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Steve Humble

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Steve Humble[1]
The National Centre for Excellence in the Teaching of Mathematics in the North East of England

Is randomness merely the human inability to recognise a pattern that may in fact exist?

The three activities described show how order can be found in seemly random activities. The author has found that by using these activities on randomness, his students have developed a greater understanding of mathematical pattern and sequence. The teaching mathematical concepts in this way, engages and reinforces learning. It puts the ideas learnt into a setting and allows time for those ideas to be developed without any of the maths hang-ups which can sometimes occur in the classroom. By taking the maths beyond the classroom, we can more clearly illustrate the connections between the real world and what they are studying in school. In so doing students and teachers alike are enthused by the wealth of resources they have all around them in their own environments.

Understanding if events are random or have some underlying structure is a fascinating area of mathematics, filled with great discoveries. To understand whether the present spread of swine flu throughout the world has some structure, or is just random pockets of disease, will save lives. If there is a pattern, then finding this could enable countries to stop its spread. In the 1920s mathematicians Kermack and McKendrick (1) pioneered work into understanding if a set of results was randomly generated or had some underlying pattern. One of the first uses of these techniques was to predict the spread of disease.
As humans we find it very hard to deal with randomness. Psychologists call it Confirmation of Bias – with a new idea we attempt to prove it correct not wrong.
For example given the data 2,4,6 and asked to guess the rule, most people would say the numbers go up in 2`s in this pattern and so the next would be 8 and 10. Yet equally well it could be that the pattern is increasing and so the next numbers are 7 and 12.
The Philosopher Francis Bacon said “the human understanding, once it has adopted an opinion, collects any instances that confirm it, and though the contrary instances may be more numerous, either does not notice them or else rejects them, in order that this opinion will remain unshaken” (2)

[1] drmaths@hotmail.co.uk

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The concept of randomness is merely an attempt to characterise and distinguish types of sequences which confuse most people. It seems almost irrelevant to think about how it has been generated: flip of a coin, Geiger counter or practical joker. What matters is the effect on those who see it.
Which is more random, a string of heads or tails, or alternating heads and tails?

A game that was created by Walter Penney in 1969 (3), and is based on observing the occurrence of groups of heads and tails when repeatedly throwing a coin.
Let your opponent (1st player) select any sequence of three coins and then, referring to the table below, you choose the relevant 2nd players choice next to it according to the chart. You then record a sequence of coin throws looking for one of your three coin sequences in the long chain of throws, such as HTHTHHHHTHHHTTTTHTHH. The winner is the person whose pattern appears first.

1st player's choice	2nd player's choice	Odds in favour of 2nd player
HHH	THH	7 to 1
HHT	THH	3 to 1
HTH	HHT	2 to 1
HTT	HHT	2 to 1
THH	TTH	2 to 1
THT	TTH	2 to 1
TTH	HTT	3 to 1
TTT	HTT	7 to 1

At first glance you would think that the game is completely fair and not biased in any way, but in fact whatever sequence is selected by your opponent, you can always select a sequence which is more likely to appear first.

The maths of this order from apparent randomness can be seen by looking at the following three cases:

- If your opponent chooses HHH, you then choose THH (as in the table). The one time in eight that the first three tosses of the coin is HHH, your opponent wins straight away. Yet in all other cases, if HHH is not in the first three tosses of the coin, then THH will occur first.
- If your opponent chooses HHT, you then choose THH. The chance that HHT occurs first is conditional on either getting HHT or HHHT or HHHHT etc

$$\text{P(HHT before THH)} = \frac{1}{8}+\frac{1}{16}+\frac{1}{32}+.... = \frac{1/8}{\left(1-1/2\right)} = \frac{1}{4}$$

Hence $\text{P(THH before HHT)} = 1-\frac{1}{4}=\frac{3}{4}$

- If your opponent chooses HTH, you then choose HHT
  let x=P(HHT comes before HTH).
  Ignore any leading Ts, and if we look to see what happens after the first H. Half the time the next throw is H, and then HHT is more likely to occur before HTH.
  Half the time the next throw is T, but if this is followed by another T, we are back to the beginning, hence you can write $x=\frac{1}{2}+\frac{1}{2}\cdot\frac{1}{2}\cdot x$
  This gives, $\frac{3}{4}x=\frac{1}{2}$, so $x=\frac{2}{3}$

