The Mathematics Enthusiast

Volume 7
Number 2 *Numbers 2 & 3*

Article 12

7-2010

The history of mathematics as a pedagogical tool: Teaching the integral of the secant via Mercator's projection

Nick Haverhals

Matt Roscoe

The history of mathematics as a pedagogical tool: Teaching the integral of the secant via Mercator's projection

Nick Haverhals[1] *& Matt Roscoe*[2]
Dept of Mathematical Sciences
The University of Montana

Abstract: *This article explores the use of the history of mathematics as a pedagogical tool for the teaching and learning of mathematics. In particular, we draw on the mathematically pedigreed but misunderstood development*[i] *of the Mercator projection and its connection to the integral of the secant function. We discuss the merits and the possible pitfalls of this approach based on a teaching module with undergraduate students. The appendices contain activities that can be implemented as an enrichment activity in a Calculus course.*

Keywords: conformal mapping; history of mathematics; integrals; Mercator projection; rhumb lines; secant function; undergraduate mathematics education

Introduction

There is no shortage of research advocating the use of history in mathematics classrooms (Jankvist, 2009). Wilson & Chauvot (2000) lay out four main benefits of using the history of mathematics in the classroom. Its inclusion "sharpens problem-solving skills, lays a foundation for better understanding, helps students make mathematical connections, and highlights the interaction between mathematics and society" (Wilson & Chauvot, 2000, p. 642). Bidwell (1993) also recognizes the ability of history to humanize mathematics. His article opens with description of mathematics instruction treated as an island students perceive as "closed, dead, emotionless and all discovered" (p. 461). By including the history of mathematics, "we can rescue students from the island of mathematics and relocate them on the mainland of life that contains mathematics that is open, alive, full of emotion, and always interesting" (p. 461). Marshall & Rich (2000) argue that the history of mathematics can be a facilator for the reform called for by the NCTM.

In addition to the benefits mentioned above, Jankvist (2009) identifies more gains that can be had by using the history of mathematics. Among them are increased motivation (that can be found in generating interest and excitement) and decreased intimidation - through the realization that the mathematics is a human creation and that its creators struggled as they do. Jankvist (2009) also mentions history as a pedagogical tool that can give new perspectives and insights into material and even can serve as a guide to the difficulties students may encounter as they learn a particular mathematical topic. Marshall & Rich (2000) conclude their article by saying:

[1] nicolas.haverhals@umconnect.umt.edu
[2] roscoem@mso.umt.edu

> To sum up, history has a vital role to play in today's mathematics classrooms. It allows students and teachers to think and talk about mathematics in meaningful ways. It demythologizes mathematics by showing that it is the creation of human beings. History enriches the mathematics curriculum. It deepens and broadens the knowledge that students construct in mathematics class. (p. 706)

This is by no means a comprehensive summary of the research advocating the use of history in mathematics. All of the research cited above, particularly Jankvist (2009), provides many more sources. Bidwell (1993) also mentions three ways of using history in the classroom. The first is an *anecdotal display*, which features the display of pictures of famous mathematicians or historical facts in the classroom. The second is to *inject anecdotal material as the course is presented*. Here, Bidwell is referring to making historical references to coursework while it is being covered. Barry (2000), however, cautions against letting the use of history limited to the use of anecdotes. The third use mentioned is to *make accurate developments of topics a part of the course*. This third use best describes the remaining contents of this article.

Background of this research

The current research evolved out of an assignment given to two of the authors (N. Haverhals & M. Roscoe) in a graduate level history of mathematics class at the University of Montana[ii] which mutated into the study that is currently being reported. The remainder of the article is a reporting of this research.

Methodology

The study sought to investigate the merits of employing a historical approach through the teaching and learning of the topic of the integral of the secant, a topic that is common to most second semester calculus courses at both the high school and university level. The integral of the secant played a key role in the development of the Mercator map in the 16^{th} and 17^{th} centuries. The map was a critical tool during the age of discovery due to the fact that it was a conformal projection of the globe onto the plane, that is, it projected the globe in such a manner as to preserve angles (at the cost of distorting lengths and areas). This property allowed mariners to navigate across large expanses of featureless ocean by following compass bearings that the map provided.

In preparation for the investigation, the authors conducted a review of pertinent literature. In particular, we sought material treating the subject of the Mercator projection that was easily translated into an educational setting where the integral of the secant is taught *through* the historical reenactment of its discovery. Furthermore, we wanted to construct a unit that could be realistically included in a traditional calculus course. Since these courses typically allow for little divergence from firmly established traditional content, we decided that the unit had to be brief, able to be employed in a single class meeting.

After reading a number of articles and several educational units which dealt with the role of the integral of the secant in the Mercator projection we set out to design and create our own activity. It was decided that two documents would be produced. The first document consisted of a "take home" primer on the Mercator projection (which can be found in Appendix 1). This document "set the stage" for the investigation. In it we gave a brief description of the problem of conformal projection and motivated the historical need for such a map during the age of discovery. We included new terms such as "rhumb line", "loxodrome" and "conformal" as well as introduced the key historical figure in the development of the map, namely, Gerhardus Mercator. We also included an example of the important role of the conformal map by demonstrating how a seaman's bearing changes for a non-conformal plane projected map leading to errors in navigation.

The second document that we produced was conceived as the "in-class" exploration of the integral of the secant (Appendix 2). In this document, we hoped to lead students through a "historical reenactment" of the discovery of the integral of the secant motivated by a desire for mathematical description of the Mercator projection. The document first asked students to reason about the horizontal scaling of latitudes and then went on to describe the "mechanical integration" that was carried out by Edward Wright which determined the vertical conformal scaling. Students were asked carryout and compare the accuracy of two such approximating integrations. A proof of the closed form of the integral was provided with several missing steps and students were asked to complete the traditional proof. Finally, a number of extensions to the in class exploration asked students to investigate the way that distance is distorted by the projection.

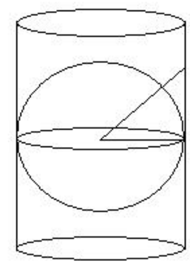
A sample of 16 undergraduate students consisting of 9 males and 7 females participated in the study. The students were all mathematics majors who had completed their calculus sequence. Students were given the "take home" document one week before being asked to complete the "in-class" document. A period of two hours was scheduled for the in-class portion. Students completed the exercise in groups of two. The authors of the study circulated about the classroom, answering questions. At the end of the period, the completed documents were collected.

One week after participation in the classroom investigation into the historical account of the integral of the secant, two groups of four students each were chosen for separate case study analysis. One group of four students was interviewed to probe for affective reaction to the educational activity. These students were asked the following questions.

1. Describe what you learned in the activity on the historical approach to the integral of the secant.
2. How was the activity different from a typical mathematics class?
3. Here is the calculus textbook that we use here at the University of Montana. This is the presentation for the integral of the secant. How does it differ from the historical presentation of the integral of the secant that was presented last week?
4. Did the activity change the way that you view mathematical discovery?
5. Did the activity change the way that you view learning mathematics?
6. Would you say that you were more or less motivated to complete the traditional proof of the integral of the secant after having placed its discovery in a historical context?
7. Does including mathematics history make mathematics more meaningful? How?

Student response to these questions was audio recorded and transcribed for analysis. A second group of four students was shown a false physical model of the Mercator projection (chosen because of its commonality in supposedly "explaining" the projection). These students were asked to disprove the physical model using the knowledge that they had acquired through participation in the educational activity on the Mercator projection. Specifically, these students were shown the following:

> A common misconception about the Mercator projection involves a physical model where the globe is projected onto a cylinder tangent to its radius through "illumination" of the globe from its center. Use the figure at the right to find the vertical stretching factor to determine whether or not this physical model gives rise to the Mercator projection.



Student response to this prompt was audio recorded and transcribed for analysis.

