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Problem Posing from the Foundations of Mathematics

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Abstract: *This reflective paper develops a repertoire of questions for teachers to use in their classrooms during episodes of mathematics discussions with and among students. These questions are motivated by an examination of questions posed by Wittgenstein in Zettel, and are connected to underlying tacit assumptions about mathematics, most of which lie subtly below the generally accepted milieu of math-talk. Once classrooms norms have been established to encourage participation by all students in a democratic and just classroom environment, these questions can be used effectively to stimulate meaningful discourse. These questions provide important examples of problem posing designed to encourage student reflection.*

Keywords: classroom environment; discourse; history of mathematics; problem posing; mathematical discourse; tacit knowledge; Wittgenstein; *Zettel*

Introduction

Recent developments in mathematics education research have shown that creating active classrooms, posing and solving cognitively challenging problems, promoting reflection, meta-cognition and facilitating broad ranging discussions, enhances students' understanding of mathematics at all levels. The associated discourse is enabled not only by the teacher's expertise in the content area, but also by what the teacher says, what kind of questions the teacher asks, and what kind of responses and participation the teacher expects and negotiates with the students. Teacher expectations are reflected in the social and socio-mathematical norms established in the classroom. But not all teachers are adept at asking the appropriate questions in a way that enhances learning. Often, they have not had experience in classroom settings or as team teachers where this kind of active dialogic discourse has been observable, nor have they had opportunities to practice it themselves. How can one short-cut the process of equipping teachers with the perspectives and skills they need to respond in routine situations when probing and challenging questions are called for? If teachers were to have a repertoire (which some teachers eventually gain through trial and error) of questions and insights, they would perhaps improve the level of cognitive demand and intellectual stimulation in their classrooms for all students, even for those whose classroom participation is infrequent, hesitant or uncomfortable.

To elucidate this kind of repertoire I have gone back into the historical record of mathematical discussions and found, in *Zettel* (Wittgenstein, 1967), a rich source of short phrases or aphorisms which I have matched with probing questions that relate to the foundations of mathematics. This

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resource offers suggestions that correlate with discussions in mathematics classes that I have taught, observed, participated in and analyzed during the last two decades.

Background

Wittgenstein was a major force behind the scenes of some of the most important developments in the philosophy of mathematics (as he was also for the disciplines of Psychology, Philosophy, Logic, Linguistics and Ethics.) (Marion, 1998) During the twentieth century there were so many innovations in science and philosophy that the explanation of the foundations of mathematics by Ludwig Wittgenstein most often goes completely unacknowledged. Wittgenstein matriculated into Cambridge just as Russell & Whitehead's *Principia Mathematica* was printed, and his was one of the first critical minds to study it and see its fundamental weaknesses (Russell & Whitehead, 1956). He was credited with having convinced Bertrand Russell that his joint work was seriously flawed (Ramsey, 1925). Wittgenstein never published a counter explanation, but, his explanation of the foundations of mathematics can be found scattered in pieces and bits in several of his writings some 50 years after he began that study, after he had been a teacher and lecturer for many years, and published only after his death in 1951 (Marion, 1998). One goal of Wittgenstein's effort was to challenge the mindset people have about mathematics, and open a door to new ways of thinking. In this essay I have extracted and put together into topical categories his expressions about mathematics that were published in his collection entitled *Zettel* (Wittgenstein, 1967). I have taken *Zettel* as a resource, and from each paragraph identified probing questions and hypothetical challenges that will give teachers a repertoire of questions and knowledge about mathematical reasoning and reflection, and suggestions on how best to stimulate mathematical explanations and justification.

This paper is intended to help teachers prepare to make meaningful discourse happen in their own classrooms. Where I have paraphrased or expanded, I have endeavored to preserve the sense and uniqueness of Wittgenstein's insightful comments. This paper is not an overview identifying which of Wittgenstein's concepts and explanations were most influential on mathematicians and in the development of mathematics both at Cambridge and more broadly around the world. Nor is it an attempt to identify what is missing in *Zettel* that might be needed to complete the discussion of the foundations of mathematics. Much of that discussion can be found in other books compiled from Wittgenstein's writings and published posthumously (Wittgenstein, 1978). I have resisted the temptation to say too much about what Wittgenstein is trying to uncover with his probing questions and aphorisms. The purpose of this paper is to gather in one place a collection of insightful questions and probes that could be used to help teachers pose problems for and with their students.

As testimony to Wittgenstein's relevance in teaching mathematics, his writing and teachings have inspired many authors and contemporary experts. Anna Sfard (2008) pays homage to both Lev S. Vygotsky and Ludwig Wittgenstein:

> …[T]wo giants whose shoulders proved wide enough to accommodate legions of followers and a wide variety of interpreters … [They] continue to inspire new ideas even as I am writing these lines. This, it seems, is due to one important feature their writings have in common: rather than provide information, they address the reader as a partner in thinking; rather than presenting a completed edifice with all the scaffolding removed,

> they extend an invitation for a guided tour of the construction site; rather than present firm convictions, they share the 'doubt that comes after belief.' These two writers had a major impact on my thinking; I can only hope they had a similar effect on my ability to share it. (pp. xxi-xxii)

Whereas the writings of Vygotsky are widely studied by pre-service teachers and educators, Wittgenstein is widely quoted but seldom studied. I hope that this paper contributes to correcting that deficit.

Wittgenstein usually uses general or surrogate subjects (i.e. colors, words) rather than discipline specific topics as examples, and it is up to the reader to see and say "Aha… that also explains what is at the basis of mathematics." Some of the following quotes are to be viewed in this way; others are more obviously applicable to the discussion of the foundations of mathematics. I have laid these side by side (quotes from Wittgenstein are in italics) to invite development of a provocative discussion about the foundations of mathematics, as well as to prompt provocative questions for teachers to use productively when planning and implementing instruction. After a brief introduction and historical background, what follows is organized into eight categories - Emerging Potentialities; Practice Applying Rules; Origins in Nature; Mathematical Procedures; Cultural Implications; Teaching relationships; Familiarity with Proofs; and Philosophy of Mathematics - to provide a source of questions that are perhaps most germane to the reader's present teaching needs.

