchers in Discrete Mathematics and in its Teaching. I see that, most of the time, they work with iconic representations, and not with matrices. They find numerous theorems. And also a lot of consistent situations for the students to work (colors, Euler paths, etc.). But it's different from what you do, and one completes the other".

Despite the fact that matrix representation is more abstract than the graphical one, it opened the way to new mathematical knowledge, through manipulation both of old knowledge coming from another field and of the usage of a CAS. A great diversity of situations could not have been presented using graphical representation only (see Lesser and Tchoshanov 2005). Actually we may view the working sequence as a two-step activity:

a. Switching from the iconic representation to the matrix representation, according to Peschek (2005), as "one abstracts relationships from the (reference) context and presents them with symbols, thus outsourcing the problem in the formal-operative system of mathematics".
b. Outsourcing (part of the) operative knowledge to the computer.

Graphs, multigraphs (whether oriented or not) are defined as abstract objects, namely a pair of sets with a suitable property linking them (see Rosen 1999). We have here two presentations for a graph:

- The graphical presentation is visual/iconic (Lesser and Tchoshanov 2005) and acts as "stimuli on the senses" (Janvier et al. 1993).
- The other representation is algebraic. It is a symbolic representation enabling manipulations.

As noted by Lesser and Tchoshanov (2005), a single type of representation does not insure student learning and performance. In many occurrences, a graphical representation is used to "encode" more abstract properties. That is the case with the study of a function: the first steps, namely finding the domain and the possible symmetries, computing limits, derivatives, checking domains where the function is monotonous, where there are (eventually) extremal points and/or points of inflection, are then encoded into a graphical representation. It happens that this representation has to be fractioned into pieces, because an impossibility to represent all the special features in one graph (see Dana-Picard 2005). Here the *symbolic-algebraic representation* is a useful tool for the student to *understand the graphical situation* and to gain a more profound insight.

3. The switching direction.

We wish to emphasize an interesting aspect of the work. As mentioned in first section, similarity of matrices is a topic which had been taught in a previous course, but a lack of time enabled the teacher to give only a small number of examples. Students' personal work suffered also of this lack of time for practice. A similar situation occurs generally for the study of graph isomorphisms, but for a slightly different reason. The amount of necessary computations increases very fast with the number of vertices in the graph. So the teacher may decide either to limit himself/herself to examples of graphs with only a few vertices, or to present larger graphs but showing only the results. In both cases, students do not acquire practical skills; they have no real opportunity to improve their operative knowledge. Activities built on the switching between representations, iconic and algebraic, supported by CAS, enabled to really build new mathematical knowledge in both domains, Graph Theory and Linear Algebra, simultaneously. This enhances the fact that each topic can be viewed as belonging to a cognitive neighborhood of the other. Generally bridges are built in one direction, from topic A to topic B, but here the bridge between the two topics is traveled in both directions when switching from iconic representation to algebraic representation and conversely.

The CAS provided the help ``for reasoning by fostering the development of ... experimental reasoning style" (Sinclair et al. 2006). This appears through the intertwining of reflective thinking and application of operative knowledge during the sessions. A difference appears with the human support: not only the CAS assistance does not fade with time, but the new computing skills become an integral part of the new mathematical knowledge.

In the second activity, different CAS may give different outputs when displaying the eigenvalues. For the given square matrices of order 8, Derive gives seven different eigenvalues, inviting the student to understand that one of them must be a double eigenvalue. Note that other packages may give a more detailed output, including the multiplicities of the eigenvalues. We have here an example of the *double reference* evoked by Artigue (1997, page 152): on the one hand, the computer ``understands" the input in a way which can be different from the students' intention, on the other hand the mathematical meaning of the output can be different of what the student expects when he/she writes the same thing. See also (Lagrange, 2000).

Students working with a CAS become progressively acquainted with swapping between various representations: algebraic, numerical and graphical. For a given object, different representations can be provided by the CAS itself. Functions of one real variable are a well documented example, with numerical representation (a table of values), graphical representation and generally algebraic representation (a "closed form" such as $f(x) = \dots$) . The main problem is developing students' ability to link representations; see Pierce (2001). Prior to the activities described in this paper, the students had to solve a couple of exercises in reversed directions: a) write the adjacency matrix of a graph (resp. directed graph, given in iconic form, b) draw a picture of a graph whose adjacency matrix is given.

The activities performed here by the students have a different aspect. The main originality lies in the fact that the link is not oriented from an algebraic representation towards a graphical one as in most problems on one-real-variable functions, but in the other direction. Graphs are given by graphical representations and the representation used for checking the existence of an isomorphism between two graphs is purely algebraic. This may be technically trivial, but from a conceptual point of view, it is not trivial for the students: the work requires reversing the direction of the switch between representations. The students' hesitations reveal their level of ability to deal with a matrix representation instead of a graphical one. Previous working sessions revealed the difficulty for students to link matrices to graphs and graphs to matrices (including oriented graphs, i.e. links towards non-symmetric matrices) but helped with removing the obstacles. The CAS provided assistance, and students showed increasing operative knowledge.