Framework

While the use of history in mathematics classrooms is widely supported, it is probably safe to say that implementation is not so widely seen. Man-Keung Siu (2007) provides the following list of 16 *unfavorable factors* that contribute to the lack of history in mathematical classes:

(1) "I have no time for it in class!"
(2) "This is not mathematics!"
(3) "How can you set question on it in a test?"
(4) "It can't improve the student's grade!"
(5) "Students don't like it!"
(6) "Students regard it as history and they hate history class!"
(7) "Students regard it just as boring as the subject mathematics itself!"
(8) "Students do not have enough general knowledge on culture to appreciate it!"
(9) "Progress in mathematics is to make difficult problems routine, so why bother to look back?"
(10) "There is a lack of resource material on it!"
(11) "There is a lack of teacher training in it!"
(12) "I am not a professional historian of mathematics. How can I be sure of the accuracy of the exposition?"
(13) "What really happened can be rather tortuous. Telling it as it was can confuse rather than to enlighten!"
(14) "Does it really help to read original texts, which is a very difficult task?"
(15) "Is it liable to breed cultural chauvinism and parochial nationalism?"
(16) "Is there any empirical evidence that students learn better when history of mathematics is made use of in the classroom?"

The list was compiled by Siu for the purpose of collecting the views of mathematics educators.

The authors took used their experience in preparing and administering their Mercator map activity to address each of these factors. The list was divided into groups of related items and these sub-lists form the next four sections.

A Philosophical Response to Three Unfavorable Factors

Many of Siu's (2007) unfavorable factors for the use of history of mathematics in classroom teaching are tied to philosophical questions concerning the nature of mathematics and mathematics instruction. That is, in response to the query of, "Why don't you use the history of mathematics in your classroom?" teachers often disclose personal beliefs about mathematics and how it should be taught. Specifically the following list of three of Siu's unfavorable factors fit this description:

(1) "I have no time for it in class!"
(2) "This is not mathematics!"
(9) "Progress in mathematics is to make difficult problems routine, so why bother to look back?"

By stating that there is no time for a historical approach to the teaching of mathematics in the classroom, teachers reveal personal beliefs that the history of mathematics is peripheral to other content matter in the subject which are given higher priority in classrooms where time is a limited commodity. This is especially the case in the modern American setting where student performance in mathematics on state and federally mandated tests is directly tied to school funding which places direct pressure on mathematics teachers to produce students who are computationally proficient in arithmetic, geometry, algebra and the like.

The statement "this is not mathematics" is a rejection of the history of mathematics as traditional mathematical classroom content. Here, the personal philosophy of mathematics might be seen as one which draws a clear line between that which is *history* and that which is *mathematics* thereby promoting a vision of mathematics that is at once highly specialized while also strictly compartmentalized from other areas of study.

Finally, the statement equating progress in mathematics with making "difficult problems routine" is a firm expression of a philosophy of mathematics which can be best described as one which seeks to avoid the complexities associated with the historical development of the subject in favor of routines, algorithms and memorized procedures.

While many authors have written about the role of personal philosophies in the teaching of mathematics (Thom, 1973; Hersh, 1986; Ball, 1988; etc), perhaps Paul Ernest's (1988) framework of philosophies of mathematics provide the most succinct and streamlined approach to the subject. Ernest identifies three psychological systems of beliefs about mathematics each with components addressing the nature of mathematics, the nature of mathematics learning and the nature of mathematics teaching.

Ernest identifies the *instrumentalist view*. Here the conception of mathematics is one of an "accumulation of facts, rules and skills to be used in the pursuance of some external end" (Ernest, 1988, p. 2). Mathematics is then thought of as a useful collection of unrelated rules and facts. The teacher's role is then envisioned as an *instructor* who promotes skills mastery and correct performance in his or her students through strict adherence to curricular materials. The student of mathematics fulfills a role characterized by compliant behavior leading to the mastery of mathematical content, namely, the rules, skills and mathematical procedures presented by the teacher.

Ernest secondly describes the *Platonist view* of mathematics. Here the conception of mathematics is one of a "static but unified body of certain knowledge" (Ernest, 1988, p.2) which is discovered (not created) by humans through mathematical investigation. Thus mathematics is inherent to the world in which we live. It is a "universal language" which exists independently from human knowledge or awareness of the subject. The teacher's role is then envisioned as an *explainer,* tasked with the promotion of conceptual understanding in his or her students as well as a presentation of mathematics as a unified system of knowledge. The student then learns mathematics through reception of mathematical knowledge. Proficiency is demonstrated through student presentation of knowledge possession, usually taking the form of variations of the same sorts of problems presented by the teacher during instruction.

Finally, Ernest describes the *problem solving view* of mathematics. Mathematics is conceived of as "a dynamic, continually expanding field of human creation and invention, a cultural product" (Ernest, 1988, p.2). Here mathematics is seen as a process rather than product, a means of inquiry rather than a static field of knowledge. As a human created body of knowledge, mathematics is envisioned as uncertain and open to refutation and revision. The teacher's role is then taken as *facilitator* and is tasked with the confident presentation of problems. The student then learns mathematics through the act of problem solving, actively constructing knowledge through investigation. Proficiency in mathematics is equated with autonomous problem solving and even problem posing.

Placing the historical approach to the integral of the secant into this philosophical framework it seems apparent that our approach to this common calculus topic seems most strongly associated with the problem solving view of mathematics. The activity was presented to the student group with little more than an introduction concerning the problem of mapping a spherical globe onto a planar map. The questions posed were largely open-ended and lacked any algorithmic approach. The role of the teacher (here, the authors) was one of facilitator. Students were expected to construct their own knowledge through active investigation and group collaboration.

Perhaps more notable is the fact that the presentation of the discovery of the integral of the secant, as a necessary component of a conformal projection of the globe, can be thought of as a historical argument for the problem solving view of mathematics. Indeed, the first map presented by Mercator was produced without the aid of the integral – Mercator produced his map through geometric construction (Rickey, 1980). The map was improved upon by Wright through "mechanical integration" and the use of tables of values of the secant taken at one minute intervals (Sachs, 1987). Finally, the actual exact value for the integral of the secant was discovered by Henry Bond through the keen observation that Wright's sums seemed to agree with tables of values of

$$\ln\left|\tan\left(\frac{\theta}{2}+\frac{\pi}{4}\right)\right|,$$

which can be shown to equal

$$\ln\left|\sec\theta+\tan\theta\right|,$$

the value of the integral of the secant that is presented in modern calculus textbooks today. Certainly this presentation of the subject presents a notion of mathematics that is "dynamic", "continuously expanding" and "open to refutation and revision" as Ernest's problem solving approach describes.

If we place each of the three of Siu's unfavorable factors listed above into Ernest's framework of mathematical philosophies it seems apparent that these objections are most closely aligned with the instrumentalist view of mathematics. "I have no time for it in class" seems to imply a classroom where the teacher's role is taken as instructor (note the use of "I" instead of "we"). "This is not mathematics" seems to reject historical lessons on the basis that they do not promote any specific "skill". Finally, the instrumentalist approach is especially apparent in the last comment that identifies "progress in mathematics" as making "difficult problems routine" which presents a truly procedural philosophy of mathematics and mathematics instruction.

While philosophical debate over the true nature mathematical knowledge continues, educators from both Platonist and problem-solving perspectives level criticism directed at the instrumentalist approaches to mathematics education. Indeed, Thompson (1992) notes that none of the philosophical models of mathematics education have "been the object of more criticism by mathematics educators than the model following most naturally from an instrumentalist perspective" (p.136). Critics of the approach argue that computational proficiency is not necessarily a measure of mathematical understanding and point to studies that document impoverished notions of mathematics by students who display satisfactory performance on routine tasks (Schoenfeld, 1985). Proponents of the problem solving view also object that the instrumentalist approach denies the student the opportunity to "construct" their own mathematical knowledge thereby disallowing the student true understanding of the structure in mathematics which is discovered through active investigation.

It seems evident that these three unfavorable factors for the use of a historical approach to the integral of the secant are actually subtle philosophical arguments concerning the nature of mathematics and mathematics instruction. When placed within Ernest's framework of philosophies of mathematics it is apparent that the incorporation of such an activity into instruction on the topic most closely aligns with the problem-solving view of mathematics, while the reasons not to incorporate such an activity align most closely with the instrumentalist view of mathematics. Perhaps the debate is best concluded through deictic example by imagining a classroom where historical approaches to mathematics are strictly forbidden. In such a world, the student would come away from a mathematics lesson with little notion of where mathematics comes from or how it is developed. There would be no sense of mathematics driven by both practical necessity and human curiosity, both of which play into the story surrounding the integral of the secant. Finally, there would be little appreciation for those that have given us the wealth of knowledge that we now enjoy or biographical inspiration to further the science.