Emerging Potentialities

Important mathematical understanding does not begin in the teen years when students are for the first time exposed to Euclidean Geometry and proofs beginning with definitions and axioms. Long before this, playtime experimentation with objects and quantities prepares each child's mind to receive more formal instruction in the abstractions that we know as numbers, counting, arithmetic and all parts of mathematics. But children are seldom encouraged to reflect on the progression from play to formal mathematical reasoning.

> *700. Why do we count? Has it proved practical? Do we have the concepts we have, e.g. our psychological concepts, because it has proved advantageous? –And yet we do have certain concepts on that account, we have introduced them on that account.*
>
> *701. At any rate the difference between what are called propositions of mathematics and empirical propositions comes to light if one reflects whether it makes sense to say: 'I wish twice two were five!'*

With these questions, Wittgenstein prompts us to think about what is special about mathematics. When scientists wish, they develop hypotheses; when writers wish, they develop fiction; when doctors wish, they look for alternative cures. In mathematics, wishing is akin to conjecturing, a playfulness in asking "What if?", and "Why?", and "Why not?", and "Why does it have to be this way?" Taking these questions as a beginning point, I will show how Wittgenstein builds an explanation of the language-game of mathematics and how it is not only a different (but in important ways parallel) activity to the way we think and live. This language-game[2] is a form of

[2] Mathematicians often create their own vocabulary by giving special meanings to ordinary terms and phrases. Giving specialized meaning to old terms allows mathematicians to say things that might otherwise be difficult to say. But this can be confusing to novices who assume the "ordinary language" meaning.

being and doing that is interlaced with our language and our daily activities. As such, it is also an intellectual feature of our lives that is identifiable as a separate, dynamic activity.

A historical perspective can be useful, and here Wittgenstein suggests reflection on one's own personal history as it relates to things mathematical.Questions teachers might ask students: How did you learn to count? When did you begin to do arithmetic? Can you rephrase this (any particular) discussion into a mathematical proposition? Why is this a mathematical proposition? Can you wish it to be different, or is it somehow determinate? Can you wish a certain kind of result? Can you influence the result with your will? Do you take what you get?

By the time a child develops the notion of ***conservation of numbers*** the ability to think and talk mathematically is normally well underway (Piaget, 1952.) Researchers have identified rudimentary mathematical thinking in babies as early as two months old (Dehaene, 1997) which suggests that the basic sense about "number" is inherited and perhaps instinctive in humans. Elementary problem solving, a child's propensity to experiment and make quantitative choices, can be witnessed very early by all observant parents.

> *103. …--But what should be called 'making trials' and 'comparisons' can in turn be explained only by giving examples, and these examples will be taken from our life or from a life that is like ours.*
>
> *104. If he has made some combination in play or by accident and he now uses it as a method of doing this and that, we shall say he thinks. –In considering he would mentally review ways and means. But to do this he must already have some in stock. Thinking gives him the possibility of perfecting his methods. Or rather: He 'thinks' when, in a definite kind of way, he perfects a method he has. [Marginal note: What does the search look like?]*
>
> *105. It could also be said that a man thinks when he learns in a particular way.*

Our shared childhood cultural development prepares us to develop (or inhibits) our capacity to deal with, think about, and operate with numbers and mathematical symbols. Our intuition (or considering, as Wittenstein describes it), guides us. Educators can learn to encourage its development in students and teachers.

Questions: How can you make a conjecture and use your knowledge to verify your claim? How might you pose "what if?" questions? How can you vary the parameters or values to see a generalizable result? What personal experience have you had that shows you that this result is valid? Can you predict the result of your work by estimating, rounding off and achieving a quick guess? How does your answer compare to the estimate? What do you think will happen if …?
And of course counting and speaking with numbers has a long tradition even in many prehistoric cultures. For example, the Ishango bone, a piece of bone notched with what are believed to be mathematically significant tally marks, was discovered in the small African fishing village of Ishango, on the border of Zaire and Uganda. It is one of the oldest known mathematical artifacts, dating to 20,000 BC. (Zaslavsky, 1979).

> *143. We might say: in all cases what one means by 'thought' is what is alive in the sentence. That without which it is dead, a mere sequence of sounds* [i.e., mathematical symbols] *or written shapes…*

> *144. How words* [or symbols] *are understood is not told by words alone* [explanation of meaning].

This suggests not just a vital, but essential role for pictures, graphs, images, symbols, manipulatives and other visual displays in mathematics that help to explain and illuminate mathematical ideas.

There were well developed cultures in California such as Yahi (Kroeber, 1976) from which we have first-hand accounts of how numbers and mathematical thinking were passed on as a sacred, mystical tradition comparable to the way some contemporary people use Numerology. For most of us mathematics is the language of science and is interwoven into our mundane and routine lives. But superstitions surrounding and containing numbers persist in many cultures, including our own.

Questions: Why is 3 a lucky number? Why is thirteen considered an unlucky number in many cultures? Why is 4 an unlucky number in Japan? Can you give other examples of how number words are used in superstitious ways, or, of how numbers are used in popular culture, and imbued with special meaning? What is special about prime numbers? Or perfect numbers?

Mathematics teachers may find it interesting and rewarding to watch students in the beginning stages as they progress through ever more complex applications. They go through false starts of understanding, and then go back and review until it is completely integrated into their mental framework. Even the average student can go back to the beginning of the book half way through the course and do the "easier" mathematics with greater facility than during the first days. Although many students seem to struggle, they learn something, about both the content and the tacit rules of operation and construction.

> *153. It somehow worries us that the thought in a sentence is not wholly present at any one moment. We regard it as an object which we are making and have never got all there, for no sooner does one part appear than another vanishes.*

This is the kind of mental thinking that can take place as students learn mathematics, or as someone develops mathematics even in the most fundamental settings. It helps to explain why not all students do their homework correctly each day. Understanding is elusive and sometimes tenuous.

Questions: How can you repeat this and say it in a different way? How do we incorporate knowledge of irrational numbers, imaginary numbers and exponents, for example, as part of our operational system? What rule should we use to factor a polynomial?

