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MODULES, FIELDS OF DEFINITION, AND THE CULLER-SHALEN NORM

Charles Walter Katerba

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By

CHARLES WALTER KATERBA

B.S., Northern Arizona University, Flagstaff, AZ, 2011
M.A., University of Montana, Missoula, MT, 2013

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Approved by:

Sandy Ross, Associate Dean of the Graduate School
Graduate School

Dr. Eric Chesebro, Chair
Mathematical Sciences

Dr. Kelly McKinnie
Mathematical Sciences

Dr. Karel Stroethoff
Mathematical Sciences

Dr. Nikolaus Vonessen
Mathematical Sciences

Dr. Nate McCrady
Physics and Astronomy

Katerba, Charles, Doctorate of Philosophy, May 2017 Mathematics

Modules, Fields of Definition, and the Culler-Shalen norm

Committee Chair: Eric Chesebro, Ph.D.

Culler-Shalen theory uses the algebraic geometry of a 3-manifold's $\mathrm{SL}_2(\mathbb{C})$-character variety to construct essential surfaces in the manifold. There are module structures associated to the coordinate ring of the character variety which are intimately related to essential surface construction. When these modules are finitely generated, we derive a formula for their rank that incorporates the variety's field of definition and the Culler-Shalen norm.

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Chapter 1

Introduction

This thesis investigates a module-theoretic perspective on Culler and Shalen's character variety techniques for constructing essential surfaces in 3-manifolds. Essential surfaces encode important topological information about a 3-manifold and the techniques of Culler and Shalen provide very general tools for their construction. Chesebro noticed a connection between between an infinite collection of module structures on the coordinate ring of a 3-manifold's character variety and the construction of essential surfaces via Culler-Shalen theory [5]. For a given 3-manifold, these modules are often finitely generated and free, but explicit computation of their rank had only been achieved through case-by-case calculations. In Chapter 2, we derive a formula for their rank that involves two 3-manifold invariants: the Culler-Shalen norm and the variety's field of definition. This formula allows us to compute the rank of each of the finitely generated modules associated to a 3-manifold in many situations.

In the remainder of this chapter, we recall the relevant definitions and ideas from 3-manifold topology, commutative algebra, and algebraic geometry. We then review character varieties, one way they give rise to essential surfaces in 3-manifolds, and discuss the Culler-Shalen norm. The final section describes Chesebro's module-theoretic perspective on essential surface detection and motivates the work in Chapter 2.

1.1 3-manifolds

A *knot manifold* is a compact, irreducible, orientable 3-manifold whose boundary consists of a single irreducible torus. Knots in S^3 (i.e. smooth embeddings $S^1 \to S^3$) provide many examples of knot manifolds by taking the complement of an open tubular neighborhood of the knot. These, however, are not the only examples of knot manifolds.

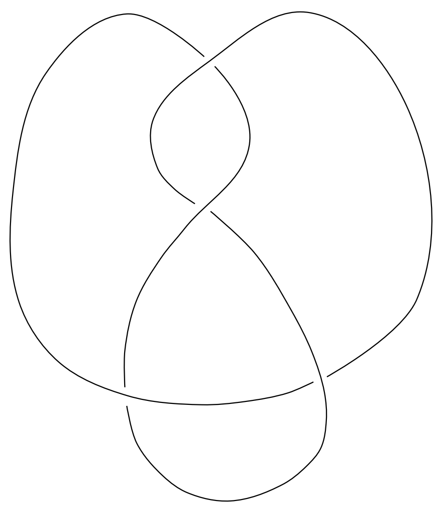

Figure 1.1: The figure-eight knot.

In studying the structure of a manifold, one frequently looks to the codimension one submanifolds which encode pertinent topological information. For instance, the existence of non-trivial closed curves on a torus distinguish it from the plane. Essential surfaces play a similar role in 3-manifold topology.

Take a knot manifold N. Let Σ be a non-empty, connected, orientable surface properly embedded in N. We say Σ is *essential* if

1. The homomorphism $\pi_1(\Sigma) \to \pi_1(N)$ induced by inclusion is injective.

2. Σ is not homeomorphic to S^2.

3. Σ is not freely homotopic into ∂N.

A properly embedded surface is *essential* if each connected component is. Essential surfaces in N may have empty or non-empty boundary. We say N is *small* if every essential surface in N has non-empty boundary. The exterior of the figure-eight knot in S^3, for example, is a small knot manifold. The image below shows a diagram of a knot whose exterior in S^3 contains a closed essential surface.

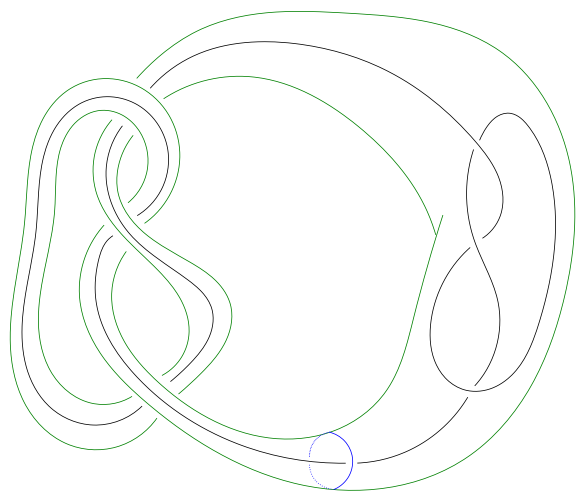

Figure 1.2: An essential torus in a knot exterior.

Essential surfaces in a knot manifold N can be complicated, so we often simplify matters by examining their boundaries. A *slope* on ∂N is the unoriented isotopy class of a simple closed curve on ∂N. The boundary components of an essential surface in N are non-trivial and parallel curves on ∂N, so they represent the same slope. Thus, we say a slope α is a *boundary slope* if there is an essential surface Σ in N such that the components of $\Sigma \cap \partial N$ represent α.

The character variety techniques of Culler and Shalen use algebraic geometry to find essential surfaces in 3-manifolds. In practice, these tools are quite good at determining the boundary slopes of a knot manifold.

1.2 Commutative Algebra

Fix an integral domain R and a subring $S \subseteq R$. Then R is naturally an S-module and an S-algebra. We say R is a *finite extension of* S, or that R *is finite over* S, if R is a finitely generated S-module. On the other hand, if R is only finitely generated as an S-algebra, we say R is *finitely generated over* S. If there is some $\alpha \in R$ such that R is generated by α as an S-algebra, the extension R/S is *simple* and $\alpha \in R$ is a *primitive element.*

The notion of integrality connects finite and finitely generated ring extensions. Recall that an element $r \in R$ is *integral over* S if r is the root of a monic polynomial with coefficients in S. The set

$$\overline{S} = \{r \in R \mid r \text{ is integral over } S\}$$

is a subring of R called *the integral closure of* S *in* R. If $S = \overline{S}$, then S is *integrally closed in* R and if $\overline{S} = R$, we say R *is integral over* S. If R is a finitely generated integral extension of S, then R is actually finite over S [2, Corollary 5.2].

We define the *rank of* R *over* S to be

$$\operatorname{rk}_S(R) = \begin{cases} \infty & \text{if there is an infinite } S\text{-linearly independent subset of } R \\ \max|\mathcal{L}| & \text{where } \mathcal{L} \text{ ranges over all } S\text{-linearly independent subsets of } R \end{cases}$$

When R is finite over S and free as an S-module, the rank of R over S is also equal to the minimum cardinality of all spanning sets (cf. [15, Section 2.2]).

Now take a field F and a subfield $E \subseteq F$. In this case, the rank of F over E is often called the *degree of the extension* and is denoted $[F : E]$. We will refer to integral field extensions as *algebraic extensions.*

A subset $D \subseteq F$ is *algebraically dependent over* E if there is a finite subset $\{d_1, \dots, d_n\} \subseteq D$

and a nonzero polynomial $f(z_1, \ldots, z_n) \in E[z_1, \ldots, z_n]$ such that $f(d_1, \ldots, d_n) = 0$. If D is not algebraically dependent, then it is *algebraically independent.* The transcendence degree of F over E, denoted $\operatorname{tr.deg}_E F$, is the cardinality of a maximal algebraically independent subset of F. Note that $\operatorname{tr.deg}_E F = 0$ if and only if F is an algebraic extension of E.

Let $\operatorname{Aut}(F/E)$ denote the group of field automorphisms of F which fix E. For a subgroup H of $\operatorname{Aut}(F/E)$, the *fixed field of* H is the subfield $\{a \in F \mid \sigma(a) = a \ \ \forall \sigma \in H\}$.

1.3 Algebraic geometry

In this section we review the algebraic geometry needed in Chapter 2. First, recall that if J is an ideal in $R = \mathbb{C}[z_1, \ldots, z_n]$, then the *zero set of* J is

$$Z(J) = \{P \in \mathbb{C}^n \mid f(P) = 0 \quad \forall f \in J\}.$$

A subset X of $\mathbb{C}^n$ is *algebraic* if there is some ideal J of R such that $X = Z(J)$. For an algebraic set $X \subseteq \mathbb{C}^n$, let

$$I(X) = \{f \in \mathbb{C}[z_1, \ldots, z_n] \mid f(P) = 0 \quad \forall P \in X\}$$

and $\mathbb{C}[X] = R/I(X)$ be the *ideal* and *coordinate ring of* X. The elements of $\mathbb{C}[X]$ are *regular functions on* X since each $f \in \mathbb{C}[X]$ determines a well-defined Zariski-continuous function $f\colon X \to \mathbb{C}$.

If J is any ideal such that $Z(J) = X$, then $I(X)$ is equal to the radical of J by Hilbert's Nullstellensatz [13, Theorem I.1.3A]; that is,

$$\sqrt{J} = \{f \in \mathbb{C}[z_1, \ldots, z_n] \mid f^r \in J \quad \text{for some } r \in \mathbb{N}\} = I(X).$$

We say X is *irreducible* if $I(X)$ is a prime ideal and *reducible* otherwise. If X is reducible, then $I(X)$ is a radical ideal [13, Corollary I.1.4], so we may write $I(X) = \cap_1^m P_i$ where the P_i's are distinct prime ideals of R and no P_j is properly contained in P_i (cf. [9, Ch. 4, Sec. 7, Corollary 10] and [2, Chapter 4]). Then each $X_i := Z(P_i)$ is an irreducible algebraic subset of X and we call X_i an *irreducible component of* X.

If X is irreducible, then $\mathbb{C}[X]$ is an integral domain and its field of fractions $\mathbb{C}(X)$ is called the *field of rational functions on* X. Each $f/g \in \mathbb{C}(X)$ is a *rational function on* X since f/g determines a well-defined map $f/g\colon U \to \mathbb{C}$ for some Zariski-open subset $U \subseteq X$.

Suppose X has irreducible components $X_1, \ldots, X_k$ with $k \geq 1$. Each $\mathbb{C}(X_i)$ has finite transcendence degree over $\mathbb{C}$, so we define the *dimension of* X to be

$$\dim X = \max_i \operatorname{tr.deg}_{\mathbb{C}} \mathbb{C}(X_i).$$

When $\dim X = 1$, we call X a *curve*.