If we have no time for history in mathematics instruction then we have abandoned crucial sources of inspiration and insight. If history of mathematics is not mathematics then mathematics is without a story: alien to the student, not of this world. And if mathematics is meant to "make difficult problems routine" then we should expect our students to excel only in that which is "routine" which certainly will not equip them with the tools to adapt in a changing world.

Student Responses to Unfavorable Factors

Several of Siu's (2007) unfavorable factors for the use of the history mathematics in classroom teaching relate to teacher's beliefs regarding student's opinions about the use of such materials in the classroom. The following four, in particular, fit this description:

(5) "Students don't like it!"
(6) "Students regard it as history and they hate history class!"
(7) "Students regard it just as boring as the subject mathematics itself!"
(8) "Students do not have enough general knowledge on culture to appreciate it!"

Each of these reasons for not incorporating the history of mathematics into mathematics instruction proceeds from the standpoint of the student and argues against its incorporation into the mathematics classroom on two fronts.

The first three factors (5, 6, and 7) seem to argue that the inclusion of the history of mathematics in the mathematics classroom has a negative (or negligible) outcome on student motivation in the subject. Students who do not like or even "hate" the history of mathematics are likely to be unmotivated and even repelled by its inclusion in the classroom. If historical approaches to mathematical topics, such as the integral of secant, are "just as boring" as a more traditional approach then, it is argued, such approaches are perceived as a waste of a teacher's valuable classroom and preparation time. These three factors seem to argue that the benefits of historical approaches to mathematical topics do not outweigh the costs that such approaches require of the teacher in terms of research, planning and implementation.

The last factor (8) is, perhaps, more severe than the first three. For here there is a tone of cultural superiority on the behalf of the teacher. The student is perceived as culturally deficient in their ability to perceive and understand mathematics when it is placed in a historical context. There is a tone of "teacher knows best" what is "good for the student" in terms of the lessons of history.

Our study, which placed the integral of the secant in a historical context by examining the development of the Mercator projection map, found evidence which dispels the factors provided by Siu which are outlined above. During the implementation of the unit, students displayed

intense curiosity in the mathematics behind the projection. With some instructional guidance, all student groups were able to successfully finish the unit in the two hour classroom time that was allotted for the experiment.
In a follow up to the activity, four students were chosen at random for interview which was conducted one week after the classroom meeting in which the experiment had been conducted. Each student was asked the following questions:

1. Describe what you learned in the activity on the historical approach to the integral of the secant.
2. How was the activity different from a typical mathematics class?
3. Here is the calculus textbook that we use in this mathematics department. This is the presentation for the integral of the secant. How does it differ from the historical presentation of the integral of the secant that was presented last week?
4. Did the activity change the way that you view mathematical discovery?
5. Did the activity change the way that you view learning mathematics?
6. Would you say that you were more or less motivated to complete the traditional proof of the integral of the secant after having placed its discovery in a historical context?
7. Does including mathematics history make mathematics more meaningful? How?

Transcripts from these interviews were analyzed for evidence for or against the merits of Siu's factors outlined above. All four interviewees were found to respond favorably to the historical approach to the integral of the secant. Consider the following response from student 1 to question 4:

> I think that it's unbelievable, first of all. That people find these connections…and the fact that these guys did it without the tools that I have now. I mean, Mercator doing this, not perfectly, but pretty good, pretty good, having an idea, um, I just think that it's really cool that they know there's an answer. That these guys are so intelligent that they know something's up…and through their own intuition and through their own work they get there…and that has to be the greatest feeling ever for these guys. So, it gives more respect to anyone that has discovered something that we use or even something that we don't use in our Calc books…it definitely gave me more respect for these guys. It's unbelievable that they did these things…(Student 1)

Certainly the response to the question displays a sense of wonder at the use of mathematics in the Mercator projection. Words such as "cool" and "unbelievable" and "respect" are used in describing the historical discovery of the integral in the making of the map. In response to question 3, student 1 comments:

> And, again, you can show me this and I am going to accept it, 'cause it's in my Calc book and we have no choice but to accept it and memorize it…but that is completely different than starting with this [points to historical approach to integral of secant activity]…starting with integration and ending with the natural log of the secant of x plus the tangent of x. So, for me and the way that I think and the way that I enjoy school, it

> was helpful, and I can even imagine seeing myself start out learning about integration with this example (Student 1)

Again, student 1 responds favorably to the activity placing it in the category of activities that the student "enjoys" in school and calling it "helpful". Student number 2 also expressed a positive reaction to the activity. In response to question 5 the student comments:

> I found it that it made me feel that my work was more important than it usually is. And the fact that usually when you do a problem, you get an answer, and think you're done, but, there really is no point to it that you see…I mean…when you're taking…you're doing integration by parts, it's like, okay, what are we ever going to use this for? And so, you do all this work and you never see really ever where it applies…they'll try to do stuff and…I mean it's really, really basic and it doesn't really apply, but, if they could take examples and show where its used, the historical context, it makes it feel as if you're kind of working along side of those people when they were actually doing the work hundreds of years ago…you went through and saw what they did, and so it gives a level of importance that isn't usually ever there…you know…that was valuable. (Student 1)

Here the student contrasts "traditional" approaches to common calculus topics (integration by parts) with the historical approach to the integral of the secant and describes how the historical approach lends a "level of importance" to an otherwise mundane mathematical topic. Student 3 in response to question 7 echoes this sentiment:

> It's nice to be able to first learn about the secant and then they'll show you what it's used for…then it makes a lot more sense because you have been exposed to it already and you're already kind of familiar to it. It's nicer to see people apply it to their life and situations. (Student 3)

Here we see the characterization of the historical approach as adding a real world "applied" aspect to instruction which is positively characterized as "nice" by the student. Finally, student 4, in response to questions 1 and 7, comments that:

> I liked the historical context. It helps me put things in perspective. It's cool that people were applying integration before integration was codified. It also illustrated the closer and closer estimations using smaller rectangles better than my Calculus study of Riemann sums… understanding how anything, especially math, is related to real world problems and solving them makes me more motivated to understand the methodology and consider broader applications of the problem solving technique. (Student 4)

And so, our analysis of student response to the historical approach to the integral of the secant is unanimous in its approval of the educational technique. Rather than Siu's suggestion that they hate it, find it boring and non-motivating, our study group reported that they "enjoy" it and characterize the approach as "cool" and "useful" commenting that they are "more motivated" and "interested" in mathematics which is given an added "level of importance" when it is couched in a historical context. Furthermore, our data shows that students *do* have enough cultural maturity to appreciate the approach. There is a sense that the accomplishment of a conformal map was an "unbelievable" achievement won through great intelligence with "the tools that they had" before the advent of calculus through mechanical integration. There is evidence of an understanding that these early map makers were "applying integration before integration was codified" which displays the student's ability to imagine a mathematical culture before the invention of the calculus. Finally, there is a sense that the student is "working along side of those people when they were actually doing the work hundreds of years ago" which certainly indicates a level of cultural respect and admiration.

A Logistical Response to Unfavorable Factors

As expected from a list as comprehensive as Siu's, a number of the factors that discourage teachers from employing the history of mathematics deal with very practical matters. This list of unfavorable factors relating to practicality is divided into two groups: logistics and preparation.

The following list is comprised of the factors the authors would describe as logistical in nature. These are factors that might discourage even those who are inclined to include the history of mathematics in their teaching. Each factor in the list will be addressed individually, from the perspective of the authors and through the lens of creating and implementing the teaching module. The list of logistical factors, determined by the authors, is as follows:

(3) "How can you set question on it in a test?"
(4) "It can't improve the student's grade!"
(13) "What really happened can be rather tortuous. Telling it as it was can confuse rather than to enlighten!"
(14) "Does it really help to read original texts, which is a very difficult task?"

The first factor on the list, (3), is one that most teachers would probably agree needs to be addressed before using the history of mathematics in their classes. Rarely is anything presented by a teacher deemed unimportant. However, it is generally necessary to test students on material in order for them to see it as important. At the same time, asking students to be accountable for historical and/or biographical information is likely to re-enforce the sorts of ideas responsible for factor (2).

