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Abstract: *Infinity has been one of the more difficult concepts for humanity to grasp. A major component of the research on mathematics education related to infinity has been the study of student's conceptions and reasoning about calculus subjects, particularly limits and series. Some related studies are about Cantor's ordinal and cardinal infinity. However since most students at the high school and college level are unfamiliar with symbolic representations and terminology, such as a set theoretic approach, a context (generally geometric) is used for investigating notions of infinity indirectly. In this paper we report on a study on the constructs of PhD Students about the notion of infinity. The aim of our study was to gain insight into the constructs about infinity held by PhD students and to investigate the effects of a graduate level set theory course on their informal models. We also propose repertory grid methodology as a way of capturing the constructs of students and argue that this methodology can help us to learn further details about the understanding of infinity.*

Keywords: Infinity; Potential infinity; Actual Infinity; Set Theory; Repertory Grid techniques; Teaching and learning set theory; Ordinality; Cardinality.

"*It seems that the various strategies that were used in the learning unit "Finite and Infinite Sets" did indeed enable the students to progress towards acquiring intuitions which are consistent with the theory they learned. However, we lack the means to evaluate these effects systematically. In order to proceed in devising instructional strategies that take into account the intuitive background of the learners we need to develop means to measure "degrees of intuitiveness*" Dina Tirosh (1991)

DIFFERENT CONCEPTS OF INFINITY

Today there are different concepts of infinity which are generally accepted by the mathematical community in spite of the rejections of many mathematicians since Aristotle.

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From the early history of mathematics two basic concepts of infinity have been actual and potential infinity. Fischbein (2001) identified actual infinity as what our intelligence finds difficult, even impossible to grasp: the infinity of the world, the infinity of the number of points in a segment, the infinity of real numbers as existing, as given etc. According to him when we mention the concept of potential infinity, we deal with a dynamic form of infinity. We consider processes, which are, at every moment, finite, but continue endlessly. For example we cannot conceive the entire set of natural numbers, but we can conceive the idea that after every natural number, there is another natural number.

Dubinsky et al. (2005) proposed an APOS analysis of conceptions of infinity. They suggested that interiorizing infinity to a process corresponds to an understanding of potential infinity, while encapsulating an object corresponds to actual infinity. For instance, potential infinity could be described by the process of creating as many points as desired on a line segment to account for their infinite number, whereas actual infinity would describe the infinite number of points on a line segment as a complete entity. Tall (2001) categorized the concepts of infinity as natural and formal infinities. He wrote: "Concepts of infinity usually arise by reflecting on finite experiences and imagining them extended to the infinite."

Tall (2001) referred to such personal conceptions as natural infinities. Natural concepts of infinity are developed from experiences in the finite world. He suggested that in the twentieth century to rationalize inconsistencies different formal conceptions of infinity are built by formal deduction. Contrary to the Cantorian counting paradigm that leads to Cardinal number theory, Tall (1980) suggested an alternative framework for interpreting intuitions of infinity by extrapolating the measuring properties of number. When the cardinal infinity asserts that any two line segments have the same number of points, regardless of length, a measuring intuition of infinity asserts that the longer of two line segments will have a "larger" infinite number of points. For example, if the line segment CD is twice as long as the line segment AB then CD has twice as many points as AB. Tall (1980) called this notion "measuring infinity" and suggested that it is a reasonable and natural interpretation of infinite quantities for students. He stated that "Infinite measuring numbers are part of a coherent number system extending the real numbers, including both infinitely large and infinitely small quantities. So it is consistent with non-standard analysis." Tall argued that experiences of infinity that children encounter are more related to the notion of infinite measuring number and are closer to the modern theory of non-standard analysis than to cardinal number theory. However Monaghan (2001) suggested that this can only be described as a possible trend in older children's thought.

Since measuring infinity can be thought in the non-standard view, a well-organized view to different conceptions of infinity comes from Tirosh. In her study about teaching Cantorian theory, Tirosh (1991) categorized the term infinity in different contexts as potential infinity (representing a process that is finite and yet could go on for as long as is desired), actual infinity (in the sense of the cardinal infinity of Cantor or ordinal infinity, also in the sense of Cantor, but this time representing correspondences between ordered sets) and non-standard infinity (which arises in the study of non-standard analysis, and unlike the others, admits all the operations of arithmetic, including division to give infinitesimals). These themes about the conceptions of infinity which were observed before guided this study and repertory grid methodology was applied in the light of these identified conceptions.

RESEARCH ON UNDERSTANDING INFINITY

In most of the research about infinity, students have been given some tasks generally related with calculus and geometry subjects (limit, series, straight line, point etc.) and researchers have looked for students' difficulties in understanding different mathematical concepts as infinitesimal calculus, series, limits etc. Particularly the focus is on the difficulties in understanding actual infinity (Borasi, 1985; Falk, 1986; Fischbein, Tirosh & Hess 1979; Jirotkova & Littler, 2003; Jirotkova & Littler, 2004; Monaghan, 2001; Petty, 1996; Sierpinska, 1987; Taback, 1975; Tall, 1980)

Petty (1996) attempted to explore the role that reflective abstraction plays in the individual's construction of knowledge about infinity and infinite processes. He gave four undergraduate elementary education students some tasks involving iterative processes generally from calculus and geometry. Comparing 0.999… and 1, folding a rectangular sheet of paper continuously and drawing polygons with an increasing number of sides inside a circle are some examples of the tasks. Students are interviewed as they attempted to resolve problematic situations involving infinity and infinite processes. Based upon four detailed student case studies, he found that as students begin to relate their solution activity to previous tasks, higher levels of reflective activity could be observed.

In Piagetian-constructivist parlances some models have been developed to determine the understanding levels of limit and infinity. In order to clarify the role of reflective abstraction in his study Petty (1996) used one of these models developed by Robert (1982) and Sierpinska (1987). This model involves three stages:

Stage 1. Static Concept of Limit

a. The individual's perception of a limit is in finite terms.
b. For the individual, infinity does not exist; everything is finite and definite.
c. If infinity does exist, all that is bounded must be finite and definite.

Stage 2. Dynamic Concept of Limit

a. The individual has a perception of limit as a continuous unending process; the limit is a value which is approached yet never attained.
b. For this individual, infinity exists and involves recognition of potential infinity as opposed to actualized infinity. It has a contextual aspect and may involve a transitional phase.

Stage 3. Actualized Infinity

a. The individual has conceptualized infinity as a mathematical object.
b. Infinity is treated as a whole and definite object.
c. The individual believes that it is possible to predict the outcome of an infinite process.

Based on the study, Petty altered the three-stage model to a four-stage model by separating third stage. He wrote:

> "If a student acknowledges that the infinite repetition of 0,999… is identical to 1, this individual is accepting that an infinite process has a predictable outcome. However, this does not infer that this same individual has conceptualized infinity as a mathematical object. Whereas, this individual will perhaps have attained the Actualized Infinity level, it would not be sufficient evidence to suggest that this person is able to perform operations on or with their conception of infinity."

The four stages of this model are: the Static Level, the Dynamic Level, the Actualized Infinity Level, and Infinity as a Mathematical Object Level. He found that none of the students interviewed for this study had attained either the third or fourth level of conceptual development. Again in the Piagetian view Dubinsky et al. (2005) applied APOS theory to suggest a new explanation of how people might think about the concept of infinity. They proposed cognitive explanations, and in some cases resolutions, of various dichotomies, paradoxes, and mathematical problems involving the concept of infinity. These explanations are expressed in terms of the mental mechanisms of interiorization and encapsulation. As we mentioned before, they characterized the process and object conceptions of infinity by APOS theory that correspond to an understanding of potential and actual infinity. In his article about Cantors' cardinal and ordinal infinities, Jahnke (2001) emphasized that Cantor was the first to use the concept of pair wise correspondence to distinguish meaningfully and systematically between the sizes of infinite sets. Dependent on this usage, many studies involve Cantors' cardinal and ordinal infinity concern the comparison of infinite quantities (Fischbein, Tirosh & Hess 1979; Duval, 1983; Tirosh, 1985; Martin & Wheeler, 1987; Sierpinska, 1987; Tirosh and Tsamir, 1996; Tsamir, 2001).

One such study is Tirosh's (1985) dissertation in which she gave 1381 students (in the age range 11-17 years) 32 mathematical problems related with a comparison of infinite quantities. In each of these problems two infinite sets, with which the students were relatively familiar, were given. The students were asked to determine whether the two sets were equivalent and to justify their answers. She found that students' responses to the problems were relatively stable across the age group.

The main argument used by the students to justify their claim that two sets have the same number of elements was "All infinite sets have the same number of elements". For example 80 % of the students claimed that there is an infinite number of natural numbers and also an infinite number of points in a line. The claim that two infinite sets were not equivalent was justified by one of the following three arguments:

1."A proper subset of a given set contains fewer elements than the set itself." For example, 51% of the students used this argument claimed that there are fewer positive even numbers than natural numbers.

