[The Mathematics Enthusiast](https://scholarworks.umt.edu/tme)

Volume 7
Number 2 *Numbers 2 & 3* Article 5

7-2010

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Recommended Citation

Mamolo, Ami (2010) "Polysemy of symbols: Signs of ambiguity," *The Mathematics Enthusiast*: Vol. 7 : No. 2 , Article 5.
Available at: https://scholarworks.umt.edu/tme/vol7/iss2/5

Polysemy of symbols: Signs of ambiguity

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Abstract: *This article explores instances of symbol polysemy within mathematics as it manifests in different areas within the mathematics register. In particular, it illustrates how even basic symbols, such as '+' and '1', may carry with them meaning in 'new' contexts that is inconsistent with their use in 'familiar' contexts. This article illustrates that knowledge of mathematics includes learning a meaning of a symbol, learning more than one meaning, and learning how to choose the contextually supported meaning of that symbol.*

Keywords: mathematical registers; polysemy; symbols; transfinite arithmetic

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The Montana Mathematics Enthusiast, **ISSN 1551-3440, Vol. 7, nos.2&3**, pp.247- 262

Ambiguity in mathematics is recognized as "an essential characteristic of the conceptual development of the subject" (Byers, 2007, p.77) and as a feature which "opens the door to new ideas, new insights, deeper understanding" (p.78). Gray and Tall (1994) first alerted readers to the inherent ambiguity of symbols, such as 5 + 4, which may be understood both as *processes* and *concepts*, which they termed *procepts*. They advocated for the importance of flexibly interpreting procepts, and suggested that "This ambiguous use of symbolism is at the root of powerful mathematical thinking" (Gray and Tall, 1994, p.125). A flexible interpretation of a symbol can go beyond process-concept duality to include other ambiguities relating to the diverse meanings of that symbol, which in turn may also be the source of powerful mathematical thinking and learning. This article considers cases of ambiguity connected to the context-dependent definitions of symbols, that is, the *polysemy* of symbols.

A polysemous word can be defined as a word which has two or more different, but related, meanings. For example, the English word 'milk' is polysemous, and its intended meaning can be determined by the context in which it is used. Mason, Kniseley, and Kendall (1979) observed that word polysemy in elementary school reading tasks was a source of difficulty – students demonstrated a tendency to identify the common meaning of words, despite being presented contexts in which an alternative meaning was relevant. Durkin and Shire (1991) discussed several instances of polysemous words within the mathematics classroom. They noted confusion in children's' understanding of expressions that had both mathematical and familiar 'everyday' meanings. In resonance with Mason, Kniseley, and Kendall (1979), Durkin and Shire found that "when children misidentified the meaning of an ambiguous word in a mathematical sentence, the sense they chose was often the everyday sense" (1991, p.75).

In addition to potential confusion between a word's 'everyday' meaning and its specialized meaning within mathematics, learners are also often faced with polysemous terms within the mathematics register. Zazkis (1998) discussed two examples of polysemy in the mathematics register: the words 'divisor' and 'quotient'. These words were problematic for a group of prospective teachers when confusion about their meanings could not be resolved by considering context – both meanings arose within the same context. In the case of 'divisor', attention to subtle changes in grammatical form was necessary to resolve the confusion. In the case of 'quotient', a conflict between familiar use and precise mathematical definition needed to be acknowledged and then resolved. Zazkis relates to the mathematics register Durkin and

Shire's (1991) suggestion that enriched learning may ensue from monitoring, confronting and 'exploiting to advantage' ambiguity.

I would like to continue the conversation on polysemy within the mathematics register, and extend its scope to consider the polysemy of mathematical symbols. This article examines the polysemy of the '+' symbol as it manifests in different areas within the mathematics register. The article begins with a reminder of the 'familiar' – addition and addends in the case of natural numbers – as well as a brief look at an example where meanings of symbols are extended within the sub-register of elementary school mathematics. Following that, I focus on two instances where meanings of familiar symbols are extended further: the first involves modular arithmetic, while the second involves transfinite arithmetic. I chose to focus on these cases for two reasons: (i) the extended meanings of symbols such as '$a + b$' contribute to results that are inconsistent with the 'familiar', and (ii) they are items in pre-service teacher mathematics education.