I invite you to try Wittgenstein's exercise which relates to the development of mathematics:

> *310. Imagine you had to describe how humans learn to count (in the decimal system, for example). You describe what the teacher says and does and how the pupil reacts to it. What the teacher says and does will include e.g., words and gestures which are supposed to encourage the pupil to continue a sequence; and also expressions such as 'now he can count'. Now should the description which I give of the process of teaching and learning include, in addition to the teacher's words, my own judgment: the pupil can count now,*

> *or: now the pupil has understood the numeral system? If I do not include such a judgment in the description –is it incomplete: And if I do include it, am I going beyond pure description: --Can I refrain from that judgment, giving as my ground: 'That is all that happens'?*

Thus, do we think that the evidence for learning is just the accomplishment of the task, not the process developed, or the explanation that is given? Can he/she still do it next week? How does that relate to participation in the classroom, when a few bright students consistently raise their hands first, or dominate in group activity? We should encourage the bright students, give them praise; but should we give only them the extra challenge of justifying, proving or generalizing? Or is it better to give them an additional application?
Questions: That looks excellent, tell us how you did that? Can you justify your work? How might you generalize your results? Is there another way to look at this problem?

Practice in Applying Rules
Yet the words for numbers and operations that match symbols, and our facility with these at ever more sophisticated levels, have their origin in the way we are taught.

> *145…If the sign is an order, we translate it into action by means of rules* [e.g., + for addition, or] *or tables. It does not get as far as an impression, like that of a picture; nor are stories written in this language.*
> *146. In this case one might say: 'Only in the system has the sign any life.'*

Mathematics is in many ways a different language, woven into our ordinary language (or form of life) that some master and with which others struggle for their entire lives. It is easy to imagine telling someone something about the measurement of a shirt or a hat using mathematical terms and realizing by looking at the expression on their face, or hearing their questions, that they do not have facility with the mathematics being used, that they are outside the system, do not feel at ease participating in that language game.

Questions: Why do we use Greek letters as symbols in mathematics? Can you write a sentence that restates what this sign signifies, what it is telling you to do? What is an algorithm? Can you identify and emphasize the key vocabulary words (in a new mathematical concept)? Who developed this symbol, and when did it come into popular mathematical vernacular? What problems were being posed and why was it important to solve them? When do we use this word to mean something else? And is this the best word you can choose for this mathematical use?

Wittgenstein gives a story of how people develop and use rules which provides some insight into the foundations of mathematics and their transfer to new generations.

> *292. …If an order were given us in code with the key for translating it into English, we might call the procedure of constructing the English form of the order 'derivation of what we have to do from the code' or 'derivation of what executing the order is'. If on the other hand we act according to the order, obey it, here too in certain cases one may speak of derivation of the execution.*
> *293. I give the rules of a game* [e.g., matrix algebra]. *The other party makes a move, perfectly in accord with the rules, whose possibility I had not foreseen, and which spoils*

> *the game, that is, as I had wanted it to be. I now have to say: 'I gave bad rules; I must change or perhaps add to my rules.'*
> *So in this way have I a picture of the game in advance?* [Sometimes yes, sometimes no – perhaps we are inventing new mathematics*] In a sense: Yes.*
> *It was surely possible, for example, for me not to have foreseen that a quadratic equation need have no real root.*

When Wittgenstein uses examples from mathematics that help explain his point it is perhaps easier to see how he intended all of his discussions to also apply to the foundations of mathematics even when it is ostensibly about language or thinking. The concept of following rules is an essential ingredient of Wittgenstein's notion of the foundations of mathematics.

Questions: What rule are you following here? And why is this important to follow? Why is it important in addition to align numbers with the decimal points in a vertical column? When you find a mistake in your procedure or result, what rule have you violated? How would you explain how you can expand this compact notation? Why do you simplify improper fractions?
It seems reasonable to support the idea that teaching and training is how mathematics is transferred from generation to generation, but how did the first mathematician develop his or her techniques of solving problems such as telling time or measuring distances? Telling time using an analog clock is a rather advanced mathematical activity, judging from the many intermediate steps of knowledge development that are needed. Examining the roots of learning mathematics is useful both for its own sake and as an activity of hypothesizing about how mathematics came to be and how it developed step by step. It is now well acknowledged that mathematical knowledge is socially constructed (Sfard, 2008; Vygotsky 1962, 1978).

> *412. Am I doing child psychology? –I am making a connexion(sic) between the concept of teaching and the concept of meaning.*
> *413. One man is a convinced realist, another a convinced idealist and* [each] *teaches his children accordingly. In such an important matter as the existence or non-existence of the external world they don't want to teach their children anything wrong…*
> *419. Any explanation has its foundation in training. (Educators ought to remember this.)*

Their training, their personal beliefs and competency in mathematics together determine how teachers approach the task of teaching. What determines how well students develop a meaningful understanding of mathematical concepts? If mathematics is truly a distinguishable language-game, does a thorough understanding change the person's life in a way that no other knowledge does, e.g., history?

Questions: Is the context of mathematics important to you, or do you simply enjoy engaging in the game, puzzle, or algorithm? Do you need to see real-world applications of mathematics in order for it to make sense and be meaningful and useful to you? Would you learn mathematics more easily if it were connected to your daily life and activities? Write a short story to illustrate this mathematics.

Clearly there are a lot of linguistic and cultural skills that come into play when we set out to teach and learn mathematics. Consider this simple act of mimicking a certain procedure:

> *305. 'Do the same.' But in saying this I must point to the rule. So its application must already have been learnt. For otherwise what meaning will its expression have for him?*
> *306. To guess the meaning of a rule, to grasp it intuitively, could surely mean nothing but: to guess its application. And that can't now mean: to guess the kind of application; the rule for it. Nor does guessing come in here.*

Students who guess about rather than understand the structure of the application of the rules of mathematics often commit the error of false generalization; they apply rules to algorithms where these do not apply. And they end up believing that mathematics is akin to magic, or at least mysteriously complicated.

Questions: Why can you do the same in this case? Why can you apply the same algorithm here? Are you able to transfer your knowledge to a new situation? Can you generalize? Can you extend your result? Try to look at different ways of adding and multiplying fractions, and see the results obtained when these rules are applied differently. How can you decide whether this alternative algorithm is valid?

It is obvious that at each step along the way rules and applications have to be developed, assessed, revisited, refreshed, refined and verified before they can be fully understood.