2."A bounded set contains fewer elements than an unbounded set." For example, 12 % of the students used this argument claimed that the number of the points in a square is greater than that in a line segment.

3."A linear set contains more elements than a two dimensional set." For example, 38 % of the students used this argument claimed that there are more points in a square than in a line segment.

She found that a small percentage of the students (less than 1 %) intuitively employed the notion of 1-1 correspondence and the intuitive criteria that the students used to compare infinite quantities and theorems of set theory were inconsistent with each other.

In order to examine the impact of the "Finite and infinite sets" learning unit on high-school students' understanding of actual infinity, Tirosh and her colleagues taught this unit to students aged 15-16 in four tenth-grade classes. Questionnaires employed after the lessons showed that 86 % of the students acquired the basic concepts in the learning unit. Based on the study, she claimed that learning unit on infinite sets may be introduced without particular difficulty starting from the tenth grade.

Duvals' (1983), Tirosh and Tsamirs' (1996) observations were related to representations of the same comparison-of-infinite-sets task. They studied students by giving different representations concerning the equivalency of two infinite sets and evaluated their responses. They found that students decisions as to whether two given infinite sets have the same number of elements largely depended on the representation of the infinite sets in the task.

Tirosh (1991) also summarized the results of some late psycho-didactical studies about Cantorian set theory. According to her research, she found that:

1. There are profound contradictions between the concept of actual infinity and our intellectual schemes, which are naturally adapted to finite objects and finite events. Consequently, some of the properties of cardinal infinity, such as the fact that $\aleph_0 + 1 = \aleph_0$ and $2\aleph_0 = \aleph_0$ are very difficult for many of us to swallow.
2. Intuitions of actual infinity are very resistant to the effects of age and of school-based instruction.

3. Intuitions of actual infinity are very sensitive to the conceptual and figural context of the problem posed.
4. Students possess different ideas of infinity which largely influence their ability to cope with problems that deal with actual infinity. These ideas are usually based on the notion of potential infinity.
5. The experiences that children encounter with actual infinity rarely relate to the notion of transfinite cardinal numbers. But they do have increasing experiences in school of quantities which grow large or small.

In her study, Tsamir (2001) described a research-based activity which encourages students to reflect on their thinking about infinite quantities and to avoid contradictions by using only one criterion, one-to-one correspondence, for comparing infinite quantities. According to several studies that involve tasks concerning the comparison of infinite sets, Tsamir (2001) says that when students are presented with tasks involving comparison of infinite sets, they often use four criteria to determine whether a given pair of infinite sets are equivalent: the part-whole criterion (e.g. "A proper subset of a given set contains fewer elements than the set itself"), the single infinity criterion (e.g. "All infinite sets have the same number of elements, since there is only one infinity"), the 'infinite quantities-are-incomparable` criterion and the one-to-one correspondence criterion (as in Fischbein, Tirosh & Hess 1979; Duval, 1983; Borasi, 1985; Martin & Wheeler, 1987; Moreno & Waldegg, 1991; Sierpinska, 1987; Tirosh & Tsamir, 1996)

Even if Cantorian set theory is the most commonly set theory of infinity, it has a limited place in math curricula. Topics in Set theory require the full knowledge of the concept of (actual) infinity and they have been thought to be difficult for students. Thus pedagogical studies in this area have been generally with upper secondary and university students and even in this setting it has been hard for researchers to investigate students' understanding about all concepts of infinity, potential, actual, non-standard etc. In the remainder of this paper, we report on our study with mathematicians who are PhD students, in which we look for their constructs according to all known concepts of infinity as reported in extant studies. We claim that we can get more clear results about understandings of infinity by studying with mathematicians who can live in the actual world of infinity.

REPERTORY GRIDS

Repertory Grid Techniques is a methodology originally developed by Kelly as a research tool to explore people's personalities in terms of his Personal Construct Psychology (Kelly, 1955). It can be thought as a type of structured interview technique that guides individual's ability to compare elements to elicit constructs (attitudes, category making, assessing criteria and probably some personal tacit knowledge). This technique attempts to elicit constructs which the subject uses to give meaning to his or her world, and to have the subject rate items (elements) of their experience in terms of these elicited constructs (Williams, 2001). Although its original use was to investigate constructs about people, recent applications have included events, situations and abstract ideas. Pope and Keen (1981) point out that the technique has evolved into "methodology involving high flexible techniques and variable applications" (p. 36) Many researchers and writers of handbooks have explained these techniques and applications (Cohen et al, 1994; Fransella & Bannister, 1977).

Williams (2001) used repertory grid methodology together with a predicational view of human thinking to describe the informal models of the limit concept held by two college calculus students. In his thesis, he explained the three aspects that answer the question "Why Repertory Grid Methodology is appropriate for his study?"

1. The underlying theory, namely personal construct psychology, is essentially Kantian in general orientation and dependence on a dialectical viewpoint and hence in harmony with logical learning theory.
2. The method attempts to have subjects themselves express the constructs they use, rather than having an observer interpret from their protocols the thought processes they employ.
3. Repertory Grid Techniques retain the flavor of qualitative methodology while at the same time providing the opportunity for quantitative analysis. Techniques for such analysis are widely used and easily available.

This methodology is particularly based upon the notion of construct. Methodologically, Kelly (1955) described a construct as "a way in which two elements are similar and contrast with the third" (p.61). To elicit a construct we need at least two elements (or statements). Elements can be people at students' environment or events, things that are related to student. Relatives of student, mathematical meanings are some examples. Elements are chosen to represent the area in which construing is to be investigated. If it is interpersonal relationships, the elements may well be people (Fransella & Bannistar, 1977). A common method of eliciting a construct is to present subjects with three items chosen from a list of interests and ask them to designate how two of the items are similar, and therefore different, from the third. In other cases, subjects may be asked to simply compare and contrast two items as in this study. In either case the result is a construct with two poles, each one represented by one of the contrasting items (Williams, 2001). To fill the rating form of the repertory grid, after the elicited constructs are written on the grid, the subject rates the elements. The Grid is a matrix of elements (items) by constructs with ratings of the elements in terms of how well the constructs apply to each other.

OVERVIEW OF THE STUDY

In this study 4 mathematicians with experience in their understanding of the notion of infinity from Gazi University, Turkey were chosen as the subjects. These mathematicians were PhD students enrolled in set theory course in the first semester of the 2006/2007academic year. All students except one of them graduated from math departments of education faculties in Ankara. One of them is graduated from the math department of science and art faculty again in Ankara. Students had no prior experience in Cantorian set theory but they had many experiences concerning infinite sets and infinity. Although we tried to report the common constructs between all of the students, this report focuses on 2 of the 4 students from the original study whom we will call the first and the second student. We thought that they are good examples of the study.

The set theory course consisted of 30 lessons spread over 15 weeks and was subdivided into sections finite and infinite sets, countable and uncountable sets, Zorns' lemma, cardinality, ordinality etc. A special attempt was made, through out the lessons, to interact with the students' intuitive background in regard to infinity and to change their primary intuitive reactions. During each lesson, students' definitions of infinity were explored and discussed, so that their personal constructs of infinity modified and refined over 15 weeks. During the lessons, students were asked to respond to a series of tasks aimed at moving their informal concepts of potential infinity toward an actual and cantorian infinity. The tasks students worked on included quizzes that are composed of set theory infinity problems in a typical textbook (Schaums & Halms) that assess students' technical competence; discussing contrasting opinions about infinity as voiced by the students and the instructor; working problems which are also asked to picture different concepts on a paper, for example picturing a 1-1 correspondence or cardinality on a paper. The focus was on meeting students with different concepts of infinity. Beside the interviews, all of the lessons were recorded by the researcher and the questioning and interviewing that surrounded these tasks formed another corpus of data against the repertory grids from which we elicited construct relationships.

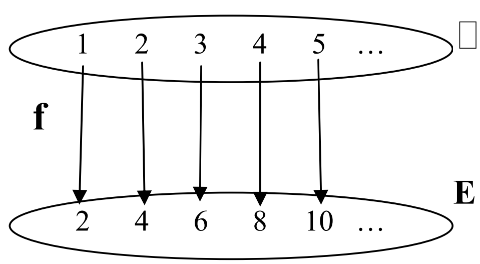
In order to investigate the changes of informal models initial and final repertory grids were elicited before and after the course. The data reported here is primarily from these repertory grids with clarifying evidence coming from transcripts of the lessons.

Eliciting elements (statements): A series of 12 written statements about infinity were used in the elicitation of a repertory grid. They were based on themes obtained from students' statements and past studies about infinity. We emphasized four themes in this study and prepared the statements (elements) according to these themes. These themes are; potential infinity (**P**), actual infinity (**A**) which can be Cantors' cardinal (**Card. A)** or ordinal **(Ord. A**) infinity, non-standard infinity (**N**) and measuring infinity (**M**).The aim was to provide students with statements in which each of different concepts of infinity were present. They were made as broad as possible to encourage the elicitation of constructs. The statements were also designed to differ in their degree of showing the concepts of infinity; some more closely approximated the potential infinity and others are more actual … They were subjected to pilot testing with other mathematicians, master degree students and were substantially rewritten. They were also reviewed by two university lecturers (Prof. Dr. and Assoc. Prof.) who give set theory course to ensure that all different concepts of infinity occurred in the statements and no important themes were missed. The statements are listed below.