The trick, then, is to create assessment questions that use the skills developed in the course of using the history of mathematics. This can come in a number of forms. The first and most obvious is the form that assessment generally comes in. Often, when new material is presented in a class, the teacher will lecture for some period of time and then leave the students to practice the skills taught for the remainder of the class and on the assigned homework. However, this practice usually only matches a small portion of the lecture time – when the teacher presents examples, usually occurring at the end of the lecture. Typically the practice problems assigned does not match a bulk of the lecture – the part when the teacher explains, justifies or proves the technique or material to be taught. Thus if this portion of lecture time is spent presenting (explaining, justifying or proving) the material from a historical perspective, little need be changed in the way that students are assessed.

This response to factor (3) may easily be criticized, however, on the grounds that it still promotes the sort of "drill and kill" mentality that the inclusion of the history of mathematics is largely meant to discourage. If the goal for including the history of mathematics in the classroom is to get away from this mentality and promote the development of other skills (such as problem solving) then assessment should reflect this aim. For this, the preparers of the historical content need to get creative.

For our study, a problem very much related to the Mercator projection was chosen and given to four students during a follow-up interview a week after the study. The students were asked to evaluate the validity of a commonly used physical characterization of the Mercator projection. The illustration (see methodology section above) was described as follows: students were asked to imagine a semi-transparent globe sitting snuggly in a cylinder with a light bulb glowing in the center of it. Light shines through the globe and "shadows" are cast by land masses on the globe. These shadows become the placement of the land masses on the cylinder, which is then sliced and laid flat to form the map. While this characterization does share some properties with the Mercator projection (the poles of the globe can never be projected and the stretching increases with increases in latitude), it is actually a different (non-conformal) projection. As the students were able to deduce, the factor of stretching in this alternative representation is a tangent function which is not equal to

$$\ln\left|\sec\theta + \tan\theta\right|,$$

the factor of stretching found in the exploration.

While the students definitely needed some nudging to get them started, once the ball was rolling all four were able to deduce that the representation would not yield the Mercator projection. The authors feel that a question such as this would make for a legitimate test question. Granted, the students did need encouragement and some may not have known where to start if the saw it on a test they had to work out on their own. However, the authors believe that this is due in large part to the fact that the students are rarely asked to perform this type of task. If mathematics was taught from more of a historical perspective, they would be more accustomed to problem-solving and therefore would be more flexible in their thinking. It is worth noting that the first part of the Mercator exploration asked the students to find the factor of stretching for arbitrarily chosen latitude. The potential assessment question had the students do the same thing from a different perspective – so it indeed was assessing a skill they used in the activity.

The second item on this sub-list of unfavorable factors, (4), deals with students' grades. Like before, the way in which using the history of mathematics in the classroom affects students' grades depends on how it is used. If it is used simply as an alternative lecturing format with little or no change in assessment then it is possible that students see no benefit in terms of grade. However, the history of mathematics can be used as a guide for how students learn mathematics. This can be seen in the difficulties students have in learning particular mathematical ideas (Moreno-Armella & Waldegg, 1991; Jankvist, 2009) and in students' conceptions of mathematical proof (Bell, 1976; Almeida, 2003). By using history as a guide for how students learn, it is possible that instruction could be improved.

This can even be taken a step further. Fawcett (1938/1966) describes a high school geometry class which, it could be argued, was set up in a fashion that mimics the historical development of Euclidean geometry. Under the guidance and supervision of the teacher and through class discussion and consensus, the experimental geometry class created their own textbooks consisting of definitions, axioms and theorems. The main goal of the experiment was to improve the students' knowledge of mathematical proof, a goal that was achieved. It should be noted, however, that the students also outperformed a control class on a standardized geometry test administered state-wide. This was despite the fact that the experimental class covered less material than the control class. Some of this uncovered material showed up on the standardized test, but the students in the experimental class were flexible enough to deal with material new to them.

The last two unfavorable factors in this section, (13) and (14), are quite similar and will be addressed together. Basically, they are both speaking to the fact that dealing with historical mathematics can be quite difficult. Much of the time, this is true. While modern day mathematicians can often handle the mathematical content associated with historical mathematical documents, other barriers to understanding exist. One stumbling block stems from the fact that the first solution to a problem is rarely the most elegant or straightforward to understand, as mentioned in factor (13). Language (terminology) and notation are two other major obstacles, referred to in (14).

The authors believe, however, that the module provided serves as an example that these concerns can be addressed. Although the authors made every effort to make the activity historically accurate, much of the historical difficulty was described, rather than recreated. Students were asked to mimic the process of mechanical integration (before they likely realized that was what they were doing) used historically but to a far less accurate, but more user friendly, degree. Based on student responses this served the intended purpose, as each group was able to recognize that smaller intervals gave better approximations. This was a necessary insight to understand the link between the Mercator projection and the integral of the secant. Also, students were told about the "lucky accident" that resulted in the discovery of the closed form for the integral of secant; they were not expected to find it on their own. Relieving the students of unnecessary difficulty does not mean they are left with nothing to do on their own. As is mentioned in the module, the original proof for the validity in question was extremely laborious and difficult. The students were then guided through an alternative (and later) historical proof – one that allowed the use of methods familiar to the students from their pre-calculus and calculus classes.

The amount of editing of historical material virtually eliminates the factor (14) from the students' perspective. The only original material that made it into the final teaching module was quotes carefully chosen to provide historical context (or humor, as the case may be). Factor (14) is not yet eliminated from the content-preparer's perspective. However, this will be addressed in the next section.

A Response to Unfavorable Class Preparation Factors

Factor (14) raises a completely different issue from the teacher's point of view. If the students can be shielded from difficult to read original texts, are not the teachers responsible for doing the shielding? Not completely, as was seen by the authors in the preparation of the teaching module. This issue is tied into the next three factors that Siu mentions. They are:

(10) "There is a lack of resource material on it!"

(11) "There is a lack of teacher training in it!"
(12) "I am not a professional historian of mathematics. How can I be sure of the accuracy of the exposition?"

The bulk of the material that made its way into the activity came from journal articles or other teaching modules relating to the topic. In this way, the authors were not responsible for dealing with the difficult task of reading original material. Rather, they were free to concentrate on preparing the material in such a way as to be appropriate for their students.

This speaks to factor (10) as well. In preparing the module, the authors found more than enough resource material on the topic. It is true that not all of the material was deemed suitable by the authors for their targeted students. However, the materials found did provide enough for a complete, coherent teaching module to be put together. It should be noted that the authors acknowledge the possibility that factor (10) has not been interpreted as Siu intended. It is possible that what is being referred to is a lack of ready-to-use materials that can be implemented by teachers with little or no modification. The authors can not speak to this concern directly. Although some of the materials used were indeed designed to be used without modification, as mentioned, none were deemed appropriate for the students who were to see it. This was of no concern to the authors, however, because the creation of the module was an end in and of itself. Appropriate, ready-to-use materials were not sought. Instead, enough material was collected to complete the activity and that is all. It is unclear whether or not a completed module that was appropriate for the students in question could have been found.

The experience the authors had while completing this activity also helps dispel factor (11). While the module was originally meant to be part of a history of mathematics course the authors were taking at the time, no skills were explicitly taught in the class that lent themselves to its creation. What was gained from the class, however, was an appreciation for and interest in the history of mathematics. This new motivation, coupled with the authors' existing mathematical skills, was sufficient to see them through to the completion of the project. As the activity was designed for students taking Calculus II, the authors feel that it (or something similar) could have been created by any teacher with a grasp of Calculus II material and the desire and interest to do so. Thus, in general, a lack of training in the history of mathematics need not be a deterrent for those teachers who wish to use it in their classrooms.

The last factor to be addressed in this section, (12), that will be addressed in this section is related to (11). One may get the feeling that since he or she lacks training in the history of mathematics, they may be ill-equipped to judge the accuracy of sources. The authors were able to alleviate this concern through the use of articles from reputable scholarly journals. The use of such journals assures the readers (content-producers) that the materials have been peer-reviewed. That way, the burden of verification is placed on professionals and the teachers preparing the material can concentrate on making it appropriate for and useful to their students.

A Response to the Final Two Unfavorable Factors

The authors thought that the last two factors did not relate closely with the others and will be addressed briefly here. They are:

(15) "Is it liable to breed cultural chauvinism and parochial nationalism?"
(16) "Is there any empirical evidence that students learn better when history of mathematics is made use of in the classroom?"