This article presents an argument that suggests that the challenges learners face when dealing with polysemous terms (both within and outside mathematics) are also at hand when dealing with mathematical symbols by starting with 'obvious' and well-known illustrations of symbol polysemy in order to prepare the background to analogous but not-so-obvious observations. It focuses on cases where acknowledging the ambiguity in symbolism and explicitly identifying the precise, context-specific, meaning of that symbolism go hand-in-hand with understanding the ideas involved.

Building on the familiar: from natural to rational

The main goal of this section is to establish some common ground with respect to 'familiar' meanings of symbols of addition and addends. In the subsequent sections, the meanings of these symbols will be extended in different ways, dependent on context. Their extensions will be explored so as to highlight ambiguity in meanings which can be problematic for learners should it go unacknowledged.

Since experiences with symbols in mathematics often start with the natural numbers, it seems fitting that this paper should start there as well. Natural numbers may be identified with cardinalities[2], or 'sizes', of finite sets – where '1' is the symbol for the cardinality of a set with a single element, '2' the symbol for the cardinality of a set with two elements, and so on. With

[2] Natural numbers may also be identified with ordinals; however addition of ordinals is not commutative (Hrbacek and Jech, 1999), and thus in doing so one loses a fundamental property of natural number arithmetic.

such a definition, addition over the set of natural numbers may be defined as the operation which determines the cardinality of the union of two disjoint sets (Hrbacek and Jech, 1999; Levy, 1979). As noted earlier, a symbol such as '1+2' can be considered a *procept*, and as such may be viewed as both the *process* of adding two numbers and also the *concept* of the sum of two numbers. For the purposes of this paper, it is enough to restrict attention to the concept of '1+2' (and hereafter all other arithmetic expressions), though the process of '1+2' is no less polysemous.

A more formal definition of addition over the set of natural numbers, $\mathbb{N}$, can be written as the following:

- if A and B are two disjoint sets with cardinalities *a, b* in $\mathbb{N}$, then the sum $a + b$ is equal to the cardinality of the union set of A and B, that is, the set ($A \cup B$).

Table 1 below summarizes the meanings of the symbols '1', '2', and '1+2' when considered within the context of natural number addition:

Symbol	Meaning in context of natural numbers
1	Cardinality of a set containing a single element
2	Cardinality of a set containing exactly two elements
1+2	Cardinality of the union set

Table 1: Summary of familiar meaning in $\mathbb{N}$

Sensitivity towards various meanings attributed to arithmetic symbols is endorsed by teacher preparation guides and texts, such as Van de Walle and Folk's *Elementary and Middle School Mathematics*, which notes that "each of the [arithmetic] operations has many different meanings" and that "Care must be taken to help students see that the same symbol can have multiple meanings" (2005, p.116). Van de Walle and Folk highlight as an example the 'minus sign', which they observe has a broader meaning than 'take away'. However, they seem to take for granted that their readers are familiar with exact mathematical meaning of arithmetic symbols. For instance, they introduce addition as a 'big idea' which "names the whole in terms of the parts" (p.115), but without explicitly defining addition over the natural numbers, nor distinguishing conceptually natural number addition from, say, rational number addition. Rather, they recommend that "the same ideas developed for operations with whole numbers should apply to operations with fractions. Operations with fractions should begin by applying these same ideas to fractional parts" (p.244). This advice has dubious implications both conceptually and pedagogically when we consider the definition of natural (and whole) numbers as cardinalities of sets. Rational numbers do not have an analogous definition as cardinalities, and indeed, the idea that a set might contain ½ or ¼ of an element is not meaningful. Instead, rational numbers may be described as numbers that can be represented as a ratio $v : w$, where v and w are integers.

Campbell (2006) warns against conflating whole number and rational number arithmetic, and suggests that merging the two ideas may be the root of both conceptual and procedural difficulties during an individual's transition from arithmetic to algebra. Campbell identifies a source for this confusion as the

> "relatively recent development in the history of mathematics that has logically subsumed whole (and integer) numbers as a formal subset of rational (and real) numbers. This development appears to have motivated and encouraged some serious pedagogical mismatches between the historical, psychological, and formal development of mathematical understanding" (2006, p.34)

Campbell asserts that the set of natural (and whole) numbers are not a subset of the set of rational numbers, but rather are isomorphic to a subset of the rational numbers. As such, this distinction is significant as it carries with it separate definitions for the set of natural numbers (and its corresponding arithmetic operations) and the subset of the rational numbers to which it is isomorphic. In particular, although the symbols appear the same, their meaning in this new context is different, as illustrated in Table 2.