========================ELEMENTS==============================

Statement 1 (Themes -A, P):

Since the number of elements in any set is bigger than the number of elements in its proper subset, the number of elements in Natural Numbers Set $\mathbb{N}$ is bigger than the number of elements in Even Numbers Set **E**.



However, because of there exists an injection f:N → E, f(x) = 2x between N and E, it can be thought that the number of elements in N is equal to the number of elements in E which is a proper subset of N. From this point of view S (E) = S ($\mathbb{N}$) and both sets have infinite number of elements.

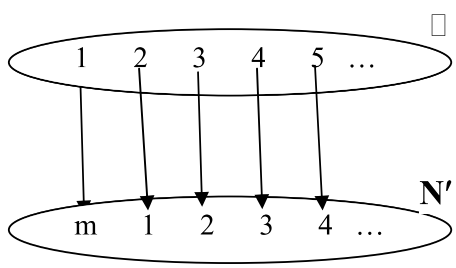
Understanding infinity as a number causes this contradiction here. Whereas infinity is not a number, it is a concept that shows the inexhaustibility of elements in the set. The symbol ∞ is used in subjects like limit instead of numerical quantities that we can not express, in fact there is no number which is called as infinite.

Statement 2 (Themes P, M):

$\lim_{n\to\infty}\left(\frac{n^2}{n^2+1}\right)=1$. Because as n tends to infinity $\left(\frac{n^2}{n^2+1}\right)\to\frac{\infty}{\infty}=1$. In the same way, it can be found a similar result for $\lim_{n\to\infty}\left(\frac{n^5}{(1,1)^n}\right)$. The only difference here is that the denominator is growing faster and goes to infinity more quickly. '$n\to\infty$' expression states that n becomes bigger continuously, however '$n\to-\infty$' expression states that n becomes smaller continuously. Infinity is used to express the greatness of an inexhaustible quantity, but it can not be mentioned about any infinite number or different infinities.

Statement 3 (Theme A):

If the element m which is not a natural number is added to Natural Numbers Set, there will be a 1-1 mapping g between Natural Numbers Set and the new constructed set **N′** = {m,1,2,3,…} that is defined at the following. So the number of elements in N′ is equal to the number of elements in $\mathbb{N}$.



$$g(x)=\begin{cases} m, & \text{if } x=1 \\ x-1, & \text{if } x>1 \end{cases}$$

If number of elements in these sets are added we come to a conclusion S($\mathbb{N}$) = S(**N′**) = S($\mathbb{N}$) + 1 =1+S($\mathbb{N}$). If the number of elements in the infinite set N is shown by **a**, the statement will be **a** = **a**+1 = 1+**a**. We must not think infinite numbers like finite ones. Maybe this statement seems to us strange because of our prejudices belong to finite numbers.

Statement 4 (Theme P):

There are an infinite number of points on a line. Also Natural Numbers Set has an infinite number of elements. Since it can not be mentioned about different infinite numbers, the number of points on a line is equal to the number of elements in Natural Numbers Set. We can show this by a 1-1 mapping between natural numbers and all of points on a line

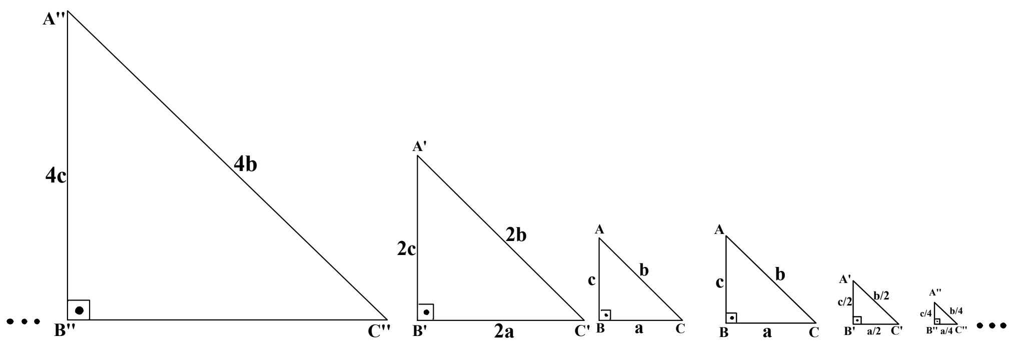
Statement 5 (Theme A):

In the open interval (0, 1) there is infinite number of real numbers and infinite number of rational numbers. Since $A=\{x \mid x \in (0,1) \text{ and } x\in\mathbb{Q}\}$ and B = (0, 1), $A \subset B$. So the infinite number that shows the number of real numbers in the interval (0, 1) is bigger than the infinite number that shows the number of rational numbers. Even if there are different finite numbers, also there are different infinite numbers. But infinite numbers must not be thought as finite numbers. The arithmetic operations valid for finite numbers are not valid for infinite numbers in the same way.

Statement 6 (Themes N, M):

Since $[0, 1] \subset [0, 2]$ for A=[0, 2] and B=[0, 1], S(A) > S(B). Same thing is valid for the numbers of points that are exist on different line segments which have different lengths. The number of points on a line segment of length 4 cm is bigger than the number of points on a line segment of length 3 cm.

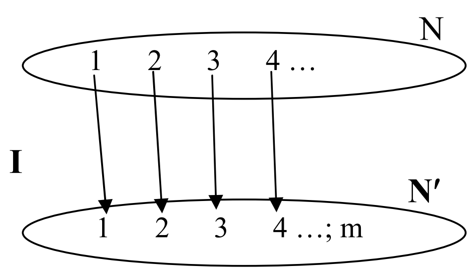
Statement 7 (Themes N, M):



As above, suppose that new right triangles are continuously formed by doubling or getting half of the lengths of the sides of $\overset{\Delta}{ABC}$ right triangle. If this process lasts forever in both ways, we obtain two triangles which have areas infinitely small and infinitely large at the end. Areas of these triangles would be never zero or very big number. These kind of quantities are expressed by infinite and infinitely small numbers that are obtained by the extension of real numbers.

Statement 8 (Theme Ord. A):

Suppose that the ordered set **N′** = {1, 2, 3, … ; m} was formed by adding a non natural number m to the Natural Numbers Set and in **N′** m is in order after all natural numbers. At the following **I: N → N′**, **I (x) = x** identity function is the insertion mapping of **N** into **N′** that preserves order.

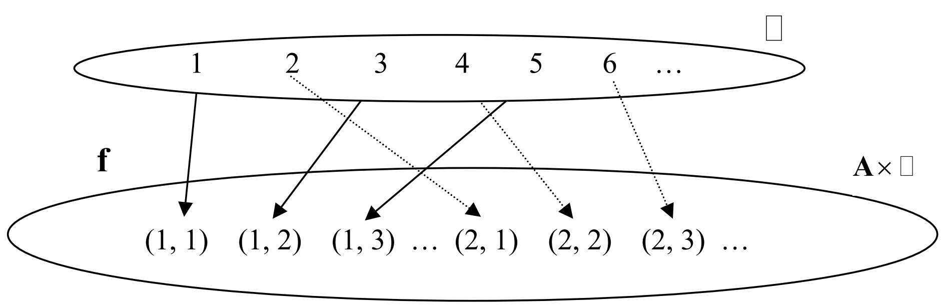


Although these sets are equivalent infinite sets, they are different. Any mapping between these sets that preserves order as $x < y \rightarrow f(x) < f(y)$ can not be both 1-1 and surjective. When an element is added to an infinite set, although the number of elements (cardinality) does not change, functions that preserves order give different meaning to these sets. Thus it can be said that different infinite sets that have different orders and different numbers represent these sets but they have the same number of elements (cardinality).

<u>Statement 9 (Theme Card. A):</u>

For A={1, 2} and B={a, b, c}, $A\times B$={(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)} and S $(A\times B)$ = S(A)**.**S(B). In the same way for A={1, 2} and $\mathbb{N}$ ={1, 2, 3, 4,…}, $A\times\mathbb{N}$ ={(1, 1), (1, 2), (1, 3), …; (2, 1), (2, 2), (2, 3), …}. According to the mapping f defined at the following, it can be seen that the number of elements in $(A\times\mathbb{N})$ is equal to the number of elements in $\mathbb{N}$.

$$f(x)=\left\{\begin{array}{ll}(1, x) & \text{, eğer } x=2k-1 \text{ ve } k\in\mathbb{N}^{+} \text{ ise}\\ (2, x) & \text{, eğer } x=2k \text{ ve } k\in\mathbb{N}^{+} \text{ ise}\end{array}\right\}$$



Since S(Ax$\mathbb{N}$) = S(A)**.**S($\mathbb{N}$), **a** = 2**.a**

<u>Statement 10 (Themes -N, P):</u>

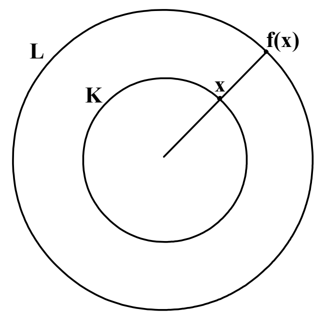
A number α which is smaller than all positive real numbers is called infinitely small. But no number α on the real line could be 'arbitrarily small'. If $r > \alpha > 0$ for all positive real numbers r, then $(\alpha/2) > 0$ is positive and even smaller than α and we can not call α an infinitesimal. As a result, infinitesimal and infinite concepts which are used to express only very small and very big numbers. We can not mention about these kind of numbers.