> *307. …The application of a rule can be guessed only when there is already a choice between different applications.*

This statement points to the prior experience and cultural learning from which the student benefits but which is often taken for granted. If this is missing then the teacher has to supply some kind of equivalent explanation or experience.

> *308. We might in that case also imagine that, instead of 'guessing the application of the rule,' he invents it. Well, what would that look like? Ought he perhaps to say 'Following the rule +1 may mean writing 1, 1+1, 1+1+1, and so on'? But what does he mean by that? For the 'and so on' presupposes that one has already mastered a technique.*

Here Wittgenstein is most explicit about discussing the foundations of mathematics, that these are based on the idea that "…one has already mastered a technique." This is in part what it means when we say that mathematical knowledge is socially constructed. A good deal of mathematical technique is imbedded in our mathematical language. It is developed as tacit knowledge as well, but in learning mathematics we take it out, organize it and put it into a new language-game of its own called doing mathematics. When we begin geometry we already have ideas about a point and a line, but learn new concepts, and also new definitions that explain and clarify familiar concepts.

Questions: Where did the concept of zero develop, and why? What would life be like without negative numbers? How did you know what to do to find the next and n^{th} term in a sequence? What does n stand for?

Origins in Nature

The suggestion that mathematics is merely a collection of rules to be applied, rather than truths of nature, however, is disputed by most mathematicians who think mathematics is fundamentally about patterns, and is taken from nature.

> *293 (cont.) Thus the rule leads me to something of which I say: 'I did not expect this pattern: I imagined a solution always like this…'*

And thus our facility with mathematics grows as does the field of mathematics itself.

> *294. In one case we make a move in an existent game, in the other we establish a rule of the game. Moving a piece could be conceived in these two ways: as a paradigm for the future moves, or as a move in an actual game.*

Since the debate about whether mathematics is discovered or invented is long-standing, when does it make sense to suggest that Plato with his 'forms' was correct? The fact that mathematics is essentially impervious to this reflective debate simply goes to show how knowledge of mathematics leads (one way or the other) to an identifiably separate form of life.

Questions: Take me step by step through the process you used to get that result. Is it possible that some mathematics is both discovered in nature and some invented by humans? Do the processes you used and your results hold for any such mathematical proposition…? Can you generalize? Can you justify this? Can you restate the problem for me in your own words? Can you write a proof?

At the very beginning stages of developing the capacity to do mathematics our training takes many forms. But one must look at Wittgenstein's use of the word "training" in its broadest sense, and not just as if one set out to train a horse. This training might include activities that Wittgenstein describes in this somewhat different context of learning language, Through this "training" we discover that mathematics connects to and explains reality; it is not fiction or about arbitrary shapes; further it involves repetition, trial and error and learning by physical manipulations.

> *195. Let us imagine a kind of puzzle picture: there is not one particular object to find; at first glance it appears to us as a jumble of meaningless lines*, [or an undecipherable mathematical formula or a tricky story problem] *and only after some effort do we see it as, say, a picture of a landscape. –What makes the difference between the look of the picture before and after the solution? It is clear that we see it differently the two times. But what does it amount to to say that after the solution the picture means something to us, whereas it meant nothing before?*

Mathematics often organizes and gives order to nature. This principle of pattern recognition is the basis for repetition in teaching, as well as being basic to discovery learning in various forms. This repetitive process is said to lead to "endorsable" mathematical statements when by following well-defined rules anyone is able to come to the same result (Sfard, 2008; Lakatos, 1976).

Questions: How does this relate to your personal experience? Have you seen or done this before? How does this connect with your prior knowledge? Given a sequence of numbers, how do you find the next number? What is your strategy? How is learning mathematics analogous to and different from learning a language? Can you apply this same rule in a problem that has bigger numbers, more complications, symbols instead of numbers? At what point did you figure it out, or gain understanding? Did it just suddenly make sense?

Is the result obtained in ‡195 based on previous rehearsal, or training? Have you acquired the system or algorithm in your mental framework, where the signs and numbers have useful and specific meanings?

> *228. Explain to someone what the position of the clock-hands that you have just noted down is supposed to mean: the hands of this clock are now in this position. –The awkwardness of the sign in getting its meaning across, like a dumb person who uses all sorts of suggestive gestures –this disappears when we know that it all depends on the system to which the sign belongs. We wanted to say: only the thought can say it, not the sign.*

Wittgenstein often tries to tease his reader into what might be thought of as a traditional mental cramp i.e. "…only thought can say it…" This explains how his writings become misunderstood when one mistakes his pedagogy for doctrine. He simply wants to say that there is a good deal of fundamental cultural background and training that gets absorbed prior to one's learning to tell time, for example.

Questions: From where did you get your concept of time passing? What other kinds of clocks do you know about? How can other mechanisms be used to create and communicate [i.e. spherical geometry] information? How do signs get their meanings i.e. ÷, ×, ∞? In what other part of life do signs have special meanings? Describe the different lengths of hands on the clock and how they are used for different kinds of time. When did you first learn this? Who first developed time as a concept? Why is time so important? Why are there 24 hours in a day?

Mathematical Procedures

What does it take to know we have solved a problem correctly, followed a rule, identified a characteristic of nature, or developed a useful model? This is certainly an important detail:

> *196. We can also put this question like this: What is the general mark of the solution's having been found?*

When faced with problems and challenges of increasing complexity, whether caused by curiosity or necessity, humankind did develop counting, geometry and mathematical techniques for solving problems, over periods of thousands of years in several geographically isolated locations around the globe. We spoke earlier of endorsable results to mathematical theorems and propositions. This discussion of verification and justification progresses from this into a usable and repeatable procedure. As mathematics grows, this takes us well beyond the exposition of the foundations of mathematics.

Questions: What shall we take as proper or sufficient explanation, justification? At each level of mathematics what constitutes a proof? What preparation gives students a better capability to do detailed proofs later? Can you take the solution and put it back into the original problem to verify it? What constitutes thinking mathematically?

Cultural Implications

There have been many studies that have shown how success in learning mathematics can be culturally biased. We criticize standardized tests for having cultural bias that can interfere with students' understanding of the input, the instructional language, as well as the elements of a problem, making it difficult for students to demonstrate their mathematical knowledge.