Symbol	Meaning in context of rational numbers
1	A ratio of integers equivalent to 1:1
2	A ratio of integers equivalent to 2:1
1+2	A ratio of integers equivalent to 3:1

Table 2: Summary of extended meaning in $\mathbb{Q}$

Campbell suggests that although the

> "standard view… is to claim that young children are simply not developed or experienced enough to grasp the various abstract distinctions and relations to be made between whole number and rational number arithmetic… it may be the case that the cognitive difficulties in children's understanding of basic arithmetic is a result of selling short their cognitive abilities" (2006, p.34).

Thus, although it may seem cumbersome to distinguish between $1 \in \mathbb{N}$ and 1 (or 1.0) $\in \mathbb{Q}$, where $\mathbb{Q}$ symbolizes the set of rational numbers, it is conceptually important. In a broad context, the operation of addition may be considered as a binary function, and as such, its definition depends on the domain to which it applies. Recalling Table 1, we may add another row:

Symbol	Meaning in context of natural numbers
1	Cardinality of a set containing a single element
2	Cardinality of a set containing exactly two elements
1+2	Cardinality of the union set
+	Binary operation over the set of natural numbers

Table 1B: Summary of familiar meaning in $\mathbb{N}$

It is useful for purposes of clarity in this paper to distinguish between different definitions of the addition symbol as they apply to different domains. The symbol $+_N$ will be used to represent addition over the set of natural numbers, $+_Z$ to represent addition over the set of integers, and $+_Q$ to represent addition over the set of rational numbers. $+_N$ and $+_Q$ have, to apply Zazkis's (1998) phrase, the 'luxury of consistency' – despite the different definitions, $1 +_N 2 = 3$ and $1 +_Q 2 = 3$. However, if we consider summing non-integer rational numbers, there are pedagogical consequences for neglecting the distinction between natural number addition and rational number addition. In particular with respect to motivating and justifying the specific algorithms applicable to computations with fractions, and also with respect to interpreting student error. A classic error such as $\frac{1}{2}+\frac{1}{3}=\frac{2}{5}$ may be seen as a reasonable interpretation of Van de Walle and Folk's (2005) advice of applying whole number operations to fractional parts. Without distinction, $\frac{1}{2}+\frac{1}{3}$ is, for a learner, equivalent to $\frac{1}{2}+_N\frac{1}{3}$. This latter expression is logically problematic: as a binary function, $+_N$ is applicable only to elements in its domain – the set of natural numbers – in which the fractions $\frac{1}{2}$ and $\frac{1}{3}$ are not. $\frac{1}{2}+\frac{1}{3}=\frac{2}{5}$ may be viewed as an algorithm that restricts the function $+_N$ to elements of its domain (the two numerators, and the two denominators). Adequate knowledge of addition as an operation whose properties depend upon the domain to which it applies, offers teachers a powerful tool to address the inappropriateness of this improvised algorithm.

The following sections build on the idea of addition as a domain-dependent binary operation. They explore examples of two domains for which a 'luxury' of consistency is absent: (i) the set {0, 1, 2} and (ii) the class of (generalised) cardinal numbers. When clarification is necessary, the notation $+_3$ will be used to represent addition over the set {0, 1, 2} (i.e. modular arithmetic with base 3), and $+_\infty$ will be used to represent addition over the class of cardinal numbers (i.e. transfinite arithmetic). The sections take a close look at familiar and not-so-familiar examples of domains for which an understanding develops hand-in-hand with an understanding of the associated arithmetic operations.

Extending the familiar: an example in modular arithmetic

Modular arithmetic is one of the threads of number theory that weaves its way through elementary school to university mathematics to teachers' professional development programs – it is introduced to children in 'clock arithmetic', it is fundamental to concepts in group theory, and it is a concept that has helped teachers develop both their mathematical and pedagogical content knowledge. This section considers the context of group theory. It takes as a generic example the group $\mathbb{Z}_3$ – the group of elements {0, 1, 2} with the associated operation of addition modulo 3.