Let $X \subseteq \mathbb{C}^n$ be an irreducible agebraic set and suppose $\dim X = m$. $I(X)$ can be generated by a finite number of polynomials $f_1, \ldots, f_r \in R$ by the Hilbert basis theorem. For a point $P \in X$, the *Jacobian of* X *at* P is the $r \times n$ matrix

$$J_P = \begin{pmatrix} \frac{\partial f_1}{\partial z_1}(P) & \cdots & \frac{\partial f_1}{\partial z_n}(P) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_r}{\partial z_1}(P) & \cdots & \frac{\partial f_r}{\partial z_n}(P) \end{pmatrix}$$

and we say that X is *nonsingular at* P if the rank of J_P is $n - m$. X is *nonsingular* if it is nonsingular at every point.

Take another algebraic set $Y \subseteq \mathbb{C}^m$. A function $f\colon X \to Y$ is a *regular* (resp. *rational*) *map* if there are regular (resp. rational) functions $f_1, \ldots, f_m$ on X such that $f = (f_1, \ldots, f_m)$. Note that a rational map $f\colon X \to Y$ may not be defined on all of X.

Let $f\colon\ X \to Y$ be a regular map between irreducible affine algebraic sets. We call f *dominant* if the Zariski-closure of the image of X is all of Y. A dominant regular map f induces a field monomorphism $f^*\colon\ \mathbb{C}(Y) \to \mathbb{C}(X)$ by precomposing a rational function on Y with f. The *degree of* f is the degree of the field extension $[\mathbb{C}(X) : f^*(\mathbb{C}(Y))]$. The induced homomorphism f^* restricts to an embedding $f^*\colon\ \mathbb{C}[Y] \to \mathbb{C}[X]$, so $\mathbb{C}[X]$ is naturally a $f^*(\mathbb{C}[Y])$-module. We say a dominant regular map $f\colon\ X \to Y$ is *finite* if $\mathbb{C}[X]$ is a finite extension of $f^*(\mathbb{C}[Y])$. Note that, equivalently, f is finite if and only if $\mathbb{C}[X]$ is an integral extension of $f^*(\mathbb{C}[Y])$ since $\mathbb{C}[X]$ is a finitely generated extension of $\mathbb{C}$.

Finite degree dominant regular maps are not necessarily finite. If $X \subset \mathbb{C}^2$ is the curve cut out by $xy - 1$, then the projection onto the first coordinate has degree one since $\mathbb{C}(X) \cong \mathbb{C}(x)$. However, $\mathbb{C}[X]$ is not a finite extension of $\mathbb{C}[x]$ since $y \in \mathbb{C}[X]$ is not integral over $\mathbb{C}[x]$.

1.3.1 Fields of definition and conjugate varieties

Suppose $X \subseteq \mathbb{C}^n$ is an irreducible algebraic set and let $k \subseteq \mathbb{C}$ be a subfield. If the ideal $I(X)$ can be generated by polynomials with coefficients in k, we call k a *field of definition for* X and say that X *is defined over* k. X has a unique *minimal field of definition* k_0 such that whenever X is defined over k we have $k_0 \subseteq k$ [16, Ch. III, Thm. 7]. In general, the field of definition of an algebraic set depends on its embedding in affine space. For instance, the curves cut out of $\mathbb{C}^2$ by $x - y$ and $x - iy$ are isomorphic algebraic sets, but the first is defined over $\mathbb{Q}$ while the second has minimal field of definition $\mathbb{Q}[i]$.

The group $\mathrm{Aut}(\mathbb{C}/k)$ acts on $\mathbb{C}^n$ by applying $\sigma \in \mathrm{Aut}(\mathbb{C}/k)$ to each coordinate of a point; for $P \in \mathbb{C}^n$ let P^σ denote $\sigma \cdot P$. Similarly, $\mathrm{Aut}(\mathbb{C}/k)$ acts on $R = \mathbb{C}[z_1, \ldots, z_n]$ by applying $\sigma \in \mathrm{Aut}(C/k)$ to the coefficients of a polynomial $p \in R$; let $p^\sigma \in R$ denote the resulting polynomial.

If k is a subfield of $\mathbb{C}$ containing the minimal field of definition k_0 for X, then the action of $\mathrm{Aut}(\mathbb{C}/k)$ on $\mathbb{C}^n$ restricts to an action on X. To see this, suppose $I(X)$ is generated by polynomials $f_1, \ldots, f_r \in k_0[z_1, \ldots, z_n]$. Then for each $\sigma \in \mathrm{Aut}(\mathbb{C}/k)$, $f_i^\sigma = f_i$. If $P \in X$, then $f_i(P) = 0$, so $f_i(P^\sigma) = 0$ and $P^\sigma \in X$. A similar argument shows that the action of $\mathrm{Aut}(\mathbb{C}/k)$ descends to a well-defined action on the coordinate ring $\mathbb{C}[X]$.

On the other hand, suppose k_0 is a finite extension of the field k. Then an automorphism $\sigma \in \mathrm{Aut}(\mathbb{C}/k)$ may not act on X. Instead, σ may carry X to another irreducible algebraic set $X^\sigma = \{P^\sigma \mid P \in X\}$. The ideal of X^σ is simply

$$I(X^\sigma) = \{f^\sigma \mid f \in I(X)\}.$$

With this mind, we say that an irreducible algebraic set $Y \subset \mathbb{C}^n$ is *conjugate to X over k* if there is some automorphism $\sigma \in \mathrm{Aut}(\mathbb{C}/k)$ such that $Y = X^\sigma$. The irreducible algebraic sets in the collection $\{X^\sigma \mid \sigma \in \mathrm{Aut}(\mathbb{C}/k)\}$ are the *conjugate varieties to X over k*. By [16, Ch. III, Thm. 10], there are precisely $[k_0 : k]$ varieties in $\mathbb{C}^n$ conjugate to X over k.

1.4 Representation and character varieties

This section describes the construction of the $\mathrm{SL}_2(\mathbb{C})$-representation and character varieties for a finitely generated group Γ. Our exposition draws heavily from both [22] and [17].

A *representation of* Γ *into* $\mathrm{SL}_2(\mathbb{C})$ is a group homomorphism $\rho\colon \Gamma \to \mathrm{SL}_2(\mathbb{C})$ and the *character of* ρ is the function $\chi\colon \Gamma \to \mathbb{C}$ given by $\chi(\gamma) = \operatorname{trace} \rho(\gamma)$. A representation $\rho\colon \Gamma \to \mathrm{SL}_2(\mathbb{C})$ is *reducible* if there is a non-trivial proper vector subspace of $\mathbb{C}^2$ fixed by $\rho(\Gamma)$; we say ρ is *irreducible* otherwise.

Fix a finitely generated group Γ and a presentation $\langle \gamma_1, \ldots, \gamma_n \mid \{r_j\}_{j\in J}\rangle$ for Γ. For $1 \le i \le n$, let x_i, y_i, z_i, and w_i be variables with $x_iw_i - y_iz_i = 1$. Set

$$A_i = \begin{pmatrix} x_i & y_i \\ z_i & w_i \end{pmatrix} \quad \text{so that} \quad A_i^{-1} = \begin{pmatrix} w_i & -y_i \\ -z_i & x_i \end{pmatrix}$$

and substitue A_i and A_i^{-1} for γ_i and γ_i^{-1} in the relator r_j. This gives 4 polynomials

$$\begin{pmatrix} R_{j1} & R_{j2} \\ R_{j3} & R_{j4} \end{pmatrix}$$

in $4n$ variables with integer coefficients. Identify $M_2(\mathbb{C})$ and $\mathbb{C}^4$ with the vector space map

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto (a, b, c, d)$$

then identify $(M_2(\mathbb{C}))^n$ with $\mathbb{C}^{4n}$ similarly. The set of representations $\{\Gamma \to \mathrm{SL}_2(\mathbb{C})\}$ is in one-to-one correspondence with the algebraic subset $R(\Gamma) \subseteq \mathbb{C}^{4n}$ cut out by the set of polynomials

$$\{R_{j1} = R_{j4} = 1, R_{j2} = R_{j3} = 0\}_{j \in J} \cup \{x_iw_i - y_iz_i - 1\}_{i=1}^{n}$$

since a function $\{\gamma_i\} \to \mathrm{SL}_2(\mathbb{C})$ taking $\gamma_i \mapsto B_i$ extends to a representation if and only if substituting B_i for γ_i in each relator r_j yields the identity matrix. We call $R(\Gamma)$ the *representation variety of* Γ. From the discussion above, note that $R(\Gamma)$ is defined over $\mathbb{Z}$. While the collection of polynomials we use to define $R(\Gamma)$ may be infinite, the Hilbert basis theorem implies that the ideal of $R(\Gamma)$ is generated by finitely many polynomials.

We now use the representation variety of Γ to construct the character variety of Γ. The elements of Γ naturally determine regular *trace functions* on $R(\Gamma)$: for each $\gamma \in \Gamma$ define $I_\gamma\colon R(\Gamma) \to \mathbb{C}$ by $I_\gamma(\rho) = \operatorname{trace} \rho(\gamma)$. From the above discussion, I_γ can be expressed as a polynomial function in $4n$ variables with coefficients in $\mathbb{Z}$. Define $T(\Gamma)$ to be the ring with unity generated by the trace functions on $R(\Gamma)$ in $\mathbb{C}[R(\Gamma)]$. Culler and Shalen [11] showed that $T(\Gamma)$ is generated by the set

$$\{I_V \mid V = \gamma_{i_1} \cdots \gamma_{i_k},\ 1 \le k \le n,\ 1 \le i_1 < i_2 < \cdots < i_k \le n\}.$$

Label the elements of this set $I_{V_1}, \ldots, I_{V_N}$ and define a regular map $t\colon R(\Gamma) \to \mathbb{C}^N$ by $t(\rho) = (I_{V_1}(\rho), \ldots, I_{V_N}(\rho))$. The image of t is a closed algebraic subset of $\mathbb{C}^N$ (cf. [11] or [12]). Moreover, since $R(\Gamma)$ and each I_{V_i} can be defined by polynomials with integer coefficients, $X(\Gamma) := t(R(\Gamma))$ is defined over $\mathbb{Z}$. There is a natural bijection between the points of $X(\Gamma)$ and the characters of $\mathrm{SL}_2(\mathbb{C})$-representations of Γ since the trace functions $\{I_{V_i}\}_1^N$ generate $T(\Gamma)$. Thus we call $X(\Gamma)$ the *character variety of* Γ.