> *201. For someone who has no knowledge of such things a diagram representing the inside of a radio receiver will be a jumble of meaningless lines. But if he is acquainted with the apparatus and its function, that drawing will be a significant picture for him.*

Having a broad frame of reference helps students learn mathematics and mathematics contributes to this cultural growth of students. Once a method to solve a problem was developed, it became part of an expanding form of life that was often shared in an "interactive sphere." The history of □ gives a clue to how ubiquitous and specialized mathematical thinking has been. We have to also learn (and we share) verification skills that are practical and applicable in our lives as we progress with our mathematical education..

Questions: Does that hold true every time you do this…? For every number? What does it mean to use sample data that are representative of the whole population? How can you extrapolate your results? What kind of visualization skills can you employ to see this shape differently? Is mathematics a "universal" language across national borders? Can you verify your result best with a sketch, graph, or diagram?

The skill of verification, and motivation for it, is fundamental to the development of the language-game of mathematics. Developing this is also connected to developing socio-mathematical norms in the classroom, and with practice these will become a matter of course.

> *309. We copy the numeral from 1 to 100, say, and this is the way we infer, think.*
> *I might put it this way: If I copy the numerals from 1 to 100 –how do I know that I shall get a series of numerals that is right when I count them? And here what is a check on what? Or how am I to describe the important empirical fact here? Am I to say experience teaches that I always count the same way? Or that none of the numerals gets lost in copying? Or that the numerals remain on the paper as they are, even when I don't watch them? Or all these facts? Or am I to say that we simply don't get into difficulties? Or that almost always everything seems all right to us?*
> *This is how we think. This is how we act. This is how we talk about it.*

This is how the foundations of mathematical verification skills get sorted out, often by being told: "Do this, don't do that." Or by self discovery. When students learn to contribute their own reasoning and justifications they take ownership of the mathematics they are learning, and this new language becomes part of who they are.

Questions: How can you do that a different way? Can you explain your work step by step? How can you look at this operation and find the mistake or error? What role does neatness have in preventing errors and in developing a clear form of communication?

There is the expectation, unspoken in most cases, that we use mathematics to connect to, understand and explain the world around us, not only as an abstract or academic activity. We mathematize our world in so many ways. (Sfard, 2008) How fast is the wind blowing? How much food (how many calories) should we (or do we) eat in one sitting? We celebrate birthdays and count the days to the next major holiday. We score competitive events, compile performance statistics, and race the clock to meet deadlines. Our whole economy is built on the use of coinage, and measured in billions and trillions. It is no secret and no coincidence that mathematics has become the most widely spoken and only "universal" language of humanity. Someone who is ignorant of mathematics gets left out at some level.

> *695. The understanding of a mathematical question. How do we know if we understand a mathematical question?*
> *A question –it may be said—is a commission. And understanding a commission means: knowing what one has got to do. Naturally, a commission can be quite vague – e.g., if I say 'Bring him something that'll do him good'. But that may mean: think about him, about his state etc. in a friendly way and then bring him something corresponding to your sentiment towards him.*
> *696. A mathematical question is a challenge. And we might say: it makes sense, if it spurs us on to some mathematical activity.* [E.g., making change, buying groceries.]
> *697. We might then also say that a question in mathematics makes sense if it stimulates the mathematical imagination.*
> *698. Translating from one language into another is a mathematical task, and the translation of a lyrical poem, for example, into a foreign language is quite analogous to a mathematical problem. For one may well frame the problem 'How is this joke (e.g.) to be translated (i.e. replaced) by a joke in the other language?' and this problem can be solved; but there was no systematic method of solving it.*

When we translate our activities, patterns or problems from the real world into mathematical operations, how well have we done this? (Mathematical modeling and applications.)

Questions: Does it work and make some sense compared to what we are looking at…? Does this make sense, have we noticed, interpreted and translated reality accurately? Can you unpack this definition into plain language? Can you give an example? What is the proof all about? What does it say to us? What if you change this… ?

Teachers have a tendency to think in learned patterns, avoiding the assumptions inside, below or prior to these theories or customary operations. Is it helpful to know that much of mathematics and learning about mathematics is connected to our culture and to the thinking processes we have developed? Is this a benefit or a hindrance?

> *375. These are the fixed rails along which all our thinking runs, and so our judgment and action goes according to them too.*
> *382. In philosophizing we may not terminate a disease of thought. It must run its natural course, and slow cure is all important. (That is why mathematicians are such bad philosophers*.)

Wittgenstien's humor and biases aside, teachers and learners do have to expect to develop new skills apart from the way they learned mathematics. If we just keep doing it the same way, pedagogy will not improve.

Teaching Relationships

When operating mathematically, do we use the same mental equipment that is used for other linguistic (or cultural) activities? Is the thinking and remembering of music any different? How much does it depend on a certain propensity for logical thinking or common sense? Why are some people said to be "mathematically inclined?" i.e. Gardner's measure of Logical/Mathematical (multiple) intelligence? (Gardner, 2003)

> *666. I can display my good memory to someone else and also to myself. I can subject myself to an examination. (Vocabulary, dates)* [And in mathematics.]
> *667. But how do I give myself an exhibition of remembering?* [is knowing different?] *Well, I ask myself 'How did I spend this morning?' and give myself an answer. –But what have I really exhibited to myself? Remembering? That is, what it's like to remember something? – Should I have exhibited remembering to someone else by doing that?*
> *668. Forgetting the meaning of a word* [or of a symbol] *and then remembering it again. What sorts of processes go on here? What does one remember, what occurs to one, when one recalls what the French word 'peut-etre' means?*
> *669. If I am asked 'Do you know the ABC?' and I answer "Yes' [*Can you count to 20?] *I am not saying that I am now going through the ABC in my mind, or that I am in a special mental state that is somehow equivalent to the recitation of the ABC.*
> *672. …This calculation takes one and a half minutes; but how long does being able to do it take? And if you can do it for an hour, do you keep on starting afresh?*

Here Wittgenstein is again trying to tease the reader into a mental cramp, or theory to answer these hypothetical quandaries e.g. "…how long does being able to do it take?" Scientists now know that different parts of the brain are involved in different tasks, but all of those parts are components, interconnected and at the disposal of our total thinking apparatus. Similar elements and considerations are involved in doing mathematics that are identifiable in our ordinary language i.e. grammar, syntax, iconography, vocabulary… and the mental activities that are engaged flow together and we usually just take these for granted. Experienced teachers and many text books provide learning strategies, and It is a useful teaching practice to solicit strategies from students and make learning strategies explicit.