The reasoning behind the other cases follows in a similar way, by putting the first last. You can see from the table that your choice as the second player has a greater chance of appearing before your opponents in each case. This is why on average the second player should win over a group of say ten games.
As well as looking at the theory, students should be encouraged to play the game. This practical aspect of mathematical development is often overlooked in education and often leads to a richer understanding of the subject. (4)
A variation on Penney's Game is **The Humble Randomness Game** and uses a pack of ordinary playing cards. The game follows the same format using Red and Black cards, instead of Heads and Tails. Yet due to the finite number of cards in a pack you can show that the second players chance of winning is greatly increased.

When we know that the event is random how can we deal with choices?

The Game of Googol was invented by John Fox in 1958 (5). This game is played by asking someone to take as many slips of paper as he pleases, and on each slip write a different positive number. The number can be as small or large as they please, hence the name googol.
The slips are then turned face down and shuffled. You then, one at a time, turn the slips face up. The aim is to stop turning when you come to the number that you guess

to be the largest of the series. At no point can you go back and pick a previously turned slip. If at the end you have turn over all the slips, then you must select the final slip.

You may think that the chance of finding the correct slip is $\frac{1}{N}$, with N being the number of candidate slips. Yet this is far from true if you use a clever strategy.
Regardless of the number of slips in the Game of Googol, the probability of picking the largest number, using a good strategy, is around 37% or $1/e$

The game has generated many interesting applications, such as how to optimise the selection of your BEST partner, how to select the BEST job applicant and which is the BEST motel to stay at. (6)
A new application suggested by the author is **How to Optimise Your BEST Buy in the Sales**. The work of Psychologists and experimental economists has shown that people tend to stop searching too soon. Given you have 20 shops to visit how do you know when to make the purchase?
With the Game of Googol strategy you would first visit 7 shops, making a note of the BEST bargain up to this point. Then use this "BEST bargain so far" as a reference for future shopping. Once you find a better BEST bargain than the one you found in the first 7 shops, you buy it.
Here is how the theory works. Given N shops, select the BEST bargain in each shop you visit. Reject an initial number of r shops, and then choose the first BEST bargain which is better than all of the ones so far.

BEST bargain is in one of N shops each with a chance $\frac{1}{N}$

If BEST bargain is in the first r th shops, it is rejected, but if it is in the r+1th shop it is certain to be selected.
If in the r+2 th shop, you cannot be sure if it is selected or not. It will only be chosen BEST, if best so far is in the initial r shops.

Chance of this is $\frac{r}{r+1}$

If in the r+3 th shop, it will only be selected if best so far is in initial r, the chance of this is $\frac{r}{r+2}$ and so on.

$$P(r) = \sum_{k=r}^{N} \text{ P(K th bargain is the BEST).P(K th bargain is selected/it is the BEST)}$$
$$= \frac{1}{N}\left(1+\frac{r}{r+1}+\frac{r}{r+2}+\frac{r}{r+3}+\ldots+\frac{r}{N-1}\right)$$
$$= \frac{r}{N}\sum_{k=r}^{N-1}\frac{1}{k}$$

Since $\sum_{k=r}^{N-1}\frac{1}{k} \approx \int_{r}^{N-1}\frac{1}{k}dk = \ln\left(\frac{N-1}{r}\right)$ due to Euler-Maclaurin with approximate error of $\frac{1}{2}\left(f(N-1)+f(r)\right)$

Hence $P(r) \approx \frac{r}{N}\ln\left(\frac{N-1}{r}\right)$

To find the maximum, differentiate and set equal to zero

$$\frac{d}{dr}P(r) = \frac{1}{N}\ln\left(\frac{N-1}{r}\right) - \frac{r}{N}\left(\frac{1}{r}\right) = 0$$

$$\Rightarrow e = \frac{N-1}{r} \text{ or } \frac{r}{N-1} = e$$

In the short run, chance may seem to be volatile and unfair. Considering the misconceptions, inconsistencies, paradoxes and counter intuitive aspects of probability, it is not a surprise that as a civilization it has taken us a long time to develop some methods to deal with this. In antiquity, chance mechanisms, such as coins, dice and cards were used for decision making and there was a strong belief in the fact that God or Gods controlled the outcome. Even today, some people see chance outcomes as fate or destiny – "that which was meant to be"