Statement 11 (Theme A):

For circumferences $K= \{(x, y) \mid x^2 + y^2 = 1\}$ and $L= \{(x, y) \mid x^2 + y^2 =4\}$, even if their radiuses' lengths are different, the numbers of points that construct these circumferences are equal to each other. At the following, for every point on the small circumference you can find a point on the big circumference to map. In polar coordinates as you see, you can map every point $(\cos\alpha, \sin\alpha)$ on K to the point $(2\cos\alpha, 2\sin\alpha)$ on L.

$f: K \to L$,

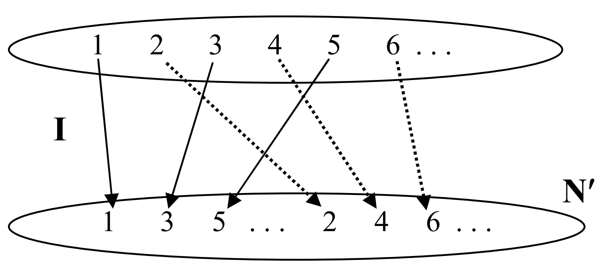
$f(x, y)=\left\{(2\cos\alpha, 2\sin\alpha) \mid x=\cos\alpha,\ y=\sin\alpha,\ 0<\alpha<2\pi\right\}$



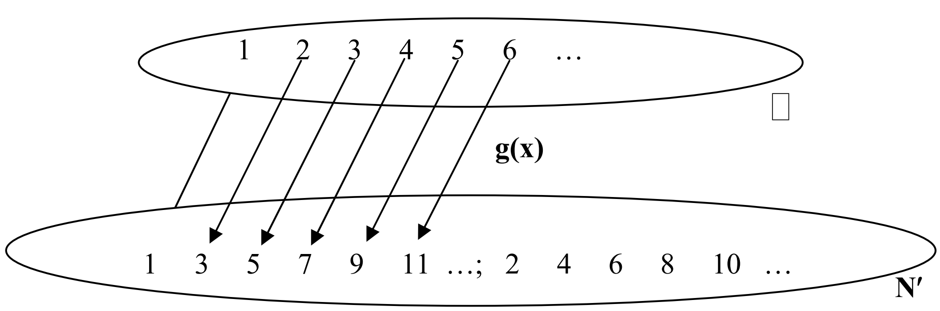
By a 1-1 mapping f, you can see that numbers of the points on two circumferences are equal to each other. That is to say the number of elements in K is equal to the number of elements in L. They have the same cardinality.

Statement 12 (Themes Card. A, Ord. A):

At the following, **I(x) = x** identity function is a 1-1 mapping between the infinite sets $\mathbb{N} = \{1, 2, 3, 4, 5, 6, \dots\}$ and $\mathbb{N} = N' = \{1, 3, 5, \dots; 2, 4, 6, \dots\}$ which has a different order.



Thus, both of sets have infinite elements and numbers of elements in these sets are equal. But, there can be also an injective mapping between these sets that preserves order which is defined as $g: \mathbb{N} \to N'$, $g(x) = 2x - 1$.



In this condition, although both of them are infinite sets, **S(N′) > S(□)** because there are enough elements until 2 in **N′** for 1-1 mapping with N. In these examples, f does not preserve order but g preserves order as $x < y \Leftrightarrow g(x) < g(y)$. As a result, according to conservation of order, you can meet with different results that are related to infinite sets.

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Eliciting Constructs: At the beginning of the course and at the end of the course, constructs were elicited by structured interviews using these 12 infinity statements. A subject was given two of the statements and asked to describe how the two statements were alike or different. Since the statements (elements) are very long, dyad method was chosen for this study. The subject responded with verbal descriptions of what was alike or different, which the interviewer recorded. Clarification was asked for as needed such as when the verbal descriptions of two constructs seemed to be the same. The interviewer chose a few of the subject's words to use as label for the emergent construct and, when offered by the subject, a label to stand for the opposite construct. Also subjects were allowed to omit some constructs which are meaningless for them. Every attempt was made to ensure that the construct elicited were not suggested to the student. When all similarities and differences between one pair of statements were written as constructs, all 12 statements were rated by the subject on a 5-point scale. One pole is arbitrarily assigned a rating of 1, it's opposite 5. This procedure was then repeated for a second pair of statements, and the process continued until the interviewer and the subject both agreed that they were unlikely to elicit new constructs.

Pairs of statements were presented to all subjects in the same order. The pairs (**(1-7), (2-5), (3-8), (6-11), (7-10), (4-11), (9-12), (4-5))** were chosen because they seemed to represent different concept of infinity in twos. The number of constructs elicited varied across subjects, and not all the statement pairs elicited new constructs. Subjects were also asked to rate all 12 statements on two constructs supplied by the researcher: whether the statement was true and whether the statement is good according to subject. The interviews typically lasted about an hour, with the subjects examining five to eight pairs of statements. At the end of the procedure a matrix of comparisons was produced, usually with scored ratings for each element in terms of the elicited constructs. It is a matrix in which each row represented one construct and consisted of a series of 12 rankings (one of each statement) on a 5-point scale. A rank of "5" for a statement indicated that the emergent (left-hand) pole of the construct applied strongly to that statement, whereas a rank of "1" implied that the opposite pole of the construct applied strongly to the statement (Williams, 2001).

Analysis Of Grids: There are myriad methods of analyzing the grids by hand or by computer. We analyzed the grids by a simple hand method, comparing rating patterns in successive rows by calculating the difference scores. The ratings for each element were subtracted and the differences for the row were added to provide a difference score in which low numbers indicate close relationships between constructs and high scores indicate negative relationships. The actual range of possible difference scores would depend on the number of elements and the size of the rating scales used (Bannister and Mair, 1968). In this instance the possible range of difference scores is from 0 to 48. The construct pairs which have the difference scores 0 and 48 are at given below.

Emergent Construct	ELEMENETS (STATEMENTS)												Opposite Construct
	1	2	3	4	5	6	7	8	9	10	11	12	
True	1	1	4	3	4	3	3	3	5	2	3	5	False
Infinity is a concept	1	1	4	3	4	3	3	3	5	2	3	5	Infinity is not a concept
Differences	0	0	0	0	0	0	0	0	0	0	0	0	***Difference Score:0***

Emergent Construct	ELEMENETS (STATEMENTS)												Opposite Construct
	1	2	3	4	5	6	7	8	9	10	11	12	
True	1	1	1	1	1	1	1	1	1	1	1	1	False
Infinity is a number	5	5	5	5	5	5	5	5	5	5	5	5	Infinity is not a number
Differences	4	4	4	4	4	4	4	4	4	4	4	4	***Difference Score:48***

After we had obtained a construct relationship score for each construct pair, we used elementary linkage analysis to identify the conceptual organization. It is a cluster analysis technique that is one method of grouping or clustering together correlation coefficients which show similarities among a set of variables (Cohen et al., 2000). At the end, we prepared a construct relationship figure for each student, involving clusters and difference scores between construct pairs which indicate strengths of the relations. These figures gave us information about the understanding features of different concepts of infinity in an organized way.

RESULTS AND DISCUSSION

In this section, we evaluate the grids of two students as an example and discuss in detail the understandings of infinity held by the students. We also discuss how their understandings evolved over the set theory course. With all of the students, a clear image for infinity emerged during the interviews.

THE FIRST STUDENT

First Student's Initial View Of Infinity: The first Student had graduated from a Faculty of Education in Turkey. He was a successful student like others. His last course, related with our research subject, was the construction of real number system which had been given one semester before. He participated in the full activity. Some related comments are provided to give the reader a sense of how the constructs were elicited which were eventually used to rank all 12 items.