Questions: Why is it important to learn your multiplication facts until you own them? What is math fluency? Why is it helpful to be able to convert certain key fractions to percentages from memory? Does speed and accuracy in doing the algorithms of arithmetic predict future math competency? What is the best way to learn this procedure so that you will remember it? How do you remember what to do with this (or write this) proof? Why is it important for you to have a clear understanding of definitions, axioms, and be able to apply them in proving theorems?

By now it should be clear that a facility with arithmetic and with all other mathematics requires training and motivation to learn.

> *355. ...If we teach a human being such-and-such a technique by means of examples, --that he then proceeds like this and not like that in a particular new case, or that in this case he gets stuck, and thus that this and not that is the 'natural' continuation for him: this of itself is an extremely important fact of nature.*

How do we develop and expand our logical-mathematical skills? It is important in teaching to understand proximal levels of knowledge. This is also important in the discussion of developing proofs. Proving in a subtle way involves a meta-language and techniques that are outside the operation of the mathematics in question, and yet connected as a "new case."

Questions: Can you give me an example of … from your experience? How did you solve this challenging problem? Explain how you thought about it. Can you draw a picture of this? How do we develop visualization skills? Can you check this conjecture with several examples to determine whether it is likely true or not?

Wittgenstein's discussion of how mathematics is usually a matter of accepted and learned conventions, takes on a subtle touch when he uses an example involving taste.

> *366. Confusion of tastes: I say 'This is sweet', someone else 'This is sour' and so on. So someone comes along and says: 'you have none of you any idea what you are talking about. You no longer know at all what you once called a taste.' What would be the sign of our still knowing? (Connects with a question about confusion in calculating.)*
> *367. But might we not play a language-game even in this 'confusion'? – But is it still the earlier one?—*

Teachers know that there is often more than one way to solve a problem.

> *373. Concepts other than [those] akin to ours might seem very queer to us* [e.g., performing arithmetic in base 8]*; deviations from the usual in an unusual direction.*

Questions: How many different ways can you write a statement of division? Can you look at this alternative way of solving this problem and decide whether it is valid? You can visualize when we graph in two and three dimensions, but what happens in your mind when you have to use four and more dimensions in a problem? Can you count and perform arithmetic in base 8, or use numbers to write in code? Is mathematics a science? What did you think about as you solved that problem?

Teachers might suggest to students to read over their homework assignment when they receive it, make sure they understand it all, and just let it ferment in the brain for a few days. In this way, the problem runs its course and a solution often appears unbidden. What does it mean to know mathematics? Why is scientific notation useful?

> *387. I want to say: an education quite different from ours might also be the foundation for quite different concepts.*
> *388. For here life would run on differently. –What interests us would not interest them. Here different concepts would no longer be unimaginable. In fact, this is the only way in which essentially different concepts are imaginable.*

Textbooks change and often different textbooks use different notation or different statements of the same theorems, definitions, and concepts. Is it possible to imagine a different form of mathematics in the same way one could imagine using different number bases?

Questions: Does deciphering codes involve using special techniques? How does guessing contribute to solving mathematical problems? How does the development of the computer give rise to a whole different kind of mathematics previously unimaginable? Why did you solve it this way? Can there be different geometries? IS the mathematics of infinity the same as regular mathematics?

In learning the fundamentals of mathematics, one comes to know early on that symbols are specific, and learns how to use each of them in so many different circumstances, and the list grows with each new problem solved.

> *333. 'Red is something specific' –that would have to mean the same as: 'That is something specific' –said while pointing to something red. But for that to be intelligible, one would have already to mean our concept of 'red', to mean the use of that sample.*

Substitute the equal sign '=' for "red" in this sentence, and we see how understanding the operation of a simple symbol has at its root some understanding of abstractions and the use of symbols and the '…use of that sample.'

> *334. I can indeed obviously express an expectation at one time by the words 'I'm expecting a red circle' and at another by putting a coloured(sic) picture of a red circle in the place of the last few words. But in this expression there are not two things corresponding to the two separate words 'red' and 'circle'. So the expression in the second language is of a completely different kind.*

When we think of the symbol □ it has a meaning not only because of what we have otherwise or previously learned about it, so we can use it without reinventing it every time, but also on a deeper or more fundamental level associated with our ability and capacity to use symbols at all:

> *336. …The important question here is never: how does he know what to abstract from? but: how is this possible at all? or: what does it mean?*

Questions: What does the number one '1' stand for? Are numbers abstractions in the same way that other symbols are? Or in a different way? Which symbols are representative such as "£" or "∑"? Which contain rules or instructions?

In reading Wittgenstein it is easy to get caught up in the actual content of his examples, rather than retaining the idea that color, for example, is a surrogate to develop the philosophical concept he is driving at. Here he reaffirms this idea.

> *347. The fact that we calculate with certain concepts and not with others only shews(sic) how various in kind conceptual tools are (how little reason we have here ever to assume uniformity). [Marginal Note: On propositions about colours that are like mathematical ones e.g. Blue is darker than white. On this Goethe's Theory of Colour.]*

Here one might reflect on the lack of uniformity between using cardinal versus ordinal numbers, or the multiplicity of ways of representing division. And calculations are performed using cardinal numbers, 1, 2, 3, 4…, not 1^{st}, 2^{nd}, 3^{rd}, 4^{th}…, but also the letters, *x, y, n*… are used and are fundamental to mathematics.

Questions: How can color be important in developing mathematical concepts? Why is red used to show a deficit or a loss? How effective are Roman numerals for mathematical manipulations?

Once the process of learning and reproducing proofs begins, mathematics is viewed in a somehow different way, as from the outside looking in, as suggested, with ordinary academic 'input language' as a meta-language. Concepts such as generalization and justification become recognized as important.