4. Consolidation and routinization of previous knowledge.

In the same fashion we had to be careful when speaking about scaffolding, we must be careful if we wish to deal with *consolidation*. Both are very personal and apply to individuals, one student at a time. Here our study relies on the dynamic of a group of students. Each student has his/her own pace of acquisition of new mathematical knowledge, and consolidation should be checked with each student separately. The above classroom activities do not provide enough individual data.

Nevertheless, the observation of the group reveals various components of consolidation among those enumerated by Dreyfus and Tsamir (2004): immediacy, self-evidence, confidence, flexibility and awareness. For example, along the different activities, there were more and more immediate reactions to questions, either immediate answers (revealing also self-evidence) or immediate and correct outsourcing of the work to the computer. This last point is part of the ability to switch between different representations of the graphs (flexibility). From the beginning, Vered showed enough self-confidence to answer and ask, but for others like Naomi, the first intervention appeared during the third activity.

The activities revealed also the following fact: at the beginning, the students did not achieve for symmetric matrices and their diagonalization the routinization mentioned (and requested) by Artigue (1997). In Section II, we mentioned the lack of time at the end of the Linear Algebra course, which provoked a shortage in solved examples. Even for low dimensions, a lot of computations are needed, looking for eigenvalues and eigenvectors, inverting matrices, and so on. Hand computations are very unilluminating and both educators and students are reluctant to do them. The students had here an opportunity to make full computations of eigenvalues and eigenvectors, and sometimes of the diagonalization of a matrix. An important progress towards the requested routinization has been made as a byproduct of the activities. Moreover they had an opportunity to deal with a concrete problem involving these tools. The CAS was a facilitator, making examples of higher dimension possible to treat, thus enabling students to acquire an extended operative knowledge, and at the same time more mathematical insight.. The CAS has not been used as a black box, but rather as an assistant in a process of *reasoned instrumentation*. We meet Elbaz-Vincent's requirements (2005) about ``the necessity of developing specific classroom activities and specific exercise sheets, ..., showing clearly the value of the CAS either as a platform for experimentation or as an assistant ..."

CAS-assisted work had another side effect. For non isomorphic graphs, the following cases can appear:

a. The adjacency matrices have different characteristic polynomials.

b. The adjacency matrices have the same characteristic polynomial, whence the same eigenvalues with the same multiplicities, but one of the matrices is diagonalizable and the other one is not.

At the beginning, the command **eigenvalues** has been used without reference to the characteristic polynomial. The necessity to obtain more information, and to know how to interpret the output, has revealed the necessity of another command. During the activities, a black box has been opened and examined.

5. The role of CAS: further characteristics.

The assistance provided by the CAS is useful only if the students are able "to plan correct operations and to interpret results intelligently" (Fey 1990, quoted by Pierce 2001). Two remarks made by students emphasize this issue:

(i) In the second activity, Shira's remark on the number of eigenvalues is important. It has been provoked by Derive's output, where the eigenvalues are given, without mention of their respective multiplicities.

(ii) The meaning of Vered's claim "we did it" is non trivial. She noted that, despite the regular usage of a CAS to provide explicit numerical results, this time the actual eigenvalues of the matrices were quite irrelevant. The important issue was the comparison between the two sets of eigenvalues. Vered has understood that the fact that the eigenvalues are not the same is the important issue.

There are not so many opportunities to convince students that either the precise or approximate values of results are not the only interesting output. In this study, we found a couple of occurrences where the precise values of the matrix eigenvalues were not interesting. The point was in the comparison between the sets of eigenvalues. The outsourcing of the computations has an effect beyond the computations themselves. The CAS assisted activities described in section III are an example of the claim by Cuoco and Goldenberg (1996): "...we are talking about using technology in support of the hard thinking, not for performing the low-level details". More than acting as a calculator, the CAS worked here as an *assistant to reflection*.

Acknowledgement

The author wishes to thank Denise Grenier, from Grenoble (France) for valuable remarks, and for fruitful conversations at ICME 11 in Monterrey, Mexico.

References

Anghileri, J. (2006). Scaffolding practices that enhance mathematics learning, *Journal of Mathematics Teacher Education* 9, 33-52.

Arcavi A. (2003): The role of visual representations in the learning of mathematics, *Educational Studies in Mathematics* 52, 215-241.

Artigue, M. (1997). Le logiciel Derive comme révélateur de phénomènes didactiques liés à l'utilisation d'environnements informatiques pour l'apprentissage, *Educational Studies in Mathematics* 33 (2), 133-169.