Infinity is a number- Infinity is a concept (S7 - S10) "In fact I think that there is something as an infinity concept. But Should I think it as a number? ... According to me infinitesimal numbers exist. If I can not call something infinity then also I can not call something infinitesimal."
Densities of infinite sets are different (S1-S7) "Their densities are different. In the same way there are two infinities. ∞ / ∞ = 1… (S3-S8) (Are there different infinities?) Eee .Their densities can be different. To be countable or not to be. B is denser. Infinity is again the same infinity. Statement 8 is different."
1-1 corespondence can be made between infinite sets (S3-S8) "I thought Why can not I map them, I believe Statement 3. Same thing must be done in Statement 8. Here although their numbers of elements are different … (Do natural numbers finish?) No…I don not know that 1-1 correspondence can be made between them."
Infinite sets can be classified as countable and uncountable (S2-S5) "Their element numbers are infinite. Infinity means that I can count it in a way. I can count rational numbers, but I can not count real numbers. I separate them in this way"
Infinıty can be ruled (S11) "… It likes infinity that I map with R but I can not hold it, it slides."
The element numbers of infinite sets are same : If I subtract infinity from infinity the result will be again infinity, infinity does not end. In fact that I think being element number infinite and being cardinal set infinite are the same thing. I think that their number of elements are equal.

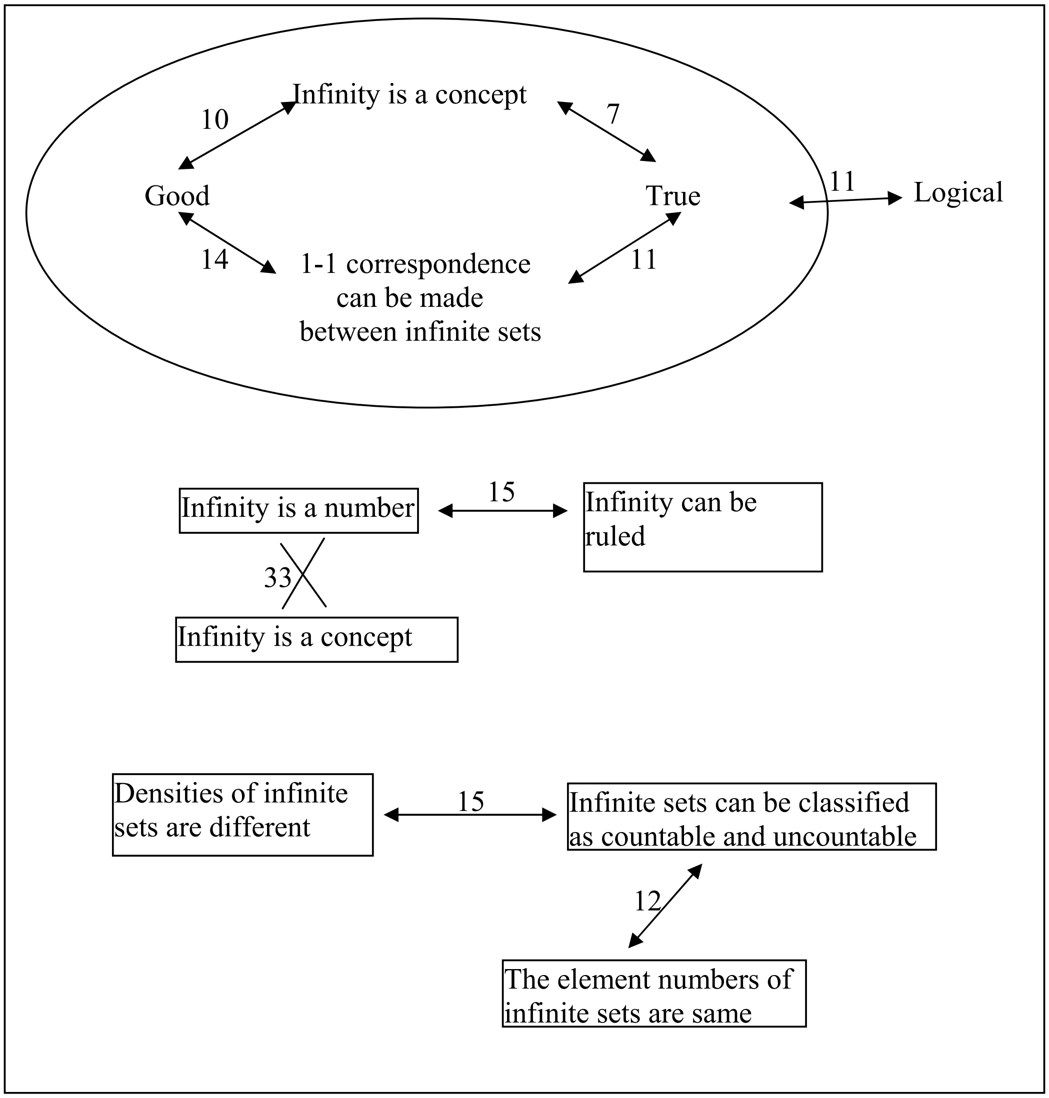
In the statement 3, we asked the students what can be written instead of "**a**". When he was asked about "**a**" that shows the number of natural numbers, he said "∞".Maybe the limit $\lim_{n\to\infty}\left(\frac{n^2}{n^2+1}\right)=1$ effected him that he found the result of $\lim_{n\to\infty}\left(\frac{n^5}{(1,1)^n}\right)$ as 1. He showed his doubts sometimes. For example, at the beginning he had said true for S6. After he had compared it with S11, he said, "it is not true". Since that was the first time that he met with cardinal and ordinal infinity, the interviewer described the meaning of ";" that is used to describe ordinality of elements and the condition of m (limit ordinal) which has no predecessor. At the end of the grid elicitation session, the following grid was formed by rating constructs.

First Student's Initial Repertory Grid

Construct Emergent Pole "5"	1	2	3	4	5	6	7	8	9	10	11	12	Construct Opposite Pole "1"
Infinity is a number	1	5	1	3	4	5	2	5	1	1	2	1	Infinity is not a number
Densities of infinite sets are different	1	3	1	5	2	1	3	2	1	4	1	1	Densities of infinite sets are not different
1-1 correspondence can be made between infinite sets	5	3	5	1	2	4	3	1	5	2	5	2	1-1 correspondence can not be made between infinite sets
Infinite sets can be classified as countable and uncountable	3	3	3	5	3	3	3	3	3	3	3	3	Infinite sets can not be classified as countable and uncountable
Infinity can be ruled	1	1	1	1	3	3	1	1	1	1	3	1	Infinity can not be ruled
The element numbers of infinite sets are same	5	3	5	5	5	3	2	3	3	5	5	2	The element numbers of infinite sets are not same.
Infinity is a concept	5	2	5	1	4	1	5	1	2	3	5	4	Infinity is not a concept
Logical	5	4	5	1	1	1	5	2	3	1	5	2	Illogical
True	5	3	5	1	1	1	5	2	2	2	5	3	False
Good	5	4	5	3	2	1	5	3	3	2	5	4	Bad

Although the primary focus of this report is on the relationship figure, we can note a few obvious relationships. He ranked the statements 1, 3, 7, 11 as "true, logical, good and representing that infinity is a concept". In addition these statements have generally high rankings on the construct "1-1 correspondence can be made between infinite sets". Whereas the elicited constructs in a repertory grid give an idea of the major meanings used to understand a subject, the relationship figure of the constructs shows how those meanings are related. First student's initial construct relationship figure is given next, the numbers in the figure show the difference scores.

First Student's Initial construct relationship figure



As anyone can understand from the first cluster at the top, his constructs "Infinity is a concept" and "1-1 correspondence can be made between infinite sets" are near the "good" and "true" constructs. He finds them logical.

In comparison to other students who participated in the study, "Infinity is a number" and "Infinity can be ruled" constructs are very near. According to students' constructs, "ruling infinity" and "being infinity a number" are two core meanings which are related to each other. In this study generally we see this kind of cluster in all of the figures. This is an important intuition of human being. According to the researchers, it shows the thinking of controlling infinity like using numbers to control and to understand other things. And according to him, being infinity a number and being infinity a concept are opposite constructs.

We think that the last relation between the constructs "Densities of infinite sets are different" and "Infinite sets can be classified as countable and uncountable" depends on his mathematical background. According to our observations in the lessons, students try to deal with the contradictory nature of infinite sets by using the "countable" and "uncountable" statements. For example, first student tried to explain different infinities by classifying infinite sets as "countable" and "uncountable" sets. Jahnke (2001), in his article about Cantor's cardinal and ordinal infinities, pointed out that "The discovery of countable and non-countable sets was a motive for Cantor to define the concept of cardinal number." In conjunction with his view, our findings also pointed the importance of countability and uncountability to swallow the idea of different infinities.

His construct relationship figure represents the view of a mathematician who did not meet with the set theory and different concepts of infinity. We see the effects of the real number theory and signs of the potential infinity in it.

First Student's Emerging View Of Infinity: First student attended all of the lessons and participated in the discussions. His answers to some questions during the lessons are in the following.