> *171. But isn't understanding shewn(sic) e.g. in the expression with which someone reads the poem, sings the tune*? [or completes the math problem?] *Certainly, but what is the experience during the reading? About that you would just have to say: you enjoy and understand it if you hear it well read, or feel it well read in your speech-organs.*
> *172. Understanding a musical phrase* [or proof] *may also be called understanding a language.*

This kind of aesthetic is often related to observing the quality and elegance of a well written proof. And being able to write the proof and derive it, shows an understanding well beyond simple performance or even beyond explanation. How do teachers share their interest and passion for mathematics with their students?

Questions: What are the different techniques you use to write proofs? When do you need to provide a proof of a mathematical operation or formulation?

Is understanding mathematically akin to "understanding a musical phrase" or just another move in a specialized language-game?

Familiarity with Proofs

But still students often struggle and lose the meaning of some of the terms, or use them in the wrong way, or misunderstand their instructions or the prescribed techniques.

> *183. The man I call meaning-blind will understand the instruction 'Tell him he is to go to the bank—I mean the river bank,' but not 'Say the word bank and mean the bank of a river'. What concerns this investigation is not the forms of mental defect that are found among men; but the possibility of such forms. We are interested, not in whether there are men incapable of a thought of the type: 'I was then going to …' –but in how the concept of such a defect should be worked out.*
>
> *If you assume that someone cannot do this, how about that? Are you supposing he can't do that either? –Where does this concept take us? For what we have here are of course paradigms.*

How do students understand the expressions the teacher of a proofs class uses? What is the prospect for discourse in finding meaning through conversation and exchange of ideas among students? It seems that all one's skills with language and mathematics go into developing an acceptable proof; it all comes together in the mind with the suggested rules and 'paradigms' or it doesn't. And for so many people; why doesn't it?

Questions: Do you learn mathematics better in your first language? When should you use mathematical induction? When contradiction? What is the role of definitions? When you don't follow instructions, does that mean you can't do the mathematics next time? How do you learn to write a mathematical proof?

In a certain way that is not always clear to a teacher, the practice of and reasons for doing proofs are quite foreign to some students. And of course over the years the requirements of what it takes to make a good proof have changed; certainly Euclid's explanations have been revised and improved a little. The standards of proof have changed; the conclusions are sometimes different because the postulates are different. The discovery of non-Euclidean geometry throws a whole new set of wrinkles into the discussion.

393. It is easy to imagine and work out in full detail events which, if they actually came about, would throw us out in all our judgments.

If I were sometime to see quite new surroundings from my window instead of the long familiar ones, if things, humans and animals were to behave as they never did before, then I should say something like 'I have gone mad'; but that would merely be an expression of giving up the attempt to know my way about. And the same thing might befall me in mathematics. It might e.g. seem as if I kept on making mistakes in calculating, so that no answer seemed reliable to me.

But the important thing about this for me is that there isn't any sharp line between such a condition and the normal one.

This kind of imagined dissonance can make students dislike mathematics. There are special procedures to use when solving problems (or when attempting to write proofs); once these are learned there will be more tools available for solving some problems, but that is not always sufficient to allow generalizations to be made to other problems.

Questions: Explain the steps you took; what was your thinking? Why do you think your solution is correct and makes sense? What will you do differently next time?

If students fail to learn and do proofs, is it because they are not fully conversant or fluent with the language-game of mathematics?

185. It's just like the way some people do not understand the question 'What colour has the vowel a *for you?' –If someone did not understand this, if he were to declare it was nonsense—could we say he did not understand English, or the meaning of the individual words 'colour', 'vowel' etc.?*

On the contrary: Once he has learned to understand these words, then it is possible for him to react to such questions 'with understanding' or 'without understanding'.

And to rephrase: Once one has learned to understand a certain proof and the strategy at stake, then it is possible to react with understanding and give explanations, show competence by recognizing when a statement about the proof (or an incorrect or incomplete proof) has been given incorrectly. We have thus mastered the language-game.

186. Misunderstanding –non-understanding. Understanding is effected [and affected] *by explanation; but also by training.*

It can be a valuable teaching technique to give students incorrect or invalid proofs, and have them identify what is wrong, or distinguish valid from invalid proofs. Such exercises are useful pedagogical tools not only to assess the students' grasp of the proof and the mathematics in question, but also to help students learn the language of proof writing.

Questions: What do you mean when you use these words…? Does this necessarily follow? Is this always true? What are the applications of this?

Is it always possible to show reasoning, or is intuition a valuable asset in doing mathematics? The essential motivation for doing and understanding proofs is to accept the quality of 'knowing' that is fundamental to the language-game of mathematics.

> *408. But isn't there a phenomenon of knowing, as it were quite apart from the sense of the phrase 'I know'? Is it not remarkable that a man can know something, can as it were have the fact within him? –But that is a wrong picture.—For, it is said, it's only knowledge if things really are as he says. But that is not enough. It mustn't be just an accident that they are. For he has got to know that he knows: for knowing is a state of his own mind; he cannot be in doubt or error about it –apart from some special sort of blindness. If then knowledge that things are so is only knowledge if they really are so; and if knowledge is in him so that he cannot go wrong about whether it is knowledge; in that case, then, he is also infallible about things being so, just as he knows his knowledge; and so the fact which he knows must be within him just like the knowledge.*
>
> *And this does indeed point to one kind of use for 'I know'. "I know that it is so' then means: I know that it is so', then means: It is so, or else I'm crazy.*
>
> *So: when I say, without lying: 'I know that it is so', then only through a special sort of blindness can I be wrong.*

This is the kind of certainty and conviction that can be looked for in the language-game of mathematics with respect to proofs.

Questions: Does this mathematics fit and connect in some important way to reality? Does your process follow all the well established rules for arithmetic, factoring, algebra, induction, logic, contradiction or deduction etc.? Can the solutions be verified?

The proper application and execution of proofs, algorithms, and formulas ought to have a certain kind of elegance and grace, as well as a certain degree of convincing.

> *410. A person can doubt only if he has learnt certain things; as he can miscalculate only if he has learnt to calculate. In that case it is indeed involuntary.*

If one were making mistakes on purpose, that is like lying or fraud in other contexts.

Questions: When is it okay to say this, do this, conclude this? When does that statement have meaning or relevance? What kind of explanation would it take to convince you that this proposition is true?