The algebro-geometric structure of $R(\Gamma)$ and $X(\Gamma)$ is independent of our choice of presentation for Γ. For instance, if $X(\Gamma)$ is a curve, then the genera of its irreducible components are invariants of Γ. We sketch an argument for $X(\Gamma)$ and a similar argument will work for $R(\Gamma)$. Say $\langle \delta_1, \ldots, \delta_m \mid \{s_i\}_{i \in I}\rangle$ is another presentation for Γ. Let $R'(\Gamma)$ be the algebraic subset of $\mathbb{C}^{4m}$ corresponding to our new presentation for Γ coming from the above construction. Suppose the ring $T'(\Gamma)$ generated by the trace functions on $R'(\Gamma)$ is generated by the elements of $\{I_{W_i}\}_{i=1}^M$ and let $X'(\Gamma)$ be the image of the map $t'\colon R'(\Gamma) \to \mathbb{C}^M$ given by $t'(P) = (I_{W_1}(P), \ldots, I_{W_M}(P))$. If $\mathcal{X}$ is the set of characters of $\mathrm{SL}_2(\mathbb{C})$-representations of Γ, then we have natural bijections $\phi\colon \mathcal{X} \to X(\Gamma)$ and $\psi\colon \mathcal{X} \to X'(\Gamma)$. We claim that the functions

$$\psi \circ \phi^{-1}\colon X(\Gamma) \to X'(\Gamma) \quad \text{and} \quad \phi \circ \psi^{-1}\colon X'(\Gamma) \to X(\Gamma)$$

are regular maps. Take an isomorphism

$$\Omega\colon \langle \gamma_1, \ldots, \gamma_n \mid \{r_j\}_{j \in J}\rangle \to \langle \delta_1, \ldots, \delta_m \mid \{s_i\}_{i \in I}\rangle$$

of groups. For each W_i pick a word w_i in the γ_j's such that $\Omega(w_i) = W_i$. Since $\{I_{V_i}\}_{i=1}^N$ generates $T(\Gamma)$ as a ring, we can write each I_{w_i} as a finite sum of products of the I_{V_j}'s with integer coefficients. Thus, in coordinates, the function $\psi \circ \phi^{-1}$ is given by

$$\psi \circ \phi^{-1} = (I_{w_1}, \ldots, I_{w_M})$$

and from this we see that it is not only regular, but defined over $\mathbb{Z}$. An identical argument shows that $\phi \circ \psi^{-1}$ is also regular and defined over $\mathbb{Z}$. The functions $\phi \circ \psi^{-1}$ and $\psi \circ \phi^{-1}$ are inverse to each other, so $X(\Gamma)$ and $X'(\Gamma)$ are isomorphic as algebraic sets.

When N is a knot manifold, we define the *representation and character varieties of* N to be $R(N) = R(\pi_1(N, x_0))$ and $X(N) = X(\pi_1(N, x_0))$ for some base point $x_0 \in N$. Note that choosing a different base point for the fundamental group of N gives rise to a different presentation for the fundamental group. Since the algebraic geometry of $R(N)$ and $X(N)$ is independent of presentation, we suppress the choice of base point from our notation in what follows.

We include the next example to show that character varieties of knot manifolds can be computed concretely.

Example 1. Let N be the exterior of the figure-eight knot in S^3. There is a presentation

$$\langle a, b \mid ab^{-1}aba^{-1} = b^{-1}aba^{-1}b \rangle$$

for $\pi_1(N)$. The discussion above shows that we may take I_a, I_b, and I_{ab} as coordinates on $X(N)$. Since a and b are conjugate in $\pi_1(N)$ and trace is invariant under conjugation, we have $I_a = I_b$. The Caley-Hamilton formula implies that $I_{ab} = I_a I_b - I_{ab^{-1}}$. Thus, we may embed $X(N)$ in $\mathbb{C}^2$ using I_a and $I_{ab^{-1}}$ as coordinates.

Take an irreducible representation $\rho\colon\ \pi_1(N) \to \mathrm{SL}_2(\mathbb{C})$. After conjugation, we may assume

$$\rho(a) = \begin{pmatrix} Y & 1 \\ 0 & 1/Y \end{pmatrix} \quad \text{and} \quad \rho(b) = \begin{pmatrix} Y & 0 \\ 2-t & 1/Y \end{pmatrix}$$

where $Y, t \in \mathbb{C}$, $Y \neq 0$, and $t \neq 2$. Note that $I_a(\rho) = \operatorname{trace}\rho(a) = Y + 1/Y$ and $I_{ab^{-1}}(\rho) = t$ Applying this assignment for a and b to each side of the relation in our presentation and taking the difference gives

$$\begin{pmatrix} 0 & 1 + (\frac{1}{Y^2} + Y^2)(1-t) - t + t^2 \\ (2-t)(1 + (\frac{1}{Y^2} + Y^2)(1-t) - t + t^2) & 0 \end{pmatrix}.$$

Since ρ is a representation, the entries in the above matrix must be 0. Notice that $I_a(\rho)^2 = Y^2 + 1/Y^2 - 2$, so $Y^2 + 1/Y^2 = I_a(\rho)^2 - 2$. Thus, if we set $y = I_a$, the characters of irreducible representations $\pi_1(N) \to \mathrm{SL}_2(\mathbb{C})$ are naturally in 1-1 correspondence with the planar curve $X \subseteq \mathbb{C}^2$ defined by the polynomial $-1 + y^2 + t - y^2t + t^2$.

Now take a reducible representation $\rho\colon \pi_1(N) \to \mathrm{SL}_2(\mathbb{C})$. Since we are working over $\mathrm{SL}_2(\mathbb{C})$, the character of ρ is the character of an abelian representation. Abelian representations are diagonal, so we may assume

$$\rho(a) = \rho(b) = \begin{pmatrix} Y & 0 \\ 0 & 1/Y \end{pmatrix}$$

which implies $I_{ab^{-1}}(\rho) = 2$. Thus, the characters of reducible representations of $\pi_1(N)$ are in 1-1 correspondence with the plane curve cut out by $(2-t)$. In particular, the character variety $X(N)$ of N contains 2 irreducible components and is defined by the polynomial

$$(2-t)(1 + y^2 + t - y^2t + t^2).$$

Recall from Section 1.3 that an algebraic set's minimal field of definition depends on its embedding in general. It turns out that the fields of definition of the irreducible components of $X(\Gamma)$ are independent of our choice of presentation. To see this, take an irreducible component $X' \subseteq X'(\Gamma)$. Since $X'(\Gamma)$ is defined over $\mathbb{Z}$, the minimal field of definition for X' is a number field k_0. The isomorphism $\phi \circ \psi^{-1}\colon X'(\Gamma) \to X(\Gamma)$ is defined over $\mathbb{Z}$, so $\phi \circ \psi^{-1}(X')$ is an irreducible component of $X(\Gamma)$ with minimal field of definition k_0. We will use this fact in

Chapter 2, so we record this observation with the following proposition.

Proposition 1. *[17, Proposition 3.1] Let Γ be a finitely presented group and X an irreducible component of $X(\Gamma)$. Then the minimal field of definition of X is a number field k_0 that does not depend on the presentation used to compute $X(\Gamma)$.*

Recently there has been some research on the fields of definition of the irreducible components in character varieties of knot manifolds. Long and Reid [17] investigated which number fields can arise as the minimal field of definition of an irreducible component of a knot manifold's character variety and constructed a family of knot manifolds whose character varieties contained irreducible components defined over arbitrarily large extensions of $\mathbb{Q}$. In a similar vein, Paoluzzi and Porti [20] showed that some symmetries of knots in S^3 give rise to components in character varieties whose fields of definition are non-trivial extensions of $\mathbb{Q}$.

1.5 Essential surfaces from character varieties

Let N be a knot manifold. In this section, we describe Culler and Shalen's character variety techniques for constructing essential surfaces in N. These tools use the behavior of the trace functions on $X(N)$ and Tits-Bass-Serre theory to get an action of $\pi_1(N)$ on a simplicial tree. A construction due to Stallings uses this action to build non-empty essential surfaces in N.

Fix a presentation $\langle \gamma_1, \ldots, \gamma_n \mid \{r_j\}_{j \in J} \rangle$ for $\pi_1(N)$ and compute both $R(N)$ and $X(N)$ with this presentation. Take an irreducible curve $X \subseteq X(N)$. The *normalization of* X is a non-singular affine curve $\overline{X}$ and a dominant degree 1 regular map $\phi\colon \overline{X} \to X$ such that the coordinate ring of $\overline{X}$ is equal to the integral closure of $\mathbb{C}[X]$ in $\mathbb{C}(X)$. Up to isomorphism there is a unique smooth projective curve $\widetilde{X}$ and a birational map $\iota\colon \widetilde{X} \dashrightarrow \overline{X}$ whose inverse is defined on all of $\overline{X}$. The elements of $\mathcal{I}(X) = \widetilde{X} - \iota^{-1}(\overline{X})$ are the *ideal points of* X. Note that $\mathbb{C}(\widetilde{X}) = \mathbb{C}(\overline{X}) = \mathbb{C}(X)$ since $\phi \circ \iota$ is birational.

For each point $P \in \widetilde{X}$ there is a natural valuation on $\mathbb{C}(X)$. Take a rational map $f \in \mathbb{C}(\widetilde{X})$. $\widetilde{X}$ is a complex smooth projective curve, so we may think of f as a meromorphic function on a Riemann surface. Thus, the function $v_P\colon \mathbb{C}(X) \to \mathbb{Z}$ defined by

$$v_P(f) = \begin{cases} -(\text{order of the pole of } f \text{ at } P) & \text{if } f(P) = \infty \\ 0 & \text{if } f(P) \in \mathbb{C} - \{0\} \\ \text{order of the zero of } f \text{ at } P & \text{if } f(P) = 0 \end{cases}$$

is a valuation on $\mathbb{C}(X)$. This valuation is called *the valuation associated to* P.

Recall the map $t\colon R(N) \to X(N)$ from Section 1.4 which takes a representation to its character. Pick an irreducible component R of $t^{-1}(X)$ so that $t(R) = X$. The restriction $t\colon R \to X$ induces an inclusion $\mathbb{C}(X) \hookrightarrow \mathbb{C}(R)$. The fields $\mathbb{C}(R)$ and $\mathbb{C}(X)$ are both finitely generated extensions of $\mathbb{C}$, so $\mathbb{C}(R)$ is certainly a finitely generated extension of $\mathbb{C}(X)$. Thus, the following lemma provides a way to extend valuations on $\mathbb{C}(X)$ to $\mathbb{C}(R)$.

Lemma 2. *[1, Lemma 1.1] Let K be a finitely generated extension of a field F and $v\colon F \to \mathbb{Z}$ a valuation of F. There is a valuation w on K and a positive integer d such that $w \mid F = d \cdot v$*

Fix an ideal point $P \in \mathcal{I}(X)$, let $v = v_P$ be the valuation associated to P, and take an extension w of v to $\mathbb{C}(R)$ given by Lemma 2. Work of Tits, Bass, and Serre (cf. [21] or [22]) associates to the valuation w a simplicial tree T on which $\mathrm{SL}_2(\mathbb{C}(R))$ acts *without inversions*; that is, if $A \in \mathrm{SL}_2(\mathbb{C}(R))$ fixes an edge e of T, then A also fixes the endpoints of e.

Our goal is an action of $\pi_1(N)$ on T, so we define a representation $\mathcal{P}\colon \pi_1(N) \to \mathrm{SL}_2(\mathbb{C}(R))$. First, fix a generator γ_i from our presentation for $\pi_1(N)$. Recalling the construction of $R(\Gamma)$ from Section 1.4, we see that there are four coordinate functions x_i, y_i, z_i, w_i on $R(\Gamma)$ corresponding to the generic assignment

$$\gamma_i \mapsto \begin{pmatrix} x_i & y_i \\ z_i & w_i \end{pmatrix}.$$

Thus, there is a natural function $\mathcal{P}\colon \{\gamma_i\}_{i=1}^n \to \mathrm{SL}_2(\mathbb{C}(R))$. From the construction of $R(\Gamma)$, $\mathcal{P}$ extends to a representation of $\pi_1(N)$. We call $\mathcal{P}$ the *tautological representation of* Γ. Since $\mathrm{SL}_2(\mathbb{C}(R))$ acts on T, $\mathcal{P}$ induces an action of $\pi_1(N)$ on T.