Within group theory the meanings of symbols such as 0, 1, 2, +, and 1+2 are extended from the familiar in several ways. As an element of $\mathbb{Z}_3$, the symbol 0 is short-hand notation for the congruence class of 0 modulo 3. That is, it is taken to mean the set consisting of all the integral multiples of 3: {… -6, -3, 0, 3, 6, …}. Similarly, the symbol 1 represents the congruence class of 1 modulo 3, which consists of the integers which differ from 1 by an integral multiple of 3, and 2 represents the congruence class of 2 modulo 3, which consists of the integers which differ from 2 by an integral multiple of 3. The symbol '+' also carries with it a new meaning in this context: it is defined as addition modulo 3. As Dummit and Foote (1999) caution:

> "we shall frequently denote the elements of $\mathbb{Z}/n\mathbb{Z}$ [or $\mathbb{Z}_n$] simply by {0, 1, … n-1} where addition and multiplication are *reduced mod* [modulo] n. It is important to remember, however, that the elements of $\mathbb{Z}/n\mathbb{Z}$ are *not* integers, but rather collections of usual integers, and the arithmetic is quite different" (p.10, emphasis in original)

Pausing for a moment on the symbol '1+2', we might explore just how different the meaning of addition modulo 3 is from the 'usual integer' addition. Since the symbols '1' and '2' (in this context) represent the congruence classes {… -5, -2, 1, 4, 7, …} and {… -4, -1, 2, 5, 8,…}, respectively, the sum '1+2' must also be a congruence class. Dummit and Foote (1999) define the sum of congruence classes by outlining its computation. In the case of 1+2 (modulo 3), we may compute the sum by taking *any representative integer* in the set {… -5, -2, 1, 4, 7, …} and *any representative integer* in the set {… -4, -1, 2, 5, 8,…}, and summing them in the 'usual integer way' (i.e. with the operation $+_Z$). Having completed this, the next step is to determine the final result: the congruence class containing the integral sum of the two representative integers. Defined in this way, addition modulo 3 does not depend on the choice of representatives taken for '1' and '2'. Thus, recalling the notation introduced in the previous section, sample computations to satisfy this definition include:

$$\begin{aligned} 1 +_3 2 &= (1 +_Z 2) \text{ modulo } 3 \\ &= (1 +_Z 5) \text{ modulo } 3 \\ &= (-2 +_Z -1) \text{ modulo } 3 \end{aligned}$$

all of which are equal to the congruence class 0.

Laden with new meaning, these symbols pose a challenge for students who must quickly adjust to a context where the complexity of such compact notation is taken for granted, and where inconsistencies arise between the symbols' specialized meaning and their 'familiar', 'usual' meaning. Table 3 below summarizes the meanings of the symbols '1', '2', and '1+2', and '+' when considered within the context of $\mathbb{Z}_3$:

Symbol	Meaning in context of $\mathbb{Z}_3$
1	Congruence class of 1 modulo 3: {… -5, -2, 1, 4, 7, …}
2	Congruence class of 2 modulo 3: {… -4, -1, 2, 5, 8,…}
1+2	Congruence class of (1+2) modulo 3: {…, -3, 0, 3, …}
+	Binary operation over set {0, 1, 2}; addition modulo 3

Table 3: Summary of extended meaning in $\mathbb{Z}_3$

The process of adding congruence classes by adding their representatives is a special case of the more general group theoretic construction of a quotient and quotient group – central ideas in algebra, and ones which have been acknowledged as problematic for learners (e.g. Asiala et al., 1997; Dubinsky et al., 1994).These concepts are challenging and abstract, and are made no less accessible by opaque symbolism. As in the case with words, the extended meaning of a symbol can be interpreted as a metaphoric use of the symbol, and thus may evoke prior knowledge or experience that is incompatible with the broadened use. In a related discussion of the challenges learners face when the meaning of a term is extended from everyday language to the mathematics register, Pimm (1987) notes that "the required mental shifts involved can be extreme, and are often accompanied by great distress, particularly if pupils are unaware that the difficulties they are experiencing are not an inherent problem with the idea itself" (p.107) but instead are a consequence of inappropriately carrying over meaning from one register to the other. A similar situation arises as learners must stretch and revise their understanding of a symbol within the mathematics register – an important mental shift that is taken for granted when clarification of symbol polysemy remains tacit.