Philosophy of Mathematics

The processes and the mental activities, the forms of life, are what are fundamental to mathematics, in a way of speaking, because these are what give rise to the relevance, development and perpetuation of mathematics.

> *702. If one considers that 2+2=4 is a proof of the proposition 'there are even numbers', one sees how loosely the word 'proof' is used here. The proposition 'there are even numbers' is supposed to proceed from the equation 2+2=4! –And what is the proof of the existence of prime numbers? –The method of reduction to prime factors. But in this method nothing is said, not even about 'prime numbers'.*
>
> *703. 'To understand sums in the elementary school the children would have to be important philosophers; failing that, they need practice'.*

In this case Wittgenstein seems to be making the argument that it is through practice and learning definitions, that we learn about even numbers and prime numbers rather than through some kind of proof, and certainly this is the normal sequence of events. We learn to play, invent, and conjecture with prime numbers, even numbers, long before we understand their significance. This kind of proof would come possibly in a graduate school mathematics classroom, not in the first grade. We are not born with language or mathematics, or if we were we would all speak the same language.

Questions: Can you say that in a different way? How do you know you are right?

The following ending comments in *Zettel* are not just miscellany; rather these serve as interesting challenge questions about how one comes to understand the foundations of mathematics. Initially mathematics is a practical tool, producing clear and correct answers to quantitative real-world questions; however in the processes of proving and generalizing, the context is often removed, and the mathematics is used in what might seem to be an unusual way. Both activities, applications and proofs, are parts of mathematics.
704. Russell and Frege take concepts as, as it were, properties of things. But it is very unnatural to take the words man, tree, treatise, circle, as properties of a substrate.
Bertrand Russell admitted the limitations of his own analysis in his final comment: "After some twenty years of very arduous toil, I came to the conclusion that there was nothing more that I could do in the way of making mathematical knowledge indubitable…" (Russell).

Questions: Who was the first person to use this… math this way? What was Descartes' most important contribution?

Being conversant in the history of mathematics is usually emphasized late in the learning process for want of time. Certainly this can be enrichment for any unit in mathematics. Positing a problem or conjecture, then providing the answer in the form of a summary of an historical solution takes mathematics beyond the walls of the traditional calculation-based classroom.

> *705. Dirichlet's conception of a function is only possible where it does not seek to express an infinite rule by a list, for there is no such thing as an infinite list.*
> *706. Numbers are not fundamental to mathematics.*

Then what is mathematics? Is it our intellectual nature that equips us with culture, our training, our ability to follow rules, and our linguistic capacity? These are examples of quandaries that learners at any level can be encouraged to discuss (in age appropriate ways) which will give good practice in justification and reasoning, and make their mathematics training relevant to their lives.

Questions: Read this explanation and describe why it is true or false. How would you teach or explain this to someone else? How does your tacit knowledge help you?

Research into the fundamentals of mathematics and number theory can be introduced at different stages of mathematics education, not just at the college level.

> *707. The concept of the 'order' of the rational numbers, e.g., and of the impossibility of so ordering the irrational numbers. Compare this with what is called an 'ordering' of digits. Likewise the difference between the 'co-ordination' of one digit (or nut) with another and the 'co-ordination' of all whole numbers with the even numbers; etc. Everywhere distortion of concepts.*

708. There is obviously a method of making a straight-edge. This method involves an ideal, I mean an approximation-procedure of unlimited possibility, for this very procedure is the ideal. [What does it mean to create a paradigm?]

Or rather: only if there is an approximation-procedure of unlimited possibility can (not must) the geometry of this procedure be Euclidean.

709. To regard a calculation as an ornament, is also formalism, but of a good sort.

710. A calculation can be regarded as an ornament. A figure in a plane may fit another one or not, may be taken with other ones in various ways. If further the figure is coloured, there is a further fit according to colour. (Colour is only another dimension).

It is also possible to pose questions that match students' abilities and talents.

Question: What is the role of aesthetics in mathematics?

Summary:

This discussion of relevant quotations from *Zettel* that relate to mathematics and the foundations of mathematics is intended to serve as a resource that may stimulate insights into how mathematics came to be, and how reasoning, discourse and thinking about proofs can be stimulated. This is not intended to be an exhaustive discussion of the foundations of mathematics, but rather a source for questioning to provoke discussions of learning strategies, justifications, verifications and insights in pedagogical settings. There is more that can be said, but all explanations must come to an end, and hopefully this is a good beginning.

References

Note: All the numbered lines come from *Zettel*, by Ludwig Wittgenstein.

Dehaene, S. (1997). *The number sense: How the mind creates mathematics*. New York: Oxford University Press.

Gardner, H, (2003). Frames of mind, the theory of multiple intelligences. New York: Basic Books.

Kroeber, T. (1976). *Ishi in two worlds: A biography of the last wild Indian in North America*. Los Angeles: University of California Press.

Lakatos, I. (1976). *Proofs and refutation*s. Cambridge: Cambridge University Press.

Marion, M. (1998). *Wittgenstein, finitism, and the foundations of mathematics*. Oxford: Clarendon Press.

Piaget, J. (1952). *The origins of intelligence of the child.* London: Routledge and Kegan Paul.

Ramsey, F. P., 1925. The foundations of mathematics**.** Originally published in the *Proceedings of the London Mathematical Society, Ser.* 2, 25(5), 338-384.

Russell, B. (1956). *Portraits from memory, and other essays*. New York: Simon and Schuster.

Sfard, A. (2008). *Thinking as communicating, Human development, the growth of discourses, and mathematizing*. New York: Cambridge University Press.

Vygotsky, L.S. (1962). *Thought and languag*e. Cambridge, MA: MIT Press.

Vygotsky, L.S. (1978). *Mind in society*. Cambridge, MA: Harvard University Press.

Wittgenstein, L. (1978) *Remarks on the foundations of mathematics*, translated by G.E.M. Anscombe. Cambridge, MA: MIT Press.

Wittgenstein, L. (1967). *Zettel*. G.E.M. Anscombe & G.H. von Wright, Eds. Translated by G.E.M. Anscombe. Berkeley and Los Angeles: University of California Press.

Zaslavsky, C. (1979) *Africa counts: Number and pattern in African culture*, L. Hill.