There is an inclusion $\mathbb{C}[X] \hookrightarrow \mathbb{C}(\widetilde{X})$, so we may view each I_γ as rational function on $\widetilde{X}$. P is an ideal point of the curve X and a finite collection of trace functions are coordinates on X, so there must be an element $\gamma \in \pi_1(N)$ such that $I_\gamma(P) = \infty$. In particular, $v(I_\gamma) < 0$, so

$$w(\operatorname{tr}\mathcal{P}(\gamma)) = d \cdot v(I_\gamma) < 0.$$

Hence, by [22, Property 5.4.2], no point in T is fixed by all of $\pi_1(N)$. Finally, applying Stallings' essential surface construction (cf. [22, Chapter 2]) to this action of $\pi_1(N)$ on T gives a non-empty essential surface in N. We say that an essential surface in N obtained from the construction described above is *associated to the ideal point* P. The essential surfaces in N associated to some ideal point of X are *detected by* X.

Ignoring base points as usual, the inclusion $\partial N \to N$ induces a monomorphism $\pi_1(\partial N) \to \pi_1(N)$ whose image is well-defined up to conjugation. Identify $\pi_1(\partial N)$ with a representative of this conjugacy class of subgroups. For $\gamma \in \pi_1(\partial N)$, the function I_γ is well-defined on $R(N)$ and $X(N)$ since trace is invariant under conjugation. A slope α on ∂N corresponds to a pair $\{\gamma^{\pm 1}\}$, so the function $I_\alpha = I_\gamma$ is well defined since trace is invariant under inversion. For $\gamma \in \pi_1(\partial N)$, we call I_γ a *peripheral trace function.*

There can be many essential surfaces associated to an ideal point $P \in \mathcal{I}(X)$. The following proposition shows that we may determine the boundary slopes of essential surfaces associated to some ideal points by examining the behavior of peripheral traces functions at each ideal point.

Proposition 3. *[11, Proposition 1.3.8] Let N be a knot manifold and suppose $X \subseteq X(N)$ is an irreducible curve. Fix an ideal point $P \in \mathcal{I}(X)$ and let Σ be an essential surface in N associated to P.*

1. *If $v(I_\alpha) \geq 0$ for every slope α on ∂N, then Σ may be selected to have empty boundary.*
2. *On the other hand, there is a unique slope α with $v(I_\alpha) \geq 0$. In this case, α is a boundary slope and every component of $\partial\Sigma$ represents α.*

Proposition 3 divides the set of ideal points of X into two classes; we refer to the ideal points from the first and second part of Proposition 3 as *type one* and *type two ideal points* respectively. If Σ is an essential surface associated to a type two ideal point, we say that the slope represented by $\partial\Sigma$ is *strongly detected by the ideal point.*

Type two ideal points are very well-understood. In fact, the Culler-Shalen norm of [10], which we will describe in Section 1.6, determines precisely which boundary slopes of a knot manifold are strongly detected by the type two ideal points of X. On the other hand, little seems to be known about type one ideal points. For example, it is currently unknown whether every closed essential surface in a knot manifold is detected by an ideal point of a curve in $X(N)$. The analogous question has been answered for type two ideal points: Chesebro and Tillmann found an infinite family of knot manifolds with a boundary slope that is not strongly detected by an ideal point [6].

Question 4. *Suppose N is a knot manifold. Is every closed essential surface in N detected by an ideal point of a curve in $X(N)$?*

1.6 The Culler-Shalen norm

As usual, let N be a knot manifold. Suppose $X \subseteq X(N)$ is irreducible curve. In this section, we describe a norm on $H_1(\partial N) \cong \mathbb{R}^2$ associated to X and the norm's connection to the boundary slopes of N that are strongly detected by ideal points of X. Our summary follows [22]. In Chapter 2 we will connect the Culler-Shalen norm to the rank of certain modules associated to essential surface detection.

Proposition 5. *[10, Section 1.4] Let N be a knot manifold. Fix an irreducible curve $X \subseteq X(N)$. Suppose the trace function I_γ is non-constant on X for every $\gamma \in \pi_1(\partial N)$. Then there is a norm $||-||$ on $H_1(\partial N, \mathbb{R})$ such that*

$$||\gamma|| = \deg I_\gamma$$

for every $\gamma \in H_1(\partial N, \mathbb{Z})$.

Sketch of proof. First, since $\pi_1(\partial N)$ is abelian, the Hurewicz homomorphism $\pi_1(\partial N) \to H_1(\partial N, \mathbb{Z})$ is an isomorphism. We identify $H_1(\partial N, \mathbb{Z})$ with a subgroup of $\pi_1(N)$ by composing the inverse of the Hurewicz map with the monomorphism $\pi_1(\partial N) \to \pi_1(N)$ selected in the previous section. Take a smooth projective model $\phi\colon\ \widetilde{X} \dashrightarrow X$. Let $P_1, \ldots, P_k \in \widetilde{X}$ be the ideal points of X and suppose v_i is the valuation on $\mathbb{C}(X)$ associated to P_i . If $\gamma \in H_1(\partial N, \mathbb{Z})$, the degree of I_γ is equal to the sum of the order of the poles of I_γ at each P_i since $\widetilde{X}$ is smooth [19, Proposition 3.17]. For each P_i define a function $p_i\colon\ H_1(\partial N, \mathbb{Z}) \to \mathbb{Z}$ by

$$p_i(\gamma) = \begin{cases} 0 & I_\gamma(P_i) \in \mathbb{C} \\ \text{order of the pole of } I_\gamma \text{ at } P_i & \text{otherwise} \end{cases}$$

Using the fact that $H_1(\partial N, \mathbb{Z})$ is abelian, one can realize p_i as the absolute value of a homomorphism $l_i\colon\ H_1(\partial N, \mathbb{Z}) \to \mathbb{Z}$. Viewing $H_1(\partial N, \mathbb{Z})$ as a lattice in $H_1(\partial N, \mathbb{R})$, extend l_i to an

$\mathbb{R}$-linear functional on $H_1(\partial N, \mathbb{R})$. Now define $||-|| : H_1(\partial N, \mathbb{R}) \to \mathbb{R}$ by

$$||x|| = \sum_{i=1}^{n} |l_i(x)|$$

and note that the triangle inequality holds since the l_i's are linear. Since I_γ is non-constant for every $\gamma \in H_1(\partial N, \mathbb{Z})$, each I_γ has a pole at at least one ideal point. Hence $||v|| = 0$ if and only if $v = 0$ and $||-||$ is a norm on $H_1(\partial N, \mathbb{R})$. □

If there is some $\gamma \in H_1(\partial N, \mathbb{Z})$ such that I_γ is constant, the function $||-||$ is merely a semi-norm. Thus, curves satisfying the hypotheses of Proposition 5 are called *norm curves* and $||-||$ is called *the norm associated to X*.

Norm curves arise often in practice. For example, work of Thurston shows that if the interior of N admits a finite volume hyperbolic metric, then any component of $X(N)$ containing the character of a discrete faithful representation is a norm curve.

We now describe the connection between the norm associated to X and the boundary slopes of N strongly detected by X. Suppose $J = \{P_1, \ldots, P_r\}$ is the set of type two ideal points of X and that $P_i \in J$ detects a slope represented by $\gamma_i \in H_1(\partial N, \mathbb{Z})$. Then $I_{\gamma_i}(P_i) \in \mathbb{C}$, so γ_i is in the kernel of the functional l_i. In fact, since P_i is a type two ideal point, the kernel of l_i is the line L_i spanned by γ_i in $H_1(\partial N, \mathbb{R})$. The lines $L_1, \ldots, L_r$ divide the plane $H_1(\partial N, \mathbb{R})$ into $2r$ segments. On a fixed segment each l_i is either strictly positive or strictly negative and hence $|l_i|$ is equal to either l_i or $-l_i$. In particular, when restricted to a fixed segment, $||-||$ is a linear functional. Thus, the collection of points in a fixed segment with norm 1 is a line. These observations imply the following theorem.

Theorem 6. *[10, Section 1.4] Let $X \subset X(N)$ be a norm curve. Then the unit ball of the norm associated to X is a finite-sided polygon whose vertices are rational multiples of the boundary slopes strongly detected by the ideal points of X*

1.7 From ideal points to module structures on $\mathbb{C}[X]$

We can now set up the framework for Chapter 2. Throughout this section, let $X \subseteq \mathbb{C}^n$ be an irreducible affine curve with normalization $\phi\colon \overline{X} \to X$ and smooth projective model $\iota\colon \widetilde{X} \dashrightarrow \overline{X}$.

For a point $P \in \widetilde{X}$, let $\mathcal{O}_{\widetilde{X},P}$ denote the set of rational functions on $\widetilde{X}$ which are regular at P. Then $\mathcal{O}_{\widetilde{X},P}$ is a subring of $\mathbb{C}(\widetilde{X})$ called the *local ring of* $\widetilde{X}$ *at* P. Take a dominant regular map $f\colon X \to Y$ with Y an irreducible affine variety. If

$$(f \circ \phi \circ \iota)^*(\mathbb{C}[Y]) \subseteq \mathcal{O}_{\widetilde{X},P}$$

for some idea point $P \in \mathcal{I}(X)$ we say f has a *hole at* P. Chesebro discovered a connection between the finiteness of f and the presence of holes.

Theorem 7. *[5, Theorem 2.4] A dominant regular map $\phi\colon X \to Y$ is finite if and only if* $\dim Y = 1$ *and ϕ has no hole.*

Take a knot manifold N and suppose $X \subseteq X(N)$. We briefly summarize Chesebro's work in [5] which connects essential surface detection to module structures on $\mathbb{C}[X]$. This perspective has the advantage of providing new insight into the case when X detects a closed essential surface.

Let $\partial\colon X \to X(\partial N)$ be the regular map defined by restricting characters of representations $\pi_1(N) \to \mathrm{SL}_2(\mathbb{C})$ to $\pi_1(\partial N)$. Let ∂X be the Zariski-closure of $\partial(X)$.

Theorem 8. *[5, Theorem 3.3] The regular map $\partial\colon X \to \partial X$ is finite if and only if X does not detect a closed essential surface.*

Each trace function $I_\gamma\colon X \to \mathbb{C}$ induces a homomorphism I_γ^* on coordinate rings which is injective if I_γ is non-constant since $\dim X = 1$. Identify the image of the affine line's coordinate ring under I_γ^* with the ring $\mathbb{C}[I_\gamma]$ generated by I_γ in $\mathbb{C}[X]$. Then $\mathbb{C}[X]$ is naturally a $\mathbb{C}[I_\gamma]$-module for each $\gamma \in \pi_1(N)$.

Take a subring $R \subset \mathbb{C}$ with unity and let $T_R(X)$ denote the R-algebra generated by 1 and the coordinate functions on X in $\mathbb{C}[X]$. Since the trace functions are defined over $\mathbb{Z}$, $T_R(X)$ is an $R[I_\gamma]$-module.

Let $\mathcal{S}$ denote the set of slopes on ∂N and recall that for each $\alpha \in \mathcal{S}$ there is a well-defined trace function I_α. For any subring of R of $\mathbb{C}$ define a function $\operatorname{rk}_X^R \colon \mathcal{S} \to \mathbb{Z} \cup \{\infty\}$ by

$$\operatorname{rk}_X^R(\alpha) = \operatorname{rk}_{R[I_\alpha]}(T_R(X)).$$

The following was the main result of [5].