Beyond the familiar: an example in transfinite arithmetic

Transfinite arithmetic may be thought of as an extension of natural number arithmetic – its addends (transfinite numbers) represent cardinalities of finite or *infinite* sets. Transfinite arithmetic poses many challenges for learners, not the least of which involves appreciating the idea of 'infinity' in terms of cardinalities of sets. Before one may talk meaningfully about polysemy and ambiguity in transfinite arithmetic, it is important to first develop some ideas about 'infinity as cardinality', which is where this section will begin.

Infinity is an example of a term which is polysemous both across and within registers. The familiar association of infinity with endlessness is extended into the mathematics register in areas such as calculus where the idea of *potential infinity* is indispensible. Potential infinity may be thought of as an inexhaustible process – one for which each step is finite, but which continues indefinitely. In calculus for example, the idea of limits which 'tend to' infinity relates the notion of an on-going process that is never completed. This extension across registers preserves some of the meaning connected to the colloquial use of the term 'infinity', however it is distinct from intuitions which, say, connect infinity to endless time or to the all-encompassing (see Mamolo and Zazkis, 2008). Within the mathematics register, the term 'infinity' is extended further to the idea of *actual infinity*, which is prevalent in the field of set theory. Actual infinity is thought of as a completed and existing entity, one that encompasses the potentially infinite. The set of natural numbers is an example of an actually infinite entity – it contains infinitely many elements and, as a set, exists despite the impossibility of enumerating all of its elements. The cardinality of the set of natural numbers is another instance of actual infinity; it is also the smallest transfinite number.

Transfinite numbers are generalised natural numbers which describe the cardinalities of infinite sets. As implied, infinite sets may be of different cardinality: the set of natural numbers, for example, has a different cardinality than the set of real numbers, though both contain infinitely many elements. Cardinalities of two infinite sets are compared by the existence or non-existence of a one-to-one correspondence between the sets. Two sets share the same cardinality if and only if every element in the first set may be 'coupled' with exactly one element in the second set, and vice versa. This is a useful approach, and I will return to it when illustrating properties of transfinite arithmetic. The point I am trying to make here is that the concept of a transfinite number, which intuitively may be thought of as an 'infinite number', requires extending beyond the familiar idea of infinity as endless (and thus unsurpassable). Also, in resonance with Pimm's (1987) observation regarding negative and complex numbers, the concept of a transfinite number "involves a metaphoric broadening of the notion of number itself" (p.107). In this case, the broadening includes accommodating some arithmetic properties which are both unfamiliar and unintuitive.

As in the case with arithmetic over the set of natural numbers, transfinite arithmetic involves determining the cardinality of the union of two disjoint sets. The crucial distinction is of course that at least one of these sets must have infinite magnitude – its cardinality must be equal to a transfinite number. To illustrate some of the distinctive properties of transfinite arithmetic consider, without loss of generality, the cardinality of the set of natural numbers, denoted by the symbol $\aleph_0$. Imagine adding to the set of natural numbers, $\mathbb{N}$, a new element, say β. This union set $\mathbb{N} \cup \{\beta\}$ has cardinality equal to $\aleph_0 + 1$ – there is nothing new here. However, each element in $\mathbb{N}$ can be 'coupled' with exactly one element in $\mathbb{N} \cup \{\beta\}$, and vice versa. By definition, two infinite sets have the same cardinality if and only if they may be put in one-to-one correspondence, thus the cardinality of $\mathbb{N}$ is equal to the cardinality of $\mathbb{N} \cup \{\beta\}$. As such, $\aleph_0 = \aleph_0 + 1$. Similarly, it is possible to add an arbitrary natural number of elements to the set of natural numbers and not increase its cardinality, that is $\aleph_0 = \aleph_0 + \upsilon$, for any $\upsilon \in \mathbb{N}$, and further $\aleph_0 + \aleph_0 = \aleph_0$.

This 'tutorial' in transfinite arithmetic is relevant to the discussion on polysemy as it illustrates how the symbol '+' in this context is quite distinct in meaning from addition over the set of natural numbers. Whereas with '$+_N$' adding two numbers always results in a new (distinct) number, with '$+_\infty$' there exist non-unique sums. Further, since the concept of a set of numbers must be extended to the more general 'class' of transfinite numbers, the symbol '1' in the expression '$\aleph_0 + 1$' also takes on a slightly new meaning since it must be considered more generally as a class (rather than set) element[3]. Extended meanings connected to transfinite arithmetic are summarized in Table 4:

Symbol	Meaning in context of transfinite arithmetic
1	Cardinality of the set with a single element; class element
$\aleph_0$	Cardinality of $\mathbb{N}$; transfinite number; 'infinity'

[3] For distinction between set and class, see Levy (1979).