It is in this world that the magician lives and it is these beliefs that he uses to help to create illusions. One such magic trick is to claim to know the position of all the cards in a random pack. A famous version was created by Si Stebbins in 1898. Stebbins was an American vaudeville performer who developed a system which requires you to arrange the cards and suits in a sequence. Each subsequent card in the sequence has a value three more than the previous one and the suits rotate in Clubs, Hearts, Spades and Diamonds order (known as "CHaSeD" order). This arrangement allows the magician to know the identity of a chosen card by glimpsing the next card, or determining the exact position of any card in the pack by a mathematical calculation, although many other properties of the system are known and have been applied to different card tricks.

3c 6h 9s Qd 2c 5h 8s Jd Ac 4h 7s 10d Kc
3h 6s 9d Qc 2h 5s 8d Jc Ah 4s 7d 10c Kh
3s 6d 9c Qh 2s 5d 8c Jh As 4d 7c 10h Ks
3d 6c 9h Qs 2d 5c 8h Js Ad 4c 7h 10s Kd

Given that
1=Ace
11=Jack
12=Queen
13=King

Notice that these four groups of thirteen cards have a number of patterns. When you put these groups together as a pack you can then cut the deck as many times as you wish as the cyclic order is still persevered.

Stebbins admitted, in one of his books, entitled Stebbins' Legacy to Magicians (1935) to having developed the system from the Spanish magician Salem Cid. Yet if we look back in history there have been examples of similar tricks to this being used. Variations in which the cards progress by five, four or three have been seen as far

back as early 17th century in books by the Portuguese writer Gaspar Cardoso de Sequeira and the Spanish writer Minguet. (7)
There are many developments you can make to this trick such as:

- When a spectator chooses a card from the deck the magician can easily find their card by looking for a break in the pattern
- Asking a spectator to cut the pack at any point and then by glancing at the bottom card you can know their chosen card.

The problem with the Stebbins system from a mathematical point of view is that it is very predictable and does not look like a random collection of cards.

1s, 4d, 3s,10c, 7c, 11s, 8s, 12c, 13d, 4c, 2d, 10h, 6s, 6d,
9d, 5c, 5h, 4s, 13h, 2c, 9c, 4h, 1c, 6h, 7h, 10d, 8d, 2s, 7s,
9s, 2h, 8h, 13c, 3d, 13s, 1h, 5s, 3h, 11d, 11h, 9h, 3c, 12s,
11c, 10s, 5d, 6c, 8c, 1d, 7d, 12h, 12d

s=spade (1)
h=heart (2)
c=club (3)
d=diamond (4)
1=Ace
11=Jack
12=Queen
13=King

This collection of cards is also in a sequence and by knowing the previous card you can determine the next just as with Stebbins arrangement. This sequence is better if you want to show the spectator the cards before you start the trick as it looks random. Good questions to ask students are:

Can you see the pattern?
How could you find a pattern, in such a seemly random collection of cards?

This is an interesting question from the point of view of trying to discover hidden secrets in our world. When a mathematician first sets out to try and discover how something works they may start from just this point with a collection of data which they believe holds a pattern and yet looks completely random.

To start to solve this problem a good thing to do is to look at small numbers in the pattern first.
1s gives 4d
2s gives 7s
3s gives 10c
Three times the card value plus one gives the next card value. Then can you find similar patterns with other suits?

Once you have found the connections between the valves you will then need to discover what the pattern is between the suits.
1s gives 4d Spade gives Diamond
1h gives 5s Heart gives Spade

1c gives 6h Club gives Heart

Students enjoy discovering hidden patterns and this work develops naturally into why we need to find order in a seemly random world.

Maurice Kendal points out that, man is in his childhood and is still afraid of the dark. Few prospects are darker than the future subject to blind chance! (8)

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