Theorem 9. *[5, Theorem 1.2] Let X be an irreducible component of $X(N)$ and $R \in \{\mathbb{Z}, \mathbb{Q}, \mathbb{C}\}$.*

1. *X detects a closed essential surface if and only if rk_X^R is constant with value ∞.*
2. *Otherwise, $\operatorname{rk}_X^R(\alpha) = \infty$ if and only if X strongly detects the slope α.*

When $R = \mathbb{C}$, Theorem 9 essentially follows from Theorem 8 and Proposition 3. On the other hand, the proofs for when $R = \mathbb{Q}$ or $\mathbb{Z}$ are substantially more involved and incorporate ideas from [4] and [8]. Theorem 9 leads naturally to the following questions.

Question 10. *For a fixed slope α, what is the relationship between $\operatorname{rk}_X^R(\alpha)$ for $R \in \{\mathbb{Z}, \mathbb{Q}, \mathbb{C}\}$?*

In the final section of [5], Chesebro calculated the value of $\operatorname{rk}_X^R(\alpha)$ for a collection of knot manifolds and a handful of slopes. His techniques were computationally demanding and involved Gröbner bases. It appears difficult to apply his techniques to compute $\operatorname{rk}_X^R(\alpha)$ for every slope $\alpha \in \mathcal{S}$. Thus, we ask:

Question 11. *Given a knot manifold N and an irreducible curve $X \subseteq X(N)$, can we calculate $\operatorname{rk}^R_X(\alpha)$ for every $\alpha \in \mathcal{S}$?*

We will spend the bulk of the next chapter investigating these questions. We will derive a formula for $\operatorname{rk}^k_X(\alpha)$ for many subfields k of $\mathbb{C}$ which involves the Culler-Shalen norm from Section 1.6 to the minimal field of definition of X. This formula will allow us to answer Question 11 affirmatively for many specific knot manifolds.

Chapter 2

Fields of definition and the Culler-Shalen norm

Fix a small knot manifold N whose character variety $X(N)$ contains an irreducible norm curve X. Let $\mathcal{S}$ denote the set of slopes on ∂N. Suppose the minimal field of definition of X is the number field k_0 and recall that k_0 is independent of the presentation used to compute $X(N)$. Let $||-|| : H_1(\partial N, \mathbb{R}) \to \mathbb{R}$ be the Culler-Shalen norm associated to X from Section 1.6. The following is the main result of this thesis.

Theorem 12. *Suppose $\alpha \in \mathcal{S}$ is not a boundary slope of N.*

1. *For every subfield $k_0 \subseteq k \subseteq \mathbb{C}$, $\mathrm{rk}\,^k_X(\alpha) = ||\alpha||$ and*
2. *For all subfields $\mathbb{Q} \subseteq k \subseteq k_0$, $\mathrm{rk}\,^k_X(\alpha) = [k_0 : k] \cdot ||\alpha||$.*

This theorem follows from purely algebro-geometric considerations, so we divide this chapter into two sections. The first section will be devoted to algebraic geometry and the second will apply our work to the character variety of N.

2.1 Algebraic geometry

Throughout this section, fix an irreducible curve $X \subset \mathbb{C}^n$. Let $\phi\colon \mathbb{C}[z_1, \ldots, z_n] \to \mathbb{C}[X]$ be natural quotient induced by the inclusion $X \hookrightarrow \mathbb{C}^n$. We are primarily concerned with curves in the character varieties of knot manifolds, so we assume that the minimal field of definition k_0 of X is a number field (see Proposition 1).

The proof of Theorem 12 involves studying the trace functions associated to slopes which are not boundary slopes of a small knot manifold. By Theorem 9, these trace functions are finite regular maps. Recall from Section 1.4 that each trace function can be represented by a polynomial with integer coefficients. With these observations in mind, fix a polynomial $g \in \mathbb{Z}[z_1, \ldots, z_n]$ such that $f = \phi(g) \in \mathbb{C}[X]$ is finite as a regular function on X for the remainder of this section.

Since X is an irreducible curve and f is finite, $f\colon X \to \mathbb{C}$ must be non-constant and hence dominant. Then f induces a field monomorphism between the field of rational functions on the affine line and $\mathbb{C}(X)$. The coordinate ring of and field of rational functions on the affine line are isomorphic to $\mathbb{C}[s]$ and $\mathbb{C}(s)$ respectively and the homomorphism induced by f takes $s \mapsto f$ in $\mathbb{C}(X)$. Thus, the subring $\mathbb{C}[f]$ and subfield $\mathbb{C}(f)$ of $\mathbb{C}(X)$ are isomorphic to $\mathbb{C}[s]$ and $\mathbb{C}(s)$.

We have $\mathbb{C} \subset \mathbb{C}(f) \subset \mathbb{C}(X)$, so

$$\operatorname{tr.deg}_{\mathbb{C}} \mathbb{C}(X) = \operatorname{tr.deg}_{\mathbb{C}(f)} \mathbb{C}(X) + \operatorname{tr.deg}_{\mathbb{C}} \mathbb{C}(f)$$

and $\dim X = \dim \mathbb{C} = 1$, so $\operatorname{tr.deg}_{\mathbb{C}(f)} \mathbb{C}(X) = 0$. In particular, $\mathbb{C}(X)$ is an algebraic extension of $\mathbb{C}(f)$. $\mathbb{C}(X)$ is a finitely generated extension of $\mathbb{C}$ and hence of $\mathbb{C}(f)$ as well. Thus, $\mathbb{C}(X)$ is actually a finite extension of $\mathbb{C}(f)$ [2, Corollary 5.2] and the degree $d = [\mathbb{C}(X) : \mathbb{C}(f)]$ of f is finite.

The field $\mathbb{C}(f)$ has characteristic 0, so the primitive element theorem implies that the extension $\mathbb{C}(X)/\mathbb{C}(f)$ is simple. A few of our proofs in this section will rely on a careful choice of primitive elements, so we record a key ingredient of the proof of the primitive element theorem from [18, Theorem 5.1] in the theorem below.

Theorem 13 (The primitive element theorem)**.** *Let E be a field with characteristic 0 and let F be a finite extension of E. If F can be generated by $a, b \in F$ over E, then $F = E[a + \lambda b]$ for all but finitely many $\lambda \in E$.*

Remark. The statement of Theorem 13 does not give a standard formulation of the primitive element theorem. It does, however, imply one of these formulations: by inductively applying Theorem 13 to an arbitrary finite extension F of E one can show that F is a simple extension of E.

The next lemma relates the degree d of f to the rank of $\mathbb{C}[X]$ over $\mathbb{C}[f]$.

Lemma 14. *$\mathbb{C}[X]$ is a finitely generated rank d free $\mathbb{C}[f]$-module.*

Proof. Since f is finite, $\mathbb{C}[X]$ is finitely generated $\mathbb{C}[f]$-module. $\mathbb{C}[f]$ is a principle ideal domain and $\mathbb{C}[X]$ is torsion-free over $\mathbb{C}[f]$ since X is irreducible. Hence, by [3, Theorem 2.7.6], $\mathbb{C}[X]$ is a free $\mathbb{C}[f]$-module. Let r denote the rank of $\mathbb{C}[X]$ over $\mathbb{C}[f]$.

Let $\mathcal{B}$ be any subset of $\mathbb{C}[X]$. Clearing denominators transforms any linear dependence relation among the elements of $\mathcal{B}$ over $\mathbb{C}(f)$ into a dependence relation over $\mathbb{C}[f]$, so a maximal $\mathbb{C}[f]$-linearly independent subset of $\mathbb{C}[X]$ is linearly independent in $\mathbb{C}(X)$ over $\mathbb{C}(f)$. Hence $r \leq d$.

$\mathbb{C}(X)$ is a finite extension of $\mathbb{C}(f)$, so by Theorem 13, we may choose a polynomial $p \in \mathbb{C}[z_1, \ldots, z_n]$ such that $\mathbb{C}(X) = \mathbb{C}(f)[\phi(p)]$. The minimum polynomial for $\phi(p)$ over $\mathbb{C}(f)$ has degree d, so the set $\{\phi(p)^j\}_{j=0}^{d-1} \subset \mathbb{C}[X]$ is linearly independent over $\mathbb{C}(f)$. In particular, $d \leq r$. $\square$

Recalling the notation from Section 1.7, note that $f \in T_k(X)$ for every subfield k of $\mathbb{C}$ since $g \in \mathbb{Z}[z_1, \ldots, z_n]$ and $f = \phi(g)$. Thus, each $T_k(X)$ is a $k[f]$-module. We will prove the following theorem as a sequence of lemmas and propositions that use elementary Galois theory. Our approach is motivated by the observation that the two norm curves in [5, Example 7.16] are conjugate varieties.

Theorem 15. *Let k be a subfield of $\mathbb{C}$.*

1. *If $k \supseteq k_0$, then $T_k(X)$ is a free $k[f]$-module with rank d.*
2. *If $k \subseteq k_0$, then $T_k(X)$ is a free $k[f]$-module with rank $[k_0 : k] \cdot d$.*

The next proposition gives the first part of Theorem 15.

Proposition 16. *For every field k containing k_0, $T_k(X)$ is a rank d free $k[f]$-module*

Proof. By the remark following Corollary 2.5 in [5] we may pick a free basis $\mathcal{B} = \{b_j\}_{j=1}^d$ for $\mathbb{C}[X]$ over $\mathbb{C}[f]$ which lies in $T_{\mathbb{Z}}(X)$. Then $\mathcal{B}$ is linearly independent over $k[f] \subseteq \mathbb{C}[X]$.

We claim that $\mathcal{B}$ spans $T_k(X)$ over $k[f]$. Take $h \in T_k(X)$. There are unique $p_j \in \mathbb{C}[f]$ so that $h = \sum p_j b_j$ in $\mathbb{C}[X]$. Since $k_0 \subseteq k$, the group $\mathrm{Aut}(\mathbb{C}/k)$ acts naturally on $\mathbb{C}[X]$ (see Subsection 1.3.1). If $\sigma \in \mathrm{Aut}(\mathbb{C}/k)$, then

$$h = \sigma \cdot h = \sum p_j^\sigma b_j \quad \Rightarrow \quad 0 = \sum (p_j - p_j^\sigma) b_j.$$

Thus $p_j = p_j^\sigma$ for every $\sigma \in \mathrm{Aut}(\mathbb{C}/k)$. Since k is perfect and $\mathbb{C}$ is algebraically closed, the fixed field of $\mathrm{Aut}(\mathbb{C}/k)$ is k [18, Theorem 9.29]. This implies that $p_j \in k[f]$, so $\mathcal{B}$ spans $T_k(X)$ over $k[f]$. In particular, $T_k(X)$ is a rank d free $k[f]$-module. $\square$

The proof of the second part of Theorem 15 is more involved. The basic idea is this: if $k \subseteq k_0$, then there are precisely $[k_0 : k]$ varieties conjugate to X over k [16, Theorem III.10]. The finite regular function $f\colon X \to \mathbb{C}$ extends to a finite regular function on each conjugate variety and the rank of their coordinate rings over $\mathbb{C}[f]$ is d. If $\mathfrak{X}$ is the union of the conjugate varieties to X over k, we show the rank of $\mathbb{C}[\mathfrak{X}]$ over $\mathbb{C}[f]$ is $d \cdot [k_0 : k]$. Next, we show that $T_k(\mathfrak{X})$ is isomorphic to $T_k(X)$ as both rings and $k[f]$-modules and the result follows by an argument similar to the proof of Proposition 16.