The first of these, (15), speaks to the potential that the history of mathematics has to create a classroom setting that is not agreeable to the teacher. It is possible that the history of mathematics could be used to create a narrow view of the development of mathematics. This narrow view, in turn, may lead to the impression that a select group of peoples alone were responsible for (and therefore good at) mathematics. The can easily be avoided by the careful inclusion of mathematics from many different cultures. While the contributions of the ancient Greek and later European cultures are well known, they are not the only wells from which to draw. The articles referenced by Katz (1995) and Wang (2009) serve as examples of articles that describe methods developed by other cultures.

The last unfavorable factor, (16), is on to which the authors can not respond. To their knowledge, there is no convincing empirical evidence that students learn better when the history of mathematics is used in the classroom. However, the student responses gathered from this article suggest that students would welcome the inclusion of the history of mathematics – and more enthusiastic students generally make for better learners.

Conclusion

Siu has done the mathematics educational community a service by playing the role of devil's advocate in maintaining a list of popular reasons why teachers do not use historical approaches in mathematics education. His list provided the framework for analysis of this educational experiment on the historical approach to the integral of the secant in the development of the Mercator projection map.

We found that several of Siu's unfavorable factors could be characterized as subtle philosophical statements regarding the nature of mathematics and mathematics instruction. When viewed within the framework of Ernest's (1988) philosophies of mathematics it is apparent that these objections most closely align with an instrumentalist view of mathematical knowledge and mathematical instruction. This view equates computational proficiency with mathematical understanding and is subject of much criticism in denying true mathematical understanding. In contrast, the historical approach employed in this study placed problem solving at the heart of instruction. By taking a historical approach to the subject, students learn that the closed form of the integral of the secant was "needed" to mathematically explain the Mercator projection. A historical approach allows for crucial sources of inspiration, insight and motivation which are missing from strict instrumentalist approaches, seen in this light, any argument against historical approaches can be seen as an argument in favor of an impoverished notion of mathematics.

A number of Siu's unfavorable factors were characterized as teacher statements regarding negative student predispositions to historical approaches in the mathematics classroom. Analysis of our data disproves these notions. Student response to the activity was universally positive thus affirming the approach from a student standpoint and dispelling misapplied characterizations commonly held by teachers.

There were unfavorable factors that were seen as logistical concerns. Our unit demonstrates that each of these concerns can be overcome. We were able to creatively "set a question on a test" to the historical approach. We feel that, in the area of problem solving, the historical approach does "help student's grades" by endowing them with a richer and more meaningful understanding of the process of mathematical meaning making. Finally, student difficulty in confronting historical text can be alleviated by careful and thoughtful presentation that is at once historically accurate while educationally streamlined toward an intended goal, in this case, an understanding of the integral of the secant.

In terms of Siu's unfavorable classroom preparation factors, our study has shown to dispel many of the commonly espoused concerns. We encountered ample resources that aided the creation of the educational unit. No special teacher training was required. Lastly we appealed to reputable journals to insure accuracy of historical exposition, thus, an educator need not be a professional historian of mathematics in order to create educational materials which teach mathematical concepts from a historical standpoint.

While we acknowledge the concerns of cultural chauvinism and parochial nationalism raised by Siu, we feel that an evenhanded approach to historical topics in mathematics education may lead to quite the opposite outcome. Here the "historically educated" student of mathematics might come to an awareness of the great cultural and national diversity that has contributed to the development of the subject.

Finally, in response to Siu's assertion that there is a lack of empirical evidence that supports historical approaches to mathematics in terms of improving student understanding we stand by the fact that student response to the unit pointed to greater interest and enthusiasm in the subject, which, we assert, are prerequisites to deep and meaningful learning in mathematics.

References

Almeida, D. (2003). Engendering proof attitudes: Can the genesis of mathematical knowledge teach us anything? *International Journal of Mathematical Education in Science and Technology, 34*(4), 479 – 488.

Ball, D. (1988). Unlearning to teach mathematics. *For the Learning of Mathematics*, 8(1), 40-48.

Barry, D.T. (2000). Mathematics in Search of History. *Mathematics Teacher*, *93*(8), 647- 650.

Bell, A. W. (1976). A study of pupils' proof-explanations in mathematical situations. *Educational Studies in Mathematics, 7*(1/2), 23 – 40.

Bidwell, J. K. (1993). Humanize Your Classroom with The History of Mathematics. *Mathematics Teacher*, 86, 461-464.

Ernest, P. (1988). *The impact of beliefs on the teaching of mathmatics.* Paper presented at ICME VI, Budapest, Hungary.

Fawcett, H. P. (1966). *The nature of proof: A description and evaluation of certain procedures used in senior high school to develop an understanding of the nature of proof.* New York, NY: AMS Reprint Company.

Hersh, R. (1986). Some proposals for revising the philosophy of mathematics. In T. Tymoczko (Ed.), *New directions in the philosophy of mathmatics* (pp. 9-28). Boston: Birkhauser.

Jankvist, U. T. (2009) A characterization of the "whys" and "hows" of using history in mathematics education. *Educational Studies in Mathematics, 71*(3), 235 – 261.

Katz, V. J. (1995). Ideas of calculus in Islam and India. *Mathematics Magazine, 68*(3), 163-174.

Marshall, G. L., & Rich, B. S. (2000). The Role of History in a Mathematics Class. *Mathematics Teacher*, 93 (8), 704-706.

Moreno-Armella, L., & Waldegg, G. (1991). The Conceptual Evolution of Actual Mathematical Infinity. *Educational Studies in Mathematics*, *22*, 211–231.

Rickey, V. F. & Tuchinsky, P. M. (1980). An application of geography to mathematics: history of the integral of the secant. *Mathematics Magazine*, 50(3), 162-166.

Sachs, J. M. (1987). A curious mixture of maps, dates, and names. *Mathematics Magazine*, 60(3), 151-158.

Schoenfeld, A. (1985). *Mathematical problem solving*. San Diego, CA: Academic Press, Inc.

Siu, M.-K. (2007). No, I don't use history of mathematics in my class. Why? In F. Furinghetti, S. Kaijser, & C. Tzanakis (Eds.), *Proceedings HPM2004 & ESU4* (revised edition, pp. 268–277). Uppsala: Uppsala Universitet.

Sriraman, B., Roscoe, M., & English, L. (2010). Politicizing Mathematics Education: Has Politics gone too far? Or not far enough? In B. Sriraman & L. English (Eds), Theories of Mathematics Education: Seeking New Frontiers (pp. 621-638), Springer, Berlin/Heidelberg.

Thom, R. (1973). Modern mathematics: Does it exist?. In A. G. Howson (Ed.), *Developments in mathematical education: Proceedings on the Second International Congress on Mathematics Education* (pp. 194-209). Cambridge: Cambridge University Press.

Thompson, A. G. (1992). Teachers' beliefs and conceptions: a synthesis of research. In D. A. Grouws Ed., *Handbook of research on mathematics teaching and learning* (pp. 127-146). New York: Macmillan.

Wang, Y. (2009). Hands-on mathematics: Two cases from ancient Chinese mathematics. *Science & Education*, *18*(5), 631 – 640.

Wilson, P.S., & Chauvot, J. B. (2000). Who? How? What? A Strategy for Using History to Teach Mathematics. *Mathematics Teacher*, 93(8), 642-645.

