The Mathematics Enthusiast

Volume 6 | Number 3 Article 14

7-2009

SUM OF "N" CONSECUTIVE INTEGERS

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Sum of n Consecutive Numbers

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Theorem
For all n, it is always possible to find at least one sum of n consecutive numbers with an equivalent sum of $n-1$ consecutive numbers**?**

Until recently I did not realise that this wonderful pattern existed.

1+2=3
4+5+6=7+8
9+10+11+12=13+14+15
16+17+18+19+20=21+22+23+24
etc

My first thoughts on reading this connection in O'Shea's[1] book about number curiosities, was why had I not read about it before? It is simply beautiful. O'Shea writes but a few lines on it, and then moves on to his next strange fact. This made me wonder if the pattern would always be true, and here is my proof that it is.

Note that the LHS always starts with a square number. This will always be true, as square numbers occur in the natural number system as follows; 1, 4, 9, 16, 25, 36… with a common difference of 3, 5, 7, 9…..which you can see fits the pattern above. Therefore we can say that each line will always start with a square number. The first few lines in the pattern can be shown to be true, hence it can be proven that the pattern is true for all natural numbers, by considering that the next k th line in the pattern is true.

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The Montana Mathematics Enthusiast, **ISSN 1551-3440, Vol. 6, no.3**, pp.495- 496

$$k^2 + (k^2 + 1) + (k^2 + 2) + ... + (k^2 + k) = (k^2 + k + 1) + (k^2 + k + 2) + ... + (k^2 + k + k)$$

To show that LHS equal RHS, collect the k, k^2 terms on both sides

$$k^3 + k^2 + (1) + (2) + ... + (k) = k^3 + (k + 1) + (k + 2) + ... + (k + k)$$

Then collect $(1 + 2 + 3 + ... + k) = \frac{1}{2}k(k + 1)$ on both sides, giving

$$k^3 + k^2 + \tfrac{1}{2}k(k + 1) = k^3 + \tfrac{1}{2}k(k + 1) + (k) + (k) + ... + (k)$$

$$k^3 + k^2 + \tfrac{1}{2}k(k + 1) = k^3 + \tfrac{1}{2}k(k + 1) + k(k)$$

Therefore true for k, hence true for all and proof of the above theorem.

Corollary

In each line in this natural number pattern, we find a triangular, square and cube number sequence.

Angel Proof – "as if at a glance"

Between each pair of square numbers there are $2n$ numbers, n on the LHS and n on the RHS of the above pattern. By adding n to each of the n numbers on the LHS we obtain the RHS.

Reference

[1]The magic numbers of the professor by Owen O'Shea & Underwood Dudley [Page 135] Published by The Mathematical Association of America 2007