With this sketch in mind, we begin with a slightly more general set up. Fix a collection of distinct irreducible curves $\mathfrak{X} = X_1 \cup \cdots \cup X_m \subset \mathbb{C}^n$. Let $\phi\colon \mathbb{C}[z_1, \dots, z_n] \to \mathbb{C}[\mathfrak{X}]$ and $\phi_i\colon \mathbb{C}[z_1, \dots, z_n] \to \mathbb{C}[X_i]$ denote the natural quotients. Set $f_i = \phi_i(g)$, $f = \phi(g)$, and suppose that f_i is a finite, dominant regular function on X_i with degree d_i.

We begin our investigation of the rank of $\mathbb{C}[\mathfrak{X}]$ over $\mathbb{C}[f]$ with a lemma. Thinking geometrically the lemma essentially says that we can pick a "direction" such that a regular map $\mathfrak{X} \to \mathbb{C}^2$ whose first coordinate is f projects the X_i's onto distinct curves in $\mathbb{C}^2$.

Lemma 17. *There is a polynomial $p \in \mathbb{C}[z_1, \dots, z_n]$ such that*

1. *$p_i = \phi_i(p)$ is a primitive element of $\mathbb{C}(X_i)$ over $\mathbb{C}(f_i)$, and*
2. *the minimum polynomial $m_i(t) \in \mathbb{C}(f_i)[t]$ for p_i in $\mathbb{C}(X_i)$ is a monic irreducible polynomial of degree d_i with coefficients in $\mathbb{C}[f_i]$, and*
3. *for each i there is a unique polynomial $q_i(s,t) \in \mathbb{C}[s,t]$ such that $m_i(t) = q_i(f_i, t)$ and*

$$q_i(s,t) = q_j(s,t) \iff i = j.$$

Proof. We prove the lemma in the case that $m = 2$. The general result follows by induction. Theorem 13 gives a polynomial $r_i \in \mathbb{C}[z_1, \dots, z_n]$ such that $\phi_i(r_i)$ is primitive in $\mathbb{C}(X_i)$ over

$\mathbb{C}(f_i)$ for each i. Thus

$$\begin{aligned} \mathbb{C}(X_1) &= \mathbb{C}(f_1)[\phi_1(r_1)] &= \mathbb{C}(f_1)[\phi_1(r_1), \phi_1(r_2)] \\ \text{and} \quad \mathbb{C}(X_2) &= \mathbb{C}(f_2)[\phi_2(r_2)] &= \mathbb{C}(f_2)[\phi_2(r_2), \phi_2(r_1)]. \end{aligned}$$

By Theorem 13 we can pick $\lambda \in \mathbb{C}^\times$ so that

$$\mathbb{C}(X_1) = \mathbb{C}(f_1)[\phi_1(r_1 + \lambda r_2)] \quad \text{and} \quad \mathbb{C}(X_2) = \mathbb{C}(f_2)[\phi_2(r_2 + \lambda^{-1} r_1)].$$

In particular, if $p = r_1 + \lambda r_2 \in \mathbb{C}[z_1, \ldots, z_n]$, then $p_i = \phi_i(p)$ is primitive in $\mathbb{C}(X_i)$ over $\mathbb{C}(f_i)$ for each i.

The regular function $f_i\colon X_i \to \mathbb{C}$ is finite, so $\mathbb{C}[X_i]$ is a finitely generated $\mathbb{C}[f_i]$-module. Hence, by [2, Proposition 5.1], $\mathbb{C}[X_i]$ is integral over $\mathbb{C}[f_i]$. Since $\mathbb{C}[f_i]$ is integrally closed in $\mathbb{C}(f_i)$, the minimum polynomial $m_i(t) \in \mathbb{C}(f_i)[t]$ of p_i in $\mathbb{C}(X_i)$ is monic, irreducible, and has coefficients in $\mathbb{C}[f_i]$ [2, Proposition 5.15]. Since p_i is primitive in $\mathbb{C}(X_i)$, $m_i(t)$ has degree $d_i = [\mathbb{C}(X_i) : \mathbb{C}(f_i)]$.

Now consider the regular map $(f_i, p_i)\colon X_i \to \mathbb{C}^2$. The regular function f_i is dominant so (f_i, p_i) is non-constant, which implies that the Zariski-closure of its image is a curve Y_i. The closure of the image of an irreducible algebraic set is irreducible, so Y_i is the zero set of a single irreducible polynomial $q_i(s,t) \in \mathbb{C}[s,t]$. By construction, $q_i(g,p) \in I(X_i)$ and hence $q_i(f_i,t)$ is divisible by $m_i(t)$ in $\mathbb{C}(f_i)[t]$. But $q_i(s,t)$ is irreducible, so we must have $q_i(f_i,t) = c \cdot m_i(t)$ for some $c \in \mathbb{C}^\times$. Thus, after multiplying by a scalar, we may assume $q_i(s,t)$ is monic of degree d_i in t. The uniqueness of $q_i(s,t)$ now follows from this choice of normalization and the irreducibility of $q_i(s,t)$.

Lastly, we show that we can adjust p so that $q_1(s,t) \neq q_2(s,t)$ if necessary. X_1 and X_2 are distinct irreducible curves, so there is a polynomial $r \in I(X_1) - I(X_2)$. The ideal $I(X_2)$ is prime, so $\lambda r \in I(X_1) - I(X_2)$ for every $\lambda \in \mathbb{C}^\times$. Note that $m_2(t) = q_2(f_2,t)$ has at most $d_1 = d_2$ roots in $\mathbb{C}(X_2)$, so for all but finitely many $\lambda \in \mathbb{C}^\times$, $m_2(\phi_2(p + \lambda r)) \neq 0$ in $\mathbb{C}[X_2]$. Moreover, avoiding another finite collection of scalars, we may choose $\lambda \in \mathbb{C}^\times$ so that $p_i' = \phi_i(p + \lambda r)$ is primitive in $\mathbb{C}(X_i)$ over $\mathbb{C}(f_i)$.

If $q_1(s,t) = q_2(s,t)$, then the curves Y_1 and Y_2 are equal. Since $\lambda r \in I(X_1)$, $p_1' = p_1$ in $\mathbb{C}[X_1]$ and the closure of the image of the regular map $(f_1, p_1')\colon X_1 \to \mathbb{C}^2$ is Y_1. On the other hand, $m_2(p_2') \neq 0$ in $\mathbb{C}(X_2)$, so the image of the map $(f_2, p_2')\colon X_2 \to \mathbb{C}^2$ is an irreducible curve Y_2' distinct from $Y_1 = Y_2$. Hence there is a polynomial $q_2'(s,t)$ that is monic of degree d_2 in t and irreducible with $q_2'(s,t) \neq q_1(s,t)$ such that $q_2'(f_2, t)$ is the minimum polynomial for p_2' in $\mathbb{C}(X_2)$ over $\mathbb{C}(f_2)$. □

We can now relate the rank of $\mathbb{C}[\mathfrak{X}]$ over $\mathbb{C}[f]$ to that of the $\mathbb{C}[X_i]$'s over $\mathbb{C}[f_i]$. Notice that $\mathbb{C}[X_i]$ is a finitely generated free $\mathbb{C}[f]$-module by defining $f \cdot h = f_i h$ for each $h \in \mathbb{C}[X_i]$. The rank of $\mathbb{C}[X_i]$ over $\mathbb{C}[f]$ is equal to that of $\mathbb{C}[X_i]$ over $\mathbb{C}[f_i]$.

Proposition 18. *$\mathbb{C}[\mathfrak{X}]$ is a free $\mathbb{C}[f]$-module with rank $D = \sum_1^m d_i$.*

Proof. Consider the homomorphism $\mathbb{C}[\mathfrak{X}] \to \oplus_1^m \mathbb{C}[X_i]$ given by $\phi(h) \mapsto (\phi_i(h))_{i=1}^m$ for $h \in \mathbb{C}[z_1, \ldots, z_n]$. Since $I(\mathfrak{X}) = \cap_1^m I(X_i)$, it is well-defined and injective. It is easy to see that the homomorphism is $\mathbb{C}[f]$-linear. By Lemma 14 and our observations above, $\oplus_1^m \mathbb{C}[X_i]$ is a finitely generated free $\mathbb{C}[f]$-module. In particular, $\mathbb{C}[\mathfrak{X}]$ is finitely generated and free with rank at most D over $\mathbb{C}[f]$.

Fix a polynomial $P \in \mathbb{C}[z_1, \ldots, z_n]$ given by Lemma 17. Set $p_i = \phi_i(P)$ and $p = \phi(P)$. Let $q_i(s,t) \in \mathbb{C}[s,t]$ be the polynomial from the third part of the lemma and define $q(s,t) = \prod_1^m q_i(s,t)$. Note that $q(s,t)$ is monic of degree D in t with no repeated factors. By construction, $q_i(g, P) \in I(X_i)$ and hence $q(g, P) \in I(\mathfrak{X})$.

As in the proof of Lemma 17, let Y_i denote the Zariski closure of the image of the regular map

$(f_i, p_i)\colon X_i \to \mathbb{C}^2$ and let $\mathfrak{Y} = Y_1 \cup \cdots \cup Y_m$. Then

$$Y_i = Z(q_i(s,t)) \quad \text{and} \quad \mathfrak{Y} = Z(q(s,t))$$

Since $q(s,t)$ has no repeated factors, the ideal it generates in $\mathbb{C}[s,t]$ is radical. Hence $I(\mathfrak{Y})$ is principle and generated by $q(s,t)$. Let ψ_i and ψ denote the natural quotients from $\mathbb{C}[s,t]$ to $\mathbb{C}[Y_i]$ and $\mathbb{C}[\mathfrak{Y}]$ respectively. Since $(f_i, p_i)\colon X_i \to Y_i$ is dominant, the induced homomorphism $\mathbb{C}[Y_i] \to \mathbb{C}[X_i]$, which takes $\psi_i(s) \mapsto f_i$ and $\psi_i(t) \mapsto p_i$, is injective.

Claim. The $\mathbb{C}$-algebra homomorphism $F\colon \mathbb{C}[\mathfrak{Y}] \to \mathbb{C}[\mathfrak{X}]$ defined by $F(\psi(s)) = f$ and $F(\psi(t)) = p$ is injective.

The proof of the proposition follows readily from this claim. We show that the set $\{p^j\}_{j=0}^{D-1}$ is linearly independent in $\mathbb{C}[\mathfrak{X}]$. Suppose we have $r_j(s) \in \mathbb{C}[s]$ such that $\sum_0^{D-1} r_j(f)p^j = 0$ in $\mathbb{C}[\mathfrak{X}]$. Then, by the claim, $\sum_0^{D-1} r_j(s)t^j \in I(\mathfrak{Y})$ which implies that $q(s,t)$ divides $\sum_0^{D-1} r_j(s)t^j$. But q is monic of degree D in t, so each $r_j(s)$ must be 0.