Cuoco, A. and Goldenberg, E. P. (1996). A role for technology in mathematics education, *Journal of Education* 178(2), 15-32.

Cvetkovič D.M., Doob M. and Sachs H. (1995). *Spectra of Graphs*, Joann Ambrosius Barth Verlag, Heidelberg.

Dana-Picard Th. (2005a). Technology assisted discovery of conceptual connections within the cognitive neighborhood of a mathematical topic}, *Proceedings of* CERME 4, M. Bosch (ed), San Feliu de Guixols (Spain).

Dana-Picard Th. (2005b): Enhancing conceptual insight: plane curves in a computerized learning environment, *International Journal of Technology in Mathematics Education* 12 (1), 33-43.

Dana-Picard Th. and Kidron I. (2008). Exploring the phase space of a system of differential equations: different mathematical registers, to appear in the International Journal of Science and Mathematics Education.

Dreyfus T. and Tsamir P. (2004). Ben's consolidation of knowledge structures about infinite sets, *Journal of Mathematical Behavior* 23 (3), 271-300.

Duval R. (1999), Representation, vision and visualization: cognitive functions in mathematical thinking. Basic issues for learning. In F. Hitt & M. Santos (Eds.), *Proceedings of 21st Conference of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol.1, pp. 3-26). Cuernavaca, Mexico: PME-NA.

Elbaz-Vincent P. (2005). A CAS as an assistant to reasoned instrumentation, in D. Guin, K. Ruthven & L. Trouche (Eds.), *The Didactical Challenge of Symbolic Calculators: Turning a Computational Device into a Mathematical Instrument*, Mathematics Education Library 36. New York: Springer Verlag.

Ferrari P.L. (2003) Abstraction in Mathematics, *Philosophical Transactions of the Royal Society London* B 358, 1225-1230, available: DOI 10.1098/rstb.2003.1316

Fey, J.T. (1990). Quantity, in L.A. Steen (Ed.), *On the Shoulder of Giants: New Approaches to Numeracy*, 61-94, Washington: National Academy Press.

Fischer R. (1991). Mathematik und gesellschaftlicher Wandel, *Journal für Mathematik Didaktik*, 4, 323-345.

Grenier, D. (2008). Some specific concepts and tools of Discrete Mathematics, ICME 11, TSG 15, Monterrey (Mexico). Available: http://tsg.icme11.org/document/get/754

Hobsbaum A., Peters S. and Sylva K. (1996). Scaffolding in reading discovery, *Oxford Review of Education*, 22 (1), 17-35.

Janvier, C., Girardon, C., and Morand, J. (1993), Mathematical symbols and representations, in Wilson P. (Ed.), *Research ideas for the classroom: High school mathematics*, 79-102, NCTM.

Lagrange, J.-B. (2000). L'intégration d'instruments informatiques dans l'enseignement: une approche par les techniques, *Educational Studies in Mathematics*, 43, 1-30.

Lesser L. and Tchoshanov M. (2005), The Effect of Representation and Representational Sequence on Students' Understanding, *Proceedings of the 27th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education,* VA: Roanoke.

Peschek, W. and Schneider, E. (2001). How to identify basic knowledge and basic skills? Features of modern general education in mathematics, The *International Journal of Computer Algebra in Mathematics Education*, 8 (1), 7-22.

Peschek, W. (2005). the impact of CAS on our understanding of mathematics education. Retrieved on 20/11/2006 from CAME 2005 - The Fourth CAME Symposium Web site: http://www.lonklab.ac.uk/came/events/CAME4/CAME4-topic2-Peschek-paper.pdf

Pierce, R. (2001). Algebraic Insight for an Intelligent Partnership with CAS, in: H. Chick, K. Stacey, J. Vincent and J. Vincent (Eds.) *The Future of the Teaching and Learning of Algebra, Proceedings of the 12th ICMI* 3, 732-739, Melbourne: the University of Melbourne.

Presmeg N. (2006), A semiotic view of the role of imagery and inscriptions in mathematics teaching and learning, in Novotna J., Moraova H., Stehlikova N. (Edts) *Proceedings 30th Conference of the International Group for the Psychology of Mathematics Education,* Vol.1, 19-34, Prague: PME.

Rosen, K. (1999*). Discrete Mathematics with Applications*, Boston: McGraw Hill.

Selden, A. (2005). New developments and trends in tertiary mathematics education: Or, more of the same? International *Journal of Mathematics Education in Science and Technology*, 36 (203), 131-147.

Sinclair, N., Liljedahl, P. and Zazkis, R. (2006). A colored window on pre-service teachers' conceptions of rational numbers, International *Journal of Computers for Mathematical Learning*, 11, 177-203.

Skiena, S. (1990). *Implementing Discrete Mathematics: Combinatorics and Graph Theory with Mathematica.* Reading, MA: Addison-Wesley.