$\aleph_0 + 1$	Cardinality of the set $\mathbb{N} \cup \beta$; equal to $\aleph_0$
$+$	Binary operation over the class of transfinite numbers

Table 4: Summary of extended meaning in transfinite arithmetic

A specific challenge related to the polysemy of + in this context derives from the existence of non-unique sums, a consequence of which is indeterminate differences. Explicitly, since $\aleph_0 = \aleph_0 + \upsilon$, for any $\upsilon \in \mathbb{N}$, then $\aleph_0 - \aleph_0$ has no unique resolution. As such, the familiar experience that 'anything minus itself is zero' does not extend to transfinite subtraction. This property is in fact part and parcel to the concept of transfinite numbers. Identifying precisely the context-specific meaning of these symbols ('$+_\infty$' and '$-_\infty$') can help solidify the concept of transfinite numbers, while also deflecting naïve conceptions of infinity as simply a 'big unknown number' by emphasizing that transfinite numbers are different from 'big numbers' since they have different properties and are operated upon (arithmetically) in different ways.

In this section, to address issues of polysemy of symbols, it was necessary to first glance at the polysemy of the term infinity. It is a complex concept that can encompass different connotations across and within different registers. Within mathematics, it is difficult to think of infinity – even in the context of transfinite numbers – without imagining that well-known symbol '∞'. Informally, the symbol '$\aleph_0 + 1$' might be thought of as '$\infty + 1$'. This informal symbolism suggests the idea of adding 1 to a 'concept' rather than a 'set number', of adding 1 to endlessness. Notwithstanding the formal use of '$\aleph_0$', an intuition of '∞' may persist (if only tacitly), carrying with it all sorts of inappropriate associations.

Concluding Remarks

This article examined instances of symbol polysemy within mathematics. The intent was to illustrate how even basic symbols, such as '+' and '1', may carry with them meaning that is inconsistent with their use in 'familiar' contexts. It focused on cases where acknowledging the ambiguity in symbolism and explicitly identifying the precise (extended) meaning of that symbolism go hand-in-hand with developing an understanding of the ideas involved. While this article focused on particular examples of distinguishing among the symbolic notation for arithmetic over the set of natural numbers, rational numbers, equivalence classes, and transfinite cardinals is fundamental to appreciating the subtle (and not-so-subtle) differences among the elements of those sets, this argument has broader application. I suggest that the challenges learners face when dealing with polysemous terms (both within and outside mathematics) are also at hand when dealing with polysemous symbols. Just as knowledge of languages such as English include "learning a meaning of a word, learning more than one meaning, and learning how to choose the contextually supported meaning" (Mason et al., 1979, p.64), knowledge of mathematics includes learning a meaning of a symbol, learning more than one meaning, and learning how to choose the contextually supported meaning of that symbol. Further, echoing Pimm's (1987) advice and extending its scope to include mathematical symbols:

> "If … certain conceptual extensions in mathematics [are] not made abundantly clear to pupils, then specific meanings and observations about the original setting, whether intuitive or consciously formulated, will be carried over to the new setting where they are often inappropriate or incorrect" (p.107).

Sfard (2001) suggests that symbols – such as the ones discussed here, but also in a more general sense – are not "mere auxiliary means that come to provide expression to pre-existing, pre-formed thought" but rather are "part and parcel of the act of communication and thus of cognition" (p.29). As such, attending to the polysemy of symbols, either as a learner, for a learner, or as a researcher, may expose confusion or inappropriate associations that could otherwise go unresolved. Research in literacy suggests that students "may rely on context when a word does not have a strong primary meaning to them but will choose a common meaning, violating the context, when they know one meaning very well" (Mason et al., 1979, p.63).

Further research in mathematics education is needed to establish to what degree analogous observations apply as students begin to learn '+' in new contexts.

Acknowledgements

I would like to thank Rina Zazkis for her insightful comments and encouragement on an early draft of this paper.

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