Question: How many cardinals are there?
Answer: **C**
Question: What about cardinality of natural numbers and cardinality of real numbers ?
Answer: …It is logical to think that they are equal.
Question: What about the equality $\aleph_0 . \aleph_0 = \aleph_0$?
Answer: There is multiplication operation in the set of natural numbers. It is closed under multiplication. I find the equality $\aleph_0 . \aleph_0 = \aleph_0$ logical, because it does not jump to **C**.
(Thinking $\aleph_0$ as a natural number. It is hard to consider natural numbers totally)

The lessons offered the first student to refine his thinking on different concepts of infinity and give him the opportunity to meet with many examples of cantor's cardinal and ordinal infinity. It is observed that he had some information about axiom of choice and the definition of infinite sets from the previous courses. But at the beginning he was very mindful because of the statements like "$\aleph_0 + 1 = \aleph_0$" or "$2.\aleph_0 = \aleph_0$". As he encountered different conflicting conditions, he gained the opportunity to examine and to refresh his knowledge about infinity.

We assessed the effects of the learning unit on students' formal and intuitive understandings of infinity. After the lessons finished, the researcher interviewed with him again and a second repertory gird was elicited. During the final interview the following constructs were elicited.

<u>*There are different infinities - Infinity is a magnitude*</u>: (S4 – S5) "I see from this statement that there are different infinite numbers….(S6 – S11) I say that it is bigger. But it seems to me that [0, 1] and [0, 2] same. As a concept 3 cm and 4 cm states different infinities? …
(S2-S5) According to me infinity and different infinities exist. I consider them like a magnitude not like a number.
<u>*Infinity is a concept - Infinity is a number:*</u> (S1) Yes, there is no a thing named infinity. I understood infinity as a representative, as a statement or a concept that represents something…. It seems to me like a concept… (S3 – S8) Infinite numbers should not be thought as finite numbers. They should be thought as a different concept.
<u>*Infinity can be ruled:*</u> (S1 - S7) There is something that I can not rule. It has no end, but, I …

<u>*There is conservation of order:*</u> (S9 – S12) In our minds, It begins from the smallest and goes towards the bigger one. In our minds there is natural ordering… Here, since we do not care conservation of order there is cardinality. Because of this, a = 2.a
<u>*1-1 correspondence can be made between infinite sets:*</u> S4 is not true, 1-1 correspondence can not be made …
<u>*Cardinality exists:*</u> (S1 – S7) According to their number of elements there are more natural numbers than even numbers. We skip 3, 5, 7, … but they have the same cardinality.
<u>*Ordinality exists:*</u> (S3 – S8) Sets have the same ordinality and cardinality …
<u>*Countable infinity exists:*</u> (S2 – S5) I can count rational numbers but I can not count real numbers. In one of them there is uncountable infinity. In fact same operations are not valid but same operations are valid for infinite numbers partially…

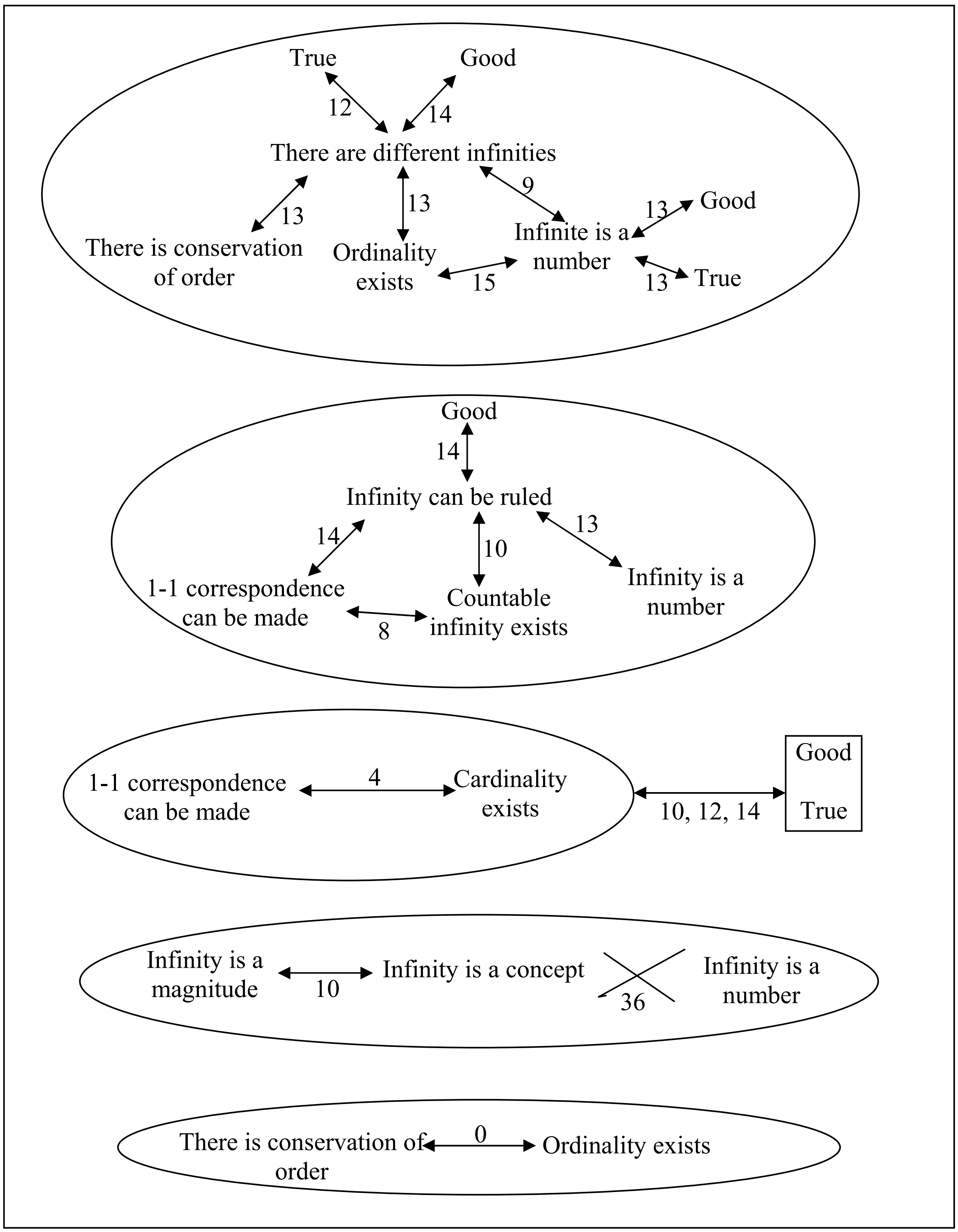
In the statement 3, again we asked the students what can be written instead of "**a**". When he was asked about "a" that shows the number of natural numbers, he said that "In the first interview maybe I used to write "∞" but now I prefer "$\aleph_0$"." His final repertory grid and construct relationship figure are on the following.

First Student's Final Repertory Grid

Construct Emergent Pole "5"	1	2	3	4	5	6	7	8	9	10	11	12	Construct Opposite Pole "1"
There are different infinities	1	1	5	1	5	3	4	5	2	1	2	5	There are not different infinities
Infinity is a concept	5	5	1	1	1	3	3	2	1	5	3	2	Infinity is not a concept
Infinity can be ruled	2	3	4	4	5	3	3	3	3	2	3	4	Infinity can not be ruled
Infinity is a number	1	1	5	5	5	3	5	5	5	1	3	5	Infinity is not a number
Infinity is a magnitude	5	4	2	3	2	3	4	2	2	4	2	3	Infinity is not a magnitude
There is conservation of order	1	1	1	1	1	1	3	5	1	1	1	5	There is not conservation of order
1-1 correspondence can be made between infinite sets	5	3	5	4	3	3	3	5	5	3	5	5	1-1 correspondence can not be made between infinite sets
Cardinality exists	5	3	5	5	4	5	3	5	5	3	5	5	Cardinality does not exist
Ordinality exists	1	1	1	1	1	1	3	5	1	1	1	5	Ordinality does not exist
Countable infinity exists	4	3	5	4	3	3	3	3	4	3	1	5	Countable infinity does not exist
True	1	2	5	1	5	1	3	5	5	3	5	5	False
Good	2	2	5	2	5	1	3	5	5	3	5	5	Bad

As it can be seen from the rankings in the grid, polarization between the constructs increased. The numbers used to rank are generally 1, 3 and 5.

First Students' Final construct relationship figure



One of the major changes in his construct relationship figure is the presence of a new cluster (There are different infinities, Infinity is a number, There is conservation of order, Ordinality exists, Good, True) emerged which attaches the constructs "true" and "good" with the constructs "There are different infinities" and "Infinite is a number". And as a natural consequence of the set theory lessons the constructs "1-1 correspondence can be made" and "Cardinality exists" were linked with each other which are very near to "good" and "true" constructs. As we saw in other students' construct relationships, it was seen through his constructs that, "There is conservation of order" and "ordinality exists" have the same meaning for him. Another observation about ordinality is that there is no relationship directly between the constructs related with ordinality and the constructs "true" and "good". In the lessons we observed that understanding ordinality was more difficult than understanding cardinality for students. Anyway the cardinal numbers are generally before the ordinal numbers in set theory books as Cantor's published papers. Again the relation can be seen between the constructs "Infinity can be ruled" and "Infinity is a number". We think that intuitionally, he tries to rule infinity by undertaking it as a number. Maybe this intuition was also motive for Cantor. At the end of the set theory lessons this cluster was developed by the constructs "1-1 correspondence can be made" and "Countable infinity exists". Again we observe the opposition between the constructs "Infinity is a concept" and "Infinity is a number". Generally for all students being infinity a concept and being infinity a number were different things.