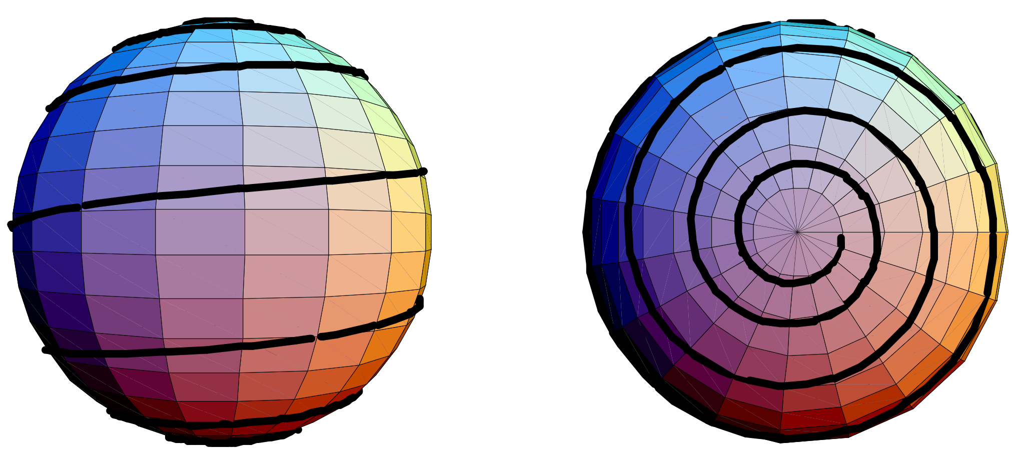
Appendix 1
Student take home

Mercator's World Map
A Historical Approach to the Integral of the Secant

Suppose that you are tasked with navigating a ship that is to travel from a point in Europe to the "New World" recently discovered across the Atlantic Ocean. How would you navigate the vessel using only 16th century technology? Most mariners during the age of discovery steered their ships along lines of constant bearing using a magnetic compass. A path of constant bearing on the globe is called a "rhumb" line named for the Spanish *rumbo* meaning "way" or "direction". This concept of a path of constant bearing was later named a loxodrome from the Latin *loxos* signifying "slant" and *drome* signifying "running". So, most mariners of the 16th century travelled paths across the ocean that we know call loxodromes.

On the globe a loxodrome intersects all north-south lines of constant longitude at the same angle. Parallels, or lines of constant latitude, are therefore loxodromes because they intersect all north-south lines of constant longitude (meridians) at right angles. Early sailors, cartographers and later mathematicians realized that these paths of constant bearing became spiral-like curves whenever the direction chosen was not due east or west. This effect is due to the fact that as a rhumb line moves north the distance separating meridians grows closer and thus the line must turn away from the pole to maintain the heading.

Figure 1: *Two views of a typical rhumb line, a path of constant bearing, on a globe. All rhumb lines, except paths of constant latitude, create spiral paths on the globe differing only in slope.*



The spiraling nature of lines of constant bearing created the need for a special kind of map in which a sailor could draw a line from his present location to his objective and measure the bearing by determining the angle that is formed by the path and the meridians that are crossed in route. Such a map was presented to the world in 1569 by Gerhardus Mercator and is today known as the Mercator projection. The map signified a gigantic improvement over previous plane projection maps and is still widely used in navigation today.

Figure 2: *A Mercator Projection Map*

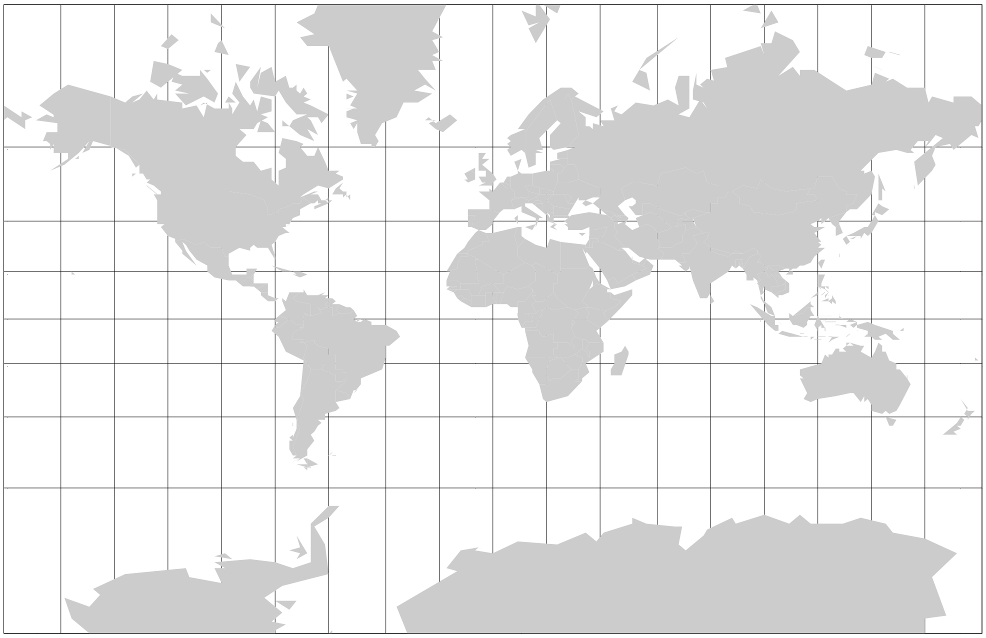
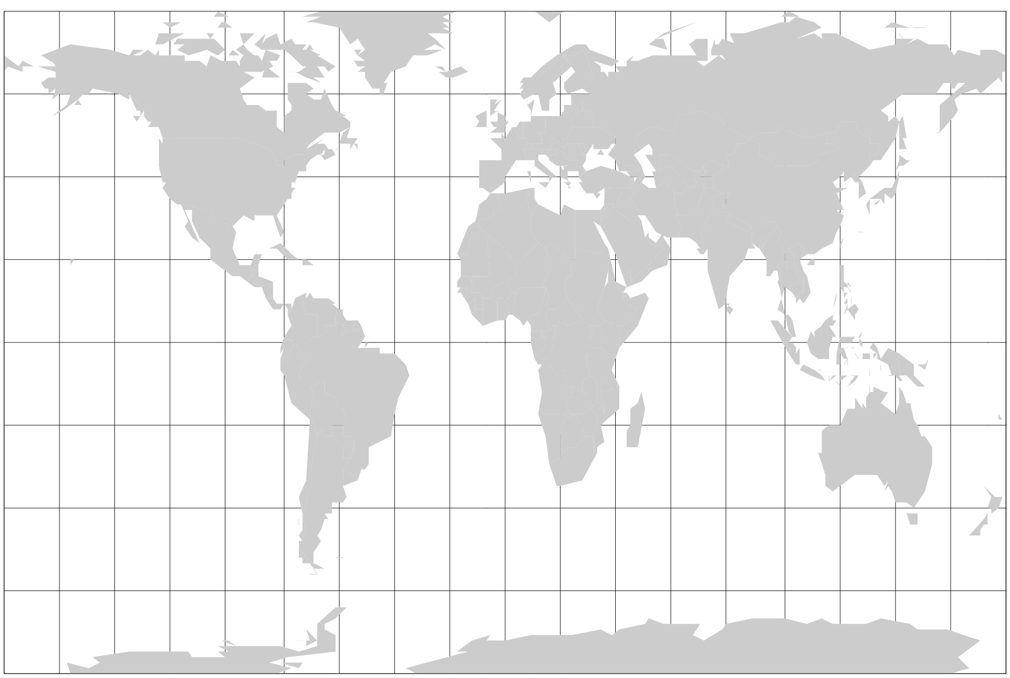


Figure 3: *A Plane Projection Map*



Close inspection of the two maps reveals that the plane projection has evenly spaced lines of constant latitude. In contrast, the distance between lines of constant latitude grows as a function of distance from the equator in Mercator's version.

In order to better understand the effect of Mercator's special scaling, consider a path on the globe that carries a seaman from Colon, Panama to Land's End, England. Using a magnetic compass (or the North Star) a sailor can successfully make such a trip by following a rhumb line that leaves Colon at a bearing of approximately 56° from true north. Such a path of travel over the globe then crosses all meridians at this same angle and thus scribes a spiraling loxodrome across the surface of the globe. If we were to plot the path of such a journey on both the Mercator and plane projection maps we would find that only on the Mercator projection would such a journey actually cross all meridians at an angle of 56° from true north thus correctly directing the sailor to his home (figure 4). On the plane projection map such a journey crosses all meridians at an angle of 60° from true north (figure 5). If we plot a course that leaves Colon at 60° from true north we find that our sailor is erroneously directed to France as indicated on the Mercator map (figure 6).