To prove the claim, first note that if $u - v \in I(\mathfrak{Y})$ for some $u, v \in \mathbb{C}[s,t]$, then q divides $u - v$ and so $u(g,P) - v(g,P) \in I(\mathfrak{X})$. Hence F is well-defined. Now take $r(s,t) \in \mathbb{C}[s,t]$ with $r(g,P) \in I(\mathfrak{X})$. Then $r(g,P) \in I(X_i)$ for each i. The homomorphism $\mathbb{C}[Y_i] \to \mathbb{C}[X_i]$ induced by f_i is injective, so $r(s,t) \in I(Y_i)$. Thus, each $q_i(s,t)$ divides $r(s,t)$. But the q_i's are distinct and irreducible, so $q(s,t)$ divides $r(s,t)$. Hence $r(s,t) \in I(\mathfrak{Y})$, which proves the claim. □

We now return to our proof of Theorem 15. Take subfield k of the number field k_0. There are $m = [k_0 : k]$ varieties in $\mathbb{C}^n$ conjugate to X over k [16, Ch. III, Thm. 10]. Label them $X_1, \ldots, X_m$ with $X_1 = X$ and set $\mathfrak{X} = \cup X_i$. $\mathfrak{X}$ is invariant under the action of $\mathrm{Aut}(\mathbb{C}/k)$ on $\mathbb{C}^n$, so $\mathfrak{X}$ is defined over k [16, Ch. III, Thm. 7]. Recall the notation $f = \phi(g)$ and $f_i = \phi_i(g)$ and the $\mathbb{C}[f]$-module structure on each $\mathbb{C}[X_i]$ from the preamble to Proposition 18.

Lemma 19. *As a $\mathbb{C}[f]$-module, $\mathbb{C}[X_i]$ is free with rank d and $\mathbb{C}[\mathfrak{X}]$ is free with rank $m \cdot d$.*

Proof. Let $\sigma_i \in \mathrm{Aut}(\mathbb{C}/k)$ be any automorphism such that $\sigma_i \cdot X_1 = X_i$. As an automorphism of $\mathbb{C}[z_1, \ldots, z_n]$, σ_i descends to an isomorphism $\sigma_i^* \colon \mathbb{C}[X_1] \to \mathbb{C}[X_i]$ taking $\phi_1(h) \mapsto \phi_i(h^\sigma)$ for $h \in \mathbb{C}[z_1, \ldots, z_n]$. Then σ_i^* is $\mathbb{C}[f]$-linear since $\sigma_i \cdot f_1 = f_i$ and $f \cdot p = f_i p$ for $p \in \mathbb{C}[X_i]$. Thus $\mathbb{C}[X_i]$ is free with rank d over $\mathbb{C}[f]$.

By Proposition 18, $\mathbb{C}[\mathfrak{X}]$ is free with rank $m \cdot d$ over $\mathbb{C}[f]$. □

Lemma 20. *As rings, k-algebras, and $k[f]$-modules, $T_k(X)$ is isomorphic to $T_k(\mathfrak{X})$.*

Proof. Composing the inclusion and the natural quotient

$$k[z_1, \ldots, z_n] \hookrightarrow \mathbb{C}[z_1, \ldots, z_n] \twoheadrightarrow \mathbb{C}[X]$$

gives a surjection $k[z_1, \ldots, z_n] \to T_k(X)$ with kernel $I_k(X) := I(X) \cap k[z_1, \ldots, z_n]$, so

$$T_k(X) \cong k[z_1, \ldots, z_n]/I_k(X)$$

and similarly, $T_k(\mathfrak{X}) \cong k[z_1, \ldots, z_n]/I_k(\mathfrak{X})$.

We claim $I_k(X) = I_k(\mathfrak{X})$. Let $J = \mathbb{C} \cdot I_k(X)$ be the ideal generated by $I_k(X)$ in $\mathbb{C}[z_1, \ldots, z_n]$. Notice that $J \subset I(X)$. For each $1 \leq i \leq m$ take $\sigma_i \in \mathrm{Aut}(\mathbb{C}/k)$ with $\sigma_i \cdot I(X) = I(X_i)$. By construction, J is defined over k, so $\sigma_i \cdot J = J$. Thus, $J \subseteq I(X_i)$ for each i and hence $J \subseteq I(\mathfrak{X}) = \cap_1^m I(X_i)$. Unwinding the definitions we have

$$I_k(X) = J \cap k[z_1, \ldots, z_n] \subseteq (\cap_1^m I(X_j)) \cap k[z_1, \ldots, z_n] = I_k(\mathfrak{X})$$

and the equality follows since $I(\mathfrak{X}) \subseteq I(X)$.

Thus $T_k(X) \cong T_k(\mathfrak{X})$ and the isomorphism is $k[f]$-linear since $f = \phi(g)$ with $g \in \mathbb{Z}[z_1, \ldots, z_n]$. □

Theorem 15 readily follows from the work above. The first part of the theorem is Proposition 16. The second part of the theorem follows from Lemmas 19 and 20 as well as an argument identical to the proof of Proposition 16 applied to $T_k(\mathfrak{X}) \subseteq \mathbb{C}[\mathfrak{X}]$.

2.2 Character varieties of knot manifolds

In this section we prove Theorem 12 and discuss applications. Let N be a small knot manifold. Suppose $X \subseteq X(N)$ is an irreducible norm curve whose minimal field of definition is the number field k_0. Let $||-||$ denote the Culler-Shalen norm associated to X.

Proof of Theorem 12. Suppose $\alpha \in \mathcal{S}$ is not a boundary slope of N. N is small, so $I_\alpha \colon X \to \mathbb{C}$ is finite by Theorem 9. Since $||\alpha|| = \deg I_\alpha = [\mathbb{C}(X) : \mathbb{C}(I_\alpha)]$ [10, Section 4], the first part of Theorem 15 implies that

$$\operatorname{rk}{}_X^k(\alpha) = ||\alpha||$$

whenever k is a field containing k_0. Since k_0 is a finite extension of $\mathbb{Q}$, the second part of Theorem 15 implies

$$\operatorname{rk}{}_X^k(\alpha) = [k_0 : k] \cdot ||\alpha||$$

for any subfield k of k_0. □

Theorem 12 answers Question 10 for $R = \mathbb{C}$ and $\mathbb{Q}$, but not $\mathbb{Z}$. It turns out, however, that we get no new information from $\operatorname{rk}{}_X^{\mathbb{Z}}$. Recall that if k is a number field, the *ring of integers of* k, denoted $\mathcal{O}_k$, is the integral closure of $\mathbb{Z}$ in k. By Theorem 9, if α is not a strongly detected by X, then $\operatorname{rk}{}_X^{\mathcal{O}_k}(\alpha) < \infty$. We can use Theorem 12 to be explicit about the values of $\operatorname{rk}{}_X^{\mathcal{O}_k}$.

Corollary 21. *Take a number field k. Suppose $\alpha \in \mathcal{S}$ is not strongly detected by X.*

1. *If $k \supseteq k_0$ and α, then $\operatorname{rk}{}_X^{\mathcal{O}_k}(\alpha) = ||\alpha||$.*

2. *If* $k \subseteq k_0$ *and* α*, then* $\operatorname{rk}_X^{\mathcal{O}_k}(\alpha) = [k_0 : k]\,||\alpha||$

Proof. It is easy to see that $\operatorname{rk}_X^{\mathcal{O}_k}(\alpha) \geq \operatorname{rk}_X^{k}(\alpha)$. On the other hand, recall that the field of fractions of $\mathcal{O}_k$ is equal to k. Take any subset $\mathcal{B}$ of $T_{\mathcal{O}_k}(X)$ and a linear dependence relation among the elements of $\mathcal{B}$ over $k[I_\alpha]$. Write the elements of k as fractions in $\mathcal{O}_k$. Clearing denominators transforms this into a dependence relation over $\mathcal{O}_k$. Hence, if $\mathcal{B}$ is a maximal linearly indpendent subset of $T_{\mathcal{O}_k}(X)$, it is also linearly independent in $T_k(X)$ and so $\operatorname{rk}_X^{k}(\alpha) = \operatorname{rk}_X^{\mathcal{O}_k}(\alpha)$. The corollary follows from Theorem 12. □

Remark. Our definition of the rank of a module differs from Chesebro's in [5] when the base ring is not a principle ideal domain. Thus, Corollary 21 does not contradict Theorem 7.17 in [5], where Chesebro shows that $T_{\mathbb{Z}}(X)$ may not be projective over $\mathbb{Z}[I_\alpha]$ and computes the minimal number of generators for $T_{\mathbb{Z}}(X)$ over $\mathbb{Z}[I_\alpha]$ for a specific knot manifold. In this case, the minimum number of generators is not equal to the rank, so this quantity is more interesting than rank when working over $\mathbb{Z}$. In general, however, the minimal number of generators of a non-projective module is much more difficult to compute than its rank. We are currently investigating knot manifolds whose associated modules are not projective.

Our next application of Theorem 12 answers Question 11. To describe it, we briefly review the A-polynomial from [7] and use the exposition from [4, Page 3]. Fix a basis $\{\mu, \lambda\}$ for $\pi_1(\partial N)$. Consider the collection Δ of representations $\rho\colon \pi_1(N) \to \mathrm{SL}_2(\mathbb{C})$ such that $\chi_\rho \in X$ and $\rho(\pi_1(\partial N))$ is upper triangular. Then the Zariski-closure of the set

$$\left\{(m,l) \in \mathbb{C}^2 \mid \exists \rho \in \Delta \text{ such that } \rho(\mu) = \begin{pmatrix} m & \star \\ 0 & m^{-1} \end{pmatrix} \text{ and } \rho(\lambda) = \begin{pmatrix} l & \square \\ 0 & l^{-1} \end{pmatrix}\right\}$$

is an irreducible algebraic curve $E \subseteq \mathbb{C}^2$. The defining equation $A(L,M)$ of E is *the A-polynomial of N associated to X*.

Write $A(L, M) = \sum a_{ij} L^i M^j$. The *Newton polygon* P of $A(L, M)$ is the convex hull of the set $\{(i, j) \in \mathbb{R}^2 \mid a_{ij} \neq 0\}$ in $\mathbb{R}^2$. It turns out that the slopes of the edges of P are precisely the boundary slopes strongly detected by X (cf. [7] and [8]). More specifically, a slope $\alpha = \{\mu^{\pm p}\lambda^{\pm q}\}$ is strongly detected by X if and only if P has an edge with slope p/q.

Recall that the unit ball of $||-||$ is a finite sided polygon in $H_1(\partial N, \mathbb{R}) = \mathbb{R}^2$ whose corners are rational multiples of the boundary slopes strongly detected by X. Boyer and Zhang discovered that $A(L, M)$ determines the Culler-Shalen norm associated to X by demonstrating that the Newton polygon P is dual to the unit ball of $||-||$ [4, Section 8]. In particular, they showed that the Culler-Shalen norm of a slope can be realized as a sum involving the corners of P. Thus, their work implies the following corollary to Theorem 12.

Corollary 22. *Suppose $X \subseteq X(N)$ is a norm curve. If the A-polynomial of N associated to X is known, then one can calculate $\mathrm{rk}_X^{\mathbb{C}}(\alpha)$ for every slope α on ∂N.*

The A-polynomial has been calculated for many knot manifolds, including a few infinite families (cf. [14]). Marc Culler has also established a database of A-polynomials which you can find at this link: `http://homepages.math.uic.edu/~culler/Apolynomials/`. Moreover, the A-polynomial is, in principle, computable for any knot manifold using classic elimination theory. Our next example demonstrates how Corollary 22 works in practice.

Example 2. Let N be the exterior of the figure eight knot in S^3. Using the presentation and notation from Example 1, let X be the curve cut out of $\mathbb{C}^2$ by $\Phi = 1 + y^2 + t - y^2 t + t^2$. The interior of N admits a finite volume hyperbolic metric and X contains the character of a discrete faithful representation, so X is a norm curve.