In summary, in his final construct relationship figure we observed the effects of the set theory lessons to his mental model. We also saw that some of the relations between his constructs did not change but they were developed with new constructs. We see the evidence of actual infinity in his figure. But as it can be seen from the distances between the constructs, even after a fifteen-week learning unit we can not say that his new figure is totally appropriate to actual or any concept of infinity. In the following quote from his interview he refers to an obstacle about the ordinality of infinite sets.

> "It is hard to think ordered infinite sets. What is in our mind is that, it begins from the small, goes towards the big. There is the natural ordering in human's mind."

THE SECOND STUDENT

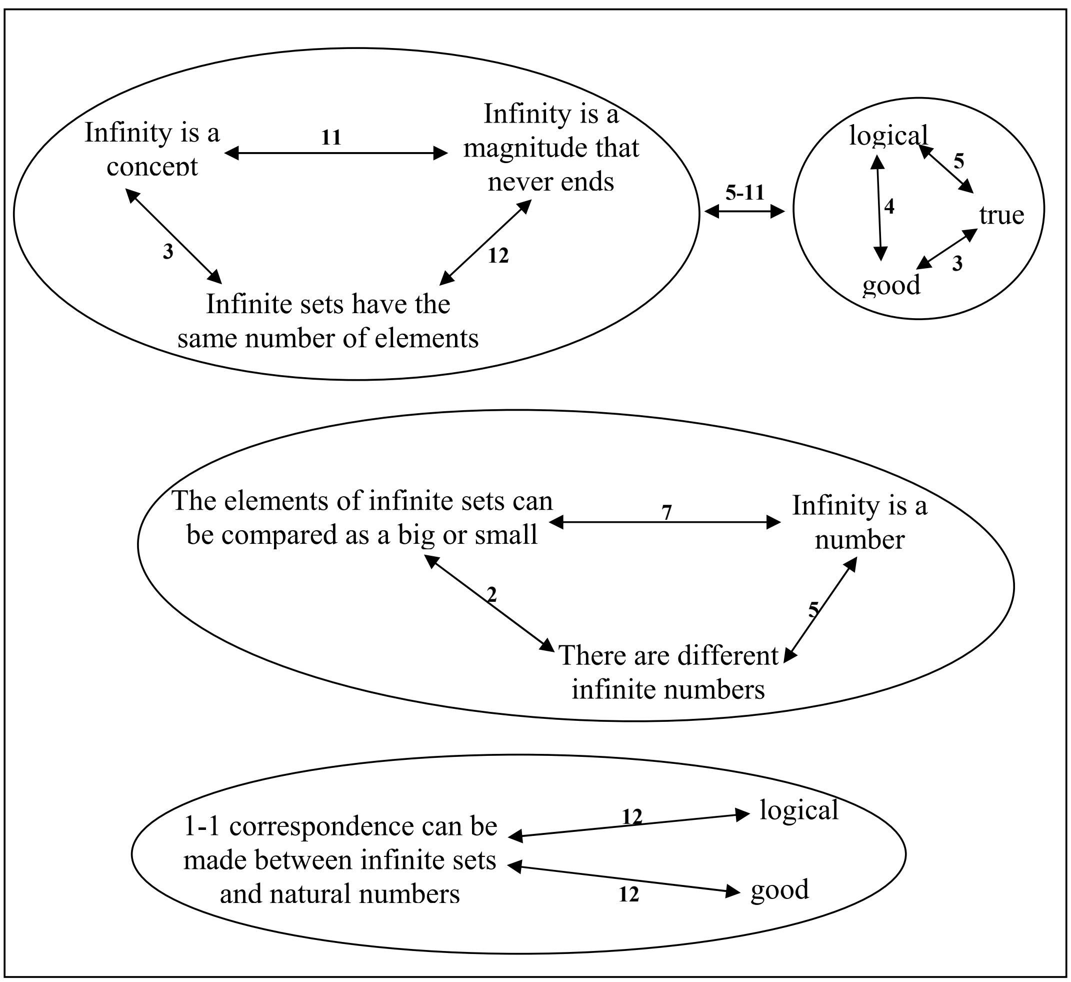
Second Student's Initial View Of Infinity: In order to illustrate the relationships between constructs about infinity more, we will look briefly at a second student. He had graduated from another faculty of education in Turkey. He had a less knowledge about infinite sets and it was difficult for him to explain the statements. The following constructs were elicited from the second student during the initial interview:

The elements of infinite sets can be compared as a big or small: (S1 – S7) In the first the infinite sets are compared. We are trying to compare the according to their numbers. Then it is said that since 1-1 correspondence can be made, the number of elements is infinite but infinite is not a number we must not mix them. Ooo but we say infinite at the end…
Infinity is a number: (S-7) In the 7th statement it is said that it can never zero or a very big number… It is mentioned about very big and very small numbers…
Infinity is a concept: (S7 – S10) We see here infinite and infinitesimal statements as a concept…

<u>*Infinity is a magnitude that never ends:*</u> (S2 – S5) There is a contradiction here… In the second infinite is described as a magnitude that never ends. Then in the other it is said that there are not any different infinite numbers or infinities.
<u>*1-1 correspondence can be made between infinite sets and natural numbers - Infinite sets have the same number of elements:*</u> (S6 – S11) " So both of these sets have the same number of elements. We see this by 1-1 correspondence… They have the same number of elements. There is no contradiction… Maybe we can make a 1-1 correspondence between line segments which have different lengths…
<u>*There are different infinite numbers:*</u> (S2 – S5) "Here It is said that as if there are different finite numbers also there are different infinite numbers… Both of the sets are infinite, but there are also irrational numbers in R…"

In the statement 3, we asked the students what can be written instead of "**a**". When he was asked about "a" that shows the number of natural numbers, he said "∞" like the first student. He had no clear idea about the result of $\lim_{n\to\infty}\left(\frac{n^5}{(1,1)^n}\right)$. During the first interview his comments were generally repeating of the statements and gave a little information about his understanding of this contradictory subject. But his elicited initial construct relationship figure was more confirmative. It is on the following.

Second Student's Initial construct relationship figure



First cluster on the top (“Infinity is a concept”- “Infinity is a magnitude that never ends”- “Infinite sets have the same number of elements”) is a clear sign of potential infinity which are also linked with the constructs “logical”, “true” and “good”. This cluster also indicates the “inexhaustible capacity of infinity” very well that Fischbein (2001) discussed.

“A second aspect, related to the intuitive interpretation of infinity refers to what one may call the “inexhaustible capacity of infinity” Infinity appears intuitively as being equivalent with inexhaustible, that is, if one continues the process of division indefinitely, all the points can be reached. In our opinion, this interpretation of infinity is the essential reason for which, intuitively, there is only one kind, one level of infinity.”

According to the figure, it is also an important detail that the constructs “logical”, “true” and “good” do not have the same meaning for him. For him, a thing that is true can not be fully logical. The constructs “infinity is a concept” and “Infinite sets have the same number of elements” are closer to one another than other constructs. That kind of relation is also observed in other student’s figures. Thinking infinity as a concept also implies the one infinity which indicates the number of elements in all infinite sets. In the second cluster the constructs “The elements of infinite sets can be compared as a big or small”, “Infinity is a number” and “There are different infinite numbers” are linked. As it can be understood from the lessons and other student’s construct relationships the researcher thinks that discussing infinity as a total quantity or a number is directly related with accepting different infinities. It can be seen that the construct “1-1 correspondence can be made between infinite sets and natural numbers” is the indication of countability that are close to the constructs “logical” and “good”. In summary the second student differs from the first in having a clear image of potential infinity.

Second Student’s Emerging View Of Infinity: First student attended all of the lessons like the first student. He had not spoken much in the lessons but the dialog on the following can give us a reason about how did he participate in the discussions.

Question: How many countable infinite set are there?
Student: Infinite.
Question: How many countable infinite cardinality are there?
Student: One
Question: How many uncountable infinite cardinality are there?
Student: Infinite, aah I do not know.
Question: What about the cardinals $\aleph_0 < c <$ Card P(R) < Card P(P(R)) <… What can be the union of these cardinalities? (P(R) indicates the power set of real numbers) Is this the biggest one?
Student: I can take the power set of union…
Question: Then, how many cardinality are there?
Student: Infinitely many, uncountable and infinitely many.