Figure 4: *Our Seaman's Journey on the Mercator Projection Map: Directs Seaman to a Bearing of 56 Degrees East of True North*

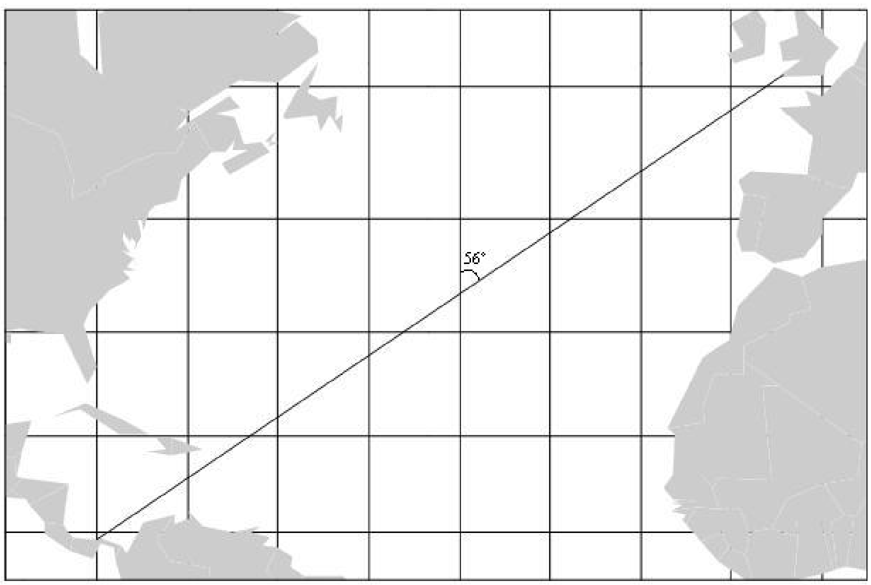


Figure 5: *Our Seaman's Journey on the Plane Projection Map: Directs Seaman to a Bearing of 60 Degrees East of True North*

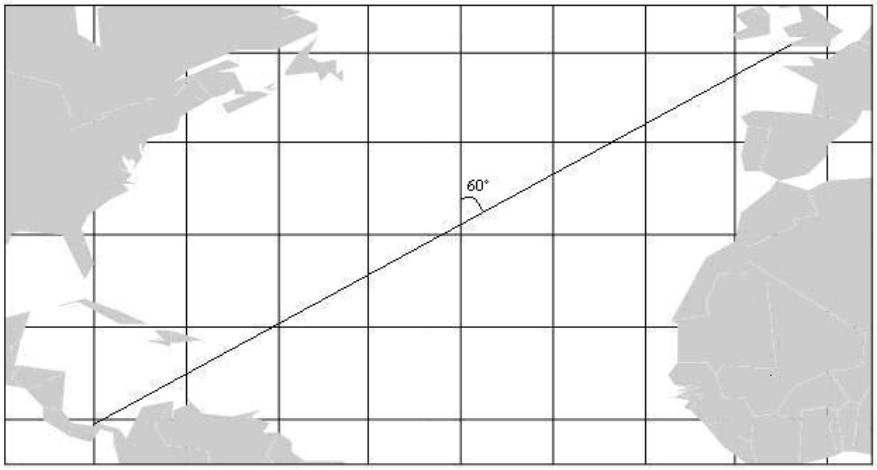
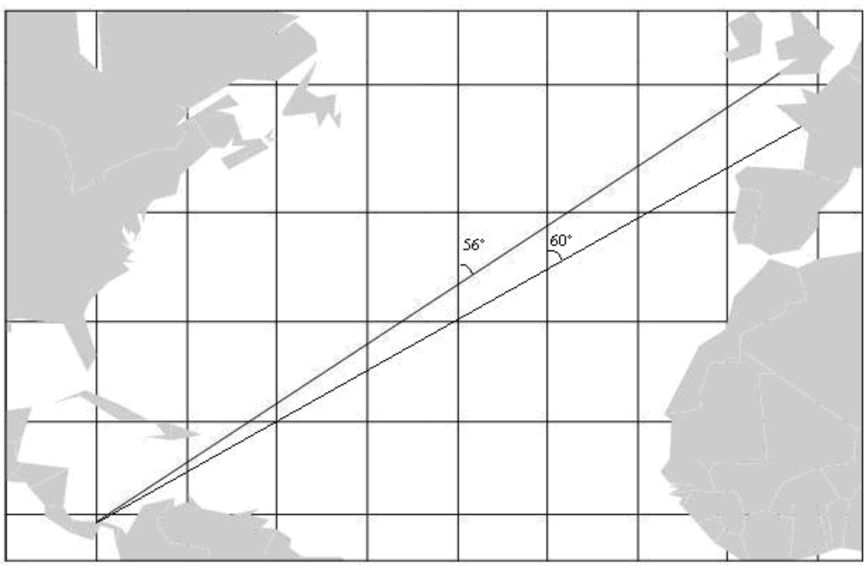


Figure 6: *Our Seaman's Journey on the Mercator Projection Map: Bearing 60 Degrees and Bearing 56 Degrees East of True North*



So, it becomes apparent that the Mercator projection provides the seaman with a much more useful tool where a line of constant compass direction corresponds to a straight line on the map which making it possible for a 16th century seaman to determine the correct line of bearing to follow in order to arrive at the intended destination.

From a mathematical point of view, Mercator's projection is *conformal,* meaning that the projection from the globe onto the plane preserves angles. It should be apparent that the projection does not preserve distances. It is a mathematical fact that any projection from the sphere onto the plane cannot preserve both of these quantities, but that is another story best saved for another day. How did Mercator decide on his special scaling? Mercator himself comments on this scaling in the legend of the map of 1596:

> In view of these things, I have given to the degree of latitude from the equator towards the poles, a gradual increase in the length proportionate to the increase of the parallels beyond the length which they have on the globe, relative to the equator. (Sachs, 1987)

Mercator created his special map using a compass and straight edge but mathematicians of the era challenged "any one or more persons that have a mind to engage" to mathematically describe the scaling that produced the successful map. (Rickey, 1980)

References

Alexander, J. (2004). Loxodromes: a rhumb way to go. *Mathematics Magazine*, 77(5), 349-356.

Carslaw, H. S. (1924). The story of Mercator's map. *The Mathematical Gazette*, 12(168), 1-7.

Comap. (N.D.). Calculus of the Mercator map. Derive Lab Manual for Calculus. Houghton Mifflin Co., New York, NY.

Rickey, V. F. & Tuchinsky, P. M. (1980). An application of geography to mathematics: history of the integral of the secant. *Mathematics Magazine*, 50(3), 162-166

Sachs, J. M. (1987). A curious mixture of maps, dates, and names. *Mathematics Magazine*, 60(3), 151-158.

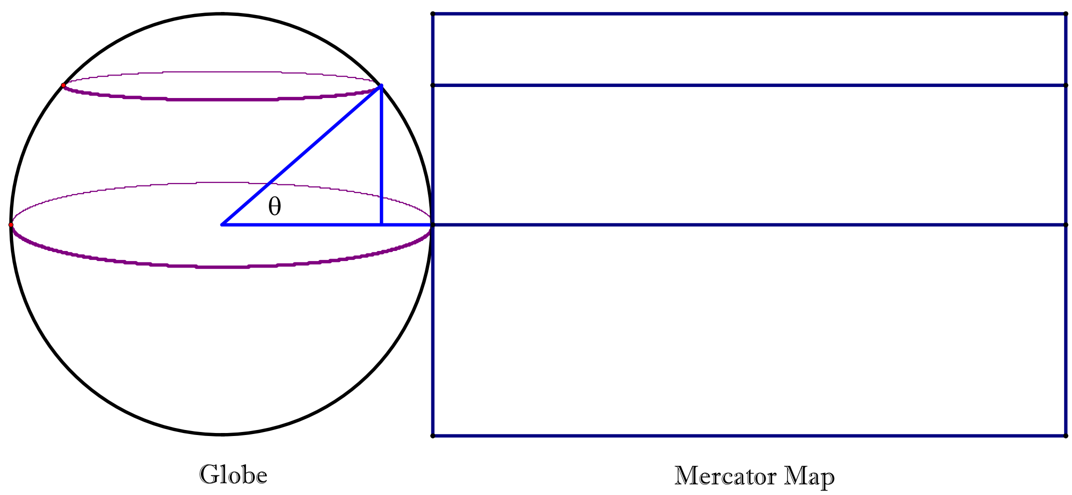
Tuchinsky, P. M. (1987). Mercator's world map and the calculus. In *UMAP modules in undergraduate mathematics and its applications*. Lexington, MA: COMAP.