We first compute the A-polynomial of N associated to X. Set $\mu = a$ and $\lambda = b^{-1}aba^{-2}bab^{-1}$. Then $\{\mu, \lambda\}$ forms a basis for a peripheral subgroup of $\pi_1(N)$. Recall that $I_\mu(y, t) = y$. One can compute a formula for $I_\lambda(y, t)$ by repeatedly applying the Cayley-Hamilton formula and using elementary trace identities. Dividing this formula by Φ using the multi-variable division algorithm with respect to the lexicographic order $y > t$ shows that $I_\lambda = -y^4 + 5y^2 - 2$ on X.

Now compute a Gröbner basis $\mathcal{G}$ for the ideal generated by $\{\Phi, l - I_\lambda\}$ in $\mathbb{C}[l, y, t]$ with respect to the lexicographic order $l > y > t$. Then

$$\mathcal{G} \cap \mathbb{C}[y, l] = \{2 + l - 5y^2 + y^4\}$$

and, by the elimination theorem [9, Ch. 3, Sec. 1, Thm 2], the closure of the image of the regular map $(I_\mu, I_\lambda)\colon X \to \mathbb{C}^2$ is defined by $\Psi(l, y) = 2 + l - 5y^2 + y^4$. The polynomial $M^4L \cdot \Psi(L + L^{-1}, M + M^{-1})$ is irreducible. Moreover, from the definition, we see that the A-polynomial of N associated to X is

$$\begin{aligned} A(L, M) &= M^4L \cdot \Psi(L + L^{-1}, M + M^{-1}) \\ &= L^2M^4 + LM^8 - LM^6 - 2LM^4 - LM^2 + L + M^4. \end{aligned}$$

The Newton polygon of $A(L, M)$ is given in the figure below.

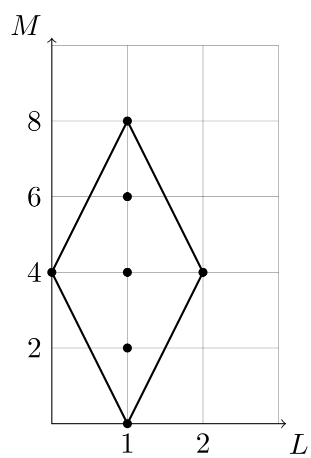

Figure 2.1: The Newton polygon of the figure eight knot's A-polynomial.

Thus, the boundary slopes of N strongly detected by X are represented by $\mu^{\pm 4}\lambda$ in $\pi_1(N)$. By [4, Proposition 8.2], if α is a slope on ∂N represented by $\mu^p\lambda^q$, then $||\alpha|| = |4q - p| + |4q + p|$.

X is defined over $\mathbb{Q}$, so by Corollary 22

$$\operatorname{rk}_X^k(\alpha) = \begin{cases} \infty & p = \pm 4 \text{ and } q = 1 \\ |4q-p| + |4q+p| & \text{otherwise} \end{cases}$$

for every subfield k of $\mathbb{C}$.

In Chapter Seven of [5], Chesebro calculated values of the rank functions associated to certain knot manifolds at various slopes by finding free bases for the corresponding modules. Finding such bases in general appears to be a difficult task. Our proof of Theorem 15 provides a straightforward method for finding a regular function $p \in \mathbb{C}[X]$ such that $\mathcal{P} = \{p^j\}_0^{d-1}$ is a maximal linearly independent subset of $\mathbb{C}[X]$ over $\mathbb{C}[I_\alpha]$. Naively, one might hope that $\mathcal{P}$ is actually basis for $\mathbb{C}[X]$. Our next example serves two purposes: we show how one can construct the set $\mathcal{P}$ in practice, then demonstrate that $\mathcal{P}$ is not a basis. After a fair amount of experimentation, it appears that $\mathcal{P}$ will rarely span $\mathbb{C}[X]$ over $\mathbb{C}[I_\alpha]$.

Example 3. We continue using the set up from Example 2. The slope represented by $\lambda \in \pi_1(N)$ is not strongly detected by X and X is defined over $\mathbb{Q}$, so $T_k(X)$ is a rank 8 free $k[I_\lambda]$-module for any subfield k of $\mathbb{C}$ by Example 2.

By the primitive element theorem, $ay + bt$ is primitive in $\mathbb{C}(X)$ over $\mathbb{C}(I_\lambda)$ for all but finitely many choices $a, b \in \mathbb{Z}$ since $\mathbb{C}(X) = \mathbb{C}(I_\lambda)[y, t]$. Using the elimination theorem to compute the polynomial $q(s, t)$ given in the third part of Lemma 17, we see that we must avoid $(a, b) = (0, \pm 1)$ and $(\pm 1, 0)$, but $y - t$ works. Hence, by the proof of Theorem 12, the $k[I_\lambda]$-submodule M_k generated $\{(y-t)^j\}_0^7$ in $T_k(X)$ has maximal rank over $k[I_\lambda]$ for every subfield k of $\mathbb{C}$. Note that since $M_{\mathbb{C}}$ is generated by powers of $y - t$, it is closed under multiplication and hence is equal to the $\mathbb{C}$-subalgebra of $\mathbb{C}[X]$ generated by I_λ and $y - t$.

Claim. $M_k \neq T_k(X)$.

It suffices to prove the claim for $k = \mathbb{C}$, for if $M_k = T_k(X)$ for some subfield $k \subset \mathbb{C}$, $T_{\mathbb{Z}}(X) \subset$

$M_k \subset M_{\mathbb{C}}$. Since $M_{\mathbb{C}}$ is a $\mathbb{C}$-algebra, this implies $M_{\mathbb{C}} = \mathbb{C}[X]$.

Define a regular map $F\colon X \to \mathbb{C}^2$ by $F(y,t) = (I_\lambda, y-t)$. By the elimination theorem [9, Ch. 3, Sec. 1, Thm 2], $Y = \overline{F(X)}$ is cut out by

$$p(l,r) = r^8 + \sum_0^7 q_j(l) r^j \in \mathbb{Z}[r,l]$$

for certain polynomials $q_j(l) \in \mathbb{Z}[l]$. Computing a Gröbner basis for the ideal generated by $\{p, \partial p/\partial l, \partial p/\partial r\}$ in $\mathbb{C}[l,r]$ with respect to the lexicographic order $r > l$ shows that there are 7 points in Y such that $\partial p/\partial l$ and $\partial p/\partial r$ both vanish. In particular, Y is singular at these seven points since the Jacobian of Y is zero.

It is quick to check that the curve X is nonsingular and that the regular map F has degree one. Thus, $F\colon X \to Y$ is the normalization of Y. Since Y is singular, $F^*(\mathbb{C}[Y])$ is a proper subset of $\mathbb{C}[X]$. Lastly, note that $F^*(l) = I_\lambda$ and $F^*(r) = y - t$, so $F^*(\mathbb{C}[Y]) = M_{\mathbb{C}}$ as $\mathbb{C}$-algebras and hence $M_{\mathbb{C}} \neq \mathbb{C}[X]$.

Recall that if the slope α is strongly detected by X or if X detects a closed essential surface in N, then $\mathrm{rk}_X^{\mathbb{C}}(\alpha) = \infty$ by Theorem 9. In light of Theorem 12, it is natural to wonder if we can recover the Culler-Shalen norm from a module structure and distinguish strongly detected boundary slopes in these cases. We answer both questions affirmatively.

Proposition 23. *Suppose N is a knot manifold such that $X(N)$ contains a norm curve X. Fix a slope $\alpha \in \mathcal{S}$. The rank of the integral closure of $\mathbb{C}[I_\alpha]$ in $\mathbb{C}(X)$ as a module over $\mathbb{C}[I_\alpha]$ is equal to $||\alpha||$.*

Proof. Since X is a norm curve, the regular function $I_\alpha\colon X \to \mathbb{C}$ is non-constant and hence dominant. The argument in our preamble to Lemma 14 implies the degree of I_α finite even if I_α is not a finite map. Hence, by the Finiteness of Integral Closure [13, Theorem I.3.9A], the integral closure C of $\mathbb{C}[I_\alpha]$ in $\mathbb{C}(X)$ is a finite extension of $\mathbb{C}[I_\alpha]$. Since $\mathbb{C}(X)$ is a finite extension of $\mathbb{C}(I_\alpha)$, the field of fractions of C is equal to $\mathbb{C}(X)$. Finally, the proof of Lemma 14 implies that the rank of C over $\mathbb{C}[I_\alpha]$ is equal to $[\mathbb{C}(X) : \mathbb{C}(I_\alpha)] = ||\alpha||$. □

Next we will show that module structures can distinguish strongly detected boundary slopes, even in the presence of detected closed essential surfaces. As usual, suppose $X \subseteq X(N)$ is a norm curve and recall the regular map $\partial\colon\ X \to X(\partial N)$ from Section 1.7. Since X is a norm curve, the Zariski-closure ∂X of $\partial(X)$ must be an irreducible curve.

For each slope $\alpha \in \mathcal{S}$ there is a regular function $J_\alpha\colon\ \partial X \to \mathbb{C}$ such that $I_\alpha = J_\alpha \circ \partial$ since the regular map ∂ takes a character $\chi \in X$ to its restriction to $\pi_1(\partial N)$. Note that each J_α is a non-constant regular function on ∂X.

Proposition 24. *Fix a slope α on ∂N. $\mathbb{C}[\partial X]$ is a finitely generated $\mathbb{C}[J_\alpha]$-module if and only if α is not strongly detected by X. Moreover, if α is not strongly detected by X, then*

$$\operatorname{rk}{}^{\mathbb{C}}_{\partial X}(\alpha) = \frac{||\alpha||}{\deg \partial}$$

Proof. Suppose α is strongly detected by X. Then there is an ideal point $P \in \mathcal{I}(X)$ such that $I_\alpha(P) \in \mathbb{C}$ and $I_\beta(P) = \infty$ for every other slope β by Proposition 3. The map $\partial\colon\ X \to \partial X$ extends to a regular map between smooth projective models $\widetilde{\partial}\colon\ \widetilde{X} \to \widetilde{\partial X}$ [13, Proposition I.6.8] and our previous observations imply that $Q = \widetilde{\partial}(P)$ must be an ideal point of ∂X with $J_\alpha(Q) \in \mathbb{C}$. In other words, J_α has a hole at Q. By [5, Corollary 2.4], $\mathbb{C}[\partial X]$ is not a finitely generated $\mathbb{C}[J_\gamma]$-module.

Conversely, suppose that $\mathbb{C}[\partial X]$ is a finitely generated $\mathbb{C}[J_\alpha]$-module. If α is strongly detected by X, then, as above, there is an ideal point $P \in \mathcal{I}(X)$ such that $I_\alpha(P) \in \mathbb{C}$ and $Q = \widetilde{\partial}(P)$ must be ideal point of ∂X. Since $I_\alpha = J_\alpha \circ \partial$, J_α must have a hole at Q. This is a contradiction, since our assumptions imply that J_α has no holes. Hence α is not strongly detected by X.

If α is not strongly detected, then $\operatorname{rk}^{\mathbb{C}}_{\partial X}(\alpha) = \deg J_\alpha$ by Lemma 14. By the tower theorem for finite field extensions,

$$||\alpha|| = \deg I_\alpha = [\mathbb{C}(X) : \mathbb{C}(I_\alpha)] = \deg \partial \cdot \deg J_\alpha \quad \Rightarrow \quad \operatorname{rk}^{\mathbb{C}}_{\partial X}(\alpha) = \frac{||\alpha||}{\deg \partial}. \qquad \square$$

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