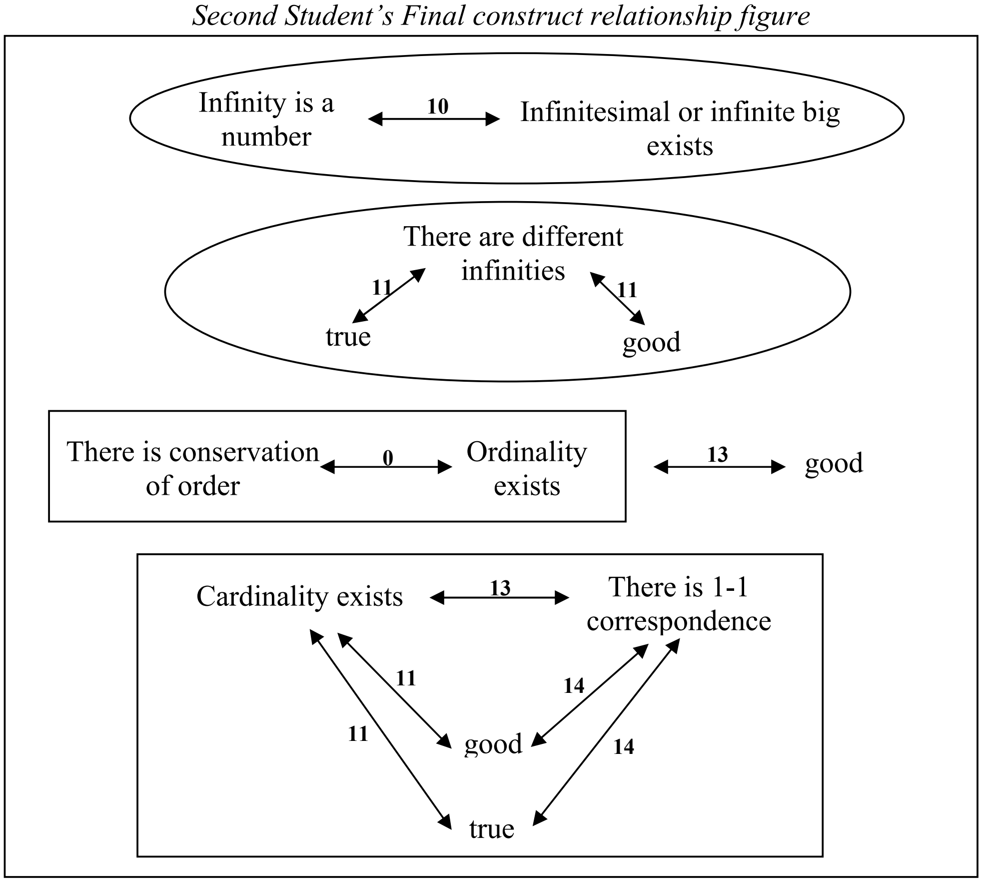
The following constructs were elicited from the second student during the last interview:

<u>*Infinity is a concept - Infinity is a number:*</u> (S1-S7) “...when we think it as a number or a concept we can think different sets…” (S7 - S10) It is mentioned about infinitesimal and infinite big. These are also concepts like anything we found. Yes, I can not say it is only this

… In the statement 7 it means that it can not be zero or very big number… (S4-S10) It is said that It can not be mentioned about different infinite numbers …
There are different infinities: (S1-S7) I agree with statement 7, there is not only one infinity, there can be bigger infinities. When we mention about infinity, we should not think the same thing … (S2) In the statement 2 it is said that we can not mention about different infinities, I do not agree with it. (S4 – S5) I do not agree with the statement " It can not be mentioned about different infinities". Statement 4 is wrong."
There is conservation of order: (S9-S12) "If I take into account order, this is like an excess…" (S3 – S8) "In the statement 8 I add this to the end. I see here ordinality. Since I add it to the end, since I made it w+1, even if I never see m, I can make 1-1 correspondence, I see the importance of conservation of order…"
Ordinality exists-Cardinality exists: (S3 - S8) "…since its' cardinality is infinite , sorry we do not say infinite anymore, since its' cardinality is $\aleph_0$, nothing change even if I add one element. Of course ordinality is also different…In the statement 8, I add this to the end. I see here ordinality…
Infinitesimal or infinite big exists: (S7-S10) "Extension of real numbers can be obtained by infinitesimal and infinite big. Thus…"
There is 1-1 correspondence: (S1-S7) "Card (R) = Card (C). Because we can make 1-1 mapping between them. The paragraph at the below is not but the part that point the equality of natural numbers and even numbers is okey. Because we can make 1-1 correspondence between them…" (S6-S11) "I can set up 1-1 correspondence between them…"

He answered the question about "a" that shows the number of natural numbers by "$\aleph_0$" like the first student. He had still no idea about the result of $\lim_{n\to\infty}\left(\frac{n^5}{(1,1)^n}\right)$. Even if the constructs are not very close to each other, His construct relationship figure on the following indicates the movement to the non-standard and actual infinity.

Second Student's Final construct relationship figure



First cluster indicates the relationship between the constructs "Infinity is a number" and "Infinitesimal or infinite big exists" that refers the statements about non-standard infinity. We can see the result of meeting him different concepts of infinity during the fifteen week period by looking the second cluster ("There are different infinities"- "true" – "good"). The relationships between these constructs are not very strong but we think that the idea of different infinities occurred in his mental model about infinity. Like the first student's constructs after the lessons, the construct "Ordinality exists" is linked with the construct "There is conservation of order". They have nearly the same meaning for him. The cluster at the bottom indicates the main requirement of cardinality very well, even if the relationships between the constructs are not very strong. Since the cardinality of a set is defined by its equivalence class which is built with 1-1 correspondence between equivalent sets, it is a natural cluster that shares the conceptual space of a mathematician.

CONCLUSION

This paper presents the elicited constructs of PhD students concerning infinity and the effects of the set theory lessons to their constructs and concept images. It also offers insights into individual thinking of young mathematicians about infinity.

For the Cantor, the transfinite ordinals and the actual infinite are creations just as legitimate as the complex numbers had been ages ago (Jahnke, 2001). Although Cantor tried to legitimize his findings by arguing the modern theory of complex functions and others, his theory is not very easy to swallow intuitively even for mathematicians. Understandings that came from their past and environment are continuing to be cognitive obstacles for students. As Tirosh says "…our primary intuitions are not adapted to the notion of cardinal infinity. Thus, it would seem to require a considerable effort to develop appropriate "secondary intuitions" (i.e., intuitions which are acquired through educational intervention) of the notion of cardinal infinity."

Students generally have an understanding of potential infinity and after the lessons we saw emerging evidence of actual infinity. According to their model, Sierpinska (1987) and Petty (1996) had found that undergraduate students generally are at static and dynamic levels. We suggest that this is not also different for graduate students. Considerable effort is required to reach the higher levels of the model. One important result is the tendency of interpreting different infinities which indicate element numbers of different infinite sets by countability and uncountability of sets. We suggest that students intuitively think this when they think of the total numbers of real and natural numbers. Of course it is also directly related with their mathematical background as a mathematician. This finding is considerably parallel with the suggestion of Jahnke (2001) that is "The discovery of countable and non-countable sets was a motive for Cantor to define the concept of cardinal number."

Another important result is about teaching of set theory subjects. We observed that students found ordinality harder than cardinality as studies at the past (Tirosh, 1991). We suggest that when students meet with Cantorian theory, even if they are mathematicians, meeting them with cardinality will be more suitable. Since there is not a single concept of infinity, our goal was not to impose the Cantorian infinity as a single one. But we were expecting them to distinguish the nature of thinking intuitively and mathematically. After the study we observed that they were wary of responding intuitively to our questions.

In his paper on "Predications of the Limit Concept: An Application of Repertory Grids", Williams (2001) argues that "Certainly no one method can capture the richness and variety that characterizes human thinking. Any method will necessarily obscure some aspects as it highlights others". However we found that as in the William's study, the repertory grid methodology was useful to capture students' understandings about the related subject. Looking for construct relationships by analyzing repertory grids provided us fundamental models and metaphors about infinity which gives further details. We think that it is a very suitable method, particularly in this subject to pass students' cognitive defenses. For example, even if the second student was not an active participant, we could learn something about his understanding by eliciting his constructs. Instead of giving some questions as a task or forcing them to discuss about any subject, evaluating and ranking different statements which everybody can do was more applicable. The technique also allowed us to capture the growth after the lessons. Of course a change had to be expected after a fifteen-week period, but observing the changes on the constructs about infinity systematically was the outcome of the methodology. We should also note the verbal data obtained by the analysis of interviews was utilized. We think that the combination of infinity concept and repertory grid technique was an original and useful in relation to existing studies on infinity. It can help us to determine the place of different concepts of infinity in our math curricula and looking at different age groups can provide us a model for the understanding of infinity concept.

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[i] This article was independently reviewed by two reviewers external and not affiliated with the editors or the editorial board of the journal.

[ii] The syntax of the English in this paper varies occasionally in its usage of tense. The reader should bear in mind that the study was conducted in Turkish and the data was then translated and reported in English.