Appendix 2
Mercator in class activity

Mercator's World Map
A Historical Approach to the Integral of the Secant

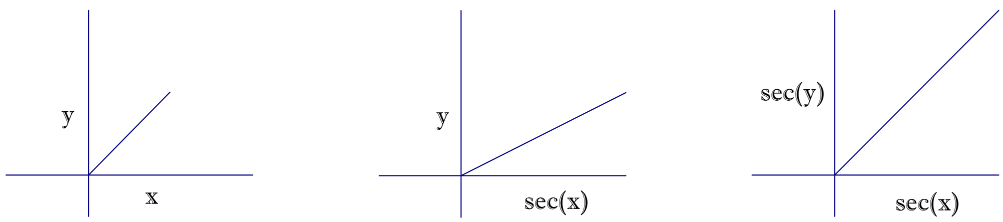
Mercator wrote, "In making this representation of the world we had…to spread on the plane the surface of the sphere in such a way that the positions of places shall correspond on all sides with each other both in so far as true direction and distance are concerned and as concerns correct longitudes and latitudes…With this intention we have had to employ a new proportion and a new arrangement of the meridians with reference to the parallels…It is for these reasons that we have progressively increased the degrees of latitude towards each pole in proportion to the lengthening of the parallels with reference to the equator." (Rickey, 1980)

Using the figure provided below determine the function that governs, "The lengthening of the parallels with reference to the equator." That is, given a parallel at latitude θ determine the function $f(\theta)$ that tells us how the latitude lines must be stretched horizontally in order to appear equal in length to the equator.



In the previous quote Mercator comments, "…It is for these reasons that we have progressively increased the degrees of latitude towards each pole in proportion to the lengthening of the parallels with reference to the equator…" Mercator determined this vertical scaling through compass constructions. It was not until 1610 that Edward Wright, a Cambridge professor of mathematics and a navigational consultant to the East India Company, described a mathematical way to construct the Mercator map which produced a better approximation than the original. In 1599 he published *Errors in Navigation Detected and Corrected*. Wright argued that in order to preserve angles on the Mercator projection, the vertical scaling factor had to be the same as the horizontal scaling factor. To visualize this phenomena imagine a 45° angle drawn on a small portion of a globe. Recall that when this region gets projected to the plane, it gets stretched in the horizontal direction by an amount that depends on the latitude. Notice what happens. The angle as projected is no longer 45°. In order for the angle to be preserved, a stretch must occur in the vertical direction that matches the horizontal stretch.

Angle on Globe **Horizontal Scaling** **Conformal Scaling**



Wright also realized that the correct interval of placement of a parallel on the Mercator projection was the result of the addition of any subintervals into which it could be divided. To this end, Wright made a table of secants taken at a common interval, added these results and then multiplied by the interval widths to determine the location of a particular parallel on the map. So, if the location of the 60th parallel is desired and an interval width of 10° is used then Wright would have performed the following:

Table of Secants	**Multiply by Interval Width**	**Location on Mercator Map**
Secant 10° = 1.0154 Secant 20° = 1.0642 Secant 30° = 1.1547 Secant 40° = 1.3054 Secant 50° = 1.5557 Secant 60° = 2.0000 Total = 8.0954	10° × 8.0954 = 80.954	Place the 60th parallel at a location that is 80.954° north of the equator.

Use Wright's method to determine the location of each of the following parallels using an interval length of 5 degrees.

Latitude on the Globe	Location on Mercator Projection
15°	
30°	
45°	
60°	
75°	
90°	

You should notice that the location of the 60^{th} parallel that you just calculated is different than the one that was calculated in the example that proceeded. Which placement produces a more accurate map? How do you know?

What difficulty did you encounter in determining the placement of 90° Latitude? What is this location on the Globe? What are the implications for the Mercator Map?

Historically, Wright's table of secants had an interval width of one minute or one sixtieth of a degree. Describe mathematically, using modern notation, the process that Wright is carrying out in determining the vertical scaling of the Mercator projection. Is Wright's method exact? How could it be improved?

As you have probably discovered, the exact mathematical explanation for Wright's technique in developing the vertical scaling for the Mercator projection hinges on a closed form for the integral of the secant. In Wright's time this result was still some 50 years from being discovered. However, with Wright's charts at his disposal, Henry Bond in the 1640s had a very lucky accident. Bond, who fancied himself a teacher of navigation and mathematics, compared Wright's table to a table of values in which the tangent function was composed with the natural logarithm. This led him to conjecture that the closed form for the integral of the secant equaled

$$\ln\left|\tan\left(\frac{\theta}{2}+\frac{\pi}{4}\right)\right|,$$

which can be shown to equal

$$\ln\left|\sec\theta+\tan\theta\right|.$$

The first proof of the integral of the secant was provided in 1668 by James Gregory. Edmund Halley commented on the proof, "The excellent Mr. James Gregory in his *Exercitationes Geometricae*, published Anno 1668, which he did not, without a long train of consequences and complication of proportions, whereby the evidence of the demonstration is in a great measure lost, and the reader wearied before he attain it." (Rickey, 1980). And so we avoid this proof and instead offer guidance through a proof offered by Isaac Barrow. Complete the missing steps in the proof

$$\int \sec\theta\, d\theta =$$

$$=$$

$$= \int \frac{\cos\theta}{(1-\sin\theta)(1+\sin\theta)}\, d\theta$$

$$=$$

$$=$$

$$=$$

$$= \frac{1}{2}\int \frac{\cos\theta}{1-\sin\theta} + \frac{\cos\theta}{1+\sin\theta}\, d\theta$$

$$=$$

$$=$$

$$= \frac{1}{2}\left[-\ln|1-\sin\theta| + \ln|1+\sin\theta|\right] + c$$

$$=$$

$$= \frac{1}{2}\ln\left|\frac{(1+\sin\theta)^2}{(\cos\theta)^2}\right| + c$$

$$=$$

$$= \ln|\sec\theta + \tan\theta| + \mathrm{c}$$

The angle θ in the previous integral assumes a radian measure. If θ is measured in degrees then a change of variables will yield the following:

$$\int_0^\theta \sec\theta \; d\theta = \frac{180}{\pi} \ln\left|\sec\theta + \tan\theta\right|$$

Use this result to determine the exact location of each of the following parallels.

Latitude on the Globe	Location on Mercator Projection
15°	
30°	
45°	
60°	
75°	
90°	

How do these placements compare to those found earlier in the exercise?

EXTENSIONS

1. How do distances vary on the Mercator map relative to latitude? Consider travelling parallel to the equator at various latitudes on the map. Consider travelling perpendicular to the equator at various latitudes on the map.

2. How do areas vary on the Mercator map relative to latitude? What happens to the area of a region as one moves farther from the equator? Give examples.

[i] The mathematics behind the Mercator map has nothing to do with the way the map ended up being used for political purposes. A number of critical theorists who have no idea of the mathematics behind the map run around saying "the map was purposefully made that way" . Gerhardus Mercator (1512-1594) created the map for navigational purposes with the goal of preserving conformality, i.e., angles of constant bearing crucial for plotting correct navigational courses on charts . In other words a line of constant bearing on a Mercator map is a rhumb line on the sphere. Conformality as achieved by Mercator with his projection came at the price of the distortion that occurred when projecting the sphere onto a flat piece of paper. The history of the map is also linked to the limitations of the Calculus available at that time period, and the difficulty of integrating the secant function (see Carslaw, 1924). Mercator himself comments, "…It is for these reasons that we have progressively increased the degrees of latitude towards each pole in proportion to the lengthening of the parallels with reference to the equator…" Mercator determined this vertical scaling through compass constructions. It was not until 1610 that Edward Wright, a Cambridge professor of mathematics and a navigational consultant to the East India Company, described a mathematical way to construct the Mercator map which produced a better approximation than the original (Sriraman, Roscoe & English, 2010).

[ii] Math 606 was a topics course in the history of mathematics taught by Professor Bharath Sriraman in Spring 2009. One of the assignments in the course was to take Carslaw's (1924) paper and rewrite in such a way as to make it readable by modern students. In an effort to get more mileage out of the work to be done, the students asked if it would be possible to turn the assignment into something that could be used in the future – namely an activity designed for use in a calculus classroom. Dr. Sriraman allowed for the change and arranged for the activity to be completed by undergraduate students who had completed Honors Calculus II. He also encouraged us to use the opportunity to perform some research.